

Conjugacy classes of maximal cyclic subgroups of metacyclic p -groups

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Abstract

In this paper, we set $\eta(G)$ to be the number of conjugacy classes of maximal cyclic subgroups of a finite group G . We compute $\eta(G)$ for all metacyclic p -groups. We show that if G is a metacyclic p -group of order p^n that is not dihedral, generalized quaternion, or semi-dihedral, then $\eta(G) \geq n - 2$, and we determine when equality holds.

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1 Introduction

Unless otherwise stated, all groups in this paper are finite, and we will follow standard notation from [6]. As in [3] and [4], we set $\eta(G)$ to be the number of conjugacy classes of maximal cyclic subgroups of a group G . For $p = 2$, we have that $\eta(G) = 3$ when G is a dihedral 2-group, a generalized quaternion 2-group, or a semi-dihedral group. In [1], the second and third authors along with Yiftach Barnea and Mikhail Ershov have shown that for every prime $p \geq 5$ there are infinitely many p -groups with $\eta = p + 2$ and for $p = 3$ there are infinitely many 3-groups with $\eta = 9$. This answers negatively Question 5.0.9 from [9] which asked whether $\eta(G)$ grows with the order of G when G is a p -group and p is odd.

On the other hand, it is rare for this to occur. Indeed, the only 2-groups (in fact the only p -groups) that have $\eta = 3$ are the Klein 4-group, the dihedral groups, the generalized quaternion groups, and the semi-dihedral groups. To

see this, we know that $\eta(G) \geq \eta(G/G')$ (see [3]), and for p -groups $\eta(G/G') \geq p + 1$ when G/G' is not cyclic (see [4]). Thus, $\eta = 3$ can only occur when $p = 2$. Also, in [4], we show that $\eta(G/G') = 3$ if and only if $G/G' \cong C_2 \times C_2$. It is well known that if G is a 2-group of order at least 8 and $|G : G'| = 4$, then G is either dihedral, generalized quaternion, or semi-dihedral. (See Problem 6B.8 of [6].)

Now, dihedral groups, generalized quaternion groups, and semi-dihedral groups are examples of metacyclic groups. I.e., groups G with a normal subgroup N so that N and G/N are both cyclic groups. This motivated us to investigate the invariant η for all metacyclic p -groups. Indeed this project began before the results of [1] were known and we were originally curious as to whether we would find another family of metacyclic p -groups with fixed η . However, we prove the following:

Theorem 1.1 *Let G be a metacyclic p -group of order p^n that is not a dihedral group, generalized quaternion group, or semi-dihedral group. Then $\eta(G) \geq n - 2$.*

In fact, we compute $\eta(G)$ for every metacyclic p -group G . Thus, we list the metacyclic p -groups where equality occurs in Theorem 1.1. King in [7] gave a description of all metacyclic p -groups. We will give this description of these groups in Section 3. In particular, King divided the metacyclic p -groups into two families of groups which he called *positive type* and *negative type*. The negative type groups only occur when $p = 2$, so if p is an odd prime, then all of the metacyclic p -groups are of positive type. We have the following result for the metacyclic groups of positive type.

Theorem 1.2 *Let G be a metacyclic group of positive type. Then $\eta(G) = \eta(G/G')$.*

We note that Rogério in [8] has a formula to compute $\eta(A)$ for an abelian group A . His formula involves the Euler ϕ -function and a second number theoretic function. When G is a metacyclic abelian p -group, we prove in [4] a formula for $\eta(G)$ that is only in terms of the sizes of the direct factors of G . Notice in Theorem 1.2 that G/G' will be a metacyclic abelian p -group, and so, our formula will compute $\eta(G/G')$ and hence, $\eta(G)$.

When G is a metacyclic p -group of negative type, it is not usually the case that $\eta(G)$ and $\eta(G/G')$ are equal. However, we will find that there usually

is a proper quotient whose value of η equals $\eta(G)$. We will also see for most metacyclic groups of negative type that the formula for η is dependent on the formula for η that we found for the metacyclic abelian p -groups.

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2 Preliminaries

In our preprint [3], we prove two results that we need in this paper. The first is a criteria for determining when the quotient of a p -group G has the same value for η as $\eta(G)$. Given a prime p , we set $G^{\{p\}} = \{g^p \mid g \in G\}$. I.e., $G^{\{p\}}$ is the set of p -th powers in G .

Theorem 2.1 *Let N be a normal subgroup of the p -group G . Then $\eta(G/N) \leq \eta(G)$. Furthermore, $\eta(G/N) = \eta(G)$ if and only if $N \subseteq G^{\{p\}}$ and for all $x \in G \setminus G^{\{p\}}$ every element of xN is conjugate to a generator of $\langle x \rangle$. In particular, if $\eta(G/N) = \eta(G)$, then $G^{\{p\}}$ is a union of N -cosets and $G^{\{p\}}N = G^{\{p\}}$.*

This second Proposition relates $\eta(G)$ to the number of G -orbits of maximal cyclic subgroups of a normal subgroup.

Proposition 2.2 *Let N be a normal subgroup of a group G and let $\eta^*(N)$ be the number of G -orbits on the N -conjugacy classes of maximal cyclic subgroups of N . Then $\eta(G) \geq \eta^*(N)$. In particular,*

- (i) *if N is central in G , then $\eta(G) \geq \eta(N)$.*
- (ii) *if $|G : N| = k$, then $\eta(G) \geq \eta(N)/k$.*

Let p be a prime, and let a and b be positive integers. We take $k = \max(a, b)$ and $l = \min(a, b)$. We set $g_p(a, b) = p^{(l-1)}((k-l)(p-1) + p + 1)$. In [4], we prove the following lemma.

Lemma 2.3 *If p is a prime and a and b are positive integers so that $G = C_{p^a} \times C_{p^b}$, then $g_p(a, b) = \eta(G)$.*

We close this section with an easy lemma that computes g_2 for small values and gives a lower bound for larger values. We remark that when $p = 2$, this function is much easier to work with.

Lemma 2.4 *Suppose $k \geq l$. Then the following hold:*

1. *If $l = 1$, then $g_2(k, 1) = k + 2$.*
2. *If $l = 2$, then $g_2(k, 2) = 2(k + 1)$.*
3. *If $l = 3$, then $g_2(k, 3) = 4k$.*
4. *If $l \geq 4$, then $g_2(k, l) \geq 4k + 2l$.*

Proof. We have $g_2(a, b) = g_2(k, l) = 2^{l-1}(k - l + 3)$. Conclusions (1), (2), and (3) are immediate. We focus on (4). Begin with $g_2(4, 4) = 24$; so the result holds for $g_2(4, 4)$. Next, $g_2(l, l) - 6l = 3 \cdot 2^{l-1} - 6l$ is clearly increasing when $l \geq 3$. Thus, we have $g_2(l, l) \geq 4l + 2l$ when $l \geq 3$. Let $k = l + m$ for $m \geq 0$. Then $g_2(k, l) = g_2(l + m, m) = 2^{l-1}(m + 3)$ and $4k + 2l = 4(l + m) + 2l = 6l + 4m$. Fixing $l \geq 4$, we note that $2^{l-1}(m + 3) - 6l - 4m$ will be an increasing function in m . We conclude that $g_2(k, l) \geq 4k + 2l$ for $l \geq 4$. \square

3 Metacyclic p -Groups

For the rest of the paper, we will focus on metacyclic p -groups. A finite metacyclic p -group can be described as follows. This description is taken from [7],

$$G_p(\alpha, \beta, \epsilon, \delta, \pm) = \langle x, y \mid x^{p^\alpha} = 1, y^{p^\beta} = x^{p^{\alpha-\epsilon}}, x^y = x^r \rangle$$

where $r = p^{\alpha-\delta} + 1$ (positive type) or $r = p^{\alpha-\delta} - 1$ (negative type). The integers $\alpha, \beta, \delta, \epsilon$ satisfy $\alpha, \beta > 0$ and δ, ϵ nonnegative, furthermore $\delta \leq \min\{\alpha - 1, \beta\}$ and $\delta + \epsilon \leq \alpha$. When G has negative type, only $\epsilon = 0$ or 1 occur. For p odd

$$G \cong G_p(\alpha, \beta, \epsilon, \delta, +).$$

In other words, the negative type only occurs when $p = 2$; when p is odd, only the positive type occurs. Metacyclic 2-groups can be of either positive type or negative type. We note that dihedral, semi-dihedral and generalized quaternion groups are all of negative type.

If $p = 2$, then in addition $\alpha - \delta > 1$ and

$$G \cong G_2(\alpha, \beta, \epsilon, \delta, +) \text{ or } G \cong G_2(\alpha, \beta, \epsilon, \delta, -).$$

Note, the above presentation does not guarantee nonisomorphic groups for different parameters (see [2]). However, the parameters do determine some structural information about G . For example, $|G| = p^{\alpha+\beta}$ and $G' = \langle x^{p^{\alpha-\delta}} \rangle$ if G is of positive type and $G' = \langle x^2 \rangle$ if G is of negative type. All elements of G can be written as $y^b x^a$ for some integers a and b . Also if G is of positive type then $Z(G) = \langle x^{p^\delta}, y^{p^\delta} \rangle$ and $|Z(G)| = p^{\alpha+\beta-2\delta}$, if G is of negative type $Z(G) = \langle x^{2^{\alpha-1}}, y^{2^{\max\{1,\delta\}}} \rangle$, [2, Prop. 2.5]. Note that if G is of positive type and $\delta = 0$, then G will be abelian.

As we mentioned above, the dihedral groups, the generalized quaternion groups, and the semi-dihedral groups are the only p -groups G that satisfy $\eta(G) = 3$. These are also precisely the 2-groups of maximal class. We have also mentioned that they are metacyclic. In terms of our notation, the dihedral groups are $G_2(\alpha, 1, 0, 0, -)$, the generalized quaternion groups are $G_2(\alpha, 1, 1, 0, -)$, and the semi-dihedral groups are $G_2(\alpha, 1, 0, 1, -)$.

For Lemmas 3.1 and 3.3, we are writing $G_p(\alpha, \beta, \epsilon, \delta, \pm)$ as $G_p(\alpha, \beta, \epsilon, \delta, \gamma)$ where we take $\gamma = +$ when G is of positive type and $\gamma = -$ when G is of negative type. We consider quotients of G . Note that this lemma would not be well defined if $\delta = 0$ and would not say anything if $\delta = 1$.

Lemma 3.1 *Suppose G is $G_p(\alpha, \beta, \epsilon, \delta, \gamma)$ with $\delta \geq 2$. Then $N = \langle x^{p^{\alpha-\delta+1}} \rangle$ is a normal subgroup of G and G/N is isomorphic to*

$$G_p(\alpha - \delta + 1, \beta, (\epsilon - \delta + 1)^*, 1, \gamma)$$

where $(\epsilon - \delta + 1)^* = \epsilon - \delta + 1$ when $\epsilon \geq \delta - 1$ and $(\epsilon - \delta + 1)^* = 0$ when $\epsilon < \delta - 1$.

Proof. Set $Z = \langle x^{p^{\alpha-1}} \rangle \leq Z(G)$. We first prove that G/Z is isomorphic to $G_p(\alpha - 1, \beta, \epsilon - 1, \delta - 1, \gamma)$ when $\epsilon \geq 1$ and $G_p(\alpha - 1, \beta, 0, \delta - 1, \gamma)$ when $\epsilon = 0$. We know that $G/Z = \langle xZ, yZ \rangle$ where xZ has order $p^{\alpha-1}$. Observe that $(yZ)^{p^\beta} = y^{p^\beta} Z = x^{p^{\alpha-\epsilon}} Z$. When $\epsilon \geq 1$, we have

$$x^{p^{\alpha-\epsilon}} Z = x^{p^{(\alpha-1)-(\epsilon-1)}} Z$$

and when $\epsilon = 0$, we have

$$x^{p^{\alpha-\epsilon}} Z = x^{p^\alpha} Z = Z.$$

Also,

$$(xZ)^{yZ} = x^y Z = x^{p^{\alpha-\delta} + \gamma} Z = x^{p^{(\alpha-1)-(\delta-1)+\gamma}} Z.$$

Hence, G/Z satisfies the hypotheses for $G_p(\alpha-1, \beta, \epsilon-1, \delta-1, \gamma)$ when $\epsilon \geq 1$ and $G_p(\alpha-1, \beta, 0, \delta-1, \gamma)$ when $\epsilon = 0$.

We know that $X = \langle x \rangle$ is a cyclic, normal subgroup of G . Observe that N is contained in X and so is characteristic. This implies that N is normal in G . Observe that $Z \leq N$ and we have shown that $G/Z \cong G_p(\alpha-1, \beta, \epsilon-1, \delta-1, \gamma)$ or $G_p(\alpha-1, \beta, 0, \delta-1, \gamma)$. If $\delta = 2$, then $N = Z$, and we have the desired result. Otherwise, we have $\delta \geq 3$. Using induction, we have $G/N \cong (G/Z)/(N/Z)$ is isomorphic to either

$$G_p((\alpha-1)-(\delta-1)+1, \beta, (\epsilon-1)-(\delta-1)+1, 1, \gamma) \cong G_p(\alpha-\delta+1, \beta, \epsilon-\delta+1, 1, \gamma)$$

or

$$G_p((\alpha-1) - (\delta-1) + 1, \beta, 0, 1, \gamma) \cong G_p(\alpha - \delta + 1, \beta, 0, 1, \gamma). \quad \square$$

We consider the metacyclic groups of positive type and use Theorem 2.1 and Lemma 2.3. Thus, we first analyze G/G' .

Lemma 3.2 *Suppose $G = G_p(\alpha, \beta, \epsilon, \delta, +)$.*

(i) *If $\delta \geq \epsilon$ or $\delta < \epsilon$ and $\alpha \geq \beta + \epsilon$, then $G/G' = C_{p^{\alpha-\delta}} \times C_{p^\beta}$.*

(ii) *If $\delta < \epsilon$ and $\alpha < \beta + \epsilon$, then $G/G' = C_{p^{\alpha-\epsilon}} \times C_{p^{\beta+\epsilon-\delta}}$.*

Proof. Now $G' = \langle x^{p^{\alpha-\delta}} \rangle$, so $|G'| = p^\delta$. Also $|G| = p^{\alpha+\beta}$, so $|G : G'| = p^{\alpha+\beta-\delta}$.

If $\delta \geq \epsilon$, then $\langle y \rangle \cap G' = \langle x^{p^{\alpha-\epsilon}} \rangle = \langle x \rangle \cap \langle y \rangle$. We see that xG' has order $p^{\alpha-\delta}$, and yG' has order p^β and $G/G' = \langle xG' \rangle \times \langle yG' \rangle$ yielding the desired result.

Now suppose $\delta < \epsilon$. In this case, we see that $G' < \langle x^{p^{\alpha-\epsilon}} \rangle = \langle x \rangle \cap \langle y \rangle$. We see that xG' has order $p^{\alpha-\delta}$ and yG' has order $p^{\beta+\epsilon-\delta}$. Since $G' < \langle x \rangle \cap \langle y \rangle$, we do not have that G/G' is a direct product of $\langle xG' \rangle$ and $\langle yG' \rangle$. We see that G/G' is abelian and generated by xG' and yG' , so every element of G/G' has order $\leq \max\{p^{\alpha-\delta}, p^{\beta+\epsilon-\delta}\}$. If $\alpha \geq \beta + \epsilon$, then $\alpha - \delta \geq \beta + \epsilon - \delta$. In this case, xG' has the largest order of any element in G/G' , and so we get $G/G' = C_{p^{\alpha-\delta}} \times C_{p^\beta}$ since $|G/G'| = p^{\alpha+\beta-\delta}$. On the other hand, if $\alpha < \beta + \epsilon$, then $\alpha - \delta < \beta + \epsilon - \delta$. In this case, yG' has the largest order of any element

in G/G' and we get $G/G' = C_{p^{\alpha-\epsilon}} \times C_{p^{\beta+\epsilon-\delta}}$. \square

Given an element $g \in G$, we write $\text{cl}(g)$ to denote the conjugacy class of g in G .

Lemma 3.3 *Let $G = G_p(\alpha, \beta, \epsilon, \delta, \gamma)$. If $g = y^{pl+a}x^m$ for integers l, m , and a so that $a \in \{1, \dots, p-1\}$, then $\text{cl}(g) = gG'$.*

Proof. We first claim that $G = \langle x, g \rangle$. We know that $G = \langle x, y \rangle$. Obviously, $\langle x, g \rangle \leq G$. Observe that $y^{pl+a} = gx^{-m} \in \langle x, g \rangle$. Since the order of y is a power of p , this implies that $y \in \langle x, g \rangle$. We conclude that $G = \langle x, y \rangle \leq \langle x, g \rangle \leq G$. This proves the claim.

Because $\langle x \rangle$ is normal in G , we obtain $G = \langle x \rangle \langle g \rangle$. Observe that $\langle g \rangle \leq C_G(g)$. By Dedekind's lemma (see Lemma X.3 on page 328 of [6]), it follows that $C_G(g) = (C_G(g) \cap \langle x \rangle) \langle g \rangle = C_{\langle x \rangle}(g) \langle g \rangle$. Since x centralizes x^m , we have

$$C_{\langle x \rangle}(g) = C_{\langle x \rangle}(y^{pl+a}x^m) = C_{\langle x \rangle}(y^{pl+a}) = C_{\langle x \rangle}(y) = \langle x^{p^t} \rangle,$$

where $t = \delta$ if $\gamma = +$ and $t = \alpha - 1$ when $\gamma = -$. We see that $C_G(g) = \langle g, x^{p^t} \rangle$. We deduce that

$$|G : C_G(g)| = |\langle x \rangle : \langle x^{p^t} \rangle| = p^t = |G'|.$$

Since $\text{cl}(g) \subseteq gG'$, we conclude that $\text{cl}(g) = gG'$. \square

Given a group G and a prime p , we define $G^p = \langle G^{\{p\}} \rangle$. I.e., G^p is the subgroup generated by $G^{\{p\}}$. In a similar fashion, we define $G^4 = \langle g^4 \mid g \in G \rangle$. Following the literature, we say that a finite p -group G is *powerful* if (i) $G' \leq G^p$ when p is odd and (ii) $G' \leq G^4$ when $p = 2$. If G is a powerful p -group, then it is known that $G^p = G^{\{p\}}$, i.e. the set of p -powers of elements of G is equal to the subgroup the p -powers generate. (See Section 2 of [5] and in particular Propostion 2.6 of that citation.)

We claim that metacyclic p -groups of positive type are powerful. Let G be $G_p(\alpha, \beta, \epsilon, \delta, +)$, then $G' = \langle x^{p^{\alpha-\delta}} \rangle$. As $\alpha - \delta \geq 1$ it follows immediately that G is powerful when p is odd. For $p = 2$, we note that $\alpha - \delta \geq 2$ so again we have that G is powerful.

When G is of positive type, we extend Lemma 3.3.

Lemma 3.4 *Let $G = G_p(\alpha, \beta, \epsilon, \delta, +)$ and $g \in G \setminus G^{\{p\}}$. Then $\text{cl}(g) = gG'$.*

Proof. Let $g \in G$ then $g = y^n x^m$ for some integers n and m . As G is powerful, it follows that if $g \in G \setminus G^{\{p\}}$, then $g \notin G^p$, and thus, one of n and m is not divisible by p . When n is not divisible by p , we obtain the conclusion by Lemma 3.3.

We now suppose that $g = y^n x^m$ where m is not divisible by p . We want to prove that $\text{cl}(g) = gG'$. We know that $\text{cl}(g) \subseteq gG'$. It suffices to prove that $|\text{cl}(g)| \geq |gG'| = |G'| = p^\delta$. On the other hand, we know that y acts as an automorphism of order p^δ on $\langle x \rangle$, so x has p^δ distinct images under powers of y . Thus, if $1 \leq a, b \leq p^\delta$, then $x^{y^a} = x^{y^b}$ if and only if $a = b$. Since m is coprime to p , we see that $(x^{y^a})^m = (x^{y^b})^m$ if and only if $a = b$. Hence, we have that $g^{y^a} = g^{y^b}$ if and only if $(y^n x^m)^{y^a} = (y^n x^m)^{y^b}$ and this occurs if and only if $a = b$. We deduce that g has at least p^δ distinct conjugates under $\langle y \rangle$ and so $|\text{cl}(g)| \geq p^\delta$ as desired. This proves the lemma. \square

We now prove that if G is metacyclic of positive type, then $\eta(G) = \eta(G/G')$. Combining this fact with Lemmas 2.3 and 3.2, we are able to compute $\eta(G)$ for all primes p .

Corollary 3.5 *Suppose G is $G_p(\alpha, \beta, \epsilon, \delta, +)$. Then $\eta(G) = \eta(G/G')$.*

Proof. As G is powerful, by Theorem 2.1, we need to show that for all $g \in G \setminus G^{\{p\}}$ every element of gG' is conjugate to a generator of $\langle g \rangle$, this follows from Lemma 3.4. \square

For the record, we explicitly record the value of $\eta(G)$ when G is a metacyclic group of positive type.

Corollary 3.6 *Suppose G is $G_p(\alpha, \beta, \epsilon, \delta, +)$.*

- (i) *If $\delta \geq \epsilon$ or $\delta < \epsilon$ and $\alpha \geq \beta + \epsilon$, then $\eta(G) = g_p(\alpha - \delta, \beta)$.*
 - (a) *If $\beta \leq \alpha - \delta$, then $\eta(G) = p^{\beta-1}((\alpha - \delta - \beta)(p - 1) + p + 1)$.*
 - (b) *If $\beta > \alpha - \delta$, then $\eta(G) = p^{\alpha-\delta-1}((\beta - \alpha + \delta)(p - 1) + p + 1)$.*
- (ii) *If $\delta < \epsilon$ and $\alpha < \beta + \epsilon$, then $\eta(G) = g_p(\alpha - \epsilon, \beta + \epsilon - \delta) = p^{\alpha-\epsilon-1}((\beta - \alpha + 2\epsilon - \delta)(p - 1) + p + 1)$.*

Proof. Using Corollary 3.5, we have $\eta(G) = \eta(G/G')$. If $\delta \geq \epsilon$ or $\delta < \epsilon$ and $\alpha \geq \beta + \epsilon$, then in view of Lemma 3.2, we see that $G/G' = C_{p^{\alpha-\delta}} \times C_{p^\beta}$

and $\eta(G) = g_p(\alpha - \delta, \beta)$. The remainder of (i) follows from the definition of g_p . Suppose $\delta < \epsilon$ and $\alpha < \beta + \epsilon$. Applying Lemma 3.2, we see that $G/G' = C_{p^{\alpha-\epsilon}} \times C_{p^{\beta+\epsilon-\delta}}$. Observe that $\alpha < \beta + \epsilon$ yields $\alpha - \epsilon < \beta < \beta + \epsilon - \delta$ as we are assuming $\delta < \epsilon$. In light of the definition of g_p , we obtain conclusion (ii). \square

When G is metacyclic of positive type, we show that $\eta(G) \geq \alpha + \beta$.

Corollary 3.7 *If G is $G_p(\alpha, \beta, \epsilon, \delta, +)$, then $\eta(G) \geq \alpha + \beta$.*

Proof. We consider separately the cases given in Corollary 3.6. We use the fact that $2^{\beta-1} \geq \beta$ for β a positive integer. First, (i)(a), where $\alpha - \delta \geq \beta$,

$$\begin{aligned} \eta(G) &= p^{\beta-1}((\alpha - \delta - \beta)(p - 1) + p + 1) \\ &\geq 2^{\beta-1}(\alpha - \delta - \beta + 3) \\ &\geq \alpha - \delta - \beta + 3\beta \\ &\geq \alpha + \beta \end{aligned}$$

since $\beta \geq \delta$.

Now, case (i)(b), so $\alpha - \delta < \beta$. First assume $\alpha - \delta > 1$, then

$$\begin{aligned} \eta(G) &= p^{\alpha-\delta-1}((\beta - \alpha + \delta)(p - 1) + p + 1) \\ &\geq 2^{\alpha-\delta-1}(\beta - \alpha + \delta + 3) \\ &\geq 2(\beta - \alpha + \delta) + 3(\alpha - \delta) \\ &= 2\beta + (\alpha - \delta) \\ &\geq \beta + \alpha + (\beta - \delta) \\ &\geq \beta + \alpha \end{aligned}$$

since $\beta \geq \delta$. If $\alpha - \delta = 1$ then $p \geq 3$, also note $\alpha = 1 + \delta \leq 1 + \beta$. So, we have

$$\eta(G) \geq 2(\beta - \alpha + \delta) + 4 = 2\beta + 2 > \beta + \alpha.$$

Case (ii) follows similarly to (i)(a), we have $\alpha - \epsilon \leq \beta + \epsilon - \delta$,

$$\begin{aligned} \eta(G) &= p^{\alpha-\epsilon-1}((\beta - \alpha + 2\epsilon - \delta)(p - 1) + p + 1) \\ &\geq 2^{\alpha-\epsilon-1}(\beta - \alpha + 2\epsilon - \delta + 3) \\ &\geq \beta - \alpha + 2\epsilon - \delta + 3(\alpha - \epsilon) \\ &= \beta + \alpha + (\alpha - \epsilon - \delta) \\ &\geq \beta + \alpha \end{aligned}$$

since $\alpha \geq \delta + \epsilon$. \square

4 Metacyclic Groups of Negative Type

The goal of this section is to compute η when G is a metacyclic group of negative type. We begin by looking at quotients of G . We begin with a preliminary lemma that is useful in understanding the quotients.

Using the notation of Section 3 and applying Theorem 2.1, we have that if $G = G_2(\alpha, \beta, \epsilon, \delta, -)$ with $\delta \geq 1$ and $N = \langle x^{2^{\alpha-\delta+1}} \rangle$, then $\eta(G) \geq \eta(G/N)$. We now show that in fact this is an equality. We remind the reader that $\alpha - \delta \geq 2$ when $p = 2$.

We now prove the promised equality between $\eta(G)$ and $\eta(G/N)$.

Theorem 4.1 *Let $G = G_2(\alpha, \beta, \epsilon, \delta, -)$ where $\delta \geq 1$. Then $\eta(G) = \eta(G/N)$ where $N = \langle x^{2^{\alpha-\delta+1}} \rangle$.*

Proof. Note that N does not make sense if $\delta = 0$; so that it is why we assume $\delta \geq 1$. Also, if $\delta = 1$, then $N = 1$; so the conclusion is trivial in this case. Hence, we will assume $\delta \geq 2$.

We first prove that $\eta(G) = \eta(G/Z)$ where $Z = \langle x^{2^{\alpha-1}} \rangle$. Recall from Theorem 2.1 that to prove $\eta(G) = \eta(G/Z)$, we need to prove that $Z \subseteq G^{\{2\}}$ and if $g \in G \setminus G^{\{2\}}$, then every element of gZ is conjugate to a generator of $\langle g \rangle$. Observe that $Z \subseteq G^{\{2\}}$. Since $x^{2^{\alpha-1}}$ is the only nonidentity element of Z , it suffices to prove that if $g \notin G^{\{2\}}$, then $\langle g \rangle$ and $\langle gx^{2^{\alpha-1}} \rangle$ are conjugate. We know from [2] that $G' = \langle x^2 \rangle$.

We prove the claim by working by induction on δ . We begin with the case that $\delta = 2$. We know that $x^y = x^{2^{\alpha-2}-1}$. It follows that

$$(x^2)^y = (x^y)^2 = (x^{2^{\alpha-2}-1})^2 = x^{2^{\alpha-1}-2} = (x^{-2})x^{2^{\alpha-1}}.$$

Observe that this yields that $(x^{-2})^y = x^2x^{2^{\alpha-1}}$. Using this fact and the observation that $x^{2^{\alpha-1}}$ is central, we then have

$$(x^2)^{y^2} = (x^{-2}x^{2^{\alpha-1}})^y = x^2x^{2^{\alpha-1}}x^{2^{\alpha-1}} = x^2.$$

It follows that x^2 and y^2 commute. Let $A = \langle x^2, y^2 \rangle$, and observe that $G' \leq A$, so A is a normal, abelian subgroup of G .

We know that every element of G has the form $y^k x^m$ where $0 \leq k \leq 2^\beta - 1$ and $0 \leq m \leq 2^\alpha - 1$ are integers. Notice that if 4 divides both k and m , then $g \in A^{\{2\}} \subseteq G^{\{2\}}$. Also, $x^2, y^2 \in G^{\{2\}}$.

If $g = y^{2l+1} x^m$ for integers l and m , then we can appeal to Lemma 3.3 to see that g is conjugate to $gx^{2^{\alpha-1}}$ and so, $\langle g \rangle$ and $\langle gx^{2^{\alpha-1}} \rangle$ are conjugate, as desired.

Since $x^y = x^{2^{\alpha-2}-1}$, we have $x^{y^2} = x^{2^{2\alpha-4}-2^{\alpha-1}+1}$. Since $\delta \geq 2$, we know that $\alpha \geq 4$ (this is using the fact that $\alpha - \delta \geq 2$), so $2\alpha - 4 \geq \alpha$. Hence, we have $x^{y^2} = x^{-2^{\alpha-1}+1}$. In addition, $x^{2^{\alpha-1}}$ has order 2, so $x^{-2^{\alpha-1}} = x^{2^{\alpha-1}}$. Thus, we have shown $x^{y^2} = x^{2^{\alpha-1}+1}$.

Suppose now that $g = y^{2l} x^{2h+1}$ for integers l and h . From above, we have $g^{y^2} = (y^{2l} x^{2h+1})^{y^2} = y^{2l} (x^{y^2})^{2h+1} = y^{2l} (x^{2^{\alpha-1}+1})^{2h+1} = y^{2l} x^{2h+1} x^{2^{\alpha-1}} = gx^{2^{\alpha-1}}$.

We deduce that $\langle g \rangle$ and $\langle gx^{2^{\alpha-1}} \rangle$ are conjugate, as desired.

We have shown that $x^{y^2} = xx^{2^{\alpha-1}}$. This implies that $x^{-1}y^{-2}x = x^{2^{\alpha-1}}y^{-2}$. Inverting, we obtain $(y^2)^x = y^2x^{2^{\alpha-1}}$. Now, suppose that $g = y^{2l}x^{2h}$. We can assume from above that either l is odd or h is odd. Assume first that l is odd. We have

$$g^x = (y^{2l}x^{2h})^x = ((y^2)^x)^l x^{2h} = (y^2x^{2^{\alpha-1}})^l x^{2h} = y^{2l}x^{2h}x^{2^{\alpha-1}} = gx^{2^{\alpha-1}}.$$

We obtain $\langle g \rangle$ and $\langle gx^{2^{\alpha-1}} \rangle$ are conjugate, as desired.

We are left with the case that $g = y^{4l}x^{2(2h+1)}$ for integers h and l . We claim that $g \in G^{\{2\}}$. Notice that there is an integer k so that $\langle g \rangle = \langle y^{4k}x^2 \rangle$ and that $g \in G^{\{2\}}$ if and only if $y^{4k}x^2 \in G^{\{2\}}$. We show that $y^{4k}x^2 \in G^{\{2\}}$. We have $x^{y^2} = xx^{2^{\alpha-1}}$. It follows that $xy^2 = y^2xx^{2^{\alpha-1}}$ and

$$(y^{2k}x)^2 = y^{2k}xy^{2k}x = y^{2k}y^{2k}xx^{2^{\alpha-1}k}x = y^{4k}x^2x^{2^{\alpha-1}k}.$$

When k is even, we see that $(y^{2k}x)^2 = y^{4k}x^2$. Now assume that k is odd. We have

$$\begin{aligned} (y^{2k}xx^{2^{\alpha-2}})^2 &= y^{2k}xx^{2^{\alpha-2}}y^{2k}xx^{2^{\alpha-2}} = y^{2k}y^{2k}xx^{2^{\alpha-1}k}xx^{2^{\alpha-2}2} \\ &= y^{4k}x^2x^{2^{\alpha-1}(k+1)} = y^{4k}x^2. \end{aligned}$$

Note that we are using the fact that $x^{2^{\alpha-2}}$ commutes with both x and y^2 here. Thus, this yields $g \in G^{\{2\}}$. We conclude for all elements $g \in G \setminus G^{\{2\}}$ that g and $gx^{2^{\alpha-1}}$ are conjugate and we have proved that $\eta(G) = \eta(G/Z)$ when $\delta = 2$.

We now assume that $\delta > 2$. Let $M = \langle x^2, y \rangle$. Since $x^y = x^{2^{\alpha-\delta}-1}$, we see that $(x^2)^y = (x^{2^{\alpha-\delta}-1})^2 = (x^2)^{2^{(\alpha-1)-(\delta-1)-1}}$. Also, $y^{2^\beta} = x^{2^{\alpha-\epsilon}} = (x^2)^{2^{(\alpha-1)-\epsilon}}$. Observe that $(x^2)^{2^{(\alpha-1)-1}} = x^{2^{\alpha-1}}$. We conclude that $M = G_2(\alpha-1, \beta, \epsilon, \delta-1, -)$. Let $g \in G \setminus G^{\{2\}}$. If $g \in M$, then $g \in M \setminus M^{\{2\}}$. By induction, we have that g is conjugate to $g(x^2)^{2^{(\alpha-1)-1}}$, and so, g and $gx^{2^{\alpha-1}}$ are conjugate. Thus, we may assume that $g \notin M$. This implies that $g = y^l x^{2m+1}$ for integers l and m . We know that y induces an automorphism of $\langle x \rangle$ of order 2^δ . It follows that $y^{2^{\delta-1}}$ induces an automorphism of $\langle x \rangle$ of order 2. Since $\delta \geq 3$, we know that this automorphism is a square. It is not difficult to see that $x \mapsto xx^{2^{\alpha-1}}$ is the unique automorphism of $\langle x \rangle$ that has order 2 and is a square. Hence, we see that $x^{y^{2^{\delta-1}}} = xx^{2^{\alpha-1}}$. We conclude that $g^{y^{2^{\delta-1}}} = (y^l x^{2m+1})^{y^{2^{\delta-1}}} = y^l (x^{y^{2^{\delta-1}}})^{2m+1} = y^l (xx^{2^{\alpha-1}})^{2m+1} = y^l x^{2m+1} x^{2^{\alpha-1}} = gx^{2^{\alpha-1}}$. This completes the proof of the claim that $\eta(G) = \eta(G/Z)$.

We now work to prove $\eta(G) = \eta(G/N)$. We work by induction on δ . If $\delta = 2$, then $N = Z$, and the above claim yields the result. We assume that $\delta \geq 3$. We have that $\eta(G) = \eta(G/Z)$. By induction, $\eta(G/Z) = \eta((G/Z)/(N/Z))$, and the First Isomorphism Theorem implies that $G/N \cong (G/Z)/(N/Z)$, so $\eta(G/N) = \eta((G/Z)/(N/Z))$, and we have the desired equality. \square

In light of Theorem 4.1 and Lemma 3.1, we see that if we can compute η for $G_2(\alpha, \beta, \epsilon, \delta, -)$ when $\delta = 0, 1$, then we can compute η for all metacyclic 2-groups of negative type. There are a number of cases to consider when $\delta = 0$ or 1, and then using these cases, we will compute η when $\delta \geq 2$. Recall that the dihedral 2-groups are the groups of the form $G_2(\alpha, 1, 0, 0, -)$, the generalized quaternion 2-groups are of the form $G_2(\alpha, 1, 1, 0, -)$, and the semi-dihedral groups are of the form $G_2(\alpha, 1, 0, 1, -)$. Also, it is known that $G_2(\alpha, \beta, 1, 0, -)$ and $G_2(\alpha, \beta, 1, 1, -)$ are isomorphic for all $\alpha \geq 3$ and $\beta \geq 2$. Since $\delta \leq \beta$, it follows that dihedral, generalized quaternion, and semi-dihedral are the only groups of negative type where $\beta = 1$.

Thus, we need to analyze the negative metacyclic 2-groups of type

$$G_2(\alpha, \beta, \epsilon, \delta, -)$$

with $\beta \geq 2$. We recall a few facts about the classification of such groups. In particular, for negative type ϵ is either 0 or 1 only. Also the parameters satisfy: $\alpha \geq \delta + 2$ and $\beta \geq \delta$ when $\epsilon = 0$ and $\beta \geq \delta + 1$ when $\epsilon = 1$.

When $\delta = 0$ or 1, there is a particular abelian normal subgroup M of G . For this subgroup M , we determine which maximal cyclic subgroups of M

are maximal in G and how many maximal cyclic subgroups of G lie outside of M . This yields the following result. Recall that $\eta^*(M)$ is the number of G -orbits on the M -conjugacy classes of maximal cyclic subgroups of M .

Proposition 4.2 *Suppose G is $G_2(\alpha, \beta, \epsilon, \delta, -)$ where $\delta = 0$ or 1 and $\beta \geq 2$. Let $M = \langle x, y^2 \rangle$. Then M is a normal abelian subgroup of G and the following holds:*

- (i) *If $\delta = 0$, then $\eta(G) = \eta^*(M) + 1$ and every maximal cyclic subgroup of M is maximal cyclic in G except $\langle y^2 \rangle$.*
- (ii) *If $\delta = 1$, then $\eta(G) = \eta^*(M)$ and every maximal cyclic subgroup of M is maximal cyclic in G except $\langle y^2 \rangle$ and $\langle y^2 x^{2^{\alpha-1}} \rangle$.*

Proof. As M is a subgroup of index 2 in G it follows that M is normal in G . Let $Y = \langle y^2 \rangle$. Observe that y^2 centralizes $\langle x \rangle$ and is obviously central in $\langle y \rangle$; so $Y = \langle y^2 \rangle$ is central in G . Now, M is central-by-cyclic, so M is abelian.

We now prove that there are exactly two conjugacy classes of maximal cyclic subgroups of G outside of M . Since $\langle x \rangle$ is normal in G and $G = \langle x \rangle \langle y \rangle = \langle x \rangle \langle xy \rangle$, we see that $C_G(\langle y \rangle) = C_{\langle x \rangle}(\langle y \rangle) \langle y \rangle = \langle x^{2^{\alpha-1}} \rangle \langle y \rangle$ and $C_G(\langle xy \rangle) = \langle x^{2^{\alpha-1}} \rangle \langle xy \rangle$. It follows that both $\langle y \rangle$ and $\langle xy \rangle$ lie in conjugacy classes of size $|\langle x \rangle : \langle x^{2^{\alpha-1}} \rangle| = 2^{\alpha-1}$. It is not difficult to see now that every cyclic subgroup of G outside of M is conjugate to either $\langle y \rangle$ or $\langle xy \rangle$.

(i) For $\delta = 0$ we show that every maximal cyclic subgroup of M is a maximal cyclic subgroup of G except $\langle y^2 \rangle$ which lies in exactly 2 different conjugacy classes of maximal cyclic subgroups of G , namely $\langle y \rangle$ and $\langle xy \rangle$.

Observe that yY acts on M/Y inverting every element. Thus, M/Y is a cyclic subgroup of index 2 in G/Y . We have $(yY)^2 = Y$, so G/Y is a dihedral group. It follows that if $g \in G \setminus M$, then $(gY)^2 = Y$ and so, $g^2 \in Y$. Hence, Y is the only maximal cyclic subgroup of M that is not maximal cyclic in G . Notice that $Y \leq \langle y \rangle$. Also, we know that $\langle yY \rangle$ and $\langle xyY \rangle$ are in different conjugacy classes of subgroups of G/Y , so $\langle y \rangle$ and $\langle xy \rangle$ are in different conjugacy classes of G . Since $x^y = x^{-1}$, so $xy = yx^{-1}$. It follows that $(yx)^2 = yxyx = y(yx^{-1})x = y^2$.

(ii) For $\delta = 1$ we show that the only maximal cyclic subgroups of M that are not maximal in G are $\langle y^2 \rangle$ and $\langle y^2 x^{2^{\alpha-1}} \rangle$. Again there are exactly 2 different conjugacy classes of maximal cyclic subgroups outside of M given by $\langle y \rangle$ and $\langle xy \rangle$. Note that $\langle y \rangle$ contains $\langle y^2 \rangle$ and $\langle xy \rangle$ contains $\langle y^2 x^{2^{\alpha-1}} \rangle$.

Note that M/Y is cyclic in G/Y of order 2^α . Also, $(yY)^2 = Y$ and $(xY)^{yY} = x^{2^{\alpha-1}-1}Y = (xY)^{2^{\alpha-1}-1}$. It follows that G/Y is isomorphic to a semi-dihedral group. Let $Z = \langle x^{2^{\alpha-1}}, Y \rangle$, and observe that $Z/Y = Z(G/Y)$. Notice that if $g \in G \setminus M$, then $(gY)^2 \in Z/Y$. This implies that $g^2 \in Z$. Observe that $\langle y^2 \rangle$ and $\langle y^2 x^{2^{\alpha-1}} \rangle$ are central (and hence normal) in G . It follows that the square of any conjugate of y will be y^2 . Since $\delta = 1$, we have $x^y = x^{2^{\alpha-1}-1}$, so $xy = yx^{2^{\alpha-1}-1}$. We have $(yx)^2 = yxyx = y(yx^{2^{\alpha-1}-1})x = y^2 x^{2^{\alpha-1}}$. This implies that the square of any conjugate of xy will be $y^2 x^{2^{\alpha-1}}$. Hence, any other subgroup of M that is maximal cyclic in M will be maximal cyclic in G . \square

We now work to compute η for the groups with negative type and δ equal to 0 or 1. We will first handle the case when $\epsilon = 0$ and $\beta = 2$. For the following lemma recall that $\alpha \geq \delta + 2$ when $p = 2$, so when $\delta = 1$ we must have $\alpha \geq 3$.

Lemma 4.3 *Suppose G is $G_2(\alpha, 2, 0, \delta, -)$. Then*

- (i) $\eta(G) = \alpha + 3$ if $\delta = 0$ and
- (ii) $\eta(G) = \alpha + 2$ if $\delta = 1$.

Proof. Following Proposition 4.2, we take $M = \langle x, y^2 \rangle$; so M is abelian. We have $M \cong C_{2^\alpha} \times C_2$ and $\eta(M) = \alpha + 2$ by Lemma 2.3. We claim that all subgroups of M are normal in G . To see this, note that if K is a subgroup of M then (1) K is a subgroup of $\langle x \rangle$, (2) $K = \langle x^a, y^2 \rangle$ for some integer $1 \leq a \leq 2^\alpha - 1$ or (3) $K = \langle x^a y^2 \rangle$ for some integer $1 \leq a \leq 2^\alpha - 1$. When $\delta = 0$, we know that $x^y = x^{-1}$, so $(x^a)^y = (x^a)^{-1}$ and $(x^a y^2)^y = (x^a y^2)^{-1}$ for every integer a . When $\delta = 1$, we have $(x^a)^y = x^{a(2^{\alpha-1}-1)}$. The observation is that $\langle x^a \rangle = \langle x^{a(2^{\alpha-1}-1)} \rangle$, $\langle x^a, y^2 \rangle = \langle x^{a(2^{\alpha-1}-1)}, y^2 \rangle$, and $\langle x^a y^2 \rangle = \langle x^{a(2^{\alpha-1}-1)} y^2 \rangle$. This proves the claim. Therefore $\eta^*(M) = \eta(M)$ and the result follows from Proposition 4.2. \square .

We continue with the case where $\epsilon = 0$. We now consider the case that $\beta \geq 3$. Recall that $g_p(a, b) = p^{(l-1)}((k-l)(p-1) + p + 1)$ where p a prime, a and b are positive integers, and we take $k = \max(a, b)$ and $l = \min(a, b)$. Recall also that $g_p(a, b) = \eta(C_{p^a} \times C_{p^b})$. The following can be viewed as an improvement on Proposition 2.2(ii).

Theorem 4.4 *Suppose G is $G_2(\alpha, \beta, 0, \delta, -)$ with $\beta \geq 3$. As previously let $M = \langle x, y^2 \rangle$. Then the following hold:*

1. *If $\delta = 1$, then $\eta(G) = \eta(M)/2 + 2 = g_2(\alpha, \beta - 1)/2 + 2$.*
2. *If $\delta = 0$, then $\eta(G) = \eta(M)/2 + 3 = g_2(\alpha, \beta - 1)/2 + 3$.*

Proof. Note that we are assuming δ is 0 or 1. As in Proposition 4.2, we let $M = \langle x, y^2 \rangle$; so it follows that M is abelian. In particular, since we are assuming that $\epsilon = 0$, we have $M \cong \langle x \rangle \times \langle y^2 \rangle = C_{2^\alpha} \times C_{2^{\beta-1}}$. Using Lemma 2.3, we obtain $\eta(M) = g_2(\alpha, \beta - 1)$. Let k be the maximum of α and $\beta - 1$ and let l be the minimum of α and $\beta - 1$; so that $\eta(M) = g_2(\alpha, \beta - 1) = 2^{l-1}(k - l + 3)$. We now work to prove that $\eta^*(M) = g_2(\alpha, \beta - 1)/2 + 2$. Once this is done, then we will have the conclusion via Proposition 4.2.

It is not difficult to see that $\langle x \rangle$, $\langle y^2 \rangle$, and $\langle y^2 x^{2^{\alpha-1}} \rangle$ are maximal cyclic subgroups of M that are normal in G . We claim that $\langle y^{2(2^{\beta-2})} x \rangle$ is a maximal cyclic subgroup of M that is normal in G . It is easy to see that it is maximal cyclic. When $\delta = 0$, we see that $(\langle y^{2(2^{\beta-2})} x \rangle)^y = \langle y^{2(2^{\beta-2})} x^{-1} \rangle = \langle (y^{2(2^{\beta-2})} x)^{-1} \rangle$, and when $\delta = 1$, we have $(\langle y^{2(2^{\beta-2})} x \rangle)^y = \langle y^{2(2^{\beta-2})} x^{2^{\alpha-1}} \rangle = \langle (y^{2(2^{\beta-2})} x)^{2^{\alpha-1}} \rangle$. This proves that it is normal in G .

We will prove that all the other maximal cyclic subgroups of M will be in conjugacy classes of size 2 in G . Thus, $\eta^*(M) = (\eta(M) - 4)/2 + 4 = \eta(M)/2 - 2 + 4 = g_2(\alpha, \beta - 1)/2 + 2$.

Let C be a maximal cyclic subgroup of M . It is not difficult to see that C will be generated by an element of the form $y^{2l}x$ or one of the form y^2x^l . When $\delta = 0$, we have that $(y^{2l}x)^y = y^{2l}x^{-1}$ and $(y^2x^l)^y = y^2x^{-l}$. For C to be normal, we need this conjugate to be in C . When the generator is $y^{2l}x$, we need $y^{2l}x^{-1} = (y^{2l}x)^k = y^{2lk}x^k$ for some integer k . This implies that $y^{2l-2lk} = x^{k+1}$. Since $\epsilon = 0$, we have that $y^{2l-2lk} = x^{k+1} = 1$. We see that we must have 2^α dividing $k + 1$ and 2^β must divide $2l(1 - k)$. Thus, there is an integer r so that $k + 1 = 2^\alpha r$, and thus, $k = 2^\alpha r - 1$. We obtain that $2^{\beta-1}$ must divide $l(1 - (2^\alpha r - 1)) = l(2 - 2^\alpha r) = 2l(1 - 2^{\alpha-1}r)$. Since we know that $\alpha \geq 2$, this implies that $2^{\beta-2}$ must divide l . It follows that $\langle x \rangle$ and $\langle y^{2^{\beta-1}}x \rangle$ are the only two maximal cyclic subgroups of M that are normal in G that are generated by an element of the form $y^{2l}x$ when $\delta = 0$.

When the generator is y^2x^l , we need $y^2x^{-l} = (y^2x^l)^k = y^{2k}x^{lk}$ for some integer k . This implies that $y^{2-2k} = x^{lk+l} = 1$. This implies that 2^β divides $2(1 - k)$ and so, $2^{\beta-1}$ divides $1 - k$. Hence, there is an integer r so that $1 - k = r2^{\beta-1}$, and hence, $k = 1 - r2^{\beta-1}$. We see that 2^α divides $l(1 + k) =$

$l(1 + (1 - r2^{\beta-1})) = l(2 - r2^{\beta-1}) = 2l(1 - r2^{\beta-2})$. Since $\beta \geq 3$, we deduce that $2^{\alpha-1}$ must divide l . It follows that $\langle y^2 \rangle$ and $\langle y^2 x^{2^{\alpha-1}} \rangle$ are the only maximal cyclic subgroups of M that are normal in G that are generated by an element of the form $y^2 x^l$ when $\delta = 0$. This proves the result when $\delta = 0$.

Now we suppose that $\delta = 1$. Recall that $\alpha \geq \delta + 2$, so $\alpha \geq 3$. We have that $(y^{2l} x)^y = y^{2l} x^{2^{\alpha-1}-1}$ and $(y^2 x^l)^y = y^2 x^{l(2^{\alpha-1}-1)}$. For C to be normal, we need this conjugate to be in C . Suppose the generator is $y^{2l} x$. We need $y^{2l} x^{2^{\alpha-1}-1} = (y^{2l} x)^k = y^{2lk} x^k$ for some integer k . This implies that $y^{2l-2lk} = x^{k-2^{\alpha-1}+1} = 1$. We deduce that 2^α must divide $k - 2^{\alpha-1} + 1$, and so, there is an integer r so that $k - 2^{\alpha-1} + 1 = 2^\alpha r$. We obtain $k = 2^\alpha r + 2^{\alpha-1} - 1$. We have that 2^β divides $2l(1 - k) = 2l(1 - 2^\alpha r - 2^{\alpha-1} + 1)$. It follows that $2^{\beta-2}$ divides $l(1 - 2^{\alpha-1} r - 2^{\alpha-2})$. Since $\alpha \geq 3$, we see that $2^{\beta-2}$ divides l . We conclude that $\langle x \rangle$ and $\langle y^{2^{\beta-1}} x \rangle$ are the only two maximal cyclic subgroups of M that are normal in G that are generated by an element of the form $y^{2l} x$ when $\delta = 1$.

When the generator is $y^2 x^l$, we need $y^2 x^{l(2^{\alpha-1}-1)} = (y^2 x^l)^k = y^{2k} x^{lk}$ for some integer k . We see that $y^{2-2k} = x^{lk-l(2^{\alpha-1}-1)} = 1$. It follows that 2^β divides $2(1 - k)$, and so, $2^{\beta-1}$ divides $1 - k$. There is an integer r so that $1 - k = 2^{\beta-1} r$ which yields $k = 1 - 2^{\beta-1} r$. We now determine that 2^α divides $l(k - 2^{\alpha-1} + 1) = l(1 - 2^{\beta-1} r - 2^{\alpha-1} + 1) = 2l(1 - 2^{\beta-2} r - 2^{\alpha-2})$. Since $\alpha \geq 3$ and $\beta \geq 3$, we have that $2^{\alpha-1}$ divides l . We conclude that $\langle y^2 \rangle$ and $\langle y^2 x^{2^{\alpha-1}} \rangle$ are the only maximal cyclic subgroups of M that are normal in G that are generated by an element of the form $y^2 x^l$ when $\delta = 1$. This proves the result when $\delta = 1$. \square

In this next corollary, recall that $\delta \leq \beta$, so when $\beta = 2$, we must have $\delta = 2$. We are able to use Theorem 4.4 to compute η for groups of negative type where $\delta \geq 2$.

Corollary 4.5 *Suppose G is $G_2(\alpha, \beta, \epsilon, \delta, -)$ with $\delta \geq 2$, then*

1. $\eta(G) = \alpha - \delta + 3 = \alpha + 1$ if $\beta = 2$.
2. $\eta(G) = g_2(\alpha - \delta + 1, \beta - 1)/2 + 2$ if $\beta \geq 3$.

Proof. By Theorem 4.1, we have that $\eta(G) = \eta(G/N)$ where $N = \langle x^{2^{\alpha-\delta+1}} \rangle$. Applying Lemma 3.1, we see that $G/N \cong G_2(\alpha - \delta + 1, \beta, 0, 1, -)$. Using Lemma 4.3, we see that $\eta(G/N) = \alpha - \delta + 1 + 2 = \alpha + 3 - \delta$ when $\beta = 2$. Since $2 \leq \delta \leq \beta = 2$, we see that $\delta = 2$, and so, $\eta(G) = \alpha + 1$. When $\beta \geq 3$, we

apply Theorem 4.4 to see that $\eta(G) = \eta(G/N) = g_2(\alpha - \delta + 1, \beta - 1)/2 + 2$. \square

We now compute η for groups of negative type with $\delta = 0$ and $\epsilon = 1$. We first handle the case where $\beta = 2$.

Lemma 4.6 *Suppose G is $G_2(\alpha, 2, 1, 0, -)$ then $\eta(G) = \alpha + 2$.*

Proof. Define $M = \langle x, y^2 \rangle$. By Proposition 4.2, we know that M is a normal abelian subgroup of G . First note that $(x^{2^{\alpha-2}}y^2)^2 = x^{2^{\alpha-1}}y^4 = x^{2^{\alpha-1}}x^{2^{\alpha-1}} = x^{2^\alpha} = 1$. Thus, $M = \langle x \rangle \times \langle x^{2^{\alpha-2}}y^2 \rangle \cong C_{2^\alpha} \times C_2$ and $\eta(M) = \alpha + 2$. Consideration of the maximal cyclic subgroups of M shows that all are normal except $\langle (1, x^{2^{\alpha-2}}y^2) \rangle$ and $\langle (x^{2^{\alpha-1}}, x^{2^{\alpha-2}}y^2) \rangle$ which are conjugate in G via y . To see that these two subgroups are conjugate, observe that M has three subgroups of order 2 and that $\langle x^{2^{\alpha-1}} \rangle = \langle y^{2^\beta} \rangle$ is central in G and that $Z(G)$ is cyclic. Either y normalizes both of the other two subgroups of order 2 or it permutes them. However, if y were to normalize them, they would be normal in G and since they have order 2, that would imply that they would be central in G . This however would contradict the fact that the center of G is cyclic. Thus $\eta^*(M) = \alpha + 1$. The result follows from Proposition 4.2. \square

We continue with the groups of negative type where $\delta = 0$ and $\epsilon = 1$. We next consider $\beta \geq 3$ and $\alpha = 2$.

Lemma 4.7 *Suppose G is $G_2(2, \beta, 1, 0, -)$ with $\beta \geq 3$. Then $\eta(G) = \beta + 2$.*

Proof. Define $M = \langle x, y^2 \rangle$. By Proposition 4.2, we know that M is a normal abelian subgroup of G . Note that $(xy^{2^{\beta-1}})^2 = x^2y^{2^\beta} = x^2x^2 = x^4 = 1$. So $M = \langle xy^{2^{\beta-1}} \rangle \times \langle y^2 \rangle \cong C_2 \times C_{2^\beta}$ and $\eta(M) = \beta + 2$. Consideration of the maximal cyclic subgroups of M shows that all are normal except for $\langle (xy^{2^{\beta-1}}, 1) \rangle$ and $\langle (xy^{2^{\beta-1}}, y^{2^\beta}) \rangle$ which are conjugate in G via y . The proof that these two subgroups are conjugate is similar to the proof of Lemma 4.6. In particular, $Z(G)$ is cyclic, M has three subgroups of order 2, and if y normalized these two subgroups, then it would centralize them and contradict the fact that $Z(G)$ is cyclic. Thus $\eta^*(M) = \beta + 1$. The result follows from Proposition 4.2. \square

We conclude by computing η when $\delta = 0$, $\epsilon = 1$, $\alpha \geq 3$, and $\beta \geq 3$. Note this also covers the cases $\delta = 1$, $\epsilon = 1$ and $\alpha, \beta \geq 3$.

Theorem 4.8 *Suppose G is $G_2(\alpha, \beta, 1, 0, -)$ with $\alpha \geq 3$ and $\beta \geq 3$. Let $M = \langle x, y^2 \rangle$.*

1. *If $\alpha \geq \beta$, then $\eta(G) = \eta(M)/2 + 3 = g_2(\alpha, \beta - 1)/2 + 3$.*
2. *If $\alpha < \beta$, then $\eta(G) = \eta(M)/2 + 3 = g_2(\alpha - 1, \beta)/2 + 3$.*

Proof. As in Proposition 4.2, we let $M = \langle x, y^2 \rangle$; so it follows that M is abelian. We know that $|M| = 2^{\alpha+\beta-1}$, that x has order 2^α and y^2 has order 2^β . Suppose $\alpha \geq \beta$, then $M \cong C_{2^\alpha} \times C_{2^{\beta-1}}$, and so $\eta(M) = g_2(\alpha, \beta - 1)$. Let $w = y^2 x^{2^{\alpha-\beta}}$. Observe that $w^{2^{\beta-2}} = (y^2 x^{2^{\alpha-\beta}})^{2^{\beta-2}} = y^{2^{\beta-1}} x^{2^{\alpha-2}} \notin \langle x \rangle$ and $w^{2^{\beta-1}} = (y^2 x^{2^{\alpha-\beta}})^{2^{\beta-1}} = y^{2^\beta} x^{2^{\alpha-1}} = x^{2^{\alpha-1}} x^{2^{\alpha-1}} = 1$. It follows that $M = \langle x \rangle \times \langle w \rangle$.

If $\beta \geq \alpha + 1$, then $M \cong C_{2^{\alpha-1}} \times C_{2^\beta}$, and so $\eta(M) = g_2(\alpha - 1, \beta)$. Let $u = y^{2^{\beta-\alpha+1}} x$. We compute $u^{2^{\alpha-2}} = (y^{2^{\beta-\alpha+1}} x)^{2^{\alpha-2}} = y^{2^{\beta-1}} x^{2^{\alpha-2}} \notin \langle y \rangle$ and $u^{2^{\alpha-1}} = (y^{2^{\beta-\alpha+1}} x)^{2^{\alpha-1}} = y^{2^\beta} x^{2^{\alpha-1}} = x^{2^{\alpha-1}} x^{2^{\alpha-1}} = 1$. We deduce that $M = \langle u \rangle \times \langle y \rangle$.

In both cases, we will show that $\eta^*(M) = \eta(M)/2 + 2$, and we obtain the conclusion by applying Proposition 4.2. Notice that a maximal cyclic subgroup of M will be generated either by an element of the form $y^{2l}x$ for some integer l or by an element of the form y^2x^l for some integer l . Observe that $\langle x \rangle$ and $\langle y^2 \rangle$ are maximal cyclic subgroups of M that are normal in G .

We next show that $\langle y^{2^{\beta-1}}x \rangle$ and $\langle y^2x^{2^{\alpha-2}} \rangle$ are normal subgroups in G . Since M is abelian and has index 2 in G , it suffices to show that y normalizes these subgroups. We compute $(y^{2^{\beta-1}}x)^y = y^{2^{\beta-1}}x^{-1} = (y^{2^{\beta-1}}x)^{-1}$. Since y conjugates the generator of $\langle y^{2^{\beta-1}}x \rangle$ to its inverse, this implies that $\langle y^{2^{\beta-1}}x \rangle$ is normal in G .

We now turn to $\langle y^2x^{2^{\alpha-2}} \rangle$. We begin with the observation that $(y^2x^{2^{\alpha-2}})^4 = y^8$. Since $\beta \geq 3$, we see that $x^{2^{\alpha-1}} = y^{2^\beta} \in \langle y^2x^{2^{\alpha-2}} \rangle$. Conjugating yields $(y^2x^{2^{\alpha-2}})^y = y^2x^{-2^{\alpha-2}}$. Note that $x^{-2^{\alpha-2}} = x^{2^{\alpha-2}}x^{2^{\alpha-1}}$. We have $(y^2x^{2^{\alpha-2}})^y = y^2x^{2^{\alpha-2}}x^{2^{\alpha-1}}$. Since both $y^2x^{2^{\alpha-2}}$ and $x^{2^{\alpha-1}}$ lie in $\langle y^2x^{2^{\alpha-2}} \rangle$, we conclude that $(y^2x^{2^{\alpha-2}})^y$ lies in $\langle y^2x^{2^{\alpha-2}} \rangle$. We deduce that $\langle y^2x^{2^{\alpha-2}} \rangle$ is normal in G .

We prove that the remaining maximal cyclic subgroups of M lie in orbits of size 2. We have noted that a maximal cyclic subgroup C of M will have a generator of the form $y^{2l}x$ or of the form y^2x^l for some integer l . If C has a generator of the form $y^{2l}x$, then for C to be normal we need $(y^{2l}x)^y = y^{2l}x^{-1} \in C$. This implies that $y^{2l}x^{-1} = (y^{2l}x)^k$ for some integer k . We have $y^{2l-2lk} = x^{k+1} = u \in \langle x \rangle \cap \langle y^2 \rangle = \langle x^{2^{\alpha-1}} \rangle$. Hence, u is either 1 or $x^{2^{\alpha-1}}$. If

$u = 1$, then 2^α divides $k+1$ and $2^{\beta+1}$ divides $2l(1-k)$. We see that there is an integer r so that $k+1 = 2^\alpha r$, and hence, $k = 2^\alpha r - 1$. This implies that $2^{\beta+1}$ divides $2l(1-k) = 2l(1-2^\alpha r+1) = 4l(1-2^{\alpha-1}r)$. Since $\alpha \geq 2$, this yields $2^{\beta-1}$ divides l . When $u = x^{2^{\alpha-1}}$, we obtain that $k+1 \equiv 2^{\alpha-1} \pmod{2^\alpha}$. Hence, there is an integer r so that $k+1 = 2^{\alpha-1} + r2^\alpha$, and so, $k = 2^{\alpha-1} + r2^\alpha - 1$. We see that $2l(1-k) \equiv 2^\beta \pmod{2^{\beta+1}}$. This implies that $2^{\beta+1}$ divides $2l(1-k) - 2^\beta = 2l(1-2^{\alpha-1} - r2^\alpha + 1) - 2^\beta = 4l(1-2^{\alpha-2} - r2^{\alpha-1}) - 2^\beta$. We deduce that $2^{\beta-2}$ divides l . We conclude that $\langle x \rangle$ and $\langle y^{2^{\beta-1}}x \rangle$ are the only maximal cyclic subgroups of M having the form $\langle y^{2^l}x \rangle$ that are normal in G .

We now suppose that C has a generator of the form y^2x^l . We need $(y^2x^l)^y = y^2x^{-l} \in C$. Hence, we have that $y^2x^{-l} = (y^2x^l)^k = y^{2k}x^{lk}$ for some integer k . We have $y^{2-2k} = x^{lk+l} = u$. As in the previous paragraph, we see that u is either 1 or $x^{2^{\alpha-1}}$. If $u = 1$, then we have that $2^{\beta+1}$ divides $2(1-k)$, and so, there is an integer r so that $1-k = 2^\beta r$. We determine that 2^α divides $l(k+1) = l(1-2^\beta r+1) = 2l(1-2^{\beta-1}r)$. It follows that $2^{\alpha-1}$ divides l . Now, suppose that $u = x^{2^{\alpha-1}}$. We must have that $2(1-k) \equiv 2^\beta \pmod{2^{\beta+1}}$ and $l(k+1) \equiv 2^{\alpha-1} \pmod{2^\alpha}$. Hence, there is an integer r so that $2(1-k) = 2^\beta + 2^{\beta+1}r$. This implies that $k = 1 - 2^{\beta-1} - 2^\beta r$. We then obtain that 2^α divides $l(k+1) - 2^{\alpha-1} = l(1 - 2^{\beta-1} - 2^\beta r + 1) - 2^{\alpha-1} = 2(l(1 - 2^{\beta-2} - 2^{\beta-1}r) - 2^{\alpha-2})$. This implies that $2^{\alpha-1}$ divides $l(1 - 2^{\beta-2} - 2^{\beta-1}r) - 2^{\alpha-2}$. Hence, there is an integer s so that $l(1 - 2^{\beta-2} - 2^{\beta-1}r) - 2^{\alpha-2} = 2^{\alpha-1}s$. This leads to $l(1 - 2^{\beta-2} - 2^{\beta-1}r) = 2^{\alpha-1}s + 2^{\alpha-2} = 2^{\alpha-2}(2s + 1)$. This yields $2^{\alpha-2}$ divides l . Observe that $x^{2^{\alpha-1}} = y^{2^\beta}$, and so, $\langle y^2x^{2^{\alpha-1}} \rangle = \langle y^2 \rangle$. We deduce that $\langle y^2 \rangle$ and $\langle y^2x^{2^{\alpha-2}} \rangle$ are the only maximal cyclic subgroups of M having the form $\langle y^2x^l \rangle$ that are normal in G ,

We now see that the number of G -orbits of maximal cyclic subgroups of M is $(\eta(M) - 4)/2 + 4 = \eta(M)/2 - 2 + 4 = \eta(M) + 2$, which completes the proof of the result. \square

We close by proving that when G is metacyclic of minus type that is not dihedral, generalized quaternion, or semi-dihedral, then $\eta(G) \geq \alpha + \beta - 2$ and we determine when equality occurs. We first handle when δ equals 0 or 1. In this case, we have $\eta(G) \geq \alpha + \beta$.

Proposition 4.9 *Suppose $G = G_2(\alpha, \beta, \epsilon, \delta, -)$ with $\delta = 0$ or 1 and $\beta \geq 2$. Then $\eta(G) \geq \alpha + \beta$.*

Proof. (i) Suppose $\epsilon = 0$. Denote $l = \min(\alpha, \beta - 1)$ and $k = \max(\alpha, \beta - 1)$.

First, consider $l \geq 3$. Then $\beta \geq 4$ and by Theorem 4.4 and Lemma 2.4

$$\eta(G) \geq g_2(\alpha, \beta - 1)/2 + 2 \geq 2k + 2 \geq \alpha + \beta.$$

Next, assume $l = 2$. So $\beta \geq 3$ and by Theorem 4.4 and Lemma 2.4

$$\eta(G) \geq g_2(\alpha, \beta - 1)/2 + 2 = k + 3 \geq \alpha + \beta.$$

Finally, set $l = 1$. As $\alpha \geq 2$, we have $\beta = 2$. The result follows from Lemma 4.3.

(ii) Now suppose $\epsilon = 1$. Assume $\alpha \geq \beta$, then $l = \min(\alpha, \beta - 1) = \beta - 1$ and $k = \max(\alpha, \beta - 1) = \alpha$. If $l \geq 3$, then $\beta \geq 4$ and $\alpha \geq 4$, so we can assume $\delta = 0$. Applying Theorem 4.8 and Lemma 2.4 yields

$$\eta(G) = g_2(\alpha, \beta - 1)/2 + 3 \geq 2k + 3 \geq \alpha + \beta.$$

If $l = 2$, then $\beta = 3$, and we again appeal to Theorem 4.8 to obtain

$$\eta(G) = g_2(\alpha, \beta - 1)/2 + 3 = g_2(k, 2)/2 + 3 = k + 4 \geq \alpha + \beta.$$

If $l = 1$, then $\beta = 2$. If $\alpha = 2$ then $\delta = 0$ and if $\alpha \geq 3$ we can assume $\delta = 0$. Thus we apply Lemma 4.6.

Finally, suppose $\epsilon = 1$ and $\alpha < \beta$. We set $l = \min(\alpha - 1, \beta) = \alpha - 1$ and $k = \max(\alpha - 1, \beta) = \beta$. When $l \geq 3$, we apply Theorem 4.8 and Lemma 2.4 to get

$$\eta(G) = g_2(\alpha - 1, \beta)/2 + 3 \geq 2k + 3 \geq \alpha + \beta.$$

If $l = 2$, then $\alpha = 3$ and $\beta > 3$. Apply Theorem 4.8 with Lemma 2.4 to give

$$\eta(G) = g_2(\beta, 2)/2 + 3 = \beta + 4 \geq \alpha + \beta.$$

If $l = 1$, then $\alpha = 2$ and $\delta = 0$, the result follows from Lemma 4.7. \square

We now have the case where $\delta \geq 2$.

Proposition 4.10 *Suppose $G = G_2(\alpha, \beta, \epsilon, \delta, -)$ with $\delta \geq 2$. Then $\eta(G) \geq \alpha + \beta - 2$. Equality holds if and only if $\beta = \delta$ and either (i) $\beta = 3$ or (ii) $\beta \geq 4$ and $\alpha - \beta = 2$.*

Proof. Set $l = \min(\alpha - \delta + 1, \beta - 1)$ and $k = \max(\alpha - \delta + 1, \beta - 1)$. We consider various cases according to the value of l .

First, suppose $l \geq 4$. Then by Corollary 4.5 and Lemma 2.4

$$\begin{aligned}
\eta(G) &= g_2(\alpha - \delta + 1, \beta - 1)/2 + 2 \\
&= g_2(k, l)/2 + 2 \geq 2k + l + 2 \\
&= (k + l) + k + 2 \\
&\geq \alpha - \delta + \beta + \beta - 1 + 2 \\
&\geq \alpha + \beta + 1
\end{aligned}$$

since $\delta \leq \beta$.

Now consider $l = 3$. We use Corollary 4.5 and Lemma 2.4 to find an exact value for $\eta(G)$.

$$\eta(G) = g_2(\alpha - \delta + 1, \beta - 1)/2 + 2 = g_2(k, 3)/2 + 2 = 2k + 2.$$

If $\alpha - \delta + 1 > \beta - 1 = 3$, then $\delta \leq 4$ and

$$\eta(G) = 2(\alpha - \delta + 1) + 2 = \alpha + (\alpha - \delta + 2) + (-\delta + 2) > \alpha + \beta - 2.$$

On the other hand, when $\beta - 1 \geq \alpha - \delta + 1 = 3$, we obtain $\beta \geq 4$ and $\alpha - \delta = 2$, so $\alpha - 2 \leq \beta$ and

$$\eta(G) = 2(\beta - 1) + 2 = 2\beta \geq \beta + \alpha - 2$$

with equality if and only if $\beta = \delta$.

Next suppose $l = 2$. Since $\alpha - \delta + 1 \geq 2 + 1 = 3$, we must have $\beta = 3$. Applying Corollary 4.5 and Lemma 2.4,

$$\begin{aligned}
\eta(G) &= g_2(\alpha - \delta + 1, \beta - 1)/2 + 2 = g_2(k, 2)/2 + 2 \\
&= k + 3 = \alpha - \delta + 4 \\
&\geq \alpha + 1 = \alpha + \beta - 2
\end{aligned}$$

with equality if and only if $\delta = 3 = \beta$.

Lastly consider $l = 1$. In this case $\beta = 2$ and the result follows from Corollary 4.5. \square

References

- [1] Y. Barnea, R. D. Camina, M. Ershov, M. L. Lewis, On groups that can be covered by conjugates of finitely many cyclic or procyclic subgroups, Preprint.
- [2] J. R. Beuerle, An elementary classification of finite metacyclic p -groups of class at least 3, *Algebra Colloquium* 12:4 (2005) 553-562.
- [3] M. Bianchi, R. D. Camina, M. L. Lewis, E. Pacifici, Conjugacy classes of maximal cyclic subgroups, submitted for publication, arXiv:2201.05637.
- [4] M. Bianchi, R. D. Camina, M. L. Lewis, Conjugacy classes of maximal cyclic subgroups and nilpotence class of p -groups, to appear in *Bull. Aust. Math. Soc.*, doi:10.1017/S0004972722000211. arXiv:2201.05642.
- [5] J. D. Dixon, M. P. F. du Sautoy, A. Mann, D. Segal, Analytic pro- p -groups, London Mathematical Society Lecture Note Series, 157. Cambridge University Press, Cambridge, 1991.
- [6] I. M. Isaacs, Finite Group Theory, Graduate Studies in Mathematics, Volume 92, American Mathematical Society, Providence, Rhode Island, 2008.
- [7] B. King, Presentations of metacyclic groups, *Bull. Austral. Math. Soc.* 8 (1973) 103-131.
- [8] J. R. Rogério, A note on maximal coverings of groups, *Comm. Algebra* **42**(10) (2014), 4498-4508.
- [9] T. W. von Puttkamer, On the Finiteness of the Classifying Space for Virtually Cyclic Subgroups, PhD thesis.

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