# A NOTE ON TRIGONOMETRIC APPROXIMATIONS OF BESSEL FUNCTIONS OF THE FIRST KIND, AND TRIGONOMETRIC POWER SUMS 

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#### Abstract

I reconsider the approximation of Bessel functions with finite sums of trigonometric functions, in the light of recent evaluations of Neumann-Bessel series with trigonometric coefficients. A proper choice of the angle allows for an efficient choice of the trigonometric sum. Based on these series, I also obtain straightforward non-standard evaluations of new parametric sums with powers of cosine and sine functions.


## 1. Introduction

Bessel functions are among the most useful and studied special functions. Analytic expansions exist for different regimes [18], and numerical algorithms for their precise evaluation [15 21] [13. Their simplest approximations are polynomials [1] [12] 14] and finite trigonometric sums, that can be advantageous in applications 5].
Let's consider $J_{0}$. Several trigonometric sums appeared in the decades, sometimes being rediscovered. These very simple ones

$$
\begin{align*}
& J_{0}(x) \simeq \frac{1}{4}\left[1+\cos x+2 \cos \left(\frac{\sqrt{2}}{2} x\right)\right]  \tag{1}\\
& J_{0}(x) \simeq \frac{1}{6}\left[1+\cos x+2 \cos \left(\frac{1}{2} x\right)+2 \cos \left(\frac{\sqrt{3}}{2} x\right)\right] \tag{2}
\end{align*}
$$

have errors $\epsilon=J_{0}-J_{0}^{\text {approx }}$ with power series (the marvel of Mathematica)

$$
\epsilon(x)=-\frac{x^{8}}{2^{8} \cdot 20560}\left(1-\frac{x^{2}}{36}+\ldots\right), \quad \epsilon(x)=-\frac{x^{12}}{2^{12} \cdot 239500800}\left(1-\frac{x^{2}}{52}+\ldots\right)
$$

In practice, an error less than 0.001 is achieved for $x \leq 3$ or $x \leq 5.9$. These approximations were obtained by Fettis with the Poisson formula [11. Rehwald [20] and later Waldron [24, Blachman and Mousavinezhad 6] and [3] used the strategy of truncating to the first term Neumann-Bessel series like

$$
J_{0}(x)+2 J_{8}(x)+2 J_{16}(x)+\ldots=\frac{1}{4} \cos \left[1+\cos x+2 \cos \left(\frac{\sqrt{2}}{2} x\right)\right]
$$

that can be obtained from the Bessel generating function. The examples correspond to $n=4,6$ of eq. 19 in [2]:

$$
\begin{equation*}
J_{0}(x)+2 \sum_{k=1}^{\infty}(-)^{k n} J_{2 k n}(x)=\frac{1}{n} \sum_{\ell=0}^{n-1} \cos \left(x \cos \frac{\pi}{n} \ell\right) \tag{3}
\end{equation*}
$$

and the errors reflect the behaviour $J_{2 n}(x) \approx(x / 2)^{2 n}$ of the first neglected term, but with much larger denominators.

[^0]The truncation yields $J_{0}$ as a sum of cosines that corresponds to the evaluation of the Bessel integral $J_{0}(x)=\int_{0}^{\pi} \frac{d \theta}{\pi} \cos (x \cos \theta)$ with the trapezoidal rule with $n$ nodes [22] [4] [23. Increasing $n$ increases accuracy: $n=15$ is a formula by Fettis [11] with 8 cosines (instead of 15 , symmetries of the roots of unity reduce the number of terms):

$$
\begin{equation*}
J_{0}(x) \simeq \frac{1}{15} \cos x+\frac{2}{15} \sum_{k=1}^{7} \cos \left(x \cos \frac{k \pi}{15}\right) \tag{4}
\end{equation*}
$$

The error now is order $x^{30} \times 10^{-42}$ and less than $10^{-6}$ for $x<15$.
In this note I reconsider the approximations for $J_{0}$ in the light of new NeumannBessel trigonometric series in ref. [17]. They extend the series (3) by including an angular parameter, that is chosen to kill the term with $J_{2 n}$, so that the truncation involves the next-to-next term $J_{4 n}$ of the series.
The same strategy on appropriate series is then used for Bessel functions $J_{n}$ of low order, that are discussed in section 3 .
In section 4, I show that the series give in very simple way some parametric sums of powers of cosines and sines. Some are in the recent literature (Jelitto [10], 2022) while the following ones, to my knowledge, are new:

$$
\sum_{\ell=0}^{n} \sin ^{p}\left(\frac{\theta+2 \pi \ell}{n}\right) \sin _{\cos }\left(q \frac{\theta+2 \pi \ell}{n}\right) \quad(p, q=0,1, \ldots)
$$

## 2. The Bessel function $J_{0}$

Consider the Neumann trigonometric series eq. 11 in [17]:

$$
\begin{equation*}
J_{0}(x)+2 \sum_{k=1}^{\infty}(-)^{k n} J_{2 k n}(x) \cos (2 k n \theta)=\frac{1}{n} \sum_{\ell=0}^{n-1} \cos \left[x \cos \left(\theta+\frac{\pi}{n} \ell\right)\right] \tag{5}
\end{equation*}
$$

The approximations (1), (2) and (4) are obtained with $\theta=0, n=4,6,15$, and neglecting functions $J_{8}, J_{12}, J_{30}$ and higher orders. However they are not optimal. The advantage of eq. (5) is the possibility to choose the angle $\theta=\pi / 4 n$ to kill all terms $J_{2 n}, J_{6 n}$, etc. Then:

$$
\begin{equation*}
J_{0}(x)-2 J_{4 n}(x)+2 J_{8 n}(x)-\ldots=\frac{1}{n} \sum_{\ell=0}^{n-1} \cos \left(z \cos \frac{1+4 \ell}{4 n} \pi\right) \tag{6}
\end{equation*}
$$

An expansion for $J_{0}$ results, again, by neglecting the other terms.
Some examples:

- $n=2$. It is $J_{0}(x)=\frac{1}{2}\left[\cos \left(x \cos \frac{\pi}{8}\right)+\cos \left(x \sin \frac{\pi}{8}\right)\right]+\epsilon_{2}(x)$. If we neglect the error, the first zero occurs at $\pi \sqrt{2-\sqrt{2}}=2.4045\left(j_{0,1}=2.4048\right)$.
- $n=3$. The approximation has three cosines:

$$
\begin{align*}
& J_{0}(x)=\frac{1}{3}\left[\cos \left(x \frac{1}{\sqrt{2}}\right)+\cos \left(x \frac{\sqrt{3}-1}{2 \sqrt{2}}\right)+\cos \left(x \frac{\sqrt{3}+1}{2 \sqrt{2}}\right)\right]+\epsilon_{3}(x)  \tag{7}\\
& \epsilon_{3}(x)=\frac{x^{12}}{2^{12} \cdot 239500800}\left[1-\frac{x^{2}}{52}+\frac{x^{4}}{52 \cdot 112}-\frac{x^{6}}{52 \cdot 112 \cdot 180}+\ldots\right] .
\end{align*}
$$

Remarkably, the first powers of the error are opposite of those for the expansion eq. 22, that would involve 6 terms if not for the degeneracy of the roots of unity.


Figure 1. The Bessel function $J_{0}$ (thick) and the trigonometric expansion (9). The difference increases with $x$; it is less than $10^{-9}$ for $x<8$ and $10^{-3}$ for $x<15$.

The half-sum of (2) and (7),

$$
\begin{array}{r}
J_{0}(x) \simeq \frac{1}{12}\left[1+\cos x+2 \cos \left(\frac{1}{2} x\right)+2 \cos \left(\frac{\sqrt{3}}{2} x\right)+2 \cos \left(x \frac{1}{\sqrt{2}}\right)\right. \\
\left.+2 \cos \left(x \frac{\sqrt{3}-1}{2 \sqrt{2}}\right)+2 \cos \left(x \frac{\sqrt{3}+1}{2 \sqrt{2}}\right)\right] \tag{8}
\end{array}
$$

has error $\epsilon(x)=-\frac{x^{24}}{5.2047} \times 10^{-30}\left[1-\frac{x^{2}}{100}+\frac{x^{4}}{20800}-\ldots\right]$.

- $n=6$ gives a precision similar to the sum (8):

$$
\begin{align*}
J_{0}(z)= & \frac{1}{6}\left[\cos \left(x \cos \frac{\pi}{24}\right)+\cos \left(x \cos \frac{3 \pi}{24}\right)+\cos \left(x \cos \frac{5 \pi}{24}\right)\right. \\
& \left.+\cos \left(x \sin \frac{\pi}{24}\right)+\cos \left(x \sin \frac{3 \pi}{24}\right)+\cos \left(x \sin \frac{5 \pi}{24}\right)\right]+\epsilon_{6}(x) . \tag{9}
\end{align*}
$$

The error has power expansion $\epsilon_{6}(x)=\frac{x^{24}}{5.2047} \times 10^{-30}\left[1-\frac{x^{2}}{100}+\ldots\right]$.

- $n=8$ is a sum of 8 cosines and compares with the formula (4) by Fettis. The two approximations are different but with the same number of terms (because $\theta=0$ produces degenerate terms) and similar precision.


## 3. Bessel functions $J_{n}$.

- $\mathbf{J}_{1}$ is evaluated via $J_{1}=-J_{0}^{\prime}$. Eq. (8) gives:

$$
\begin{array}{r}
J_{1}(x) \simeq \frac{1}{12}\left[\sin x+\sin \left(\frac{1}{2} x\right)+\sqrt{3} \sin \left(\frac{\sqrt{3}}{2} x\right)+\sqrt{2} \sin \left(x \frac{1}{\sqrt{2}}\right)\right. \\
\left.+\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right) \sin \left(x \frac{\sqrt{3}-1}{2 \sqrt{2}}\right)+\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right) \sin \left(x \frac{\sqrt{3}+1}{2 \sqrt{2}}\right)\right] \tag{10}
\end{array}
$$

with error $\epsilon(x) \simeq(x / 20)^{23} \times 3.87 \times\left[1-\frac{13}{1200} x^{2}+\ldots\right]$.

- $\mathbf{J}_{\mathbf{2}}, \mathbf{J}_{\mathbf{4}}$ can be evaluated with the following identity (the real part of eq.(5) in [17):

$$
\begin{array}{r}
J_{p}(x)+\sum_{k=1}^{\infty}\left[J_{k n+p}(x)+(-1)^{k n+p} J_{k n-p}(x)\right] \cos (k n \theta) \\
=\frac{1}{n} \sum_{\ell=0}^{n-1} \cos \left[x \sin \left(\theta+\frac{2 \pi \ell}{n}\right)+p\left(\theta+\frac{2 \pi \ell}{n}\right)\right] \tag{11}
\end{array}
$$

Because of the term $J_{n-p}$, we take $2 p<n$. With $y=\frac{\pi}{2 n}$ :

$$
J_{p}(x)-(-1)^{p} J_{2 n-p}(x)+\ldots=\frac{1}{n} \sum_{\ell=0}^{n-1} \cos \left[x \sin \left(\frac{1+4 \ell}{2 n} \pi\right)+p \frac{1+4 \ell}{2 n} \pi\right]
$$

If only $J_{p}$ is kept, the approximation depends on the parity of $p$ :

$$
J_{p}(x) \simeq\left\{\begin{align*}
\cos \left(p \frac{1+4 \ell}{2 n} \pi\right) \times \frac{1}{n} \sum_{\ell=0}^{n-1} \cos \left[x \sin \left(\frac{1+4 \ell}{2 n} \pi\right)\right] & p \text { even }  \tag{12}\\
-\sin \left(p \frac{1+4 \ell}{2 n} \pi\right) \times \frac{1}{n} \sum_{\ell=0}^{n-1} \sin \left[x \sin \left(\frac{1+4 \ell}{2 n} \pi\right)\right] & p \text { odd }
\end{align*}\right.
$$

$p=2, n=6$, give the short formula

$$
\begin{equation*}
J_{2}(x) \simeq \frac{1}{2 \sqrt{3}}\left[\cos \left(x \sin \frac{\pi}{12}\right)-\cos \left(x \cos \frac{\pi}{12}\right)\right] \tag{13}
\end{equation*}
$$

with error $\epsilon(x)=2.69114 \times(x / 10)^{-10}\left[1-\frac{x^{2}}{44}+\ldots\right]$. The first zero is evaluated $\frac{2}{3} \pi \sqrt{6} \simeq 5.1302\left(j_{2,1}=5.13562\right)$. A better approximation is $n=8, y=\frac{\pi}{16}$ :

$$
\begin{align*}
J_{2}(x) \simeq & \frac{1}{4} \cos \left(\frac{\pi}{8}\right)\left[\cos \left(x \sin \frac{\pi}{16}\right)-\cos \left(x \cos \frac{\pi}{16}\right)\right]  \tag{14}\\
& +\frac{1}{4} \sin \left(\frac{\pi}{8}\right)\left[\cos \left(x \cos \frac{5 \pi}{16}\right)-\cos \left(x \sin \frac{5 \pi}{16}\right)\right]
\end{align*}
$$

with error $\epsilon(x)=7.00119 \times 10^{-16} x^{14}\left[1-\frac{x^{2}}{60}+\ldots\right] ; \epsilon(5)=3 \times 10^{-6}, \epsilon(8)=0.0010$.
For $J_{4}$ we select $p=4, n=8, \theta=\frac{\pi}{16}$. Now the lowest neglected term is $J_{12}$ :

$$
\begin{equation*}
J_{4}(x) \simeq \frac{\sqrt{2}}{8}\left[\cos \left(x \sin \frac{\pi}{16}\right)+\cos \left(x \cos \frac{\pi}{16}\right)-\cos \left(x \sin \frac{5 \pi}{16}\right)-\cos \left(x \cos \frac{5 \pi}{16}\right)\right] \tag{15}
\end{equation*}
$$

The error is less that $10^{-3}$ at $x<6.3$.

- $\mathbf{J}_{\mathbf{3}}, \mathbf{J}_{\mathbf{5}}$. A useful sum for odd-order Bessel functions is eq.(17) in [17):

$$
\begin{equation*}
\sum_{k=0}^{\infty}(-)^{n+k} J_{(2 n+1)(2 k+1)}(x) \cos [(2 k+1) \theta]=\sum_{\ell=0}^{2 n} \frac{\sin \left[x \cos \left(\frac{\theta+2 \pi \ell}{2 n+1}\right)\right]}{2(2 n+1)} \tag{16}
\end{equation*}
$$

The angle $\theta=\frac{\pi}{6}$ cancels $J_{6 n+3}, J_{14 n+7}$ etc. and gives the approximation

$$
\begin{equation*}
J_{2 n+1}(x) \simeq \frac{(-1)^{n}}{\sqrt{3}} \sum_{\ell=0}^{2 n} \frac{\sin \left[x \cos \frac{1+12 \ell}{12 n+6} \pi\right]}{2 n+1} \tag{17}
\end{equation*}
$$

that neglects $J_{10 n+5}$ etc. With $n=1$ and $n=2$ we obtain:

$$
\begin{align*}
& J_{3}(x) \simeq-\frac{1}{3 \sqrt{3}}\left[\sin \left(x \cos \frac{\pi}{18}\right)-\sin \left(x \sin \frac{2 \pi}{9}\right)-\sin \left(x \sin \frac{\pi}{9}\right)\right]  \tag{18}\\
& J_{5}(x) \simeq \frac{1}{5 \sqrt{3}}\left[\sin \left(x \cos \frac{\pi}{30}\right)+\sin \left(x \sin \frac{\pi}{15}\right)-\sin \left(\frac{\sqrt{3}}{2} x\right)\right. \\
& \left.-\sin \left(x \sin \frac{4 \pi}{15}\right)+\sin \left(x \cos \frac{2 \pi}{15}\right)\right] . \tag{19}
\end{align*}
$$

The expansion for $J_{3}$ has error $\epsilon=2.33373 \times 10^{-17} x^{15}\left[1-\frac{x^{2}}{64}+\ldots\right]$. The second one has error $\epsilon=1.92134 x^{25} \times 10^{-33} \times\left[1-\frac{x^{2}}{104}+\ldots\right]$.

## 4. Trigonometric identities

The Neumann-Bessel series here used provide sums of powers of sines and cosines. They arise by expanding in powers of $x$ the Bessel functions in the series,

$$
J_{n}(x)=\sum_{k=0}^{\infty}(-1)^{k} \frac{(x / 2)^{n+2 k}}{k!(k+n)!}
$$

and the trigonometric functions in the sum of the series.


Figure 2. The Bessel function $J_{3}$ (thick) and the approximation 18.) The difference is $\epsilon(6)=6 \times 10^{-6}, \epsilon(8)=.0003, \epsilon(10)=.0045$.

- Consider the series eq. (5). At threshold powers $x^{2 n}, x^{4 n}$ etc. new Bessel functions $(-)^{n} 2 J_{2 n} \cos (2 n \theta), 2 J_{4 n} \cos (4 n \theta)$ etc. enter a term in the sum of cosines.

$$
\begin{align*}
& \frac{1}{n} \sum_{\ell=0}^{n-1}\left[\cos \frac{\theta+\ell \pi}{n}\right]^{2 k}=  \tag{20}\\
& = \begin{cases}\frac{1}{4^{k}}\binom{2 k}{k} & 0 \leq k<n \\
\frac{1}{4^{k}}\left[\binom{2 k}{k}+2\binom{2 k}{k-n} \cos (2 \theta)\right] & n \leq k<2 n \\
\frac{1}{4^{k}}\left[\binom{2 k}{k}+2\binom{2 k}{k-n} \cos (2 \theta)+2\binom{2 k}{k-2 n} \cos (4 \theta)\right] & 2 n \leq k<3 n \\
\ldots & \ldots\end{cases}
\end{align*}
$$

By replacing $\theta$ with $\theta+n \frac{\pi}{n}$ we obtain:

$$
\begin{align*}
& \frac{1}{n} \sum_{\ell=0}^{n-1}\left[\sin \frac{\theta+\ell \pi}{n}\right]^{2 k}=  \tag{21}\\
& \left.=\left\{\begin{array}{ll}
\frac{1}{4^{k}}\binom{2 k}{k} & 0 \leq k<n \\
\frac{1}{4^{k}} \\
\left.\frac{1}{4^{k}}\left[\begin{array}{c}
2 k \\
k
\end{array}\right)+(-)^{n} 2\binom{2 k}{k-n} \cos (2 \theta)\right] & n \leq k<2 n \\
\ldots & \ldots \\
k
\end{array}\right)+(-)^{n} 2\binom{2 k}{k-n} \cos (2 \theta)+2\binom{2 k}{k-2 n} \cos (4 \theta)\right] \\
& \ldots
\end{align*}
$$

Examples:
$\frac{1}{9} \sum_{\ell=0}^{8}\left[\sin \frac{\theta+\ell \pi}{9}\right]^{20}=\frac{1}{4^{10}}\left[\binom{20}{10}-2\binom{20}{1} \cos (2 \theta)\right]$.
$\frac{1}{n} \sum_{\ell=0}^{n-1}\left[\cos \left(\frac{1+6 \ell}{6 n} \pi\right)\right]^{2 n}=\frac{1}{4^{n}}\left[\binom{2 n}{n}+1\right], \quad \frac{1}{n} \sum_{\ell=0}^{n-1}\left[\cos \left(\frac{1+4 \ell}{4 n} \pi\right)\right]^{4 n}=\frac{1}{16^{n}}\left[\binom{4 n}{2 n}-2\right]$.
For $\theta=0$ and $\theta=\frac{\pi}{2}$ these identities are eqs. 4.4.2 in [19], 2.1 and 2.2 (together with several other non-parametric sums) in [9. The series had also been studied in [16]. Parametric averages on the full circle were recently evaluated by Jelitto [10], with a different method.

- With the Neumann series (16) we obtain:

$$
\begin{align*}
& \frac{1}{2 n+1} \sum_{\ell=0}^{2 n}\left[\cos \frac{\theta+2 \pi \ell}{2 n+1}\right]^{2 k+1}=  \tag{22}\\
& = \begin{cases}0 & 2 n+1 \leq 2 k+1<3(2 n+1) \\
\frac{1}{4^{k}}\binom{2 k+1}{k-n} \cos \theta & 1 \leq 2 k+1<2 n+1 \\
\frac{1}{4^{k}}\left[\binom{(2 k+1}{k-n} \cos \theta+\binom{2 k+1}{k-3 n-1} \cos (3 \theta)\right] & 3(2 n+1) \leq 2 k+1<5(2 n+1) \\
\ldots & \ldots\end{cases}
\end{align*}
$$

The sums of even powers of cosines are obtained from the series eq. 16 in [17]:

$$
\left.\begin{array}{c}
J_{0}(x)+2 \sum_{k=1}^{\infty}(-)^{k} J_{(4 n+2) k}(x) \cos (2 k \theta)=\sum_{\ell=0}^{2 n} \frac{\cos \left[x \cos \frac{\theta+2 \pi \ell}{2 n+1}\right]}{2 n+1} \\
\left.=\left\{\begin{array}{ll}
2 n+1 & \sum_{\ell=0}^{2 n}\left[\cos \frac{\theta+2 \pi \ell}{2 n+1}\right]^{2 k}= \\
\frac{1}{4^{k}}\binom{2 k}{k} & 0 \leq k<2 n+1 \\
\left.\frac{1}{4^{k}}\left[\begin{array}{l}
2 k \\
4^{k} \\
k
\end{array}\right)+2\binom{2 k}{k-2 n-1} \cos (2 \theta)\right] \\
\ldots & 2 n+1 \leq k<4 n+2 \\
k
\end{array}\right)+2\binom{2 k}{k-2 n-1} \cos (2 \theta)+2\binom{2 k}{k-4 n-2} \cos (4 \theta)\right]  \tag{24}\\
4 n+2 \leq k<6 n+3
\end{array}\right] .
$$

Example: $\left(\cos \frac{\theta}{3}\right)^{12}+\left(\cos \frac{\theta+\pi}{3}\right)^{12}+\left(\cos \frac{\theta+2 \pi}{3}\right)^{12}=\frac{3}{4^{6}}\left[\binom{12}{6}+2\binom{12}{3} \cos (2 \theta)+2 \cos (4 \theta)\right]$. The formulae with sines are obtained by shifting the parameter $\theta$.

- Now let's consider the sum eq. 11 with $p<n-p$. The equations are new and are easier to state if we distinguish the parity of $n$ and of $p$.

Case $\mathbf{n}=\mathbf{2 m}$ and $\mathbf{p}=\mathbf{2 q}$. Eq. 11 now is:

$$
\begin{aligned}
& \frac{1}{2 m} \sum_{\ell=0}^{2 m-1} \cos \left[x \sin \left(\frac{\theta+\pi \ell}{m}\right)\right] \cos \left[2 q \frac{\theta+\pi \ell}{m}\right]=J_{2 q}(x)+ \\
& +\left[J_{2 m-2 q}(x)+J_{2 m+2 q}(x)\right] \cos (2 \theta)+\left[J_{4 m-2 q}(x)+J_{4 m+2 q}(x)\right] \cos (4 \theta)+\ldots
\end{aligned}
$$

Separation of even and odd parts in $x$, and expansion in $x$ give:

$$
\frac{1}{2 m} \sum_{\ell=0}^{2 m-1}\left[\sin \frac{\theta+\pi \ell}{m}\right]^{2 k+1} \sin \left(2 q \frac{\theta+\pi \ell}{m}\right)=0, \quad \forall k
$$

This result is obvious as the sum from 0 to $m-1$ is opposite of the rest of the sum. The symmetry is used also in the other result:

$$
\begin{align*}
& \frac{1}{m} \sum_{\ell=0}^{m-1}\left[\sin \frac{\theta+\pi \ell}{m}\right]^{2 k} \cos \left(2 q \frac{\theta+\pi \ell}{m}\right)=  \tag{25}\\
& =\frac{(-)^{q}}{4^{k}} \begin{cases}0 & k<q \\
\binom{2 k}{k-q} & q \leq k<m-q \\
\binom{2 k}{k-q}+(-1)^{m}\binom{2 k}{k-m+q} \cos (2 \theta) & m-q \leq k<m+q \\
\left.\binom{2 k}{k-q}+(-1)^{m}\left[\begin{array}{c}
2 k \\
k-m+q
\end{array}\right)+\binom{2 k}{k-m-q}\right] \cos (2 \theta) & m+q \leq k<2 m-q \\
\ldots & \cdots\end{cases}
\end{align*}
$$

Case $\mathbf{n}=\mathbf{2 m}, \mathbf{p}=\mathbf{2 q}+1$. Eq. 11 becomes:

$$
\begin{aligned}
& -\frac{1}{2 m} \sum_{\ell=0}^{2 m-1} \sin \left[x \sin \left(\frac{\theta+\pi \ell}{m}\right)\right] \sin \left[(2 q+1) \frac{\theta+\pi \ell}{m}\right]=J_{2 q+1}(x)+ \\
& +\left[-J_{2 m-2 q-1}(x)+J_{2 m+2 q+1}(x)\right] \cos (2 \theta)+\left[J_{4 m-2 q-1}(x)+J_{4 m+2 q+1}(x)\right] \cos (4 \theta)+\ldots
\end{aligned}
$$

The non trivial result is:

$$
\begin{align*}
& \frac{1}{m} \sum_{\ell=0}^{m-1}\left[\sin \frac{\theta+\pi \ell}{m}\right]^{2 k+1} \sin \left[(2 q+1) \frac{\theta+\pi \ell}{m}\right]=  \tag{26}\\
& =\frac{(-)^{q}}{2^{2 k+1}} \begin{cases}0 & k<q \\
\binom{2 k+1}{k-q} & q \leq k<m-q-1 \\
\binom{2 k+1}{k-q}+(-1)^{m}\binom{2 k+1}{k+m-q} \cos (2 \theta) & m-q-1 \leq k<m+q \\
\binom{2 k+1}{k-q}+(-1)^{m}\left[\binom{2 k+1}{k+m-q}+\binom{2 k+1}{k-m-q}\right] \cos (2 \theta) & m+q \leq k<2 m-q-1 \\
\ldots & \ldots\end{cases}
\end{align*}
$$

Example: $\frac{1}{5} \sum_{\ell=0}^{4} \sin ^{13}\left(\frac{\pi \ell}{5}\right) \sin \left(\frac{3 \pi \ell}{5}\right)=-\frac{1}{2^{13}}\left[\binom{13}{5}-\binom{13}{10}-\binom{13}{0}\right]=-\frac{125}{1024}$.

## Case $n=2 m+1$ and $p=2 q$ :

$$
\begin{align*}
& \frac{1}{2 m+1} \sum_{\ell=0}^{2 m} \cos \left[x \sin \left(\frac{\theta+2 \pi \ell}{2 m+1}\right)\right] \cos \left(2 q \frac{\theta+2 \pi \ell}{2 m+1}\right)=J_{2 q}(x)+\left[J_{4 m+2-2 q}(x)+J_{4 m+2+2 q}(x)\right] \cos (2 \theta)+\ldots \\
& \frac{1}{2 m+1} \sum_{\ell=0}^{2 m} \sin \left[x \sin \left(\frac{\theta+2 \pi \ell}{2 m+1}\right)\right] \sin \left(2 q \frac{\theta+2 \pi \ell}{2 m+1}\right)=\left[J_{2 m+1-2 q}(x)-J_{2 m+1+2 q}(x)\right] \cos \theta+\ldots \\
& \frac{1}{2 m+1} \sum_{\ell=0}^{2 m}\left[\sin \frac{\theta+2 \pi \ell}{2 m+1}\right]^{2 k} \cos \left(2 q \frac{\theta+2 \pi \ell}{2 m+1}\right)=  \tag{27}\\
& =\frac{(-)^{q}}{4^{k}} \begin{cases}0 & k<q \\
\binom{2 k}{k-q} & q \leq k<2 m+1-q \\
\binom{2 k}{k-q}-\binom{2 k}{k+q-1-2 m} \cos (2 \theta) & 2 m+1-q \leq k<2 m+1+q \\
\left.\binom{2 k}{k-q}-\left[\begin{array}{c}
2 k \\
k+q-1-2 m
\end{array}\right)+\binom{2 k}{k-q-1-2 m}\right] \cos (2 \theta) & 2 m+1+q \leq k<4 m+2-q \\
\ldots & \ldots\end{cases} \\
& \frac{1}{2 m+1} \sum_{\ell=0}^{2 m}\left[\sin \frac{\theta+2 \pi \ell}{2 m+1}\right]^{2 k+1} \sin \left(2 q \frac{\theta+2 \pi \ell}{2 m+1}\right)=  \tag{28}\\
& =\frac{(-)^{q+m+1}}{2^{2 k+1}} \begin{cases}0 & k<m-q \\
\binom{2 k+1}{k-m+q} \cos \theta & m-q \leq k<m+q \\
{\left[\binom{2 k+1}{k-m+q}-\binom{2 k+1}{k-m-q}\right] \cos \theta} & m+q \leq k<3 m+1-q \\
\ldots & \ldots\end{cases}
\end{align*}
$$

Case $\mathbf{n}=\mathbf{2 m}+1$ and $p=2 q+1$ :

$$
\begin{gathered}
\frac{1}{2 m+1} \sum_{\ell=0}^{2 m} \cos \left[x \sin \frac{\theta+2 \pi \ell}{2 m+1}\right] \cos \left[(2 q+1) \frac{\theta+2 \pi \ell}{2 m+1}\right]=\left[J_{2 m-2 q}(x)+J_{2 m+2 q+2}(x)\right] \cos \theta+\ldots \\
-\frac{1}{2 m+1} \sum_{\ell=0}^{2 m} \sin \left[x \sin \frac{\theta+2 \pi \ell}{2 m+1}\right] \sin \left[(2 q+1) \frac{\theta+2 \pi \ell}{2 m+1}\right]=J_{2 q+1}(x)+ \\
+\left[-J_{4 m-2 q+1}(x)+J_{4 m+2 q+3}(x)\right] \cos (2 \theta)+\ldots
\end{gathered}
$$

$$
\begin{align*}
& \frac{1}{2 m+1} \sum_{\ell=0}^{2 m} \sin \left[\frac{\theta+2 \pi \ell}{2 m+1}\right]^{2 k} \cos \left[(2 q+1) \frac{\theta+2 \pi \ell}{2 m+1}\right]=  \tag{29}\\
& =\frac{(-)^{m+q}}{4^{k}} \begin{cases}0 & k<m-q \\
\binom{2 k}{k-m+q} \cos \theta & m-q \leq k<m+q+1 \\
\left.\left[\begin{array}{c}
2 k \\
k-m+q
\end{array}\right)-\binom{2 k}{k-m-q-1}\right] \cos \theta & m+q+1 \leq k<3 m+q+2 \\
\ldots & \ldots\end{cases} \\
& \frac{1}{2 m+1} \sum_{\ell=0}^{2 m} \sin \left[\frac{\theta+2 \pi \ell}{2 m+1}\right]^{2 k+1} \sin \left[(2 q+1) \frac{\theta+2 \pi \ell}{2 m+1}\right]=  \tag{30}\\
& =\frac{(-)^{q}}{2^{2 k+1}} \begin{cases}0 & k<q \\
\binom{2 k+1}{k-q} & q \leq k<2 m-q \\
\binom{2 k+1}{k-q}-\binom{2 k+1}{k+q-2 m} \cos (2 \theta) & 2 m-q \leq k<2 m+q+1 \\
\left.\binom{2 k+1}{k-q}-\left[\begin{array}{c}
2 k+1 \\
k+q-2 m
\end{array}\right)+\binom{2 k+1}{k-q-1-2 m}\right] \cos (2 \theta) & 2 m+q+1 \leq k<4 m-q+1 \\
\ldots & \ldots\end{cases}
\end{align*}
$$

Data availability. Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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