Bifurcation into spectral gaps for strongly indefinite Choquard equations

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Abstract: We consider the semilinear elliptic equations

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * |u|^p)\,|u|^{p-2}u + \lambda u \quad \text{for } \ x \in \mathbb{R}^N, \\ u(x) \to 0 \ \text{ as } \ |x| \to \infty, \end{cases}$$

where I_α is a Riesz potential, $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$, $N \geq 3$, and V is continuous periodic. We assume that 0 lies in the spectral gap (a, b) of $-\Delta + V$. We prove the existence of infinitely many geometrically distinct solutions in $H^1(\mathbb{R}^N)$ for each $\lambda \in (a, b)$, which bifurcate from b if $\frac{N+\alpha}{N} < p < 1 + \frac{2+\alpha}{N}$. Moreover, b is the unique gap-bifurcation point (from zero) in $[a, b]$. When $\lambda = a$, we find infinitely many geometrically distinct solutions in $H^2_{loc}(\mathbb{R}^N)$. Final remarks are given about the eventual occurrence of a bifurcation from infinity in $\lambda = a$.

Keywords: Choquard equation; Schrödinger-Newton equation; Bifurcation into spectral gaps.

MSC(2010): 35Q40, 35Q55, 47J35.

1 Introduction

The purpose of this paper is to study bifurcation into spectral gaps for a class of nonlinear and nonlocal Schrödinger equations with periodic potential. More precisley, let us consider the following equation

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * |u|^p)\,|u|^{p-2}u + \lambda u \quad \text{for } \ x \in \mathbb{R}^N, \\ \quad u(x) \to 0 \ \text{ as } \ |x| \to \infty, \end{cases} \tag{1.1}$$

where $N \geq 3$, $p \in [2, \frac{N+\alpha}{N-2})$, I_α ($\alpha \in (0, N)$) is the Riesz potential defined for every $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = A_\alpha |x|^{\alpha-N}, \quad \text{where } A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha \pi^{\frac{N}{2}} \Gamma(\frac{\alpha}{2})} \ \text{ and } \ \Gamma \text{ is the Gamma function}$$

and $V(x)$ is the external potential, assumed to be, in our case, continuous periodic. We recall that the choice of the constant A_α ensures the semigroup property

$$I_\alpha * I_\beta = I_{\alpha+\beta}, \quad \forall \alpha, \beta > 0 \text{ such that } \alpha + \beta < N. \tag{1.2}$$

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In the relevant physical case $N = 3, \alpha = N - 2, p = 2$ equation (1.1) takes its origin from the so called Choquard-Pekar equation

$$-\Delta u + u = \left(I_2 * |u|^2\right) u \quad \text{for} \quad x \in \mathbb{R}^3$$

which appears in various contexts, modeling the quantum polaron at rest [15, 33] and then used by Choquard in 1975, as pointed out by Lieb [24], to study steady states of the one-component plasma approximation in the Hartree–Fock theory. The Choquard-Pekar equation is also known as the Schrödinger-Newton equation in models coupling the Schrödinger equation of quantum physics together with nonrelativistic Newtonian gravity. Lieb in [24] proved the existence and uniqueness of positive solutions to the Choquard equation by using rearrangement techniques. Multiplicity results were then obtained by Lions [25] by means of a variational approach. A broad literature has been recently developed and we refer to [9, 10, 13, 45] for an up to date, though non exhaustive bibliography. We also refer to [29] and references therein for an extensive survey on the topic.

Whereas Choquard type problems have been widely investigated in the nonperiodic case, the case of periodic (nonconstant) potentials $V(x)$ is much less studied: in this case the compactness issue is more difficult to handle due to the invariance of the equation under the action of the noncompact group induced by translation in the coordinate directions. It is well known (see, e.g., [35]) that the spectrum of the self-adjoint operator $-\Delta + V$ in $L^2(\mathbb{R}^N)$ is purely continuous and may contain gaps, i.e. open intervals free of spectrum.

The corresponding *local* problem has been widely investigated, both for its physical applications and its mathematical interest. In the case $\lambda = 0$, the location of 0 in the spectrum of $-\Delta + V$ determines the geometry of the associated energy functional. There are many results available when $\inf \sigma(-\Delta + V) > 0$ (the *positive definite* case) or when 0 lies in a gap of the spectrum (the *strongly indefinite* case), see, for example, [11, 31, 3, 8, 43] or the monograph [5] and references therein. An interesting situation occurs when 0 coincides with one of the borderline points of the spectral gap. As we will see later, this case is much more difficult to be approached, and several questions seem not yet solved. We refer to [6] and further generalizations proved in [47, 44, 26, 37, 4], which are, to the best of our knowledge, the only papers dealing with this case.

The parameter dependent situation $\lambda \neq 0$ can be discussed replacing the potential $V(x)$ by $V(x) - \lambda$. In this context, an interesting physical and mathematical issue is establishing the existence of branches of solutions u_λ converging towards the trivial solution as λ approaches some point λ_0 of the spectrum, a so called *bifurcation point*. An intriguing situation is the so called *gap-bifurcation*, that is, bifurcation occurring at boundary points of the spectral gaps, when $V(x)$ is a periodic potential: the existence of nontrivial solutions reveals the presence of bound states whose "energy" $\lambda \in \mathbb{R}$ lies in gaps of the spectrum of the Schrödinger operator $-\Delta + V$. These bound states are created by the nonlinear perturbation. A first systematic approach was obtained in [20, 17], see also [39, 40] and references therein.

In particular, in [17], Heinz, Küpper and Stuart applied some of their previous abstract results on bifurcation theory (see references in the paper) to study the following *model* problem

$$-\Delta u + V(x)u = r(x)|u|^\sigma u + \lambda u, \quad \text{in} \quad \mathbb{R}^N,$$

where $r \in L^\infty(\mathbb{R}^N)$ and nonnegative a.e. on $\mathbb{R}^N$, with periodic continuous (nonconstant) potential $V(x)$. They proved that, if $0 < \sigma < \frac{4}{N-2}$ ($\sigma > 0$ for $N = 2$) and $[a, b]$ is a spectral gap for the Schrödinger operator $-\Delta + V$, then there is a nontrivial solution for any $\lambda \in (a, b)$. Furthermore, if $r(x)$ is constant and $0 < \sigma < 4/N$, they proved that b is a gap-bifurcation (from zero) point. In [21, 41] it is also shown that no bifurcation from 0 can occur in a, and that the condition $0 < \sigma < 4/N$ is necessary for the displayed equation.

What is known in the *nonlocal* case, that is, for the Choquard equation (1.1)? The literature is much less developed. As in the local framework, in the particular case $\lambda = 0$ the geometry of the functional depends on the location of 0 in the spectrum of $-\Delta + V$, and, consequently, the suitable approach.
When $V > 0$ we are in the positive definite case, where the spectrum lies in $(0, +\infty)$: Ackermann [1] obtained a nontrivial solution taking advantage of the linking straucture, as well as infinitely many geometrically distinct solutions, in the case of odd nonlinearities.
When the potential V changes sign we are in the strongly indefinite case, and the spectrum consists of a union of closed intervals. If $\lambda = 0$ is in a gap of the (essential) spectrum, the existence of at least one nontrivial solution has been proved by Buffoni, Jeanjean and Stuart in [8], for the physical case $N = 3, \alpha = N - 2, p = 2$, and by Ackermann [1], always in dimension $N = 3$ but for a more general convolution kernel $W(x)$ and nonlinearity $f(u)$. The higher dimensional case has been recently approached by Qin, Rădulescu and Tang [34], who showed the existence of ground state solutions without assuming the Ambrosetti-Rabinowitz type condition for $f(u)$.
The parameter dependent case and the related bifurcation analysis, instead, seem not to have been considered in literature. The first goal of the present paper is to address this issue. More precisely, if V is a potential satisfying the following conditions:

(V_1) $V \in C(\mathbb{R}^N)$ is $1-$periodic in $x_1, ...x_N$;

(V_2) 0 lies in a gap of the spectrum of the Schrödinger operator $-\Delta + V$, that is

$$\sup[\sigma(-\Delta + V) \cap (-\infty, 0)] := a < 0 < b := \inf[\sigma(-\Delta + V) \cap (0, \infty)],$$

we will prove

(1) the existence of infinitely many *geometrically distinct* solutions $u_\lambda \in H^1(\mathbb{R}^N)$ for any $\lambda \in (a, b)$;

(2) the convergence towards 0 of these solutions u_λ, as $\lambda \to b^-$, for some values of p.

Before giving the precise statements, we need to introduce the following definition.

Definition 1.1. *Suppose that u_1, u_2 solve* (1.1)*; if $\mathcal{O}(u_1) \neq \mathcal{O}(u_2)$, we say that u_1, u_2 are geometrically distinct, where, $\mathcal{O}(u)$ denotes the orbit of u_0 with respect to the action of $\mathbb{Z}^N$:*

$$\mathcal{O}(u_0) := \{\tau_k u_0 : k \in \mathbb{Z}^N\}, \quad (\tau_k u)(x) := u(x + k)$$

We are now able to introduce our first result.

Theorem 1.2. *Let $N \geq 3$, $\alpha \in (0, N)$, $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ and $(V_1) - (V_2)$ hold. Then, for each $\lambda \in (a, b)$, there exist infinitely many geometrically distinct solutions $u_\lambda \in H^1(\mathbb{R}^N)$ of* (1.1)*.*

The restrictions on the parameter p follow from the variational approach: it guarantees differentiability properties of the energy functional. It is remarkable the appearance, in the class of the Choquard type problems, of a lower nonlinear restriction.

The next result concerns the bifurcation from the right boundary point b of the spectral gap, where we assume the following

Definition 1.3. *A point $\lambda \in \mathbb{R}$ is called a gap-bifurcation (from zero) point for* (1.1) *if there exists a sequence $\{(\lambda_n, u_n)\}$ of solutions of* (1.1) *such that $\lambda_n \in \rho(-\Delta+V)$, $\lambda_n \to \lambda$ and $\|u_n\|_{H^1(\mathbb{R}^N)} \to 0$ as $n \to \infty$, where $\rho(-\Delta+V) = \mathbb{R}\setminus\sigma(-\Delta+V)$ is the resolvent set for $-\Delta+V$.*

Theorem 1.4. *Let $N \geq 3$, $\alpha \in (0,N)$, $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ and $(V_1)-(V_2)$ hold. For any $\lambda \in (a,b)$, let u_λ be a solution of equation* (1.1) *obtained in Theorem 1.2, then*

$$\Phi_\lambda(u_\lambda) = O((b-\lambda)^{\frac{2p-Np+N+\alpha}{2p-2}}) \to 0 \quad as \quad \lambda \to b^-$$

Moreover, if $\frac{N+\alpha}{N} < p < \frac{N+\alpha+2}{N}$, then

$$\|u_\lambda\|_{H^1} = O((b-\lambda)^{\frac{2-Np+N+\alpha}{4p-4}}) \to 0 \quad as \quad \lambda \to b^-.$$

Moreover, b is the only possible gap-bifurcation (from zero) point for (1.1) *in* $[a,b]$.

The proofs are based on variational methods applied to the functional

$$\Phi_\lambda(u) = \frac{1}{2}Q_\lambda(u) - \frac{1}{2p}J(u), \tag{1.3}$$

where

$$J(u) := \int_{\mathbb{R}^N} (I_\alpha * |u|^p)\,|u|^p dx \tag{1.4}$$

The standard assumptions $\alpha \in (0,N)$ and $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ imply that $\Phi_\lambda : H^1(\mathbb{R}^N) \to \mathbb{R}$ is of class $\mathcal{C}^1$ and that critical points of Φ_λ are weak solutions of (1.1). By assumption (V_2), we have $H^1 = E_\lambda^- \oplus E_\lambda^+$ corresponding to the decomposition of the spectrum: we will apply a generalized linking theorem due to Kryszewski-Szulkin [22], inspired by [6, 1]. We emphasize that some of the results stated in the above theorems could also be obtained by applying abstract results in bifurcation theory (for example, from the results stated in Section 5 of [40]). However, verifying the validity of all the assumptions, in particular the so-called condition $T(\delta)$ is not trivial, and it would require the proofs of some intermediate lemmas. Hence, we prefer to give here a self-contained approach, referring to the cited papers for further generalizations.

We then address the interesting case $\lambda = a$, that is we study the Choquard-Schrödinger equation in a right boundary point of the spectrum of the Schrödinger operator. As in the local framework, this location of λ causes a loss of completeness in the decomposition of the domain of the Schrödinger operator. Inspired by Bartsch and Ding [6], we find infinitely many geometrically distinct solutions for (1.1) which lie in $H^2_{loc}(\mathbb{R}^N)$ but not necessarily in $H^1(\mathbb{R}^N)$ (see Theorem 1.5): the presence of the nonlocal term adds an extra difficulty in the choice of a suitable space framework and in the analysis of (PS) sequences. The results are stated in the following

Theorem 1.5. *Let $N \geq 3$, $\alpha \in (0,N)$, $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ and $(V_1)-(V_2)$ hold. For $\lambda = a$, there exist infinitely many geometrically distinct solutions for* (1.1) *in $H^2_{loc}(\mathbb{R}^N)$.*

The paper is organized as follows. In Section 2, we introduce the variational framework and the main variational tools. In Section 3 we address the existence result when $\lambda \in (a,b)$, proving Theorem 1.2. In Section 4, we prove the bifurcation from zero at the right boundary point b, Theorem 1.4. In Section 5 we consider the delicate case $\lambda = a$, proving Theorem 1.5.

As stated in Theorem 1.4, b is the *only* possible gap-bifurcation point for 0 in $[a,b]$: what about the branches of solutions u_λ, as $\lambda \to a^+$? This is an intriguing question, which seems still open also in the local case. Some perspectives will be proposed in the final Section 6.

We make use of the following notation:

- $|\cdot|_p$ denotes the usual norm of the space $L^p(\mathbb{R}^N)$.
- $C, C_i, i = 1, 2, \cdots$, will be repeatedly used to denote various positive constants whose exact values are irrelevant.
- $B_R(x_0)$ denotes the ball $\{x \in \mathbb{R}^N : |x - x_0| \leq R\}$.
- $o(1)$ denotes the infinitesimal as $n \to +\infty$.
- For the sake of simplicity, integrals over the whole $\mathbb{R}^N$ will be often written $\int$.
- For the sake of simplicity, we often omit the constant A_α in I_α.

2 The functional framework

In this section we present the functional settings and regularity properties of the functionals defined in (1.3), (1.4). We start recalling a classical inequality which turns out to be a main tool in our arguments.

Proposition 2.1. *[23] (Hardy–Littlewood–Sobolev inequality.) Let t, $r > 1$ and $0 < \alpha < N$ with $\frac{1}{t} + \frac{1}{r} = 1 + \frac{\alpha}{N}$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a constant $C(N, \alpha, t, r)$, independent of f, h, such that*

$$|I_\alpha * h|_{t'} \leq C(N, \alpha, t, r)|h|_r \tag{2.1}$$

and

$$\int (I_\alpha * h)\, f \ dx \leq C(N, \alpha, t, r)|f|_t|h|_r, \tag{2.2}$$

where $|\cdot|_s$ denotes the $L^s(\mathbb{R}^N)$-norm for $s \in [1, \infty]$, and t' denotes the conjugate exponent such that $\frac{1}{t'} + \frac{1}{t} = 1$.

Let us now prove some properties of the Riesz potential I_α.

Proposition 2.2. *Let $\alpha \in (0, N)$ and $f, g \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Then*

$$\int (I_\alpha * f) g dx = \int \left(I_{\frac{\alpha}{2}} * f\right)\left(I_{\frac{\alpha}{2}} * g\right) dx. \tag{2.3}$$

Moreover,

$$\int (I_\alpha * f) f dx = \int \left(I_{\frac{\alpha}{2}} * f\right)^2 dx \geq 0 \tag{2.4}$$

and

$$\int (I_\alpha * f) g dx \leq \left[\int (I_\alpha * f) f dx\right]^{\frac{1}{2}} \left[\int (I_\alpha * g) g dx\right]^{\frac{1}{2}}. \tag{2.5}$$

Proof. By the semigroup property of the Riesz potential, (1.2), we have

$$\int (I_\alpha * f) g dx = \int (I_{\frac{\alpha}{2}} * I_{\frac{\alpha}{2}} * f) g dx = \int\int\int I_{\frac{\alpha}{2}}(y) I_{\frac{\alpha}{2}}(x - z - y) f(z) g(x) dx dy dz.$$

Let $z = x' - y$, since $I_{\frac{\alpha}{2}}$ is even, we get

$$\int (I_\alpha * f) g dx = \int\int\int I_{\frac{\alpha}{2}}(y) I_{\frac{\alpha}{2}}(x' - x) f(x' - y) g(x) dx dy dx' = \int (I_{\frac{\alpha}{2}} * f)(I_{\frac{\alpha}{2}} * g) dx.$$

Thus (2.3) holds. (2.4) is obviously from (2.3).

Using (2.3) and the Hölder inequality, we obtain

$$\begin{aligned}\int (I_\alpha * f) g dx &= \int \left(I_{\frac{\alpha}{2}} * f\right)\left(I_{\frac{\alpha}{2}} * g\right) dx \\ &\le \left[\int \left(I_{\frac{\alpha}{2}} * f\right)^2 dx\right]^{\frac{1}{2}} \left[\int \left(I_{\frac{\alpha}{2}} * g\right)^2 dx\right]^{\frac{1}{2}} \\ &= \left[\int (I_\alpha * f) f dx\right]^{\frac{1}{2}} \left[\int (I_\alpha * g) g dx\right]^{\frac{1}{2}}.\end{aligned}$$

This completes the proof. □

Using the semigoup property of the Riesz potential, the nonlocal term (1.4) in the energy functional can be written also as

$$J(u) := \int (I_\alpha * |u|^p)\, |u|^p dx = \int \left(I_{\frac{\alpha}{2}} * |u|^p\right)^2 dx.$$

Our aim is now to define a *natural* energy space associated with the energy functional Φ_λ. As noted in the Introduction, the choice will depend on the location of λ with respect to the spectrum of the Schrödinger operator. In what follows we state some properties of the nonlocal term $J(u)$, postponing to specific subsections the definitions and descriptions of the functional frameworks. As part of the energy functional, the analysis of $J(u)$ has been performed in several papers approaching variationally Choquard type equations. We mainly refer to the survey [29], and to [27].

It is easy to observe that J is well defined on $H^1(\mathbb{R}^N)$, for any $\frac{N+\alpha}{N} \le p \le \frac{N+\alpha}{N-2}$: combining the Hardy–Littlewood–Sobolev inequality (2.2) and the Sobolev inequality yields

$$J(u) \le C|u|^{2p}_{\frac{2Np}{N+\alpha}} \le C\|u\|^{2p}_{H^1}. \tag{2.6}$$

Furthermore, in [27] the authors noted that $J(u)$ is naturally settled in the so called *Coulomb spaces* $\mathcal{Q}^{\alpha,p}$, defined as the vector spaces of measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that $J(u)$ is finite. They also proved that the quantity

$$\|u\|_{\mathcal{Q}^{\alpha,p}} := \left(\int_{\mathbb{R}^N} \left|I_{\frac{\alpha}{2}} * |u|^p\right|^2 dx\right)^{\frac{1}{2p}} \tag{2.7}$$

defines a norm, which will guarantees the convexity of the functional J. Hence, inequality (2.6) corresponds to the embedding $H^1 \subset L^{\frac{2Np}{N+\alpha}} \subset \mathcal{Q}^{\alpha,p}$. The paper [27] then introduces and carefully studies the Couloumb-Sobolev spaces and regularity properties of J in this framework, which differs from ours. Therefore, for the sake of completeness we state and prove some useful properties of J in H^1, even if some of them could be deduced by known results in literature.

Lemma 2.1. *The functional $J : H^1(\mathbb{R}^N) \mapsto \mathbb{R}$ satisfies the following properties:*

(*i*) *J is continuous and weakly sequentially lower semi-continuous.*

(*ii*) *For all $u, v \in H^1(\mathbb{R}^N)$, there is $C > 0$ such that*

$$\langle J'(u), v\rangle \le J(u)^{1-\frac{1}{2p}} J(v)^{\frac{1}{2p}} \le C\|u\|^{2p-1}_{H^1}\|v\|_{H^1}. \tag{2.8}$$

Moreover, J' is weakly sequentially continuous.

(iii) J is even and convex, and for all $u, w \in H^1(\mathbb{R}^N)$

$$J(u+w) \geq 2^{1-2p} J(u) - J(w).$$

Proof. (i) By (2.6), J is is well defined. Let $u_n \to u$ in $H^1(\mathbb{R}^N)$, then by the Hardy–Littlewood–Sobolev inequality and the elementary inequality $\big||a|^p - |b|^p\big| \leq |a-b|^p$,

$$|J(u_n) - J(u)| \leq |u_n - u|_{\frac{2Np}{N+\alpha}} |u_n|_{\frac{2Np}{N+\alpha}} + |u_n - u|_{\frac{2Np}{N+\alpha}} |u|_{\frac{2Np}{N+\alpha}} \to 0,$$

as $n \to \infty$. Thus J is continuous.

Now let $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. We can assume (up to a subsequence) that $u_n \to u$ a.e. in $\mathbb{R}^N$. By Fatou's Lemma,

$$J(u) = \int\int \lim_{n\to\infty} \frac{|u_n(x)|^p |u_n(y)|^p}{|x-y|^{N-\alpha}} dxdy \leq \liminf_{n\to\infty} J(u_n)$$

Thus J is weakly sequentially lower semi-continuous.

(ii) For any $u, v \in H^1(\mathbb{R}^N)$

$$\langle J'(u), v\rangle = 2p \int (I_\alpha * |u|^p)\, |u|^{p-2} uv dx$$

By (2.3), (2.4) and Hölder inequality we have

$$\begin{aligned}
\left|\int (I_\alpha * |u|^p)|u|^{p-2}uvdx\right| &\leq \left(\int (I_\alpha * |u|^p)|u|^p dx\right)^{\frac{1}{2}} \left(\int \left|I_{\frac{\alpha}{2}} * (|u|^{p-2}uv)\right|^2 dx\right)^{\frac{1}{2}} \\
&\leq \left(\int (I_\alpha * |u|^p)|u|^p dx\right)^{\frac{1}{2}} \left(\int \left|I_{\frac{\alpha}{2}} * |u|^p\right|^{2-\frac{2}{p}} \cdot \left|I_{\frac{\alpha}{2}} * |v|^p\right|^{\frac{2}{p}}\right)^{\frac{1}{2}} \\
&\leq \left(\int (I_\alpha * |u|^p)|u|^p dx\right)^{\frac{1}{2}} \left[\int \left|I_{\frac{\alpha}{2}} * |u|^p\right|^2 dx\right]^{\frac{1}{2}-\frac{1}{2p}} \left[\int \left|I_{\frac{\alpha}{2}} * |v|^p\right|^2 dx\right]^{\frac{1}{2p}} \\
&= \left(\int (I_\alpha * |u|^p)|u|^p dx\right)^{\frac{1}{2}} \left[\int (I_\alpha * |u|^p)|u|^p dx\right]^{\frac{1}{2}-\frac{1}{2p}} \left[\int (I_\alpha * |v|^p)|v|^p dx\right]^{\frac{1}{2p}} \\
&\leq [J(u)]^{1-\frac{1}{2p}} [J(v)]^{\frac{1}{2p}} \qquad (2.9)
\end{aligned}$$

Hence, J' is well defined. Let us now prove that J' is weakly sequentially continuous. Let us first show that if $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, then

$$\int (I_\alpha * |u_n|^p)|u_n|^{p-2}u_n v dx - \int (I_\alpha * |u_n|^p)|u|^{p-2}uvdx \to 0 \text{ as } n \to \infty \qquad (2.10)$$

Indeed, since $v \in L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$, for any $\varepsilon > 0$ there is $R > 0$ such that $|v|_{L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N \setminus B_R)} \leq \varepsilon$. Then, by (2.5)

$$\begin{aligned}
&\left|\int (I_\alpha * |u_n|^p)\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) v\chi_{\mathbb{R}^N\setminus B_R} dx\right| \leq \left|\int (I_\alpha * |u_n|^p)|u_n|^p dx\right|^{1/2} \cdot \\
&\quad \cdot \left|\int \left(I_\alpha * \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) v\chi_{\mathbb{R}^N\setminus B_R}\right)\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) v\chi_{\mathbb{R}^N\setminus B_R} dx\right|^{1/2} \\
&\leq C|u_n|^{1/2}_{\frac{2Np}{N+\alpha}} \cdot \left|\int \left(I_\alpha * \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) v\chi_{\mathbb{R}^N\setminus B_R}\right)\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) v\chi_{\mathbb{R}^N\setminus B_R} dx\right|^{1/2} \qquad (2.11)
\end{aligned}$$

The estimate of the right hand side splits in two different cases - note that $\{u_n\}$ is bounded in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$:

- If $p \geq 2$, then we apply the following inequality, which holds for any $a, b \in \mathbb{R}$:

$$\left||a|^{p-2}a - |b|^{p-2}b\right| \leq C_p(|a| + |b|)^{p-2}|a-b|$$

which yields

$$\begin{aligned}
&\left|\int (I_\alpha * \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) v\chi_{\mathbb{R}^N\setminus B_R}) \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) v\chi_{\mathbb{R}^N\setminus B_R} dx\right|^{1/2} \\
&\leq C\left|\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) v\chi_{\mathbb{R}^N\setminus B_R}\right|_{\frac{2N}{N+\alpha}} \leq C_p\left|(|u_n| + |u|)^{p-2}|u_n - u||v|\chi_{\mathbb{R}^N\setminus B_R}\right|_{\frac{2N}{N+\alpha}} \\
&\leq C_p\left|(|u_n| + |u|)\right|^{p-2}_{\frac{2Np}{N+\alpha}} |u_n - u|_{\frac{2Np}{N+\alpha}} |v|_{L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N\setminus B_R)} \leq C\varepsilon
\end{aligned}$$

- If $1 < p < 2$, then we apply the following inequality, which holds for any $a, b \in \mathbb{R}$:

$$\left||a|^{p-2}a - |b|^{p-2}b\right| \leq C_p|a-b|^{p-1}$$

which yields

$$\begin{aligned}
&\left|\int (I_\alpha * \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) v\chi_{\mathbb{R}^N\setminus B_R}) \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) v\chi_{\mathbb{R}^N\setminus B_R} dx\right|^{1/2} \\
&\leq C\left|\left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) v\chi_{\mathbb{R}^N\setminus B_R}\right|_{\frac{2N}{N+\alpha}} \leq C_p\left||u_n - u|^{p-1}|v|\chi_{\mathbb{R}^N\setminus B_R}\right|_{\frac{2N}{N+\alpha}} \\
&\leq C_p|u_n - u|^{p-1}_{\frac{2Np}{N+\alpha}} |v|_{L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N\setminus B_R)} \leq C\varepsilon
\end{aligned}$$

In both cases, we obtain that for any fixed v and for any $\varepsilon > 0$ there is R such that the right hand side of (2.11) is less then $C\varepsilon$. By the same argument, on the ball B_R we have:

- if $p \geq 2$

$$\begin{aligned}
&\left|\int (I_\alpha * \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) v\chi_{B_R}) \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) v\chi_{B_R} dx\right|^{1/2} \\
&\leq C_p\left|(|u_n| + |u|)\right|^{p-2}_{\frac{2Np}{N+\alpha}} |u_n - u|_{L^{\frac{2Np}{N+\alpha}}(B_R)} |v|_{\frac{2Np}{N+\alpha}} \leq C\varepsilon
\end{aligned}$$

if $n \geq n_\varepsilon$, since $u_n \rightharpoonup u$ in H^1 and $p < \frac{N+\alpha}{N-2}$;

- if $1 < p < 2$, again,

$$\begin{aligned}
&\left|\int (I_\alpha * \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) v\chi_{B_R}) \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) v\chi_{B_R} dx\right|^{1/2} \\
&\leq C_p|u_n - u|^{p-1}_{L^{\frac{2Np}{N+\alpha}}(B_R)} |v|_{\frac{2Np}{N+\alpha}} \leq C\varepsilon
\end{aligned}$$

if $n \geq n_\varepsilon$, since $u_n \rightharpoonup u$ in H^1 and $p < \frac{N+\alpha}{N-2}$.

Combining the above cases yields (2.10). Now, by (2.1) the Riesz potential I_α defines a linear continuous map from $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ to $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$. Thus $(I_\alpha * |u_n|^p) \rightharpoonup (I_\alpha * |u|^p)$ in $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$, so that

$$\int (I_\alpha * |u_n|^p)|u|^{p-2}uv dx \to \int (I_\alpha * |u|^p)|u|^{p-2}uv dx \text{ as } n \to \infty. \tag{2.12}$$

since $|u|^{p-2}uv \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Combining (2.10) and (2.12) implies that J' is weakly sequentially continuous.

(iii) Obviously, J is even. Proposition 2.1 in [27] proves that $\|u\|_{\mathcal{Q}^{\alpha,p}}$ is a norm, so that it is a convex functional. Since $J(u) = \|u\|^{2p}_{\mathcal{Q}^{\alpha,p}}$, it is also convex. By convexity and the $2p$-homogeneity of J, we have

$$J(u+v) \leq \frac{1}{2}(J(2u) + J(2v)) \leq 2^{2p-1}(J(u) + J(v))$$

□

We end stating a version of the nonlocal Brézis-Lieb property, see Lemma 2.4 in [28] or the survey [29] and references therein.

Lemma 2.2. *Let $N \geq 3$, $0 < \alpha < N$ and $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$. Let $\{u_n\} \subset L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ and $u_n \rightharpoonup u$ in $L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$, then*

$$\int (I_\alpha * |u_n|^p)|u_n|^p dx - \int (I_\alpha * |u_n - u|^p)|u_n - u|^p dx \to \int (I_\alpha * |u|^p)|u|^p dx$$

as $n \to \infty$.

In the following subsections we will give the details of the functional framework, which depends on the two different cases, λ in the spectral gap (a, b), or λ in the right borderline point of the spectrum, that is $\lambda = a$.

2.1 The case $\lambda \in (a, b)$

Let (a, b) denote a spectral gap as defined in assumption $(V2)$. For any $\lambda \in [a, b]$, let $S_\lambda := -\Delta + V - \lambda$. Under condition (V_1), the operator S_λ is self-adjoint in $L^2(\mathbb{R}^N)$ with domain $D(S_\lambda) = H^2(\mathbb{R}^N)$, and the spectrum $\sigma(S_\lambda)$ is purely absolutely continuous and bounded from below. Let $\left(P_{\lambda,\nu} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)\right)_{\nu\in\mathbb{R}}$ denote the spectral family of S_λ. Setting $L^-_\lambda := P_{\lambda,0}L^2(\mathbb{R}^N)$ and $L^+_\lambda := (Id - P_{\lambda,0})L^2(\mathbb{R}^N)$ we have the decomposition $L^2(\mathbb{R}^N) = L^-_\lambda \oplus L^+_\lambda$, where $L^-_\lambda \subset H^2(\mathbb{R}^N)$ since the spectrum of S_λ is bounded from below.

Let E_λ be the completion of $D(\sqrt{|S_\lambda|}) = H^1(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{E_\lambda} := \left|\sqrt{|S_\lambda|}u\right|_2 = \left(\int_{-\infty}^{+\infty} |\nu| d(P_{\lambda,\nu}u, u)_2\right)^{\frac{1}{2}},$$

where $|S_\lambda|$ is the absolute value of S_λ, such that

$$|S_\lambda|u = \begin{cases} S_\lambda u & \text{if } u \in D(S_\lambda) \cap L^+_\lambda; \\ -S_\lambda u & \text{if } u \in D(S_\lambda) \cap L^-_\lambda. \end{cases}$$

Clearly E_λ is a Hilbert space with inner product

$$(u, v)_{E_\lambda} = \left(\sqrt{|S_\lambda|}u, \sqrt{|S_\lambda|}v\right)_2.$$

E_λ can be orthogonally decomposed as $E_\lambda = E_\lambda^- \oplus E_\lambda^+$, according to the decomposition of $\sigma(S_\lambda)$. We shall write $u = u^- + u^+$ with $u^\pm \in E_\lambda^\pm$ for $u \in E_\lambda$, and

$$\|u^+\|_{E_\lambda}^2 = (S_\lambda u^+, u^+)_2 = \int |\nabla u^+|^2 + (V(x) - \lambda)|u^+|^2 dx,$$
$$\|u^-\|_{E_\lambda}^2 = -(S_\lambda u^-, u^-)_2 = -\int |\nabla u^-|^2 + (V(x) - \lambda)|u^-|^2 dx.$$

For brevity, we denote $S_0 = S$, $P_{0,\nu} = P_\nu$, $E_0 = E$. Note that, by assumptions $(V_1) - (V_2)$, $E = H^1(\mathbb{R}^N)$ and $\|\cdot\|_E$ is equivalent to the usual norm of $H^1(\mathbb{R}^N)$.

Let $Q_\lambda : E_\lambda \to \mathbb{R}$ be the quadratic form

$$Q_\lambda(u) := \int |\nabla u|^2 + (V(x) - \lambda)u^2 dx.$$

Then

$$Q_\lambda(u) = (S_\lambda u, u)_2 = \|u^+\|_{E_\lambda}^2 - \|u^-\|_{E_\lambda}^2 \ ,$$

and Q_0 is negative definite on E^- and positive definite on E^+ respectively, that is,

$$Q_0(u^-) \le -\alpha_0 \|u^-\|_{H^1}^2, \quad Q_0(u^+) \ge \beta_0 \|u^+\|_{H^1}^2$$

for all $u^- \in E^-$ and $u^+ \in E^+$. Moreover,

$$Q_0(u) = Q_0(u^- + u^+) = Q_0(u^-) + Q_0(u^+),$$

and the borderline points of the spectral gap (a, b) can be characterized as

$$a = \sup_{u^- \in E^-, |u^-|_2 = 1} Q_0(u^-) < 0 < \inf_{u^+ \in E^+, |u^+|_2 = 1} Q_0(z) = b.$$

The same spectral splitting holds for any $\lambda \in (a, b)$. This is made precise by the following lemma.

Proposition 2.3. *([42, Lemma 2]) Let the spectral gap (a, b) be given as in assumption (V_2). Let $\lambda \in (a, b)$. Then*

$$Q_\lambda(u^-) = -\|u^-\|_{E_\lambda}^2 \le -\alpha_\lambda \|u^-\|_{H^1}^2, \quad Q_\lambda(u^+) = \|u^+\|_{E_\lambda}^2 \ge \beta_\lambda \|u^+\|_{H^1}^2$$

for all $u^- \in E_\lambda^-$ and $u^+ \in E_\lambda^+$, where

$$\alpha_\lambda := \begin{cases} \alpha_0 \left(1 - \dfrac{\lambda}{a}\right) & \text{if } \lambda \le 0, \\ \alpha_0 & \text{if } \lambda > 0, \end{cases}$$

$$\beta_\lambda := \begin{cases} \beta_0 \left(1 - \dfrac{\lambda}{b}\right) & \text{if } \lambda > 0, \\ \beta_0 & \text{if } \lambda \le 0. \end{cases}$$

Consequently,

$$\|u\|_{E_\lambda}^2 = Q_\lambda(u^+) - Q_\lambda(u^-) \ge \frac{1}{2} \min\{\alpha_\lambda, \beta_\lambda\} \|u\|_{H^1}^2.$$

As a consequence of Proposition 2.3, for any $\lambda \in (a, b)$ it holds $E_\lambda = E = H^1(\mathbb{R}^N)$.

2.2 The case $\lambda = a$

The case $\lambda = a$ requires a deeper analysis. In this situation $0 \in \sigma(S_a)$ is a right boundary point of $\sigma(S_a)$, where $S_a = -\Delta + V - a$. Since the spectrum of S_a restricted to L_a^+ is contained in $[b-a, +\infty)$, which is bounded away from 0, the norm $\|\cdot\|_{E_a}$ is equivalent to the H^1-norm on E_a^+. However, 0 is contained in the spectrum of S_a restricted to L_a^-, hence the norm $\|\cdot\|_{E_a}$ is weaker than the H^1-norm on E_a^-. Moreover, $H^1(\mathbb{R}^N) \cap L_a^- = L_a^-$ is not complete with respect to $\|\cdot\|_{E_a}$: indeed, since $0 \in \sigma(S_a)$ is a continuous spectrum point, there is a sequence $\{u_n\} \subset D(S_a)$ such that $|u_n|_2 = 1$ and $S_a u_n \to 0$, so that $\|u_n\|_{E_a} \to 0$.

Furthermore, when $\lambda = a$, $J(u)$ is no more well-defined on E_a. To overcome these difficulties, we are going to define a new space E_{HL} such that there are continuous embeddings $H^1(\mathbb{R}^N) \subset E_{HL} \subset E_a$. Let us recall the definition of the Coulomb norm (2.7)

$$\|u\|_{\mathcal{Q}^{\alpha,p}} := \left(\int (I_\alpha * |u|^p)\, |u|^p \right)^{\frac{1}{2p}} = \left(\int \left(I_{\frac{\alpha}{2}} * |u|^p \right)^2 \right)^{\frac{1}{2p}}$$

$\|\cdot\|_{\mathcal{Q}^{\alpha,p}}$ is a norm on $L_a^- \subset H^1(\mathbb{R}^N)$. Further, for any $u \in E_a^-$, we have

$$0 \le \|u\|_{E_a}^2 = -\int (|\nabla u|^2 + (V(x) - a)u^2) dx$$

which implies

$$|\nabla u|_2 \le C|u|_2, \quad \forall u \in E_a^-.$$

Then, by Hardy-Littlewood-Sobolev inequality and Gagliardo-Nirenberg inequality, we have

$$\|u\|_{\mathcal{Q}^{\alpha,p}} \le C|u|_{\frac{2Np}{N+\alpha}} \le C|\nabla u|_2^{\frac{Np-N-\alpha}{2p}} |u|_2^{\frac{2p-Np+N+\alpha}{2p}} \le C|u|_2, \quad \forall u \in E_a^-.$$

Let us define

$$\|u\|_{E_{\mathcal{Q}}} := \|u\|_{E_a} + \|u\|_{\mathcal{Q}^{\alpha,p}}$$

where, for the sake of brevity, we omit the indexes a, α, p. Let $E_{\mathcal{Q}}^-$ be the completion of L_a^- with respect to $\|\cdot\|_{E_{\mathcal{Q}}}$ and set $E_{\mathcal{Q}} := E_{\mathcal{Q}}^- \oplus E_a^+$ Then $E_{\mathcal{Q}}$ is the completion of $H^1(\mathbb{R}^N)$ with respect to $\|\cdot\|_{E_{\mathcal{Q}}}$ due to $E_a^+ \sim L_a^+$. Clearly $(E_{\mathcal{Q}}, \|\cdot\|_{E_{\mathcal{Q}}})$ is a Banach space and $J(u)$ is well-defined on $E_{\mathcal{Q}}$.

Remark 2.3. *Let us stress the main difference between the local setting proposed in [6] and our choice. Mimicking [6], we could choose as new space $E_{\frac{2Np}{N+\alpha}}$, defined as the completion of $H^1(\mathbb{R}^N)$ with respect to the norm*

$$\|\cdot\|_{E_a} + |\cdot|_{\frac{2Np}{N+\alpha}}$$

However, although the nonlinear term $(I_\alpha * |u|^p)|u|^p$ is well-defined in $E_{\frac{2Np}{N+\alpha}}$ by Hardy-Littlewood-Sobolev inequality, we are not able to prove that the (PS) sequences $\{u_n\}$ are bounded in $E_{\frac{2Np}{N+\alpha}}$. The main reason is that $\int (I_\alpha * |\cdot|^p)|\cdot|^p$ cannot control any Lebesgue norm $|\cdot|_{L^\mu(\mathbb{R}^N)}$, and we cannot count on $\|\cdot\|_{E_a}$ either, because the norm $\|\cdot\|_{E_a}$ is weaker than $\|\cdot\|_{H^1}$ in the singular case. We overcome this difficulty taking into account the nonlocal nature of our problem in the construction of the space $E_{\mathcal{Q}}$, which turns out to be embedded into H^1_{loc}, as we will prove in Lemma 2.6 below.

Let us prove some basic properties of the space $E_{\mathcal{Q}}$.

Lemma 2.4. *$H^1(\mathbb{R}^N) \subset E_{\mathcal{Q}} \subset E_a$ and all norms $\|\cdot\|_{E_a}$, $\|\cdot\|_{H^1}$, $\|\cdot\|_{E_{\mathcal{Q}}}$ are equivalent on E_a^+.*

Proof. The embedding $E_{\mathcal{Q}} \subset E_a$ is obvious. By Sobolev embedding and Hardy-Littlewood-Sobolev inequality, for any $u \in H^1(\mathbb{R}^N)$, we have

$$\|u\|_{H^1} \geq C|u|_{\frac{2Np}{N+\alpha}} \geq C\|u\|_{\mathcal{Q}^{\alpha,p}},$$

On the other hand, we have $\|u\|_{H^1} \geq C\|u\|_{E_a}$. Therefore $\|u\|_{H^1} \geq C\|u\|_{E_{\mathcal{Q}}}$. Thus $H^1(\mathbb{R}^N) \subset E_{\mathcal{Q}}$. For $u \in E_a^+$, we know that $\|\cdot\|_{H^1}$ and $\|\cdot\|_{E_a}$ are equivalent. Thus

$$\|u\|_{H^1} \leq C\|u\|_{E_a} \leq C\|u\|_{E_{\mathcal{Q}}}.$$

This completes the proof. □

We briefly recall that a norm $\|\cdot\|_X$ on a linear space is said *uniformly convex* if for any $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that for any $x, y \in X$ with $\|x\|_X = \|y\|_X = 1$ and $\|x - y\|_X \geq \varepsilon$, then $\|\frac{x+y}{2}\|_X \leq 1 - \delta_\varepsilon$. In [27], Proposition 2.8, the authors prove the following property:

Lemma 2.5. *[Proposition 2.8 in [27]] Let $\alpha \in (0, N)$ and $p > 1$. Then $\|\cdot\|_{\mathcal{Q}^{\alpha,p}}$ is a uniformly convex norm.*

Consequently, $E_{\mathcal{Q}}$ is a reflexive Banach space.
As already said in the introduction, the location of λ on the right borderline point of the spectrum prevents the embedding of E_a^- in H^1. Nevertheless, we can recover a partial regularity, stated in the following Lemma.

Lemma 2.6. *$E_{\mathcal{Q}}^-$ embeds continuously into $H^1_{loc}(\mathbb{R}^N)$, and hence compactly into $L^t_{loc}(\mathbb{R}^N)$ for $2 \leq t < 2^*$. Moreover, $S_a u \in L^2$ for $u \in E_{\mathcal{Q}}^-$, and $E_{\mathcal{Q}}^-$ embeds continuously into $H^2_{loc}(\mathbb{R}^N)$.*

Proof. Let us first prove the embedding of $E_{\mathcal{Q}}^-$ in $H^1_{loc}(\mathbb{R}^N)$. Let $u \in E_{\mathcal{Q}}^-$. Since L_a^- is dense in $E_{\mathcal{Q}}^-$, we can choose a sequence $\{u_n\}_{n\in\mathbb{N}}$ in L_a^- with $\|u_n - u\|_{E_{\mathcal{Q}}} \to 0$, as $n \to \infty$.
For any fixed $R \in \mathbb{R}^+$ and for any $x \in B_{R/2}(0)$, we have $B_R(x) \supset B_{R/2}(0)$ so that

$$\begin{aligned}\|u\|^{2p}_{\mathcal{Q}^{\alpha,p}} &= \int |u(x)|^p \int \frac{|u(y)|^p}{|x-y|^{N-\alpha}} dy \geq \frac{1}{R^{N-\alpha}} \int |u(x)|^p \left[\int_{B_R(x)} |u(y)|^p dy\right] dx \\ &\geq \frac{1}{R^{N-\alpha}} \int_{B_{R/2}(0)} |u(x)|^p \left[\int_{B_R(x)} |u(y)|^p dy\right] dx \\ &\geq \frac{1}{R^{N-\alpha}} \int_{B_{R/2}(0)} |u(x)|^p \left[\int_{B_{R/2}(0)} |u(y)|^p dy\right] dx = \frac{1}{R^{N-\alpha}} \left(\int_{B_{R/2}(0)} |u(x)|^p dx\right)^2 \end{aligned} \quad (2.13)$$

Therefore, $u \in L^p_{loc}(\mathbb{R}^N)$.
Given a bounded domain $\Omega \subset \mathbb{R}^N$, let us take a function $\eta \in C_0^\infty(\mathbb{R}^N)$ with $\eta \equiv 1$ in Ω. Then for any $v \in L_a^- \subset H^2(\mathbb{R}^N)$,

$$-\Delta(\eta v)\eta v = \eta^2 \cdot (-\Delta v) \cdot v + v^2 \cdot (-\Delta \eta)\eta - 2\nabla(\eta v) \cdot v \nabla \eta + 2|\nabla \eta|^2 v^2,$$

so that we get

$$\begin{aligned}\int |\nabla(\eta v)|^2 dx &\leq \langle S_a v, \eta^2 v\rangle - \int (V(x) - a)\eta^2 v^2 dx + \int v^2 \cdot (-\Delta\eta)\eta dx + \\ &+ \frac{1}{2}\int |\nabla(\eta v)|^2 dx + 4\int |\nabla \eta|^2 v^2 dx \leq C\|v\|^2_{E_a} + C|\eta v|_2^2 + \frac{1}{2}\int |\nabla(\eta v)|^2 dx \end{aligned} \quad (2.14)$$

so that:

- if $p \geq 2$, combining (2.13) with (2.14) yields immediately

$$\int |\nabla(\eta v)|^2 dx \leq C\|\eta v\|^2_{E_a} + C|\eta v|^2_2 \leq C\|\eta v\|^2_{E_a} + C|\eta v|^2_p \leq C\|\eta v\|^2_{E_a} + C\|\eta v\|^2_{\mathcal{Q}^{\alpha,p}}$$

  where C depends on Ω.

- if $p < 2$ we combine the interpolation inequality with Young inequality and (2.13):

$$|\eta v|^2_2 \leq C|\eta v|^{2\theta}_{2^*}|\eta v|^{2(1-\theta)}_p \qquad \text{where} \quad \theta = \frac{N(2-p)}{2N - p(N-2)}$$
$$\leq \varepsilon|\nabla(\eta v)|^2_2 + C_\varepsilon|\eta v|^2_p \leq \varepsilon|\nabla(\eta v)|^2_2 + C_\varepsilon\|\eta v\|^2_{\mathcal{Q}^{\alpha,p}}$$

  Inserting this last inequality into (2.14) and choosing ε small enough we obtain, again,

$$\int_\Omega |\nabla(\eta v)|^2 dx \leq C\|\eta v\|^2_{E_a} + C\|\eta v\|^2_{\mathcal{Q}^{\alpha,p}}$$

  where C depends on Ω.

We have then proved that, for any $v \in L^-_a \subset H^2(\mathbb{R}^N)$

$$\int_\Omega |\nabla(\eta v)|^2 dx \leq C\|\eta v\|^2_{E_a} + C\|\eta v\|^2_{\mathcal{Q}^{\alpha,p}}$$

where Ω is any bounded domain and $\eta \in C^\infty_0(\mathbb{R}^N)$ with $\eta \equiv 1$ in Ω. Applying this inequality to the above sequence $\{u_n\}$ we obtain that $\{u_n\}$ is a Cauchy sequence in $H^1(\Omega)$, and hence $u \in H^1_{loc}$. Thus $E^-_{\mathcal{Q}}$ embeds continuously into $H^1_{loc}(\mathbb{R}^N)$.
Now we can follow the same lines of the proof of [6, Lemma 2.1] to show that $S_a u \in L^2$ and $u \in H^2_{loc}(\mathbb{R}^N)$. For the convenience of the reader, we give the details. Since $\inf \sigma(S_a) := -\theta > -\infty$ we have

$$|S_a(u_n - u_m)|^2_2 = \int_{-\theta}^0 \nu^2 d|P_{a,\nu}(u_n - u_m)|^2_2 \leq -\theta \int_{-\theta}^0 \nu d|P_{a,\nu}(u_n - u_m)|^2_2$$
$$= \theta \left| |S_a|^{\frac{1}{2}}(u_n - u_m) \right|^2_2 = \theta\|u_n - u_m\|^2_{E_a}.$$

Therefore $\{S_a u_n\}$ is a Cauchy sequence in L^2 and it follows that $S_a u_n \to S_a u$ in L^2.
For $r > 0$, $\varepsilon > 0$ and $y \in \mathbb{R}^N$, by Calderon-Zygmund inequality [16, Theorem 9.11] we have

$$\|u_n - u_m\|_{H^2(B_r(y))} \leq C(r,\varepsilon)\left(|u_n - u_m|_{L^2(B_{r+\varepsilon}(y))} + |S_a(u_n - u_m)|_{L^2(B_{r+\varepsilon}(y))}\right).$$

This implies $u \in H^2_{loc}(\mathbb{R}^N)$. □

Remark 2.7. *A space closely related to ours has been introduced by Ruiz in [36] in the more relevant physical case $N = 3, \alpha = 2$.*
We observe that another possible choice for the functional setting in the case $\lambda = a$ could be a variant of the Coulomb-Sobolev spaces introduced in [27].

3 Existence of solutions for $\lambda \in (a, b)$.

The aim of this section is to prove Theorem 1.2. As discussed in the previous section, if $a < \lambda < b$, then $\|\cdot\|_{E_\lambda}$ is equivalent to $\|\cdot\|_{H^1}$ and $E_\lambda = H^1(\mathbb{R}^N)$.
Due to the geometry of the functional Φ_λ, the main tool to find nontrivial critical points will be the following generalized Linking Theorem [6, 12].
Theorem 3.1 (Generalized Linking Theorem [22])**.** *Let X be a real Hilbert space. Suppose that $\Phi \in C^1(X, \mathbb{R})$ satisfies the following conditions:*

(i) There exists a bounded self-adjoint linear operator $L : X \mapsto X$ and a functional $\Psi \in C^1(X, \mathbb{R})$ which is bounded below, weakly sequentially lower semi-continuous with $\Psi' : X \mapsto X$ weakly sequentially continuous and such that

$$\Phi(u) = \frac{1}{2}\langle Lu, u\rangle - \Psi(u).$$

(ii) There exist a closed separable L-invariant subspace Y of X and a positive constant α such that

$$\langle Lu, u\rangle \leq -\alpha\|u\|_X^2 \text{ for } u \in Y \quad \text{and} \quad \langle Lu, u\rangle \geq \alpha\|u\|_X^2 \text{ for } u \in Z := Y^\perp.$$

(iii) There are constants $\kappa, \rho > 0$ such that $\Phi(u) \geq \kappa$ for $u \in Z$ and $\|u\|_X = \rho$.

(iv) Let $\zeta \in Z \setminus \{0\}$. Then there exists $R > \rho$ (R depending on ζ) such that $\Phi(u) \leq 0$ for any $u \in \partial M$, where

$$M := \{u = u^- + s\zeta : u^- \in Y, s \geq 0 \quad \text{and} \quad \|u\|_X \leq R\}.$$

Then there exists a Palais-Smale sequence $\{u_n\}$ such that

$$\Phi(u_n) \to c \in [\kappa, \sup \Phi(M)] \quad \text{and} \quad \Phi'(u_n) \to 0, \quad \text{as } n \to \infty.$$

Let us now verify the our functional Φ_λ verifies the linking structure of the above theorem, assumptions (iii) and (iv).

Lemma 3.2. *For any $\lambda \in (a, b)$, there exist $\kappa(\lambda)$, $\rho > 0$ such that for any $u \in E_\lambda^+ \cap \partial B_\rho(0)$ it results that* $\inf_{u \in E_\lambda^+, \|u\|_{E_\lambda} = \rho} \Phi_\lambda(u) := \kappa(\lambda) > 0$.

Proof. By Proposition 2.3, for any $u \in E_\lambda^+ \setminus \{0\}$ we have $\|u^+\|_{H^1}^2 \leq \frac{1}{\beta_\lambda}\|u^+\|_{E_\lambda}^2$. Then, by Sobolev embedding and Hardy-Littlewood-Sobolev inequality, we have

$$\Phi_\lambda(u) \geq \frac{1}{2}\|u^+\|_{E_\lambda}^2 - C(N, \alpha, p)\|u^+\|_{H^1}^{2p} \geq \frac{1}{2}\|u^+\|_{E_\lambda}^2 - \frac{C(N, \alpha, p)}{\beta_\lambda^p}\|u^+\|_{E_\lambda}^{2p}.$$

Let $\rho = \frac{1}{2}\left(\frac{\beta_\lambda^p}{2C(N,\alpha,p)}\right)^{\frac{1}{2p-2}}$; since $p > 1$, we have

$$\kappa(\lambda) := \Phi_\lambda(u)|_{E_\lambda^+ \cap \partial B_\rho(0)} \geq \left(\frac{1}{8} - \frac{1}{2^{2p+1}}\right)\left(\frac{\beta_\lambda^p}{2C(N, \alpha, p)}\right)^{\frac{1}{p-1}} > 0. \tag{3.1}$$

□

Lemma 3.3. *Let Z_0 be a finite dimensional subspace of E_a^+. Then $\Phi_\lambda(u) \to -\infty$ as $\|u\|_{E_\lambda} \to \infty$ in $E_\lambda^- \oplus Z_0$.*

Proof. Following [1, Lemma 4.2], for $\beta \in (0,1)$, we set $\gamma = \sin(\arctan \beta) \in (0,1)$ and

$$K = \{u \in E_\lambda : u^+ \in Z_0, \|u^+\|_{E_\lambda} \geq \gamma, \|u\|_{E_\lambda} = 1\}.$$

Then there is $\{u_n\} \subset K$ with $\lim\limits_{n\to\infty} J(u_n) = \inf J(K) =: \delta \geq 0$. Since K is bounded we may assume that $u_n \rightharpoonup u \in E_\lambda$ such that $u_n^+ \to u^+$ in Z_0. Clearly $\|u^+\|_{E_\lambda} \geq \gamma$ and $u \neq 0$. Since J is weakly sequentially lower semi-continuous in E_λ, we have $\delta \geq J(u) > 0$.

Let $u \in E_\lambda^- \oplus Z_0$ satisfy $\|u\|_{E_\lambda} \geq 1$. We have two cases.

- If $\|u^+\|_{E_\lambda}/\|u^-\|_{E_\lambda} \geq \beta$ we have

$$\|u^+\|_{E_\lambda}/\|u\|_{E_\lambda} = \sin\arctan(\|u^+\|_{E_\lambda}/\|u^-\|_{E_\lambda}) \geq \gamma$$

and therefore $u/\|u\|_{E_\lambda} \in K$. By $J(u) = J(u/\|u\|_{E_\lambda})\|u\|_{E_\lambda}^{2p}$ and the definition of δ we obtain $J(u) \geq \delta\|u\|_{E_\lambda}^{2p}$ and

$$\Phi_\lambda(u) \leq \frac{1}{2}\|u\|_{E_\lambda}^2 - \frac{\delta}{2p}\|u\|_{E_\lambda}^{2p}.$$

- If $\|u^+\|_{E_\lambda}/\|u^-\|_{E_\lambda} < \beta$ we have

$$\Phi_\lambda(u) \leq \frac{1}{2}(\|u^+\|_{E_\lambda}^2 - \|u^-\|_{E_\lambda}^2) \leq -\frac{1-\beta^2}{2(1+\beta^2)}\|u\|_{E_\lambda}^2. \tag{3.2}$$

For $\|u\|_{E_\lambda}$ large we find in either case that (3.2) is satisfied, and the claim is proved since $\beta^2 < 1$. □

By Lemma 2.1 and by Lemma 3.3 Φ_λ satisfies all the conditions in Theorem 3.1, for any $\lambda \in (a,b)$. Thus, there exists a Palais-Smale sequence $\{u_n\}$ at level c_λ,

$$c_\lambda \in [\kappa(\lambda), \sup_{u\in E_\lambda^-\oplus\mathbb{R}^+\zeta} \Phi_\lambda], \tag{3.3}$$

where $\kappa(\lambda) > 0$ is a constant that depends on λ. Moreover, by Proposition 2.3 and (3.1), we have

$$\kappa(\lambda) \geq \left(\frac{1}{8} - \frac{1}{2^{2p+1}}\right)\left(\frac{\beta_\lambda^p}{2C(N,\alpha,p)}\right)^{\frac{1}{p-1}} \to 0, \quad \text{as } \lambda \to b^-.$$

In the following lemma we verify the boundedness of any (PS) sequence.

Lemma 3.4. *If $\{u_n\}$ is a $(PS)_{c_\lambda}-$sequence for Φ_λ. Then $\|u_n\|_{E_\lambda}$ are bounded.*

Proof. Let n large such that $\Phi_\lambda(u_n) \leq c_\lambda + 1$ and $\|\Phi_\lambda'(u_n)\|_{E_\lambda} \leq 1$. Then

$$c_\lambda + 1 + \frac{1}{2}\|u_n\|_{E_\lambda} \geq \Phi_\lambda(u_n) - \frac{1}{2}\langle\Phi_\lambda'(u_n), u_n\rangle = \left(\frac{1}{2} - \frac{1}{2p}\right) J(u_n). \tag{3.4}$$

By (2.9) and (3.4), we have

$$\begin{aligned}\|u_n^+\|_{E_\lambda}^2 &= \langle\Phi_\lambda'(u_n), u_n^+\rangle + \int (I_\alpha * |u_n|^p)|u_n|^{p-2}u_n u_n^+ dx \\ &\leq 1\cdot\|u_n^+\|_{E_\lambda} + J(u_n)^{1-\frac{1}{2p}} J(u_n^+)^{\frac{1}{2p}} \leq \|u_n^+\|_{E_\lambda} + C(\lambda)(1+\|u_n\|_{E_\lambda})^{1-\frac{1}{2p}}\|u_n^+\|_{E_\lambda}\end{aligned}$$

Thus

$$\|u_n^+\|_{E_\lambda}^2 \le C(\lambda)(1+\|u_n\|_{E_\lambda})^{2-\frac{1}{p}},$$

which together with

$$\|u_n^-\|_{E_\lambda}^2 \le -2\Phi_\lambda(u_n) + \|u_n^+\|_{E_\lambda}^2$$

implies that

$$\|u_n\|_{E_\lambda}^2 = \|u_n^+\|_{E_\lambda}^2 + \|u_n^-\|_{E_\lambda}^2 \le C(\lambda)(1+\|u_n\|_{E_\lambda})^{2-\frac{1}{p}}.$$

Since $2-\frac{1}{p}<2$, $\|u_n\|_{E_\lambda}$ is bounded. □

By the previous arguments we have obtained a $(PS)_{c_\lambda}$-sequence $\{u_n\}$ which is bounded in E_λ. Then by using Lions' concentration compactness principle [46, Lemma 1.21] and the invariance of Φ_λ under the action of $\mathbb{Z}^N$, we get a nontrivial weak solution for (1.1). Similar to [1], by using Theorem 4.2 in [6], the existence of infinitely many geometrically distinct solutions can be obtained in a similar way. Thus we have proved Theorem 1.2.

4 Bifurcation from zero when $\lambda \to b^-$.

In this section we prove Theorem 1.4, that is, the bifurcation phenomenon occurring on the left borderline point of the spectrum, for some values of p, extending to the nonlocal case the results present in [17, 42].
Since $b \in \sigma(-\Delta+V)$, we know that there exists a Bloch wave Ψ in $H^2_{\text{loc}}(\mathbb{R}^N)\cap C^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ that satisfies $-\Delta\Psi + V\Psi = b\Psi$ (see [14]). Ψ is uniformly almost-periodic (UAP) in the sense of Besicovitch [7]. The essential tool is a nonlocal version of the Riemann-Lebesgue lemma and the estimate of the nonlocal part of the functional $\Phi_\lambda(\Psi_{(b-\lambda)^{-1/2}})$:

$$\int\int \frac{|\Psi_{(b-\lambda)^{-1/2}}(x)|^p|\Psi_{(b-\lambda)^{-1/2}}(y)|^p}{|x-y|^{N-\alpha}}dxdy,$$

where the testing vectors $\Psi_{(b-\lambda)^{-1/2}}$ are constructed from the Bloch wave Ψ of the linear Schrödinger operator.

To any uniformly almost-periodic (UAP) function $f:\mathbb{R}^N \to \mathbb{C}$ is associated a mean-value, $M(f)$, which may be defined by

$$M(f) = \lim_{T\to\infty} \frac{1}{T^N}\int_0^T \cdots \int_0^T f(x)dx_1\cdots dx_N.$$

We recall here the classical Riemann-Lebesgue lemma.
Proposition 4.1. *([17]) Let $f:\mathbb{R}^N\to\mathbb{C}$ be a uniformly almost-periodic (UAP) function and let $g\in L^1(\mathbb{R}^N)$. Then*

$$\lim_{T\to\infty}\int f(Tx)g(x)dx = M(f)\int g(x)dx.$$

For $R\in(0,+\infty)$, let us set

$$\Psi_R(x) := R^{-\frac{N}{2}}\eta\left(\frac{x}{R}\right)\Psi(x)$$

where $\eta \in C_0^\infty(\mathbb{R}^N;[0,1])$ equals 1 on $B(0,1)$. Then, $\Psi_R \in H^2(\mathbb{R}^N)\cap C^1(\mathbb{R}^N)$.
It is easy to see from Proposition 4.1 that for all $\gamma\in[1,+\infty)$,

$$\lim_{R\to\infty} R^{\frac{N}{2}-\frac{N}{\gamma}}|\Psi_R|_\gamma = [M(\Psi^\gamma)]^{\frac{1}{\gamma}}|\eta|_\gamma.$$

The following proposition states a nonlocal version of the Riemann-Lebesgue lemma, which is an easy consequence of the classical one.

Lemma 4.1. *Let $f : \mathbb{R}^N \to \mathbb{C}$ he a uniformly almost-periodic (UAP) function and let $g \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. Then*

$$\lim_{T\to\infty} \int\int \frac{f(Tx)g(x)f(Ty)g(y)}{|x-y|^{N-\alpha}} dxdy = [M(f)]^2 \int\int \frac{g(x)g(y)}{|x-y|^{N-\alpha}} dxdy.$$

Proof. Since

$$\left| \int\int \frac{g(y)g(x)}{|x-y|^{N-\alpha}} dxdy \right| \leq \left(\int |g|^{\frac{2N}{N+\alpha}} dx \right)^{\frac{N+\alpha}{N}} < \infty$$

and $f(Tx)f(Ty)$ is a uniformly almost-periodic (UAP) function, then by Proposition 4.1, we get the conclusion. □

Let us now apply the above lemma to estimate the functional J tested on the Bloch wave Ψ:

$$\begin{aligned} J(\Psi_R) = \int\int \frac{|\Psi_R(x)|^p |\Psi_R(y)|^p}{|x-y|^{N-\alpha}} dxdy &= R^{-Np} \int\int \frac{|\eta\left(\frac{x}{R}\right)\Psi(x)|^p |\eta\left(\frac{y}{R}\right)\Psi(y)|^p}{|x-y|^{N-\alpha}} dxdy \\ &= R^{N+\alpha-Np} \int\int \frac{|\eta(x)\Psi(Rx)|^p |\eta(y)\Psi(Ry)|^p}{|x-y|^{N-\alpha}} dxdy, \end{aligned}$$

then by Lemma 4.1, we get

$$\lim_{R\to\infty} R^{Np-N-\alpha} J(\Psi_R) = [M(|\Psi|^p)]^2 J(\eta). \tag{4.1}$$

Now, for $\lambda \in (a,b)$, let $R(\lambda) := \frac{1}{\sqrt{b-\lambda}}$. From [42], we know that

$$\|P_0 \Psi_{R(\lambda)}\|_{H^1} = O(b-\lambda) \text{ as } \lambda \to b. \tag{4.2}$$

By the Hardy–Littlewood–Sobolev inequality and Sobolev inequality, we have

$$J(P_0\Psi_{R(\lambda)}) \leq C|P_0\Psi_{R(\lambda)}|^{2p}_{\frac{2Np}{N+\alpha}} \leq C\|P_0\Psi_{R(\lambda)}\|^{2p}_{H^1},$$

which together with (4.2) implies that

$$J(P_0\Psi_{R(\lambda)}) = O(|b-\lambda|^{2p}) \text{ as } \lambda \to b. \tag{4.3}$$

Let us define

$$\zeta_\lambda := (Id - P_0)\Psi_{R(\lambda)} \in E^+$$

By Lemma 2.1-(iii), we have

$$J(\zeta_\lambda) = J(\Psi_{R(\lambda)} - P_0\Psi_{R(\lambda)}) \geq 2^{1-2p} J(\Psi_{R(\lambda)}) - J(P_0\Psi_{R(\lambda)})$$

Since $p < \frac{N+\alpha}{N-2}$, we also have $2p - \frac{Np-N-\alpha}{2} > 0$; hence, combining the last inequality with (4.1) and (4.3) yields

$$\liminf_{\lambda\to b} (b-\lambda)^{-\frac{Np-N-\alpha}{2}} J(\zeta_\lambda) \geq 2^{1-2p}[M(|\Psi|^p)]^2 J(\eta) > 0 \tag{4.4}$$

On the other hand, we have from [39, 42] that

$$Q_\lambda(\zeta_\lambda) = O(b-\lambda) \quad \text{as} \quad \lambda \to b^- \tag{4.5}$$

We are now ready to prove the first part of Theorem 1.4, that is an estimate for the critical level c_λ found in (3.3), as $\lambda \to b^-$.

Proposition 4.2. $c_\lambda = O((b-\lambda)^{\frac{2p-Np+N+\alpha}{2p-2}}) \to 0$ *as* $\lambda \to b^-$.

Proof. By (3.3), we have

$$\kappa(\lambda) \le c_\lambda \le \sup_{v\in E_\lambda^-, s\ge 0} \Phi_\lambda(v+s\zeta_\lambda) = \sup_{v\in E_\lambda^-, s\ge 0} \left[\frac{1}{2}Q_\lambda(v) + \frac{1}{2}s^2 Q_\lambda(\zeta_\lambda) - \frac{1}{2p}J(v+s\zeta_\lambda)\right],$$

where $Q_\lambda(v) \le 0$, and from Lemma 2.1-(iii)

$$J(v+s\zeta_\lambda) > s^{2p}\left(2^{1-2p}J(\zeta_\lambda) - J(v/s)\right).$$

Let us now prove that

$$J(v/s) \le C\alpha_\lambda^{-p}|Q_\lambda(\zeta_\lambda)|^p. \tag{4.6}$$

Indeed, since

$$\sup_{v\in E^-,\ s>0} \Phi_\lambda(v+s\zeta_\lambda) \ge c_\lambda > 0,$$

we can restrict our attention to the couples (v,s) satisfying $\Phi_\lambda(v+s\zeta_\lambda) \ge 0$ and $s>0$ Then by Proposition 2.3, we get

$$\Phi_\lambda(v+s\zeta_\lambda) \ge 0 \Rightarrow Q_\lambda(v) + s^2 Q_\lambda(\zeta_\lambda) \ge 0 \Rightarrow Q_\lambda(\zeta_\lambda) \ge \alpha_\lambda \|v/s\|_{H^1}^2.$$

On the other hand, by the Hardy–Littlewood–Sobolev inequality and the Sobolev inequality, we have

$$J(v/s) \le C(N,p,\alpha)\,|v/s|^{2p}_{\frac{2Np}{N+\alpha}} \le C\,\|v/s\|^{2p}_{H^1}$$

Combining the two above inequalities yields (4.6).
From (4.6), we deduce

$$\Phi_\lambda(v+s\zeta_\lambda) \le \frac{1}{2}s^2 Q_\lambda(\zeta_\lambda) - \frac{1}{2p}s^{2p}\left[2^{1-2p}J(\zeta_\lambda) - C\alpha_\lambda^{-p}|Q_\lambda(\zeta_\lambda)|^p\right]$$

Since

$$\frac{N+\alpha}{N} \le p < \frac{N+\alpha}{N-2} \Rightarrow 0 < \frac{Np-N-\alpha}{2} < p,$$

then by (4.1), (4.4) and (4.5), for λ approaching b sufficiently ($\lambda < b$),

$$L(\zeta_\lambda) := \frac{1}{2p}\left[2^{1-2p}J(\zeta_\lambda) - C\alpha_\lambda^{-p}|Q_\lambda(\zeta_\lambda)|^p\right] > 0,$$

and

$$L(\zeta_\lambda) = O((b-\lambda)^{\frac{Np-N-\alpha}{2}})$$

Therefore,

$$\begin{aligned} c_\lambda \le \sup_{s>0}\left(\frac{1}{2}s^2 Q_\lambda(\zeta_\lambda) - s^{2p}L(\zeta_\lambda)\right) &= (p-1)(2p)^{-\frac{2p}{2p-2}} Q_\lambda(\zeta_\lambda)^{\frac{2p}{2p-2}} L(\zeta_\lambda)^{-\frac{2}{2p-2}} \\ &= O((b-\lambda)^{\frac{2p-Np+N+\alpha}{2p-2}}) \end{aligned}$$

Since

$$\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2} \Rightarrow \frac{2p-Np+N+\alpha}{2p-2} > 0,$$

we have the final consequence

$$c_\lambda \to 0 \text{ as } \lambda \to b^-.$$

□

Let $\{u_n\}$ be a Palais-Smale sequence at level c_λ such that $u_n \rightharpoonup u_\lambda$ in $H^1(\mathbb{R}^N)$. The weak limit u_λ is a critical point of Φ_λ. By Lemma 2.2 (the Brézis-Lieb lemma of nonlocal type), we have

$$\begin{aligned} c_\lambda &= \Phi_\lambda(u_n) - \frac{1}{2}\langle \Phi'_\lambda(u_n), u_n\rangle + o(1) = \frac{p-1}{2p}\int (I_\alpha * |u_n|^p)\,|u_n|^p dx + o(1) \\ &= \frac{p-1}{2p}\int (I_\alpha * |u_\lambda|^p)\,|u_\lambda|^p dx + \frac{p-1}{2p}\int (I_\alpha * |u_n - u_\lambda|^p)\,|u_n - u_\lambda|^p dx + o(1) \\ &\geq \frac{p-1}{2p}\int (I_\alpha * |u_\lambda|^p)\,|u_\lambda|^p dx + o(1) \\ &= \Phi_\lambda(u_\lambda) - \frac{1}{2}\langle \Phi'_\lambda(u_\lambda), u_\lambda\rangle + o(1) = \Phi_\lambda(u_\lambda) + o(1). \end{aligned} \tag{4.7}$$

Combining this estimate with Proposition 4.2 we can prove the second part of Theorem 1.4.

Proposition 4.3. *When $\lambda \to b^-$, $\|u_\lambda\|_{H^1} = O(\sqrt{c_\lambda/\beta_\lambda})$, and in particular*

$$\|u_\lambda\|_{H^1} = O((b-\lambda)^{\frac{2-Np+N+\alpha}{4p-4}}) \to 0,$$

if $\frac{N+\alpha}{N} \leq p < 1 + \frac{2+\alpha}{N}$.

Proof. Let us apply (4.7) to get a relationship between c_λ and u_λ:

$$c_\lambda \geq \Phi_\lambda(u_\lambda) = \Phi_\lambda(u_\lambda) - \frac{1}{2}\langle \Phi'_\lambda(u_\lambda), u_\lambda\rangle = \frac{p-1}{2p} J(u_\lambda)$$

Decompose u_λ as $u_\lambda = u_\lambda^- + u_\lambda^+$ with $u_\lambda^- \in E_\lambda^-$, $u_\lambda^+ \in E_\lambda^+$. Proposition 2.3 and $\Phi'_\lambda(u_\lambda) = 0$ imply that

$$\begin{aligned} \beta_\lambda \|u_\lambda^+\|_{H^1}^2 + \alpha_\lambda \|u_\lambda^-\|_{H^1}^2 &\leq Q_\lambda(u_\lambda^+) - Q_\lambda(u_\lambda^-) \\ &= \frac{1}{2}\langle Q'_\lambda(u_\lambda), u_\lambda^+ - u_\lambda^-\rangle = \int (I_\alpha * |u_\lambda|^p)|u_\lambda|^{p-2}u_\lambda(u_\lambda^+ - u_\lambda^-)dx \\ &= J(u_\lambda) - 2\int (I_\alpha * |u_\lambda|^p)|u_\lambda|^{p-2}u_\lambda u_\lambda^- dx \end{aligned} \tag{4.8}$$

From (2.8) we have

$$\int (I_\alpha * |u_\lambda|^p)|u_\lambda|^{p-2}u_\lambda u_\lambda^- dx \leq C\,[J(u_\lambda)]^{1-\frac{1}{2p}}\,\|u_\lambda^-\|_{H^1},$$

so that, by (4.8) and Young inequality, we get

$$\beta_\lambda \|u_\lambda^+\|_{H^1}^2 + \frac{\alpha_\lambda}{2}\|u_\lambda^-\|_{H^1}^2 \leq \frac{2p}{p-1}c_\lambda + C c_\lambda^{2-\frac{1}{p}} \tag{4.9}$$

Now let $\lambda \to b^-$: then $\alpha_\lambda = \alpha_0$, $\beta_\lambda = \frac{\beta_0}{b}(b-\lambda)$ and $c_\lambda \to 0$. By (4.9), we have

$$\lim_{\lambda\to b^-} \frac{(b-\lambda)\|u_\lambda\|_{H^1}^2}{c_\lambda} \leq C$$

Therefore, by Proposition 4.2, we obtain

$$\|u_\lambda\|_{H^1} = O(\sqrt{c_\lambda/\beta_\lambda}) = O((b-\lambda)^{\frac{2-Np+N+\alpha}{4p-4}})$$

Moreover, if $\frac{N+\alpha}{N} \leq p < 1 + \frac{2+\alpha}{N}$, then $\frac{2-Np+N+\alpha}{4p-4} > 0$ and $\lim_{\lambda\to b^-}\|u_\lambda\|_{H^1} = 0$. □

Combining Proposition 4.2 and Proposition 4.3, we have proved the first and the second part part of Theorem 1.4. Now, let us complete the proof, verifying that b is the only possible gap-bifurcation point for (1.1) in $[a, b]$.

Let $u_\lambda = u_\lambda^- + u_\lambda^+$ be a nontrivial weak solution of (1.1). Then by testing (1.1) with u_λ^+ and u_λ^-, we have

$$Q_\lambda(u_\lambda^+) = \int (I_\alpha * |u_\lambda|^p)|u_\lambda|^{p-2}u_\lambda u_\lambda^+ \quad \text{and} \quad Q_\lambda(u_\lambda^-) = \int (I_\alpha * |u_\lambda|^p)|u_\lambda|^{p-2}u_\lambda u_\lambda^-$$

which implies directly

$$Q_\lambda(u_\lambda^+) - Q_\lambda(u_\lambda^-) = \frac{1}{2p}\langle J'(u_\lambda), u_\lambda^+ - u_\lambda^-\rangle = \frac{1}{2p}\langle J'(u_\lambda), 2u_\lambda^+ - u_\lambda\rangle$$

Since $J : H^1(\mathbb{R}^N) \mapsto \mathbb{R}$ is even and convex, $\langle J'(u_\lambda), 2u_\lambda^+ - u_\lambda\rangle \leq J(2u_\lambda^+) - J(u_\lambda)$. Then, by Hardy–Littlewood–Sobolev inequality and Sobolev inequality, we get

$$Q_\lambda(u_\lambda^+) - Q_\lambda(u_\lambda^-) \leq \frac{1}{2p}J(2u_\lambda^+) - \frac{1}{2p}J(u_\lambda) \leq \frac{2^{2p}}{2p}J(u_\lambda^+) \leq \frac{2^{2p}}{2p}\|u_\lambda^+\|_E^{2p}$$

By Proposition 2.3, we have

$$\beta_\lambda\|u_\lambda^+\|_{H^1}^2 + \alpha_\lambda\|u_\lambda^-\|_{H^1}^2 \leq \frac{2^{2p}}{2p}\|u_\lambda^+\|_{H^1}^{2p} \quad \Longrightarrow \quad \|u_\lambda^+\|_{H^1}^{2p-2} \geq \frac{2p\beta_\lambda}{2^{2p}}$$

Therefore, by the definition of β_λ, b is the only possible gap-bifurcation point for (1.1) in $[a, b]$.

5 The case $\lambda = a$: existence of H^1_{loc}-solutions.

In this section we focus on the most delicate case, that is, when $\lambda = a$, the right borderline point of the spectrum of our Schrödinger operator. We will prove Theorem 1.5.

In Section 2 we defined the space E_Q and proved some of its properties. Although J is well-defined on E_Q, E_Q is not a Hilbert space and, due to the location of λ, Φ_a does not present a linking structure as required by the generalized Linking Theorem 3.1 (in particular, condition (ii) is not verified). To overcome this problem, we use an approximation argument like [6]. For each $j \in \mathbb{N}$ we set

$$E_j^- := P_{a,-1/j}L^2(\mathbb{R}^N) = P_{a,-1/j}L_a^- \subset L_a^- \subset E_a^-$$

and

$$E_j := E_j^- \oplus E_a^+ \subset E_a$$

Then the spectrum of S_a restricted to each E_j is bounded away from 0, and we have

$$\|\cdot\|_{E_a} \sim \|\cdot\|_{H^1} \text{ on } E_j$$

Let

$$Q_j := P_{a,-1/j} + (Id - P_{a,0}) : E_a \mapsto E_j$$

denote the orthogonal projection. Then we have for any $u \in H^1(\mathbb{R}^N)$:

$$Q_j u \to u \text{ as } j \to \infty, \text{ with respect to } \|\cdot\|_{E_a} \text{ and } |\cdot|_t,\ 2 \leq t < 2^*$$

For each $j \in \mathbb{N}$, let $\Phi_j := \Phi_a|_{E_j}$, $J_j := J|_{E_j}$, where $E_j = E_j^- \oplus E_a^+$, $E_j^- = P_{a,-1/j}L^2(\mathbb{R}^N)$. Obviously, $\Phi_j, J_j \in C^1(E_j, \mathbb{R})$ and for $u, v \in E_j$,

$$\langle J_j'(u), v\rangle = \int (I_\alpha * |u|^p)|u|^{p-2}uvdx,$$
$$\langle \Phi_j'(u), v\rangle = \langle S_a u, v\rangle - \int (I_\alpha * |u|^p)|u|^{p-2}uvdx$$

Definition 5.1. *A sequence $\{u_n\}_{n\in\mathbb{N}}$ is said to be a $\widetilde{(PS)}_c$-sequence for Φ_a with respect to $(E_{j_n}, \|\cdot\|_{E_a})$, some $c \in \mathbb{R}$, if*
(i) $u_n \in E_{j_n}$ with $j_n \to \infty$ as $n \to \infty$;
(ii) $\Phi_a(u_n) \to c$ as $n \to \infty$;
(iii) $\|\Phi_{j_n}'(u_n)\|_{E_a} \to 0$ as $n \to \infty$.

Let us first prove the boundedness of the $\widetilde{(PS)}_c$-sequences.

Lemma 5.2. *If $\{u_n\}$ is a $\widetilde{(PS)}_c$-sequence for Φ_a, then $\|u_n\|_{E_a}$ and $\|u_n\|_{\mathcal{Q}^{\alpha,p}}$ are bounded or equivalently, $\|u_n\|_{E_\mathcal{Q}}$ is bounded.*

Proof. Let n large such that $\Phi_a(u_n) \le c+1$ and $\|\Phi_{j_n}'(u_n)\|_{E_a} \le 1$, then

$$c + 1 + \frac{1}{2}\|u_n\|_{E_a} \ge \Phi_a(u_n) - \frac{1}{2}\langle \Phi_{j_n}'(u_n), u_n\rangle = \left(\frac{1}{2} - \frac{1}{2p}\right)|u_n|_{HL}^{2p} \tag{5.10}$$

Note that by (2.9), we have

$$\left|\int (I_\alpha * |u_n|^p)|u_n|^{p-2}u_n u_n^+ dx\right| \le \|u_n\|_{\mathcal{Q}^{\alpha,p}}^{2p-1}\|u_n^+\|_{\mathcal{Q}^{\alpha,p}} \tag{5.11}$$

Thus, by (5.10)-(5.11), we have

$$\begin{aligned}\|u_n^+\|_{E_a}^2 =& \langle \Phi_{j_n}'(u_n), u_n^+\rangle + \int (I_\alpha * |u_n|^p)|u_n|^{p-2}u_n u_n^+ dx \\ \le& 1 \cdot \|u_n^+\|_{E_a} + \|u_n\|_{\mathcal{Q}^{\alpha,p}}^{2p-1}\|u_n^+\|_{\mathcal{Q}^{\alpha,p}} \\ \le& \|u_n^+\|_{E_a} + [C(1+\|u_n\|_{E_a})]^{1-\frac{1}{2p}}\|u_n^+\|_{\mathcal{Q}^{\alpha,p}}\end{aligned} \tag{5.12}$$

Since $\|u_n^+\|_{E_a}$ and $\|u_n^+\|_{H^1}$ are equivalent for $u_n^+ \in E_a^+$, we have

$$\|u_n^+\|_{\mathcal{Q}^{\alpha,p}} \le C|u_n^+|_{\frac{2Np}{N+\alpha}} \le C\|u_n^+\|_{H^1} \le C\|u_n^+\|_{E_a}$$

Then by (5.12), we have

$$\|u_n^+\|_{E_a}^2 \le \|u_n^+\|_{E_a} + [C(1+\|u_n\|_{E_a})]^{1-\frac{1}{2p}}\|u_n^+\|_{E_a} \quad \Longrightarrow \quad \|u_n^+\|_{E_a}^2 \le [C(1+\|u_n\|_{E_a})]^{2-\frac{1}{p}},$$

which together with

$$\|u_n^-\|_{E_a}^2 \le -2\Phi_a(u_n) + \|u_n^+\|_{E_a}^2$$

implies that

$$\|u_n\|_{E_a}^2 = \|u_n^+\|_{E_a}^2 + \|u_n^-\|_{E_a}^2 \le C + [C(1+\|u_n\|_{E_a})]^{2-\frac{1}{p}}$$

Since $2 - \frac{1}{p} < 2$, $\|u_n\|_{E_a}$ is bounded, hence, applying (5.10) once more we obtain that $\|u_n\|_{\mathcal{Q}^{\alpha,p}}$ is bounded. □

Let us note that $\Phi_j \in C^1(E_j, \mathbb{R})$ has the form $\Phi_j(u) = \frac{1}{2}\langle S_a u, u\rangle_{E_a} - J(u)$. From Lemma 2.1 and the fact that $\|\cdot\|_{E_a}$ and $\|\cdot\|_{H^1}$ are equivalent on E_j, we can deduce that $J \in C^1(E_j, \mathbb{R})$ is bounded below, weakly sequentially lower semicontinuous and $\nabla_{E_a} J : E_j \mapsto E_j$ is weakly sequentially continuous. Obviously, the functional Φ_j satisfies the conditions $(i)-(ii)$ in Theorem 3.1. Following the same lines as for the proofs of Lemma 3.2 and Lemma 3.3, we can verify that the functional Φ_j satisfies the linking structure, that is, conditions $(iii) - (iv)$ in Theorem 3.1 as stated in the following

Lemma 5.3. *There exist κ, $\rho > 0$ such that for any $u \in S_\rho^+ := E_a^+ \cap \partial B_\rho(0)$ it results that* $\inf \Phi_j(S_\rho^+) := \kappa > 0$.

Lemma 5.4. *Let Z_0 be a finite dimensional subspace of E_a^+. Then $\Phi_j(u) \to -\infty$ as $\|u\|_{E_a} \to \infty$ in $E_j^- \oplus Z_0$.*

For the sake of brevity, we omit the two proofs.

Setting $X := E_j$, $Y := E_j^-$ and $Z := E_a^+$, by Lemma 5.3 and Lemma 5.4, Φ_j satisfies all the assumptions of Theorem 3.1. Consequently, for any j there exists a sequence $\{v_m^j\}_{m\in\mathbb{N}}$ in E_j such that $\Phi_j'(v_m^j) \to 0$ and $\Phi_j(v_m^j) \to c_j \in [\kappa, \sup \Phi_j(M)]$ as $m \to \infty$, where $\kappa > 0$ is defined in Lemma 5.3, and M is defined as

$$M := \{u = u^- + s\zeta : u^- \in E_j^-, s \geq 0 \ \text{ and } \ \|u\|_{E_a} \leq R\}$$

For $m(j)$ large we therefore have

$$\|\Phi_j'(v_m^j(j))\|_{E_a} + |c_j - \Phi_j(v_{m(j)}^j)| < \frac{1}{j}.$$

Since

$$\sup_M \Phi_j(u) \leq \frac{1}{2} \sup_M \left[\|u^+\|_{E_a}^2 - \|u^-\|_{E_a}^2\right] \leq \frac{1}{2} \sup_M \left[\|u\|_{E_a}^2\right] \leq \frac{1}{2} R^2,$$

there is a subsequence c_{j_n} such that $c_{j_n} \to c \in [\kappa, \frac{1}{2}R^2]$. The sequence $u_n := v_{m(j_n)}^j$ is then a $\widetilde{(PS)}_c$-sequence as required. By Lemma 5.2, $\{u_n\}$ is bounded in $E_\mathcal{Q}$. Since $E_\mathcal{Q}$ is a reflexive Banach space by Lemma 2.5, up to a subsequence we have $u_n \rightharpoonup u$ in $E_\mathcal{Q}$.

Let us now show that $u \neq 0$. We claim that for any $r > 0$ there exists a sequence $\{y_n\}$ in $\mathbb{R}^N$ and $\delta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} u_n^2 dx \geq \delta. \tag{5.13}$$

Indeed, if not, then by Lions' concentration compactness principle [46, Lemma 1.21],

$$u_n \to 0 \text{ in } L^q(\mathbb{R}^N) \ \forall q \in (2, 2^*), \quad \text{as } n \to \infty.$$

Then, by Hardy-Littlewood-Sobolev inequality, we have

$$J(u_n) \leq C|u_n|_{\frac{2Np}{N+\alpha}}^{2p} \to 0, \quad \text{as } n \to \infty. \tag{5.14}$$

On the other hand, we have

$$0 < c = \Phi_j(u_n) - \frac{1}{2}\langle \Phi_j'(u_n), u_n\rangle + o(1) = \left(\frac{1}{2} - \frac{1}{2p}\right) J(u_n) + o(1),$$

which contradicts (5.14). Thus, (5.13) holds. Now we choose $k_n \in \mathbb{Z}^N$ such that

$$|k_n - y_n| = \min\{|k - y_n| : k \in \mathbb{Z}^N\}$$

and set $v_n := u_n(\cdot + k_n)$. Using (5.13) and the invariance of $E_{j_n}, E^{\pm}$ under the action of $\mathbb{Z}^N$ we see that $v_n \in E_{j_n}$ and

$$\int_{B_{r+\sqrt{N}/2}(0)} v_n^2 dx \geq \frac{\delta}{2}. \tag{5.15}$$

Moreover, $\|v_n\|_{E_a} = \|u_n\|_{E_a}$ and $\|v_n\|_{\mathcal{Q}^{\alpha,p}} = \|u_n\|_{\mathcal{Q}^{\alpha,p}}$, hence $\|v_n\|_{E_{\mathcal{Q}}}$ is bounded. Lemma 2.5 yields the existence of a subsequence (which we continue to denote by $\{v_n\}$) such that $v_n \rightharpoonup u$ weakly in $E_{\mathcal{Q}}$. Then by Lemma 2.6, $v_n \to u$ strongly in $L^2_{loc}(\mathbb{R}^N)$. Clearly (5.15) implies $u \neq 0$.

Let $v \in C_0^\infty(\mathbb{R}^N)$ be any test function. By Hardy-Littlewood-Sobolev inequality and Hölder inequality, we see that

$$\begin{aligned}\int (I_\alpha * |v_n|^p)|v_n|^{p-2} v_n (Id - Q_{j_n}) v dx \leq & |v_n^p|_{\frac{2N}{N+\alpha}} |v_n^{p-1}(Id - Q_{j_n})v|_{\frac{2N}{N+\alpha}} \\ \leq & |v_n|^p_{\frac{2Np}{N+\alpha}} |v_n|^{p-1}_{\frac{2Np}{N+\alpha}} |(Id - Q_{j_n})v|^p_{\frac{2Np}{N+\alpha}}\end{aligned}$$

The right hand side converges to 0 as $n \to \infty$. Now

$$\begin{aligned}&\langle S_a v_n, v\rangle_{E_a} = \langle S_a v_n, Q_{j_n} v\rangle_{E_a} \\ &= \langle \Phi_a'(v_n), Q_{j_n} v\rangle + \int (I_\alpha * |v_n|^p)|v_n|^{p-2} v_n v dx - \int (I_\alpha * |v_n|^p)|v_n|^{p-2} v_n (Id - Q_{j_n}) v dx\end{aligned}$$

and therefore, letting $n \to \infty$, we have

$$\int (\nabla u \cdot \nabla v + (V(x) - a) uv) dx = \langle S_a u, v\rangle_{E_a} = \int (I_\alpha * |u|^p)|u|^{p-2} uv dx$$

This shows that u is a weak solution for (1.1).

We end this section by proving the multiplicity result for (1.1): it will be a consequence of Theorem 4.2 in [6], see also [1]. Let us first recall the definition of $(PS)_I$-attractor:

Definition 5.5. *Let $\Phi : X \mapsto \mathbb{R}$, denote $\Phi_a^b = \{u \in X : a \leq \Phi(u) \leq b\}$. Given an interval $I \subset \mathbb{R}$, call a set $\mathcal{A} \subset X$ a $(PS)_I$-attractor if for any $(PS)_c$-sequence $\{u_n\}$ with $c \in I$, and any $\varepsilon, \delta > 0$ one has $u_n \in U_\varepsilon(\mathcal{A} \cap \Phi_{c-\delta}^{c+\delta})$ provided n is large enough.*

Theorem 5.6. *([6]). Let X be a reflexive Banach space with the direct sum decomposition $X = X^- \oplus X^+$, $u = u^- + u^+$ for $u \in X$, and suppose that X^- is separable. If Φ satisfies the following hypotheses:*

(Φ_1) $\Phi \in C^1(X, \mathbb{R})$ *is even and* $\Phi(0) = 0$.

(Φ_2) *There exist $\kappa, \rho > 0$ such that $\Phi(z) \geq \kappa$ for every $z \in X^+$ with $\|z\|_X = \rho$.*

(Φ_3) *There exists a strictly increasing sequence of finite-dimensional subspaces $Z_n \subset X^+$ such that $\sup \Phi(X_n) < \infty$ where $X_n := X^- \oplus Z_n$, and an increasing sequence of real numbers $r_n > 0$ with $\Phi(X_n \setminus B_{r_n}) < \inf \Phi(B_\rho)$.*

(Φ_4) *$\Phi(u) \to -\infty$ as $\|u^-\|_X \to \infty$ and $\|u^+\|_X$ bounded.*

(Φ_5) *$\Phi' : X_w^- \oplus X^+ \to X_w^*$ is sequentially continuous, and $\Phi : X_w^- \oplus X^+ \to \mathbb{R}$ is sequentially upper semi-continuous, where X_w^- denote the space X^- with the weak topology.*

(Φ_6) *For any compact interval $I \subset (0,\infty)$ there exists a $(PS)_I-$attractor $\mathcal{A}$ such that*

$$\inf\{\|u^+ - v^+\|_X : u, v \in \mathcal{A},\ u^+ \neq v^+\} > 0.$$

Then there exists an unbounded sequence (c_n) of positive critical values.

Write

$$\mathcal{K} =: \{u \in E_{\mathcal{Q}} : \Phi_a'(u) = 0\}$$

for the set of critical points. Let $\mathcal{F}$ consist of arbitrarily chosen representatives of the orbits in $\mathcal{K}$ under the action of $\mathbb{Z}^N$. By the evenness of Φ_a we can also assume that $\mathcal{F} = -\mathcal{F}$. To prove that there are infinitely many geometrically distinct solutions of (1.1), setting $X := E_{\mathcal{Q}}$, $X^- := E_{\mathcal{Q}}^-$ and $X^+ := E_{\mathcal{Q}}^+$, it suffices to prove that hypotheses $(\Phi_1) - (\Phi_6)$ in Theorem 5.6 are satisfied for Φ_a. (Φ_1) is obvious. Since $\|\cdot\|_{E_{\mathcal{Q}}}$ is equal to $\|\cdot\|_{H^1}$ on $E_{\mathcal{Q}}^+$, then by similar arguments of Lemma 5.3, (Φ_2) holds. Since J is weakly sequentially lower semi-continuous in $E_{\mathcal{Q}}$, then by similar arguments of Lemma 5.4, (Φ_3) holds. Condition (Φ_4) holds since $J \geq 0$.

The embedding $E_{a_w}^- \oplus E_a^+ \hookrightarrow E_{aw}$ is sequentially continuous. Therefore, by Lemma 2.1, J' is sequentially continuous on $E_{a_w}^- \oplus E_a^+$, and the same holds for Φ_a'. For the same reason J is sequentially lower semi-continuous on $E_{a_w}^- \oplus E^+$. Moreover $\|\cdot\|_{E_{\mathcal{Q}}}$ is sequentially lower semi-continuous on $E_{a_w}^-$. These facts together give (Φ_5).

The rest is the proof of (Φ_6).

Lemma 5.7. *There is $\beta > 0$ such that for any $u \in \mathcal{K} \setminus \{0\}$ we have $\Phi_a(u) \geq \beta$.*

Proof. Observe that for any $u \in E_a^-$,

$$\Phi_a(u) = -\frac{1}{2}\|u^-\|_{E_a}^2 - \frac{1}{2p}\|u^-\|_{\mathcal{Q}^{\alpha,p}}^{2p} \leq 0,$$

but for $u \in \mathcal{K} \setminus \{0\}$,

$$\Phi_a(u) = \Phi_a(u) - \frac{1}{2}\langle \Phi_a'(u), u\rangle = \left(\frac{1}{2} - \frac{1}{2p}\right)\|u\|_{\mathcal{Q}^{\alpha,p}}^{2p} > 0$$

Therefore,

$$(\mathcal{K} \setminus \{0\}) \cap E_a^- = \emptyset$$

Now, let $u \in \mathcal{K} \setminus \{0\}$. First we show that $\|u\|_{E_{\mathcal{Q}}}$ is bounded away from 0. By Lemma 2.4, we know that the norms $\|\cdot\|_{E_{\mathcal{Q}}}$, $\|\cdot\|_{E_a}$ and $\|\cdot\|_{E_{\mathcal{Q}}}$ are equivalent on the space E_a^+, therefore we only need to prove $\|u\|_{E_a} = \|u^+\|_{E_a} \geq C > 0$. If $\|u^+\|_{E_a} \leq 1$, by $\langle \Phi_a'(u), u\rangle = 0$ and (2.8),

$$\|u\|_{E_a}^2 = \langle J'(u), u\rangle \leq C\|u\|_{E_a}^{2p-1}\|u\|_{E_a},$$

and therefore

$$\|u\|_{E_a} \leq C\|u\|_{E_a}^{2p-1}$$

This shows that $\|u\|_{E_a} \geq C > 0$ for some independent constant C.

We have

$$\Phi_a(u) = \Phi_a(u) - \frac{1}{2}\langle \Phi_a'(u), u\rangle = \left(\frac{1}{2} - \frac{1}{2p}\right) J(u)$$

If $J(u) \geq 1$ we have an independent positive lower bound for $\Phi_a(u)$. If $J(u) \leq 1$, by (2.8) it follows that

$$\|u\|_{E_a}^2 = \langle J'(u), u\rangle \leq CJ(u)^{1-\frac{1}{2p}}\|u\|_{E_a},$$

and thus

$$\|u\|_{E_a} \leq CJ(u)^{1-\frac{1}{2p}}.$$

Therefore

$$\Phi_a(u) \geq C > 0$$

for some independent C since $\|u\|_{E_a}$ is bounded away from 0 on $\mathcal{K} \setminus \{0\}$ as shown above. □

Lemma 5.8. *The* $(PS)_c$ *sequence* $\{u_n\}$ *satisfying*

$$\Phi_a(u_n) \to c, \quad \|\Phi_a'(u_n)\|_{(E_{\mathcal{Q}})^*} \to 0, \quad \text{as } n \to \infty.$$

is bounded in $E_{\mathcal{Q}}$*.*

Proof. The proof is similar to Lemma 5.2, we omit it. □

In the following lemma β denotes the constant given by Lemma 5.8.

Lemma 5.9. *For* $c \in \mathbb{R}$ *let* $\{u_n\} \subset E_{\mathcal{Q}}$ *be a* $(PS)_c-$*sequence for* Φ_a*. Then either* $c = 0$ *and* $u_n \to 0$ *or* $c \geq \beta$ *and there are* $k \in \mathbb{N}$, $k \leq [c/\beta]$*, and for each* $1 \leq i \leq k$ *a sequence* $\{k_{i,n}\}_n \subset \mathbb{Z}^N$ *and a function* $v_i \in E_{HL} \setminus \{0\}$ *such that, after extraction of a subsequence of* $\{u_n\}$,

$$\left\| u_n - \sum_{i=1}^{k} \tau_{k_{i,n}} v_i \right\|_{E_{\mathcal{Q}}} \to 0,$$

$$\Phi_a\left(\sum_{i=1}^{k} \tau_{k_{i,n}} v_i\right) \to \sum_{i=1}^{k} \Phi_a(v_i) = c,$$

$$|k_{i,n} - k_{j,n}| \to \infty \quad \text{for } i \neq j,$$

$$\Phi_a'(v_i) = 0 \quad \text{for all } i.$$

Proof. The proof follows the same lines as for the proof of Lemma 4.5 in [1], so we omit it. □

Given any compact interval $I \subset (0, \infty)$ with $d = \max I$ we set $k = [d/\beta]$ and

$$[\mathcal{F}, k] = \left\{ \sum_{i=1}^{j} \tau_{m_i} v_i : 1 \leq j \leq k, m_i \in \mathbb{Z}^N, v_i \in \mathcal{F} \right\}$$

By Lemma 5.9, $[\mathcal{F}, k]$ is a $(PS)_I-$attractor.

Since the projections $P_{a,0}, Id - P_{a,0}$ commute with the action of $\mathbb{Z}^N$ on E_{HL}, it is easy to get (Φ_6), see [38, Prop. 2.57]. Therefore, by Theorem 5.6, we get infinitely many geometrically distinct solutions. This concludes the proof of Theorem 1.5.

6 Final remarks

In this section, we discuss the case $\lambda \to a^+$. What may happen if a is in the spectrum of $-\Delta + V$, or, equivalently, if 0 is in the spectrum of $-\Delta + V - a$? It is commonly believed that in this case well localized solutions of the corresponding local Schrödinger equation do not exist (see [32], section 7). Indeed, if u solves the local equation

$$-\Delta u + V(x)u = f(x,u) + a \cdot u$$

then, by setting $W(x) := -f(x,u)/u$, u is also a solution of

$$-\Delta u + (V(x) + W(x))u = a \cdot u$$

If $W(x) \to 0$ as $|x| \to \infty$, we are in front of a periodic Schrödinger operator perturbed by a decaying potential. If $W(x)$ decays sufficiently fast, it defines a relatively compact perturbation of the Schrödinger operator, so that the essential spectrum does not change (see [18], Theorem 5.35), and a turns out to be an eigenvalue lying in the essential spectrum. So, the problem of the non existence of well localized solutions (at least H^1) is strictly related to the non existence of embedded eigenvalues for periodic Schrödinger operators perturbed by decaying potentials. This problem seems far from being solved in its generality. Some interesting results have been proved for the one-dimensional case (Hill's equation). In this case, it is known that the endpoints of the continuous spectrum for Hill'sequation can be characterized by special eigenvalues of associated eigenvalue problems with periodic, respectively anti-periodic boundary conditions. In particular it was shown that the branches of solutions (bifurcating both from zero of from infinity) exist globally, and only over the gaps they consist of square-integrable solutions. Hence they "disappear" within the L^2-setting when reaching the spectrum and reoccur when reaching the gap again (see [19, 2, 20]).

The same question arises in the nonlocal case. Furthermore, what about the behaviour of the branches of solutions u_λ, as $\lambda \to a^+$? We can prove that if there exists no H^1-solution for (1.1) in the left borderline point of the spectrum, $\lambda = a$, then the branches of solutions bifurcate from ∞ in a:

Proposition 6.1. *Let $N \geq 3$, $\alpha \in (0, N)$, $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ and $(V_1) - (V_2)$ hold. Let us assume that problem*

$$\begin{cases} -\Delta u + V(x)u = (I_\alpha * |u|^p)\,|u|^{p-2}u + \lambda u \quad \text{in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right) \end{cases} \tag{6.1}$$

has for $\lambda = a$ only the trivial solution. For any $\lambda \in (a, b)$, let u_λ be a solution obtained in Theorem 1.2, then

$$\|u_\lambda\|_{H^1} \to +\infty \quad \text{as} \quad \lambda \to a^+. \tag{6.2}$$

Moreover, a is the only possible gap-bifurcation (from infinity) point for (1.1) *in* $[a, b]$.

Proof. We prove (6.3) by contradiction. Suppose that

$$\limsup_{\lambda \to a^+} \|u_\lambda\|_{H^1} < +\infty.$$

Take a sequence $\{\lambda_n\} \subset [a, b]$ such that $\lim\limits_{n \to +\infty} \lambda_n = a$. Then $\{u_{\lambda_n}\}$ is bounded in $H^1(\mathbb{R}^N)$ and

$$\Phi_{\lambda_n}(u_{\lambda_n}) = c_{\lambda_n}, \quad \Phi'_{\lambda_n}(u_{\lambda_n}) = 0.$$

Further, by (3.1) we know that for $\lambda_n < 0$

$$c_{\lambda_n} \geq \kappa(\lambda_n) \geq \left(\frac{1}{8} - \frac{1}{2^{2p+1}}\right)\left(\frac{\beta_{\lambda_n}^p}{2C(N,\alpha,p)}\right)^{\frac{1}{p-1}} = \left(\frac{1}{8} - \frac{1}{2^{2p+1}}\right)\left(\frac{\beta_0^p}{2C(N,\alpha,p)}\right)^{\frac{1}{p-1}} > 0. \tag{6.3}$$

Take a subsequence of $\{u_{\lambda_n}\}$ so that $u_{\lambda_n} \rightharpoonup u$ as $n \to +\infty$. Since Φ_a' is weakly sequentially continuous, for any $v \in H^1(\mathbb{R}^N)$ we have

$$\langle \Phi_a'(u), v\rangle = \langle \Phi_a'(u_{\lambda_n}), v\rangle + o(1) = \langle \Phi_{\lambda_n}'(u_{\lambda_n}), v\rangle + o(1).$$

Thus u is a weak solution of (6.1). We assert that $u \not\equiv 0$. Indeed, if for any $r > 0$

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} u_{\lambda_n}^2 dx = 0, \tag{6.4}$$

by Lions' concentration compactness principle [46, Lemma 1.21],

$$u_{\lambda_n} \to 0 \text{ in } L^q(\mathbb{R}^N)\ \forall q \in (2, 2^*).$$

Then, by the Hardy-Littlewood-Sobolev inequality, we have

$$J(u_{\lambda_n}) \leq C|u_{\lambda_n}|_{\frac{2Np}{N+\alpha}}^{2p} \to 0,$$

which combined with

$$c_{\lambda_n} = \Phi_{\lambda_n}(u_{\lambda_n}) - \frac{1}{2}\langle \Phi_{\lambda_n}'(u_{\lambda_n}), u_{\lambda_n}\rangle = \left(\frac{1}{2} - \frac{1}{2p}\right) J(u_n)$$

implies $c_{\lambda_n} \to 0$. This contradicts with (6.3).

Therefore (6.4) does not hold, so that there is an $r > 0$ and a sequence $\{y_n\}$ such that

$$\lim_{n\to\infty} \int_{B_r(y_n)} u_{\lambda_n}^2 dx = \delta > 0$$

We can assume, without loss of generality, that for any n

$$\int_{B_r(y_n)} u_{\lambda_n}^2 dx \geq \frac{\delta}{2}$$

Now we choose $k_n \in \mathbb{Z}^N$ such that $|k_n - y_n| = \min\{|k - y_n| : k \in \mathbb{Z}^N\}$: note that $|k_n - y_n| \leq \sqrt{N}/2$. Set $v_n := \tau_{k_n} u_{\lambda_n} = u_{\lambda_n}(\cdot + k_n)$. Hence we have

$$\int_{B_{1+\sqrt{N}/2}(0)} v_n^2 dx \geq \frac{\delta}{2}. \tag{6.5}$$

Moreover, $\|v_n\|_{H^1} = \|u_{\lambda_n}\|_{H^1}$ is also bounded in $H^1(\mathbb{R}^N)$. Thus, up to a subsequence, $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$ and $v_n(x) \to v(x)$ almost everywhere in $\mathbb{R}^N$. By Sobolev compact embedding, $v_n \to v$ strongly in $L^2_{loc}(\mathbb{R}^N)$, where $v \not\equiv 0$ thanks to (6.5). By the invariance of Φ_a under the action of $\mathbb{Z}^N$, we have

$$\Phi_a'(v) = \Phi_a'(u) = 0.$$

Thus we get a nontrivial weak solution v for (1.1), which contradicts our assumption.

Moreover, since $\lim_{\lambda\to d} \|u_\lambda\|_{H^1} < \infty$ for any $d \in (a, b)$ by Theorem 1.2, a is the only possible gap-bifurcation (from infinity) point for (1.1) in $[a, b]$. This completes the proof. □

Remark 6.1. *It is easy to observe that, when $V \equiv 0$, problem* (6.1) *has for $\lambda \geq 0$ only the trivial solution. Indeed, if $V \equiv 0$ then $\sigma(-\Delta + V) = [0, +\infty)$. If u is a solution of* (6.1)*, by testing the equation against u, we obtain the identity*

$$\int |\nabla u|^2 dx - \lambda \int |u|^2 dx - \int (I_\alpha * |u|^p)|u|^p dx = 0. \tag{6.6}$$

Moreover, we have the Pohožaev identity [30, Theorem 3]

$$\frac{N-2}{2} \int |\nabla u|^2 dx - \frac{N}{2}\lambda \int |u|^2 dx - \frac{N+\alpha}{2p} \int (I_\alpha * |u|^p)|u|^p dx = 0. \tag{6.7}$$

Combining (6.6) *and* (6.7)*, we have*

$$\left(\frac{N-2}{2} - \frac{N+\alpha}{2p}\right) \int |\nabla u|^2 dx = \lambda \left(\frac{N}{2} - \frac{N+\alpha}{2p}\right) \int |u|^2 dx. \tag{6.8}$$

Since $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$,

$$\frac{N-2}{2} - \frac{N+\alpha}{2p} < 0 \quad \text{and} \quad \frac{N}{2} - \frac{N+\alpha}{2p} > 0.$$

Then if $\lambda \geq 0$ we have from (6.8) *that $u \equiv 0$.*

Acknowledgments
We are grateful to C.A. Stuart for pointing out several interesting ideas related to open problems presented in the last Section.
The first author is supported by the National Natural Science Foundation of China(No. 11901532).
The third author is partially supported by Gnampa-Indam.

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