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Dislocations in a layered elastic medium with applications to fault detection

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Abstract. We consider a model for elastic dislocations in geophysics. We model a portion of the Earth's crust as a bounded, inhomogeneous elastic body with a buried fault surface, along which slip occurs. We prove well-posedness of the resulting mixed-boundary-value-transmission problem, assuming only bounded elastic moduli. We establish uniqueness in the inverse problem of determining the fault surface and the slip from a unique measurement of the displacement on an open patch at the surface, assuming in addition that the Earth's crust is an isotropic, layered medium with Lamé coefficients piecewise Lipschitz on a known partition and that the fault surface satisfies certain geometric conditions. These results substantially extend those of the authors in [Arch. Ration. Mech. Anal. 263, 71–111 (2020)].

Keywords. Dislocations, elasticity, Lamé system, well-posedness, inverse problem, uniqueness

1. Introduction

The focus of this work is an analysis of both the forward or direct problem, as well as the inverse problem, for a model of buried faults in the Earth's crust. Specifically, we prove well-posedness of the direct problem, assuming only L^∞ elastic coefficients, and uniqueness in the inverse problem, under additional assumptions, which are motivated by the ill-posedness of the inverse problem and are not overly restrictive for the applications we are concerned about.

We model the Earth's crust as a layered, inhomogeneous elastic medium, and the fault as an oriented, open surface S immersed in this elastic medium and not reaching the surface (the case of *buried* or *blind faults*), along which there can be slippage of the rock. Faults can have any orientation with respect to the surface: horizontal, vertical, or oblique.

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When slip occurs, we speak of elastic *dislocations*. Mathematically, the slip is given by a non-trivial jump in the elastic displacement across the fault, represented by a non-zero vector field $\boldsymbol{g}$ on S. The surface of the Earth can be assumed traction free, that is, no load is bearing on it, while on the fault itself one can assume that the jump in the traction is zero, that is, the loads on the two sides of the fault balance out. (We refer the reader to [15, 17] for instance for a mathematical treatment of elasticity.)

The *direct* or *forward* problem consists in finding the elastic displacement in the Earth's crust induced by the slip on the fault. The *inverse* problem consists in determining the fault surface S and the slip $\boldsymbol{g}$ from measurements of surface displacement. The inverse problem has important applications in seismology and geophysics. The surface displacement can be inferred from Synthetic Aperture Radar (SAR) and from Global Positioning System (GPS) arrays monitoring (see e.g. [20, 37, 45, 46]).

In the so-called *interseismic* period, that is, the (usually long) period between earthquakes, one can make a quasi-static approximation and work within the framework of elastostatics. In seismology, the assumption of small deformations is generally a good approximation away from active faults, and therefore linear elastostatics is typically employed. Near active faults, and especially during earthquakes, the so-called *co-seismic* period, more accurate models assume the rock is viscoelastic. However, a rigorous analysis of these more complex, non-linear models is still essentially missing. We plan to address non-linear and non-local models in future work.

The study of elastic dislocations is classical in the context of isotropic, homogeneous, linear elasticity, when the surface S is assumed to be of a particular simple form, that is, a rectangular fault that has a not-too-big inclination angle with respect to the unperturbed, flat Earth's surface. (We refer to [16, 36] and references therein for a more in-depth discussion.) In this case, modeling the Earth's crust as an infinite half-space, there exists an explicit formula for the displacement field induced by the slip on the fault, due to Okada [27] (see also [25]). To our knowledge, there are few works that tackle the forward problem in the case of non-homogeneous regular coefficients and more realistic geometries for the fault. Indeed, the problem is intrinsically singular along the fault, where non-standard transmission conditions are imposed. A variational formulation of the problem for a bounded domain was introduced in [41].

In [9], we proved well-posedness of the direct problem for elastic dislocations, assuming the Earth's crust is an infinite half-space, the elastic coefficients are Lipschitz continuous, and the surface S is also of Lipschitz class. We also established uniqueness in the inverse problem from one measurement of surface displacement on an open patch, under some additional assumptions on the geometry of the fault and the slip, namely we took S to be a graph with respect to an arbitrary, but given, coordinate system, we assumed that S has at least one corner singularity, and that $\boldsymbol{g}$ is tangential to S. The main difficulties in that work were twofold. On the one hand, we had to work with suitably weighted Sobolev spaces in order to control the slow decay of solutions at infinity. On the other, we allowed slips that do not vanish anywhere on S. Then the solution at the boundary of the fault may develop singularities, for instance in the case of constant slip and a rectangular fault, for which logarithmic blow-up at the vertices exists, as noted already by Okada [27].

These potential singularities are unphysical and do not allow for a variational approach to well-posedness. Instead, owing to the regularity of the coefficients, we used a duality argument for an equivalent source problem. We also established a double-layer-potential representation for the solution. Uniqueness for the inverse problem was obtained using unique continuation, again owing to the regularity of the coefficients.

The main focus of this work is to generalize the results in [9] to a more realistic set-up. We model a portion of the Earth's crust, where the fault is located, as a Lipschitz bounded domain Ω, which includes the case of polyhedral domains, relevant to numerical implementations and applications. The direct problem consists in solving a mixed-boundary-value-transmission problem for the elasticity system in Ω, given in equation (6). On the buried part of the boundary of Ω, which we call Σ, we impose homogeneous Dirichlet boundary conditions, that is, zero displacement. Such boundary conditions model the situation where the relative motion of rock formations is small away from the fault as compared to that near the fault itself, except at the surface of the Earth due to the traction-free assumption there. This assumption implicitly includes that Ω is large enough compared to the size of the active portion of the fault where the slippage occurred, so our model is not well suited for large, active faults. A non-zero displacement on Σ can also be imposed and other types of boundary conditions on Σ can be treated, such as inhomogeneous Neumann boundary conditions, modeling the load bearing on the rock formations at the boundary from nearby formations. We assume that the Earth's crust is a layered elastic medium, a common assumption in geophysics, that is, we assume that the elastic coefficients are piecewise regular, but may jump across a known partition of Ω (see Figure 1), and impose standard transmission conditions at the interfaces of the partition. This set-up has been considered in the literature to model dislocations in geophysics (see for example [32, 35, 43]). Furthermore, posing the problem in a bounded domain lends itself naturally to a numerical implementation that does not utilize boundary integral equations, but instead uses a variational formulation for the problem [6, 7].

In this work, we assume that the slip $\boldsymbol{g}$ vanishes at the boundary of the fault. We are therefore modeling the case of an *unlocked fault patch* on only a part of the fault. By unlocked fault patch we mean a part of the fault surface where the rocks of the two sides of

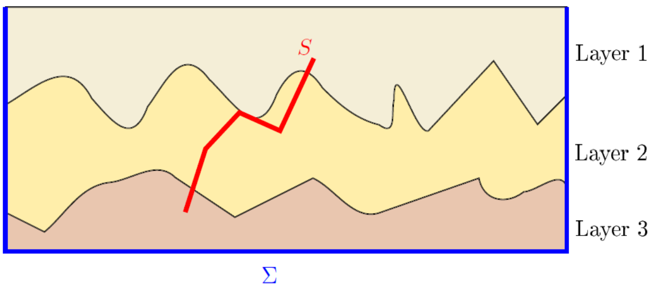

Fig. 1. An example of the geometrical setting. A section of a layered medium with S, the dislocation surface, and with Σ, the buried part of Ω.

the fracture have slipped freely relative to one another. It is observed that most faults have a distribution of locked and unlocked patches. The quasi-static approximation can be used for so-called *aseismic creeping faults*, that is, faults where the rock slowly slips without major seismic events. There are known creeping faults in major populated areas of Japan and California, for instance. Creeping faults are also relevant for *microseismicity*, which indicates frequent seismic activity of small amplitude. (Among the vast literature on the subject, we refer the reader to [18, 26, 33, 34].) The support of $\boldsymbol{g}$ can still be the entire fault surface, a situation that arises in the inverse problem. Then a variational solution exists for problem (6), constructed by solving suitable auxiliary Neumann and mixed-boundary-value problems, after [2]. For the direct problem, well-posedness holds if the elasticity tensor $\mathbb{C}$ is an (anisotropic) bounded, strongly convex tensor.

For the inverse problem, we require more. On the one hand, we need to guarantee that unique continuation for the elasticity system holds. This can be achieved by assuming that Ω is partitioned into finitely many Lipschitz subdomains and assuming that $\mathbb{C}$ is isotropic with Lamé coefficients that are Lipschitz continuous in each subdomain (see [4, 11, 12, 14] where a similar approach has been used to determine internal properties of an elastic medium from boundary measurements). On the other hand, the uniqueness proof, which uses an argument by contradiction, can be guaranteed to hold when S is a graph with respect to an arbitrary, but chosen, coordinate system. This assumption is again not too restrictive in the geophysical context and allows for an arbitrary orientation of the surface, horizontal, vertical, or oblique (see Remark 4.2). Differently than in [9], however, due to the fact that the slip vanishes on the boundary of S, one does not need to assume S has a corner singularity or assume a specific direction for the slip field $\boldsymbol{g}$. Therefore, the results presented here are a substantial generalization of known results for both the well-posedness of the direct problem and the uniqueness of the inverse problem.

The inverse dislocation problem has been treated both within the mathematics community [44], as well as in the geophysics community (among the extensive literature we mention [8, 29, 30] and references therein). Reconstruction has been tested primarily through iterative algorithms [44], based on Newton's methods or constrained optimization of a suitable misfit functional, using either Boundary Integral methods or Finite Element methods, as well as Green's function methods to solve the direct problem. For stochastic and statistical approaches to inversion we mention [24, 42] and references therein. We do not address here the question of reconstruction and its stability (see [13, 40]). This is the focus of future work, which we plan to conduct by using appropriate iterative algorithms and solving the direct problem via discontinuous Galerkin methods (for example adapting the methods in [6, 7]).

We close this introduction with a brief outline of the paper. In Section 2, we introduce the relevant notation and the function spaces used throughout. In Section 3, we discuss the main assumptions on the coefficients and the geometry, and we address the well-posedness of the direct problem, while we discuss additional assumptions and prove uniqueness for the inverse problem in Section 4.

2. Notation and functional setting

We begin by introducing the needed notation and the functional setting for both the direct and inverse problems.

Notation: We denote scalar quantities in italics, e.g. λ, μ, ν, points and vectors in bold italics, e.g. $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$, matrices and second-order tensors in boldface, e.g. $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and fourth-order tensors in blackboard font, e.g. $\mathbb{A}, \mathbb{B}, \mathbb{C}$.

The symmetric part of a second-order tensor $\mathbf{A}$ is denoted by $\widehat{\mathbf{A}} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$, where $\mathbf{A}^T$ is the transpose matrix. In particular, $\widehat{\nabla} \boldsymbol{u}$ represents the deformation tensor. We utilize standard notation for inner products, that is, $\boldsymbol{u} \cdot \boldsymbol{v} = \sum_i u_i v_i$ and $\mathbf{A} : \mathbf{B} = \sum_{i,j} a_{ij} b_{ij}$; $|\mathbf{A}|$ denotes the norm induced by the inner product on matrices:

$$|\mathbf{A}| = \sqrt{\mathbf{A} : \mathbf{A}}.$$

Domains: Given $r > 0$, we denote the ball of radius r and center $\boldsymbol{x}$ by $B_r(\boldsymbol{x}) \subset \mathbb{R}^3$ and a circle of radius r and center $\boldsymbol{y}$ by $B'_r(\boldsymbol{y}) \subset \mathbb{R}^2$.

Definition 2.1 ($C^{k,\alpha}$ regularity of domains). Let Ω be a bounded domain in $\mathbb{R}^3$. Given k, α, with $k \in \mathbb{N}$ and $0 < \alpha \leq 1$, we say that a portion Υ of $\partial\Omega$ is of *class* $C^{k,\alpha}$ *with constants* r_0, E_0 if for any $\boldsymbol{P} \in \Upsilon$, there exists a rigid transformation of coordinates under which $\boldsymbol{P}$ is mapped to the origin and

$$\Omega \cap B_{r_0}(\mathbf{0}) = \{\boldsymbol{x} \in B_{r_0}(\mathbf{0}) : x_3 > \psi(\boldsymbol{x}')\},$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{x}' = (x_1, x_2, 0)$ identified canonically with $(x_1, x_2) \in \mathbb{R}^2$. Above, ψ is a $C^{k,\alpha}$ function on $B'_{r_0}(\mathbf{0}) \subset \mathbb{R}^2$ such that

$$\begin{aligned} \psi(\mathbf{0}) &= 0, \\ \nabla\psi(\mathbf{0}) &= \mathbf{0} \quad \text{for } k \geq 1, \\ \|\psi\|_{C^{k,\alpha}(B'_{r_0}(\mathbf{0}))} &\leq E_0. \end{aligned}$$

When $k = 0, \alpha = 1$, we also say that Υ is of *Lipschitz class with constants* r_0, E_0.

Similarly, we define a surface S to be of class $C^{k,\alpha}$ if it is locally the graph of a function ψ with the properties described above. For $k = 0$, $\alpha = 1$, the case we are interested in, since the composition of Lipschitz maps is Lipschitz, we can define a Lipschitz curve on a Lipschitz surface S (such as its boundary ∂S) by lifting a Lipschitz curve on $\mathbb{R}^2$.

Given a bounded domain $\Omega \subset \mathbb{R}^3$ such that $\overline{\Omega} := \overline{\Omega^+} \cup \overline{\Omega^-}$, where Ω^+ and Ω^- are bounded domains, we write f^+ and f^- for the restrictions of a function or distribution f to Ω^+ and Ω^-, respectively. We denote the jump of a function or vector field $\boldsymbol{f}$ across a bounded oriented surface S by $[\boldsymbol{f}]_S := \boldsymbol{f}^+_S - \boldsymbol{f}^-_S$, where $\pm$ denote the non-tangential limits from the two sides S^+ and S^- of S, where S^+ is by convention the side where the unit normal vector $\boldsymbol{n}$ points into and $\boldsymbol{n}$ is determined by the given orientation on S.

Functional setting: We use standard notation to denote the usual function spaces, e.g. $H^s(\Omega)$ denotes the L^2-based Sobolev space with regularity index $s \in \mathbb{R}$, and $C_0^\infty(\Omega)$ is the space of smooth functions with compact support in Ω.

We will need to consider trace spaces on open bounded surfaces that have a good extension property to closed surfaces containing them. (We refer to [23, 38] for an in-depth discussion). In what follows, D is a given open bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2$ or $n = 3$.

We recall that fractional Sobolev spaces on D can be defined via real interpolation, and that $H_0^s(D) := \overline{C_0^\infty(D)}^{\|\cdot\|_{H^s(D)}}$, $s \geq 0$. We also recall that $H^s(D) = H_0^s(D)$, $0 \leq s \leq 1/2$. If $s < 1/2$, it is possible to extend an element of $H^s(D)$ by zero in $\mathbb{R}^n \setminus D$ to an element of $H^s(\mathbb{R}^n)$. When $s = 1/2$ such an extension is possible for elements that are suitably weighted by the distance to the boundary, since the extension operator from $H_0^{1/2}(D)$ to $H_0^{1/2}(\mathbb{R}^n)$ is not continuous. Following Lions and Magenes [23], we introduce the space

$$H_{00}^{1/2}(D) := \{u \in H_0^{1/2}(D) : \delta^{-1/2} u \in L^2(D)\}, \tag{1}$$

where $\delta(\boldsymbol{x}) = \mathrm{dist}(\boldsymbol{x}, \partial D)$ for $\boldsymbol{x} \in D$. This space is equipped with its natural norm, i.e.

$$\|f\|_{H_{00}^{1/2}(D)} := \|f\|_{H^{1/2}(D)} + \|\delta^{-1/2} f\|_{L^2(D)},$$

which gives a finer topology than that of $H^{1/2}(D)$. Here δ can be replaced by a function $\varrho \in C^\infty(D)$ that is comparable to the distance to the boundary, in the sense that

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}_0} \frac{\varrho(\boldsymbol{x})}{\mathrm{dist}(\boldsymbol{x}, \partial D)} = d \neq 0, \quad \forall \boldsymbol{x}_0 \in \partial D, \tag{2}$$

$\varrho > 0$ in D and ϱ vanishes on ∂D (see e.g. [47, Lemma 3.6.1]). We opted for the definition above of $H_{00}^{1/2}(D)$ as we do not need to consider higher-order traces. If $v \in H_{00}^{1/2}(D)$, then its extension by zero to $\mathbb{R}^n \setminus D$ is an element of $H^{1/2}(\mathbb{R}^n)$ and the extension operator is bounded. In particular, $v = 0$ on ∂D in the trace sense. The space $H_{00}^{1/2}$ can also be identified with a real interpolation space (see e.g. [1, Chapter 7]):

$$H_{00}^{1/2}(D) = (H_0^1(D), L^2(D))_{1/2,2}.$$

We will also need to define $H_{00}^{1/2}(S)$ where S is a Lipschitz surface. We can define $H_0^{1/2}(S)$ in a standard way using partitions of unity and coordinate charts (see e.g. [1,39]). Then, we define $H_{00}^{1/2}(S)$ as in (1), where δ is the distance induced by surface area on S. In addition, Lipschitz domains are extension domains for Sobolev spaces [47], so that using a local coordinate chart, we can prove the extension property from $H_{00}^{1/2}(S)$ to $H^{1/2}(\Gamma)$, where Γ is any closed surface containing S.

Let $\mathcal{R}$ be the space of infinitesimal rigid motions in $\mathbb{R}^3$. To study the well-posedness of the direct problem, we introduce two variational spaces,

$$\mathring{H}^1(D) = \left\{\boldsymbol{\eta} \in H^1(D) : \int_D \boldsymbol{\eta} \cdot \boldsymbol{r}\, d\boldsymbol{x} = 0,\ \forall \boldsymbol{r} \in \mathcal{R}\right\}, \tag{3}$$

$$H_\Sigma^1(D) = \{\boldsymbol{\eta} \in H^1(D) : \boldsymbol{\eta}|_\Sigma = \boldsymbol{0}\}, \tag{4}$$

where Σ denotes the closure of an open subset of ∂D.

Finally, we denote the duality pairing between a Banach space X and its dual X' by $\langle\cdot,\cdot\rangle_{(X',X)}$. When clear from the context, we will omit the explicit dependence on the spaces, writing $\langle\cdot,\cdot\rangle$. We will write $\langle\cdot,\cdot\rangle_D$ for the pairing restricted to a domain D.

3. The direct problem

We first discuss the main assumptions on the dislocation surface S and the elastic tensor $\mathbb{C}$, used in the rest of the paper. Then, we study the well-posedness of the forward problem. Below, Ω is a bounded Lipschitz domain.

Assumption 1 (elasticity tensor). The elasticity tensor $\mathbb{C} = \mathbb{C}(\boldsymbol{x})$ is a fourth-order tensor satisfying the full symmetry properties

$$\mathbb{C}_{ijkh}(\boldsymbol{x}) = \mathbb{C}_{jikh}(\boldsymbol{x}) = \mathbb{C}_{khij}(\boldsymbol{x}), \quad \forall 1 \leq i,j,k,h \leq 3, \text{ and } \boldsymbol{x} \in \Omega,$$

is uniformly bounded, $\mathbb{C} \in L^\infty(\Omega)$, and is uniformly strongly convex, that is, $\mathbb{C}$ defines a positive-definite quadratic form on symmetric matrices:

$$\mathbb{C}(\boldsymbol{x})\widehat{\mathbf{A}} : \widehat{\mathbf{A}} \geq c|\widehat{\mathbf{A}}|^2, \quad \text{a.e. in } \Omega,$$

for some $c > 0$.

Assumption 2 (dislocation surface). We model the dislocation surface S by an open, bounded, oriented Lipschitz surface, with Lipschitz boundary, such that

$$\overline{S} \subset \Omega. \tag{5}$$

We assume that S can be extended to a closed Lipschitz, orientable surface Γ satisfying

$$\Gamma \cap \partial\Omega = \emptyset.$$

Moreover, we denote by Ω^- the domain enclosed by Γ and set $\Omega^+ = \Omega \setminus \overline{\Omega^-}$. We choose the orientation on S so that the associated normal $\boldsymbol{n}$ coincides with the unit outer normal to Ω^-.

In this section, we study the following mixed-boundary-value problem:

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \mathbf{0} & \text{in } \Omega \setminus \overline{S}, \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{\nu} = \mathbf{0} & \text{on } \partial\Omega \setminus \Sigma, \\ \boldsymbol{u} = \mathbf{0} & \text{on } \Sigma, \\ [\boldsymbol{u}]_S = \boldsymbol{g}, & \\ [(\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n}]_S = \mathbf{0}, & \end{cases} \tag{6}$$

where Σ is the closure of an open subset in $\partial\Omega$, $\boldsymbol{n}$ is the normal vector induced by the orientation on S (see Assumption 2), and $\boldsymbol{\nu}$ is the unit outer normal vector on $\partial\Omega$.

Ω represents a portion of the Earth's crust where the fault S lies and where both the direct and inverse problems are studied. We assume that S does not reach the boundary of Ω, which corresponds geophysically to the case of buried or blind faults. From the point of view of the inverse problem, this is the most interesting case, as there is no direct access to the fault from the boundary for monitoring. The set Σ models the buried part of the boundary of Ω. Assuming that the rock displacement is zero on Σ is justified from the geophysical point of view, as the relative motion of rock formations can be assumed much slower than rock slippage along faults. In applications, one needs to assume that Ω is large enough compared to the size of the fault for this justification to hold. The complement of Σ models the part of the boundary on the Earth's crust and hence can be taken traction free. (See e.g. [43].)

The vector field $\boldsymbol{g}$ on S models the slip along the active patch of the fault. We assume that

$$\boldsymbol{g} \in H^{1/2}_{00}(S). \tag{7}$$

Recall that elements in this space have zero trace at the boundary.

Remark 3.1. By hypothesis (see Assumption 2), S is part of a closed Lipschitz surface Γ. Then, $\boldsymbol{g} \in H^{1/2}_{00}(S)$ implies that $\boldsymbol{g}$ can be extended by zero in $\Gamma \setminus S$ to a function $\widetilde{\boldsymbol{g}} \in H^{1/2}(\Gamma)$:

$$\widetilde{\boldsymbol{g}}(\boldsymbol{x}) = \begin{cases} \boldsymbol{g}(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in S, \\ \boldsymbol{0} & \text{if } \boldsymbol{x} \in \Gamma \setminus S. \end{cases} \tag{8}$$

Remark 3.2. As discussed in the introduction, there are geophysical motivations for considering a slip $\boldsymbol{g}$ that vanishes at the boundary of the surface S (a creeping unlocked fault patch). There are also mathematical reasons for considering that class of slips. Given the minimal regularity of the coefficients, a variational formulation of the problem is the most natural one. However, it can be shown (see [9] for a discussion) that if $\boldsymbol{g}$ is an arbitrary field in $H^{1/2}(S)$, then the solution $\boldsymbol{u}$ is not necessarily in $H^1(\Omega \setminus \overline{S})$. The space $H^{1/2}_{00}(S)$ is then the optimal choice for the slip, because it consists precisely of those elements in $H^{1/2}(S)$ that can be extended by zero to $H^{1/2}(\Gamma)$, where Γ is an arbitrary Lipschitz closed surface containing S, with norm bounds on the extension and the restriction back to $\overline{S}$ [38]. Moreover, we are also interested in implementing a reconstruction algorithm. If an iterative algorithm is used, then we need to numerically solve the forward or direct problem several times. For inhomogeneous media with discontinuous coefficients, as in this work, a variational approach, such as that in FEM and DG methods, is practical. The advantage of working with Γ instead of S is that Green's formulas apply to $\Omega \setminus \Gamma$. By working with $H^{1/2}_{00}(S)$, we are then able to prove the equivalence of the problem formulation using Γ and that using S at the level of weak solutions. In the extensive literature concerning interface and boundary problems, other spaces have been considered, notably, the space $\{\boldsymbol{g} \in H^{1/2}(\Gamma) : \operatorname{supp} \boldsymbol{g} \subseteq \overline{S}\}$ [2] (we need to be able to take $\operatorname{supp} \boldsymbol{g} = \overline{S}$ for the inverse problem). It would be interesting and relevant to further investigate the relationship between this space and $H^{1/2}_{00}(S)$. However, this analysis is not the focus of our work.

By a *weak solution* of (6) we mean that

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \mathbf{0} & \text{in } (H^1_\Sigma(\Omega))', \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{\nu} = \mathbf{0} & \text{in } H^{-1/2}(\partial\Omega \setminus \Sigma), \\ \boldsymbol{u} = \mathbf{0} & \text{in } H^{1/2}(\Sigma), \\ [\boldsymbol{u}]_S = \boldsymbol{g} & \text{in } H^{1/2}_{00}(S), \\ [(\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n}]_S = \mathbf{0} & \text{in } H^{-1/2}(S). \end{cases} \tag{9}$$

The strategy we follow here is an adaptation of the procedure described in [3] to solve classical transmission problems. Given the closed surface Γ and the extension (8), we decompose Ω into two domains Ω^- and Ω^+, as in Assumption 2. Then we construct a weak solution of problem (6) by solving two boundary-value problems, one in Ω^- and one Ω^+, imposing suitable Neumann conditions on Γ. The key step in this procedure consists in identifying the proper Neumann boundary condition on Γ such that $[\boldsymbol{u}]_\Gamma = \boldsymbol{u}^+_\Gamma - \boldsymbol{u}^-_\Gamma = \widetilde{\boldsymbol{g}}$, where $\boldsymbol{u}^+_\Gamma$ and $\boldsymbol{u}^-_\Gamma$ are the traces on Γ of the solutions $\boldsymbol{u}^+$ in Ω^+ and $\boldsymbol{u}^-$ in Ω^-. In Ω^-, the solution $\boldsymbol{u}^-$ will be sought in the auxiliary space $\mathring{H}^1(\Omega^-)$ to ensure uniqueness. This choice imposes apparently artificial normalization conditions in Ω^-, which are not needed to solve the original problem (6). However, we can verify *a posteriori* that such conditions are in fact satisfied by the unique solution to the original problem.

We shall first prove some preliminary results.

Lemma 3.3. *Let $\overline{\Omega} = \overline{\Omega^+} \cup \overline{\Omega^-}$, where Ω^+ and Ω^- are defined in Assumption 2. Let*

$$\widetilde{H} := \{f \in L^2(\Omega) : f^+ \in H^1(\Omega^+),\ f^- \in H^1(\Omega^-), \text{and } [f]_{\Gamma\setminus\overline{S}} = 0\}, \tag{10}$$

where $f^+ = f\lfloor_{\Omega^+}$ and $f^- = f\lfloor_{\Omega^-}$. Then $H^1(\Omega \setminus \overline{S}) \cong \widetilde{H}$.

This result is classical (see e.g. [2] for a proof using Green's formula in Lipschitz domains, obtained in [10]). We include the proof for the reader's convenience.

Proof of Lemma 3.3. Let $f \in \widetilde{H}$ and let $\boldsymbol{\varphi} \in C^\infty(\overline{\Omega})$ with support in $\Omega \setminus \overline{S}$. We apply the Divergence Theorem in Ω^+ and Ω^-, obtaining

$$\int_{\Omega^-} \nabla f^- \cdot \boldsymbol{\varphi}\, d\boldsymbol{x} = \int_\Gamma f^- \boldsymbol{n} \cdot \boldsymbol{\varphi}^-\, d\sigma(\boldsymbol{x}) - \int_{\Omega^-} f^- \operatorname{div}\boldsymbol{\varphi}\, d\boldsymbol{x},$$

where $\boldsymbol{n}$ is the unit outer normal vector to Ω^-. Similarly

$$\int_{\Omega^+} \nabla f^+ \cdot \boldsymbol{\varphi}\, d\boldsymbol{x} = -\int_\Gamma f^+ \boldsymbol{n} \cdot \boldsymbol{\varphi}^+\, d\sigma(\boldsymbol{x}) - \int_{\Omega^+} f^+ \operatorname{div}\boldsymbol{\varphi}\, d\boldsymbol{x}.$$

Therefore, we find

$$\begin{aligned} \int_{\Omega^-} \nabla f^- \cdot \boldsymbol{\varphi}\, d\boldsymbol{x} + \int_{\Omega^+} \nabla f^+ \cdot \boldsymbol{\varphi}\, d\boldsymbol{x} = &\int_{\Gamma\setminus\overline{S}} (f^-\boldsymbol{\varphi}^- - f^+\boldsymbol{\varphi}^+) \cdot \boldsymbol{n}\, d\sigma(\boldsymbol{x}) \\ &- \int_{\Omega\setminus\overline{S}} f \operatorname{div}\boldsymbol{\varphi}\, d\boldsymbol{x}, \end{aligned}$$

noting in the terms on the right that $\boldsymbol{\varphi}$ and $\operatorname{div}\boldsymbol{\varphi}$ have compact support in $\Omega \setminus \overline{S}$ and, as an L^2 function, $f = f^+\chi_{\Omega^+} + f^-\chi_{\Omega^-}$. Moreover, $\boldsymbol{\varphi}$ is regular across $\Gamma \setminus \overline{S}$ by hypothesis and $f^+ = f^-$ on $\Gamma \setminus \overline{S}$, since $f \in \widetilde{H}$, so that

$$\int_{\Gamma\setminus\overline{S}} (f^-\boldsymbol{\varphi}^- - f^+\boldsymbol{\varphi}^+)\cdot\boldsymbol{n}\,d\sigma(\boldsymbol{x}) = \int_{\Gamma\setminus\overline{S}} \boldsymbol{\varphi}\,(f^- - f^+)\cdot\boldsymbol{n}\,d\sigma(\boldsymbol{x}) = 0.$$

Consequently,

$$\int_{\Omega\setminus\overline{S}} f\,\operatorname{div}\boldsymbol{\varphi}\,d\boldsymbol{x} = -\int_{\Omega^+}\nabla f^+\cdot\boldsymbol{\varphi}\,d\boldsymbol{x} - \int_{\Omega^-}\nabla f^-\cdot\boldsymbol{\varphi}\,d\boldsymbol{x},$$

which means that the distributional gradient of f is an L^2 function in $\Omega \setminus \overline{S}$ and agrees with

$$\nabla f^+\chi_{\Omega^+} + \nabla f^-\chi_{\Omega^-}.$$

Reversing the argument gives the opposite implication. ■

For the next lemma, we follow [3, Proposition 12.8.2], adapting that result to the case of the Lamé operator with discontinuous coefficients.

Lemma 3.4. *Let $\mathbb{C} \in L^\infty(\Omega)$, and let $\boldsymbol{\eta} \in H^1(\Omega \setminus \overline{S})$ be a weak solution of the system* $\operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{\eta}) = \boldsymbol{0}$ *in $\Omega \setminus \overline{S}$. Then $[(\mathbb{C}\widehat{\nabla}\boldsymbol{\eta})\boldsymbol{n}]_{\Gamma\setminus\overline{S}} = \boldsymbol{0}$ in $H^{-1/2}(\Gamma \setminus \overline{S})$.*

Proof. We fix a point $\boldsymbol{x}_0 \in \Gamma \setminus \overline{S}$ and we consider a ball $B_r(\boldsymbol{x}_0)$ with $r > 0$ sufficiently small so that $B_r(\boldsymbol{x}_0) \cap \overline{S} = \emptyset$ and $B_r(\boldsymbol{x}_0) \cap \partial\Omega = \emptyset$. Let $\boldsymbol{\varphi} \in H^1_0(B_r(\boldsymbol{x}_0))$. Then

$$0 = \langle \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{\eta}), \boldsymbol{\varphi}\rangle = -\int_{B_r(\boldsymbol{x}_0)} \mathbb{C}\widehat{\nabla}\boldsymbol{\eta} : \widehat{\nabla}\boldsymbol{\varphi}\,d\boldsymbol{x}. \tag{11}$$

(This identity can be established by approximating $\boldsymbol{\varphi}$ with smooth fields supported in $B_r(\boldsymbol{x}_0)$.) Next we apply Green's identities, which hold for H^1 functions, in $D^+ = B_r(\boldsymbol{x}_0) \cap \Omega^+$ and $D^- = B_r(\boldsymbol{x}_0) \cap \Omega^-$. Therefore, for all $\boldsymbol{\varphi} \in H^1_0(B_r(\boldsymbol{x}_0))$,

$$0 = -\int_{D^+} \mathbb{C}\widehat{\nabla}\boldsymbol{\eta} : \widehat{\nabla}\boldsymbol{\varphi}\,d\boldsymbol{x} - \langle (\mathbb{C}\widehat{\nabla}\boldsymbol{\eta}^+)\boldsymbol{n}, \boldsymbol{\varphi}^+\rangle_{(H^{-1/2}(\Gamma\cap\partial D^+),H^{1/2}(\Gamma\cap\partial D^+))}, \tag{12}$$

and analogously

$$0 = -\int_{D^-} \mathbb{C}\widehat{\nabla}\boldsymbol{\eta} : \widehat{\nabla}\boldsymbol{\varphi}\,d\boldsymbol{x} + \langle (\mathbb{C}\widehat{\nabla}\boldsymbol{\eta}^-)\boldsymbol{n}, \boldsymbol{\varphi}^-\rangle_{(H^{-1/2}(\Gamma\cap\partial D^-),H^{1/2}(\Gamma\cap\partial D^-))}. \tag{13}$$

Since $\boldsymbol{\varphi} \in H^1_0(B_r(\boldsymbol{x}_0))$, we have $\boldsymbol{\varphi}^+\lfloor_{\Gamma\cap B_r(\boldsymbol{x}_0)} = \boldsymbol{\varphi}^-\lfloor_{\Gamma\cap B_r(\boldsymbol{x}_0)} =: \boldsymbol{\varphi}\lfloor_{\Gamma\cap B_r(\boldsymbol{x}_0)}$. Hence, adding (12) and (13) gives

$$\begin{aligned}0 = &-\int_{D^+} \mathbb{C}\widehat{\nabla}\boldsymbol{\eta} : \widehat{\nabla}\boldsymbol{\varphi}\,d\boldsymbol{x}\\ &-\int_{D^-} \mathbb{C}\widehat{\nabla}\boldsymbol{\eta} : \widehat{\nabla}\boldsymbol{\varphi}\,d\boldsymbol{x} - \langle [(\mathbb{C}\widehat{\nabla}\boldsymbol{\eta})\boldsymbol{n}], \boldsymbol{\varphi}\rangle_{(H^{-1/2}(\Gamma\cap B_r(\boldsymbol{x}_0)),H^{1/2}(\Gamma\cap B_r(\boldsymbol{x}_0)))},\end{aligned}$$

where $[\,,]$ denotes the jump across $\Gamma \cap B_r(\boldsymbol{x}_0)$. Consequently,

$$0 = -\int_{B_r(\boldsymbol{x}_0)} \mathbb{C}\widehat{\nabla}\boldsymbol{\eta} : \widehat{\nabla}\boldsymbol{\varphi}\, d\boldsymbol{x} - \langle [(\mathbb{C}\widehat{\nabla}\boldsymbol{\eta})\boldsymbol{n}], \boldsymbol{\varphi}\rangle_{(H^{-1/2}(\Gamma\cap B_r(\boldsymbol{x}_0)), H^{1/2}(\Gamma\cap B_r(\boldsymbol{x}_0)))},$$

because, by hypothesis, both $\nabla\boldsymbol{\varphi}$ and $\nabla\boldsymbol{\eta}$ exist as L^2 functions in $B_r(\boldsymbol{x}_0)$. From (11) it follows that

$$\langle [(\mathbb{C}\widehat{\nabla}\boldsymbol{\eta})\boldsymbol{n}], \boldsymbol{\varphi}\rangle_{(H^{-1/2}(\Gamma\cap B_r(\boldsymbol{x}_0)), H^{1/2}(\Gamma\cap B_r(\boldsymbol{x}_0)))} = 0.$$

Since $\boldsymbol{\varphi}$ is an arbitrary function in $H^1_0(B_r(\boldsymbol{x}_0))$, we see that $[(\mathbb{C}\widehat{\nabla}\boldsymbol{\eta})\boldsymbol{n}]_{\Gamma\cap B_r(\boldsymbol{x}_0)} = \boldsymbol{0}$ in $H^{-1/2}(\Gamma \setminus \overline{S})$. We conclude by covering $\Gamma \setminus \overline{S}$ with a finite number of balls $B_r(\boldsymbol{x}_i)$, $i = 1, \ldots, N$. ■

We are now ready to tackle the well-posedness of problem (6). We begin by addressing the uniqueness of weak solutions.

Theorem 3.5 (Uniqueness). *Problem (6) has at most one weak solution in $H^1_\Sigma(\Omega \setminus \overline{S})$.*

Proof. Assume that there exist two solutions $\boldsymbol{u}^1, \boldsymbol{u}^2 \in H^1_\Sigma(\Omega \setminus \overline{S})$. Let $\boldsymbol{v} = \boldsymbol{u}^1 - \boldsymbol{u}^2$. From the transmission conditions on S (see (6)), we have

$$[\boldsymbol{v}]_S = \boldsymbol{0}, \quad [(\mathbb{C}\widehat{\nabla}\boldsymbol{v})\boldsymbol{n}]_S = \boldsymbol{0}.$$

Hence, by Lemma 3.3, $\boldsymbol{v} \in H^1_\Sigma(\Omega)$. It follows that $\boldsymbol{v}$ is a weak solution of the problem

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{v}) = \boldsymbol{0} & \text{in } \Omega, \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{v})\boldsymbol{\nu} = \boldsymbol{0} & \text{on } \partial\Omega \setminus \Sigma, \\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \Sigma, \end{cases} \tag{14}$$

which has a unique solution, $\boldsymbol{v} = \boldsymbol{0}$. ■

Theorem 3.6 (Existence). *There exists a weak solution $\boldsymbol{u} \in H^1_\Sigma(\Omega \setminus \overline{S})$ to problem (6).*

Proof. The strategy is to construct a weak solution of problem (6) from the solutions of two auxiliary boundary-value problems, one in Ω^- and one Ω^+, that are connected through a suitably chosen Neumann boundary condition on the interface Γ. Specifically, we consider the following Neumann boundary-value problem in $\mathring{H}^1(\Omega^-)$ (the space is chosen in order to avoid rigid motions in Ω^-):

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}^-) = \boldsymbol{0} & \text{in } \Omega^-. \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{u}^-)\boldsymbol{n} = \boldsymbol{\varphi} & \text{on } \Gamma. \end{cases} \tag{15}$$

and the following mixed-boundary-value problem in $H^1_\Sigma(\Omega^+)$:

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}^+) = \boldsymbol{0} & \text{in } \Omega^+, \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{u}^+)\boldsymbol{\nu} = \boldsymbol{0} & \text{on } \partial\Omega \setminus \Sigma, \\ \boldsymbol{u}^+ = \boldsymbol{0} & \text{on } \Sigma, \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{u}^+)\boldsymbol{n} = \boldsymbol{\varphi} & \text{on } \Gamma, \end{cases} \tag{16}$$

where Ω^- and Ω^+ are defined in Assumption 2. We denote the traces of $\boldsymbol{u}^{\pm}$ in $H^{1/2}(\Gamma)$ by $\boldsymbol{u}^{\pm}_{\Gamma}$. The key point of the proof is to identify $\boldsymbol{\varphi}$ in order to represent the solution $\boldsymbol{u}$ of (6) as $\boldsymbol{u} = \boldsymbol{u}^-\chi_{\Omega^-} + \boldsymbol{u}^+\chi_{\Omega^+}$, where χ_{Ω^-} and χ_{Ω^+} are the characteristic functions of Ω^- and Ω^+, respectively. To this end, we define the bounded Neumann–Dirichlet operators

$$N^+ : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \quad N^- : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma),$$

related to (16) and (15), respectively. Then, since $N^+\boldsymbol{\varphi} = \boldsymbol{u}^+_{\Gamma}$ and $N^-\boldsymbol{\varphi} := \boldsymbol{u}^-_{\Gamma}$, and recalling that $[\boldsymbol{u}]_{\Gamma} = \widetilde{\boldsymbol{g}}$, where $\widetilde{\boldsymbol{g}}$ is the extension of $\boldsymbol{g}$ on $\Gamma \setminus S$, as defined in (8), we need to identify $\boldsymbol{\varphi} \in H^{-1/2}(\Gamma)$ such that

$$\boldsymbol{u}^+_{\Gamma} - \boldsymbol{u}^-_{\Gamma} = (N^+ - N^-)\boldsymbol{\varphi} = \widetilde{\boldsymbol{g}}. \tag{17}$$

The invertibility of the operator $N^+ - N^-$ guarantees that $\boldsymbol{\varphi} = (N^+ - N^-)^{-1}(\widetilde{\boldsymbol{g}})$, and follows from the continuity of both the Neumann-to-Dirichlet and the Dirichlet-to-Neumann maps. The continuity is well known. We briefly outline the proof of invertibility in our setting for the reader's convenience.

First, by using the weak formulation of (15) in $\mathring{H}^1(\Omega^-)$ and (16) in $H^1_{\Sigma}(\Omega^+)$, we find a relation between the quadratic form associated to (15) and $\langle \boldsymbol{\varphi}, N^-\boldsymbol{\varphi}\rangle_{\Gamma}$, and between the quadratic form associated to (16) and $\langle \boldsymbol{\varphi}, N^+\boldsymbol{\varphi}\rangle_{\Gamma}$. Indeed, from the weak formulation of problems (15) and (16), we find that

$$\int_{\Omega^+} \mathbb{C}\widehat{\nabla}\boldsymbol{u}^+ : \widehat{\nabla}\boldsymbol{v}^+ \, d\boldsymbol{x} = -\langle \boldsymbol{\varphi}, \boldsymbol{v}^+\rangle_{\Gamma}, \quad \forall \boldsymbol{v}^+ \in H^1_{\Sigma}(\Omega^+), \tag{18}$$

as $\boldsymbol{n}$ points inwards into Ω^+, and that

$$\int_{\Omega^-} \mathbb{C}\widehat{\nabla}\boldsymbol{u}^- : \widehat{\nabla}\boldsymbol{v}^- \, d\boldsymbol{x} = \langle \boldsymbol{\varphi}, \boldsymbol{v}^-\rangle_{\Gamma}, \quad \forall \boldsymbol{v}^- \in H^1(\Omega^-). \tag{19}$$

Next, we observe that we can extend any function $\boldsymbol{v} \in H^{1/2}(\Gamma)$ to functions $\boldsymbol{v}^+ \in H^1_{\Sigma}(\Omega^+)$ and $\boldsymbol{v}^- \in H^1(\Omega^-)$, for instance by solving suitable Dirichlet problems for the Laplace operator in Ω^+ and Ω^-. Then the above identities imply

$$|\langle \boldsymbol{\varphi}, \boldsymbol{v}\rangle_{\Gamma}| \le C_{\pm}\|\boldsymbol{u}^{\pm}\|_{H^1(\Omega^{\pm})}\|\boldsymbol{v}^{\pm}\|_{H^1(\Omega^{\pm})} \le C_{\pm}\|\boldsymbol{u}^{\pm}\|_{H^1(\Omega^{\pm})}\|\boldsymbol{v}\|_{H^{1/2}(\Gamma)}.$$

Using the definition of the norm in $H^{-1/2}(\Gamma)$ as the operator norm of functionals on $H^{1/2}(\Gamma)$, it follows that

$$\|\boldsymbol{\varphi}\|_{H^{-1/2}(\Gamma)} \le C_{\pm}\|\boldsymbol{u}^{\pm}\|_{H^1(\Omega^{\pm})}. \tag{20}$$

Moreover, by choosing $\boldsymbol{v}^+ = \boldsymbol{u}^+$ in (18) and $\boldsymbol{v}^- = \boldsymbol{u}^-$ in (19), we have

$$\int_{\Omega^+} \mathbb{C}\widehat{\nabla}\boldsymbol{u}^+ : \widehat{\nabla}\boldsymbol{u}^+ \, d\boldsymbol{x} = -\langle \boldsymbol{\varphi}, N^+\boldsymbol{\varphi}\rangle_{\Gamma},$$
$$\int_{\Omega^-} \mathbb{C}\widehat{\nabla}\boldsymbol{u}^- : \widehat{\nabla}\boldsymbol{u}^- \, d\boldsymbol{x} = \langle \boldsymbol{\varphi}, N^-\boldsymbol{\varphi}\rangle_{\Gamma}.$$

Then Assumption 1, Korn's and Poincaré's inequalities (see e.g. [28]) give

$$-\langle \boldsymbol{\varphi}, N^+\boldsymbol{\varphi}\rangle_\Gamma = \int_{\Omega^+} \mathbb{C}\widehat{\nabla}\boldsymbol{u}^+ : \widehat{\nabla}\boldsymbol{u}^+ \, d\boldsymbol{x} \geq C\|\boldsymbol{u}^+\|^2_{H^1(\Omega^+)}, \tag{21}$$

$$\langle \boldsymbol{\varphi}, N^-\boldsymbol{\varphi}\rangle_\Gamma = \int_{\Omega^-} \mathbb{C}\widehat{\nabla}\boldsymbol{u}^- : \widehat{\nabla}\boldsymbol{u}^- \, d\boldsymbol{x} \geq C\|\boldsymbol{u}^-\|^2_{\mathring{H}^1(\Omega^-)}. \tag{22}$$

Therefore, by using (20) in both (21) and (22), we can establish the coercivity of the bilinear form associated to (17):

$$\|\boldsymbol{\varphi}\|^2_{H^{-1/2}(\Gamma)} \leq C\langle \boldsymbol{\varphi}, (-N^+ + N^-)\boldsymbol{\varphi}\rangle_\Gamma. \tag{23}$$

The continuity of this form follows directly from the continuity of the solution operators for (15)–(16) and the Trace Theorem. The Lax–Milgram Theorem then ensures that there exists a unique solution $\boldsymbol{\varphi} \in H^{-1/2}(\Gamma)$ such that

$$\langle \boldsymbol{\psi}, (-N^+ + N^-)\boldsymbol{\varphi}\rangle_\Gamma = \langle \boldsymbol{\psi}, -\widetilde{\boldsymbol{g}}\rangle_\Gamma, \quad \forall \boldsymbol{\psi} \in H^{-1/2}(\Gamma), \tag{24}$$

so the operator $-N^+ + N^-$ is invertible.

With this choice of $\boldsymbol{\varphi}$, problems (15) and (16) admit unique solutions $\boldsymbol{u}^- \in \mathring{H}^1(\Omega^-)$ and $\boldsymbol{u}^+ \in H^1_\Sigma(\Omega^+)$, respectively. Next, we let

$$\boldsymbol{u} = \boldsymbol{u}^-\chi_{\Omega^-} + \boldsymbol{u}^+\chi_{\Omega^+}.$$

Then $\boldsymbol{u}\lfloor_{\Omega^-} = \boldsymbol{u}^- \in H^1(\Omega^-)$, $\boldsymbol{u}\lfloor_{\Omega^+} = \boldsymbol{u}^+ \in H^1(\Omega^+)$, $\boldsymbol{u}$ is a distributional solution of $\operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \boldsymbol{0}$ in Ω^+ and Ω^-. To conclude, we show that $\boldsymbol{u}$ is a weak solution of (6). By construction, it satisfies the boundary conditions on $\partial\Omega$ in the trace sense. Again by construction,

$$[\boldsymbol{u}]_\Gamma = \boldsymbol{u}^+_\Gamma - \boldsymbol{u}^-_\Gamma = \widetilde{\boldsymbol{g}} \quad \text{in } H^{1/2}(\Gamma).$$

That is, by (8),

$$[\boldsymbol{u}]_{\Gamma\setminus\overline{S}} = \boldsymbol{0}, \quad [\boldsymbol{u}]_S = \boldsymbol{g}, \tag{25}$$

hence, by Lemma 3.3, $\boldsymbol{u} \in H^1(\Omega \setminus \overline{S})$. Moreover,

$$[(\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n}]_\Gamma = \boldsymbol{0} \quad \text{in } H^{-1/2}(\Gamma), \tag{26}$$

which follows immediately by construction. In particular, $[(\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n}]_S = \boldsymbol{0}$. Now, recalling that $\boldsymbol{u}$ is a weak solution in Ω^- and in Ω^+ and satisfies (26), reversing the steps in the proof of Lemma 3.4 we find that $\boldsymbol{u}$ is a weak solution of $\operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \boldsymbol{0}$ in $\Omega \setminus \overline{S}$. In fact, we fix a point $\boldsymbol{x}_0 \in \Gamma \setminus \overline{S}$ and we consider a ball $B_r(\boldsymbol{x}_0)$ with $r > 0$ sufficiently small such that $B_r(\boldsymbol{x}_0) \cap \overline{S} = \emptyset$. Let $\varphi \in H^1_0(B_r(\boldsymbol{x}_0))$. We apply Green's identity in $D^+ = B_r(\boldsymbol{x}_0) \cap \Omega^+$ and $D^- = B_r(\boldsymbol{x}_0) \cap \Omega^-$. Since $\boldsymbol{u}$ is a weak solution in D^+ and D^-, we get

$$0 = -\int_{D^+} \mathbb{C}\widehat{\nabla}\boldsymbol{u} : \widehat{\nabla}\boldsymbol{\varphi} \, d\boldsymbol{x} - \langle (\mathbb{C}\widehat{\nabla}\boldsymbol{u}^+)\boldsymbol{n}, \boldsymbol{\varphi}^+\rangle_{(H^{-1/2}(\Gamma\cap\partial D^+), H^{1/2}(\Gamma\cap\partial D^+))} \tag{27}$$

for all $\boldsymbol{\varphi} \in H_0^1(B_r(\boldsymbol{x}_0))$, and analogously

$$0 = -\int_{D^-} \mathbb{C}\widehat{\nabla}\boldsymbol{u} : \widehat{\nabla}\boldsymbol{\varphi}\, d\boldsymbol{x} + \langle(\mathbb{C}\widehat{\nabla}\boldsymbol{u}^-)\boldsymbol{n}, \boldsymbol{\varphi}^-\rangle_{(H^{-1/2}(\Gamma\cap\partial D^-), H^{1/2}(\Gamma\cap\partial D^-))}. \tag{28}$$

As $\boldsymbol{\varphi} \in H_0^1(B_r(\boldsymbol{x}_0))$, we have $\boldsymbol{\varphi}^+\lfloor_{\Gamma\cap B_r(\boldsymbol{x}_0)} = \boldsymbol{\varphi}^-\lfloor_{\Gamma\cap B_r(\boldsymbol{x}_0)} =: \boldsymbol{\varphi}\lfloor_{\Gamma\cap B_r(\boldsymbol{x}_0)}$ in the trace sense. Hence, adding (27) and (28) gives

$$\begin{aligned} 0 = &-\int_{D^+} \mathbb{C}\widehat{\nabla}\boldsymbol{u} : \widehat{\nabla}\boldsymbol{\varphi}\, d\boldsymbol{x} \\ &-\int_{D^-} \mathbb{C}\widehat{\nabla}\boldsymbol{u} : \widehat{\nabla}\boldsymbol{\varphi}\, d\boldsymbol{x} - \langle[(\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n}], \boldsymbol{\varphi}\rangle_{(H^{-1/2}(\Gamma\cap B_r(\boldsymbol{x}_0)), H^{1/2}(\Gamma\cap B_r(\boldsymbol{x}_0)))}, \end{aligned}$$

where $[\,,]$ denotes the jump across $\Gamma \cap B_r(\boldsymbol{x}_0)$. Then, using the fact that $[(\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n}] = \boldsymbol{0}$ on $\Gamma \cap B_r(\boldsymbol{x}_0)$ and $\widehat{\nabla}\boldsymbol{u} \in L^2(B_r(\boldsymbol{x}_0))$, we find that

$$0 = \int_{B_r(\boldsymbol{x}_0)} \mathbb{C}\widehat{\nabla}\boldsymbol{u} : \widehat{\nabla}\boldsymbol{\varphi}\, d\boldsymbol{x}.$$

Therefore, $\boldsymbol{u}$ is a weak solution in $\Omega \setminus \overline{S}$, given that $\boldsymbol{x}_0$ and $\boldsymbol{\varphi} \in H_0^1(B_r(\boldsymbol{x}_0))$ are arbitrary. ■

We note that, from the proof of the existence theorem above, a weak solution is also a variational solution in the following sense: $\boldsymbol{u} \in H_\Sigma^1(\Omega^+)$, $\boldsymbol{u} \in H^1(\Omega^-)$, $[\boldsymbol{u}]_\Gamma = \widetilde{\boldsymbol{g}}$ in $H^{1/2}(\Gamma)$ and, for every $\boldsymbol{v} \in H_\Sigma^1(\Omega)$,

$$\int_{\Omega^+} \mathbb{C}\widehat{\nabla}\boldsymbol{u} : \widehat{\nabla}\boldsymbol{v}\, d\boldsymbol{x} + \int_{\Omega^-} \mathbb{C}\widehat{\nabla}\boldsymbol{u} : \widehat{\nabla}\boldsymbol{v}\, d\boldsymbol{x} = 0. \tag{29}$$

We observe that a variational solution could also be obtained by a suitable lifting operator of the jump on Γ to $\Omega \setminus \Gamma$, analogous to that utilized in the treatment of non-homogeneous Dirichlet boundary conditions, reducing the problem to a source problem with homogeneous jump conditions on S (see for example [21, 43]).

Corollary 3.7. *There exists a unique solution* $\boldsymbol{u} \in H_\Sigma^1(\Omega \setminus \overline{S})$ *to problem* (6).

We observe that other types of boundary conditions can, in principle, be imposed on the buried part Σ of $\partial\Omega$. For example, one can impose a non-homogeneous traction there, modeling the load of contiguous rock formations on Ω itself.

Remark 3.8. The approach to proving well-posedness for (6) can be adapted to other boundary-value problems as well, such as Neumann problems with non-homogeneous boundary conditions on $\partial\Omega$. In fact, given $\boldsymbol{h} \in H^{-1/2}(\partial\Omega)$, one can show that there exists a unique solution $\boldsymbol{u}_N \in \mathring{H}^1(\Omega \setminus \overline{S})$ for the following problem:

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}_N) = \boldsymbol{0} & \text{in } \Omega \setminus \overline{S}, \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{u}_N)\boldsymbol{\nu} = \boldsymbol{h} & \text{on } \partial\Omega, \\ [\boldsymbol{u}_N]_S = \boldsymbol{g}, \\ [(\mathbb{C}\widehat{\nabla}\boldsymbol{u}_N)\boldsymbol{n}]_S = \boldsymbol{0}. \end{cases} \tag{30}$$

The proof of uniqueness in $\mathring{H}^1(\Omega \setminus \overline{S})$ follows exactly as in Theorem 3.5. For the proof of existence, we notice that due to the linearity property of (30), $\boldsymbol{u}_N$ can be decomposed as $\boldsymbol{u}_N := \mathring{\boldsymbol{u}} + \boldsymbol{w}$, where $\mathring{\boldsymbol{u}} \in \mathring{H}^1(\Omega \setminus \overline{S})$ is the unique solution to

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\mathring{\boldsymbol{u}}) = \boldsymbol{0} & \text{in } \Omega \setminus \overline{S}, \\ (\mathbb{C}\widehat{\nabla}\mathring{\boldsymbol{u}})\boldsymbol{\nu} = \boldsymbol{0} & \text{on } \partial\Omega, \\ [\mathring{\boldsymbol{u}}]_S = \boldsymbol{g}, \\ [(\mathbb{C}\widehat{\nabla}\mathring{\boldsymbol{u}})\boldsymbol{n}]_S = \boldsymbol{0}, \end{cases} \tag{31}$$

and $\boldsymbol{w} \in \mathring{H}^1(\Omega)$ is a solution to

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{w}) = \boldsymbol{0} & \text{in } \Omega \setminus \overline{S}, \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{w})\boldsymbol{\nu} = \boldsymbol{h} & \text{on } \partial\Omega, \\ [\boldsymbol{w}]_S = \boldsymbol{0}, \\ [(\mathbb{C}\widehat{\nabla}\boldsymbol{w})\boldsymbol{n}]_S = \boldsymbol{0}. \end{cases} \tag{32}$$

The proof of the existence of a solution $\mathring{\boldsymbol{u}} \in \mathring{H}^1(\Omega \setminus \overline{S})$ for (31) then follows the same ideas as in Theorem 3.6, but with the simplification that both $\boldsymbol{u}^+$ and $\boldsymbol{u}^-$ belong now to the same space $\mathring{H}^1$. Problem (32) is reduced to a standard transmission problem, hence the existence of a unique solution in $\mathring{H}^1(\Omega)$ follows easily.

4. The inverse problem: a uniqueness result

In this section we address the uniqueness for the inverse dislocation problem, which consists in identifying the dislocation S and the slip $\boldsymbol{g}$ on it from displacement measurements made at the surface of the Earth. Uniqueness will be proved under additional assumptions on the geometry and the data for problem (6). In particular, we consider a domain Ω which is partitioned into finitely many Lipschitz subdomains, we assume that the elasticity tensor is isotropic with Lamé coefficients that are Lipschitz continuous in each subdomain, and we take the dislocation surface to be a graph with respect to a fixed, but arbitrary, coordinate frame. Such assumptions are not unrealistic in the context of geophysical applications and underscore the ill-posedness of the inverse problems without additional *a priori* information.

Specifically, in addition to Assumptions 1 and 2, we assume the following:

Assumption 3 (domain and partition). We denote by $\Xi \subseteq \partial\Omega \setminus \overline{\Sigma}$ an open patch of the boundary where the measurements of the displacement field are given. Moreover, we assume that

$$\overline{\Omega} = \bigcup_{k=1}^{N} \overline{D_k},$$

where D_k, for $k = 1, \dots, N$, are pairwise non-overlapping bounded Lipschitz domains. We assume, without loss of generality, that Ξ is contained in ∂D_1.

Assumption 4 (elasticity tensor). The elasticity tensor $\mathbb{C} = \mathbb{C}(\boldsymbol{x})$ is assumed to be isotropic in each element of the partition of Ω, i.e.,

$$\mathbb{C}(\boldsymbol{x}) = \sum_{k=1}^{N} \mathbb{C}_k(\boldsymbol{x})\chi_{D_k}(\boldsymbol{x}), \quad \mathbb{C}_k(\boldsymbol{x}) := \lambda_k(\boldsymbol{x})\mathbf{I} \otimes \mathbf{I} + 2\mu_k(\boldsymbol{x})\mathbb{I}, \tag{33}$$

where $\lambda_k = \lambda_k(\boldsymbol{x})$ and $\mu_k = \mu_k(\boldsymbol{x})$, for $k = 1, \dots, N$, are the Lamé coefficients related to the subdomain D_k, and $\mathbf{I}$ and $\mathbb{I}$ are the identity matrix and the identity fourth-order tensor, respectively. Each Lamé parameter, λ_k, μ_k, for $k = 1, \dots, N$, belongs to $C^{0,1}(\overline{D_k})$, that is, there exists $M > 0$ such that

$$\|\mu_k\|_{C^{0,1}(\overline{D_k})} + \|\lambda_k\|_{C^{0,1}(\overline{D_k})} \leq M, \tag{34}$$

with $\|\cdot\|_{C^{0,1}(\overline{D_k})} = \|\cdot\|_{L^\infty(\overline{D_k})} + \|\nabla \cdot\|_{L^\infty(\overline{D_k})}$. Finally, there exist two positive constants α_0, β_0 such that

$$\mu_k(\boldsymbol{x}) \geq \alpha_0 > 0, \quad 3\lambda_k(\boldsymbol{x}) + 2\mu_k(\boldsymbol{x}) \geq \beta_0 > 0, \quad \forall \boldsymbol{x} \in \overline{D_k}, \quad k = 1, \dots, N. \tag{35}$$

These conditions ensure the uniform strong convexity of Ω.

Assumption 5 (further assumptions on the fault S). The surface S is assumed to be the graph of a Lipschitz function with respect to a given coordinate frame.

Our main result for the inverse problem is the following theorem.

Theorem 4.1. *Under Assumptions 3 and 4, let S_1, S_2 be as in Assumption 2 and such that S_1, S_2 satisfy Assumption 5 with respect to the same coordinate frame. For $i = 1, 2$, let $\boldsymbol{g}_i \in H^{1/2}_{00}(S_i)$, with $\operatorname{supp} \boldsymbol{g}_i = \overline{S}_i$, and let $\boldsymbol{u}_i$ be the unique solution of (6) in $H^1_\Sigma(\Omega \setminus \overline{S})$ corresponding to $\boldsymbol{g} = \boldsymbol{g}_i$ and $S = S_i$. If $\boldsymbol{u}_1\lfloor_\Xi = \boldsymbol{u}_2\lfloor_\Xi$, then $S_1 = S_2$ and $\boldsymbol{g}_1 = \boldsymbol{g}_2$.*

Remark 4.2. The assumptions that the surfaces S_i, $i = 1, 2$, are graphs and that they are graphs with respect to the same coordinate frame is not overly restrictive in the context of faults in geophysics. In fact, in a given geographical region faults tend to be approximately horizontal with respect to the surface of the Earth, as predominantly in *dip-slip faults*, or approximately vertical, as predominantly in *strike-slip faults*, although oblique faults can also occur, depending on the characteristics of the rock formations present (see e.g. [19, 31]). Furthermore, these assumptions exclude *a priori* the existence of internal faces common to both S_1 and S_2, when the faults enclose a bounded region of space (see case (ii) below in the proof of the theorem). In the presence of such common faces, it seems difficult to prove uniqueness and it is not at all clear, in fact, that uniqueness does hold in this case. However, it is possible to prove uniqueness under other geometric conditions on the faults that also exclude common internal faces, for example if the fault surfaces are each a union of at most two rectangular faces. In the geophysical literature, often the fault is taken to be a *single* rectangular face.

We denote by G the connected component of $\Omega \setminus \overline{S_1 \cup S_2}$ containing Ξ. By definition we have $G \subseteq \Omega \setminus \overline{S_1 \cup S_2}$. In addition, we define

$$\mathcal{G} := \partial G \setminus \partial\Omega. \tag{36}$$

Before proving Theorem 4.1, we recall the following lemma proved in [9] in the special case where Ω is a half-space. However, this result is clearly true for bounded domains as well.

Lemma 4.3. *Let S_1, S_2 be as in Assumption 2 and such that S_1, S_2 satisfy Assumption 5 with respect to the same coordinate frame. Then $\mathscr{G} = \overline{S_1 \cup S_2}$.*

Proof of Theorem 4.1. We proceed by contradiction and assume that $S_1 \neq S_2$. We first show that $\boldsymbol{w} := \boldsymbol{u}_1 - \boldsymbol{u}_2$ is identically zero in G. We can assume, without loss of generality, that Ξ is the graph of a Lipschitz function in some coordinate frame, say with respect to the z-axis. In fact, it is enough to take a possibly small open subset of Ξ instead of the entire Ξ, and then this hypothesis is always satisfied as $\partial\Omega$ is assumed globally Lipschitz. On Ξ we have

$$\boldsymbol{w} = \boldsymbol{0}, \quad (\mathbb{C}\widehat{\nabla}\boldsymbol{w})\boldsymbol{\nu} = \boldsymbol{0}.$$

Then, fixing a point $\boldsymbol{x}_0 \in \Xi$, we consider the ball $B_R(\boldsymbol{x}_0)$, where R is so small that $B_R(\boldsymbol{x}_0) \cap \partial\Omega \subseteq \Xi$; and we denote $B_R^-(\boldsymbol{x}_0) := B_R(\boldsymbol{x}_0) \cap \overline{\Omega}$ and $B_R^+(\boldsymbol{x}_0) = (B_R^-(\boldsymbol{x}_0))^C$, the complementary domain. We define

$$\widetilde{\boldsymbol{w}} := \begin{cases} \boldsymbol{w} & \text{in } B_R^-(\boldsymbol{x}_0), \\ \boldsymbol{0} & \text{in } B_R^+(\boldsymbol{x}_0). \end{cases} \tag{37}$$

We note that $\widetilde{\boldsymbol{w}} \in H^1(B_R(\boldsymbol{x}_0))$.

We observe next that, since Ξ is the graph of a Lipschitz function, the restriction of $\mathbb{C}$ on Ξ is Lipschitz as well. Then we can extend $\mathbb{C}$ to a Lipschitz elasticity tensor $\widetilde{\mathbb{C}}$ on $B_R^-(\boldsymbol{x}_0) \cup B_R^+(\boldsymbol{x}_0)$ as follows: for each $\boldsymbol{\xi}$ on the graph of Ξ, we extend $\mathbb{C}$ over $B_R^+(\boldsymbol{x}_0)$, keeping the constant value $\mathbb{C}(\boldsymbol{\xi})$ along the vertical direction of the coordinate frame. Note that this argument can be applied for each component of the tensor. Consequently, arguing as in [5], we find that $\widetilde{\boldsymbol{w}}$ is a weak solution of

$$\operatorname{div}(\widetilde{\mathbb{C}}\widehat{\nabla}\widetilde{\boldsymbol{w}}) = \boldsymbol{0} \quad \text{in } B_R(\boldsymbol{x}_0).$$

We now apply the weak continuation property [22]. In fact, since $\widetilde{\boldsymbol{w}} = \boldsymbol{0}$ in $B_R^+(\boldsymbol{x}_0)$ and since the weak continuation property holds in $B_R(\boldsymbol{x}_0)$, it follows that

$$\widetilde{\boldsymbol{w}} = \boldsymbol{0} \quad \text{in } B_R(\boldsymbol{x}_0).$$

In particular, $\boldsymbol{w} = \boldsymbol{0}$ in $B_R^-(\boldsymbol{x}_0)$. Furthermore, again applying the weak continuation property, we find that $\boldsymbol{w} = \boldsymbol{0}$ in D_1.

Next, thanks to the hypotheses on S_i, $i = 1, 2$, there exists a path-connected open subdomain of Ω that connects Ξ with every element of the partition which belongs to G. Along this path, we can always assume that the boundary of the partition is Lipschitz. Consequently, we can recursively apply the previous argument and we see that $\boldsymbol{w} \equiv \boldsymbol{0}$ in G. We then distinguish two cases:

(i) $G = \Omega \setminus \overline{S_1 \cup S_2}$;

(ii) $G \subset \Omega \setminus \overline{S_1 \cup S_2}$.

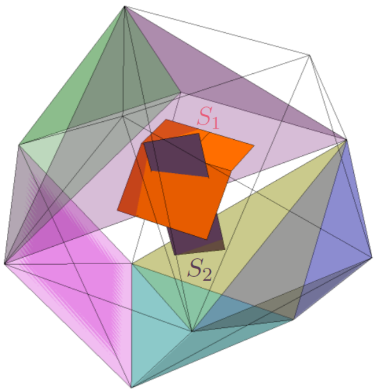

Fig. 2. An example of the geometrical setting in case (i). Graphics generated using Wolfram Mathematica©.

For case (i) (see Figure 2), by the hypothesis that the surfaces are Lipschitz and the fact that $S_1 \neq S_2$, without loss of generality, there exists a point $\boldsymbol{y} \in S_1$ such that $\boldsymbol{y} \notin \overline{S}_2$, and a ball $B_r(\boldsymbol{y})$ that does not intersect S_2, where r is sufficiently small. Hence,

$$\mathbf{0} = [\boldsymbol{w}]_{B_r(\boldsymbol{y})\cap S_1} = [\boldsymbol{u}_1]_{B_r(\boldsymbol{y})\cap S_1} = \boldsymbol{g}_1,$$

and this identity leads to a contradiction, as $\mathrm{supp}(\boldsymbol{g}_1) = \overline{S}_1$. It follows that $\overline{S}_1 = \overline{S}_2$, and consequently

$$\mathbf{0} = [\boldsymbol{w}]_{S_1} = [\boldsymbol{w}]_{S_2} \implies [\boldsymbol{u}_1]_{S_1} = [\boldsymbol{u}_2]_{S_2} \implies \boldsymbol{g}_1 = \boldsymbol{g}_2.$$

Next, we analyze case (ii) (see Figure 3). We recall that, by hypothesis, the two surfaces are Lipschitz graphs with respect to an arbitrary, but fixed, common frame. Then by Lemma 4.3 we can assume, without loss of generality, that the complement of $\overline{S_1 \cup S_2}$

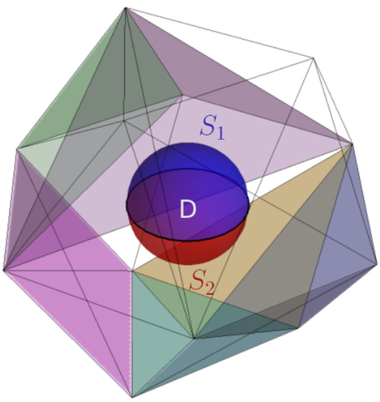

Fig. 3. An example of the geometrical setting in case (ii). The bounded connected domain D is such that $\partial D = \overline{S_1 \cup S_2}$. Graphics generated using Wolfram Mathematica©.

has only one bounded connected component, since if there are more than one, we can treat each one separately. That is, we can assume that there exists a bounded connected domain D such that $\partial D = \overline{S_1 \cup S_2}$. In this situation, in particular, $\overline{S_1}$ and $\overline{S_2}$ intersect precisely only along their common boundary $\mathcal{C} = \partial S_1 \cap \partial S_2$, which is non-empty. If there are other parts of their boundary that are not in common, they can be treated as in case (i). Then $\boldsymbol{w} = \mathbf{0}$ in a neighborhood of ∂D in $\Omega \setminus \overline{D}$, since $\boldsymbol{w} = \mathbf{0}$ in G. The continuity of the tractions $(\mathbb{C}\widehat{\nabla}\boldsymbol{u}_1)\boldsymbol{n}$ and $(\mathbb{C}\widehat{\nabla}\boldsymbol{u}_2)\boldsymbol{n}$ in the trace sense across S_1 and S_2, respectively, implies that

$$(\mathbb{C}\widehat{\nabla}\boldsymbol{w}^-)\boldsymbol{n} = \mathbf{0} \tag{38}$$

in $H^{-1/2}(\partial D)$ and hence a.e. on ∂D, where $\boldsymbol{w}^-$ indicates the function $\boldsymbol{w}$ restricted to D and $\boldsymbol{n}$ the outward unit normal to D. Moreover, $\boldsymbol{w}^-$ satisfies

$$\operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{w}^-) = \mathbf{0} \quad \text{in } D. \tag{39}$$

We conclude from (38) and (39) that $\boldsymbol{w}^-$ is in the kernel of the operator for elastostatics in $H^1(D)$, i.e., it is a rigid motion:

$$\boldsymbol{w}^- = \mathbf{A}\boldsymbol{x} + \boldsymbol{c},$$

where $\boldsymbol{c} \in \mathbb{R}^3$ and $\mathbf{A} \in \mathbb{R}^{3\times 3}$ is a skew matrix. We conclude the proof by showing that this rigid motion can only be the trivial one. By construction $\boldsymbol{w}^- = [\boldsymbol{w}]_{S_i} = \boldsymbol{g}_i$ on S_i, so in particular it must vanish along ∂S_i, i.e., on $\mathcal{C}$ due to the hypothesis $\boldsymbol{g}_i \in H^{1/2}_{00}(S_i)$. On the other hand, the set of solutions of the linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{c}$, for any given $\boldsymbol{c} \in \mathbb{R}^3$, is a one-dimensional linear subspace of $\mathbb{R}^3$, since $\mathbf{A}$ is anti-symmetric, and therefore it cannot contain a closed curve. It follows that necessarily $\mathbf{A} = \mathbf{0}$ and $\boldsymbol{c} = \mathbf{0}$. Consequently, $\boldsymbol{w}^- = \mathbf{0}$ in D, hence $[\boldsymbol{w}] = \mathbf{0}$ on ∂D. In particular, $[\boldsymbol{w}]_{S_1} = \mathbf{0} = [\boldsymbol{u}_1] = \boldsymbol{g}_1 \neq \mathbf{0}$, by the assumption that $\operatorname{supp}(\boldsymbol{g}_i) = \overline{S_i}$. We reach a contradiction, and therefore case (ii) does not occur. ∎

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