Quantifying Qubit Magic Resource with Gottesman-Kitaev-Preskill Encoding

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Quantum resource theories are a powerful framework for characterizing and quantifying relevant quantum phenomena and identifying processes that optimize their use for different tasks. Here, we define a resource measure for magic, the sought-after property in most fault-tolerant quantum computers. In contrast to previous literature, our formulation is based on bosonic codes, well-studied tools in continuous-variable quantum computation. Particularly, we use the Gottesman-Kitaev-Preskill code to represent multiqubit states and consider the resource theory for the Wigner negativity. Our techniques are useful in finding resource lower bounds for different applications as state conversion and gate synthesis. The analytical expression of our magic measure allows us to extend current analysis limited to small dimensions, easily addressing systems of up to 12 qubits.

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Identifying and quantifying which properties of quantum mechanics, or resources, are responsible for the predicted advantage of quantum computers over classical ones is the object of intense theoretical and experimental effort. Within the field of resource theories [1–10], the quantification of magic for fault-tolerant quantum computation occupies a prominent role. The leading architectures of fault-tolerant quantum computers are based on stabilizer codes [11]. In this approach, certain operations are easy to implement and constitute a nonuniversal [12] set that is fault tolerant by having a transversal implementation—such that these operations do not propagate errors within a code block [13]. This restricted set of easy or free operations called stabilizer operations includes Clifford gates, preparation of stabilizer states, and computational basis measurements. Stabilizer operations alone cannot provide quantum computational advantage, as the calculations can be efficiently classically simulated [14]. To unlock universal quantum computation, one needs to include difficult or resourceful operations—for example, through magic state injection—which naturally leads to the so-called magic state model [15]. However, the preparation of high-quality magic states typically involves costly procedures such as magic state distillation, which makes it desirable to optimize the number of magic states used for a given quantum computation—although alternative routes to magic state distillation exist [16].

The need for optimizing nonstabilizer resources has driven the area of resource theory of magic and, therefore, the definition of several magic measures for both qudit—d-dimensional—and qubit systems. Early works focused on resource theories of magic for odd-dimensional qudits [17], relying on a well-defined discrete Wigner function and its negativity [8], and its extensions to infinite dimensions [9,10]. The more defying case of multiqubit systems—for which the definition of the discrete Wigner function remains challenging [18–21]—has undergone substantial progress in recent years with the development of several magic measures. Among these, the relative entropy of magic [8], the robustness of magic [22], the dyadic negativity, the mixed state extent, and the generalized robustness [23] have been defined for general density matrices, while the stabilizer rank or extent [24–26], the stabilizer nullity, and the dyadic monotone [27] only account for the magic content of pure states. The availability of these measures has been essential to find (in few cases optimal) lower bounds in magic state distillation schemes and in non-Clifford unitary synthesis [22,23,27]. Moreover, magic monotones have inspired classical simulators of quantum computing architectures [23–26]. However, in general, it is impractical to compute these measures for large numbers of qubits ($\gtrsim 5$), and several lower bounds in the literature apply to distinct scenarios in which all easy operations are not always considered free—for instance, the measures do not always account for measurements and classical feed forward as free operations. Therefore, it is desirable to provide new measures that are practically computable for larger number of qubits, combine different quantifiers of magic to find tighter bounds applicable to general scenarios, and to identify how quantum computations can be optimized.

Here, we develop a new magic measure for multiqubit pure states by borrowing tools and notions from continuous-variable (CV) systems in the context of bosonic codes. Moreover, we connect this independently developed magic measure with the stabilizer(st) norm [22,28], and with the stabilizer Rényi entropy [29], measured recently on a quantum processor [30]. The CV framework provides a way to upgrade the st norm status to a fully-fledged measure and brings new insight on its properties and tools for its computation. We exploit the encoding of discrete-variable systems into the infinite-dimensional Hilbert space of CV systems provided by bosonic codes, the main alternative to conventional error-correction codes with growing theoretical and experimental efforts [31–34]. In contrast to all previous magic monotones, the CV mathematical formulation of our monotone allows for transferring results from CV quantum computation to magic quantification. Specifically, we map qubit systems into an infinite-dimensional Hilbert space via the Gottesman-Kitaev-Preskill (GKP) encoding [35] and then derive an expression for their Wigner logarithmic negativity (WLN) [9,10,36]. Using this expression, we develop a new magic measure that we call GKP magic. The Wigner function is well defined for CV systems and allows us to evaluate the GKP magic for significantly larger systems than previously known measures, easily reaching up to 12-qubit systems. The GKP magic properties that we prove enable the analysis of the magic resource cost in the most general scenario, where probabilistic protocols are allowed, and measurements, auxiliary qubits, and classical feed forward are free operations. In this context, we find analytical expressions of the GKP magic for relevant building blocks of quantum algorithms yielding lower bounds for the corresponding T count, a known indicator of the difficulty to implement fault-tolerant quantum circuits.

Definition of the GKP magic measure.—We consider a general n-qubit state in the $\hat{Z}$ eigenbasis, represented by the density operator

$$\hat{\rho} = \sum_{\boldsymbol{u},\boldsymbol{v}\in\mathbb{F}_2^n} \rho_{\boldsymbol{u},\boldsymbol{v}} |\boldsymbol{u}\rangle\langle\boldsymbol{v}|, \tag{1}$$

with $|\boldsymbol{u}\rangle = |u_1, \ldots, u_n\rangle$ and $|\boldsymbol{v}\rangle = |v_1, \ldots, v_n\rangle$ a tensor product of the single-qubit states in the computational basis, $\mathbb{F}_2^n$ the n-dimensional binary linear space, and $\rho_{\boldsymbol{u},\boldsymbol{v}}$ complex coefficients. With the GKP encoding in square lattices, the code words $|u_i\rangle$—with $u_i \in \{0,1\}$—correspond to the infinite superpositions of position $\hat{q}$ eigenstates

$$|u_i\rangle = \sum_{s_i=-\infty}^{\infty} |x_i = \sqrt{\pi}(u_i + 2s_i)\rangle_{\hat{q}}. \tag{2}$$

Then, the Wigner function associated to the GKP-encoded density operator of Eq. (1) is given by

$$\begin{aligned} W_{\hat{\rho}}(\boldsymbol{q},\boldsymbol{p}) &\equiv \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} d^n\boldsymbol{x} e^{i\boldsymbol{p}\boldsymbol{x}} \left\langle \boldsymbol{q}+\frac{\boldsymbol{x}}{2}\middle|\hat{\rho}\middle|\boldsymbol{q}-\frac{\boldsymbol{x}}{2}\right\rangle_{\hat{q}} \\ &= \frac{1}{(2\pi)^n}\sum_{\boldsymbol{u},\boldsymbol{v}\in\mathbb{F}_2^n}\rho_{\boldsymbol{u},\boldsymbol{v}}\prod_{i=1}^{n}\Bigg[\sum_{s_i,t_i}(-1)^{\frac{s_i}{2}(u_i-v_i-2t_i)} \\ &\quad \times \delta\left(p_i - \frac{\sqrt{\pi}}{2}s_i\right)\delta\left(q_i - \frac{\sqrt{\pi}}{2}(2t_i+u_i+v_i)\right)\Bigg], \end{aligned} \tag{3}$$

and it constitutes the foundation of the GKP magic measure.

The negativity of the Wigner function is a necessary condition for achieving exponential speedup in continuous-variable quantum computing architectures [37]. A measure of this resource is the WLN, defined as [9,10]

$$\mathcal{W}(\hat{\sigma}) = \log_2\left(\int_{-\infty}^{\infty} d^n\boldsymbol{q} d^n\boldsymbol{p} |W_{\hat{\sigma}}(\boldsymbol{q},\boldsymbol{p})|\right), \tag{4}$$

with $d^n\boldsymbol{q}$ and $d^n\boldsymbol{p}$ the n-dimensional volume differentials corresponding to $\boldsymbol{q}$ and $\boldsymbol{p}$, and $\hat{\sigma}$ a bosonic Hermitian operator.

First, we compute the negativity of the Wigner function given in Eq. (3). Since the WLN of an ideal non-normalized GKP codeword is infinite, we consider the periodicity of the Wigner function and restrict the computation to the lattice unit cell to obtain a finite value. We reduce the integration domain in Eq. (4) to a hypercube in phase space, with the domain $\mathcal{C}$ being $q_i \in [0, 2\sqrt{\pi})$, $p_i \in [0, 2\sqrt{\pi})$. Therefore, we define the WLN of one unit cell as

$$\mathcal{W}_{\mathcal{C}}(\hat{\rho}) = \log_2\left(\int_{\mathcal{C}} d^n\boldsymbol{q} d^n\boldsymbol{p} |W_{\hat{\rho}}(\boldsymbol{q},\boldsymbol{p})|\right). \tag{5}$$

The integral over the absolute values of the Wigner function can be evaluated and is obtained as

$$\int_{\mathcal{C}} d^n\boldsymbol{q} d^n\boldsymbol{p} |W_{\hat{\rho}}(\boldsymbol{q},\boldsymbol{p})| = \frac{1}{\sqrt{\pi}^n}\sum_{\boldsymbol{i},\boldsymbol{j}\in\mathbb{F}_2^n}\left|\sum_{\boldsymbol{k}\in\mathbb{F}_2^n}(-1)^{\boldsymbol{i}\cdot\boldsymbol{k}}\rho_{\boldsymbol{k},\boldsymbol{k}+\boldsymbol{j}}\right|, \tag{6}$$

with $\boldsymbol{i}\cdot\boldsymbol{k} = \sum_{j=1}^{n} i_j k_j \mod 2$ the standard binary inner product, and $\boldsymbol{k}+\boldsymbol{j}$ the bitwise sum $(\boldsymbol{k}+\boldsymbol{j})_i = k_i + j_i \mod 2$ (see Supplemental Material [38, Sec. I]).

Quantifying the cell Wigner negativity of a n-qubit GKP state in Eq. (6) allows us to define the GKP magic $\mathcal{G}(|\psi\rangle)$, a new magic monotone for pure states. This definition was first motivated by noticing that the cell WLN of encoded GKP states saturates to a constant value for encoded stabilizer states, while it is maximal for the $|T\rangle$ and $|H\rangle$ magic states [44,45]. We emphasize the generality of the previous Wigner function logarithmic negativity calculations by using ρ to denote any mixed or pure state. For the sake of clarity, since we only demonstrate the monotonic properties of the GKP magic for pure states $\rho = |\psi\rangle\langle\psi|$, we stress the difference by denoting the quantum state as $|\psi\rangle$ from now on.

Crucially, we notice that GKP-encoded pure stabilizer states contain an inherent amount of WLN in one lattice cell. Therefore, we define our GKP magic measure by subtracting the inherent cell negativity of $(2/\sqrt{\pi})^n$ to enforce that $\mathcal{G}(|\psi_S\rangle) = 0$ for $|\psi_S\rangle$ a pure stabilizer state. We provide an explicit counterexample for mixed states in [38, Sec. IV].

In the case of pure states $|\psi\rangle = \sum_{i\in\mathbb{F}_2^n} c_i|\boldsymbol{i}\rangle$, the GKP magic measure is finally obtained as

$$\mathcal{G}(|\psi\rangle) \equiv \log_2\left[\left(\frac{\sqrt{\pi}}{2}\right)^n \int_C d^n\boldsymbol{q}d^n\boldsymbol{p}|W_{|\psi\rangle\langle\psi|}(\boldsymbol{q},\boldsymbol{p})|)\right]$$

$$= \log_2\left(\sum_{\boldsymbol{i},\boldsymbol{j}\in\mathbb{F}_2^n}\left|\sum_{\boldsymbol{k}\in\mathbb{F}_2^n}\frac{(-1)^{\boldsymbol{i}\cdot\boldsymbol{k}}}{2^n}c_{\boldsymbol{k}}^*c_{\boldsymbol{k}+\boldsymbol{j}}\right|\right), \tag{7}$$

where we made use of Eq. (6).

The Wigner negativity in the argument of the logarithm is equivalent to the st norm [38, Sec. III], initially regarded as a one-way magic witness. In turn, this also implies that it is equivalent to the stabilizer Rényi entropy [29] [38, Sec. III] for $\alpha = \frac{1}{2}$. These equivalences upgrade the st norm to a fully fledged magic measure.

Using the properties of the WLN enables us to demonstrate the following properties for our GKP magic measure $\mathcal{G}$ [38, Sec. II]: (i) Invariance under Clifford unitaries: $\hat{U}_C$: $\mathcal{G}(\hat{U}_C|\psi\rangle) = \mathcal{G}(|\psi\rangle)$. (ii) Additivity: $\mathcal{G}(|\psi\rangle_A \otimes |\phi\rangle_B) = \mathcal{G}(|\psi\rangle) + \mathcal{G}(|\phi\rangle)$. (iii) Faithfulness: $\mathcal{G}(|\psi\rangle_S) = 0$ iff $|\psi_S\rangle$ is a stabilizer state. (iv) Invariance under composition with stabilizer states: $\mathcal{G}(|\psi\rangle\otimes|\phi_S\rangle) = \mathcal{G}(|\psi\rangle)$. (v) Nonincreasing under measurement in the computational basis. (vi) Nonincreasing under Clifford operations conditioned on the outcomes of computational-basis measurements.

Using our newly defined magic measure, we compute the most magic states and unitaries [38, Sec. V].

Distillation and gate synthesis.—Magic monotones play a central role in the leading approaches to fault-tolerant quantum computation, and have been used to bound the number of resourceful states for state conversion and gate synthesis [22,27]. Additionally, fundamental bounds have been found on the Gaussian conversion between GKP-encoded Hadamard eigenstates $|H\rangle$ and the logical GKP-state $|0\rangle$ in continuous-variable settings [45]. Using our GKP magic measure, we can lower bound the number of copies of a given resource state needed to implement a desired target unitary or to produce certain states when nonunitary and probabilistic protocols are allowed.

First, we address distillation protocols to extract a particular target state. We consider a stabilizer protocol [8]—a set of Clifford unitaries, composition with stabilizer states, computational basis measurements, and Pauli operations conditioned on measurement outcomes—that converts k copies of $|\psi\rangle$ to m copies of the target state $|\phi\rangle$. The GKP magic does not increase with such stabilizer protocol, and therefore, we can bound the number of input resource states by

$$k \geq m\frac{\mathcal{G}(|\phi\rangle)}{\mathcal{G}(|\psi\rangle)}, \tag{8}$$

where we have used the additive property of our measure, $\mathcal{G}(|\psi\rangle^{\otimes k}) = k\mathcal{G}(|\psi\rangle)$. We notice that this property also allows us to establish bounds even when catalyst states—loaned magic states returned at the end of the protocol—are allowed.

Moreover, we analyze probabilistic stabilizer protocols for distillation that convert k copies of an r-qubit state $|\psi\rangle$ to m copies of the s-qubit target state $|\phi\rangle$ with probability p. That is, we consider stabilizer protocols that can include postselection upon specific measurement outcomes and operations entailing partial traces that can create mixed quantum states. Despite the GKP magic monotone being only defined for pure states, its direct link with the WLN allows us to consider lower bounds of required resource states when intermediate mixed states are involved. The system's WLN restricted to the code's unit cell is additive and does not increase on average with such a probabilistic stabilizer protocol [38, Sec. IV], so that

$$k\mathcal{W}_C(|\psi\rangle) \geq pm\mathcal{W}_C(|\phi\rangle). \tag{9}$$

We can establish a lower bound on the average number of copies $\mathbb{E}[n]$ of $|\psi\rangle$ needed to distill $|\phi\rangle^{\otimes m}$ proportional to the ratio of the monotones. Since one must run the probabilistic protocol $1/p$ times to get a successful outcome, we require

$$\mathbb{E}[n] = \frac{k}{p} \geq m\frac{\mathcal{W}_C(|\phi\rangle)}{\mathcal{W}_C(|\psi\rangle)}. \tag{10}$$

Finally, for input and output pure states, the WLN per cell is directly related to the GKP magic as $\mathcal{G}(|\Psi\rangle) = \mathcal{W}_C(|\Psi\rangle) - \log_2[\mathcal{N}_0(n)]$, where we have subtracted the corresponding intrinsic logarithmic negativity per cell of a pure n-qubit stabilizer state $|\psi_S\rangle$, given by $\mathcal{N}_0(n) = (2/\sqrt{\pi})^n$. Hence, we can rewrite the bound as

$$\mathbb{E}[n] = \frac{k}{p} \geq m\frac{\mathcal{G}(|\phi\rangle) + \log_2[\mathcal{N}_0(s)]}{\mathcal{G}(|\psi\rangle) + \log_2[\mathcal{N}_0(r)]}. \tag{11}$$

This bound is strictly looser in the case of $p=1$ [38, Sec. IV].

Besides characterizing distillation protocols, magic measures have been used to bound gate synthesis. A quantum gate can be synthesized with purely unitary processes [46] or, more generally, allowing auxiliary qubits, measurements and classical feed forward [47–50]. In the field of fault-tolerant quantum computation, gates of the third level of the

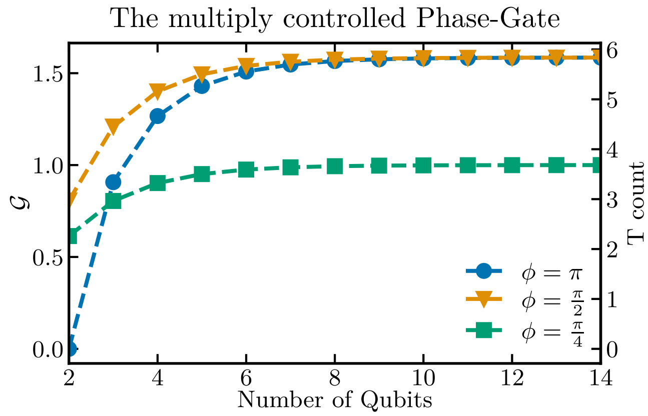

FIG. 1. The GKP magic $\mathcal{G}(|M_\phi\rangle)$, with $|M_\phi\rangle = \hat{M}_\phi|+\rangle^{\otimes n}$, where n is the number of qubits. The family of unitaries $\hat{M}_\phi$ of Eq. (13) belong to diagonal gates of the third level of the Clifford hierarchy, and the state $|M_\phi\rangle$ can be teleported to generate the corresponding gate without additional cost. Here, $\phi = \pi$ corresponds to the $C^{n-1}Z$ gate, $\phi = (\pi/2)$ to a $C^{n-1}S$ gate, and $\phi = (\pi/4)$ to a $C^{n-1}T$ gate. The GKP magic converge to a finite value for increasing numbers of qubits.

Clifford hierarchy—C_3 such that $C_{n+1} \equiv \{\hat{U}|\hat{U}\hat{P}\hat{U}^\dagger \subseteq C_n, \ \forall \ \hat{P} \in C_1\}$ [15] with the n-qubit Pauli group C_1—are the standard and most convenient non-Clifford elements to enable universal quantum computing when Clifford gates (elements in C_2) are available [15]. Although any circuit can have an equivalent teleportation gadget, C_3 gates can be implemented with the corresponding resource states and conditional operators in the Clifford group so that the state and gate costs coincide [15,51]. If the unitaries are additionally diagonal, then an explicit teleportation gadget can be given that teleports the gate $\hat{U}$ with the resource state $\hat{U}(|+\rangle)^{\otimes n} \equiv |U\rangle$, with $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ [22].

This property of the third level of the Clifford hierarchy allows us to bound the $|U\rangle$ cost (or U count) of a target unitary $\hat{U}_{\text{target}}$ if both gates $\hat{U}$ and $\hat{U}_{\text{target}}$ belong to C_3,

$$\mathcal{G}(|U\rangle^{\otimes m}) \leq \mathcal{G}(|U_{\text{target}}\rangle) \leq \mathcal{G}(|U\rangle^{\otimes m+1}). \quad (12)$$

In particular, we estimate the number of T gates (or number of $|H\rangle$ states) needed to implement different unitaries from C_3. We quantify the T count, i.e., the number of T gates needed, since the $\{\text{Clifford}, T\}$ constitutes a universal gate set [52]. Equivalently, we can measure the $|H\rangle$ cost—the number of required $|H\rangle = (|0\rangle + e^{i\pi/4}|1\rangle)/\sqrt{2}$ states—since the $|H\rangle$ magic state can be consumed to implement the non-Clifford gate T [52]. We analyze the gates characterized with the robustness of magic [22], which allowed for improved gate synthesis and proved the optimality of several circuits. The lower bound obtained with the GKP magic coincides with the lower bound given by the robustness of magic for these cases [38, Sec. V].

Moreover, we study the multiply controlled phase gates $\hat{M}_\phi$, which are from the third level of the Clifford hierarchy [53] and have a diagonal representation in computational basis

$$M_\phi = \text{diag}(1, \ldots, 1, e^{i\phi}). \quad (13)$$

They include the multiply controlled gates $C^{n-1}Z$ $(\phi = \pi)$, $C^{n-1}S$ $[\phi = (\pi/2)]$, and $C^{n-1}T$ $[\phi = (\pi/4)]$, where we use the notation for an $n-1$ times controlled G gate as $C^{n-1}G$.

We analytically derive the GKP magic value for any M_ϕ gate dimension [38, Sec. V]. Figure 1 shows that the GKP magic for different $\hat{M}_\phi$ gates converges to a finite value as the number of qubits increases. We analytically analyze this asymptotic behavior [38, Sec. V]. Furthermore, we give analytical expressions for the GKP magic for the state $|H\rangle$, the quantum adder, and the quantum Fourier transform [38, Sec. V].

Since there exist Clifford operations $\hat{U}$ and $\hat{V}$ such that $C^{n-1}X = \hat{U}C^{n-1}Z\hat{V}$, multiply controlled X and Z gates are expected to contain the same amount of magic. We confirm this numerically [38, Sec. V], and in particular, we compare our obtained T count for the C^3Z gate with known values to implement a Toffoli gate C^3X. The T count provided by our measure coincides with the count of the optimal teleportation gadget of the Toffoli gate [47], but it does not prove the optimality of the $C^{n-1}X$ gate gadget with $4(n-2)$ T gates. Notice that the optimal unitary circuit for synthesizing the Toffoli and Fredkin gates includes $7T$ gates [54], which does not contradict the optimal bound for general synthesis, where measurements and classical feed forward are allowed.

In general, we can quantify the GKP magic of any unitary by using the Choi–Jamiołkowski isomorphism [55,56], even those that are not diagonal unitaries from the third level of the Clifford hierarchy. With this correspondence, we map unitaries to quantum states suitable for our measure defined in Eq. (7) [38, Sec. V]. The GKP magic of general unitaries can be used to lower bound the resources needed for their implementation as well. For gates outside C_3, we do not expect tight bounds, as one may need to use non-Clifford gates to correct measurements in teleportation schemes or dispose of output states in probabilistic protocols.

Using the Choi–Jamiołkowski isomorphism, we calculate numerical values for gates analyzed in [54]. The largest system size we consider involves 12 qubits and corresponds to the C^5X gate. The calculated values, as well as the most magical two-qubit unitaries and the most magical three-qubit states, can be found in the Supplemental material [38, Sec. V].

GKP magic and other magic quantifiers.—It is interesting to compare the GKP magic to other magic quantifiers introduced previously in the literature. The sum negativity of a discrete Wigner function is a magic monotone for general odd qudit systems [8]. However, the extension to qubit systems remains challenging [18–21]. Even though the discrete Wigner function resembles the Wigner representation of a

GKP-encoded qubit restricted to a unit cell, the corresponding magic monotones—the sum negativity and the GKP magic—differ for the single-qubit case [38, Sec. IV], where both are valid measures.

In contrast to the stabilizer nullity and, similarly, to the robustness of magic, our measure assigns a small value close to stabilizer states. For instance, the GKP magic of the pure single-qubit state $|\psi\rangle = (|0\rangle + e^{i\phi}|1\rangle)/\sqrt{2}$ tends to zero for small angles $\phi \to 0$. This feature explains the GKP magic of the multiply controlled phase gates converging to a finite value, opposite to the stabilizer nullity for the C^nZ gate [27]. Through the GKP magic connection with the stabilizer Rényi entropy [29] and st norm [22], we conclude that it lower bounds the robustness [22] and the stabilizer nullity [38, Sec. III].

Crucially, in contrast to most other magic measures, the computation of our magic measure does not include optimization, while it allows us to find analytical expressions for n-qubit states. Also, note that the computation of GKP magic in Eq. (7) does not require any explicit CV phase space calculations.

Numerically, our measure requires three sums that scale with 2^{3n} additions of matrix elements. While, e.g., the robustness of magic could be computed for systems sizes of up to five qubits [22], or for product states with specific symmetries [57], in this Letter, we calculate the GKP magic for general states of up to $n = 12$ qubits. The computation for 12 qubits takes 962 s on one core on a laptop CPU (Intel Core i7). We expect that larger system sizes are reachable, with 12 qubits not being a hard limit. These running times and the additivity of our measure open the possibility of exploring previously unreachable system sizes.

Discussion and perspective views.—In summary, we have introduced a new additive magic measure for multiqubit pure states, the GKP magic, derived using bosonic codes—the GKP encoding—and considering the WLN in CV systems. Moreover, we have established a connection with the st norm and stabilizer Rényi entropy, with the former initially introduced solely as a one-way magic witness. Crucially, the CV framework allows us to prove the properties of our measure by transferring properties of the WLN. The convenient expression of the GKP magic in Eq. (7) allows us to lower bound the resources needed for general unitary synthesis and state conversion. In contrast to existing monotones, computing our measure does not require numerical optimization, and we can outperform previous results involving up to ≈5 qubits for general states, easily reaching 12-qubit states. Therefore, the GKP magic can be used to address general gate synthesis—where unitary operations, measurements on auxiliary systems, and classical feed forward are allowed—and lower bound unitaries and states that were out of reach previously. We also confirm existing optimal lower bounds for several unitary gates studied previously, including the Toffoli gate. Moreover, we have derived analytical expressions of our measure for multiply controlled phase gates, the quantum adder, and the quantum Fourier transform for an arbitrary number of qubits. Similar to the st norm and the robustness of magic case, we find lower bounds for any multiqubit state and the general scenario of probabilistic stabilizer protocols using the Wigner negativity per cell, well defined for mixed states.

Since the GKP encoding can be applied to qudits of any dimension, it is natural to ask whether we can define a generalized GKP magic and how it would be related to the discrete Wigner function for odd prime dimensions. Another interesting open question is whether the GKP magic quantifies the hardness of classical simulation of Clifford computation with additional resource states. Our work sheds new light on magic measures by investigating bosonic codes and CV state conversion. As such, it opens the question regarding whether other properties of finite-dimensional systems could be assessed by mapping them to infinite-dimensional ones and, thereby, bridging and transferring results from two independent areas of quantum information. Finally, the core idea of connecting concepts of CV and discrete-variable systems via bosonic codes can be interpreted as an operational blueprint for resource theories of finite-dimensional systems, beyond quantum computation.

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Correction: An update to the title during the production cycle was not implemented and has been applied.