Global Regularity to the liquid crystal flows of Q-tensor model

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Abstract. In this paper we investigate a forced incompressible Navier-Stokes equation coupled with a parabolic type equation of Q-tensors in a domain $U \subset \mathbb{R}^3$. In the case U is bounded, we prove the existence of a global strong solution when the initial data are sufficiently small, improving a result in Xiao's paper [J. Differ. Equations 2017]. The key tool of the proof is a maximum principle. Then, we establish also a result of continuous dependence of solutions on the initial data. Finally, if $U = \mathbb{R}^3$, based on a result of Du, Hu and Wang [Arch. Rational Mech. Anal. 2020], we give an interesting regularity criterium just via the $\dot{B}^{-1}_{\infty,\infty}$ norm of u and the L^∞ norm of the initial data Q_0.

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1 Introduction and main results

In this paper, we study the model for a viscous incompressible liquid crystal flow proposed by Beris and Edwards [4], which one can find in the physics literature, such as in [12, 26]. An important feature of the model is the molecular orientation of the liquid crystals described by the so-called tensor Q. More precisely, let $U \subset \mathbb{R}^3$ be a bounded domain and $t > 0$, then for any $(t, x) \in \mathbb{R}^+ \times U$, the tensor $Q(t, x)$ is a symmetric traceless matrix, namely, for any $t > 0$ and $x \in U$, $Q(t, x) \in S_0^{(3)}$, where

$$S_0^{(3)} = \{Q \in \mathbb{R}^{3\times 3};\ Q = Q^T,\ trace[Q] = 0\}.$$

Its motion is described through the velocity field $u = u(t, x),\ t \in R^+,\ x \in U \subset \mathbb{R}^3$, by the coupled system:

$$\begin{cases} \partial_t Q + (u \cdot \nabla)Q - S(\nabla u, Q) = \Gamma(\Delta Q - \mathcal{L}[\partial F(Q)]), \\ \partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla P = \operatorname{div}(\tau + \sigma), \\ \operatorname{div} u = 0, \end{cases} \tag{1.1}$$

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with

$$\mathcal{L}[A] := A - \frac{1}{3} trace[A] Id$$

denoting the projection onto the space of traceless matrices. $P(t,x)$ is the scalar pressure, $\nu > 0$ is the viscosity coefficient and $\Gamma > 0$ is the elasticity relaxation constant. Moreover,

$$S(\nabla u, Q) := (\xi\epsilon(u)+w(u))\left(Q+\frac{1}{3}Id\right)+\left(Q+\frac{1}{3}Id\right)(\xi\epsilon(u)-w(u))-2\xi\left(Q+\frac{1}{3}Id\right)trace[Q\nabla u].$$

where $\epsilon(u) := \frac{1}{2}(\nabla u + \nabla^t u)$ and $w(u) := \frac{1}{2}(\nabla u - \nabla^t u)$ are the symmetric and antisymmetric part of the velocity gradient, respectively, and $\xi \in \mathbb{R}$. The symmetric part of the additional stress tensor is given by:

$$\begin{aligned}\tau_{\alpha\beta} := &-\xi\left(Q_{\alpha\gamma}+\frac{\delta_{\alpha\gamma}}{3}\right)H_{\gamma\beta} - \xi H_{\alpha\gamma}\left(Q_{\gamma\beta}+\frac{\delta_{\gamma\beta}}{3}\right)\\ &+2\xi\left(Q_{\alpha\beta}+\frac{\delta_{\alpha\beta}}{3}\right)Q_{\gamma\delta}H_{\gamma\delta} - \partial_\beta Q_{\gamma\delta}\partial_\alpha Q_{\gamma\delta},\end{aligned} \tag{1.2}$$

and the antisymmetric part is

$$\sigma_{\alpha\beta} := Q_{\alpha\gamma}H_{\gamma\beta} - H_{\alpha\gamma}Q_{\gamma\beta}.$$

where $H := \Delta Q - \mathcal{L}[\partial F(Q)]$ and we use the Einstein summation convention. Moreover, the potential F is a function defined on symmetric matrices that will be introduced later and the term $S(\nabla u, Q)$ describes how the flow gradient rotates and stretches the order-parameter Q. The constant ξ denotes the ratio of the tumbling and aligning effects on the liquid crystals exerted by a shear flow.

Throughout the paper, we denote $A : B = A_{\alpha\beta}B_{\alpha\beta}$, $A \cdot B = trace(AB)$. Note that $A : B = A \cdot B$ for $A, B \in S_0^{(3)}$. Moreover, we use the Frobenius norm of a matrix defined by $|Q| := \sqrt{trace(Q^2)} = \sqrt{Q_{\alpha\beta}Q_{\alpha\beta}}$ for $Q \in S_0^{(3)}$. Let $|\nabla Q|^2 = \partial_\gamma Q_{\alpha\beta}\partial_\gamma Q_{\alpha\beta}$ and $|\Delta Q|^2 := \Delta Q_{\alpha\beta}\Delta Q_{\alpha\beta}$. We denote the norms of the usual Lebesgue spaces L^p by $\|u\|_{L^p}^p = \int_U |u|^p dx$, for $1 \le p < \infty$. C_i, C denote different positive constants in different places. Finally, we note $(u \otimes u)_{\alpha\beta} = u_\alpha u_\beta$, and $(\nabla Q \otimes \nabla Q)_{\alpha\beta} = \partial_\alpha Q_{\gamma\delta}\partial_\beta Q_{\gamma\delta}$.

Several competing theories exist which attempt to capture the complexity of nematic liquid crystals. In 1960s, Ericksen and his collaborators proposed the so called Ericksen-Leslie equation for the underlying flow field which is given by the Navier-Stokes equation with an additional forcing term (see [14]). This theory is very successful in many aspects and it matches well with the experimental observations. However, this model cannot describe the biaxial nematic liquid crystals. So a new and more comprehensive theory is introduced by P. G. de Gennes [11], in which he uses $N \times N$ symmetric, traceless matrix Q as the new parameter, which is called Q tensor.

As mentioned, in this paper we study the model proposed by Beris and Edwards [4]. This model has been extensively and continuously studied by many authors in recent years. In the case of polynomial potentials of the form

$$F(Q) = \frac{a}{2}|Q|^2 - \frac{b}{3}trace[Q^3] + \frac{c}{4}|Q|^4,\ a, b, c \in \mathbb{R}, \tag{1.3}$$

Paicu and Zarnescu [21] established the existence of global weak solutions in dimension $N = 2$ or 3 with $\xi = 0$. Moreover, solutions with higher order regularity and the weak-strong uniqueness for $N = 2$ is studied. Later, they extend these results to the case when $|\xi|$ is sufficiently small in

[20]. In [6], with no additional smallness assumptin on ξ, they establish the existence of global weak solution and long-time behavior of system with nonzero ξ in the two-dimensional periodic domain. The uniqueness of weak solutions to the Cauchy problem in $\mathbb{R}^2$ was proved in [9, 10].

On the other hand, concerning strong solutions, Abels, Dolzmann, and Liu [1] proved the existence of a local strong solution and global weak solution with higher regularity in time in the case of inhomogeneous mixed Dirichlet and Neumann boundary condition, in a bounded domain, without smallness assumption on ξ. Later, Liu and Wang in [19] proved the existence and uniqueness of local in time strong solutions to the system with an anisotropic elastic energy. In the paper [22], Schonbek and Shibata obtain a global strong solution in $\mathbb{R}^N, (N \geq 2)$ for more general function F. Moreover, they give an optimal decay rate for the Q tensor in L^p, $1 \leq p \leq \infty$. In the case of polynomial potentials like (1.3), Xiao [28] proved the global well posedness for the simplified model with $\xi = 0$ in a bounded domain under some assumptions on the coefficients. He constructed a solution in the maximal $L^p - L^q$ regularity class.

Regarding the decay of the solutions, it is worth pointing out that Dai et al. give a decay rate for the solutions (u, Q) with general F. Specifically, in [7] they consider functions

$$F : \mathbb{R}^{3\times 3}_{sym} \to (-\infty, \infty],$$

where $\mathbb{R}^{3\times 3}_{sym}$ is the set of 3×3 symmetric matrices. They suppose that $F \in C^2(O)$, for some open set $O \subset \mathbb{R}^{3\times 3}_{sym}$ containing the isotropic state $Q \equiv 0$. They suppose also that $Q = 0$ is the (unique) global minimum of F in O, specifically,

$$F(0) = 0, \quad F(Q) > 0 \quad \text{for any} \quad Q \in O \setminus \{0\} \tag{1.4}$$

and

$$\partial F(Q) : Q \geq 0 \quad \text{whenever} \quad Q \in B_{r_1} \quad \text{or} \quad Q \in O \setminus B_{r_2}. \tag{1.5}$$

where B_{r_1}, B_{r_2} with $r_1 < r_2$ are two balls such that

$$Q = 0 \in B_{r_1} \equiv \{|Q| < r_1\} \subset \{|Q| \leq r_2\} \subset O.$$

They show that the solutions of the system (1.1) tend to the isotropic state at the rate $(1 + t)^{-\frac{3}{2}}$ as $t \to \infty$ when $\xi = 0$. Moreover, if F is as (1.3), they obtain the exponential decay. It is worth pointing out that they introduce an interesting maximum principle. Note that any $F(Q) := \frac{a}{2}|Q|^2 - \frac{b}{3}trace[Q^3] + \frac{c}{4}|Q|^4$, with $a > 0$, satisfy the conditions (1.4), (1.5), at least in a neighborhood of 0.

In the present paper, we consider $\xi = 0$ which denotes the situation when the molecules only tumble in a shear flow but they are not aligned by the flow. Moreover, we consider the polynomial potential $F(Q) = \frac{a}{2}|Q|^2 - \frac{b}{3}trace[Q^3] + \frac{c}{4}|Q|^4$, as in the paper of Paicu and Zarnescu [21]. In this case, $\mathcal{L}[\partial F(Q)] = aQ - b\left[Q^2 - \frac{1}{3}Id\ trace(Q^2)\right] + cQ\ trace(Q^2)$, and since $\xi = 0$, the system (1.1) reduces to the following:

$$\begin{cases} \partial_t Q + (u \cdot \nabla)Q - w(u)Q + Qw(u) = \Delta Q - aQ + b\left[Q^2 - \frac{1}{3}Id\ trace(Q^2)\right] - cQ\ trace(Q^2), \\ \partial_t u + (u \cdot \nabla)u = \Delta u - \nabla P + \text{div}(\nabla Q \otimes \nabla Q) + \text{div}(Q\ \Delta Q - \Delta Q\ Q), \\ \text{div}\ u = 0, \end{cases} \tag{1.6}$$

where, without loss of generality, we set $\nu = \Gamma = 1$. Moreover, the following initial-boundary conditions are prescribed,

$$(u, Q)|_{t=0} = (u_0, Q_0), \quad (u, \partial_\nu Q)|_{\partial U} = (0, \mathbb{O}), \tag{1.7}$$

where ν is the outward normal vector on U and $\mathbb{O}$ is the zero matrix.

The main result of our paper is an improvement of the result of global existence of a strong solution in Xiao's paper [28]. In order to present Xiao's result, let us first introduce the definition of strong solution.

Definition 1.1. *For $T > 0$ and $1 < p, q < \infty$, $M_T^{p,q}$ denotes the set of triplets (u, Q, P) such that*

$$u \in C([0,T], D_{A_q}^{1-\frac{1}{p},p}) \cap L^p(0,T; W^{2,q}(U) \cap W_0^{1,q}(U)),$$
$$\partial_t u \in L^p(0,T; L^q(U)), \mathrm{div} u = 0,$$
$$Q \in C([0,T]; B_{q,p}^{3-\frac{2}{p}}) \cap L^p(0,T; W^{3,q}(U)), \quad \partial_t Q \in L^p(0,T; W^{1,q}(U))$$
$$P \in L^p(0,T; W^{1,q}(U)), \int_U P dx = 0.$$

$M_T^{p,q}$ is a Banach space with norm

$$\begin{aligned}\|(u,Q,P)\|_{M_T^{p,q}} := &\|u\|_{L^\infty([0,T], D_{A_q}^{1-\frac{1}{p},p})} + \|u\|_{L^p(0,T;W^{2,q}(U)\cap W_0^{1,q}(U))} + \|\partial_t u\|_{L^p(0,T;L^q(U))} \\ &+ \|Q\|_{L^\infty([0,T];B_{q,p}^{3-\frac{2}{p}}))} + \|Q\|_{L^p(0,T;W^{3,q}(U))} + \|\partial_t Q\|_{L^p(0,T;W^{1,q}(U))} \\ &+ \|P\|_{L^p(0,T;W^{1,q}(U))}.\end{aligned}$$

In the above definition, the space $D_{A_q}^{1-\frac{1}{p},p}$ is some fractional domain of the Stokes operator in L^q , for details see [8], Section 2.3. In short, this space consists of vectors whose $2-\frac{2}{p}$ derivatives are in L^q, divergence free, and vanish on the boundary of the domain. Moreover, we have the following embedding relation

$$D_{A_q}^{1-\frac{1}{p},p} \hookrightarrow B_{q,p}^{2(1-\frac{1}{p})} \cap L^q.$$

Besov spaces on bounded domains U can be defined as interpolation spaces as in [3],

$$B_{q,p}^{2(1-\frac{1}{p})} = (L^q, W^{2,q})_{1-\frac{1}{p},p}.$$

Notice that for the incompressible system, the pressure is still a solution up to a difference by a constant, so the condition on P in Definition 1.1 is satisfied automatically, if we replace the pressure by $P - \frac{1}{|U|}\int_U P dx$.

Definition 1.2. *Let $T \in (0, +\infty]$. The triplet (u, Q, P) is a strong solution to (1.6), (1.7) on $U \times [0, T)$ if (u, Q, P) belongs to $M_t^{p,q}$ whenever $t < T$, it satisfies the equation (1.6) in distribution sense on $U \times (0, T)$ and the initial boundary conditions (1.7).*

Recently, Xiao [28] proved the existence of a global strong solution for small initial data if one of the following conditions

$$(\mathbf{A_1}): \quad ac > \tfrac{9}{16}b^2,\ a > 0,\ c > 0, \quad \text{or} \quad (\mathbf{A_2}): \quad a = 0,$$

is satisfied. Specifically, Xiao's result is the following;

Theorem 1.1. *[28] Let U be a bounded domain in $\mathbb{R}^3$ with regular boundary. Assume that $1 < p < \infty, q > 3, u_0 \in D_{A_q}^{1-\frac{1}{p},p}, Q_0 \in B_{q,p}^{3-\frac{2}{p}} \cap W^{1,q}$. Then*

1. there exists $0 < \epsilon_0 \ll 1$ such that if

$$\|Q_0\|_{L^\infty} \le \epsilon_0,$$

then there exists some $T > 0$ *such that the system* (1.6) *and* (1.7) *has a unique local strong solution* $(u, Q, P) \in M_t^{p,q}$ *in* $U \times [0, T)$ *for any* $0 < t < T$.

2. If $(\mathbf{A_1})$ *or* $(\mathbf{A_2})$ *is satisfied, there exists a positive* δ_0, *such that if the initial data satisfies*

$$\|u_0\|_{D_{A_q}^{1-\frac{1}{p},p}} \leq \delta_0, \ \|Q_0\|_{B_{q,p}^{3-\frac{2}{p}} \cap W^{1,q}} \leq \delta_0,$$

then (1.6) *admits a unique global strong solution* $(u, Q, P) \in M_T^{p,q}$ *in* $U \times [0, T)$ *for all* $T > 0$.

We point out that $W^{1,q} \hookrightarrow L^\infty$, if $q > 3$. i.e. there exists a positive constant $C > 0$ such that $\|Q_0\|_{L^\infty} \leq C\|Q_0\|_{W^{1,q}}$. This inequality implies $\|Q_0\|_{L^\infty} \leq \epsilon_0$ if $\delta_0 \leq C^{-1}\epsilon_0$.

In his paper, inspired by the work of Danchin in [8] and Hu and Wang in [16], Xiao makes use of the maximal regularity of the Stokes operator and of the parabolic operator to get the existence of the local solution (u, Q, P) by an approximation scheme. Then to achieve the existence of the global strong solution, he uses a bootstrap argument to prove a uniform boundedness of the norm in the space $M_T^{p,q}$ of the local solution (u, Q, P). The key tool of the proof is an exponential decay of the norm $\|Q\|_{L^q}$, with $2 \leq q \leq +\infty$ which is valid under the assumption $(\mathbf{A_1})$.

Our first result is an improvement of Xiao's global existence Theorem. Indeed, we prove that the system (1.6) and (1.7) has a global strong solution under the sole hypothesis $a \geq 0$, removing the more restrictive assumptions $(\mathbf{A_1})$ and $(\mathbf{A_2})$. As in Xiao's work, the proof strategy involves showing that the norm $\|(u, Q, P)\|_{M_T^{p,q}}$ remains uniformely bounded on any interval $[0, T]$. The novelty is that we obtain the estimate thanks to a maximum principle (Lemma 2.3), inspired by a result in [7].

Notice that $F(Q) = \frac{a}{2}|Q|^2 + \frac{b}{3}trace[Q^3] + \frac{c}{4}|Q|^4, a > 0$ has the following properties. There exists a sufficiently small positive r that depends on (a, b, c) such that if

$$\mathcal{O} := \{Q \in \mathbb{R}^{3\times 3}_{sym,0} | \ \ |Q| < r\}, \tag{1.8}$$

then $Q = 0$ is the (unique) global minimum of F in $\mathcal{O}$, i.e.

$$F(0) = 0, \quad F(Q) > 0 \ \text{ for any } \ Q \in \mathcal{O} \setminus \{0\}, \tag{1.9}$$

and

$$\partial F(Q) : Q \geq 0 \ \text{ for any } \ Q \in \mathcal{O}, \tag{1.10}$$

$$F(Q) \geq \lambda |Q|^2, \ \text{ for some positive } \ \lambda \text{ and } \ \text{ for any } \ Q \in \mathcal{O}. \tag{1.11}$$

Moreover, F also satisfies

$$c_1|Q|^2 \leq \partial F(Q) : Q \leq C|Q|^2, \ \text{ for any } \ Q \in \mathcal{O} \setminus \{0\}, \tag{1.12}$$

for some $c_1 \in (0, a)$ and some positive constant C.

Using these properties we are able to prove the following Theorem.

THEOREM A. *Let* U *be a bounded domain in* $\mathbb{R}^3$ *with regular boundary. Assume that* $1 < p < \infty$, $q > 3$, $u_0 \in D_{A_q}^{1-\frac{1}{p},p}$, $Q_0 \in B_{q,p}^{3-\frac{2}{p}} \cap W^{1,q}$, *and* $a \geq 0$. *Then there exists a positive* δ_0, *such that if the initial data satisfy*

$$\|u_0\|_{D_{A_q}^{1-\frac{1}{p},p}} \leq \delta_0, \ \|Q_0\|_{B_{q,p}^{3-\frac{2}{p}} \cap W^{1,q}} \leq \delta_0,$$

then the system (1.6) *with* (1.7) *admits a unique global strong solution* $(u, Q, P) \in M_T^{p,q}$ *in* $U \times [0, T)$ *for all* $T > 0$. *Moreover, the solution satisfies*

$$\|u(t)\|_{L^2} + \|Q(t)\|_{H^1} \leq C\ exp(-dt) \quad for\ any \quad t \geq 0 \quad and\ some \quad d > 0, \tag{1.13}$$

and

$$\|Q(t)\|_{L^q} \leq \|Q_0\|_{L^q}\ exp(-c_1 t), \quad 4 \leq q \leq \infty, \ and\ for\ any\ t \geq 0. \tag{1.14}$$

with c_1 *as in* (1.12).

Remark 1.1. *Motivated by [7], we prove that if* $a > 0$ *and the domain* U *is bounded with regular boundary,* $\|Q\|_{L^q}, 1 \leq q \leq +\infty$ *has an exponential decay in time for initial data sufficiently small. Then by a bootstrap argument in a similar way as in [28], we can prove that the norm* $\|(u, Q, P)\|_{M_T^{p,q}}$ *of the strong solution is uniformly bounded on any interval* $[0, T]$ *and so the solution exists on any interval* $[0, T]$, $\forall T > 0$. *Moreover, we prove also that the norms* $\|u(t)\|_{L^2}$ *and* $\|Q(t)\|_{H^1}$ *have an exponential decay in time.*

Now, we give a result about continuous dependence of solutions.

THEOREM B. *Let* U *be a bounded domain in* $\mathbb{R}^3$ *with regular boundary. Assume that* $2 \leq p < \infty, q > 3, u_0 \in D_{A_q}^{1-\frac{1}{p},p}, Q_0 \in B_{q,p}^{3-\frac{2}{p}} \cap W^{1,q}$, *and* $\|Q_0\|_{W^{1,q}} \leq \sigma$ *for some positive* σ, *such that* $\sigma < \epsilon_0$, *with* ϵ_0 *as in Theorem 1.1. If* $(u_0^n, Q_0^n)_{n\in\mathbb{N}}$ *tends to* (u_0, Q_0) *in* $D_{A_q}^{1-\frac{1}{p},p} \times B_{q,p}^{3-\frac{2}{p}} \cap W^{1,q}$, *then there exists a positive* T *independent of* n *such that the sequence of solutions* $(u^n, Q^n)_{n\in\mathbb{N}} \in M_T^{p,q}$ *tends to the solution* (u, Q) *in* L^2 *norm. Moreover,*

$$\|Q^n - Q\|_{L^2}^2 + \|\Delta(Q^n - Q)\|_{L^2}^2 + \|u^n - u\|_{L^2}^2 \to 0, \quad uniformly \quad for \ \ t \in [0, T] \ \ when \quad n \to \infty.$$

In the last part of the paper, we consider the system (1.6), (1.7) in the whole space $U = \mathbb{R}^3$ and we study the partial regularity properties of a class of weak solutions. Results on the partial regularity of the Q tensor model have already been proved by Du, Hu and Wang [13]. Indeed, they extend the CKN Theorem [5, 17] to the Q tensor system. Later, Liu give some new partial regularity criterion in terms of the velocity gradient only in [18].

In this paper, we intend to give a new local regularity criterion assuming that $u(t)$ is small in $\dot{B}_{\infty,\infty}^{-1}$ on a suitable interval of time and the L^∞ norm of Q_0 is small. This result is inspired by a similar criterion for MHD systems in [27] and to our knowledge, it was not previously known for system (1.6). We remark that a key step in our proof will be the fact that the L^∞ norm of $Q(t)$ remains small on some interval $[0, T]$, with $T > 0$, if $\|Q_0\|_{L^\infty}$ is small enough, as it is proved in [21].

In order to introduce our results, we will give the definition of a *suitable weak solution* (see also [13] for a similar definition), which is a weak solution satisfying a local energy inequality (the definition of weak solution can be found in [13] , Definition 1.1).

Definition 1.3. *A weak solution* (u, P, Q) *with* $u \in (L^\infty(0, \infty; L^2(\mathbb{R}^3)) \cap L^2(0, \infty; H^1(\mathbb{R}^3))$, $P \in L^{\frac{3}{2}}(0, \infty; L^{\frac{3}{2}}(\mathbb{R}^3))$ *and* $Q \in (L^\infty(0, \infty; H^1(\mathbb{R}^3)) \cap L^2(0, \infty; H^2(\mathbb{R}^3)))$ *of* (1.6) *is a suitable*

weak solution of (1.6)*, if, in addition,* (u, P, Q) *satisfies the local energy inequality*

$$\begin{aligned}
&\int_{\mathbb{R}^3}(|u|^2+|\nabla Q|^2)\phi(x,t)dx+2\int_{\mathbb{R}^3\times[0,t]}(|\nabla u|^2+|\nabla^2 Q|^2)\phi(x,s)dxds\\
&\quad\le\int_{\mathbb{R}^3\times[0,t]}(|u|^2+|\nabla Q|^2)(\partial_t\phi+\Delta\phi)(x,s)dxds\\
&\quad\quad+\int_{\mathbb{R}^3\times[0,t]}[(|u|^2+2P)u\cdot\nabla\phi+2\nabla Q\otimes\nabla Q:u\otimes\nabla\phi](x,s)dxds\\
&\quad+2\int_{\mathbb{R}^3\times[0,t]}(\nabla Q\otimes\nabla Q-|\nabla Q|^2I_3):\nabla^2\phi(x,s)dxds\\
&\quad-2\int_{\mathbb{R}^3\times[0,t]}[Q,\Delta Q]\cdot u\otimes\nabla\phi(x,s)d\phi dxds\\
&\quad-2\int_{\mathbb{R}^3\times[0,t]}([w,Q]\cdot(\nabla Q\nabla\phi)+\nabla(\mathcal{L}[\partial F(Q)])\cdot\nabla Q\phi)(x,s)dxds
\end{aligned}\tag{1.15}$$

for any $0\le\phi\in C_0^\infty(\mathbb{R}^3\times(0,t])$ *and for any* $t>0$.

Our result is based on the following regularity criterion established in [13]:

Theorem 1.2. *Let us assume that* $F(Q)=\frac{a}{2}|Q|^2-\frac{b}{3}trace[Q^3]+\frac{c}{4}|Q|^4, c>0$, $U=\mathbb{R}^3$. *There exists* $\epsilon_1>0$ *such that if* $(u,Q):\mathbb{R}^3\times(0,\infty)\to\mathbb{R}^3\times S_0^{(3)}$ *is a suitable weak solution of* (1.6)*, which satisfies, for* $z_0:=(x_0,t_0)\in\Omega\times(0,\infty)$*,*

$$\limsup_{r\to 0}\frac{1}{r}\int_{Q_r(z_0)}(|\nabla u|^2+|\nabla^2 Q|^2)dxdt<\epsilon_1^2,\tag{1.16}$$

where $Q_r(z_0):=B_r(x_0)\times[t_0-r^2,t_0]$*, then* (u,Q) *is smooth near* z_0.

We recall that in the Navier-Stoke framework, many regularity criterion have been proved in the so called "critical spaces" that are some scaling invariant spaces, such as $L^3(\mathbb{R}^3)$ and $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)$ and they are related by the following embeddings:

$$L^3(\mathbb{R}^3)\hookrightarrow L^{3,q}(\mathbb{R}^3)\hookrightarrow\dot{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)\hookrightarrow\mathrm{BMO}^{-1}(\mathbb{R}^3)\hookrightarrow\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3),\quad(3<p,q<\infty).$$

Theorem C. *Let us assume that* $u_0\in\mathbf{H}$*, where* $\mathbf{H}=$ *Closure of* $\{u\in C_0^\infty(\mathbb{R}^3,\mathbb{R}^3):divu=0\}$ *in* $L^2(\mathbb{R}^3)$*, and* $Q_0\in H^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$*. Let the triplet* (u,Q,P) *be a suitable weak solution to the system* (1.6) *in* $\mathbb{R}^3\times(0,\infty)$*, with* $c>0$*, with the initial data* (u_0,Q_0)*. Then, there exist two positive constants* ε_0 *and* ε_1 *such that if* $u\in L^\infty(t_0-\tilde{\varrho}^2,t_0;\dot{B}^{-1}_{\infty,\infty})$ *satisfies, for some* $\tilde{\varrho}>0$*,*

$$\|u\|_{L^\infty(t_0-\tilde{\varrho}^2,t_0;\dot{B}^{-1}_{\infty,\infty})}<\varepsilon_0,\tag{1.17}$$

and

$$\|Q_0\|_{L^\infty(\mathbb{R}^3)}\le\varepsilon_1.\tag{1.18}$$

one gets that $(x_0,t_0)\in(\mathbb{R}^3\times\mathbb{R}^+)$ *is a regular point (namely, the solution* (u,Q) *is smooth near* (x_0,t_0)*).*

Remark 1.2. *Thanks to the Theorem 1.1 in [13], when the initial data* $u_0\in\mathbf{H}$ *and* $Q_0\in H^1(\mathbb{R}^3,S_0^{(3)})\cap L^\infty(\mathbb{R}^3,S_0^{(3)})$*, then there exists a global suitable weak solution* (u,Q,P).

The bootstrap argument will be employed in our proof. A rigorous statement of the abstract bootstrap principle can be found in T. Tao's book (see [25]).

The paper is organized as follows. In section 2, we will give some key Theorems about the maximal regularities of Stokes operator and parabolic operators, some Lemma and maximum principle. In section 3, based on this, we give the proof of the Theorem A, B, and C.

2 Preliminaries

In this section, we will recall some Lemmas and Theorems in [2, 8, 15] which will be used in the following. Define $\mathcal{W}(0,T) := W^{1,p}(0,T;W^{1,q}) \cap L^p(0,T;W^{3,q})$. Now, we recall the maximal regularity property of the parabolic and Stokes operators.

Theorem 2.1. *[2] Given $w_0 \in B_{q,p}^{3-\frac{2}{p}}$ and $f \in L^p(0,T;W^{1,q})$, the Cauchy problem*

$$\partial_t w - \Delta w = f, \quad t \in (0,T), \quad w(0) = w_0,$$

has a unique solution $w \in \mathcal{W}(0,T)$, and

$$\|w\|_{\mathcal{W}(0,T)} \le C(\|f\|_{L^p(0,T;W^{1,q})} + \|w_0\|_{B^{3-\frac{2}{p}}}),$$

where C is independent of w_0, f and T. Moreover, there exists a positive constant c_0 independent of f and T such that

$$\|w\|_{\mathcal{W}(0,T)} \ge c_0 \sup_{t\in(0,T)} \|w\|_{B_{q,p}^{3-\frac{2}{p}}}.$$

Theorem 2.2. *Let U be a bounded domain with $C^{2+\epsilon}(\epsilon > 0)$ boundary in $\mathbb{R}^3$ and $1 < p, q < \infty$. Assume that $u_0 \in D_{A_q}^{1-\frac{1}{p},p}$ and $f \in L^p(0,\infty;L^q(U))$. Then the system*

$$\begin{cases} \partial_t u - \Delta u + \nabla P = f, \\ \int_U P dx = 0, \quad \text{div } u = 0, \\ u|_{\partial U} = 0, \quad u|_{t=0} = u_0. \end{cases}$$

has a unique solution (u,P) for any $T > 0$ which satisfies

$$\begin{aligned} &\|u(T)\|_{D_{A_q}^{1-\frac{1}{p},p}} + \|(\partial_t u, \Delta u, \nabla P)\|_{L^p(0,T;L^q(U))} \\ &\quad \le C(\|u_0\|_{D_{A_q}^{1-\frac{1}{p},p}} + \|f\|_{L^p(0,T;L^q(U))}), \end{aligned}$$

where $C = C(p,q,U)$.

Next, we introduce some L^∞ estimates for the derivatives of functions $f \in L^\infty(0,T;D_{A_q}^{1-\frac{1}{p},p}) \cap L^p(0,T;W^{2,q})$.

Lemma 2.1. *[8] Let $1 < p, q < \infty$.*

(i) If $0 < \frac{p}{2} - \frac{3p}{2q} < 1$, then it holds that

$$\|\nabla f\|_{L^p(0,T;L^\infty)} \le C T^{\frac{1}{2}-\frac{3}{2q}} \|f\|^{1-\theta}_{L^\infty(0,T;D_{A_q}^{1-\frac{1}{p},p})} \|f\|^{\theta}_{L^p(0,T;W^{2,p})},$$

for some constant C depending only on U, p, q and $\frac{1-\theta}{p} = \frac{1}{2} - \frac{3}{2q}$.

(ii) If $\frac{p}{2} - \frac{3p}{2q} > 1$, then it holds that

$$\|\nabla f\|_{L^p(0,T;L^\infty)} \le C T^{\frac{1}{p}} \|f\|_{L^\infty(0,T;D_{A_q}^{1-\frac{1}{p},p})}$$

for some constant C depending only on U, p, q.

Lemma 2.2. *[16] Let $1 < p,\ r < \infty$. For $f \in W^{1,p}(0,T;L^r)$ and $f(0) \in L^r$, then for any $t \in [0,T]$,*

$$\|f\|_{L^\infty(0,t;L^r)} \le C(\|f(0)\|_{L^r} + \|f\|_{W^{1,p}(0,t;L^r)}),$$

for some constant C independent of f and T.

Before proving the existence of global strong solutions, we give a key lemma which plays an important role in completing the bootstrap argument. Motivated by the results in [7], we will prove that the local solution $(u,P,Q) \in M_T^{p,q}$ for some $T > 0$ also satisfies the following interesting maximum principle.

Lemma 2.3. *Assume that $T \in (0,+\infty)$ and that $(u,Q,P) \in M_T^{p,q}$, $1 < p < \infty$, $q > 3$ is a local strong solution to system (1.6) with (1.7) and with $a > 0$. If the initial data Q_0 satisfy $\|Q_0\|_{L^\infty} < r$ with r as in (1.8), then*

$$\|Q(t)\|_{L^\infty} \le \|Q_0\|_{L^\infty}, \tag{2.1}$$

for any $t \in (0,T)$. Moreover, if $\|Q_0\|_{L^\infty} \le \frac{r}{2}$, we have the better estimate

$$\|Q(t)\|_{L^q} \le exp(-c_1 t)\|Q_0\|_{L^q}, \quad 4 \le q \le \infty, \quad \text{for any } t \in (0,T) \tag{2.2}$$

where c_1 is the positive constant defined in (1.12).

Proof. To prove the inequality (2.1), we use the same idea as in [7]. Firstly, we rewrite the second equation of the system (1.6) as

$$\partial_t Q + u \cdot \nabla Q - \Delta Q = -\mathcal{L}[\partial F(Q)] + w(u)Q - Qw(u), \quad Q(0,\cdot) = Q_0, \tag{2.3}$$

where $\mathcal{L}[\partial F(Q)] = aQ - b\left[Q^2 - \frac{1}{3}Id\ trace(Q^2)\right] + cQtrace(Q^2)$. Note that $Q \in C([0,T];B_{q,p}^{3-\frac{2}{p}})$ and so by the following embeddings (see [3])

$$\|Q\|_{L^\infty} \le \|Q\|_{B^0_{\infty,1}} \le \|Q\|_{B^{3-\frac{2}{p}-\frac{3}{q}}_{\infty,p}} \le \|Q\|_{B^{3-\frac{2}{p}}_{q,p}},$$

the local solution $Q \in C([0,T];L^\infty)$. So we can take the scalar product of (2.3) with $2G'(|Q|^2)Q$, where $G \in C^2([0,\infty))$ is a suitable chosen function, and integrate over the domain U to obtain with some computations:

$$\frac{d}{dt}\int_U G(|Q|^2)dx + \int_U [2G'(|Q|^2)|\nabla Q|^2 + G''(|Q|^2)|\nabla |Q|^2|^2]dx = -2\int_U G'(|Q|^2)\partial F(Q) : Q dx. \tag{2.4}$$

Indeed, the equality (2.4) is a consequence of the property div $u = 0$ and the following equality

$$\begin{aligned}(w(u)Q - Qw(u)) : Q &= -(w(u)Q + Qw(u)) : Q + 2w(u)Q : Q = -(w(u)Q + Qw(u))_{\alpha\beta}Q_{\alpha\beta} \\ &\quad + 2w(u)_{\alpha z}Q_{z\beta}Q_{\alpha\beta} = 0,\end{aligned} \tag{2.5}$$

which is a consequence of the fact that the terms $(w(u)Q + Qw(u))_{\alpha\beta}Q_{\alpha\beta}$ and $(w(u)Q + Qw(u))_{\beta\alpha}Q_{\beta\alpha}$ appear in pairs and $w(u)Q + Qw(u)$ is an antisymmetric matrix while Q is a symmetric matrix. Similarly, the term $w(u)_{\alpha z}Q_{z\beta}Q_{\alpha\beta}$ and $w(u)_{z\alpha}Q_{\alpha\beta}Q_{z\beta}$ appear in pairs and $w(u)$ is an antisymmetric matrix. In order to get (2.4), we have also used the property

$$\int_U G'(|Q|^2)\mathcal{L}\partial F(Q) : Q dx = \int_U G'(|Q|^2)\partial F(Q) : Q dx,$$

since

$$2\int_U G'(|Q|^2)Q_{\alpha\beta}\frac{1}{3}trace(\partial F(Q))I_{\alpha\beta}dx = 2\int_U G'(|Q|^2)Q_{\alpha\alpha}\frac{1}{3}trace(\partial F(Q))dx = 0.$$

Now take G in (2.4) such that

$$G(z) = 0 \ \text{ for } z \in [0, \|Q_0\|^2_{L^\infty}], \quad G' \geq 0, \quad G'' \geq 0, \quad G(z) > 0 \quad \text{ for } z > \|Q_0\|^2_{L^\infty} \tag{2.6}$$

Since $\|Q_0\|_{L^\infty} < r$ and $\mathcal{O}$ is open, there exists $T_1 > 0$ small enough such that $Q(t,\cdot) \in \mathcal{O}$ for any $0 \leq t \leq T_1$. Since $G' \geq 0$, $G'' \geq 0$ and using (1.10), then (2.4) implies

$$\int_U G(|Q|^2)(t)dx \leq \int_U G(|Q_0|^2)dx = 0 \ \text{ for all } 0 \leq t \leq T_1.$$

So $G(|Q|^2(t,x)) = 0$ for a.e. $x \in U$ and thanks to (2.6), we get

$$\|Q(t)\|_{L^\infty} \leq \|Q_0\|_{L^\infty} \ \text{ for all } 0 \leq t \leq T_1. \tag{2.7}$$

Now, we let us define $\tilde{T} := \sup\{t > 0, \|Q(t)\|_{L^\infty} \leq \|Q_0\|_{L^\infty}\}$. Then we can prove that $\tilde{T} = T$ by a contradiction argument. If $\tilde{T} < T$, note $\|Q(\tilde{T})\|_{L^\infty} \leq \|Q_0\|_{L^\infty} < r$, and using the same argument as above, we get

$$\|Q(t)\|_{L^\infty} \leq \|Q_0\|_{L^\infty}, \ \text{for } \tilde{T} \leq t \leq \tilde{T} + \delta,$$

for some positive δ. But this contradicts the definition of the maximal time span $\tilde{T}$. Therefore, we get $\tilde{T} = T$ and

$$\|Q(t)\|_{L^\infty} \leq \|Q_0\|_{L^\infty} \ \text{ for all } 0 \leq t \leq T. \tag{2.8}$$

We could prove the inequality (2.2) using the inequality (2.1). But we prefer to prove it indipendently by a bootstrap argument. Firstly, we prove

$$\|Q(t)\|_{L^\infty} \leq \frac{r}{2}, \quad \forall t \in [0,T]. \tag{2.9}$$

by bootstrap argument. We assume that $\|Q(t)\|_{L^\infty} \leq r, \quad \forall t \in [0,T]$ and we prove that this implies $\|Q(t)\|_{L^\infty} \leq \frac{r}{2}, \quad \forall t \in [0,T]$, if $\|Q_0\|_{L^\infty} \leq \frac{r}{2}$. Let $G(z) = z^q, q \geq 2$ in (2.4). By simple computations, we get

$$\frac{d}{dt}\int_U |Q|^{2q}dx + \int_U [2q|Q|^{2(q-1)}|\nabla Q|^2 + q(q-1)|Q|^{2(q-2)}|\nabla|Q|^2|^2]dx = -2\int_U q|Q|^{2(q-1)}\partial F(Q) : Q dx \tag{2.10}$$

Note that the term $\int_U [2q|Q|^{2(q-1)}|\nabla Q|^2 + q(q-1)|Q|^{2(q-2)}|\nabla|Q|^2|^2]dx$ is positive and since $\|Q(t)\|_{L^\infty} \leq \frac{r}{2}$ then $Q(t) \subset \mathcal{O}$, $\forall t \in [0,T]$, and so thanks to (1.12) we have

$$\int_U q|Q|^{2(q-1)}\partial F(Q) : Q dx \geq c_1 \int_U q|Q|^{2q}dx.$$

So we get from (2.10)

$$\frac{d}{dt}\int_U |Q|^{2q}dx \leq -c_1 \int_U q|Q|^{2q}dx.$$

It follows by Gronwall's inequality that

$$\|Q(t)\|_{L^{2q}} \leq exp(-c_1 t)\|Q_0\|_{L^{2q}}, \quad q \geq 2. \tag{2.11}$$

Notice the constant c_1 is independent of q, and so when we let $q \to \infty$, we have

$$\|Q(t)\|_{L^\infty} \leq exp(-c_1 t)\|Q_0\|_{L^\infty}, \tag{2.12}$$

for any $t \in (0,T)$. Therefore if we choose $\|Q_0\|_{L^\infty} \leq \frac{r}{2}$, we get $\|Q(t)\|_{L^\infty} \leq \frac{r}{2}$, for any $t \in (0,T)$.

By bootstrap this proves the claim (2.9). Moreover, if $\|Q_0\|_{L^\infty} \leq \frac{r}{2}$ also the inequalities (2.11) and (2.12) hold and so the inequality (2.2) is proved. So the proof of the Lemma is completed. □

Remark 2.1. *We remark that in the proof of the Lemma 2.3, we prove the inequality* (2.1) *only using the properties* (1.9) *and* (1.10).

3 Global existence of a strong solution

Proof of Theorem a.

1. *Local existence of strong solution.*

By Theorem 1.1 if $\|Q_0\|_{L^\infty} \le \epsilon_0$ there exist some positive time T and a local solution $(u, Q, P) \in M_t^{p,q}$ for any $0 < t < T$.

2. Decay of the solution and energy inequalities.

In order to establish the existence of the global solution, we employ a bootstrap argument. Let (u, Q, P) be the local solution that belongs to $M_t^{p,q}$ for any $0 < t < T$ and for some positive T. As in the paper [28], we define

$$\begin{aligned} H(t) :=& \|u\|_{L^\infty(0,t;D_{A_q}^{1-\frac{1}{p},p})} + \|u\|_{L^p(0,t;W^{2,q})} + \|\nabla P\|_{L^p(0,t,L^q)} \\ &+ \|\partial_t u\|_{L^p(0,t;L^q)} + \|Q\|_{L^\infty(0,t;B_{q,p}^{3-\frac{2}{p}})} + \|Q\|_{\mathcal{W}(0,t)}, \end{aligned} \tag{3.1}$$

and

$$H_0 := \|u_0\|_{D_{A_q}^{1-\frac{1}{p},p}} + \|Q_0\|_{B_{q,p}^{3-\frac{2}{p}} \cap W^{1,q}},$$

where we recall that

$$\mathcal{W}(0,t) = W^{1,p}(0,t;W^{1,q}) \cap L^p(0,t;W^{3,q}).$$

If we are able to prove that $H(t)$ is uniformly bounded for any $t \in (0,T)$ then the solution (u, Q, P) can be extended to any interval $(0, T)$. Now, we prove that $H(t)$ is uniformly bounded on $(0, T)$. By Theorems 2.1 and 2.2 and thanks to the Poincaré inequality, we have

$$\begin{aligned} H(t) \le& C\Big(H_0 + \|u \cdot \nabla u\|_{L^p(0,t;L^q)} + \|\mathrm{div}(\nabla Q \otimes \nabla Q)\|_{L^p(0,t;L^q)} \\ &+ \|\mathrm{div}(\Delta Q Q - Q\Delta Q)\|_{L^p(0,t;L^q)} + \|u \cdot \nabla Q\|_{L^p(0,t;W^{1,q})} \\ &+ \|Q\omega - \omega Q\|_{L^p(0,t;W^{1,q})} + \| - aQ\|_{L^p(0,t;W^{1,q})} \\ &+ \|b[Q^2 - \frac{Id}{3} trace(Q^2)] - cQ trace(Q^2)\|_{L^p(0,t;W^{1,q})}\Big) \\ :=& C\big(H_0 + K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7\big). \end{aligned}$$

As in the paper [28], we can estimate $K_1 - K_5$ and K_7 in the following way:

$$\begin{aligned} K_1 &\le \|u\|_{L^\infty(0,t;L^q)} \|\nabla u\|_{L^p(0,t;L^\infty)} \\ &\le C\|u\|_{L^\infty(0,t;D_{A_q}^{1-\frac{1}{p},p})} \|u\|_{L^p(0,t;W^{2,q})} \\ &\le CH^2(t). \end{aligned}$$

Moreover,

$$\begin{aligned} K_2 &\le \|\nabla Q\|_{L^\infty(0,t;L^q)} \|\nabla^2 Q\|_{L^p(0,t;L^\infty)} \\ &\le \|Q\|_{L^\infty(0,t;W^{1,q})} \|Q\|_{L^p(0,t;W^{3,q})} \\ &\le C(\|Q_0\|_{W^{1,q}} + \|Q\|_{W^{1,p}(0,t;W^{1,q})}) \|Q\|_{\mathcal{W}(0,t)} \\ &\le C(H_0 + H(t))H(t), \end{aligned}$$

where in the third inequality we used the Lemma 2.2. Then, denoting by $\nabla^2 Q$ and $\nabla^3 Q$, respectively, the set of all the second and third order partial derivatives, by Holder inequality and Lemma 2.2, we get

$$\begin{aligned}
K_3 &\le C\|Q\|_{L^\infty(0,t;L^\infty)}\|\nabla^3 Q\|_{L^p(0,t;L^q)} + C\|\nabla Q\|_{L^\infty(0,t;L^q)}\|\nabla^2 Q\|_{L^p(0,t;L^\infty)}\\
&\le C(\|Q_0\|_{W^{1,q}} + \|Q\|_{W^{1,p}(0,t;W^{1,q})})\|Q\|_{L^p(0,t;W^{3,q})}\\
&\le C(H_0 + H(t))H(t);
\end{aligned}$$

$$\begin{aligned}
K_4 &\le \|u\|_{L^\infty(0,t;L^q)}\|\nabla Q\|_{L^p(0,t;W^{1,\infty})} + \|\nabla u\|_{L^p(0,t;L^\infty)}\|\nabla Q\|_{L^\infty(0,t;L^q)}\\
&\le C\|u\|_{L^\infty(0,t;D_{A_q}^{1-\frac{1}{p},p})}\|Q\|_{L^p(0,t;W^{3,q})}\\
&\quad + C\|u\|_{L^p(0,t;W^{2,q})}(\|Q_0\|_{W^{1,q}} + \|Q\|_{W^{1,p}(0,t;W^{1,q})})\\
&\le C(H_0 + H(t))H(t);
\end{aligned}$$

$$\begin{aligned}
K_5 &\le C\|\nabla u\|_{L^p(0,t;L^\infty)}\|Q\|_{L^\infty(0,t;W^{1,q})} + \|\nabla^2 u\|_{L^p(0,t;L^q)}\|Q\|_{L^\infty(0,t;L^\infty)}\\
&\le C(H_0 + H(t))H(t);
\end{aligned}$$

$$\begin{aligned}
K_7 &\le C\|Q\|_{L^\infty(0,t;L^\infty)}\|Q\|_{L^p(0,t;W^{1,q})} + \|Q\|^2_{L^\infty(0,t;L^\infty)}\|Q\|_{L^p(0,t;W^{1,q})}\\
&\le C(H_0^2 + H_0^3 + H^2(t) + H^3(t)).
\end{aligned}$$

It remains to deal with the term K_6. We stress that under the assumption $a > 0$, we cannot adopt the same method as in paper [28]. Let us assume that δ_0 in the hypothesis of Theorem 1.1 is small enough such that $\|Q_0\|_{L^\infty} \le C\|Q_0\|_{W^{1,q}} \le \frac{r}{2}$, with r as in (1.8). So thanks to Lemma 2.3 the solution $Q(t)$ satisfies the inequality (2.2) on $[0,T]$. Since $q > 3$ and the domain is bounded, thanks to the inequality (2.12), the embeddings $W^{1,q}(U) \hookrightarrow L^\infty(U) \hookrightarrow L^q(U),\ q > 3$ and the interpolation Gagliardo-Niremberg inequality we get

$$\begin{aligned}
K_6 := \| - aQ\|_{L^p(0,t;W^{1,q})} &\le \|Q\|_{L^p(0,t;L^q)} + \|\nabla Q\|_{L^p_t(L^q)}\\
&\le C\big(\int_0^t \|Q(s,\cdot)\|^p_{L^\infty} ds\big)^{\frac{1}{p}} + C\|Q\|^{\frac{1}{2}}_{L^p(0,t;L^q)}\|\nabla^2 Q\|^{\frac{1}{2}}_{L^p(0,t;L^q)}\\
&\le C\|Q_0\|_{L^\infty}\big(\int_0^t \exp(-pc_1 s)ds\big)^{\frac{1}{p}} + C\big(\|Q_0\|_{L^\infty}\big(\int_0^t \exp(-pc_1 s)ds\big)^{\frac{1}{p}}\big)^{\frac{1}{2}} H^{\frac{1}{2}}(t)\\
&\le CH_0 + CH_0^{\frac{1}{2}} H^{\frac{1}{2}}(t)
\end{aligned}$$

Now, by Young inequality, for any $\varepsilon > 0$

$$\begin{aligned}
&CH_0 + CH_0^{\frac{1}{2}} H^{\frac{1}{2}}(t)\\
&\le CH_0 + C\epsilon^{-1}H_0 + C\epsilon H(t).
\end{aligned} \tag{3.2}$$

Choosing ϵ small enough in (3.2) and collecting the estimates on the terms $K_1, K_2, \cdots, K_7$, one obtains

$$H(t) \le C(\hat{H}_0 + H^2(t) + H^3(t)), \tag{3.3}$$

where $\hat{H}_0 := H_0 + H_0^2 + H_0^3$ and C is a constant independent of $\hat{H}_0$ and of time t. The bootstrap argument starts with the following assumption: $H(t)$ is bounded uniformly for $t \in [0,T]$, i.e.

$$H(t) \le 4C\hat{H}_0, \quad \forall t \in [0,T], \tag{3.4}$$

for some positive constant C. If the initial data are such that $C\|Q_0\|_{W^{1,q}} \le \frac{r}{2}$ and

$$\max\{16C^2\hat{H}_0 + 64C^3\hat{H}_0^2\} \le 1$$

then it follows from the inequality (3.3) and (3.4) that

$$H(t) \leq 2C\hat{H}_0, \quad \forall t \in [0,T].$$

So we complete the proof of the bootstrap assumption (3.4). Moreover, we get that the solution is global and the decay estimates (1.14).

3. The decay of $\|u\|_{L^2}$ and $\|Q\|_{H^1}$.

Now, we will prove the inequality (1.13). As in [7] choosing $G(z) = \sqrt{z}$, by a direct computation one can check that the sum $2G'(|Q|^2)|\nabla Q|^2 + G''(|Q|^2)\big|\nabla|Q|^2\big|^2$ is nonnegative. Combining the fact that $Q(t,\cdot) \subset \mathcal{O}$ for any $t \geq 0$ and using (1.10), we get $-2\int_U G'(|Q|^2)\partial F(Q) : Q \; dx \leq 0$. From (2.4) it follows that

$$\|Q(t)\|_{L^1} \leq \|Q_0\|_{L^1}, \quad \text{for all } t > 0. \tag{3.5}$$

Finally, taking $G(z) = z$ in (2.4), we obtain

$$\frac{d}{dt}\int_U \frac{1}{2}|Q|^2 \; dx + \int_U |\nabla Q|^2 \; dx \leq -\int_U \partial F(Q) : Q dx \quad \text{for all } t > 0.$$

By (1.12), since $Q(t,\cdot) \subset \mathcal{O}$ for any $t \geq 0$, we have

$$\partial F(Q) : Q \geq c_1 |Q|^2.$$

Integrating in time, we have

$$\frac{1}{2}\|Q(t)\|_{L^2}^2 + \int_0^t \|\nabla Q\|_{L^2}^2 \; ds \leq -c_1 \int_0^t \|Q(t)\|_{L^2}^2 ds + \frac{1}{2}\|Q_0\|_{L^2}^2$$

Using the Gronwall's inequality, we obtain

$$\|Q(t)\|_{L^2}^2 \leq \|Q_0\|_{L^2}^2 exp(-2c_1 t) \quad \text{for all } t \geq 0. \tag{3.6}$$

On the other hand, thanks to the regularity of the local solution (u, Q), using the method in [21] and the boundary condition $(u, \partial_\nu Q) \mid_{\partial U} = (0, \mathcal{O})$, we get the following a priori estimate

$$\begin{aligned}\int_U \Big[\frac{1}{2}|u|^2 + \frac{1}{2}|\nabla Q|^2 + F(Q)\Big](t,x)dx + \int_s^t \int_U \left[|\nabla u|^2 + |\Delta Q - \mathcal{L}[\partial F(Q)]|^2\right](\sigma, x) \; dx \; d\sigma \\ \leq \int_U \Big[\frac{1}{2}|u|^2 + \frac{1}{2}|\nabla Q|^2 + F(Q)\Big](s,x) \; dx\end{aligned}$$

for all $t > s$ and a.e. $s \in [0, \infty)$ including $s = 0$. We can rewrite the above inequality into the differential version:

$$\frac{d}{dt}\int_U \left[\frac{1}{2}|u|^2 + \frac{1}{2}|\nabla Q|^2 + F(Q)\right] dx + \int_U \left[|\nabla u|^2 + |\Delta Q - \mathcal{L}[\partial F(Q)]|^2\right](t,x) \; dx \leq 0.$$

Using the properties (1.10) and (1.12), we prove that the energy inequality reads as follows:

$$\begin{aligned}\frac{d}{dt}\int_U \Big[\frac{1}{2}|u|^2 + \frac{1}{2}|\nabla Q|^2 + F(Q)\Big](t,x) \; dx \\ + C\int_U [\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla Q|^2 + F(Q)](t,x) \; dx \leq 0, \quad C > 0.\end{aligned} \tag{3.7}$$

Indeed, this is a consequence of the following inequality

$$\int_U |\Delta Q - \mathcal{L}[\partial F(Q)]|^2 \; dx \geq C\int_U [\frac{1}{2}|\nabla Q|^2 + F(Q)] \; dx, \quad C > 0. \tag{3.8}$$

To prove (3.8), let

$$-\Delta Q + \mathcal{L}[\partial F(Q)] = Z, \quad Z := -u \cdot \nabla Q - Qw(u) + w(u)Q - \partial_t Q,$$

and observe that, thanks to (1.12) and (1.11), we have

$$\int_U \left[\frac{1}{2}|\nabla Q|^2 + F(Q)\right] dx \leq C \int_U [|\nabla Q|^2 + \partial F(Q) : Q] dx. \tag{3.9}$$

Then, by integration by parts and since $\partial F(Q) : Q = \mathcal{L}(\partial F(Q)) : Q$, we get

$$\begin{aligned} \int_U [|\nabla Q|^2 + \partial F(Q) : Q] dx &= C \int_U Z : Q dx \\ &\leq C(\varepsilon)\|Z\|_{L^2}^2 + \varepsilon \|Q\|_{L^2}^2, \end{aligned} \tag{3.10}$$

where in the last inequality we use Hölder and Young inequalities with $\varepsilon > 0$. Finally, collecting the inequalities (3.9) and (3.10), we obtain

$$\int_U \left[\frac{1}{2}|\nabla Q|^2 + F(Q)\right] dx \leq C(\varepsilon)\|Z\|_{L^2}^2 + \varepsilon \|Q\|_{L^2}^2 \tag{3.11}$$

and since, for $\varepsilon > 0$ sufficiently small, the term $\varepsilon\|Q\|_{L^2}^2$ can be "absorbed" by $\int_U F(Q)\ dx$, thanks to (1.11) and the inequality (3.8) is proved.

Using Poincare inequality in (3.7), we obtain

$$\begin{aligned} &\frac{d}{dt}\int_U [\frac{1}{2}|u|^2 + \frac{1}{2}|\nabla Q|^2 + F(Q)](t,\cdot) dx \\ &\qquad \leq -C \int_U [\frac{1}{2}|u|^2 + \frac{1}{2}|\nabla Q|^2 + F(Q)](s,\cdot) dx, \quad C > 0. \end{aligned} \tag{3.12}$$

It follows from Gronwall's inequality that

$$\int_U \left[\frac{1}{2}|u|^2 + \frac{1}{2}|\nabla Q|^2 + F(Q)\right](t,x) dx \leq (\|u_0\|_{L^2}^2 + \|\nabla Q\|_{L^2}^2 + \|F(Q_0)\|_{L^2}^2) exp(-Ct)$$

Then, using (1.11), one has

$$\|u(t)\|_{L^2} + \|Q(t)\|_{H^1} \leq C(\|u_0\|_{L^2} + \|\nabla Q_0\|_{L^2} + \|F(Q_0)\|_{L^2}) exp(-Ct) \tag{3.13}$$

On the other hand, the inequality (1.14) has been proved in Lemma 2.3. Thus we completed the proof of Theorem a.

Proof of Theorem b**.**

Proof. Now, we prove the continuous dependence of the solutions with respect to the initial data. Since $(u_0^n, Q_0^n) \to (u_0, Q_0)$ in $D_{A_q}^{1-\frac{1}{p},p} \times B_{q,p}^{3-\frac{2}{p}} \cap W^{1,q}$ and $\|Q_0\|_{W^{1,q}} < \sigma$, then $\|Q_0^n\|_{L^\infty} \leq \|Q_0^n\|_{W^{1,q}} \leq \sigma$, for large n. So, thanks to Theorem 1.1, there exists a sequence of local solutions (u^n, Q^n) on $(0, T^n)$, for some positive T^n. Moreover, since $\|Q_0^n\|_{L^\infty} \leq \sigma$ for n large, it is possible to choose a common existence time $T > 0$. As in (3.1), we introduce the energy of any solution (u^n, Q^n, P^n)

$$\begin{aligned} H_n(t) :=& \|u^n\|_{L^\infty(0,t;D_{A_q}^{1-\frac{1}{p},p})} + \|\partial_t u^n\|_{L^p(0,t;L^q)} + \|u^n\|_{L^p(0,t;L^q)} \\ &+ \|\nabla P^n\|_{L^p(0,t;L^q)} + \|Q^n\|_{L^\infty(0,t;B_{q,p}^{3-\frac{2}{p}})} + \|Q^n\|_{\mathcal{W}(0,t)}, \end{aligned}$$

and, by similar estimates as in the proof of THEOREM A, we get that $H_n(t)$ and $H(t)$, are bounded on $[0,T]$ uniformly in n. Next, we will show the solution (u^n, Q^n) tends to the solution (u, Q) in L^2, uniformely for any $t \in [0,T]$. Denote

$$\delta^n Q := Q^n - Q, \quad \delta^n u := u^n - u, \quad \delta^n P = P^n - P.$$

Then $(\delta^n Q, \delta^n u, \delta^n P)$ verifies the following system, at least in the weak sense, with zero boundary conditions

$$\begin{cases} \partial_t \delta^n Q + \delta^n u \cdot \nabla \delta^n Q - \delta^n w(u)\delta^n Q + \delta^n Q \delta^n w(u) \\ \quad = -\delta^n u \cdot \nabla Q - u \cdot \nabla \delta^n Q - Q\delta^n w(u) - \delta^n Q w(u) + \delta^n w(u) Q + w(u)\delta^n Q \\ \quad + \Delta \delta^n Q - a\delta^n Q + b[\delta^n Q Q^n + \delta^n Q Q - \frac{Id}{3} tr(\delta^n Q Q^n + \delta^n Q Q)] \\ \quad - c[\delta^n Q tr(Q^n)^2 + Q tr(\delta^n Q Q^n + \delta^n Q Q)], \\ \partial_t \delta^n u + \delta^n u \cdot \nabla \delta^n u - \Delta \delta^n u + \nabla \delta^n P \\ \quad = -\delta^n u \cdot \nabla u - u \cdot \nabla \delta^n u - \mathrm{div}(\nabla \delta^n Q \otimes \nabla \delta^n Q) \\ \quad - \mathrm{div}(\nabla Q \otimes \nabla \delta^n Q + \nabla \delta^n Q \otimes \nabla Q) + \mathrm{div}(\delta^n Q \Delta \delta^n Q - \Delta \delta^n Q \delta^n Q) \\ \quad + \mathrm{div}(Q \Delta \delta^n Q + \delta^n Q \Delta Q - \Delta Q \delta^n Q - \Delta \delta^n Q Q), \\ \mathrm{div} \delta^n u = 0, \\ (\delta^n u, \delta^n Q)|_{t=0} = (u_0^n - u_0, Q_0^n - Q_0), \quad (\delta^n u, \partial_\nu \delta^n Q)|_{\partial U} = (0, \mathcal{O}). \end{cases} \tag{3.14}$$

Now, in the same way as in the proof of the Theorem 2.2 in [28], we multiply $-\Delta\delta^n Q + \delta^n Q$ with the first equation, then we take the trace and integrate over the space U. Then we multiply $\delta^n u$ to the second equation, integrate over the domain U and we add these two inequalities. Now by the same computations as in the proof the Theorem 2.2 in [28] (replacing Q_1 by Q^n, Q_2 by Q u_1 by u^n and u_2 by u), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\delta^n Q\|_{L^2}^2 + \|\nabla \delta^n Q\|_{L^2}^2 + \|\delta^n u\|_{L^2}^2\right) + \|\Delta \delta^n Q\|_{L^2}^2 + \|\nabla \delta^n u\|_{L^2}^2 \\ &\quad + (a+1)\|\nabla \delta^n Q\|_{L^2}^2 + a\|\delta^n Q\|_{L^2}^2 \\ &\lesssim \|\Delta \delta^n Q\|_{L^2}\|\delta^n u\|_{L^2}\|\nabla Q\|_{L^\infty} + \|\Delta \delta^n Q\|_{L^2}\|u\|_{L^\infty}\|\nabla \delta^n Q\|_{L^2} \\ &\quad + \|\delta^n u\|_{L^2}\|\delta^n Q\|_{L^2}\|\nabla Q\|_{L^\infty} + \|\Delta \delta^n Q\|_{L^2}\|\delta^n Q\|_{L^2}\|\nabla u\|_{L^\infty} + \|\nabla \delta^n u\|_{L^2}\|Q\|_{L^\infty}\|\delta^n Q\|_{L^2} \\ &\quad + |b|\|\Delta \delta^n Q\|_{L^2}\|\delta^n Q\|_{L^4}\|Q^n\|_{L^4} + |b|\|\Delta \delta^n Q\|_{L^2}\|\delta^n Q\|_{L^2}\|Q\|_{L^\infty} + |b|\|Q^n\|_{L^2}\|\delta^n Q\|_{L^4}^2 \\ &\quad + |b|\|Q\|_{L^2}\|\delta^n Q\|_{L^4}^2 + c\|\Delta \delta^n Q\|_{L^2}\|\delta^n Q\|_{L^4}\|Q^n\|_{L^8}^2 \\ &\quad + c\|\Delta \delta^n Q\|_{L^2}\|\delta^n Q\|_{L^4}\|Q\|_{L^\infty}(\|Q_n\|_{L^4} + \|Q\|_{L^4}) \\ &\quad + c\|Q\|_{L^\infty}\|\delta^n Q\|_{L^4}^2(\|Q_n\|_{L^2} + \|Q\|_{L^2}) \\ &\quad + \|\nabla \delta^n Q\|_{L^2}\|\nabla Q\|_{L^\infty}\|\nabla \delta^n u\|_{L^2} + \|\Delta Q\|_{L^\infty}\|\delta^n Q\|_{L^2}\|\nabla \delta^n u\|_{L^2} \\ &\leq \frac{1}{2}\|\Delta \delta^n Q\|_{L^2}^2 + \frac{1}{2}\|\nabla \delta^n u\|_{L^2}^2 + \frac{a}{2}\|\delta^n Q\|_{L^2}^2 \\ &\quad + C\|\delta^n u\|_{L^2}^2 \underbrace{\left(\|\nabla Q\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty}^2\right)}_{J_1} + C\|\nabla \delta^n Q\|_{L^2}^2 \underbrace{\left(\|u\|_{L^\infty}^2 + \|\nabla Q\|_{L^\infty}^2\right)}_{J_2} \\ &\quad + C\|\delta^n Q\|_{L^2}^2 \underbrace{\left(1 + \|\nabla u\|_{L^\infty}^2 + \|Q\|_{L^\infty}^2 + \|\Delta Q\|_{L^\infty}^2\right)}_{J_3} \\ &\quad + C\|\delta^n Q\|_{L^4}^2 \underbrace{\left(\|Q^n\|_{L^4}^2 + \|Q^n\|_{L^2} + \|Q\|_{L^2} + \|Q^n\|_{L^8}^4\right)}_{J_4} \\ &\quad + C\|\delta^n Q\|_{L^4}^2 \underbrace{\|Q\|_{L^\infty}\left(\|Q^n\|_{L^2} + \|Q\|_{L^2} + \|Q\|_{L^\infty}(\|Q^n\|_{L^4}^2 + \|Q\|_{L^4}^2)\right)}_{J_5}. \end{aligned}$$

Let

$$E_n(t) := \|\delta^n Q\|_{L^2}^2 + \|\Delta\delta^n Q\|_{L^2}^2 + \|\delta^n u\|_{L^2}^2,$$
$$E_n(0) := \|Q_0^n - Q_0\|_{L^2}^2 + \|\Delta Q_0^n - \Delta Q_0\|_{L^2}^2 + \|u_0^n - u_0\|_{L^2}^2,$$
$$A_n(t) := \sum_{i=1}^{5} J_i$$

Therefore, since

$$\|\delta^n Q\|_{L^4}^2 \le C\|\delta^n Q\|_{L^2}^{\frac{1}{2}}\|\nabla\delta^n Q\|_{L^2}^{\frac{3}{2}} \le C\left(\|\delta^n Q\|_{L^2}^2 + \|\nabla\delta^n Q\|_{L^2}^2\right),$$

we have

$$\frac{d}{dt}E_n(t) \le CA_n(t)E_n(t),$$

where C is indipendent of time. Since $H_n(t)$, $H(t)$ are uniformly bounded on $[0,T]$ and using the embedding relation $W^{1,q} \hookrightarrow L^\infty$, $\ q > 3$, we have that $A_n(t)$ is integrable on $[0,T]$, and $\int_0^T A_n(s)ds$ is bounded by a constant that is indipendent of n. It follows from the Gronwall inequality that for any $t \in (0,T]$

$$E_n(t) \le CE_n(0)\exp(C\int_0^t A_n(s)ds) \le CE_n(0).$$

Since $E_n(0) \to 0$, we get $E_n(t) \to 0$, as $n \to \infty$ uniformely on $[0,T]$. This ends the proof of Theorem B. □

Proof of Theorem C**.**

Proof. Firstly, we introduce some dimensionless quantities:

$$E_p(u,r) := \frac{1}{r^{5-p}}\int_{Q_r(z_0)}|u|^p dxds, \quad E_*(u,r) = \frac{1}{r}\int_{Q_r(z_0)}|\nabla u|^2 dxds,$$
$$E(u,r) := \sup_{t_0-r^2\le t<t_0}\frac{1}{r}\int_{B_r(x_0)}|u|^2dx, \quad P_{\frac{3}{2}}(P,r) := \frac{1}{r^2}\int_{Q_r(z_0)}|P|^{\frac{3}{2}}dxds,$$
$$E_*(u,\nabla Q;r) = E_*(u,r) + E_*(\nabla Q,r), \quad E(u,\nabla Q;r) := E(u;r) + E(\nabla Q;r),$$
$$E_p(u,\nabla Q;r) = E_p(u,r) + E_p(\nabla Q,r)$$

where $z_0 = (x_0,t_0) \in \mathbb{R}^3\times(0,\infty)$ and $Q_r(z_0) = Q_r(x_0,t_0) := B_r(x_0)\times[t_0-r^2,t_0]$. We will prove that the following dimensionless quantities are small:

$$E(u,\nabla Q;r) + E_*(u,\nabla Q;r) + P_{\frac{3}{2}}(P,r) \le \epsilon_1^2, \tag{3.15}$$

when r is enough small. Since $E_*(u,\nabla Q;r) = \frac{1}{r}\int_{Q_r(z_0)}|\nabla u|^2dxds + \frac{1}{r}\int_{Q_r(z_0)}|\nabla^2 Q|^2dxds$ the inequality (3.15) would imply (1.16) hand so thanks to Theorem 1.2 the solution (u,Q) is smooth near z_0. Therefore, it is enough to prove (3.15). We first remarke that in [23, 24], the following inequalities are established

$$E_3(u,r) \le C\|u\|_{L^\infty(t_0-r^2,t_0;\dot{B}^{-1}_{\infty,\infty})}^{\frac{3}{2}}\left(E^{\frac{3}{4}}(u,2r) + E_*^{\frac{3}{4}}(u,2r)\right), \tag{3.16}$$

and

$$E_3(\nabla Q,r) \le C\|\nabla Q\|_{L^\infty(t_0-r^2,t_0;\dot{B}^{-1}_{\infty,\infty})}^{\frac{3}{2}}\left(E^{\frac{3}{4}}(\nabla Q,2r) + E_*^{\frac{3}{4}}(\nabla Q,2r)\right). \tag{3.17}$$

Let $\varphi \in C_0^\infty(Q_{2r}(x_0,t_0))$ be a function such that

$$\varphi = 1 \ \text{ in } Q_r(x_0,t_0), \ \ |\nabla\varphi| \le \frac{1}{2r}, \ \ |\nabla^2\varphi| + |\varphi_t| \le \left(\frac{1}{2r}\right)^2.$$

We choose the $\phi = \varphi^2$ in the local energy inequality (1.15) and after some calculations, we get

$$\begin{aligned}
&\sup_{t_0-4r^2\le t\le t_0} \int_{\mathbb{R}^3}(|u|^2+|\nabla Q|^2)\varphi^2 dx + \int_{\mathbb{R}^3\times[t_0-4r^2,t_0]}(|\nabla u|^2+|\nabla^2 Q|^2)\varphi^2 dxdt \\
&\quad\le \int_{\mathbb{R}^3\times[t_0-4r^2,t_0]}(|u|^2+|\nabla Q|^2)(|\varphi_t||\varphi|+|\nabla\varphi|^2+|\nabla^2\varphi||\varphi|)dxdt \\
&\quad+ \int_{\mathbb{R}^3\times[t_0-4r^2,t_0]}[|u|^2+|\nabla Q|^2+|P|]|u||\nabla\varphi||\varphi|dxdt \\
&\quad+ (\|Q(t,\cdot)\|^2_{L^\infty(\mathbb{R}^3\times[t_0-4r^2,t_0])} + \|Q(t,\cdot)\|_{L^\infty(\mathbb{R}^3\times[t_0-4r^2,t_0])}) \int_{\mathbb{R}^3\times[t_0-4r^2,t_0]} |\nabla Q|^2\varphi^2 dxdt \\
&\quad+ \|Q\|_{L^\infty(\mathbb{R}^3\times[t_0-4r^2,t_0])} \int_{\mathbb{R}^3\times[t_0-4r^2,t_0]}(|\nabla u||\nabla Q|+|u||\Delta Q|)|\varphi||\nabla\varphi|dxdt.
\end{aligned} \tag{3.18}$$

In particolar, in order to establish (3.18), we use the following estimates

$$\int_{\mathbb{R}^3\times[t_0-4r^2,t_0]}[Q,\Delta Q]\cdot u\otimes\nabla\varphi^2(x,s)dxds \le \|Q\|_{L^\infty(\mathbb{R}^3\times[t_0-4r^2,t_0])}\int_{\mathbb{R}^3\times[t_0-4r^2,t_0]}|u||\Delta Q||\varphi||\nabla\varphi|dxdt$$

and

$$\begin{aligned}
\int_{\mathbb{R}^3\times[t_0-4r^2,t_0]}\nabla\mathcal{L}[\partial F(Q)]\cdot\nabla Q\varphi^2 dxds \le (&\|Q(t,\cdot)\|^2_{L^\infty(\mathbb{R}^3\times[t_0-4r^2,t_0])} \\
&+ \|Q(t,\cdot)\|_{L^\infty(\mathbb{R}^3\times[t_0-4r^2,t_0])})\int_{\mathbb{R}^3\times[t_0-4r^2,t_0]}|\nabla Q|^2\varphi^2 dxdt.
\end{aligned}$$

We now go on estimating the second term in the inequality (3.18). In the last line of (3.18), with the help of Young's inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3\times[t_0-4r^2,t_0]}(|\nabla u||\nabla Q|+|u||\Delta Q|)|\varphi||\nabla\varphi|dxdt \\
&\qquad\le \frac{1}{2\|Q\|_{L^\infty(\mathbb{R}^3\times[t_0-4r^2,t_0])}}\int_{\mathbb{R}^3\times[t_0-4r^2,t_0]}(|\nabla u|^2+|\nabla^2 Q|^2)\varphi^2 dxdt \\
&\qquad+ \|Q\|_{L^\infty(\mathbb{R}^3\times[t_0-4r^2,t_0])}\int_{\mathbb{R}^3\times[t_0-4r^2,t_0]}(|u|^2+|\nabla Q|^2)|\nabla\varphi|^2 dxdt,
\end{aligned} \tag{3.19}$$

and so the first right hand side term of the inequality (3.19) will be absorbed by the left hand side term of the inequality (3.18).

Divide by r both sides of inequality (3.18) and choose r such that $r^2 \le \frac{1}{8}$ in order to obtain

$$\frac{1}{r}\int_{\mathbb{R}^3\times[t_0-4r^2,t_0]}|\nabla Q|^2\varphi^2 dxdt \le \frac{1}{(2r)^3}\int_{B_{2r}(x_0,t_0)}|\nabla Q|^2 dxdt. \tag{3.20}$$

Then from the definition of $E(u,\nabla Q;r), E_*(u,\nabla Q;r)$ and inequalities (3.18), (3.19), (3.20), we get

$$\begin{aligned}
&E(u,\nabla Q;r)+E_*(u,\nabla Q;r) \\
&\le \frac{1}{(2r)^3}(\|Q(t,\cdot)\|^2_{L^\infty(\mathbb{R}^3\times[t_0-4r^2,t_0])} + \|Q(t,\cdot)\|_{L^\infty(\mathbb{R}^3\times[t_0-4r^2,t_0])}+1)\int_{Q_{2r}(x_0,t_0)}(|u|^2+|\nabla Q|^2)dxdt \\
&\quad+\frac{1}{(2r)^2}\int_{Q_{2r}(x_0,t_0)}|P||u|dxdt + \frac{1}{(2r)^2}\int_{Q_{2r}(x_0,t_0)}(|u|^2+|\nabla Q|^2)|u|dxdt \\
&:= I_1+I_2+I_3.
\end{aligned}$$

It is easy to check that

$$|I_1| \le (\|Q(t,\cdot)\|^2_{L^\infty(\mathbb{R}^3\times[t_0-4r^2,t_0])} + \|Q(t,\cdot)\|_{L^\infty(\mathbb{R}^3\times[t_0-4r^2,t_0])} + 1)E_3^{\frac{2}{3}}(u,\nabla Q;2r),$$

and

$$|I_2| \le E_3(u,\nabla Q;2r) + P_{\frac{3}{2}}(P,2r).$$

We estimate the term I_3 as follows:

$$\begin{aligned}|I_3| \le & E_3(u,\nabla Q;2r) + \frac{1}{(2r)^2}\bigg(\int_{Q_{2r}(x_0,t_0)}|\nabla Q|^3dxdt\bigg)^{\frac{2}{3}}\bigg(\int_{Q_{2r}(x_0,t_0)}|u|^3dxdt\bigg)^{\frac{1}{3}}\\ & \le CE_3(u,\nabla Q;2r).\end{aligned}$$

We remark that $\|Q(t,\cdot)\|_{L^\infty(\mathbb{R}^3\times[t_0-4r^2,t_0])} \le C\|Q_0\|_{L^\infty}$ see [21], so thanks to (3.16) e (3.17), we have

$$\begin{aligned}&E(u,\nabla Q;r) + E_*(u,\nabla Q;r) \le CE_3^{\frac{2}{3}}(u,\nabla Q;2r) + CE_3(u,\nabla Q;2r) + CP_{\frac{3}{2}}(P,2r)\\ &\le \|(u,\nabla Q)\|_{L^\infty(t_0-4r^2,t_0;\dot B^{-1}_{\infty,\infty})}\bigg(E^{\frac{1}{2}}(u,4r) + E_*^{\frac{1}{2}}(u,4r)\bigg)\\ &\quad + \|(u,\nabla Q)\|^{\frac{3}{2}}_{L^\infty(t_0-4r^2,t_0;\dot B^{-1}_{\infty,\infty})}\bigg(E^{\frac{3}{4}}(u,4r) + E_*^{\frac{3}{4}}(u,4r)\bigg)\\ &\quad + CP_{\frac{3}{2}}(p,2r).\end{aligned} \tag{3.21}$$

Then, we prove the following estimate of $P_{\frac{3}{2}}(P,r)$ similar to the Lemma 6.3 in paper [13].

$$P_{\frac{3}{2}}(P,r) \le C\left(\Big(\frac{\rho}{r}\Big)^2 E_3(u,\nabla Q;\rho) + \Big(\frac{r}{\rho}\Big)P_{\frac{3}{2}}(P,\rho)\right) \quad \text{when } 0<r\le\frac{\rho}{2}. \tag{3.22}$$

From the scaling invariance of all quantities, we only need to consider the case $\rho = 1$, $0 < r \le \frac{1}{2}$, $z_0 = (0,0)$. Taking the divergence of the second equation in (1.6) and using the fact [Lemma 2.3 in [13]],

$$\mathrm{div}^2[Q,H] = 0, \quad \text{in } \mathbb{R}^3,$$

we obtain

$$-\Delta P = \mathrm{div}^2[u\otimes u + \nabla Q\otimes\nabla Q]$$

Let $\eta \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function of $B_{\frac{2}{3}}(0)$ such that

$$\eta = 1, \quad \text{in } B_{\frac{2}{3}}(0), \quad \eta = 0, \quad \text{in } \mathbb{R}^3\setminus B_1(0), \quad 0\le\eta\le 1, \quad |\nabla\eta|\le 8.$$

Define the following auxiliary function:

$$P_1(x,t) = -\int_{\mathbb{R}^3}\nabla_y^2 G(x-y):\eta^2(y)[u\otimes u + \nabla Q\otimes\nabla Q](y,t)dy.$$

where G is the fundamental solution of harmonic equation. It is easy to see that the P_1 is solution of the following equation:

$$-\Delta P_1 = \mathrm{div}^2[u\otimes u + \nabla Q\otimes\nabla Q] \ \text{ in } B_{\frac{2}{3}}(0),$$

and

$$-\Delta(P-P_1) = 0 \ \text{ in } B_{\frac{2}{3}}(0).$$

We estimate the P_1 by the Calderon-Zygmund theory:

$$\begin{aligned}\|P_1\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\mathbb{R}^3)} &\le \|\eta^2|u|^2\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\mathbb{R}^3)} + \|\eta^2|\nabla Q|^2\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\mathbb{R}^3)} \\ &\le \|u\|^3_{L^3(B_1(0))} + \|\nabla Q\|^3_{L^3(B_1(0))}.\end{aligned} \tag{3.23}$$

Since $P - P_1$ is harmonic in $B_{\frac{2}{3}}(0)$, using the mean-value property, one has

$$\begin{aligned}\frac{1}{r^2}\|P - P_1\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(B_r(0))} &= \frac{1}{r^2}\int_{B_r(0)} |P - P_1|^{\frac{3}{2}} dx \\ &\le \frac{1}{r^2} r^3 \|P - P_1\|^{\frac{3}{2}}_{L^\infty(B_r(0))} \\ &\le \frac{1}{r^2} r^3 \sup_{x\in B_r(0)} \left(\frac{1}{B_{\frac{1}{8}}(x)}\int_{B_{\frac{1}{8}}(x)} |P - P_1| dx\right)^{\frac{3}{2}} \\ &\le r\int_{B_1(0)} |P - P_1|^{\frac{3}{2}} dx \\ &\le r(\|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(B_1(0))} + \|P_1\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(B_1(0))}).\end{aligned} \tag{3.24}$$

where we have used the fact $B_{\frac{1}{8}}(x) \subset B_{\frac{2}{3}}(0)$, for $x \in B_r(0)$ since $r \le \frac{1}{2}$. Integrating $\|P\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(B_1(0))}$ over $[-r^2, 0]$ and using the previous estimates (3.23), (3.24), we can show

$$\frac{1}{r^2}\int_{Q_r(0,0)} |P|^{\frac{3}{2}} dxds \le Cr P_{\frac{3}{2}}(P, 1) + C\frac{1}{r^2} E_3(u, \nabla Q; 1), \quad \text{with} \quad \rho = 1,$$

so we complete the proof of the inequality (3.22). It follows from (3.16), (3.17) and (3.22) that

$$\begin{aligned}P_{\frac{3}{2}}(P, r) &\le C\left(\left(\frac{\rho}{r}\right)^2 E_3(u, \nabla Q; \rho) + \frac{r}{\rho} P_{\frac{3}{2}}(P, \rho)\right) \\ &\le C\left(\left(\frac{\rho}{r}\right)^2 \|(u, \nabla Q)\|^{\frac{3}{2}}_{L^\infty(t_0-\rho^2, t_0; \dot{B}^{-1}_{\infty,\infty})}\left(E^{\frac{3}{4}}(u, 2\rho) + E^{\frac{3}{4}}_*(u, 2\rho)\right) + \frac{r}{\rho} P_{\frac{3}{2}}(P; 2\rho)\right) \\ &\le C\left(\left(\frac{\varrho}{r}\right)^2 \|(u, \nabla Q)\|^{\frac{3}{2}}_{L^\infty(t_0-\rho^2, t_0; \dot{B}^{-1}_{\infty,\infty})}\left(E^{\frac{3}{4}}(u, \varrho) + E^{\frac{3}{4}}_*(u, \varrho)\right) + \frac{r}{\varrho} P_{\frac{3}{2}}(P; \varrho)\right),\end{aligned} \tag{3.25}$$

where $\varrho := 2\rho$.

Set $F_1(r) := E(u, \nabla Q, r) + E_*(u, \nabla Q; r) + P_{\frac{3}{2}}(P, r)$ and $M(r) := \|(u, \nabla Q)\|_{L^\infty(t_0-r^2, t_0; \dot{B}^{-1}_{\infty,\infty})}$, $M := M(\tilde{\varrho})$. Let $\tilde{\varrho}$ be a fixed positive constant, then for r small enough, we have

$$E^{\frac{1}{2}}(u, 4r) \le (\frac{\tilde{\varrho}}{r})^{\frac{1}{2}} E^{\frac{1}{2}}(u, \tilde{\varrho}),$$

and so (3.21) and (3.25) with $\varrho = \tilde{\varrho}$ implies that for fixed $\tilde{\varrho}$ and enough small r

$$\begin{aligned}F_1(r) &\le C\Bigg(M(\frac{\tilde{\varrho}}{r})^{\frac{1}{2}}(E^{\frac{1}{2}}(u, \nabla Q; \tilde{\varrho}) + E^{\frac{1}{2}}_*(u, \nabla Q; \tilde{\varrho})) \\ &\quad + M^{\frac{3}{2}}(\frac{\tilde{\varrho}}{r})^{\frac{3}{4}}(E^{\frac{3}{4}}(u, \nabla Q; \tilde{\varrho}) + E^{\frac{3}{4}}_*(u, \nabla Q; \tilde{\varrho})) \\ &\quad + M^{\frac{3}{2}}(\frac{\tilde{\varrho}}{r})^2\left(E^{\frac{3}{4}}(u, \tilde{\varrho}) + E^{\frac{3}{4}}_*(u, \tilde{\varrho})\right) + \frac{r}{\tilde{\varrho}} P_{\frac{3}{2}}(P; \tilde{\varrho})\Bigg) \\ &\le C\left((M + M^{\frac{3}{2}})\left(\frac{\tilde{\varrho}}{r}\right)^2 (F_1^{\frac{3}{4}}(\tilde{\varrho}) + F_1^{\frac{1}{2}}(\tilde{\varrho})) + \frac{r}{\tilde{\varrho}}(F_1(\tilde{\varrho}))\right) \\ &\le C\left(\frac{r}{\tilde{\varrho}} F_1(\tilde{\varrho}) + (M + M^{\frac{3}{2}})^4\left(\frac{\tilde{\varrho}}{r}\right)^{11}\right).\end{aligned} \tag{3.26}$$

where in the last inequality we control the term $(M+M^{\frac{3}{2}})(\frac{\tilde{\varrho}}{r})^2(F_1^{\frac{3}{4}}(\tilde{\varrho})+F_1^{\frac{1}{2}}(\tilde{\varrho}))$ by $C(\frac{r}{\tilde{\varrho}})F_1(\tilde{\varrho})+C(M+M^{\frac{3}{2}})^4(\frac{\tilde{\varrho}}{r})^{11}$ by Young inequality. Note the constant C in equality (3.22) is independent on the ratio r with ρ. Set $\theta := \frac{r}{\tilde{\varrho}}$ with r small enough such that $C\theta < 1$. Then from the inequality (3.26) it follows that

$$F_1(\theta\tilde{\varrho}) \leq C\theta F_1(\tilde{\varrho}) + C\frac{(M+M^{\frac{3}{2}})^4}{\theta^{11}},$$

Iterating the above inequality, we get

$$F_1(\theta^k\tilde{\varrho}) \leq (C\theta)^k F_1(\tilde{\varrho}) + C(M+M^{\frac{3}{2}})^4(\frac{1}{\theta})^{11}\frac{1-(C\theta)^k}{1-(C\theta)} \tag{3.27}$$

Now we prove that there exists $k_0 \in N$ such that $\forall k \geq k_0$,

$$F_1(\theta^k\tilde{\varrho}) \leq \epsilon_1^2\theta^2, \tag{3.28}$$

where ϵ_1 is the constant appearing in the Theorem 1.2. We remark that the following embedding holds

$$\|\nabla Q\|_{L^\infty(t_0-r^2,t_0;\dot{B}^{-1}_{\infty,\infty})} \leq \|Q\|_{L^\infty(t_0-r^2,t_0;\dot{B}^{0}_{\infty,\infty})} \leq \|Q(t)\|_{L^\infty(t_0-r^2,t_0;L^\infty)} \leq e^{ct}\|Q_0\|_{L^\infty},$$

thanks to the property

$$\|Q(t,\cdot)\|_{L^\infty} \leq e^{ct}\|Q_0\|_{L^\infty}, \quad \forall\ t \geq 0.$$

proved in [21]. So using the hypothesis (1.17) and (1.18), we get

$$M \leq \varepsilon_0 + e^{ct_0}\|Q_0\|_{L^\infty} \leq \varepsilon_0 + e^{ct_0}\varepsilon_1.$$

Therefore, choosing k_0 large enough, and $\varepsilon_0, \varepsilon_1$ small enough, we get (3.28). For any $0 < r < \theta^{k_0}\tilde{\rho}$, we can find k_1 such that $\theta^{k_1+1}\tilde{\varrho} \leq r \leq \theta^{k_1}\tilde{\varrho}$, From the definition of F_1, one gets

$$F_1(r) \leq \frac{1}{\theta^2}F_1(\theta^{k_1}\tilde{\varrho}) \leq \epsilon_1^2.$$

Moreover, we can get the solution (u, Q) is smooth near the point $(x_0, t_0) \in \mathbb{R}^3 \times (0, +\infty)$ by Theorem 1.2. So we complete the proof of Theorem C. □

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Declarations.

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References

[1] H. Abels, G. Dolzmann, and Y. Liu, *Well-posedness of a fully coupled Navier-Stokes/Q-tensor system with inhomogeneous boundary data*, SIAM J. Math. Anal., 46 (2014), pp. 3050–3077.

[2] H. Amann, *Linear and quasilinear parabolic problems. Vol. I*, vol. 89 of Monographs in Mathematics, Birkhäuser Boston, Inc., Boston, MA, 1995. Abstract linear theory.

[3] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Grundlehren der Mathematischen Wissenschaften, No. 223, Springer-Verlag, Berlin-New York, 1976.

[4] A. N. Beris and B. J. Edwards, *Thermodynamics of flowing systems with internal microstructure*, vol. 36 of Oxford Engineering Science Series, The Clarendon Press, Oxford University Press, New York, 1994. Oxford Science Publications.

[5] L. Caffarelli, R. Kohn, and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math., 35 (1982), pp. 771–831.

[6] C. Cavaterra, E. Rocca, H. Wu, and X. Xu, *Global strong solutions of the full Navier-Stokes and Q-tensor system for nematic liquid crystal flows in two dimensions*, SIAM J. Math. Anal., 48 (2016), pp. 1368–1399.

[7] M. Dai, E. Feireisl, E. Rocca, G. Schimperna, and M. E. Schonbek, *On asymptotic isotropy for a hydrodynamic model of liquid crystals*, Asymptot. Anal., 97 (2016), pp. 189–210.

[8] R. Danchin, *Density-dependent incompressible fluids in bounded domains*, J. Math. Fluid Mech., 8 (2006), pp. 333–381.

[9] F. De Anna, *A global 2D well-posedness result on the order tensor liquid crystal theory*, J. Differential Equations, 262 (2017), pp. 3932–3979.

[10] F. de Anna and A. Zarnescu, *Uniqueness of weak solutions of the full coupled Navier-Stokes and Q-tensor system in 2D*, Commun. Math. Sci., 14 (2016), pp. 2127–2178.

[11] P.-G. De Gennes and J. Prost, *The physics of liquid crystals*, no. 83, Oxford university press, 1993.

[12] C. Denniston, D. Marenduzzo, E. Orlandini, and J. M. Yeomans, *Lattice Boltzmann algorithm for three-dimensional liquid-crystal hydrodynamics*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 362 (2004), pp. 1745–1754.

[13] H. Du, X. Hu, and C. Wang, *Suitable weak solutions for the co-rotational Beris-Edwards system in dimension three*, Arch. Ration. Mech. Anal., 238 (2020), pp. 749–803.

[14] J. L. Ericksen, *Hydrostatic theory of liquid crystals*, Arch. Rational Mech. Anal., 9 (1962), pp. 371–378.

[15] Y. Giga and H. Sohr, *Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains*, J. Funct. Anal., 102 (1991), pp. 72–94.

[16] X. Hu and D. Wang, *Global solution to the three-dimensional incompressible flow of liquid crystals*, Comm. Math. Phys., 296 (2010), pp. 861–880.

[17] F. Lin, *A new proof of the Caffarelli-Kohn-Nirenberg theorem*, Comm. Pure Appl. Math., 51 (1998), pp. 241–257.

[18] Q. Liu, *On partial regularity criterion for the co-rotational Beris-Edwards system modeling nematic liquid crystal flow*, J. Differential Equations, 301 (2021), pp. 300–329.

[19] Y. Liu and W. Wang, *On the initial boundary value problem of a Navier-Stokes/Q-tensor model for liquid crystals*, Discrete Contin. Dyn. Syst. Ser. B, 23 (2018), pp. 3879–3899.

[20] M. Paicu and A. Zarnescu, *Global existence and regularity for the full coupled Navier-Stokes and Q-tensor system*, SIAM J. Math. Anal., 43 (2011), pp. 2009–2049.

[21] ——, *Energy dissipation and regularity for a coupled Navier-Stokes and Q-tensor system*, Arch. Ration. Mech. Anal., 203 (2012), pp. 45–67.

[22] M. Schonbek and Y. Shibata, *Global well-posedness and decay for a $\mathbb{Q}$ tensor model of incompressible nematic liquid crystals in $\mathbb{R}^N$*, J. Differential Equations, 266 (2019), pp. 3034–3065.

[23] G. Seregin and D. Zhou, *Regularity of solutions to the Navier-Stokes equations in $\dot{B}^{-1}_{\infty,\infty}$*, (arXiv:1802.03600v2).

[24] ——, *Regularity of solutions to the Navier-Stokes equations in $\dot{B}^{-1}_{\infty,\infty}$*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 477 (2018), pp. 119–128.

[25] T. Tao, *Nonlinear dispersive equations*, vol. 106 of CBMS Regional Conference Series in Mathematics, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. Local and global analysis.

[26] G. Tóth, C. Denniston, and J. M. Yeomans, *Hydrodynamics of domain growth in nematic liquid crystals*, Phys. Rev. E, 67 (2003), p. 051705.

[27] Y. Wang, B. Yuan, J. Zhao, and D. Zhou, *On the regularity of weak solutions of the MHD equations in BMO^{-1} and $\dot{B}^{-1}_{\infty,\infty}$*, J. Math. Phys., 62 (2021), pp. Paper No. 091509, 13.

[28] Y. Xiao, *Global strong solution to the three-dimensional liquid crystal flows of Q-tensor model*, J. Differential Equations, 262 (2017), pp. 1291–1316.