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A Note on Generalized Quasi-Einstein and $(\lambda, n+m)$-Einstein Manifolds with Harmonic Conformal Tensor

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Abstract: Sufficient conditions for a Lorentzian generalized quasi-Einstein manifold (M, g, f, μ) to be a generalized Robertson–Walker spacetime with Einstein fibers are derived. The Ricci tensor in this case gains the perfect fluid form. Likewise, it is proven that a $(\lambda, n+m)$-Einstein manifold (M, g, w) having harmonic Weyl tensor, $\left(\nabla^j w\right)(\nabla^m w)\mathcal{C}_{jklm} = 0$ and $\nabla_l w \nabla^l w < 0$ reduces to a perfect fluid generalized Robertson–Walker spacetime with Einstein fibers. Finally, (M, g, w) reduces to a perfect fluid manifold if $\varphi = -m\nabla(\ln w)$ is a $\varphi(\text{Ric})$-vector field on M and to an Einstein manifold if $\psi = \nabla w$ is a $\psi(\text{Ric})$-vector field on M. Some consequences of these results are considered.

Keywords: $(\lambda, n+m)$-Einstein manifolds; generalized quasi-Einstein manifold; perfect fluid; torse-forming vector fields

MSC: 53C20; 53C25

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1. Introduction

A complete Riemannian manifold (M^n, g, w) satisfying

$$(\mathrm{H}^w)_{kl} = \nabla_k \nabla_l w = \frac{w}{m}(R_{kl} - \lambda g_{kl}) \quad (1)$$

is called a $(\lambda, n+m)$-Einstein manifold where H^w is the Hessian tensor of a smooth positive function w on M and R_{kl} is the Ricci tensor [1]. It is assumed that $\triangle w = \lambda w$ whenever $m = 1$ [1]. This concept widely generalizes the concept of Einstein manifolds which plays a significant role in general relativity. The Ricci tensor of those manifolds is proportional to the metric tensor. The beauty of Einstein spacetimes is that they serve as exact solutions of the Einstein field equations.

A $(\lambda, n+m)$-Einstein manifold reduces to an Einstein manifold if w is constant. A function f may be defined on M by $e^{-\frac{f}{m}} = w$ and consequently Equation (1) becomes

$$R_{kl} + \nabla_k \nabla_l f - \frac{1}{m}\nabla_k f \nabla_l f = \lambda g_{kl}. \quad (2)$$

The left-hand side is called m-Bakry–Emery–Ricci tensor Ric^m_f, $m > 0$, that is, a natural extension of the Ricci tensor. The tensor Ric^m_f is a constant multiple of the metric tensor g. Therefore, these manifolds are called m-quasi-Einstein manifolds (see [2]). The definition of m-quasi-Einstein manifolds is quite different in some articles, such as in [3,4]. More generally, a complete Riemannian manifold identified by

$$R_{kl} + \nabla_k \nabla_l f - \mu \nabla_k f \nabla_l f = \lambda g_{kl} \quad (3)$$

is called a generalized quasi-Einstein manifold (M, g, f, μ) where $\mu, \lambda \in \mathbb{R}$. A generalized quasi-Einstein manifold is Einstein if f is constant, is an m-quasi-Einstein manifold if $\mu = \frac{1}{m}$, and is a gradient Ricci soliton given that $\mu = 0$ (or m tends to infinity). This new function f will later allow us to translate results from one structure to another.

A remarkable property of $(\lambda, n+m)$-Einstein manifolds is that, for an integer m, an Einstein warped product manifold of dimension $n+m$ has a $(\lambda, n+m)$-Einstein base manifold whose dimension is n. The name $(\lambda, n+m)$-Einstein resides in this property. The significance of $(\lambda, n+m)$-Einstein manifolds is the beauty of Einstein manifolds which are warped product manifolds. Arthur Besse posted a question about the existence of non-trivial Einstein warped product manifolds. Since then, Einstein manifolds with the structure of warped product manifolds have been extensively considered in the literature. Generalized quasi-Einstein manifolds (M, g, f, μ) are very important generalizations of gradient Ricci solitons [5–7]. Furthermore, the above two structures serve as two important generalizations of Einstein's spacetime which is an exact solution to Einstein's field equations.

Quasi-Einstein manifolds were first introduced in 1991 by Defever and Deszcz [8] (see also [9–11]) as manifolds whose Ricci tensor satisfies the condition

$$R_{kl} = \alpha u_k u_l + \beta g_{kl}. \quad (4)$$

Here α, β are scalars and u_k is a 1-form [12–18]. A perfect fluid spacetime is pictured as a Lorentzian quasi-Einstein manifold given that u_k is a unit time-like vector field [14,19,20]. These manifolds emerged during the investigation of Einstein's field equation. For example, Robertson–Walker spacetimes are quasi-Einstein manifolds.

Sufficient conditions for both a Lorentzian generalized quasi-Einstein manifold (M, g, f, μ) and a Lorentzian $(\lambda, n+m)$-Einstein manifold (M, g, w) to be a generalized Robertson–Walker spacetime with Einstein fibers are derived. The Ricci tensors of both manifolds as well as the Hessian tensors $\nabla_k \nabla_l w$ and $\nabla_k \nabla_l f$ gain the perfect fluid form. Moreover, (M, g, w) reduces to a perfect fluid manifold if $\varphi = -\frac{m}{w} \nabla w$ is a $\varphi(Ric)$-vector field on M and to an Einstein manifold if $\psi = \nabla w$ is a $\psi(Ric)$-vector field on M.

2. Results

Generalized quasi-Einstein manifolds (M, g, f, μ) were studied by by several authors in recent years [21]. In [22], M. Brozos-Vasquez et al. considered conformally flat generalized quasi-Einstein manifolds and in the Lorentzian case they proved that for $\mu \neq \frac{-1}{n-2}$ and in any point where $|\nabla f| \neq 0$, M is locally isometric to a warped product $I \times_q \bar{M}$, where I is a real interval and $\bar{M}$ is an $(n-1)$-dimensional fiber of constant sectional curvature.

Here, we start our results with the following lemma for later use. It is already proved in [22] (Lemma 4).

Lemma 1. *Let (M, g, f, μ) be a generalized quasi-Einstein Lorentzian manifold with divergence-free conformal tensor. Then, $\nabla_k f$ is an eigenvector of the Hessian operator given that $\mu \neq \frac{-1}{n-2}$ and $\nabla_k f \nabla^k f < 0$.*

Proof. We follow the line of [22]. From the defining property of generalized quasi-Einstein manifolds, a covariant derivative of the Ricci tensor gives

$$\nabla_j R_{kl} = -\nabla_j \nabla_k \nabla_l f + \mu (\nabla_j \nabla_k f) \nabla_l f + \mu \nabla_k f (\nabla_j \nabla_l f).$$

By interchanging indices, we have

$$\begin{aligned} \nabla_k R_{jl} - \nabla_j R_{kl} &= R_{jkl}{}^m \nabla_m f + \mu [\nabla_j f (\nabla_k \nabla_l f) - \nabla_k f (\nabla_j \nabla_l f)] \\ &= R_{jkl}{}^m \nabla_m f + \mu \Big[\nabla_j f (-R_{kl} + \lambda g_{kl}) - \nabla_k f \Big(-R_{jl} + \lambda g_{jl}\Big)\Big]. \end{aligned}$$

If the divergence of the conformal curvature tensor vanishes, one obtains

$$\nabla_k R_{jl} - \nabla_j R_{kl} = \frac{1}{2(n-1)}\Big[(\nabla_k R) g_{jl} - (\nabla_j R) g_{kl}\Big].$$

Thus,

$$\begin{aligned} R_{jkl}{}^{m}\nabla_m f &= \mu\Big[\nabla_k f\Big(-R_{jl} + \lambda g_{jl}\Big) - \nabla_j f(-R_{kl} + \lambda g_{kl})\Big] \\ &+ \frac{1}{2(n-1)}\Big[(\nabla_k R) g_{jl} - (\nabla_j R) g_{kl}\Big]. \end{aligned} \tag{5}$$

Transvecting this by g^{jl}, it is

$$(1-\mu) R_k^m \nabla_m f = \mu[(n-1)\lambda - R]\nabla_k f + \frac{1}{2}\nabla_k R. \tag{6}$$

Multiplying this by $\nabla_l f$ yields

$$(1-\mu) R_k^m \nabla_m f \nabla_j f = \mu[(n-1)\lambda - R]\nabla_j f \nabla_k f + \frac{1}{2}\nabla_j f \nabla_k R.$$

Exchanging indices, it is

$$(1-\mu) R_j^m \nabla_m f \nabla_k f = \mu[(n-1)\lambda - R]\nabla_j f \nabla_k f + \frac{1}{2}\nabla_k f \nabla_j R.$$

Subtracting the last two equations gives

$$\frac{(1-\mu)}{n-1}\Big(R_k^m \nabla_m f \nabla_j f - R_j^m \nabla_m f \nabla_k f\Big) = \frac{1}{2(n-1)}(\nabla_j f \nabla_k R - \nabla_k f \nabla_j R). \tag{7}$$

Transvecting Equation (5) by $\nabla^l f$, one obtains

$$0 = \mu\Big[\nabla^l f \nabla_j f R_{kl} - \nabla^l f \nabla_k f R_{jl}\Big] + \frac{1}{2(n-1)}[\nabla_k R \nabla_j f - \nabla_j R \nabla_k f]. \tag{8}$$

Inserting this in Equation (7), it is inferred that

$$\frac{\mu(n-2)+1}{n-1}\Big(R_k^m \nabla_m f \nabla_j f - R_j^m \nabla_m f \nabla_k f\Big) = 0.$$

Since $\mu(n-2)+1 \neq 0$, it is

$$R_k^m \nabla_m f \nabla_j f = R_j^m \nabla_m f \nabla_k f.$$

Thus, if $\nabla_k f \nabla^k f \neq 0$, we obtain $R_k^m \nabla_m f = \eta \nabla_k f$, i.e., $\nabla_k f$ is an eigenvector of the Ricci tensor. From Equation (3), it is

$$\Big(\nabla^k f\Big)(\nabla_k \nabla_l f) = \Big(\lambda + \mu\Big(\nabla^k f \nabla_k f\Big) - \eta\Big)\nabla_l f.$$

Thus, $\nabla_k f$ is an eigenvector of the Hessian tensor and the proof is complete. □

Theorem 1. *Let (M, g, f, μ), $n \geq 3$, be a Lorentzian quasi-Einstein with harmonic Weyl tensor, i.e., $\nabla_h \mathcal{C}_{jkl}{}^{h} = 0$, and $\Big(\nabla^h f\Big)(\nabla^j f)\mathcal{C}_{jklh} = 0$. Then, if $\nabla_k f \nabla^k f < 0$ and $\mu \neq \frac{-1}{n-2}$,*

1. *the Hessian $\nabla_k \nabla_l f$ and the Ricci tensor R_{kl} gain the perfect fluid form;*
2. *the unit time-like vector*

$$u_k = \frac{\nabla_k f}{\sqrt{-(\nabla_m f)(\nabla^m f)}}$$

is a torse-forming vector field;

3. (M, g) *is a generalized Robertson–Walker spacetime with Einstein fibers.*

Proof. First, we show that the Hessian tensor and the Ricci tensor gain the perfect fluid form. From Equation (6), we have

$$\nabla_k R = 2((1-\mu)\eta - \mu(n-1)\lambda + \mu R)\nabla_k f. \tag{9}$$

This equation with Equation (5) yields

$$\nabla^m f R_{jklm} = -\mu\left[\nabla_k f R_{jl} - \nabla_j f R_{kl}\right] + \frac{(1-\mu)\eta + \mu R}{n-1}\left[(\nabla_k f)g_{jl} - (\nabla_j f)g_{kl}\right]. \tag{10}$$

Transvecting the last equation by $\nabla^j f$, it is

$$\begin{aligned} \nabla^j f \nabla^m f R_{jklm} &= \frac{(1-\mu)\eta + \mu R}{n-1}\left[\nabla_l f \nabla_k f - \left(\nabla^j f \nabla_j f\right) g_{kl}\right] \\ &\quad -\mu\left[\eta \nabla_l f \nabla_k f - \nabla^j f \nabla_j f R_{kl}\right] \end{aligned} \tag{11}$$

Now, define the unit time-like vector $u_k = \frac{\nabla_k f}{\sqrt{-(\nabla_m f)(\nabla^m f)}}$. Then, u_k is a unit time-like vector field and the above equation becomes

$$u^j u^m R_{jklm} = -\mu[\eta u_l u_k + R_{kl}] + \frac{(1-\mu)\eta + \mu R}{n-1}[u_l u_k + g_{kl}]. \tag{12}$$

The conformal curvature tensor is given by

$$\begin{aligned} \mathcal{C}_{jklm} &= R_{jklm} + \frac{1}{n-2}\left[g_{jm}R_{kl} - g_{km}R_{jl} + g_{kl}R_{jm} - g_{jl}R_{km}\right] \\ &\quad - \frac{R}{(n-1)(n-2)}\left[g_{jm}g_{kl} - g_{km}g_{jl}\right]. \end{aligned} \tag{13}$$

A straightforward calculation now gives

$$\begin{aligned} u^j u^m \mathcal{C}_{jklm} &= u^j u^m R_{jklm} + \frac{1}{n-2}\left[-R_{kl} - u_k u^j R_{jl} + g_{kl}\left(u^j u^m R_{jm}\right) - u_l u^m R_{km}\right] \\ &\quad - \frac{R}{(n-1)(n-2)}[-g_{kl} - u_k u_l]. \end{aligned}$$

Equation (12) yields

$$\begin{aligned} u^j u^m \mathcal{C}_{jklm} &= -\mu[\eta u_l u_k + R_{kl}] + \frac{(1-\mu)\eta + \mu R}{n-1}[u_l u_k + g_{kl}] \\ &\quad + \frac{1}{n-2}[-R_{kl} - \eta u_k u_l - \eta g_{kl} - \eta u_l u_k] \\ &\quad - \frac{R}{(n-1)(n-2)}[-g_{kl} - u_k u_l]. \end{aligned}$$

Thus,

$$
\begin{aligned}
u^j u^m \mathcal{C}_{jklm} &= \left(-\mu - \frac{1}{n-2}\right) R_{kl} \\
&\quad + \left(\frac{(1-\mu)\eta + \mu R}{n-1} - \frac{1}{n-2}\eta + \frac{R}{(n-1)(n-2)} \right) g_{kl} \\
&\quad + \left(-\mu\eta + \frac{(1-\mu)\eta + \mu R}{n-1} - \frac{2}{n-2}\eta + \frac{R}{(n-1)(n-2)} \right) u_k u_l.
\end{aligned}
$$

A straightforward calculation now gives the coefficient of g_{kl} as

$$
\begin{aligned}
& \frac{(1-\mu)\eta + \mu R}{n-1} - \frac{1}{n-2}\eta + \frac{R}{(n-1)(n-2)} \\
= {} & \frac{\eta}{n-1} + \frac{-\mu\eta + \mu R}{n-1} - \frac{1}{n-2}\eta + \frac{R}{(n-1)(n-2)} \\
= {} & \frac{\mu(R-\eta)}{n-1} + \frac{R-\eta}{(n-1)(n-2)} \\
= {} & \left(\mu + \frac{1}{n-2}\right)\left(\frac{R-\eta}{n-1}\right),
\end{aligned}
$$

and the coefficient of $u_k u_l$ as

$$
\begin{aligned}
& -\mu\eta + \frac{(1-\mu)\eta + \mu R}{n-1} - \frac{2}{n-2}\eta + \frac{R}{(n-1)(n-2)} \\
= {} & -\mu\eta + \frac{\eta}{n-1} + \frac{-\mu\eta + \mu R}{n-1} - \frac{2}{n-2}\eta + \frac{R}{(n-1)(n-2)} \\
= {} & \frac{-\mu\eta n + \mu\eta - \mu\eta + \mu R}{n-1} + \frac{(n-2)\eta - 2(n-1)\eta + R - \eta}{(n-1)(n-2)} \\
= {} & \mu\frac{R - n\eta}{n-1} + \frac{R - n\eta}{(n-1)(n-2)} \\
= {} & \left[\mu + \frac{1}{n-2}\right]\left(\frac{R - n\eta}{n-1}\right).
\end{aligned}
$$

That is,

$$
u^j u^m \mathcal{C}_{jklm} = \left[\mu + \frac{1}{n-2}\right]\left[-R_{kl} + \frac{R-\eta}{n-1} g_{kl} + \frac{R - n\eta}{n-1} u_l u_k\right]. \tag{14}
$$

The proof of the above result depends on Lemma 1, that is, this result is achieved only assuming $\nabla^m \mathcal{C}_{jklm} = 0$. Apparently, Equation (14) represents the Ricci tensor of an imperfect fluid with a shear tensor given by $\left[\mu + \frac{1}{n-2}\right]^{-1} u^j u^m \mathcal{C}_{jklm}$. If $u^j u^m \mathcal{C}_{jklm} = 0$, then the Ricci tensor gains the perfect fluid form

$$
R_{kl} = \frac{R-\eta}{n-1} g_{kl} + \frac{R - n\eta}{n-1} u_l u_k. \tag{15}
$$

From the defining property (3) of the generalized quasi-Einstein manifold, one simply obtain

$$
\begin{aligned}
R_{kl} + \nabla_k \nabla_l f - \mu \nabla_k f \nabla_l f &= \lambda g_{kl} \\
\frac{R-\eta}{n-1} g_{kl} + \frac{R - n\eta}{n-1} u_l u_k + \nabla_k \nabla_l f - \mu \nabla_k f \nabla_l f &= \lambda g_{kl}.
\end{aligned}
$$

Thus,

$$\nabla_k \nabla_l f = \left(\lambda - \frac{R-\eta}{n-1}\right) g_{kl} - \frac{R-n\eta}{n-1} u_l u_k + \mu \nabla_k f \nabla_l f. \tag{16}$$

The Hessian operator gains the perfect fluid form too.

A covariant derivative of $u_k = \frac{\nabla_k f}{\sqrt{-(\nabla_m f)(\nabla^m f)}}$ gives

$$\nabla_j u_k = \frac{\nabla_j \nabla_k f}{\sqrt{-(\nabla_m f)(\nabla^m f)}} + \frac{(\nabla_k f)(\nabla_j \nabla_m f)(\nabla^m f)}{\left(\sqrt{-(\nabla_m f)(\nabla^m f)}\right)^3}. \tag{17}$$

Inserting Equation (16) and $\left(\nabla^k f\right)(\nabla_k \nabla_l f) = \left(\lambda + \mu\left(\nabla^k f \nabla_k f\right) - \eta\right)\nabla_l f$ in the above equation, it is

$$\nabla_j u_k = \frac{1}{\sqrt{-(\nabla_m f)(\nabla^m f)}} \left(\lambda - \frac{R-\eta}{n-1}\right)\left[g_{kj} + u_k u_j\right].$$

Thus, u_j is a unit time-like torse-forming vector field that is also an eigenvector of the Ricci tensor. In view of Proposition 3.7 and Theorem 5.5 of the survey [17], we have that (M, g) is a generalized Robertson–Walker spacetime with Einstein fibers [23]. □

Let us discuss the converse of the above result. The Ricci tensor of a GRW spacetime is given by

$$R_{kl} = \frac{R-n\eta}{n-1} u_k u_l + \frac{R-\eta}{n-1} g_{kl} - (n-2) u^j u^m u^l \mathcal{C}_{jklm}.$$

Assume that (M, g) is, in addition, a perfect fluid spacetime, then there is a unit time-like vector field θ_l which is an eigenvector of the Ricci tensor with eigenvalue ξ and [18]

$$R_{kl} = \frac{R-n\xi}{n-1} \theta_k \theta_l + \frac{R-\xi}{n-1} g_{kl}.$$

Transvecting this equation with u^l, one obtains $u_l = \pm\theta_l$ and $\eta = \xi$. Thus, one gets

$$(n-2) u^j u^m \mathcal{C}_{jklm} = 0.$$

Thus, for $n \geq 3$, $u^j u^m \mathcal{C}_{jklm} = 0$. Finally, in [17], the second and third authors of the current work proved that a perfect fluid GRW spacetime satisfies $\nabla_h \mathcal{C}_{jkl}{}^h = 0$.

For $(\lambda, n+m)$-Einstein manifolds (M, g, w) whose identifying property is Equation (1), as stated before, we may define a function f on M by $w = e^{-\frac{f}{m}}$. It is

1. $\nabla_l w = \frac{-w}{m} \nabla_l f$ and so $\nabla_l w$ is time-like if and only if $\nabla_l f$ is time-like.
2. Let $u_k = \frac{\nabla_k f}{\sqrt{-(\nabla_m f)(\nabla^m f)}}$ and $v_k = \frac{\nabla_k w}{\sqrt{-(\nabla_m w)(\nabla^m w)}}$, then $u_k = -v_k$.
3. The Hessian tensors are related as follows

$$\nabla_k \nabla_l w = \frac{-w}{m} \nabla_k \nabla_l f + \frac{w}{m^2} \nabla_k f \nabla_l f.$$

4. The manifold (M, g, f, μ) where $\mu = \frac{1}{m}$ is a generalized quasi-Einstein manifold.

Using these notes, we can prove Theorem 2.

Theorem 2. *Let (M, g, w) be a Lorentzian $(\lambda, n+m)$-Einstein manifold having harmonic Weyl tensor $\mathcal{C}$, i.e., $\nabla_m \mathcal{C}_{jkl}{}^m = 0$, and $\left(\nabla^j w\right)(\nabla^m w)\mathcal{C}_{jklm} = 0$. If $\nabla_l w \nabla^l w < 0$, then the following conditions hold:*

1. *The Hessian $\nabla_k \nabla_l w$ and the Ricci tensor R_{kl} gain the perfect fluid form,*
2. *the unit time-like vector field $v_k = \frac{\nabla_k w}{\sqrt{-(\nabla_m w)(\nabla^m w)}}$ is torse-forming, and*

3. *(M, g) is a GRW spacetime with Einstein fibers.*

Bang-Yen Chen presented a great and simple characterization of GRW spacetimes in [24] using a time-like concircular vector field. This vector field is called Chen's vector field.

Theorem 3. *Let (M, g, w) be a $(\lambda, n+m)$-Einstein manifold, $m > 1$ be an integer, having harmonic Weyl tensor $\mathcal{C}$, i.e., $\nabla_m \mathcal{C}_{jkl}{}^m$, and $(\nabla^j w)\left(\nabla^k w\right)\mathcal{C}_{jklm} = 0$, then (M, g) reduces to a perfect fluid, $\nabla_k w$ and the Chen vector field are eigenvectors of the Ricci tensor and one of the following conditions holds:*

1. *$\nabla_k w$ and the Chen vector field ζ_k are orthogonal where the corresponding eigenvalues φ and μ satisfy*
$$\varphi = \frac{R-\mu}{n-1};$$
2. *$\nabla_k w$ and ζ_k are dependent.*

Proof. Let (M, g) be a complete simply connected Riemannian manifold having harmonic Weyl tensor $\mathcal{W}$, i.e., $\nabla_m \mathcal{C}_{jkl}{}^m$, and $(\nabla^j w)\left(\nabla^k w\right)\mathcal{C}_{jklm} = 0$. Using the above result, g is of the form
$$g = dt^2 + \omega^2(t) g_L \tag{18}$$
and $w = w(t)$ where g_L is an Einstein metric. Moreover, in [1] (Theorem 7.2), it is proved that the vector field $\xi_k = \nabla_k w$ is an eigenvector of the tensor
$$P_{kl} = \mathrm{R}_{kl} - \sigma g_{kl},$$
where
$$\sigma = \frac{1}{m-1}[(n-1)\lambda - R],$$
and R is the scalar curvature of M. Then,
$$\xi^k \mathrm{R}_{kl} = (\sigma + \theta)\xi_l, \tag{19}$$
for some scalar θ and so ξ_k is an eigenvector of the Ricci tensor with eigenvalue $\varphi = \sigma + \theta$. In [25], the author proved that a warped product of the form (18) admits a nowhere zero concircular vector field $\zeta = \omega(t)\frac{\partial}{\partial t}$, i.e., $\nabla_i \zeta_j = \rho g_{ij}$ (for definition and some interesting results, see [25]). It is clear that $\rho = \dot{\omega}(t)$. In [26] (Theorem 2.1), it is shown that ζ is an eigenvector of the Ricci tensor. Additionally, the Ricci tensor is given by
$$\mathrm{R}_{kl} = \left(\frac{R-\mu}{n-1}\right) g_{kl} + \frac{n\mu - R}{(n-1)\omega^2(t)}\zeta_k \zeta_l + + \frac{(n-2)}{\omega^2(t)}\mathcal{C}_{jklm}\zeta^j \zeta^m.$$

We note the following facts:

1. From Equation 10 in [26], it is $\mathcal{C}_{jklm}\zeta^j\zeta^m = 0$ if and only if $\mathcal{C}_{jklm}\zeta^j = 0$;
2. from Theorem 3.4 in [26], the Weyl tensor is harmonic if and only if $\mathcal{C}_{jklm}\zeta^j = 0$;
3. from Proposition 3.5 in [26], $\mathcal{C}_{jklm}\zeta^j = 0$ if and only if the Ricci tensor is quasi-Einstein.

Since the Lorentzian signature is irrelevant during their proof, we have (see also the survey [17], Theorems 3.3 and 5.5)
$$\mathrm{R}_{kl} = \left(\frac{R-\mu}{n-1}\right) g_{kl} + \frac{n\mu - R}{(n-1)\omega^2(t)}\zeta_k \zeta_l. \tag{20}$$

Now, the Ricci tensor has two eigenvectors $\zeta = \omega(t)\frac{\partial}{\partial t}$ and $\xi = \nabla w$. It is clear that $\mathrm{Ric}(\zeta, Y) = \mu g(\zeta, Y)$. However,

$$
\begin{aligned}
\xi^k \mathrm{R}_{kl} &= \left(\frac{R-\mu}{n-1}\right)\xi_l + \frac{n\mu - R}{(n-1)\omega^2(t)}\left(\xi^k \zeta_k\right)\zeta_l, \\
\varphi \xi_l &= \left(\frac{R-\mu}{n-1}\right)\xi_l + \frac{n\mu - R}{(n-1)\omega^2(t)}\left(\xi^k \zeta_k\right)\zeta_l, \\
0 &= \left(\frac{R-\mu}{n-1} - \varphi\right)\xi_l + \frac{n\mu - R}{(n-1)\omega^2(t)}\left(\xi^k \zeta_k\right)\zeta_l.
\end{aligned}
$$

Thus, we have the following cases:

1. The vector fields ζ and ξ have different eigenvalues and therefore

$$\varphi\left(\xi_l \zeta^l\right) = \xi^k \mathrm{R}_{kl} \zeta^l = \mu\left(\xi^k \zeta_k\right),$$

that is, they are orthogonal. That is,

$$
\begin{aligned}
\varphi &= \frac{R-\mu}{n-1}, \\
\xi^k \mathrm{R}_{kl} &= \left(\frac{R-\mu}{n-1}\right)\xi_l.
\end{aligned}
$$

2. The vector fields ζ and ξ have the same, eigenvalue i.e., $\mu = \varphi$.

□

It is well known that, for $n \geq 3$, the divergence of the Weyl tensor is related to the Cotton tensor $\mathcal{T}$ by the formula

$$\mathrm{div}(\mathcal{C}) = \left(\frac{n-3}{n-2}\right)\mathcal{T},$$

where

$$\mathcal{T}_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}\left[g_{jk}\nabla_i R - g_{ik}\nabla_j R\right].$$

Corollary 1. *Let (M, g, w) be a Lorentzian $(\lambda, n+m)$- Einstein manifold where the Cotton tensor vanishes, $\left(\nabla^j w\right)\left(\nabla^k w\right)\mathcal{C}_{jklm} = 0$ and $\nabla_l w \nabla^l w < 0$, then (M, g) reduces to a perfect fluid generalized Robertson–Walker spacetime.*

The vanishing of the Cotton tensor is equivalent to $\mathrm{d}\mathcal{S} = 0$, where

$$\mathcal{S}_{kl} = R_{kl} - \frac{R}{2(n-1)} g_{kl}$$

is the Schouten tensor, i.e., $\mathcal{S}$ is a Coddazzi tensor.

Corollary 2. *Let (M, g, w) be a Lorentzian $(\lambda, n+m)$-Einstein manifold where the Schouten tensor is a Coddazzi tensor, $\left(\nabla^j w\right)(\nabla^m w)\mathcal{C}_{jklm} = 0$ and $\nabla_l w \nabla^l w < 0$, then (M, g) reduces to a perfect fluid generalized Robertson–Walker spacetime.*

A vector field φ on a Riemannian manifold (M, g) is called a $\varphi(Ric)$-vector field on M if

$$\nabla_j \varphi_i = \gamma R_{ij}, \tag{21}$$

where γ is constant [27].

Theorem 4. *Let (M, g, w) be a Lorentzian $(\lambda, n+m)$-Einstein manifold. Then,*

1. *(M, g) reduces to a perfect fluid manifold if $\varphi = -\frac{m}{w}\nabla w$ is a $\varphi(Ric)$-vector field on M.*
2. *(M, g) reduces to an Einstein manifold if $\psi = \nabla w$ is a $\psi(Ric)$-vector field on M.*

Proof. Assume that $\varphi = -\frac{m}{w}\nabla w$ is a $\varphi(Ric)$-vector field on (M, g). Then,

$$\begin{aligned} \nabla_k \nabla_l w &= \frac{w}{m}\left(\frac{1}{\gamma}\nabla_j \varphi_i - \lambda g_{kl}\right), \\ \nabla_j \varphi_i &= \frac{\gamma m}{w}\nabla_k \nabla_l w + \gamma\lambda g_{kl}, \\ &= \gamma m\left(\frac{1}{w}\nabla_k \nabla_l w\right) + \gamma\lambda g_{kl}, \end{aligned}$$

$$-m\nabla_l\left[\frac{1}{w}\nabla_k w\right] = \gamma R_{kl},$$

where μ is constant. Thus,

$$\gamma R_{kl} = m\nabla_l(\ln w)\nabla_k(\ln w) - \frac{m}{w}\nabla_l \nabla_k w.$$

By using Equation (1), one obtains

$$R_{kl} = \frac{\lambda}{(\gamma+1)}g_{kl} + \frac{m}{(\gamma+1)}\nabla_l(\ln w)\nabla_k(\ln w),$$

and consequently (M, g) is a perfect fluid manifold.

Assume now that $\psi = \nabla w$ is a $\psi(Ric)$-vector field on M. Then,

$$\nabla_l \nabla_k w = \sigma R_{kl}.$$

Equation (1) implies that

$$\sigma R_{kl} = \frac{w}{m}R_{kl} - \frac{\lambda w}{m}g_{kl},$$

and hence

$$R_{kl} = \left(\frac{\lambda w}{w - m\sigma}\right)g_{kl}.$$

Thus, (M, g) is an Einstein manifold. □

A similar result holds for generalized quasi-Einstein manifolds.

Corollary 3. *Let (M, g, f, μ) be a generalized quasi-Einstein manifold. Then,*

1. *(M, g) reduces to a perfect fluid manifold if $\varphi = \nabla f$ is a $\varphi(Ric)$-vector field on M.*
2. *(M, g) reduces to an Einstein manifold if $\psi = -\mu e^{-\mu f}\nabla f$ is a $\psi(Ric)$-vector field on M.*

The reader is referred to [28,29] for the definition of warped product manifolds, to [17] for a survey on generalized Robertson–Walker spacetimes, and to [30], Chapter 16 for the definitions of harmonic Weyl tensor and some equivalent conditions.

3. Conclusions

In this short note, two Lorentzian Einstein-like structures are considered. A Lorentzian generalized quasi-Einstein manifold (M, g, f, μ) with $\nabla_h \mathcal{C}_{jkl}{}^h = 0$ and $\left(\nabla^h f\right)(\nabla^j f)\mathcal{C}_{jklh} = 0$ is shown to be a GRW spacetime with Einstein fibers if $\nabla_k f \nabla^k f < 0$ and $\mu \neq \frac{-1}{n-2}$. Likewise, a Lorentzian $(\lambda, n+m)$-Einstein manifold (M, g, w) admitting a harmonic conformal curvature tensor and $(\nabla^j w)(\nabla^m w)\mathcal{C}_{jklm} = 0$ is also a GRW spacetime with Einstein fibers if $\nabla_l w \nabla^l w < 0$. In both cases, the Ricci tensors as well as the Hessian tensors $\nabla_k \nabla_l w$ and $\nabla_k \nabla_l f$ gain the perfect fluid form. Finally, $\varphi(Ric)$-vector fields are investigated on both manifolds. (M, g, w) reduces to a perfect fluid manifold if $\varphi = -\frac{m}{w}\nabla w$ is a $\varphi(Ric)$-vector field on M and to an Einstein manifold if $\psi = \nabla w$ is a $\psi(Ric)$-vector field on M. Similar results hold on (M, g, f, μ) where $\varphi = \nabla f$ and $\psi = -\mu e^{-\mu f} \nabla f$.

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