EINSTEIN-TYPE STRUCTURES, BESSE'S CONJECTURE AND A UNIQUENESS RESULT FOR A φ-CPE METRIC IN ITS CONFORMAL CLASS

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Abstract. In this paper, we study an extension of the CPE conjecture to manifolds M which support a structure relating curvature to the geometry of a smooth map $\varphi : M \to N$. The resulting system, denoted by (φ-CPE), is natural from the variational viewpoint and describes stationary points for the integrated φ-scalar curvature functional restricted to metrics with unit volume and constant φ-scalar curvature. We prove both a rigidity statement for solutions to (φ-CPE) in a conformal class, and a gap theorem characterizing the round sphere among manifolds supporting (φ-CPE) with φ a harmonic map.

1. Introduction

The Critical Point Equation, from now on the CPE equation, is the Euler-Lagrange equation of the Hilbert-Einstein action on the space of Riemannian metrics with unit volume and constant scalar curvature on a compact manifold. It has been introduced, in the attempt to more efficiently identify Einstein metrics, by A. Besse in his treatise, [8], to which we refer for details. From now on $(M, \langle\, ,\, \rangle)$ will denote a connected Riemannian manifold of dimension $m \geq 2$. The CPE equation writes in the form

$$\text{(CPE)} \qquad \mathrm{Hess}(w) - w\left(\mathrm{Ric} - \frac{S}{m-1}\langle\, ,\, \rangle\right) = T$$

for some $w \in C^\infty(M)$ (we shall not be interested in further constraints on w, see [8]). Here, Ric, T and S denote, respectively, the Ricci, the traceless Ricci tensors and the scalar curvature of $(M, \langle\, ,\, \rangle)$.

Besse's conjecture (or at least a version of it) can be stated as follows:

Conjecture 1. *If $(M, \langle\, ,\, \rangle)$ is compact, S is constant and $w \not\equiv -1$ is a smooth solution to* (CPE) *on M, then $(M, \langle\, ,\, \rangle)$ is Einstein.*

Constant w are easily handled, as (CPE) implies that M is Einstein unless $w \equiv -1$, a case for which the constraint provided by (CPE) reduces to $S \equiv 0$. We note that, if $S \not\equiv 0$, the condition $w \not\equiv 0$ is equivalent to w being a non-constant solution of (CPE).

In order to derive (CPE) we have assumed from the very beginning that S is constant, but it is worth to observe that the mere existence of a solution w of (CPE) implies the constancy of S (this will be shown in Proposition 9 below, in a more general setting). Taking this into account, with the aid of a result of Obata [24] we may state the following form of Besse's conjecture, which up to removing the case of constant w is equivalent to the original formulation:

Conjecture 2. *If* $(M, \langle\,,\,\rangle)$ *is compact and* w *is a non-constant solution to* (CPE) *on* M*, then* $(M, \langle\,,\,\rangle)$ *is isometric to a standard sphere.*

Indeed, tracing (CPE) we get

$$\Delta w + \frac{S}{m-1} w = 0, \tag{1}$$

thus integrating by parts yields

$$\int_M \frac{S}{m-1} w^2 = \int_M |\nabla w|^2 .$$

Since w is non-constant and S is constant it follows that $S > 0$. If $(M, \langle\,,\,\rangle)$ is Einstein, (CPE) reduces to

$$\mathrm{Hess}(w) = -\frac{S}{m(m-1)} w \langle\,,\,\rangle ,$$

whence using Theorem A of [24] we obtain that M is isometric to a round sphere.

There are a number of partial results on Besse's conjecture, and we list some of them. Precisely, the conjecture is true if one of the following sets of assumptions is satisfied on M compact:

- i) $\frac{S}{m-1} \not\in \mathrm{Sp}(-\Delta)$ (cf. [8, Proposition 4.47]);
- ii) $(M, \langle\,,\,\rangle)$ is locally conformally flat and the solution w to (CPE) is not unique (Lafontaine [19]). The result was improved by removing the second assumption (LaFontaine-Rozoy [20] for $m = 3$, and Chang-Hwang-Yun [10]);
- iii) $w \ge -1$ (Hwang [17]). It is worth to observe that this result follows from the very interesting identity
$$\mathrm{div}(T(\nabla w, \cdot)^\sharp) = (1 + w)|T|^2$$
and from the fact that, if $w \not\equiv -1$, $\{x \in M : w(x) = -1\}$ has measure 0. Here $^\sharp$ is the musical isomorphism (in Lemma 10 below we shall generalize the above identity);
- iv) $\mathrm{divRiem} = 0$ (Yun-Chang-Hwang [28]);
- v) $(M, \langle\,,\,\rangle)$ is Bach flat (Qing-Yuan [25]);
- vi) $m = 4$ and div $W^+ \equiv 0$, where W^+ is the self-dual part of the Weyl tensor W (Barros-Leandro-Ribeiro [7]);
- vii) $(M, \langle\,,\,\rangle)$ is conformally Einstein (Barros-Evangelista [6]);
- viii) $m \ge 5$ and the radial Weyl curvature $i_{\nabla w} W = 0$ (Baltazar-Barros-Batista-Viana [5]);
- ix) $m \ge 3$ and condition
$$|W| \le \sqrt{\frac{m}{2(m-2)}} \left[\frac{S}{\sqrt{m(m-1)}} - 2|T| \right] \tag{2}$$
holds (Baltazar [4]);
- x) $m = 3$ and $\mathrm{Ric} \ge 0$ (He [15]).

Furthermore, in a very recent preprint, Hwang and Yu [18] showed that the CPE conjecture holds if $\langle\,,\,\rangle$ has positive isotropic curvature.

The CPE equation is strictly related to the vacuum static equation

$$\mathrm{Hess}(w) - w \left(\mathrm{Ric} - \frac{S}{m-1} \langle\,,\,\rangle \right) = 0 , \tag{VSE}$$

where we consider smooth solutions $w \not\equiv 0$. Indeed, observe that if M admits two different solutions w_0, w_1 to (CPE), then for any $t \in \mathbb{R}$ the function $w_t = (1-t)w_0 + tw_1$ solves (CPE) and its t-derivative $w_1 - w_0$ solves (VSE). Although (VSE) can be seen as the Euler-Lagrange equation of an action functional over a certain space of metrics with constant scalar curvature, similarly to (CPE) one verifies that the sole existence of a solution w of (VSE) on M implies that the scalar curvature is constant, see [11].

Suppose $m \geq 3$. A recent result of Herzlich, [16], provides a nice class of solutions to (VSE). Indeed, he shows that if X is a conformal, non-Killing vector field on $(M, \langle \, , \, \rangle)$ and the manifold is Einstein, then

$$w = \mathrm{div} X$$

is a solution to (VSE) (the non-Killing request on X is only made to prevent w from vanishing identically). In fact, when M is compact with $\partial M \neq \emptyset$, assuming the existence of X as above and constancy of S, Miao and Tam, [22], were able to prove, under some further assumptions, that if $\mathrm{div} X$ solves (VSE) then (M, g) is Einstein, providing a partial converse of Herzlich result.

An interesting problem related to the VSE equation is that of the local scalar curvature rigidity; that is, to look for domains Ω in $(M, \langle \, , \, \rangle)$ such that for each metric g inducing the same metric as $\langle \, , \, \rangle$ on $\partial\Omega$ and such that $S_g \geq S_{\langle \, , \, \rangle}$ on Ω, $H_g \geq H_{\langle \, , \, \rangle}$ on $\partial\Omega$, H the mean curvature of $\partial\Omega$ with respect to the inward pointing normal, there exists $\varepsilon > 0$ for which the condition

$$\| g - \langle \, , \, \rangle \|_{C^2(\Omega)} < \varepsilon$$

implies the existence of a diffeomorphism $\psi : \overline{\Omega} \to \overline{\Omega}$ with the property that $\langle \, , \, \rangle = \psi^* g$ and $\psi \equiv \mathrm{id}$ on $\partial\Omega$.

The problem is closely related to the well known conjecture of Min-Oo on $\mathbb{S}^m_+$, which in its full generality was disproved by Brendle, Marques and Neves, [9]. However, Hang and Wang, [14], obtained a positive answer to a weaker form of Min-Oo's conjecture, proving the scalar curvature rigidity among conformal metrics for the round hemisphere $\mathbb{S}^m_+$. The result has been recently extended by Qing and Yuan, [26], to a manifold with a solution of (VSE), that they more simply call a vacuum static space. To be more precise they proved

Theorem 3. *Let (M, g) be a complete vacuum static space with $w \not\equiv 0$ solution to* (VSE) *and scalar curvature $S_g > 0$. Assume that the open set*

$$\Omega_+ = \{x \in M : w(x) > 0\} \subseteq M$$

is relatively compact. If $\tilde{g}$ is conformal to g on M and it satisfies

$$\begin{cases} S_{\tilde{g}} \geq S_g & \text{on } \Omega_+ \\ \tilde{g} \equiv g & \text{restricted to } T\partial\Omega_+ \\ H_{\tilde{g}} \geq H_g & \text{on } \partial\Omega_+, \end{cases}$$

then $\tilde{g} \equiv g$.

Furthermore, they also prove that Ω_+ is "maximal" for the validity of the result.

Recent years saw a rising interest in manifolds whose curvatures relate to properties of a smooth map $\varphi : (M, \langle \, , \, \rangle) \to (N, \langle \, , \, \rangle_N)$ into a target Riemannian space. One of the first instances of such interplay is the work of Buzano [23], where the author investigated the Ricci flow coupled with the harmonic map flow. Solitons for the flow are characterized by the system

$$\begin{cases} \mathrm{Ric}^\varphi + \mathrm{Hess}(f) = \lambda\langle\, ,\,\rangle \\ \tau(\varphi) = \mathrm{d}\varphi(\nabla f), \end{cases} \tag{3}$$

where $\alpha, \lambda \in \mathbb{R}$, $\tau(\varphi)$ is the tension field of φ (see for instance [12]) and Ric^φ is the φ-Ricci tensor

$$\mathrm{Ric}^\varphi = \mathrm{Ric} - \alpha\varphi^*\langle\, ,\,\rangle_N\,. \tag{4}$$

See also Wang [27] for related results. For constant f, the above reduces to the Harmonic-Einstein system

$$\begin{cases} \mathrm{Ric}^\varphi = \lambda\langle\, ,\,\rangle \\ \tau(\varphi) = 0 \end{cases} \tag{5}$$

which extends the notion of Einstein manifolds to possibly nonconstant φ (as in the Einstein case, by [2, Proposition 2.15], if $m \geq 3$ then λ is necessarily constant). The interest in (5) is made even more evident if we rewrite the first identity as

$$G + \Lambda\langle\, ,\,\rangle = \alpha\overline{T}, \tag{6}$$

where G is the Einstein tensor of M,

$$\Lambda = \frac{m-2}{2}\lambda$$

and $\overline{T}$ is the stress-energy tensor[1] of the map φ:

$$\overline{T} \doteq \varphi^*\langle\, ,\,\rangle_N - \frac{|\mathrm{d}\varphi|^2}{2}\langle\, ,\,\rangle\,.$$

Notice that we did not use the fact that $\langle\, ,\,\rangle$ is Riemannian. Hence, in a Lorentzian setting, solutions to (5) with $\alpha > 0$ correspond to solutions to the Einstein field equation with cosmological constant Λ and source the wave map φ, up to a normalization constant. The fact that the left hand side of (6) is divergence free forces $\overline{T}$ to be divergence free as well, which is equivalent to the vanishing of the 1-form $\langle\tau(\varphi), \mathrm{d}\varphi\rangle_N$. The harmonicity of φ is then a sufficient condition for the compatibility of the system. Other examples and more detailed discussions can be found in [1, 2],

In what follows, we shall investigate the CPE problem in the more general setting just mentioned. To properly define the system corresponding to (CPE), first recall the obvious definitions of the φ-scalar curvature and traceless φ-Ricci tensor:

$$S^\varphi \doteq \mathrm{Tr}\,\mathrm{Ric}^\varphi = S - \alpha|\mathrm{d}\varphi|^2, \qquad T^\varphi \doteq \mathrm{Ric}^\varphi - \frac{S^\varphi}{m}\langle\, ,\,\rangle\,. \tag{7}$$

Associated to φ and α, further "curvature" tensors that we shall call φ-curvatures will be introduced below at due time. For more information we refer to [1, 2, 21] where we justify the various concepts and prove a number of results.

[1] Notice that in [2], after equation (1.5), there is a typo in the definition of the stress-energy tensor.

We formally introduce the φ-CPE equation, that we shall also call a φ-CPE structure, by requiring the existence of $w \in C^\infty(M)$ solving

$$(\varphi\text{-CPE}) \qquad \begin{cases} \operatorname{Hess}(w) - w\left(\operatorname{Ric}^\varphi - \frac{S^\varphi}{m-1}\langle\, ,\,\rangle\right) = T^\varphi \\ (1+w)\tau(\varphi) = -\mathrm{d}\varphi(\nabla w) \end{cases}$$

The system (φ-CPE) will be justified from a variational viewpoint in Section 2. We remark that, as in the case of constant φ, the mere validity of (φ-CPE) implies that S^φ is constant, see Proposition 9 below. If $w > -1$, performing the change of variable

$$f = -\log(1+w) \tag{8}$$

(φ-CPE) becomes equivalent to the Einstein-type structure

$$\begin{cases} \operatorname{Ric}^\varphi + \operatorname{Hess}(f) - \mu \mathrm{d}f \otimes \mathrm{d}f = \lambda\langle\, ,\,\rangle \\ \tau(\varphi) = \mathrm{d}\varphi(\nabla f) \end{cases} \tag{9}$$

with the choices $\mu = 1$ and

$$\lambda(x) = \frac{S^\varphi}{m-1}\left(1 - \frac{e^f}{m}\right).$$

Similarly, if $w > 0$ the φ-VSE equation

$$(\varphi\text{-VSE}) \qquad \begin{cases} \operatorname{Hess}(w) - w\left(\operatorname{Ric}^\varphi - \frac{S^\varphi}{m-1}\langle\, ,\,\rangle\right) = 0 \\ w\tau(\varphi) = -\mathrm{d}\varphi(\nabla w) \end{cases}$$

falls into the class (9) with the choices

$$f = -\log w, \qquad \mu = 1, \qquad \lambda(x) = -\frac{S^\varphi}{m-1}\frac{e^f}{m}.$$

The importance of the general Einstein-type structure (9) is evident. For instance it describes, as special cases, Ricci-harmonic solitons, Ricci solitons, generalized quasi-Einstein manifolds for $\mu = \mu(x)$ and $\lambda = \lambda(x)$, and so on. Moreover it appears quite naturally in a number of physics problems, see [1, 2, 21].

As expected the validity of (9) does not imply, in general, that S^φ is constant. Of course it does in some special cases, notably when $m \geq 3$ and (9) reduces to the harmonic-Einstein system (5). This parallels Schur's Theorem and is a simple but revealing instance pointing out that the theory of harmonic-Einstein manifolds, that is, of those Riemannian manifolds supporting a solution to (5), has many analogies with that of Einstein manifolds. For results in this direction we refer to [2].

Our first Theorem relates to Theorem 3. To state it we need some further piece of notation. Given a metric g on M we set $[g]$ to denote its conformal class. If $\varphi : (M, g) \to (N, \langle\, ,\,\rangle_N)$ and $\tilde{g} \in [g]$ we denote with a tilde quantities referred to $\tilde{g}$, and we let

$$\tilde{\varphi} \ : \ (M, \tilde{g}) \to (N, \langle\, ,\,\rangle_N), \qquad \tilde{\varphi}(x) \doteq \varphi(x).$$

Theorem 4. *Let (M, g) be a complete manifold of dimension $m \geq 3$ possessing an Einstein-type structure as in* (9) *with $\mu \in \mathbb{R}^+$ and $\lambda = \lambda(x) \in C^\infty(M)$. Assume $S^\varphi \geq 0$, and let*

$$\Omega_f = \{x \in M : f(x) < 0\}; \tag{10}$$

assume that $0 \in \mathbb{R}$ *is a regular value of* f*. Let* $\tilde{g} \in [g]$ *satisfy*

$$\begin{gathered} i)\ \tilde{S}^{\tilde{\varphi}} \geq S^{\varphi}\,; \qquad ii)\ \tilde{g} \equiv g \quad \text{restricted to } T\partial\Omega_f \\ iii)\ H_{\tilde{g}} \geq H_g \quad \text{on } \partial\Omega_f, \text{ with equality at least for one point of } \partial\Omega_f, \end{gathered} \tag{11}$$

where $H_{\tilde{g}}$ *and* H_g *are the mean curvatures of* $\partial\Omega_f \hookrightarrow \overline{\Omega_f}$ *in the direction of the inward pointing normal. Finally, let*

$$\lambda(x) \geq \frac{S^{\varphi}}{\mu m(m-1)}\left[1 + \mu(m-1) - e^{\mu f}\right] + \varepsilon \tag{12}$$

for some $\varepsilon \geq 0$*. Fix an origin* $o \in \Omega_f$ *and let* $B_r = \{x \in M : \mathrm{dist}_g(x,o) < r\}$*. Then,* $\tilde{g} \equiv g$ *on* $\overline{\Omega_f}$ *provided that either*

$$\varepsilon > 0 \qquad \text{and} \qquad \liminf_{r\to\infty} \frac{\log|B_r \cap \Omega_f|}{r} = 0 \tag{13}$$

or

$$\varepsilon = 0 \qquad \text{and} \qquad \liminf_{r\to\infty} \frac{|B_r \cap \Omega_f|}{r^2} = 0\,. \tag{14}$$

Note that, for $\mu = 1$ and

$$f = -\log(1+w), \qquad \lambda(x) = \frac{S^{\varphi}}{m-1}\left(1 - \frac{e^f}{m}\right)$$

we are exactly in the case of the φ-CPE equation and (12) is satisfied with $\varepsilon = 0$. Thus we have

Corollary 5. *Let* (M, g) *be a complete manifold of dimension* $m \geq 3$ *possessing a* φ*-CPE structure* (φ-CPE) *with* $S^{\varphi} \geq 0$*. Let*

$$\Omega_+ = \{x \in M : w(x) > 0\}\,,$$

and suppose that $0 \in \mathbb{R}$ *is a regular value of* w*. For* $\tilde{g} \in [g]$ *assume the validity of* (11) *with* $\partial\Omega_f$ *replaced by* $\partial\Omega_+$ *and suppose that, for a fixed origin* $o \in \Omega_+$,

$$\liminf_{r\to\infty} \frac{|B_r \cap \Omega_+|}{r^2} = 0\,,$$

where $B_r = \{x \in M : \mathrm{dist}_g(x,o) < r\}$*. Then,* $\tilde{g} \equiv g$ *on* $\overline{\Omega_+}$*.*

A result corresponding to Corollary 5 can be formulated for (φ-VSE), improving on Theorem 3. We leave the statement and details to the interested reader.

Remark 6. Strictly speaking Corollary 5 is a consequence of Theorem 4 in case $w > -1$ due to the change of variable (8). However, the same argument of the proof of Theorem 4 applies directly *mutatis mutandis* to Corollary 5 without requiring that $w > -1$.

We note that in Theorem 4 the open set Ω_f, corresponding to Ω_+ in Theorem 3, is not required to be relatively compact. Although the idea of the proof of Theorem 4 is basically the same of that of Theorem 3 the main difference, and in fact a bit of a subtle technical point, is that of avoiding relative compactness of Ω_f. Our aim is achieved by using a special case of an analytic result of independent interest, see Lemma 8 below.

The second theorem we are going to prove extends Baltazar's recent result mentioned above, [4], to the system (φ-CPE). At the same time, we streamline part of the proof, highlighting the role played by a Kazdan-Warner type identity in Lemma 10 below. We introduce the φ-Weyl tensor

$$W^{\varphi} = \mathrm{Riem} - \frac{1}{m-2} A^{\varphi} \owedge \langle\, , \,\rangle\,, \quad m \geq 3 \tag{15}$$

where Riem is the Riemann tensor, $\owedge$ is the "parrot" (Kulkarni-Nomizu) product and A^{φ} is the φ-Schouten tensor

$$A^{\varphi} = \mathrm{Ric}^{\varphi} - \frac{S^{\varphi}}{2(m-1)} \langle\, , \,\rangle\,. \tag{16}$$

We can express W^{φ} in terms of the usual Weyl tensor W via the equation

$$W^{\varphi} = W + \frac{\alpha}{m-2} F \owedge \langle\, , \,\rangle\,, \quad m \geq 3 \tag{17}$$

with

$$F = \varphi^{*}\langle\, , \,\rangle_N - \frac{|\mathrm{d}\varphi|^2}{2(m-1)} \langle\, , \,\rangle\,. \tag{18}$$

Note that the φ-Weyl tensor has the same symmetries of Riem, in particular, it satisfies the first Bianchi identity. However, in general it is not totally trace free.

We are ready to state our second main result. Notice that, for constant φ, (19) below becomes condition (2).

Theorem 7. *Let $(M, \langle\, , \,\rangle)$ be a compact manifold of dimension $m \geq 3$ with a φ-CPE structure as in (φ-CPE) for some non-constant function w and some $\alpha \in \mathbb{R}$. Assume that $\tau(\varphi) = 0$ and that*

$$\frac{S^{\varphi}}{2(m-1)} - \frac{\alpha}{m-2} |\mathrm{d}\varphi|^2 \geq \sqrt{\frac{m-2}{2(m-1)}} |W^{\varphi}| + \sqrt{\frac{m}{m-1}} |T^{\varphi}| \tag{19}$$

on M. Then $(M, \langle\, , \,\rangle)$ is isometric to the standard sphere $\mathbb{S}^m(\kappa) \subseteq \mathbb{R}^{m+1}$ of constant sectional curvature

$$\kappa = \frac{S^{\varphi}}{m(m-1)}\,. \tag{20}$$

Moreover, if $\alpha \neq 0$ then φ is constant.

2. Variational derivation of (φ-CPE)

We let M be compact, and denote with $\mathscr{M}$ be the set of smooth Riemannian metrics on M, endowed with the compact open C^{∞} topology. We also fix $(N, \langle\, , \,\rangle_N)$ and denote with $\mathscr{F}$ the set of smooth maps $\varphi : M \to N$, again with the compact open C^{∞} topology. We consider the functional $\mathcal{S} : \mathscr{M} \times \mathscr{F} \to \mathbb{R}$ given by

$$\mathcal{S}(g, \varphi) = \int_M S_g^{\varphi} \mathrm{d}x_g,$$

where $\mathrm{d}x_g$, S_g^{φ} are the Riemannian volume and the φ-scalar curvature of g. In this section, we explain how (φ-CPE) can be seen as the Euler-Lagrange equation of $\mathcal{S}$ restricted to the subset of metrics and maps (g, φ) with unit volume and constant φ-scalar curvature S_g^{φ}. Our treatment parallels the one in [8, pp. 127-128]. To avoid technicalities, we keep the discussion at an informal level and do not describe the function spaces used to justify the properties of the operators which we are going to consider. Given $(h,v) \in T_{(g,\varphi)}(\mathscr{M}\times\mathscr{F}) = T_g\mathscr{M}\times T_\varphi\mathscr{F}$ (notice that $T_\varphi\mathscr{F}$ can be identified with sections of φ^*TN via the exponential map), it holds

$$\Big(\mathrm{d}_{(g,\varphi)}\mathcal{S}\Big)[(h,v)] = \int_M \Big[\dot{S}^\varphi_g(h,v)\mathrm{d}x_g + S^\varphi_g \dot{\mathrm{d}x_g}(h,v) + (S^\varphi_g)'(h,v)\mathrm{d}x_g\Big]\,,$$

where the dot and prime symbols denote, respectively, differentiation with respect to g and φ at the point (g,φ). Direct computations (cf. Propositions 5.4 and 5.16 in [1]) give

$$\begin{aligned}
\dot{S}^\varphi_g(h,v) &= -\Delta_g\big(\operatorname{Tr}_g h\big) + \operatorname{div}_g\big(\operatorname{div}_g h\big) - \langle h, \operatorname{Ric}^\varphi_g\rangle_g \\
\dot{\mathrm{d}x_g}(h,v) &= \tfrac{1}{2}\operatorname{Tr}_g h\,\mathrm{d}x_g \\
(S^\varphi)'(h,v) &= -\alpha(|\mathrm{d}\varphi|^2_g)'(h,v) = -2\alpha\operatorname{div}_g\big(\langle \mathrm{d}\varphi, v\rangle_N\big) + 2\alpha\langle\tau(\varphi),v\rangle_N,
\end{aligned}$$

whence, integrating by parts,

$$\Big(\mathrm{d}_{(g,\varphi)}\mathcal{S}\Big)[(h,v)] = -\int_M \langle \operatorname{Ric}^\varphi_g - \frac{S^\varphi_g}{2}g, h\rangle_g \mathrm{d}x_g + 2\alpha\int_M \langle\tau_g(\varphi),v\rangle_N\mathrm{d}x_g.$$

Let $\mathscr{M}_1 \subset \mathscr{M}$ be the subset of metrics g with $\operatorname{vol}_g(M)=1$, and let $\mathscr{G}\subset\mathscr{M}_1\times\mathscr{F}$ be the set of pairs (g,φ) for which S^φ_g is constant. If (h,v) generates a variation (g_t,φ_t) of (g,φ) for which, up to first order, the scalar curvature $S^{\varphi_t}_{g_t}$ is constant on M for each t,

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} S^{\varphi_t}_{g_t} &= \Big(\mathrm{d}_{(g,\varphi)}S^\varphi_g\Big)[(h,v)] \\
&= -\Delta_g\big(\operatorname{Tr}_g h\big) + \operatorname{div}_g\big(\operatorname{div}_g h\big) - \langle h,\operatorname{Ric}^\varphi_g\rangle_g - 2\alpha\operatorname{div}_g\big(\langle\mathrm{d}\varphi,v\rangle_N\big) + 2\alpha\langle\tau(\varphi),v\rangle_N
\end{aligned}$$

must be constant on M, equivalently,

$$\beta_{(g,\varphi)}[(h,v)] \doteq \Delta_g\Big(\Big(\mathrm{d}_{(g,\varphi)}S^\varphi_g\Big)[(h,v)]\Big) = 0.$$

Therefore, at least formally, $T_{(g,\varphi)}\mathscr{G}$ can be seen as the set of pairs

$$(h,v)\in\ker\beta_{(g,\varphi)}\cap\big(T_g\mathscr{M}_1\times T_\varphi\mathscr{F}\big), \tag{21}$$

where $T_g\mathscr{M}_1 = \{h'\in S^2(M) : \operatorname{Tr}_g h' = 0\}$. Computing the adjoint map

$$\beta^*_{(g,\varphi)}\ :\ C^\infty(M)\to T_g\mathscr{M}_1\times T_\varphi\mathscr{F},$$

we get

$$\begin{aligned}
\int_M \langle\beta^*_{(g,\varphi)}(\eta),(h,v)\rangle_g &= \int_M \eta\beta_{(g,\varphi)}[(h,v)]\mathrm{d}x_g \\
&= \int_M \Delta_g\eta\Big(\dot{S}^\varphi_g(h,v) + (S^\varphi_g)'(h,v)\Big)\mathrm{d}x_g \\
&= \int_M \langle h, -(\Delta_g\Delta_g\eta)g + \operatorname{Hess}_g(\Delta_g\eta) - (\Delta_g\eta)\operatorname{Ric}^\varphi_g\rangle_g\mathrm{d}x_g \\
&\quad -2\alpha\int_M \operatorname{div}_g\big(\langle\mathrm{d}\varphi,v\rangle_N\big)\Delta_g\eta\mathrm{d}x_g + 2\alpha\int_M\langle\tau_g(\varphi),v\rangle_N\Delta_g\eta\mathrm{d}x_g \\
&= \int_M \langle h, -(\Delta_g\Delta_g\eta)g + \operatorname{Hess}_g(\Delta_g\eta) - (\Delta_g\eta)\operatorname{Ric}^\varphi_g\rangle_g\mathrm{d}x_g \\
&\quad +2\alpha\int_M\langle\mathrm{d}\varphi(\nabla\Delta_g\eta) + (\Delta_g\eta)\tau_g(\varphi),v\rangle_N\mathrm{d}x_g,
\end{aligned}$$

and therefore

$$\beta^*_{(g,\varphi)}(\eta) = \Big(-(\Delta_g w)g + \mathrm{Hess}_g(w) - w\mathrm{Ric}^\varphi_g, 2\alpha\big(\mathrm{d}\varphi(\nabla w) + w\tau_g(\varphi)\big)\Big)$$

where $w = \Delta_g \eta$. The tangent space $T_{(g,\varphi)}\big(\mathscr{M}_1 \times \mathscr{F}\big)$ decomposes as

$$T_g\mathscr{M}_1 \times T_\varphi\mathscr{F} = \Big(\ker \beta_{(g,\varphi)} \cap \big(T_g\mathscr{M}_1 \times T_\varphi\mathscr{F}\big)\Big) \oplus^\perp \mathrm{Im}\beta^*_{(g,\varphi)}\,.$$

Taking into account that for variations $(h, v) \in T_g\mathscr{M}_1 \times T_\varphi\mathscr{F}$ it holds

$$\Big(\mathrm{d}_{(g,\varphi)}\mathcal{S}\Big)[(h,v)] = -\int_M \langle T^\varphi, h\rangle_g \mathrm{d}x_g + 2\alpha\int_M \langle \tau_g(\varphi), v\rangle_N \mathrm{d}x_g,$$

then (g,φ) is critical for $\mathcal{S}$ with respect to variations satisfying (21) if and only if $(T^\varphi, -2\alpha\tau_g(\varphi)) \in \mathrm{Im}\beta^*_{(g,\varphi)}$, that is, if and only if there exists $w \in C^\infty(M)$ that can be written as $w = \Delta_g\eta$ (equivalently, w has mean value zero on (M,g)) and satisfies (φ-CPE).

3. Proof of Theorem 4

We shall make use of the following Liouville type theorem.

Lemma 8. *Let $(M, \langle\,,\,\rangle)$ be a complete (connected) manifold and $\Omega \subseteq M$ be an open set with non-empty boundary. Let $w \in C^1(\Omega)$, $w > 0$ and let $\psi \in C^2(\Omega)$ satisfy*

$$\text{(22)}\qquad \begin{cases} \Delta\psi + 2\langle\nabla \log w, \nabla\psi\rangle \geq c\psi & \text{on } \Omega_0 = \{x \in \Omega : \psi(x) > 0\} \neq \emptyset \\ \limsup\limits_{x\to\partial\Omega} \psi(x) \leq 0\,, \end{cases}$$

for some constant $c \geq 0$. Then, the following holds:

$$\text{(23)}\qquad \text{if } c > 0\,, \text{ then} \qquad \liminf_{r\to\infty} \frac{1}{r}\log\left(\int_{B_r\cap\Omega} w^2\psi_+^2\right) > 0\,;$$

$$\text{(24)}\qquad \text{if } c = 0\,, \text{ then} \qquad \liminf_{r\to\infty} \frac{1}{r^2}\int_{B_r\cap\Omega} w^2\psi_+^2 > 0\,.$$

Proof. Choose $\varepsilon > 0$ small enough so that

$$\Omega_\varepsilon = \{x \in \Omega : \psi(x) > \varepsilon\}$$

is non-empty, and define

$$\psi_\varepsilon = (\psi - \varepsilon)_+ = \max\{\psi - \varepsilon, 0\}\,.$$

Note that $\mathrm{supp}\psi_\varepsilon = \overline{\Omega_\varepsilon}$ does not meet $\partial\Omega$ by the second condition in (22). Since $\Omega_\varepsilon \neq \emptyset$, we can also fix $R_0 > 0$ large enough so that $B_{R_0} \cap \Omega_\varepsilon \neq \emptyset$. Given $R > R_0$, let η be a cut-off function such that

$$\eta \equiv 1 \text{ on } B_R\,, \qquad \eta \equiv 0 \text{ on } M \setminus B_{2R}\,, \qquad |\nabla\eta| \leq \frac{1}{R} \text{ on } M.$$

Let $\alpha \geq 1$ to be suitably chosen later, and consider the vector field

$$Z = \eta^{2\alpha}\psi_\varepsilon w^2 \nabla\psi\,.$$

Using the first in (22) and the fact that $\psi_\varepsilon \leq \psi$ on $\Omega_\varepsilon \subseteq \Omega_0$, we compute

$$\mathrm{div}Z \geq 2\alpha\eta^{2\alpha-1}\psi_\varepsilon w^2\langle\nabla\psi, \nabla\eta\rangle + \eta^{2\alpha}w^2|\nabla\psi|^2\mathbf{1}_{\Omega_\varepsilon} + c\eta^{2\alpha}\psi_\varepsilon^2 w^2$$

and since

$$|2\alpha\eta^{2\alpha-1}\psi_\varepsilon w^2\langle\nabla\psi,\nabla\eta\rangle| \le 2\alpha^2 w^2\eta^{2\alpha-2}\psi_\varepsilon^2|\nabla\eta|^2 + \frac{1}{2}\eta^{2\alpha}w^2|\nabla\psi|^2\mathbf{1}_{\Omega_\varepsilon}$$

from the above we deduce

$$\mathrm{div}Z \ge -2\alpha^2\eta^{2\alpha-2}\psi_\varepsilon^2 w^2|\nabla\eta|^2 + \frac{1}{2}\eta^{2\alpha}w^2|\nabla\psi|^2\mathbf{1}_{\Omega_\varepsilon} + c\eta^{2\alpha}\psi_\varepsilon^2 w^2 \,.$$

Note that Z has support contained in $\overline{B_{2R}\cap\Omega_\varepsilon}$. In particular, Z is compactly supported in the interior of Ω. We integrate on Ω and we use Hölder's inequality to obtain

(25)
$$\begin{aligned} c\int_\Omega \eta^{2\alpha}\psi_\varepsilon^2 w^2 + \frac{1}{2}\int_{\Omega_\varepsilon}\eta^{2\alpha}w^2|\nabla\psi|^2 \quad &\le \quad 2\alpha^2\int_\Omega \eta^{2\alpha-2}\psi_\varepsilon^2 w^2|\nabla\eta|^2 \\ &\le \quad 2\alpha^2\left(\int_\Omega \psi_\varepsilon^2 w^2\eta^{2\alpha}\right)^{\frac{\alpha-1}{\alpha}}\left(\int_\Omega \psi_\varepsilon^2 w^2|\nabla\eta|^{2\alpha}\right)^{\frac{1}{\alpha}} \,. \end{aligned}$$

Case $c>0$.
Having chosen R_0 such that $B_{R_0}\cap\Omega_\varepsilon \neq \emptyset$ and $R>R_0$, we have

$$\int_\Omega \psi_\varepsilon^2 w^2\eta^{2\alpha} \ge \int_{B_R\cap\Omega_\varepsilon}\psi_\varepsilon^2 w^2 > 0$$

and using the properties of η we deduce

$$\left(\frac{c}{2\alpha^2}\right)^\alpha\int_{B_R\cap\Omega_\varepsilon}\psi_\varepsilon^2 w^2 \le \int_\Omega \psi_\varepsilon^2 w^2|\nabla\eta|^{2\alpha} \le \frac{1}{R^{2\alpha}}\int_{B_{2R}\cap\Omega_\varepsilon}\psi_\varepsilon^2 w^2.$$

Defining

$$I(R) = \int_{B_R\cap\Omega_\varepsilon}\psi_\varepsilon^2 w^2,$$

we deduce the recursive relation

$$\left(\frac{2\alpha^2}{cR^2}\right)^\alpha I(2R) \ge I(R) \ge I(R_0) > 0\,.$$

We pass to logarithms with $r=2R$. Then

$$\log I(r) + \alpha\log\left(\frac{8\alpha^2}{cr^2}\right) \ge \log I(R_0)\,. \tag{26}$$

For any $r>2R_0$, this inequality holds for any $\alpha\ge 1$. For every

$$r > R_1 := \max\left\{2R_0, \frac{8}{\sqrt{c}}\right\}$$

we can choose

$$\alpha = \frac{r\sqrt{c}}{4} > 2\,,$$

so that

$$\alpha\log\left(\frac{8\alpha^2}{cr^2}\right) = -\frac{\sqrt{c}\log 2}{4}r\,.$$

With this choice of α, dividing both sides of (26) by r we get

$$\frac{1}{r}\log\int_{B_r\cap\Omega_\varepsilon}\psi_\varepsilon^2 w^2 \ge \frac{1}{r}\log I(R_0) + \frac{\sqrt{c}\log 2}{4} \qquad \forall\, r > R_1\,.$$

Since $0 \le \psi_\varepsilon \le \psi_+$ on M, we also have

$$\frac{1}{r}\log\int_{B_r\cap\Omega}\psi_+^2 w^2 \ge \frac{1}{r}\log I(R_0) + \frac{\sqrt{c}\log 2}{4} \qquad \forall\, r > R_1$$

and we obtain (23).

Case $c = 0$.
We consider (25) with $\alpha = 1$, which by our definition of η simplifies to

$$\frac{1}{2}\int_{\Omega_\varepsilon\cap B_R} w^2|\nabla\psi|^2 \le \frac{2}{R^2}\int_{\Omega_\varepsilon\cap B_{2R}}\psi_\varepsilon^2 w^2. \tag{27}$$

Assume by contradiction that (24) does not hold, that is, the liminf is zero. Letting $R \to \infty$ along a sequence $\{R_j\}$ such that $\{2R_j\}$ realizes the liminf, we deduce from (27) that $|\nabla\psi| \equiv 0$ on Ω_ε. An open-closed argument together with the definition of Ω_ε implies that either Ω_ε is empty or Ω_ε has no boundary and ψ is constant. The second possibility does not occur since $\partial\Omega_\varepsilon \ne \emptyset$ because of the boundary condition in (22). □

Proof of Theorem 4. We divide the reasoning into three steps.

Step 1. We let $u \in C^\infty(M)$, $u > 0$ be such that $\tilde{g} = u^{\frac{4}{m-2}}g$. Then, by [1], under the above conformal change of metric, we have the validity of

$$c_m\Delta u - S^\varphi u + \tilde{S}^{\tilde{\varphi}} u^{\frac{m+2}{m-2}} = 0 \tag{28}$$

with $c_m = 4\frac{m-1}{m-2}$. We define

$$v = 1 - u \tag{29}$$

and using the above, together with (11) i) we compute

$$c_m\Delta v = -S^\varphi u + \tilde{S}^{\tilde{\varphi}} u^{\frac{m+2}{m-2}} \ge -S^\varphi u\left(1 - u^{\frac{4}{m-2}}\right) = -c_m\Lambda(x)v \tag{30}$$

on Ω_f, where we set

$$\Lambda(x) = \begin{cases} \dfrac{S^\varphi u(x)[1 - u^{\frac{4}{m-2}}(x)]}{c_m[1-u(x)]} & \text{if } u(x) \ne 1 \\ \dfrac{S^\varphi}{m-1} & \text{if } u(x) = 1. \end{cases} \tag{31}$$

Notice that $\Lambda \in C^0(\overline{\Omega_f})$, and that, since $S^\varphi \ge 0$,

$$\Lambda(x) \le \frac{S^\varphi}{m-1} \qquad \text{on the set } \{v > 0\}. \tag{32}$$

To see this, simply study the function

$$y(t) = \frac{t(t^{\frac{4}{m-2}} - 1)}{c_m(t-1)} \qquad \text{on } (0,1)\,.$$

On the other hand, the validity of (11) ii) gives $v \equiv 0$ on $\partial\Omega_f$. Also, the well-known formula

$$u^{\frac{m}{m-2}}H_{\tilde{g}} = uH_g + \frac{2}{m-2}\frac{\partial u}{\partial\nu},$$

where ν is the inward pointing unit normal to $\partial\Omega_f \hookrightarrow \overline{\Omega_f}$ in the metric g, together with $u \equiv 1$ on $\partial\Omega_f$ and iii) in (11) imply

$$\frac{\partial v}{\partial \nu} = -\frac{\partial u}{\partial \nu} = \frac{m-2}{2}(H_g - H_{\tilde{g}}) \le 0 \qquad \text{on } \partial\Omega_f,$$

with equality holding at some point. Summarizing, we have

$$\begin{cases} \Delta v + \Lambda(x) v \ge 0 & \text{on } \Omega_f \\ v \equiv 0 & \text{on } \partial\Omega_f \\ \frac{\partial v}{\partial \nu} \le 0 & \text{on } \partial\Omega_f \\ \frac{\partial v}{\partial \nu}(x_0) = 0 & \text{for some } x_0 \in \partial\Omega_f\,. \end{cases} \tag{33}$$

Our goal is to prove that $v \equiv 0$ on Ω_f, which implies $u \equiv 1$ and thus $\tilde{g} = g$. To achieve the goal, we first prove that $v \le 0$ on Ω_f.

Step 2. Let

$$w = e^{-\mu f} - 1$$

and observe that $w > 0$ on Ω_f, $w \equiv 0$ on $\partial\Omega_f$. Define

$$\zeta = \frac{v}{w} \qquad \text{on } \Omega_f\,. \tag{34}$$

By the Einstein-type structure we deduce

$$\mathrm{Hess}(w) = \mu(1+w)(\mathrm{Ric}^\varphi - \lambda(x)\langle\,,\,\rangle) \tag{35}$$

so that, tracing, we obtain

$$\Delta w = \mu(S^\varphi - m\lambda(x))(1+w)\,. \tag{36}$$

Using (33) and (61) we infer

$$\begin{aligned} \Delta\zeta + 2\langle\nabla \log w, \nabla\zeta\rangle &= \frac{\Delta v}{w} - \frac{v}{w^2}\Delta w \\ &\ge \left[\frac{\mu(1+w)}{w}(m\lambda(x) - S^\varphi) - \Lambda(x)\right]\zeta \\ &= \frac{\mu(1+w)}{w}\left[m\lambda(x) - S^\varphi - \Lambda(x)\frac{1-e^{\mu f}}{\mu}\right]\zeta \qquad \text{on } \Omega_f\,. \end{aligned}$$

On the set

$$\Omega_\zeta = \{x \in \Omega_f : \zeta(x) > 0\} \equiv \Omega_f \cap \{v > 0\}$$

we have the validity of (32), hence from $\mu(1+w)/w \ge \mu$ we deduce

$$\Delta\zeta + 2\langle\nabla \log w, \nabla\zeta\rangle \ge \mu m\varepsilon\zeta \qquad \text{on } \Omega_\zeta\,.$$

Step 3. To apply Lemma 8, we shall prove that

$$\limsup_{x\to\partial\Omega_f} \zeta(x) \le 0\,.$$

This can be seen as a consequence de l'Hôpital's rule, noting that

$$v \equiv w \equiv 0, \quad \frac{\partial v}{\partial \nu} \le 0 \qquad \frac{\partial w}{\partial \nu} = -\mu\frac{\partial f}{\partial \nu} > 0 \qquad \text{on } \partial\Omega_f\,.$$

Indeed, fix $z \in \partial\Omega_f$. Since $\partial\Omega_f$ is a regular hypersurface in M, we can find a neighbourhood Σ of z in $\partial\Omega_f$ and $\delta > 0$ such that the map

$$\Psi : \Sigma \times (-\delta, \delta) \to M : (x,t) \mapsto \exp_x(t\nu)$$

is a local diffeomorphism onto a neighbourhood U of z in M, with $U \cap \Omega_f = \Psi(\Sigma \times (0,\delta))$. Since $\partial_\nu w > 0$ on $\partial\Omega$, we can fix $\delta > 0$ such that

$$\frac{\partial}{\partial t} w(\Psi(x,t)) \neq 0 \qquad \text{on } \Sigma \times (0,\delta)\,.$$

Let $\{z_n\} = \{\Psi(x_n,t_n)\} \subseteq U$ be a sequence converging to z as $n \to \infty$. Since $v \equiv 0$ and $w \equiv 0$ on Σ, by Cauchy's mean value theorem for every n we can find $\tau_n \in (0,t_n)$ such that

$$\zeta(z_n) = \frac{v(z_n)}{w(z_n)} = \frac{v(\Psi(x_n,t_n)) - v(\Psi(x_n,0))}{w(\Psi(x_n,t_n)) - w(\Psi(x_n,0))} = \frac{\partial_t v(\Psi(x_n,\tau_n))}{\partial_t w(\Psi(x_n,\tau_n))}\,.$$

Letting $n \to \infty$ we have $\Psi(x_n,\tau_n) \to z$ and then

$$\lim_{n\to\infty} \zeta(z_n) = \frac{\partial_\nu v(z)}{\partial_\nu w(z)} \leq 0$$

as claimed. Thus, applying Lemma 8, we have either the validity of (23) or (24). But since $w^2\zeta_+^2 = v_+^2 < 1$ this contradicts, respectively, assumption (13) or (14). Hence, $v \leq 0$ on Ω_f. To conclude, assume by contradiction that $v \not\equiv 0$ on Ω_f. We are in the position to apply Hopf's boundary point lemma, since the maximum of v is 0 and v satisfies the differential inequality in (33). It follows that, for any fixed $x_0 \in \partial\Omega_f$, $\frac{\partial v}{\partial \nu}(x_0) < 0$, contradicting the fourth condition in (33). $\square$

4. Proof of Theorem 7

We first prove that a φ-CPE structure has necessarily constant scalar curvature.

Proposition 9. *Let $(M,\langle\,,\rangle)$ be a connected manifold of dimension $m \geq 3$ with a φ-CPE structure*

$$\begin{cases} \mathrm{Hess}(w) - w\left(\mathrm{Ric}^\varphi - \frac{S^\varphi}{m-1}\langle\,,\rangle\right) = T^\varphi \\ (1+w)\tau(\varphi) = -\mathrm{d}\varphi(\nabla w) \end{cases} \tag{37}$$

for some $\varphi : (M,\langle\,,\rangle) \to (N,\langle\,,\rangle_N)$ and $\alpha \in \mathbb{R}$. Then, S^φ is constant on M. Furthermore, if M is compact and w is non-constant then $S^\varphi > 0$.

Proof. We prove the identity

$$\left(\frac{m-2}{m} + w\right)\nabla S^\varphi = 0 \qquad \text{on } M\,. \tag{38}$$

We take covariant derivative of the first equation in (37) to get

$$w_{ij,k} - w\left(R^\varphi_{ij,k} - \frac{S^\varphi_k}{m-1}\delta_{ij}\right) - \left(R^\varphi_{ij} - \frac{S^\varphi}{m-1}\delta_{ij}\right) w_k - R^\varphi_{ij,k} + \frac{S^\varphi_k}{m}\delta_{ij} = 0\,. \tag{39}$$

We recall the φ-Schur's identity (see equation (2.10) of [2])

$$R^\varphi_{ij,i} = \frac{1}{2}S^\varphi_j - \alpha\varphi^a_{ss}\varphi^a_j \tag{40}$$

and we trace (39) with respect to i and k to infer

$$\begin{aligned} 0 &= w_{ij,i} - wR^{\varphi}_{ij,i} + w\frac{S^{\varphi}_j}{m-1} - R^{\varphi}_{ij}w_i + \frac{S^{\varphi}}{m-1}w_j - R^{\varphi}_{ij,i} + \frac{S^{\varphi}_j}{m} \\ &= w_{ii,j} + w_t R_{tiji} - w\left(\frac{1}{2}S^{\varphi}_j - \alpha\varphi^a_{kk}\varphi^a_j\right) + \frac{1}{m-1}wS^{\varphi}_j + \frac{1}{m-1}S^{\varphi}w_j \\ &\quad - R^{\varphi}_{ij}w_i - \left(\frac{1}{2}S^{\varphi}_j - \alpha\varphi^a_{kk}\varphi^a_j\right) + \frac{S^{\varphi}_j}{m}\,. \end{aligned}$$

Tracing the first in (37) we obtain

$$\Delta w + \frac{S^{\varphi}}{m-1}w = 0\,. \tag{41}$$

Thus, using (41) and the second in (37), we deduce

$$\begin{aligned} 0 &= -\frac{1}{m-1}S^{\varphi}_j w - \frac{1}{m-1}S^{\varphi}w_j + w_t R^{\varphi}_{tj} + \alpha\varphi^a_t\varphi^a_j w_t - \frac{1}{2}S^{\varphi}_j w + \alpha\varphi^a_{kk}\varphi^a_j w \\ &\quad + \frac{1}{m-1}S^{\varphi}_j w + \frac{1}{m-1}S^{\varphi}w_j - R^{\varphi}_{tj}w_t - \frac{1}{2}S^{\varphi}_j + \alpha\varphi^a_{kk}\varphi^a_j + \frac{1}{m}S^{\varphi}_j \\ &= -\frac{1}{2}(1+w)S^{\varphi}_j + (1+w)\alpha\varphi^a_{kk}\varphi^a_j + \frac{1}{m}S^{\varphi}_j + \alpha\varphi^a_t\varphi^a_j w_t \\ &= -\frac{1}{2}\left(\frac{m-2}{m} + w\right)S^{\varphi}_j\,, \end{aligned}$$

that is, (38).

Next, we observe that (38) implies $\nabla S^{\varphi} \equiv 0$ on the open subset $U = \{w \neq -(m-2)/m\}$. On the other hand, if $x_0 \in \mathrm{Int}(M\backslash U)$, in the sense that there exists $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset M\backslash U$, from $w = -(m-2)/m$ on $B_\varepsilon(x_0)$, (41) and $m \geq 3$ we get $S^{\varphi} \equiv 0$ on $B_\varepsilon(x_0)$. Concluding, $\nabla S^{\varphi} \equiv 0$ on $U \cup \mathrm{Int}(M\backslash U)$, the complementary of which is a closed set with empty interior. Hence, $\nabla S^{\varphi} \equiv 0$ on M and thus S^{φ} is constant.

To conclude, if M is compact and w is non-constant, then integrating (41) against w and proceeding as in the Introduction we easily get that $S^{\varphi} > 0$. □

The next result is a Kazdan-Warner type obstruction that holds on every φ-CPE structure, which for convenience we write as

$$\begin{cases} w_{ji} = (1+w)T^{\varphi}_{ji} - w\frac{S^{\varphi}}{m(m-1)}\delta_{ji} \\ \varphi^a_s w_s = -(1+w)\varphi^a_{tt}\,. \end{cases} \tag{42}$$

For constant φ, the identity reduces to the formula in [17] recalled in the Introduction.

Lemma 10. *Let $(M, \langle\,,\,\rangle)$ be a manifold of dimension $m \geq 2$ with a φ-CPE structure as in (φ-CPE). Then*

$$\mathrm{div}(T^{\varphi}(\nabla w,\,\cdot\,)^{\sharp}) = \alpha(1+w)|\tau(\varphi)|^2 + (1+w)|T^{\varphi}|^2\,. \tag{43}$$

In particular, if M is compact,

$$\int_M (1+w)|T^{\varphi}|^2 = -\alpha\int_M (1+w)|\tau(\varphi)|^2\,. \tag{44}$$

Proof. We compute

$$\begin{aligned}\operatorname{div}(T^\varphi(\nabla w,\,\cdot\,)^\sharp) &= (T^\varphi_{ik}w_k)_i\\ &= T^\varphi_{ik,i}w_k + T^\varphi_{ik}w_{ki}\\ &= \left(R^\varphi_{ik,i}w_k - \frac{S^\varphi_i}{m}\delta_{ik}\right)w_k + T^\varphi_{ik}w_{ki}\,.\end{aligned}$$

Using the φ-Schur's identity and the constancy of S^φ which follows from Lemma 9,

$$\operatorname{div}(T^\varphi(\nabla w,\,\cdot\,)^\sharp) = -\alpha\varphi^a_{tt}\varphi^a_k w_k + T^\varphi_{ik}w_{ki}\,.$$

Then, the validity of (42) gives (43). Equation (44) follows immediately from (43). □

As in [3, 4], the proof of Theorem 7 depends on an integral identity, (66) below, obtained by comparing two different Bochner formulas. Before, we need to recall a few other facts and definitions. Although not strictly necessary in what follows, but to simplify notations, we introduce the linear map

$$\mathscr{W}^\varphi : S^2_0(M) \to S^2_0(M)$$

on the space $S^2_0(M)$ of traceless 2-covariant, symmetric tensors on M, defined, for $\beta = \beta_{ij}\,\theta^i\otimes\theta^j \in S^2_0(M)$, by setting

$$\mathscr{W}^\varphi(\beta) = \left[W^\varphi_{tikj} + \frac{\alpha}{2}\varphi^a_t\left(\varphi^a_i\delta_{kj} + \varphi^a_j\delta_{ki}\right)\right]\beta_{tk}\,\theta^i\otimes\theta^j\,. \tag{45}$$

Obviously indices $1 \le a, b, \cdots \le n = \dim N$ and $1 \le i, j, \cdots \le m$ refer to local orthonormal coframes respectively on N and M. Note that $\mathscr{W}^\varphi$ is well defined and self-adjoint with respect to the standard extension of $\langle\,,\,\rangle$ to $S^2_0(M)$, that we will denote with the same symbol. This is crucial for the validity of inequality (67) that we shall use later.

We let C^φ be the φ-Cotton tensor, defined as the obstruction to the φ-Schouten tensor A^φ in (16) to be Codazzi. Thus, its components in a local orthonormal coframe are given by

$$C^\varphi_{ijk} = A^\varphi_{ij,k} - A^\varphi_{ik,j}\,.$$

A calculation in [2] shows the validity of the following symmetries:

$$\begin{cases} C^\varphi_{ijk} = -C^\varphi_{ikj} \qquad \text{and thus } C^\varphi_{ikk} = 0\\ C^\varphi_{kki} = \alpha\varphi^a_{kk}\varphi^a_i\\ C^\varphi_{ijk} + C^\varphi_{jki} + C^\varphi_{kij} = 0\,.\end{cases} \tag{46}$$

Our argument to prove Theorem 7 keeps the same guidelines as [3], but with some simplifications. We split it into some lemmas. The first step is the following Bochner identity:

Lemma 11. *Let $(M, \langle\,,\,\rangle)$ be a manifold of dimension $m \ge 3$, let $\varphi : (M, \langle\,,\,\rangle) \to (N, \langle\,,\,\rangle_N)$ be a smooth map and let $\alpha \in \mathbb{R}$. Assume S^φ is constant. Then*

$$\begin{aligned}\frac{1}{2}\Delta|T^\varphi|^2 &= |\nabla T^\varphi|^2 + \frac{m}{m-2}\operatorname{Tr}(T^\varphi)^3 + \frac{1}{m-1}S^\varphi|T^\varphi|^2\\ &\quad - \langle\mathscr{W}^\varphi(T^\varphi), T^\varphi\rangle + \left(C^\varphi_{ijk}R^\varphi_{ij}\right)_k - \frac{1}{2}|C^\varphi|^2 - C^\varphi_{kki,j}R^\varphi_{ij}\,.\end{aligned} \tag{47}$$

Remark 12. From now on we indicate a 2-covariant tensor and its corresponding endomorphism with the same letter. Thus $(T^\varphi)^3$ means the composition of endomorphisms $T^\varphi \circ T^\varphi \circ T^\varphi$.

Proof. From equation (3.6) of [2] we have

$$\begin{aligned}\frac{1}{2}\Delta|T^\varphi|^2 = |\nabla T^\varphi|^2 + \frac{m-2}{2(m-1)}\operatorname{Tr}(T^\varphi \circ \operatorname{Hess}(S^\varphi)) + \frac{m}{m-2}\operatorname{Tr}(T^\varphi)^3 + \frac{S^\varphi}{m-1}|T^\varphi|^2 \\ + \operatorname{Tr}(\operatorname{div}C^\varphi \circ T^\varphi) - \langle \mathscr{W}^\varphi(T^\varphi), T^\varphi\rangle - \operatorname{Tr}(T^\varphi \circ \nabla \operatorname{Tr} C^\varphi)\end{aligned} \tag{48}$$

where we have set

$$\operatorname{div}C^\varphi = C^\varphi_{ijk,k}\,\theta^i \otimes \theta^j\,, \qquad \operatorname{Tr} C^\varphi = C^\varphi_{kki}\,\theta^i\,. \tag{49}$$

Since S^φ is constant, $\operatorname{Hess}(S^\varphi) = 0$, while using (49) we deduce

$$\operatorname{Tr}(T^\varphi \circ \nabla \operatorname{Tr} C^\varphi) = C^\varphi_{kki,j}R^\varphi_{ij} - \frac{S^\varphi}{m}C^\varphi_{kki,i} \tag{50}$$

$$\operatorname{Tr}(\operatorname{div}C^\varphi \circ T^\varphi) = C^\varphi_{ijk,k}R^\varphi_{ij} - \frac{S^\varphi}{m}C^\varphi_{ssk,k}\,. \tag{51}$$

Again using the constancy of S^φ and the relation between $R^\varphi_{ij,k}$ and C^φ_{ijk} we get

$$C^\varphi_{ijk,k}R^\varphi_{ij} = (C^\varphi_{ijk}R^\varphi_{ij})_k - \frac{1}{2}|C^\varphi|^2\,. \tag{52}$$

Inserting the above informations into (48) we conclude (47). □

Note that the validity of Lemma 11 is independent of that of the φ-CPE structure (φ-CPE). Now the idea is to make formula (47) interact with $1+w$ in order to be able to use (44). Towards this aim we observe that tracing the first equation in (42) we obtain

$$\frac{1}{2}\Delta(1+w)^2 = -\frac{1}{m-1}S^\varphi w(1+w) + |\nabla w|^2\,, \tag{53}$$

thus, when M is compact, integrating against $|T^\varphi|^2$ gives

$$\int_M |T^\varphi|^2|\nabla w|^2 = \int_M \frac{S^\varphi}{m-1}(1+w)w|T^\varphi|^2 + \frac{1}{2}\int_M (1+w)^2\Delta|T^\varphi|^2.$$

We insert (47) into the above and integrate by parts the term with $\left(C^\varphi_{ijk}R^\varphi_{ij}\right)_k$ to obtain

$$\begin{aligned}\int_M |T^\varphi|^2|\nabla w|^2 &= \int_M (1+w)^2|\nabla T^\varphi|^2 + \frac{m}{m-2}\int_M (1+w)^2\operatorname{Tr}(T^\varphi)^3 \\ &+ \frac{2}{m-1}\int_M (1+w)^2 S^\varphi|T^\varphi|^2 - \int_M (1+w)^2\langle \mathscr{W}^\varphi(T^\varphi), T^\varphi\rangle \\ &- \frac{S^\varphi}{m-1}\int_M (1+w)|T^\varphi|^2 - \frac{1}{2}\int_M (1+w)^2|C^\varphi|^2 \\ &- \int_M (1+w)^2 C^\varphi_{kki,j}R^\varphi_{ij} - 2\int_M (1+w)w_k C^\varphi_{ijk}R^\varphi_{ij}\,.\end{aligned} \tag{54}$$

So far, to get (54) we did not use (φ-CPE) in its full strength, just (53) and the constancy of S^φ. Hereafter, we shall exploit all of the assumptions in Theorem 7, that is, the validity of (42) with condition

$$\varphi^a_s w_s = -(1+w)\varphi^a_{tt} = 0. \tag{55}$$

In particular, from the constancy of S^φ and $\tau(\varphi)=0$ the φ-Schur identity becomes

$$T^\varphi_{ji,i} = R^\varphi_{ji,i} = \frac{1}{2}(S^\varphi)_j - \alpha\varphi^a_{tt}\varphi^a_j = 0. \tag{56}$$

Taking covariant derivative of the CPE equation (42) we have

$$(1+w)R^\varphi_{kj,i} = w_{kji} - w_i R^\varphi_{jk} + \frac{S^\varphi}{m-1}w_i\delta_{jk}\,. \tag{57}$$

Interchanging the role of j and i, subtracting the two identities and using Ricci commutation relations for w_{kji} we infer

$$(1+w)(R^\varphi_{kj,i} - R^\varphi_{ki,j}) = w_t R_{tkji} + \frac{S^\varphi}{m-1}(w_i\delta_{jk} - w_j\delta_{ik}) - (w_i R^\varphi_{jk} - w_j R^\varphi_{ik})\,. \tag{58}$$

Taking into account that S^φ is constant, $C^\varphi_{kji} = R^\varphi_{kj,i} - R^\varphi_{ki,j}$ and we obtain

$$(1+w)C^\varphi_{kji} = w_t R_{ijkt} + \frac{S^\varphi}{m-1}(w_i\delta_{jk} - w_j\delta_{ik}) - (w_i R^\varphi_{jk} - w_j R^\varphi_{ik})\,. \tag{59}$$

We multiply the above relation by R^φ_{kj} and take divergence to get

$$\begin{aligned}
&\left(w_j R^\varphi_{ik}R^\varphi_{kj} + w_t R_{ijkt}R^\varphi_{jk}\right)_i \\
&\quad= \left((1+w)C^\varphi_{kji}R^\varphi_{kj} + w_i\left(|\mathrm{Ric}^\varphi|^2 - \tfrac{(S^\varphi)^2}{m-1}\right) + \tfrac{S^\varphi w_j R^\varphi_{ji}}{m-1}\right)_i \\
&\quad= \left((1+w)C^\varphi_{kji}R^\varphi_{kj} + w_i|T^\varphi|^2 + \tfrac{S^\varphi w_j T^\varphi_{ji}}{m-1}\right)_i \\
&\quad= \left((1+w)C^\varphi_{kji}R^\varphi_{kj}\right)_i + \langle\nabla w, \nabla|T^\varphi|^2\rangle + \tfrac{S^\varphi}{m-1}|T^\varphi|^2,
\end{aligned} \tag{60}$$

where in the last equality we used (φ-CPE) in the form (42), its trace

$$\Delta w = -\frac{S^\varphi}{m-1}w \tag{61}$$

and the φ-Schur identity (56). We examine the left hand side of (60), that is,

$$(*) \doteq \left(w_j R^\varphi_{ik}R^\varphi_{kj} + w_t R_{ijkt}R^\varphi_{jk}\right)_i.$$

Expanding the divergence,

$$\begin{aligned}
(*) = {}& w_{ij}R^\varphi_{ik}R^\varphi_{kj} + w_j R^\varphi_{ik}R^\varphi_{kj,i} + w_j R^\varphi_{ik,i}R^\varphi_{kj} \\
&+ w_{ti}R_{ijkt}R^\varphi_{jk} + w_t R_{ijkt,i}R^\varphi_{jk} + w_t R_{ijkt}R^\varphi_{jk,i}\,.
\end{aligned}$$

Tracing the second Bianchi identity, notice that

$$R_{ijkt,i} = R^\varphi_{jt,k} - R^\varphi_{jk,t} + \alpha(\varphi^a_{jk}\varphi^a_t - \varphi^a_{jt}\varphi^a_k)\,. \tag{62}$$

Hence, using (55), (56) and (62),

$$\begin{aligned}
(*) &= w_j R^\varphi_{ik}(R^\varphi_{kj,i} - R^\varphi_{ki,j} + R^\varphi_{ki,j}) + w_t R^\varphi_{jk}(R^\varphi_{jt,k} - R^\varphi_{jk,t}) \\
&\quad + \alpha w_t R^\varphi_{jk}(\varphi^a_{jk}\varphi^a_t - \varphi^a_{jt}\varphi^a_k) + \frac{1}{2} w_t R_{ijkt}(R^\varphi_{jk,i} - R^\varphi_{ik,j}) \\
&\quad + w_{ij} R^\varphi_{ik} R^\varphi_{kj} + w_{ti} R_{ijkt} R^\varphi_{jk} \\
&= \frac{1}{2}\langle \nabla w, \nabla |\mathrm{Ric}^\varphi|^2\rangle + w_j C^\varphi_{kji} R^\varphi_{ik} + \frac{1}{2} w_t R_{ijkt} C^\varphi_{kji} \\
&\quad + w_{ij} R^\varphi_{ik} R^\varphi_{kj} + w_{ti} R_{ijkt} R^\varphi_{jk} - \alpha w_t R^\varphi_{jk}\varphi^a_{jt}\varphi^a_k + w_t R^\varphi_{jk} C^\varphi_{jtk} \\
&= 2 w_j C^\varphi_{kji} R^\varphi_{ik} + \frac{1}{2}\langle \nabla w, \nabla |T^\varphi|^2\rangle + \frac{1}{2} w_t R_{ijkt} C^\varphi_{kji} \\
&\quad + w_{ij} R^\varphi_{ik} R^\varphi_{kj} + w_{ti} R_{ijkt} R^\varphi_{jk} - \alpha w_t R^\varphi_{jk}\varphi^a_{jt}\varphi^a_k\,.
\end{aligned}$$

We next exploit (59) to remove the term $w_t R_{ijkt}$ and obtain, because of the second in (46),

$$\begin{aligned}
(*) &= 2 w_j C^\varphi_{kji} R^\varphi_{ik} + \frac{1}{2}\langle \nabla w, \nabla |T^\varphi|^2\rangle \\
&\quad + \frac{1}{2}\left[(1+w)C^\varphi_{kji} - \frac{S^\varphi}{m-1}(w_i\delta_{jk} - w_j\delta_{ik}) + (w_i R^\varphi_{jk} - w_j R^\varphi_{ik})\right] C^\varphi_{kji} \\
&\quad + w_{ij} R^\varphi_{ik} R^\varphi_{kj} + w_{ti} R_{ijkt} R^\varphi_{jk} - \alpha w_t R^\varphi_{jk}\varphi^a_{jt}\varphi^a_k \\
&= w_j C^\varphi_{kji} R^\varphi_{ik} + \frac{1}{2}\langle \nabla w, \nabla |T^\varphi|^2\rangle + \frac{1}{2}(1+w)|C^\varphi|^2 \\
&\quad + w_{ij} R^\varphi_{ik} R^\varphi_{kj} + w_{ti} R_{ijkt} R^\varphi_{jk} - \alpha w_t R^\varphi_{jk}\varphi^a_{jt}\varphi^a_k\,.
\end{aligned}$$

Using the Ricci commutation relations for the tensor Ric^φ:

$$R^\varphi_{st,ji} = R^\varphi_{st,ij} + R^\varphi_{lt} R_{lsji} + R^\varphi_{sl} R_{ltji}\,,$$

and the φ-Schur identity, which implies $R^\varphi_{ik,kt} = 0$, we deduce

$$\begin{aligned}
R^\varphi_{jk} R_{ijkt} &= R^\varphi_{jk} R_{jitk} = R^\varphi_{ik,tk} - R^\varphi_{ij} R_{jt} \\
&= R^\varphi_{ik,tk} - R^\varphi_{ij} R^\varphi_{jt} - \alpha R^\varphi_{ij}\varphi^a_j\varphi^a_t\,.
\end{aligned}$$

Plugging into the above, we get

$$\begin{aligned}
(*) &= w_j C^\varphi_{kji} R^\varphi_{ik} + \frac{1}{2}\langle \nabla w, \nabla |T^\varphi|^2\rangle + \frac{1}{2}(1+w)|C^\varphi|^2 \\
&\quad w_{ti} R^\varphi_{ik,tk} - \alpha w_{ti} R^\varphi_{ij}\varphi^a_j\varphi^a_t - \alpha w_t R^\varphi_{jk}\varphi^a_{jt}\varphi^a_k\,.
\end{aligned}$$

Differentiating (55) we get

$$w_t \varphi^a_{ti} = -w_{ti}\varphi^a_t,$$

and therefore

$$w_{ti} R^\varphi_{ij}\varphi^a_j\varphi^a_t + w_t R^\varphi_{jk}\varphi^a_{jt}\varphi^a_k = -w_t\varphi^a_{ti} R^\varphi_{ij}\varphi^a_j + w_t R^\varphi_{jk}\varphi^a_{jt}\varphi^a_k = 0.$$

Inserting into the above, we infer

$$(*) = w_j C^\varphi_{kji} R^\varphi_{ik} + \frac{1}{2}\langle \nabla w, \nabla |T^\varphi|^2\rangle + \frac{1}{2}(1+w)|C^\varphi|^2 + w_{ti} R^\varphi_{ik,tk}.$$

Plugging into (60) and rearranging, we obtain

$$\begin{aligned} &w_j C^\varphi_{kji} R^\varphi_{ik} + \frac{1}{2}(1+w)|C^\varphi|^2 + w_{ti} R^\varphi_{ik,tk} \\ &\quad = \Big((1+w) C^\varphi_{kji} R^\varphi_{kj}\Big)_i + \frac{1}{2}\langle \nabla w, \nabla |T^\varphi|^2\rangle + \frac{S^\varphi}{m-1}|T^\varphi|^2 \,. \end{aligned} \tag{63}$$

We study the term $w_{ti} R^\varphi_{ik,tk}$. Again from the φ-Schur identity and from (42),

$$\begin{aligned} w_{ti} R^\varphi_{ik,tk} &= (1+w) T^\varphi_{it} R^\varphi_{ik,tk} = (1+w) T^\varphi_{it} T^\varphi_{ik,tk} \\ &= \big((1+w) T^\varphi_{it} T^\varphi_{ik,t}\big)_k - w_k T^\varphi_{it} T^\varphi_{ik,t} - (1+w) T^\varphi_{it,k} T^\varphi_{ik,t} \,. \end{aligned}$$

Next, since S^φ is constant, $C^\varphi_{ikt} = T^\varphi_{ik,t} - T^\varphi_{it,k}$ and we deduce

$$\begin{aligned} w_{ti} R^\varphi_{ik,tk} &= \big((1+w) T^\varphi_{it} C^\varphi_{ikt}\big)_k + \frac{1}{2}\operatorname{div}\big((1+w)\nabla |T^\varphi|^2\big) \\ &\quad - w_k T^\varphi_{it} T^\varphi_{ik,t} - (1+w) T^\varphi_{it,k} T^\varphi_{ik,t} \,. \end{aligned}$$

Using the identities

$$\begin{aligned} T^\varphi_{it,k} T^\varphi_{ik,t} &= R^\varphi_{it,k} R^\varphi_{ik,t} = |\nabla \mathrm{Ric}^\varphi|^2 - \frac{1}{2}|C^\varphi|^2 = |\nabla T^\varphi|^2 - \frac{1}{2}|C^\varphi|^2 \\ T^\varphi_{it} T^\varphi_{ik,t} &= T^\varphi_{it}(T^\varphi_{it,k} - C^\varphi_{itk}) = \frac{1}{2}\big(|T^\varphi|^2\big)_k - R^\varphi_{it} C^\varphi_{itk}, \end{aligned}$$

we conclude

$$\begin{aligned} w_{ti} R^\varphi_{ik,tk} &= \big((1+w) T^\varphi_{it} C^\varphi_{ikt}\big)_k + \frac{1}{2}\operatorname{div}\big((1+w)\nabla |T^\varphi|^2\big) \\ &\quad - \frac{1}{2}\langle \nabla w, \nabla |T^\varphi|^2\rangle + w_k R^\varphi_{it} C^\varphi_{itk} - (1+w)|\nabla T^\varphi|^2 + \frac{1}{2}(1+w)|C^\varphi|^2 \,. \end{aligned}$$

Identity (63) therefore becomes

$$\begin{aligned} &w_j C^\varphi_{kji} R^\varphi_{ik} + (1+w)|C^\varphi|^2 + \big((1+w) T^\varphi_{it} C^\varphi_{ikt}\big)_k \\ &\quad + \frac{1}{2}\operatorname{div}\big((1+w)\nabla |T^\varphi|^2\big) + w_k R^\varphi_{it} C^\varphi_{itk} - (1+w)|\nabla T^\varphi|^2 \\ &\quad = \Big((1+w) C^\varphi_{kji} R^\varphi_{kj}\Big)_i + \langle \nabla w, \nabla |T^\varphi|^2\rangle + \frac{S^\varphi}{m-1}|T^\varphi|^2 \,. \end{aligned}$$

Simplifying and using that, by (46) and $\tau(\varphi) = 0$, $T^\varphi_{it} C^\varphi_{ikt} = R^\varphi_{it} C^\varphi_{ikt} = -R^\varphi_{it} C^\varphi_{itk}$,

$$\begin{aligned} &(1+w)|C^\varphi|^2 + \frac{1}{2}\operatorname{div}\big((1+w)\nabla |T^\varphi|^2\big) - (1+w)|\nabla T^\varphi|^2 \\ &\quad = 2\Big((1+w) C^\varphi_{kji} R^\varphi_{kj}\Big)_i + \langle \nabla w, \nabla |T^\varphi|^2\rangle + \frac{S^\varphi}{m-1}|T^\varphi|^2. \end{aligned}$$

Because of this last identity, we can compute

$$\begin{aligned} \frac{1}{2}\operatorname{div}\big((1+w)^2 \nabla |T^\varphi|^2\big) &= \frac{1+w}{2}\langle \nabla w, \nabla |T^\varphi|^2\rangle + \frac{1+w}{2}\operatorname{div}\big((1+w)\nabla |T^\varphi|^2\big) \\ &= -(1+w)^2 |C^\varphi|^2 + (1+w)^2 |\nabla T^\varphi|^2 \\ &\quad + 2(1+w)\Big((1+w) C^\varphi_{kji} R^\varphi_{kj}\Big)_i + \frac{3}{2}(1+w)\langle \nabla w, \nabla |T^\varphi|^2\rangle \\ &\quad + \frac{S^\varphi}{m-1}(1+w)|T^\varphi|^2. \end{aligned} \tag{64}$$

Integrating on M and using the divergence theorem,

$$\begin{aligned} 0 &= -\int_M (1+w)^2|C^\varphi|^2 + \int_M (1+w)^2|\nabla T^\varphi|^2 + \frac{S^\varphi}{m-1}\int_M (1+w)|T^\varphi|^2 \\ &\quad -2\int_M w_i(1+w)C^\varphi_{kji}R^\varphi_{kj} + \frac{3}{2}\int_M \langle (1+w)\nabla w, \nabla|T^\varphi|^2\rangle. \end{aligned}$$

Integrating by parts with the aid of (53),

$$\begin{aligned} \frac{3}{2}\int_M \langle (1+w)\nabla w, \nabla|T^\varphi|^2\rangle &= -\frac{3}{4}\int_M |T^\varphi|^2\Delta(1+w)^2 \\ &= \frac{3S^\varphi}{2(m-1)}\int_M w(1+w)|T^\varphi|^2 - \frac{3}{2}\int_M |T^\varphi|^2|\nabla w|^2 \end{aligned}$$

so we finally obtain

(65)
$$\begin{aligned} 0 &= -\int_M (1+w)^2|C^\varphi|^2 + \int_M (1+w)^2|\nabla T^\varphi|^2 + \frac{S^\varphi}{m-1}\int_M (1+w)|T^\varphi|^2 \\ &\quad -2\int_M w_i(1+w)C^\varphi_{kji}R^\varphi_{kj} + \frac{3S^\varphi}{2(m-1)}\int_M w(1+w)|T^\varphi|^2 - \frac{3}{2}\int_M |T^\varphi|^2|\nabla w|^2\,. \end{aligned}$$

We insert into (54) to remove the term with $C^\varphi_{kji}R^\varphi_{kj}$, and use $C_{kki}=0$, to get

$$\begin{aligned} \int_M |T^\varphi|^2|\nabla w|^2 &= \int_M (1+w)^2|\nabla T^\varphi|^2 + \frac{m}{m-2}\int_M (1+w)^2\operatorname{Tr}(T^\varphi)^3 \\ &\quad + \frac{2}{m-1}\int_M (1+w)^2 S^\varphi|T^\varphi|^2 - \int_M (1+w)^2\langle \mathscr{W}^\varphi(T^\varphi), T^\varphi\rangle \\ &\quad - \frac{S^\varphi}{m-1}\int_M (1+w)|T^\varphi|^2 - \frac{1}{2}\int_M (1+w)^2|C^\varphi|^2 + \int_M (1+w)^2|C^\varphi|^2 \\ &\quad - \int_M (1+w)^2|\nabla T^\varphi|^2 - \frac{S^\varphi}{m-1}\int_M (1+w)|T^\varphi|^2 \\ &\quad - \frac{3S^\varphi}{2(m-1)}\int_M w(1+w)|T^\varphi|^2 + \frac{3}{2}\int_M |T^\varphi|^2|\nabla w|^2. \end{aligned}$$

Simplifying, and using the Kazdan-Warner type identity in (44) together with assumption $\tau(\varphi)=0$, we eventually get the following integral identity:

(66)
$$\begin{aligned} 0 &= \frac{1}{2}\int_M |T^\varphi|^2|\nabla w|^2 + \frac{m}{m-2}\int_M (1+w)^2\operatorname{Tr}(T^\varphi)^3 + \frac{1}{2}\int_M (1+w)^2|C^\varphi|^2 \\ &\quad + \frac{1}{2(m-1)}\int_M (1+w)^2 S^\varphi|T^\varphi|^2 - \int_M (1+w)^2\langle \mathscr{W}^\varphi(T^\varphi), T^\varphi\rangle. \end{aligned}$$

We are now ready for the

Proof of Theorem 7. As we have already observed, in our assumptions S^φ is a positive constant and (66) holds. From the proof of Proposition 3.22 in [2] we have the validity of the following inequality:

(67)
$$\langle \mathscr{W}^\varphi(T^\varphi), T^\varphi\rangle \le \sqrt{\frac{m-2}{2(m-1)}}|W^\varphi||T^\varphi|^2 + \frac{\alpha}{m-2}|\mathrm{d}\varphi|^2|T^\varphi|^2$$

and from Okumura's lemma

(68)
$$\operatorname{Tr}(T^\varphi)^3 \ge -\frac{m-2}{\sqrt{m(m-1)}}|T^\varphi|^3\,.$$

Using (67) and (68), from (66) we infer

$$0 \geq \frac{1}{2}\int_M |T^\varphi|^2|\nabla w|^2 + \frac{1}{2}\int_M (1+w)^2|C^\varphi|^2$$
$$+\int_M (1+w)^2\left(\frac{1}{2(m-1)}S^\varphi - \sqrt{\frac{m-2}{2(m-1)}}|W^\varphi| - \frac{\alpha}{m-2}|\mathrm{d}\varphi|^2 - \sqrt{\frac{m}{m-1}}|T^\varphi|\right)|T^\varphi|^2 .$$

Observe that all terms appearing on the RHS of the above inequality are nonnegative, due to (19). Hence, they must vanish, and in particular we have

$$\int_M |T^\varphi|^2|\nabla w|^2 = 0 . \tag{69}$$

We claim that this, together with the φ-CPE equation, implies $T^\varphi \equiv 0$ on M. We postpone the proof of this claim to the subsequent Lemma 13. Assuming $T^\varphi \equiv 0$ on M, the φ-CPE equation gives

$$\mathrm{Hess}(w) = -\frac{S^\varphi}{m(m-1)}\langle\, ,\, \rangle \tag{70}$$

with $S^\varphi > 0$. But then, Theorem A of Obata, [24], implies that $(M, \langle\, ,\, \rangle)$ is isometric to $\mathbb{S}^m(\kappa)$ with κ as in (20). In particular, $S = S^\varphi$ and, when $\alpha \neq 0$, this implies $|\mathrm{d}\varphi|^2 = 0$, hence φ is constant. □

Lemma 13. *Let $(M, \langle\, ,\, \rangle)$ be a compact manifold of dimension $m \geq 3$, let $\varphi : (M, \langle\, ,\, \rangle) \to (N, \langle\, ,\, \rangle_N)$ be a smooth map and let $\alpha \in \mathbb{R}$. Assume that w is a non-constant solution to (φ-CPE) and*

$$\int_M |T^\varphi|^2|\nabla w|^2 = 0 . \tag{71}$$

Then $T^\varphi \equiv 0$ on M.

Proof. Observe first that S^φ is a positive constant by Proposition 9. Following arguments of [17], we prove that the set

$$C^w_{-1} = \{x \in M : w(x) = -1\}$$

has zero measure. Since M is compact, C^w_{-1} is compact. Let $\hat{C}_{-1}$ be the set of critical points of w in C^w_{-1}, that is,

$$\hat{C}_{-1} = \{x \in C^w_{-1} : \mathrm{d}w_x = 0\} .$$

Note that $C^w_{-1} \setminus \hat{C}_{-1}$ is a hypersurface with possibly many connected components. Writing the φ-CPE equation in the form (42), we see that if $p \in \hat{C}_{-1}$ then for $v \in T_pM$, $v \neq 0$ it holds

$$\mathrm{Hess}(w)(v,v) = \frac{S^\varphi}{m(m-1)}\langle v, v\rangle > 0$$

since $S^\varphi > 0$. Thus p is a non-degenerate critical point of s. Hence the points of $\hat{C}_{-1}$ are isolated. Since C^w_{-1} is compact, $\hat{C}_{-1}$ is a finite set. In particular, since $m \geq 3$, $C^w_{-1} \setminus \hat{C}_{-1}$ is a connected hypersurface and

$$C^w_{-1} = \hat{C}_{-1} \cup (C^w_{-1} \setminus \hat{C}_{-1})$$

has measure 0.

We now turn to the proof that $T^\varphi \equiv 0$. Since $|T^\varphi||\nabla w| \ge 0$ is a continuous function, (71) implies that $|T^\varphi||\nabla w| \equiv 0$ on M. Hence, the set

$$E = \{x \in M : T^\varphi_x \neq 0\}$$

is contained in the set $\{x \in M : \mathrm{d}w_x = 0\}$ of critical points of w. Note that E is open. Suppose, by contradiction, that $E \neq \emptyset$. Let $x \in E$ and let $U \subseteq E$ be a connected neighbourhood of x. Since $\nabla w \equiv 0$ on E, there exists a constant $c \in \mathbb{R}$ such that $w \equiv c$ on U. Since U has positive measure, by the previous observation we have $c \neq -1$. Rewriting the φ-CPE equation in the form

$$(1+w)\mathrm{Ric}^\varphi - \mathrm{Hess}(w) = \left(\frac{S^\varphi}{m} + \frac{wS^\varphi}{m-1}\right)\langle \, , \, \rangle$$

and using the fact that $w \equiv c \neq -1$ on U, we see that

$$\mathrm{Ric}^\varphi \equiv \frac{S^\varphi}{1+c}\frac{(1+c)m-1}{m(m-1)}\langle \, , \, \rangle \qquad \text{on } U\,.$$

In particular, Ric^φ is a multiple of $\langle \, , \, \rangle$ on U and then $T^\varphi \equiv 0$ on U, contradiction. □

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