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Overconvergent Modular and de Rham Sheaves and p-adic Iteration of the Gauss-Manin Connection

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Abstract

Obiettivo della tesi è la construzione di iterate p-adiche della connessione di Gauss-Manin su fasci sovra-convergenti di classi di de Rham nel caso di varietà modulari di Hilbert quando p non ramifica nel campo totalmente reale. Questa è una generalizzazione di un lavoro di Andreatta-Iovita [AI21] nel caso di curve ellittiche, usando una nuova tecnica che segna un miglioramento del loro approccio in termini di convergenza delle iterate di detta connessione.

We construct p-adic iteration of the Gauss-Manin connection on overconvergent sheaves of de Rham classes on Hilbert modular varieties in the case p is unramified in the totally real field. This is a generalization of the work of Andreatta-Iovita [AI21] in the case of elliptic curves, using a technique that marks an improvement on their approach in terms of convergence of the iterates of the connection.

Introduction

The theory of p-adic L-functions has been an important object of study due to its numerous arithmetic applications, most notably towards the Birch and Swinnerton-Dyer conjecture and its generalizations. Katz's p-adic Kronecker limit formula [Kat76] relates special values of the two variable p-adic L-function associated to a quadratic imaginary field at finite order Hecke characters to p-adic logarithms of elliptic units. The work of Bertolini-Darmon-Prasanna [BDP13] relates the central critical values of a certain p-adic Rankin L-function associated to a cusp form f and a quadratic imaginary field K to p-adic Abel-Jacobi images of *generalized Heegner cycles*. The article [Ber+14] explains why the two examples can be viewed as similar using the theory of Euler systems. The article of Darmon and Rotger [DR14] proves a p-adic Gross-Zagier formula relating the special values of the p-adic Garrett-Rankin triple product L-function attached to a triple of Hida families of modular forms to p-adic Abel-Jacobi images of certain *generalized Gross-Kudla-Schoen cycles* in the product of three Kuga-Sato varieties.

The general technique of constructing these p-adic L-functions consists of two ingredients: firstly, a theory of p-adic modular forms and secondly, a way to p-adically iterate differential operators on the space of modular forms. For example, in the case of triple product L-functions due to the work of Harris-Kudla [HK91], Ichino [Ich08] and others, one expects that the special values of classical L-functions that one wishes to p-adically interpolate is a meaningful algebraic number upto multiplication by a transcendental period. More precisely, one expects the algebraic number to be expressed as the square of a Petersson inner product of nearly holomorphic modular forms, possibly arising as the image of a holomorphic modular form under the Shimura-Maass operator. Working with Hida families, as in the case of [DR14], the p-adic analogue of the Shimura-Maass operator is the θ-operator of Serre, which acts as $\theta = q\frac{\mathrm{d}}{\mathrm{d}q}$ on q-expansions. The analogy between the Shimura-Maass operator δ and Serre's θ operator rests on the fact that they both can be viewed as the application of the Gauss-Manin connection followed by a projection to the space of modular forms. In the case of δ the projection comes from the splitting induced by the Hodge decomposition which is a special property of Kahler manifolds and in the case of θ this is given by the unit root splitting, which is a uniquely p-adic phenomenon.

Following the work of Coleman on overconvergent families [Col96], [Col97], it becomes a natural question of interest to adapt these techniques for finite slope families. The main problem in this case is the unavailability of the unit root splitting. One approach to solve this problem is to instead try to p-adically iterate the Gauss-Manin connection itself. This is the technique employed in the recent work of Andreatta-Iovita [AI21], where they construct triple product p-adic L-function attached to finite slope families. Moreover, as a consequence of working beyond the ordinary locus, they manage to construct

Katz-BDP type p-adic L-function in the case p is non-split in the quadratic imaginary field [AI19]. We describe the approach briefly here.

To talk about p-adic iteration of the Gauss-Manin connection, one needs to consider families of modular forms for p-adically varying weights, as well as families of de Rham classes for varying weights. The first object, i.e. the sheaf of p-adic modular forms has been geometrically constructed and studied for quite some time [AIP15], [AIP16a], [AIP18]. The novelty of their work has been the construction of a sheaf $\mathbb{W}_k$ that interpolates symmetric powers of H^1_{dR} of the universal elliptic curve for analytic weights k. This construction is based on the theory of *vector bundles with marked sections*. This approach has been used fruitfully by Graziani [Gra20] to define interpolation sheaves of de Rham classes, and by Aycock [Ayc20] to define the Gauss-Manin connection ∇ in the setting of Hilbert modular forms.

The article [Mol21] of Molina marks a significant improvement on this technique. One crucial step in the construction of the p-adic iteration of ∇ is the proof of its convergence. In [AI21], the authors need to carry out extremely long and complicated computations to prove this. However, in [Mol21], Molina uses a refined version of vector bundles with marked sections to simplify the computations to a large extent. The key idea of his work relies on [Mol21, Lemma 5.1] which proves that the modified integral model $\mathcal{H}_0^\sharp$ of the de Rham sheaf that one uses to define the interpolation sheaf $\mathbb{W}$, admits a splitting modulo a small power of p. Using this one can restrict to a certain well-defined open subspace in the adic geometric vector bundle with marked sections associated to $\mathcal{H}_0^\sharp$, such that the Gauss-Manin connection converges faster on the sections of this open subspace. With this the author has been able to construct triple product p-adic L-functions associated with families of quarternionic automorphic forms in Shimura curves over totally real fields in the finite slope situation.

Our work is concerned with the construction of p-adic iteration of ∇ in the case of overconvergent Hilbert modular forms as in [Gra20] and [Ayc20]. The main idea of our work is similar to that of [Mol21], in the sense that we use a refined version of vector bundles with marked sections, using a canonical splitting of our integral model of de Rham sheaf $\mathrm{H}^\sharp_{\mathcal{A}}$ modulo a small power of p, and prove that the partial Gauss-Manin connection $\nabla(\sigma)$ associated to the embedding $\sigma\colon L \to \bar{\mathbb{Q}}_p$ of our fixed totally real field L converges fast enough, and that $\nabla(\sigma)$ commutes with $\nabla(\tau)$ for $\sigma \neq \tau$. In the following, we will describe this in more detail. The thesis has three chapters. In the first chapter we implement the idea in the simpler case of elliptic modular forms, while in the second we deal with the case of Hilbert modular forms. It will be already evident in the case of elliptic modular forms that our approach gives a faster rate of convergence. We will eventually use this knowledge to deal with the higher dimensional case of Hilbert modular forms. In the introduction we have decided to describe the contents of the second chapter, as it will bring to light the specificities of the higher dimensional case as well as shed light on the key improvement in the technique.

Fix a totally real field L of degree g over $\mathbb{Q}$. Let $\mathfrak{d}$ be the different ideal of $L/\mathbb{Q}$. Fix an integer $N \geq 4$ and a prime $p \nmid N$ that is unramified in L. Fix a finite unramified extension K of $\mathbb{Q}_p$ that splits L. Assume for simplicity $p > 2$. Let Σ be the set of embeddings of L in K.

The notion of Hilbert modular forms has a slight ambiguity in the literature. Namely, one can talk either about geometric Hilbert modular forms, or about arithmetic Hilbert modular forms. This discrepancy arises from the fact that the Shimura variety associated to the group $G := \mathrm{Res}_{\mathcal{O}_L/\mathbb{Z}}\,\mathbf{GL}_2$ at the usual principal N level or μ_N level doesn't have an interpretation as a fine moduli of abelian varieties. Instead its connected components, which are parametrized by elements in the strict class group are finite étale quotients of actual moduli spaces of abelian varieties with polarization data and level structure. The geometric Hilbert modular forms are the sections of modular sheaves on the moduli scheme of abelian varieties, whereas the arithmetic Hilbert modular forms are the automorphic forms on the Shimura variety associated to G. The arithmetic Hilbert modular forms are necessary to consider for a good Hecke theory. The geometric modular forms, as the name suggests are much more suitable for use in geometric constructions. Let us make the relation between the two notions precise. Let us fix the level $K = K_1(N)$

$$K_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\hat{\mathcal{O}}_L) \mid a \equiv 1, c \equiv 0 \bmod N \right\}.$$

Then the Shimura variety $\mathrm{Sh}_K(G)$ at level K is a disjoint union of connected components indexed by the strict class group $\mathrm{Cl}^+(L)$ of L. For any $\mathfrak{c}$ coprime to p, the moduli $M(\mu_N, \mathfrak{c})$ of $\mathfrak{c}$-polarized abelian varieties with real multiplication by $\mathcal{O}_L$ and μ_N-level structure is representable by a smooth scheme, and there is a universal abelian scheme $\mathcal{A} \to M(\mu_N, \mathfrak{c})$. There is an action of the totally positive units $\mathcal{O}_L^{\times,+}$ on $M(\mu_N, \mathfrak{c})$, given simply by multiplying the polarization data by the unit. This action factors through the finite quotient $\Gamma := \mathcal{O}_L^{\times,+}/U_N^2$ where U_N is the group of units congruent to 1 mod N. This action is free and the quotient scheme is isomorphic to the connected component of $\mathrm{Sh}_K(G)$ indexed by the class $[\mathfrak{c}]$. For any weight $k \in \mathbb{Z}[\Sigma]$, let $\omega_{\mathcal{A}}^k$ be the sheaf of Hilbert modular forms of weight k. There is an action of Γ on $\omega_{\mathcal{A}}^k$ lifting the action on $M(\mu_N, \mathfrak{c})$ and arithmetic Hilbert modular forms are defined as the invariant sections for this action. (See §2.1.2.1 for details.) Henceforth everything that we discuss in the introduction will involve only the geometric modular forms.

Let $\mathfrak{W} = \mathrm{Spf}\, \mathcal{O}_L[\![(\mathcal{O}_L \otimes \mathbb{Z}_p)^\times]\!] = \mathrm{Spf}\, \Lambda$ be the formal weight space and let $\mathfrak{W}^0 = \mathrm{Spf}\, \Lambda^0$ be its connected component of the trivial character. Λ^0 is a local ring and let $\mathfrak{m}$ be its maximal ideal. Let $\mathcal{W}^0$ be the adic analytic fibre of $\mathfrak{W}^0$, and let $\mathcal{W}_p^0$ be the affinoid open where $|t| \le |p| \neq 0$ for all $t \in \mathfrak{m}$. Let $\mathfrak{W}_p^0 = \mathrm{Spf}\, \mathcal{O}^+_{\mathcal{W}_p^0}$. Let k^0 be the universal analytic weight on $\mathfrak{W}_p^0$. Due to analyticity, k^0 extends to a character on $1 + p^n \operatorname{Res}_{\mathcal{O}_L/\mathbb{Z}} \mathbb{G}_a \simeq \prod_{\sigma \in \Sigma} 1 + p^n \mathbb{G}_a$ for some n. Thus we may view k^0 as a product of characters $k^0 = \prod_{\sigma \in \Sigma} k^0_\sigma$.

Consider the p-adic completion of $M(\mu_N, \mathfrak{c})$ and let $\mathfrak{X}$ be its base change to $\mathfrak{W}_p^0$. Consider the blow-up spaces $\mathfrak{X}_r$ obtained by blowing up the Hdg ideal and taking the open where the inverse image ideal is generated by $\mathrm{Hdg}^{p^{r+1}}$. Here Hdg is the ideal generated locally by lifts of the Hasse invariant and p. These blow-up spaces are formal models for the rigid analytic overconvergent locus where $|\mathrm{Hdg}^{p^{r+1}}| \ge |p|$. The main results of this work are the following.

Theorem 1. *For suitable choice of r, there are interpolation sheaves $\mathfrak{w}_k^0$ and $\mathbb{W}_k^0$ on $\mathfrak{X}_r$, that interpolates modular forms and symmetric powers of de Rham classes for weight k^0 respectively. The sheaf $\mathbb{W}_k^0$ on $\mathfrak{X}_r$ is equipped with an increasing filtration $\{\mathrm{Fil}_i\}_{i\ge 0}$. The filtered pieces are locally free $\mathcal{O}_{\mathfrak{X}_r}$-modules and $\mathbb{W}_k^0$ is the completed colimit of $\mathrm{Fil}_i \mathbb{W}_k^0$. The zeroth filtered piece $\mathrm{Fil}_0 \mathbb{W}_k^0 = \mathfrak{w}_k^0$ coincides with the modular sheaf of weight k^0.* [§2.2, §2.3]

Theorem 2. *The Gauss-Manin connection on $H^1_{dR}(\mathcal{A})$ induces a connection ∇ on $\mathbb{W}_k^0$ over the generic fibre that satisfies Griffiths' transversality with respect to the filtration mentioned above. Moreover, let $k = \prod k_\sigma = \prod \exp(u_\sigma \log(\cdot))$ and $s = \prod s_\sigma = \prod \exp(v_\sigma \log(\cdot))$ be two analytic weights such that $u_\sigma, v_\sigma \in \mathcal{O}^+_{\mathcal{W}_p^0}$.*

Then for any p-depleted g of weight k, $\nabla^s(g)$ makes sense as an overconvergent Hilbert modular form of weight $k+2s$. [§2.4]

Let us begin by discussing our approach to proving Theorem 1. The theory of canonical subgroup tells us that there exists a canonical subgroup of level n for all $1 \leq n \leq r$. Then over the partial Igusa tower $\mathfrak{IG}_{n,r}$ that classifies trivializations $\mathcal{O}/p^n\mathcal{O} \simeq H_n^\vee$ of the dual of the canonical subgroup, one can define an integral model $\Omega_{\mathcal{A}}$ of the modular sheaf $\omega_{\mathcal{A}}$ as the submodule generated by all lifts of the image $\mathrm{dlog}(P^{\mathrm{univ}})$ of the universal generator P^{univ} of $H_n^\vee$ under the dlog map. This is an $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{n,r}}$-line bundle that is equipped with a marked section $\mathrm{dlog}(P^{\mathrm{univ}})$ by construction. Then using the theory of vector bundles with marked section one can define the modular sheaf $\mathfrak{w}_k^0$ of weight k^0 in the usual manner. This definition coincides with previous definitions in [AIP16b], [Gra20] and [Ayc20].

As mentioned before the key construction in this theory is the definition of interpolation sheaves of de Rham classes. Such a definition appears in [Gra20] and [Ayc20]. The definition due to Graziani follows closely the analogous definition in the elliptic case due to Andreatta-Iovita. One has to choose a suitable integral model $\mathrm{H}^\sharp_{\mathcal{A}}$ of $H^1_{\mathrm{dR}}(\mathcal{A})$ such that the induced Hodge filtration identifies the zeroth filtered piece with $\Omega_{\mathcal{A}}$. The definition due to Graziani then uses VBMS to define the interpolation sheaf. Our choice of the integral model $\mathrm{H}^\sharp_{\mathcal{A}}$ differs from that due to Graziani. Let $\underline{\xi}$ be the invertible $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{n,r}}$ ideal that satisfies $\Omega_{\mathcal{A}} = \underline{\xi}\omega_{\mathcal{A}}$. Let $\widetilde{HW}$ be the invertible $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{n,r}}$ ideal generated by local lifts of the Hasse-Witt matrix. The first key result towards the goal of defining the interpolation sheaf for de Rham classes is the following.

Theorem. *For suitable choice of r, n as above, there exists a locally free $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{n,r}}$ sheaf $\mathrm{H}^\sharp_{\mathcal{A}} \subset H^1_{dR}(\mathcal{A})$ of rank 2, such that the induced Hodge filtration is*

$$0 \to \Omega_{\mathcal{A}} \to \mathrm{H}^\sharp_{\mathcal{A}} \to \underline{\xi}\widetilde{HW}\omega_{\mathcal{A}}^\vee \to 0.$$

Moreover the Hodge filtration admits a canonical splitting modulo p/Hdg^{p^2} that coincides with the unit root splitting over the ordinary locus. The splitting is functorial for the lift of Verschiebung.

The proof of this result is the content of §2.2 and §2.3.1. We remark that in the case of $\mathbf{GL}_{2,\mathbb{Q}}$ we have an analogous splitting of the integral model $\mathrm{H}^\sharp_{\mathcal{E}}$ modulo a small power of p. Moreover, in this case, our $\mathrm{H}^\sharp_{\mathcal{E}}$ actually coincides with the $\mathrm{H}^\sharp_{\mathcal{E}}$ of [AI21].

Let $\mathcal{Q}$ be the kernel of the splitting in the theorem above. Using this definition of $\mathrm{H}^\sharp_{\mathcal{A}}$, we define a notion of vector bundle with marked sections and marked splitting, similar to the definition of Molina [Mol21, §4.2] as follows.

$$\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, \mathrm{dlog}(P^{\mathrm{univ}}), \mathcal{Q})(R) := \left\{ f\colon \mathrm{H}^\sharp_{\mathcal{A}}(R) \to \mathcal{O}_L \otimes R \mid f \text{ is } \mathcal{O}_L\text{-linear}, f(\mathrm{dlog}(P^{\mathrm{univ}}) = 1), f(\mathcal{Q}) = 0 \right\}$$

We prove that this vector bundle is representable. In fact as an adic space this has a very simple local description. Let s be a local lift of $\mathrm{dlog}(P^{\mathrm{univ}})$ and t be a lift of a local generator of $\mathcal{Q}$. Let β_n be the small power of p such that $\mathrm{dlog}(P^{\mathrm{univ}})$ is a section of $\Omega_{\mathcal{A}}/\beta_n$. Let $\eta = p/\mathrm{Hdg}^{p^2}$. Then locally as adic spaces,

$$\mathbb{V}_0^{\mathcal{O}_{\mathcal{L}}}(\mathrm{H}^\sharp_{\mathcal{A}}, \mathrm{dlog}(P^{\mathrm{univ}}), \mathcal{Q}) = \mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}) \left(\frac{s-1}{\beta_n}, \frac{t}{\eta} \right).$$

Here $\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}})$ is the usual vector bundle whose sections over R are $\mathcal{O}_L \otimes R$-linear maps $\mathrm{H}^\sharp_{\mathcal{A}}(R) \to \mathcal{O}_L \otimes R$. (Note since we assume that p is unramified in L it is fine to take $\mathcal{O}_L \otimes R$ as the codomain instead of $\mathfrak{d}^{-1} \otimes R$.) We then define the interpolation sheaf $\mathbb{W}_k^0$ using this refined version of VBMS and prove the rest of the statements in Theorem 1. The proof of these results concerns the rest of §2.3.

The next section §2.4 is where we define the Gauss-Manin connection on $\mathbb{W}_k^0$, and show that it can be p-adically iterated for analytic weights. The strategy is exactly similar to [AI21] and [Mol21]. Using Grothendieck's description of connections we prove the first part of Theorem 2:

Theorem. *The Gauss-Manin connection on $H^1_{dR}(\mathcal{A})$ induces a connection ∇ on $\mathbb{W}_k^0$ over the generic fibre that satisfies Griffiths' transversality with respect to the filtration mentioned above.*

The definition of p-adic iteration of ∇ follows the strategy of [AI21]. For each embedding $\sigma \in \Sigma$, the Kodaira-Spencer class corresponding to σ gives a partial connection $\nabla(\sigma)\colon \mathbb{W}_k^0[1/p] \to \mathbb{W}_{k+2\sigma}^0[1/p]$. We first study the convergence properties of $\nabla(\sigma)$ over the ordinary locus using q-expansions and local coordinates. Here we realize that $\nabla(\sigma)$ behaves exactly like the connection ∇ in the case of elliptic curves. However, owing to our use of VBMS with marked splitting, we get faster convergence estimates. This is really the key improvement to the technique of [AI21] and mimics the results obtained by Molina. In particular [AI21, Proposition 4.10] states that for any p-depleted $g \in H^0(\mathfrak{IG}_n^{\mathrm{ord}}, \mathbb{W}_k^0)$,

$$(\nabla^{p-1} - \mathrm{id})^p(g) \in pH^0(\mathfrak{IG}_n^{\mathrm{ord}}, \mathbb{W}^0).$$

Here $\mathbb{W}^0$ is the direct image of the structure sheaf of $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, \mathrm{dlog}(P^{\mathrm{univ}})) \to \mathfrak{IG}_n^{\mathrm{ord}}$. Instead, if we work with $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, \mathrm{dlog}(P^{\mathrm{univ}}), \mathcal{Q})$, and let $\mathbb{W}^0$ be the direct image of its structure sheaf, then we can prove (Corollary 1.4.1) that

$$(\nabla^{p-1} - \mathrm{id})(g) \in pH^0(\mathfrak{IG}_n^{\mathrm{ord}}, \mathbb{W}^0).$$

An exactly analogous statement holds for $\nabla(\sigma)$ for all σ in the Hilbert case. Then for any analytic weight $s = \prod_{\sigma\in\Sigma} s_\sigma = \prod \exp(v_\sigma \log(\cdot))$ and any p-depleted Hilbert modular form g of weight k, satisfying the conditions on u_σ, v_σ as mentioned in Theorem 2, we show that $\nabla(\sigma)^{s_\sigma}(g)$ defined using a formal expression

$$\nabla(\sigma)^{s_\sigma}(g) := \exp\left(\frac{v_\sigma}{p^{f_\sigma}-1}\log \nabla(\sigma)^{p^{f_\sigma}-1}\right)(g)$$

is actually the limit of a Cauchy sequence in $\mathbb{W}^0$ over the ordinary locus. Finally we manage to show that by shrinking the initial radius of overconvergence, it is possible to realize $\nabla(\sigma)^{s_\sigma}(g)$ as an overconvergent form. Moreover, the Gauss-Manin connection commutes with the U_p operator, and hence slopes are preserved. We can also check on q-expansions that $\nabla(\sigma)$ and $\nabla(\tau)$ commutes for different σ, τ. Then the definition of $\nabla^s = \prod_{\sigma\in\Sigma}\nabla(\sigma)^{s_\sigma}$ is obvious. This then proves the second part of Theorem 2.

Theorem. *Let $k = \prod k_\sigma = \prod \exp(u_\sigma \log(\cdot))$ and $s = \prod s_\sigma = \prod \exp(v_\sigma \log(\cdot))$ be two analytic weights such that $u_\sigma, v_\sigma \in \mathcal{O}^+_{\mathcal{W}_p^0}$. Then for any p-depleted g of weight k, $\nabla^s(g)$ makes sense as an overconvergent Hilbert modular form of weight $k + 2s$.*

Applications:

In their article [BF20], Blanco-Chacon and Fornea construct a twisted triple product p-adic L-function associated to two nearly ordinary families. In the special case of a real quadratic extension $L/\mathbb{Q}$, they relate the special values of their L-function to syntomic Abel-Jacobi images of *generalized Hirzebruch-Zagier cycles*. We hope that our methods can be applied to similarly construct twisted triple product p-adic L-functions associated to finite slope families. This relies on understanding the relationship between two different notions of Hecke operators – one geometric and the other using adelic q-expansions of Hilbert modular forms. In Chapter 3 we review these two notions and also define a notion of overconvergent projection in families similar to [AI21, Definition 3.36].

Further developments:

This work is a contribution to the recent trend in research in number theory trying to p-adically iterate differential operators beyond the ordinary locus [SG19], [EM21]. An interesting case is that when the ordinary locus is empty. It is clear from the description that our construction in the Hilbert case depends on a non-empty ordinary locus. So while this method will most certainly work for Siegel varieties, we need something different for more general PEL-type Shimura varieties. The article [How20] of Sean Howe might provide some insight into these cases.

While preparing this manuscript we came to know that Andrew Graham and Vincent Pilloni had reached similar results [GP] working locally on the moduli space and using the local description of our refined VBMS that we gave above. Moreover working locally they could obtain a splitting modulo any power of p, which meant that they could remove the restriction on weights for the iteration. However they did not have the notion of a global splitting.

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Chapter 1

Overconvergent Modular and de Rham Sheaves for $\mathrm{GL}_{2,\mathbb{Q}}$

1.1 The setup

Notation

Let $N \geq 4$ be an integer, p a prime coprime to N. Let $q = p$ if $p \neq 2$ and $q = 4$ otherwise.

1.1.1 The weight space

Let $\Lambda := \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!] \simeq \mathbb{Z}_p[(\mathbb{Z}/q\mathbb{Z})^\times][\![T]\!]$ be the Iwasawa algebra. Here the second isomorphism is given by sending $\exp(q) \mapsto 1 + T$. Let $\Lambda^0 = \mathbb{Z}_p[\![T]\!]$ be the quotient of Λ that sends $(\mathbb{Z}/q\mathbb{Z})^\times \mapsto 1$. Let $\mathfrak{W} := \mathrm{Spf}\,\Lambda$ and $\mathfrak{W}^0 := \mathrm{Spf}\,\Lambda^0$. The formal scheme $\mathfrak{W}$ is called the formal weight space because it satisfies the following universal property. For any p-adically complete $\mathbb{Z}_p$-algebra R,

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathbb{Z}_p\text{-alg}}(\Lambda, R) \simeq \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{Z}}(\mathbb{Z}_p^\times, R^\times).$$

The subscheme $\mathfrak{W}^0$ is the connected component of the trivial character. Let $\mathcal{W} := \mathrm{Spa}(\Lambda, \Lambda)^{\mathrm{an}}$ be the analytic adic space associated to Λ, and similarly define $\mathcal{W}^0 := \mathrm{Spa}(\Lambda^0, \Lambda^0)^{\mathrm{an}}$.

$\mathcal{W}$ is a disjoint union of copies of $\mathcal{W}^0$ indexed by $(\mathbb{Z}/q\mathbb{Z})^\times$. There is a continuous, surjective map of topological spaces $\kappa\colon |\mathcal{W}^0| \to [0, \infty]$ defined as follows. For any point $x \in \mathcal{W}^0$, let $\tilde{x}$ be the unique rank 1 generization of x. Then define

$$\kappa(x) := \frac{\log |p|_{\tilde{x}}}{\log |T|_{\tilde{x}}}.$$

Here we followed [SW20, p. 30] (but our κ is the reciprocal of loc. cit. because we will follow the notation of [AI21]). As a continuous map κ is uniquely characterized by the following property: $\kappa(x) = r$ if and only if for any rational $m/n < r$, $|p|_x^n \leq |T|_x^m$ and for any rational $m/n > r$, $|p|_x^n \geq |T|_x^m$. We note that both numerator and denominator of κ take values in $[-\infty, 0)$. Hence $\kappa(x) = 0$ iff $|T|_x = 0$

and $\kappa(x) = \infty$ iff $|p|_x = 0$. These two points correspond to the p-adic and T-adic valuations on $\mathbb{Q}_p$ and $\mathbb{F}_p(\!(T)\!)$ respectively. For any interval $I \subset [0, \infty]$, define $\mathcal{W}_I^0$ to be the interior of $\kappa^{-1}(I)$. For $I = [p^a, p^b]$ for some $a \in \mathbb{N} \cup \{-\infty\}$ and $b \in \mathbb{N} \cup \{\infty\}$,

$$\mathcal{W}_I^0 = \{x \in \mathcal{W}^0 \ : \ |p|_x \leq |T^{p^a}|_x \neq 0 \text{ and } |T^{p^b}|_x \leq |p|_x \neq 0\}.$$

There are two notable cases of I to consider: $I = [0, p^b]$ for some $b \in \mathbb{N}$ and $I = [p^a, p^b]$ for some $a \in \mathbb{N}$ and $b \in \mathbb{N} \cup \{\infty\}$. The first case gives an affinoid neighbourhood of the point $\{T = 0\}$. The adic open unit disc $\{|T| < 1\}$ over $\mathbb{Q}_p$ is $\mathcal{W}_{[0,\infty)}^0$ which has a cover by affinoids $\mathcal{W}_{[0,p^b]}^0$. We have the following description of the two cases.

$$\mathcal{W}_{[0,p^b]}^0 = \{x \in \mathcal{W}^0 \mid |T^{p^b}|_x \leq |p|_x \neq 0\} = \operatorname{Spa}\left(\Lambda^0\left\langle\frac{T^{p^b}}{p}\right\rangle\left[\frac{1}{p}\right], \Lambda^0\left\langle\frac{T^{p^b}}{p}\right\rangle\right).$$

$$\begin{aligned}\mathcal{W}_{[p^a,p^b]}^0 &= \{x \in \mathcal{W}^0 \mid |p|_x \leq |T^{p^a}|_x \neq 0 \text{ and } |T^{p^b}|_x \leq |p|_x \neq 0\} \\ &= \operatorname{Spa}\left(\Lambda^0\left\langle\frac{p}{T^{p^a}}, \frac{T^{p^b}}{p}\right\rangle\left[\frac{1}{T}\right], \Lambda^0\left\langle\frac{p}{T^{p^a}}, \frac{T^{p^b}}{p}\right\rangle\right).\end{aligned}$$

For I as above, we let $\mathcal{W}_I$ the componentwise union of $\mathcal{W}_I^0$.

We let $\Lambda_I^0 := \Gamma(\mathcal{W}_I^0, \mathcal{O}_{\mathcal{W}_I^0}^+)$ and $\Lambda_I := \Gamma(\mathcal{W}_I, \mathcal{O}_{\mathcal{W}_I}^+)$. Let $\mathfrak{W}_I^0 := \operatorname{Spf} \Lambda_I^0$ and $\mathfrak{W}_I := \operatorname{Spf} \Lambda_I$.

The affinoid adic spaces described above are adic spectra of Tate rings. We choose a pseudouniformiser for each of the two cases considered above. For $I = [0, p^b]$ we let $\alpha = p$ and for $I = [p^a, p^b]$ we let $\alpha = T$.

Analyticity of the universal character:

Let $k^{\mathrm{un}} \colon \mathbb{Z}_p^\times \to \Lambda$ be the universal character. Denote by $k^0 \colon \mathbb{Z}_p^\times \to \Lambda^0$ the character obtained by composing k^{un} with the projection onto the component of the trivial character. Let $k_I^0 \colon \mathbb{Z}_p^\times \to \Lambda_I^0$ be its restriction to $\mathcal{W}_I^0$.

Lemma 1.1.1. *For $I \subset [0, q^{-1}p^n]$, the restriction of k_I^0 to $1 + qp^{n-1}\mathbb{Z}_p$ is analytic. Thus it extends to a character*

$$k_I^0 \colon \mathcal{W}_I^0 \times \mathbb{Z}_p^\times(1 + qp^{n-1}\mathbb{G}_a^+) \to \mathbb{G}_m^+,$$

which restricts to a character

$$k_I^0 \colon \mathcal{W}_I^0 \times (1 + qp^{n+m-1}\mathbb{G}_a^+) \to 1 + qp^m\mathbb{G}_a^+.$$

Proof. See [AIP18, Proposition 2.1]. □

1.1.2 The modular curve

Let $N \geq 4$ be an integer and p a prime coprime to N. Let $Y = Y_1(N)/\mathbb{Z}_p$ be the moduli scheme of elliptic curves with level $\Gamma_1(N)$-structure. Let $X = X_1(N)$ be its smooth, proper compactification that classifies generalised elliptic curves with $\Gamma_1(N)$-level structure. Let $\mathfrak{X}$ be the p-adic completion of X and let $\pi\colon \mathcal{E} \to \mathfrak{X}$ be the universal semi-abelian scheme. Let $\omega_{\mathcal{E}}$ be the canonical extension of the sheaf of invariant differentials on the universal elliptic curve to the cusps. It is a line bundle. Let $\mathrm{H}_{\mathcal{E}}$ be the canonical extension of the relative de Rham sheaf of the universal elliptic curve to the cusps. It is a vector bundle of rank 2. $\mathrm{H}_{\mathcal{E}}$ is equipped with

1. Hodge filtration: $0 \to \omega_{\mathcal{E}} \to \mathrm{H}_{\mathcal{E}} \to \omega_{\mathcal{E}}^{\vee} \to 0$,
2. Gauss-Manin connection: $\nabla\colon \mathrm{H}_{\mathcal{E}} \to \mathrm{H}_{\mathcal{E}} \hat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \Omega^1_{\mathfrak{X}/\mathbb{Z}_p}(\log(\text{cusps}))$.

We note that away from the cusps, $\mathrm{H}_{\mathcal{E}}$ can be identified with the contravariant Dieudonné module of the p-divisible group of the universal elliptic curve.

Let $i\colon X_{\mathbb{F}_p} \hookrightarrow \mathfrak{X}$ be the closed immersion defined by $p = 0$. The Hasse invariant is a section $\mathrm{Ha} \in i^*\omega_{\mathcal{E}}^{\otimes(p-1)}$. Define the Hasse ideal to be $\underline{\mathrm{Ha}} := \mathrm{Ha} \cdot i^*\omega_{\mathcal{E}}^{\otimes(1-p)}$.

We recall the following theorem of Igusa.

Theorem 1.1.1 (Igusa)**.** *The Hasse invariant has simple zeroes.*

Proof. See [KM85, Theorem 12.4.3]. □

1.1.2.1 Frobenius and Verschiebung

For any $\mathbb{F}_p$-scheme S, the absolute Frobenius $F_{\text{abs}}\colon S \to S$ is the map of schemes that is identity on the underlying topological space and induces $x \mapsto x^p$ on the structure sheaf.

Given a map of $\mathbb{F}_p$-schemes $X \to S$, the absolute Frobenius for X and S sit in a commutative diagram as follows, simply because any map of $\mathbb{F}_p$-algebras commute with the Frobenius ring map $x \mapsto x^p$.

$$\begin{tikzcd} X \arrow[r, "F_{\text{abs}}"] \arrow[d] & X \arrow[d] \\ S \arrow[r, "F_{\text{abs}}"'] & S \end{tikzcd}$$

For any such map of $\mathbb{F}_p$-schemes $X \to S$, we denote by $X^{(p)}$ the base change of X along $F_{\text{abs}}\colon S \to S$. Thus the following diagram is Cartesian.

$$\begin{tikzcd} X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r, "F_{\text{abs}}"'] & S \end{tikzcd}$$

Then by definition of the fibre product, the absolute Frobenius of X, $F_{\text{abs}}\colon X \to X$ induces a unique morphism $F_{X/S}\colon X \to X^{(p)}$ such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[dr, "F_{X/S}"] \arrow[drr, bend left, "F_{\text{abs}}"] \arrow[ddr, bend right] & & \\
& X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\
& S \arrow[r, "F_{\text{abs}}"'] & S
\end{tikzcd}$$

Definition 1.1.1. The arrow $F_{X/S}$ as above is defined to be the relative Frobenius of X with respect to S.

Proposition 1.1.1. *For any abelian variety $A \to S$ of relative dimension g over a $\mathbb{F}_p$-scheme S, the relative Frobenius $F_{A/S}$ is an isogeny that is universally injective of degree p^g.*

Proof. [EGM, Proposition 5.15]. □

For any flat commutative group scheme G/S where S is a $\mathbb{F}_p$-scheme, one can define a homomorphism $V_{G/S}\colon G^{(p)} \to G$ of S-group schemes called the Verschiebung. For the detailed construction and definition of this map we refer the reader to §4 of [Art+65, VII_{A}] or to [EGM, §5.2]. Here we record the properties of this homomorphism, quoting from [EGM, §5.2].

Proposition 1.1.2. *Let S be a $\mathbb{F}_p$-scheme. For a flat commutative group scheme G/S, we have*

1. *$V_{G/S} \circ F_{G/S} = [p]\colon G \to G$.*
2. *If G is finite flat over S, then Verschiebung is the Cartier dual to the relative Frobenius, i.e. $V_{G/S} = (F_{G^\vee/S})^\vee$.*

Proof. [EGM, Proposition 5.19]. □

Proposition 1.1.3. *For any abelian variety $A \to S$ of relative dimension g over a $\mathbb{F}_p$-scheme S, the Verschiebung $V_{A/S}$ is an isogeny of degree p^g such that $V_{A/S} \circ F_{A/S} = [p]_A$ and $F_{A/S} \circ V_{A/S} = [p]_{A^{(p)}}$.*

Proof. [EGM, Proposition 5.20]. □

In the following we often drop the subscript from the notation of Frobenius and Verschiebung and simply write F and V respectively when the abelian variety in consideration is clear.

1.1.3 The partial Igusa tower

Fix an $I = [p^a, p^b]$. Let $\mathfrak{X}_I := \mathfrak{X} \times_{\mathrm{Spf}\, Z_p} \mathfrak{W}_I^0$. We abuse notation to denote by $\underline{\mathrm{Ha}}$ the Hasse ideal inside $\mathcal{O}_{\mathfrak{X}_I}/(\alpha)$ where we recall α is the chosen pseudouniformiser depending on I. For every $r \geq 1$, consider the inverse image of $\underline{\mathrm{Ha}}^{p^{r+1}}$ under the map $\mathcal{O}_{\mathfrak{X}_I} \to \mathcal{O}_{\mathfrak{X}_I}/(\alpha)$ and call this ideal Hdg_r. Locally on Spf $R \subset \mathfrak{X}_I$ where $\omega_{\mathcal{E}}$ admits a generator ω, and Ha admits a lift $\widetilde{\mathrm{Ha}}$ (say), let $\mathrm{Hdg} = \widetilde{\mathrm{Ha}}(E/R, \omega)$. Here E/R is the pullback of $\mathcal{E}/\mathfrak{X}_I$ to Spf R. Then $\mathrm{Hdg}_r|_{\mathrm{Spf}\, R} = (\alpha, \mathrm{Hdg}^{p^{r+1}})$.

Let $g_r \colon \mathfrak{X}_{r,I} \to \mathfrak{X}_I$ be the admissible blow-up of $\mathfrak{X}_I$ with respect to the ideal Hdg_r, where the inverse image ideal is generated locally by $\mathrm{Hdg}^{p^{r+1}}$. For any integer n with $1 \leq n \leq r$ if $I = [0, 1]$ and $1 \leq n \leq a + r$ if $I = [p^a, p^b]$, let $\lambda = \mathrm{Hdg}^{\frac{p^n-1}{p-1}}$. Note that $\frac{p}{\lambda} \in \mathcal{O}_{\mathfrak{X}_{r,I}}$.

Proposition 1.1.4. *For I, r, n as above, the semiabelian scheme $\mathcal{E} \to \mathfrak{X}_{r,I}$ admits a canonical subgroup H_n of order p^n. This is a finite, locally free subgroup scheme that satisfies the following properties:*

1. *H_n lifts $\ker F^n$ modulo p/λ,*
2. *For any α-adically complete admissible Λ_I^0-algebra R, together with a morphism $f\colon$ Spf $R \to \mathfrak{X}_{r,I}$,*
$$H_n(R) = \{s \in \mathcal{E}[p^n](R) \mid s \bmod p/\lambda \in \ker F^n\},$$
3. *Suppose $L_n = \mathcal{E}[p^n]/H_n$. Then ω_{L_n} is killed by λ and we have $\omega_{L_n} \simeq \omega_{\mathcal{E}}/\lambda\omega_{\mathcal{E}}$,*
4. *$\mathcal{E}[p^n]/H_n \simeq H_n^\vee$ through the Weil pairing and it is étale over the adic generic fibre $\mathcal{X}_{r,I}$ of $\mathfrak{X}_{r,I}$.*

Proof. For a proof of these facts about the canonical subgroup we refer the reader to [AIP18, Appendice A]. □

Definition 1.1.2. For I, r, n as above define $\mathcal{IG}_{n,r,I} \to \mathcal{X}_{r,I}$ as the adic space classifying isomorphisms $\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\sim} H_n^\vee$. Define $\mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,I}$ to be the normalisation of $\mathfrak{X}_{r,I}$ in $\mathcal{IG}_{n,r,I}$.

Proposition 1.1.5. *$\mathcal{IG}_{n,r,I} \to \mathcal{X}_{r,I}$ is an étale, Galois extension with Galois group $(\mathbb{Z}/p^n\mathbb{Z})^\times$. $\mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,I}$ is well-defined, a finite morphism and is endowed with an action of $(\mathbb{Z}/p^n\mathbb{Z})^\times$.*

Proof. The first statement is obvious. The second statement uses finiteness properties of relative normalisation of excellent rings. For the proof we refer to [AIP18, Lemme 3.2]. The Galois action of $(\mathbb{Z}/p^n\mathbb{Z})^\times$ on $\mathcal{IG}_{n,r,I}$ induces an action on $\mathfrak{IG}_{n,r,I}$ over $\mathfrak{X}_{r,I}$ by the universal property of relative normalisation. □

1.2 Splitting of de Rham sheaf

In the following we put an overline on the names of objects (semiabelian schemes, sheaves, morphisms etc.) to denote they are obtained by base change along the closed immersion $i\colon X_{\mathbb{F}_p} \hookrightarrow \mathfrak{X}$. So we have a morphism $\bar{\pi}\colon \bar{\mathcal{E}} \to X_{\mathbb{F}_p}$. Let $\bar{\omega}_{\mathcal{E}} := i^*\omega_{\mathcal{E}} = \omega_{\bar{\mathcal{E}}}$. The Verschiebung $V\colon \bar{\mathcal{E}}^{(p)} \to \bar{\mathcal{E}}$ induces a map on the Lie algebra $\mathrm{Lie}(V)\colon \omega^\vee_{\bar{\mathcal{E}}^{(p)}} \to \omega^\vee_{\bar{\mathcal{E}}}$ which gives the Hasse invariant Ha seen as an element in $\omega_{\bar{\mathcal{E}}}^{\otimes(p-1)}$. Denote by $\bar{\mathrm{H}}_{\mathcal{E}} := i^*\mathrm{H}_{\mathcal{E}}$. Let $j\colon X^{\mathrm{ord}}_{\mathbb{F}_p} \hookrightarrow X_{\mathbb{F}_p}$ be the ordinary locus, which is the open subscheme where $\underline{\mathrm{Ha}} = \mathcal{O}_{X_{\mathbb{F}_p}}$.

Let $\varphi\colon X_{\mathbb{F}_p} \to X_{\mathbb{F}_p}$ be the absolute Frobenius. The Frobenius induces a φ-linear endomorphism of $\bar{\mathrm{H}}_{\mathcal{E}}$.

Proposition 1.2.1. *Over the ordinary locus $X^{ord}_{\mathbb{F}_p}$, we have the unit root splitting which is a canonical splitting $\psi_{Frob}\colon j^*\bar{\mathrm{H}}_{\mathcal{E}} \to j^*\bar{\omega}_{\mathcal{E}}$ of the Hodge filtration that respects the Frobenius action. The kernel of ψ_{Frob} is called the unit root subspace. It is characterized by the property that it is stable under the Frobenius action and Frobenius acts invertibly on it.*

Proof. Suppose $\operatorname{Spec} R \subset X_{\mathbb{F}_p}$ is a Zariski local chart for which $\bar{\omega}_{\mathcal{E}}$ and $\bar{\mathrm{H}}_{\mathcal{E}}$ are trivial. Choose a basis $\{e, f\}$ of $\bar{\mathrm{H}}_{\mathcal{E}}$ compatible with the Hodge filtration. With respect to such a basis we can write the matrix of the Frobenius action on $\bar{\mathrm{H}}_{\mathcal{E}}$ as follows.

$$\mathrm{Frob} = \begin{pmatrix} 0 & C \\ 0 & \mathrm{Ha} \end{pmatrix}$$

Here we abuse notation to write Ha for a generator of $\underline{\mathrm{Ha}}$ obtained by evaluating the Hasse invariant at the chosen generator of $\bar{\omega}_{\mathcal{E}}$. For the change of basis of $j_*j^*\bar{\mathrm{H}}_{\mathcal{E}}(R) = \bar{\mathrm{H}}_{\mathcal{E}}(R)[1/\mathrm{Ha}]$, given by the matrix

$$P = \begin{pmatrix} 1 & C\mathrm{Ha}^{-1} \\ 0 & 1 \end{pmatrix}$$

the matrix of Frobenius becomes

$$P^{-1}\mathrm{Frob}P = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Ha} \end{pmatrix}.$$

That is, the basis of $\bar{\mathrm{H}}_{\mathcal{E}}(R)[1/\mathrm{Ha}]$ given by $\{e, (C\mathrm{Ha}^{-1})e + f\}$ satisfies the property that the matrix of Frobenius with respect to it is $P^{-1}\mathrm{Frob}P$ as above. Hence we have a splitting as claimed. □

Consider the map $\psi\colon \bar{\mathrm{H}}_{\mathcal{E}} \to j_*j^*\bar{\mathrm{H}}_{\mathcal{E}} \xrightarrow{\psi_{\mathrm{Frob}}} j_*j^*\bar{\omega}_{\mathcal{E}}$. Here the first arrow is the unit of the adjunction, which in local chart $\operatorname{Spec} R \subset X_{\mathbb{F}_p}$ as in the above proof, is simply the inclusion of $\bar{\mathrm{H}}_{\mathcal{E}}(R)$ in its localization $\bar{\mathrm{H}}_{\mathcal{E}}(R)[1/\mathrm{Ha}]$. Then let $\bar{\mathrm{H}}'_{\mathcal{E}} := \psi^{-1}\bar{\omega}_{\mathcal{E}}$. The inclusion $\bar{\omega}_{\mathcal{E}} \to \bar{\mathrm{H}}'_{\mathcal{E}}$ admits a retraction by ψ.

Lemma 1.2.1. *The sheaf $\bar{\mathrm{H}}'_{\mathcal{E}}$ sits in the following split exact sequence:*

$$0 \to \bar{\omega}_{\mathcal{E}} \to \bar{\mathrm{H}}'_{\mathcal{E}} \to \underline{\mathrm{Ha}} \cdot \bar{\omega}^{\vee}_{\mathcal{E}} \to 0 \tag{1.1}$$

Proof. Choose a local chart $\operatorname{Spec} R$ as above. Explicitly, suppose e, f form an R-basis of $\bar{\mathrm{H}}_{\mathcal{E}}$ such that e spans $\bar{\omega}_{\mathcal{E}}$ and the image of f spans $\bar{\omega}^{\vee}_{\mathcal{E}}$. Also assume that the matrix of Frobenius with respect to this basis is given as above. The map $\psi_{\mathrm{Frob}}\colon \bar{\mathrm{H}}_{\mathcal{E}}[1/\mathrm{Ha}] \to \bar{\omega}_{\mathcal{E}}[1/\mathrm{Ha}]$ sends $e \mapsto e$ and $(C\mathrm{Ha}^{-1})e + f \mapsto 0$. Then $\psi(e) = e$ and $\psi(f) = (\mathrm{Id} - P)(f) = -C\mathrm{Ha}^{-1}e$. Now Ha is a uniformiser of the local ring at any supersingular point. Moreover since the unit root splitting does not extend beyond the ordinary locus, $C\mathrm{Ha}^{-1}$ does not belong to the local ring of any supersingular point. Thus $\psi^{-1}\bar{\omega}_{\mathcal{E}} = Re \oplus R\mathrm{Ha}f$. This proves the lemma. □

Corollary 1.2.1. $\bar{\mathrm{H}}'_{\mathcal{E}} = \bar{\omega}_{\mathcal{E}} \oplus \mathrm{Frob}(\bar{\mathrm{H}}_{\mathcal{E}})$, *where by* $\mathrm{Frob}(\bar{\mathrm{H}}_{\mathcal{E}})$ *we mean the $\mathcal{O}_{X_{\mathbb{F}_p}}$-linear span of the image of* Frob.

Proof. We note that by the proof of Lemma 1.2.1, on the local chart $\operatorname{Spec} R$, we have $\psi^{-1}\bar{\omega}_{\mathcal{E}}(R) = Re \oplus R\mathrm{Ha}f$, which is the same as $Re \oplus R(Ce \oplus \mathrm{Ha}f)$. Now $R(Ce + \mathrm{Ha}f) = \mathrm{Frob}(\bar{\mathrm{H}}_{\mathcal{E}})(R)$. Hence locally we have the splitting as claimed which then glues to give a global splitting since $\mathrm{Frob}(\bar{\mathrm{H}}_{\mathcal{E}})$ is globally defined. $\square$

In the following we will construct a locally free subsheaf $\mathrm{H}'_{\mathcal{E}} \subset \mathrm{H}_{\mathcal{E}}$ over $\mathfrak{X}_{r,I}$, together with the induced Hodge filration, such that its reduction modulo a small power of p will give us the split exact sequence (1.1).

Let $i\colon \mathfrak{X}_I/(p) \hookrightarrow \mathfrak{X}_I$ be the base change of $X_{\mathbb{F}_p} \hookrightarrow \mathfrak{X}$ to $\mathfrak{X}_I$. Let $i_0\colon \mathfrak{X}_{r,I}/(p\mathrm{Hdg}^{-1}) \hookrightarrow \mathfrak{X}_{r,I}$ be the closed subscheme defined by the ideal $p\mathrm{Hdg}^{-1}$. Then we have a commutative diagram as follows:

$$\begin{tikzcd}
\mathfrak{X}_{r,I}/(p\mathrm{Hdg}^{-1}) \arrow[r, "i_0"] \arrow[d, "q"] & \mathfrak{X}_{r,I} \arrow[d, "g_r"] \\
\mathfrak{X}_I/(p) \arrow[r, "i"] & \mathfrak{X}_I
\end{tikzcd}$$

Definition 1.2.1. Define $\mathrm{H}'_{\mathcal{E}}$ to be the inverse image of $\mathrm{Hdg} \cdot \omega_{\mathcal{E}}^{\vee}$ under the projection $\mathrm{H}_{\mathcal{E}} \to \omega_{\mathcal{E}}^{\vee}$, as sheaves on $\mathfrak{X}_{r,I}$.

We have the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \omega_{\mathcal{E}} \arrow[r] \arrow[d, equal] & \mathrm{H}'_{\mathcal{E}} \arrow[r] \arrow[d] & \mathrm{Hdg} \cdot \omega_{\mathcal{E}}^{\vee} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \omega_{\mathcal{A}} \arrow[r] & \mathrm{H}_{\mathcal{E}} \arrow[r] & \omega_{\mathcal{E}}^{\vee} \arrow[r] & 0
\end{tikzcd}$$

Proposition 1.2.2. *There is an isomorphism of $\mathcal{O}_{\mathfrak{X}_{r,I}/(p\mathrm{Hdg}^{-1})}$-modules $i_0^*(\mathrm{Hdg} \cdot \omega_{\mathcal{E}}^{\vee}) \xrightarrow{\sim} q^*(\underline{\mathrm{Ha}} \cdot \bar{\omega}_{\mathcal{E}}^{\vee})$, that commutes with the induced maps to $\bar{\omega}_{\mathcal{E}}^{\vee}$. (We abuse notation to denote $q^*\bar{\omega}_{\mathcal{E}}^{\vee}$ by $\bar{\omega}_{\mathcal{E}}^{\vee}$.)*

$$\begin{tikzcd}
i_0^*(\mathrm{Hdg} \cdot \omega_{\mathcal{E}}^{\vee}) \arrow[r, "\sim"] \arrow[d] & q^*(\underline{\mathrm{Ha}} \cdot \bar{\omega}_{\mathcal{E}}^{\vee}) \arrow[d] \\
\bar{\omega}_{\mathcal{E}}^{\vee} \arrow[r, phantom, "="] & \bar{\omega}_{\mathcal{E}}^{\vee}
\end{tikzcd}$$

Proof. Let $\tilde{\omega}_{\mathcal{E}}^{\vee}$ be the inverse image of $\underline{\mathrm{Ha}} \cdot \bar{\omega}_{\mathcal{E}}^{\vee}$ under the map $\omega_{\mathcal{E}}^{\vee} \to i_* \bar{\omega}_{\mathcal{E}}$ as sheaves over $\mathfrak{X}_I$. Thus Zariski locally over $\mathfrak{X}_I$, $\tilde{\omega}_{\mathcal{E}}^{\vee} = \widetilde{\mathrm{Hdg}}\omega_{\mathcal{E}}^{\vee} + p\omega_{\mathcal{E}}^{\vee}$ for a local lift $\widetilde{\mathrm{Hdg}}$ of a generator of the Hasse ideal $\underline{\mathrm{Ha}}$. Then we note that $\mathrm{Hdg} \cdot \omega_{\mathcal{E}}^{\vee}$ is the image of the map $g_r^*\tilde{\omega}_{\mathcal{E}}^{\vee} \to g_r^*\omega_{\mathcal{E}}^{\vee}$ since $p\mathrm{Hdg}^{-1} \in \mathcal{O}_{\mathfrak{X}_{r,I}}$. The natural surjective map $i^*\tilde{\omega}_{\mathcal{E}}^{\vee} \to \underline{\mathrm{Ha}} \cdot \bar{\omega}_{\mathcal{E}}^{\vee}$ induces by pullback a surjective map $q^*i^*\tilde{\omega}_{\mathcal{E}}^{\vee} = i_0^*g_r^*\tilde{\omega}_{\mathcal{E}}^{\vee} \to q^*(\underline{\mathrm{Ha}} \cdot \bar{\omega}_{\mathcal{E}}^{\vee})$. We will show that this map factors naturally as $i_0^*g_r^*\tilde{\omega}_{\mathcal{E}}^{\vee} \to i_0^*(\mathrm{Hdg} \cdot \omega_{\mathcal{E}}^{\vee}) \to q^*(\underline{\mathrm{Ha}} \cdot \bar{\omega}_{\mathcal{E}}^{\vee})$. As surjective maps between line bundles, the second arrow will be an isomorphism. This will be the desired isomorphism.

Choose a local chart $\mathrm{Spf}\, R = U \subset \mathfrak{X}_I$ that trivializes $\omega_{\mathcal{E}}$. Let v be a generator of $\omega_{\mathcal{E}}^{\vee}$ over $\mathrm{Spf}\, R$. Abusing notation, denote by Hdg a generator of the ideal over $\mathrm{Spf}\, R$. Then $\tilde{\omega}_{\mathcal{E}\,|U}^{\vee} = \mathcal{O}_U \mathrm{Hdg} v + \mathcal{O}_U p v$. Thus there is a surjective map $\mathcal{O}_U^2 \to \tilde{\omega}^\vee_{\mathcal{E}|U}$ sending $e_1 \mapsto \mathrm{Hdg}v$ and $e_2 \mapsto pv$. Since $w := i^*(\mathrm{Hdg}v)$ is a basis of $\underline{\mathrm{Ha}} \cdot \bar{\omega}^\vee_{\mathcal{E}}$ over Spf $R/(p)$, we have an isomorphism $\mathcal{O}_{i^{-1}U} \to \underline{\mathrm{Ha}} \cdot \bar{\omega}^\vee_{\mathcal{E}|i^{-1}U}$, that sends $e_1 \mapsto w$. Let $V = g_r^{-1}U$. Then we have a surjective map $M\colon \mathcal{O}^2_{i_0^{-1}V} \to \mathcal{O}_{i_0^{-1}V}$ given by $e_1 \mapsto e_1$ and $e_2 \mapsto 0$ that induces the map $i_0^* g_r^* \tilde{\omega}^\vee_{\mathcal{E}} \to q^*(\underline{\mathrm{Ha}} \cdot \bar{\omega}^\vee_{\mathcal{E}})$. On the other hand the image of $g_r^*(\mathrm{Hdg}v)$ is a basis for $\mathrm{Hdg}\omega^\vee_{\mathcal{E}}$. Then we have a surjective map $N\colon \mathcal{O}_V^2 \to \mathcal{O}_V$ sending $e_1 \mapsto e_1$ and $e_2 \mapsto \frac{p}{\mathrm{Hdg}} e_1$ that induces the surjection $g_r^* \tilde{\omega}^\vee_{\mathcal{E}} \to \mathrm{Hdg}\omega^\vee_{\mathcal{E}}$. Now it's obvious that $M = i_0^* N$.

Summarising the above, there is a surjective map $f_0\colon \mathcal{O}^2_{i_0^{-1}V} \to i_0^* g_r^* \tilde{\omega}^\vee_{\mathcal{E}|i_0^{-1}V}$. There are isomorphisms $f_1\colon \mathcal{O}_{i_0^{-1}V} \xrightarrow{\sim} i_0^*(\mathrm{Hdg} \cdot \omega^\vee_{\mathcal{E}})_{|i_0^{-1}V}$ and $f_2\colon \mathcal{O}_{i_0^{-1}V} \xrightarrow{\sim} q^*(\underline{\mathrm{Ha}} \cdot \bar{\omega}^\vee_{\mathcal{E}})_{|i_0^{-1}V}$. There is a map $M\colon \mathcal{O}^2_{i_0^{-1}V} \to \mathcal{O}_{i_0^{-1}V}$ that induces the natural map $i_0^* g_r^* \tilde{\omega}^\vee_{\mathcal{E}|i_0^{-1}V} \to q^*(\underline{\mathrm{Ha}} \cdot \bar{\omega}^\vee_{\mathcal{E}})_{|i_0^{-1}V}$ with respect to f_0 and f_2. There is a map $i_0^* N\colon \mathcal{O}^2_{i_0^{-1}V} \to \mathcal{O}_{i_0^{-1}V}$ that induces the natural map $i_0^* g_r^* \tilde{\omega}^\vee_{\mathcal{E}|i_0^{-1}V} \to i_0^*(\mathrm{Hdg} \cdot \omega^\vee_{\mathcal{E}})_{|i_0^{-1}V}$ with respect to f_0 and f_1. Moreover, $M = i_0^* N$. This proves the proposition. ☐

Proposition 1.2.3. *The pullback of the exact sequence*

$$0 \to \omega_{\mathcal{E}} \to \mathrm{H}'_{\mathcal{E}} \to \mathrm{Hdg} \cdot \omega^\vee_{\mathcal{E}} \to 0$$

along $i_0\colon \mathfrak{X}_{r,I}/(p\mathrm{Hdg}^{-1}) \hookrightarrow \mathfrak{X}_{r,I}$ *admits a canonical splitting induced by the splitting of (1.1).*

Proof. This is immediate from Proposition 1.2.2. ☐

1.3 p-adic interpolation of modular and de Rham sheaves

Henceforth fix n a positive integer. Fix $I = [p^a, p^b]$ such that k_I^0 is analytic on $1 + p^{n-1}\mathbb{Z}_p$ and r such that H_n is defined on $\mathfrak{X}_{r,I}$. Depending on the two cases $I = [0,1]$ or $I = [p^a, p^b]$ for $a, b \in \mathbb{N}$, these conditions are satisfied if

1. $I = [0,1]$, $r \geq 2$ if $p \neq 2$ and $2 \leq n \leq r$, or $r \geq 4$ if $p = 2$ and $4 \leq n \leq r$,
2. $I = [p^a, p^b]$ with $a, b \in \mathbb{N}$, $r \geq 1$ and $r + a \geq b + 2$ if $p \neq 2$ and $b + 2 \leq n \leq a + r$, or $r \geq 2$ and $r + a \geq b + 4$ if $p = 2$, and $b + 4 \leq n \leq r + a$.

In the article [AI21], the authors construct overconvergent modular and de Rham sheaves, denoted $\mathfrak{w}_{k,I}$ and $\mathbb{W}_{k,I}$ respectively for the universal character $k\colon \mathbb{Z}_P^\times \to \Lambda_I$. These are coherent sheaves on $\mathfrak{X}_{r,I}$ that interpolate $\omega^k_{\mathcal{E}}$ and $\mathrm{Sym}^k \mathrm{H}_{\mathcal{E}}$ for any classical character $k \in \mathbb{Z}$. Moreover, let $k^f\colon (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{Z}_p^\times \to \Lambda$ be the torsion part of the character, and set $\mathfrak{w}_k^f := h_* \mathcal{O}_{\mathfrak{IG}_{i,r,I}}[k^f]$, where $h\colon \mathfrak{IG}_{1,r,I} \to \mathfrak{X}_{r,I}$ for $p \neq 2$ and $h\colon \mathfrak{IG}_{2,r,I} \to \mathfrak{X}_{r,I}$ for $p = 2$. Then $\mathfrak{w}_{k,I} = \mathfrak{w}^0_{k,I} \hat{\otimes} \mathfrak{w}^f_k$ and $\mathbb{W}_{k,I} = \mathbb{W}^0_{k,I} \hat{\otimes} \mathfrak{w}^f_k$ where $\mathfrak{w}^0_{k,I}$ is a line bundle and $\mathbb{W}^0_{k,I}$ is the completion of a filtered direct limit of locally free sheaves.

The construction of the overconvergent modular sheaf is fairly natural and stems from the following observation. Since $\omega_{\mathcal{E}}$ is a line bundle, we can view its isomorphism class as an element of $H^1(\mathfrak{X}, \mathbb{G}_m)$. Now for any classical weight k, the k-th power $\omega^k_{\mathcal{E}}$ corresponds to the image of the class of $\omega_{\mathcal{E}}$ under the map induced by change of structural group $H^1(\mathfrak{X}, \mathbb{G}_m) \xrightarrow{k} H^1(\mathfrak{X}, \mathbb{G}_m)$. Now to do the same for a weight k that is analytic on an open subgroup $1 + p^n\mathbb{Z}_p \subset \mathbb{Z}_p^\times$, one has to consider the change of structural group map induced by $k\colon 1+p^n\mathbb{G}_a \to \widehat{\mathbb{G}_m}$. In particular, one has to work with elements of $H^1(\mathfrak{X}, 1+p^n\mathbb{G}_a)$. Now it is easy to see that the group $H^1(\mathfrak{X}, 1+p^n\mathbb{G}_a)$ classifies line bundles $\mathscr{L}$ on $\mathfrak{X}$ together with an isomorphism $\mathcal{O}_{\mathfrak{X}}/(p^n) \xrightarrow{\sim} \mathscr{L}/p^n\mathscr{L}$, in other words a "marked section" modulo p^n. This motivates the study of vector bundles with marked sections. What [AI21] shows is that over $\mathfrak{IG}_{n,r,I}$, there is a line bundle $\Omega_{\mathcal{E}} \subset \omega_{\mathcal{E}}$, that is naturally constructed using the theory of canonical subgroups, comes equipped with a marked section modulo $p^n\mathrm{Hdg}^{\frac{-p^n}{p-1}}$, and moreover coincides with $\omega_{\mathcal{E}}$ over the ordinary locus. Then the definition of $\omega_{\mathcal{E}}^k$ follows by the process described above. This construction already appears in their previous works with other authors, and even for Hilbert modular forms. The novelty of [AI21] is in using the theory of vector bundles with marked sections to construct a vector bundle of rank 2, denoted $\mathrm{H}_{\mathcal{E}}^{\sharp} \subset \mathrm{H}_{\mathcal{E}}$, and using it to define p-adic interpolation of the symmetric powers of $\mathrm{H}_{\mathcal{E}}$. These sheaves are then shown to be equipped with a connection ∇, coming from the Gauss-Manin connection on $\mathrm{H}_{\mathcal{E}}$, and they prove p-adic iteration of ∇ under some restrictions on the weight of iteration and on k. However, the definition of $\mathrm{H}_{\mathcal{E}}^{\sharp}$ looks ad hoc at first sight and that leads to complicated computations in the proof of the existence of p-adic iteration of ∇.

In this section, we will show that in fact using our technique of the splitting of de Rham sheaf as developed in Section 1.2, the definition of $\mathrm{H}_{\mathcal{E}}^{\sharp}$ becomes quite natural. Moreover, this added insight allows us to give a new definition of the interpolation sheaf $\mathbb{W}_{k,I}$ which has better convergence properties for the iteration of ∇. In particular, proving p-adic iteration of ∇ with this new definition is much simpler and we can also slightly relax the condition on the weight of iteration. The main theorem of this section is the following.

Theorem. *For n, r, I as above, and k the universal weight on $\mathfrak{W}_I^0$, there are formal sheaves $\mathfrak{w}'_{k,I}$ and $\mathbb{W}'_{k,I}$ on $\mathfrak{IG}_{n,r,I}$. $\mathbb{W}'_{k,I}$ is equipped with an increasing filtration by locally free subsheaves $\mathrm{Fil}_i\mathbb{W}'_{k,I}$, such that $\mathrm{Fil}_0\mathbb{W}'_{k,I} = \mathfrak{w}'_{k,I}$. Moreover, $\mathbb{W}'_{k,I}$ is the p-adic completion of $\varinjlim_i \mathrm{Fil}_i\mathbb{W}'_{k,I}$. If $[0,1] \subset I$, then for any classical weight $m \in \mathbb{N}$ which is a point of $\mathcal{W}_I^0$, we have canonical isomorphisms $\mathfrak{w}'_{k\mapsto m,I}[1/p] \simeq \omega_{\mathcal{E}}^{\otimes m}[1/p]$ and $\mathrm{Fil}_m\mathbb{W}'_{k\mapsto m,I}[1/p] \simeq \mathrm{Sym}^m\,\mathrm{H}_{\mathcal{E}}[1/p]$, where by $k \mapsto m$ we mean specializing at the point corresponding to the classical weight m.*

Note that unlike [AI21] we do not define the interpolation sheaves over $\mathfrak{X}_{r,I}$, but rather over $\mathfrak{IG}_{n,r,I}$. This has been done primarily for simplicity, although one can define the interpolation sheaves over $\mathfrak{X}_{r,I}$ even in our theory. See Remark 1.3.2 for further explanation.

1.3.1 Vector bundles with marked sections and marked splitting

In this section we recall the general formalism of vector bundles with marked sections following [AI21, §2]. We then define a subfunctor of a vector bundle with a marked section, which we call "vector bundle with marked sections and a marked splitting" and show representability of this functor.

Let S be a formal scheme with an ideal of definition $\mathscr{I}$ which is invertible. Let $i\colon S/\mathscr{I} \hookrightarrow S$ be the closed subscheme defined by the ideal $\mathscr{I}$. Let $\mathbf{FSch}_S^{\mathscr{I}}$ be the category of formal schemes $f\colon T \to S$ such that $f^{-1}\mathscr{I}\cdot\mathcal{O}_T$ is an invertible ideal in T.

Definition 1.3.1. A formal vector bundle of rank n is a formal vector group scheme $X \to S$ which is isomorphic to $\mathbb{G}_a^n$ locally over S.

Definition 1.3.2. Let $\mathcal{E}$ be a locally free sheaf of rank n on S. The formal vector bundle $\mathbb{V}(\mathcal{E})$ of rank n

is defined as the functor on $\mathbf{FSch}_S^{\mathscr{I}}$

$$\mathbb{V}(\mathcal{E})(t\colon T \to S) := \mathcal{E}^\vee(T) = \mathrm{Hom}_{\mathcal{O}_T}(t^*\mathcal{E}, \mathcal{O}_T).$$

Lemma 1.3.1. *$\mathbb{V}(\mathcal{E})$ is representable by the formal scheme Spf $\widehat{\mathrm{Sym}^\bullet \mathcal{E}} \to S$, where the completion is with respect to the $\mathscr{I}$-adic topology. This formal scheme is a formal vector bundle of rank n. Moreover, the contravariant functor $\mathbb{V}$ defines an equivalence between locally free sheaves on S of constant rank and formal vector bundles of finite rank over S, and the equivalence preserves the notion of rank.*

Proof. [AI21, Lemma 2.2]. □

Let $\mathcal{E}$ be a locally free sheaf of rank n on S such that $\bar{\mathcal{E}} := i^*\mathcal{E}$ has sections $s_1, \ldots, s_m \in \Gamma(S/\mathscr{I}, \bar{\mathcal{E}})$ for $m \leq n$, satisfying the following two properties:

1. The subsheaf $\bar{\mathcal{E}}' \subset \bar{\mathcal{E}}$ generated by $s_1, \ldots, s_m$ is locally free,
2. $\bar{\mathcal{E}}/\bar{\mathcal{E}}'$ is locally free.

Definition 1.3.3. Let $\mathcal{E}$ be a locally free sheaf with marked sections $s_1, \ldots, s_m$ as above. The formal vector bundle with marked sections $\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)$ is defined as the functor on $\mathbf{FSch}_S^{\mathscr{I}}$

$$\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)(t\colon T \to S) := \{f \in \mathbb{V}(\mathcal{E})(T) \mid (f \bmod t^*\mathscr{I})(t^*s_i) = 1\, \forall i\}$$

Lemma 1.3.2. *$\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)$ is represented by an open formal subscheme of an admissible formal blow-up of $\mathbb{V}(\mathcal{E})$.*

Proof. This is [AI21, Lemma 2.4]. We recall the construction. $s_1, \ldots, s_m$ define an ideal $\mathscr{J} \subset \mathrm{Sym}^\bullet \mathcal{E}/\mathscr{I}$ given by $\mathscr{J} := (s_1 - 1, \ldots, s_m - 1)$. Let $\tilde{\mathscr{J}}$ be the inverse image of $\mathscr{J}$ in $\mathcal{O}_{\widehat{\mathrm{Sym}^\bullet \mathcal{E}}}$. Let $\mathbb{B}$ be the blow-up of $\mathbb{V}(\mathcal{E})$ along $\tilde{\mathscr{J}}$. Then take the $\mathscr{I}$-adic completion of the open in $\mathbb{B}$ where the inverse image ideal of $\tilde{\mathscr{J}}$ coincides with the inverse image ideal of $\mathscr{I}$. Then as shown in loc. cit. this formal scheme represents $\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)$.

In local coordinates $\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)$ can be described as follows. Let Spf $R \subset S$ be an open which trivializes $\mathcal{E}$ and $\mathscr{I}$, and let X_i be a lift of s_i for all i and extend it to a basis of $\mathcal{E}$. Then $\mathbb{V}(\mathcal{E})_{|\mathrm{Spf}\, R} =$ Spf $R\langle X_1, \ldots, X_n\rangle$. Then $\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)_{|\mathrm{Spf}\, R}$ is given by

$$\mathrm{Spf}\, R\langle \{X_i\}_{i=1}^n\rangle\langle\{\frac{X_i - 1}{\alpha}\}_{i=1}^m\rangle \simeq \mathrm{Spf}\, R\langle \{Z_i\}_{i=1}^m, X_{m+1}, \ldots, X_n\rangle$$

where α is a generator for $\mathscr{I}$, and the projection $\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m) \to \mathbb{V}(\mathcal{E})$ corresponds to the ring map that sends $X_i \mapsto 1 + \alpha Z_i$ for $1 \leq i \leq m$ and $X_i \mapsto X_i$ otherwise. □

Let $\mathcal{E}$ be a locally free sheaf of rank n with marked sections $s_1, \ldots, s_m$ as above. Suppose the short exact sequence of locally free sheaves on $S/\mathscr{I}$

$$0 \to \bar{\mathcal{E}}' \to \bar{\mathcal{E}} \to \bar{\mathcal{E}}/\bar{\mathcal{E}}' \to 0$$

admits a splitting $\psi\colon \bar{\mathcal{E}} \to \bar{\mathcal{E}}'$. Here $\bar{\mathcal{E}}'$ is the subsheaf generated by the s_i's as above. Let $\mathcal{Q} := \ker \psi$.

Definition 1.3.4. Given a locally free sheaf $\mathcal{E}$ on S of rank n, marked sections $s_1, \ldots, s_m$, and a subsheaf $\mathcal{Q} \subset \bar{\mathcal{E}}$ corresponding to a splitting as above, we define the vector bundle with marked sections and marked splitting as the functor on $\mathbf{FSch}_S^{\mathscr{I}}$

$$\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m, \mathcal{Q})(t\colon T \to S) := \{f \in \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)(T) \mid (f \bmod t^*\mathscr{I})(t^*\mathcal{Q}) = 0\}$$

Lemma 1.3.3. *$\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m, \mathcal{Q})$ is represented by an open formal subscheme of an admissible formal blow-up of $\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)$.*

Proof. The subsheaf $\mathcal{Q} \subset \bar{\mathcal{E}}$ defines an ideal in $\mathrm{Sym}^\bullet \bar{\mathcal{E}}$, viz. $\mathcal{Q}\,\mathrm{Sym}^\bullet \bar{\mathcal{E}}$. Let $\tilde{\mathcal{Q}}'$ be the inverse image of this ideal in $\widehat{\mathrm{Sym}^\bullet \mathcal{E}}$. If $f\colon \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m) \to \mathbb{V}(\mathcal{E})$ is the projection, let $\tilde{\mathcal{Q}} := f^{-1}\tilde{\mathcal{Q}}' \cdot \mathcal{O}_{\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)}$ be the inverse image ideal in $\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)$. Consider blow-up of $\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)$ along $\tilde{\mathcal{Q}}$. Let X be the $\mathscr{I}$-adic completion of the open in this blow-up where the inverse image ideal of $\tilde{\mathcal{Q}}$ coincides with the inverse image of $\mathscr{I}$. We claim that X represents $\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m, \mathcal{Q})$.

To prove this, suppose $\tilde{t}\colon T \to X$ be a lift of $t\colon T \to S$. Then certainly $\tilde{t}$ defines a point of $\mathbb{V}(\mathcal{E})$ and hence corresponds to an element $f \in \mathrm{Hom}_{\mathcal{O}_T}(t^*\mathcal{E}, \mathcal{O}_T)$. Moreover this satisfies $(f \bmod t^*\mathscr{I})(t^*s_i) = 1$. Since over X, $\mathscr{I}$ coincides with $\tilde{\mathcal{Q}}$, $(f \bmod t^*\mathscr{I})$ kills $t^*\mathcal{Q}$. Conversely, to prove that for any element $f \in \mathrm{Hom}_{\mathcal{O}_T}(t^*\mathcal{E}, \mathcal{O}_T)$ seen as a point of $\mathbb{V}(\mathcal{E})$, that sends $s_i \mapsto 1$ and kills $\mathcal{Q}$ modulo $t^*\mathscr{I}$, there exists a unique lift $\tilde{t}\colon T \to X$, it will be enough to prove this when $T = \mathrm{Spf}\, R'$ and $t\colon T \to S$ factors through an open $\mathrm{Spf}\, R \subset S$ where $\bar{\mathcal{E}}', \bar{\mathcal{E}}$ and $\bar{\mathcal{E}}/\bar{\mathcal{E}}'$ are locally free. In that case picking lifts X_i of s_i as in the proof of Lemma 1.3.2 and lifts $Y_{m+1}, \ldots, Y_n$ of a basis $y_{m+1}, \ldots, y_n$ of $\mathcal{Q}$, we can write

$$\begin{aligned} X_{|\mathrm{Spf}\, R} &\simeq \mathrm{Spf}\, R\langle \{X_i\}_{i=1}^m \{Y_j\}_{j=m+1}^n \rangle \langle \{\frac{X_i - 1}{\alpha}\}_{i=1}^m \{\frac{Y_j}{\alpha}\}_{j=m+1}^n \rangle \\ &\simeq R\langle \{Z_i\}_{i=1}^m, \{W_j\}_{j=m+1}^n \rangle \end{aligned}$$

where the projection $X \to \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)$ corresponds to $Z_i \mapsto Z_i$ and $Y_j \mapsto \alpha W_j$. Now $t^*\mathcal{E} = \oplus_{i=1}^m R'(t^*X_i) \bigoplus \oplus_{j=m+1}^n R'(t^*Y_j)$. An R'-linear map $f\colon t^*\mathcal{E} \to R'$ that satisfies $f(X_i) = 1 \bmod (\alpha)$ and $f(Y_i) = 0 \bmod (\alpha)$ can be written uniquely as

$$f = \sum_{i=1}^m (1 + \alpha r_i)(t^*X_i)^\vee + \sum_{j=m+1}^n \alpha q_j (t^*Y_j)^\vee$$

for $r_i, q_j \in R'$. Then we can define a map $R'\langle \{Z_i\}\{W_j\} \rangle$ that sends $Z_i \mapsto r_i$ and $W_j \mapsto q_j$ for all i, j. This determines a point $\tilde{t} \in X(T)$. $\square$

Lemma 1.3.4. *The functor $\mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m, \mathcal{Q})$ is functorial in tuples $(\mathcal{E}, s_1, \ldots, s_m, \mathcal{Q})$, i.e. given a map of locally free sheaves of equal rank $\rho\colon \mathcal{E} \to \mathcal{E}'$, where $\mathcal{E}$ (resp. $\mathcal{E}'$) is equipped with marked sections $s_1, \ldots, s_m$ (resp. $s'_1, \ldots, s'_m$) and a subsheaf $\mathcal{Q}$ (resp. $\mathcal{Q}'$), corresponding to a marked splitting, if $\bar{\rho}(s_i) = s'_i$ and $\bar{\rho}(\mathcal{Q}) \subset \mathcal{Q}'$, then there are natural transformations $\mathbb{V}_0(\mathcal{E}', s'_1, \ldots, s'_m, \mathcal{Q}') \to \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m, \mathcal{Q})$ and $\mathbb{V}_0(\mathcal{E}', s'_1, \ldots, s'_m) \to \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m)$ such that the following diagram commutes.*

$$\begin{tikzcd} \mathbb{V}_0(\mathcal{E}', s'_1, \ldots, s'_m, \mathcal{Q}') \arrow[r] \arrow[d] & \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m, \mathcal{Q}) \arrow[d] \\ \mathbb{V}_0(\mathcal{E}', s'_1, \ldots, s'_m) \arrow[r] & \mathbb{V}_0(\mathcal{E}, s_1, \ldots, s_m) \end{tikzcd}$$

Proof. Follows easily from the definition. □

1.3.2 The sheaves $\Omega_{\mathcal{E}}$ and $\mathrm{H}^{\sharp}_{\mathcal{E}}$

The trivialization of $H_n^\vee$ over $\mathcal{IG}_{n,r,I}$ induces an equality of groups $H_n^\vee(\mathfrak{IG}_{n,r,I}) = H_n^\vee(\mathcal{IG}_{n,r,I}) \simeq \mathbb{Z}/p^n\mathbb{Z}$. Let P^{univ} be the image of $1 \in \mathbb{Z}/p^n\mathbb{Z}$ in $H_n^\vee(\mathfrak{IG}_{n,r,I})$. We have a $\mathcal{O}_{\mathfrak{IG}_{n,r,I}}$-linear map

$$\begin{array}{ccccc} & & \omega_{\mathcal{E}} & & \\ & & \big\downarrow & & \\ H_n^\vee(\mathfrak{IG}_{n,r,I}) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathfrak{IG}_{n,r,I}} & \xrightarrow{\mathrm{dlog}\otimes 1} & \omega_{H_n} & \xrightarrow{\sim} & \omega_{\mathcal{E}}/p^n\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}\omega_{\mathcal{E}}. \end{array}$$

Recall the definition of the dlog map: a point $P \in H_n^\vee(R)$ defines a group homomorphism $\gamma_P \colon H_{n,R} \to \mathbb{G}_{m,R}$. Then $\mathrm{dlog}(P) := \gamma_P^*(dt/t)$ where dt/t is the canonical differential of $\mathbb{G}_m$.

Definition 1.3.5. Define $\Omega_{\mathcal{E}}$ to be the inverse image under $\omega_{\mathcal{E}} \to \omega_{H_n}$ of the image of $\mathrm{dlog}\otimes 1$. We call this the modified modular sheaf.

The cokernel of $\mathrm{dlog}\otimes 1$ is killed by $\mathrm{Hdg}^{\frac{1}{p-1}}$ [AIP18, Proposition A.3]. Hence $\Omega_{\mathcal{E}} = \mathrm{Hdg}^{\frac{1}{p-1}}\omega_{\mathcal{E}}$ is a line bundle. There is a canonical isomorphism

$$H_n^\vee(\mathfrak{IG}_{n,r,I}) \otimes \mathcal{O}_{\mathfrak{IG}_{n,r,I}}/p^n\mathrm{Hdg}^{\frac{-p^n}{p-1}} \xrightarrow{\sim} \Omega_{\mathcal{E}}/p^n\mathrm{Hdg}^{\frac{-p^n}{p-1}}.$$

Let $\beta_n := p^n\mathrm{Hdg}^{\frac{-p^n}{p-1}}$.

Letting $s = \mathrm{dlog}(P^{\text{univ}})$, we have a line bundle with a marked section $(\Omega_{\mathcal{E}}, s)$. Thus we have a morphism of formal schemes $\nu \colon \mathbb{V}_0(\Omega_{\mathcal{E}}, s) \to \mathfrak{IG}_{n,r,I}$.

1.3.2.1 Formal group action on $\mathbb{V}_0(\Omega_{\mathcal{E}}, s)$

The formal scheme $\mathbb{V}_0(\Omega_{\mathcal{E}}, s)$ carries an action of the formal group $\mathfrak{T} := 1 + \beta_n\mathbb{G}_a$ over $\mathfrak{IG}_{n,r,I}$, which realizes it as a torsor.

The action is described on points as follows. Let $(\rho, f) \in \mathbb{V}_0(\Omega_{\mathcal{E}}, s)(R)$ be an R-valued point. Here $\rho \colon \mathrm{Spf}\, R \to \mathfrak{IG}_{n,r,I}$ is a morphism of formal schemes, and $f \in \mathrm{Hom}_R(\rho^*\Omega_{\mathcal{E}}, R)$. By definition $(f \bmod \beta_n)(\rho^* s) = 1$. For any point $t \in 1 + \beta_n R$, let $t * (\rho, f) := (\rho, tf)$. Then clearly (ρ, tf) defines an element of $\mathbb{V}_0(\Omega_{\mathcal{E}}, s)(R)$.

On local coordinates this action can be described in the following manner. We choose local coordinates as described in Lemma 1.3.2, i.e. pick a lift X of the marked section S over an open $\mathrm{Spf}\, R \subset \mathfrak{IG}_{n,r,I}$ that trivializes $\Omega_{\mathcal{E}}$. Then we have the following cartesian diagram.

$$\begin{array}{ccccc} \mathfrak{IG}_{n,r,I} & \longleftarrow & \mathbb{V}(\Omega_{\mathcal{E}}) & \longleftarrow & \mathbb{V}_0(\Omega_{\mathcal{E}}, s) \\ \big\uparrow & & \big\uparrow & & \big\uparrow \\ \mathrm{Spf}\, R & \longleftarrow & \mathrm{Spf}\, R\langle X\rangle & \xleftarrow[X\mapsto 1+\beta_n Z]{} & \mathrm{Spf}\, R\langle Z\rangle \end{array}$$

The action of $\mathfrak{T}$ on Z is such that $t * (1 + \beta_n Z) = t(1 + \beta_n Z)$. In other words, $t * Z = \frac{t-1}{\beta_n} + tZ$.

Recall we defined $\mathrm{H}'_{\mathcal{E}}$ (Definition 1.2.1) as a locally free subsheaf of $\mathrm{H}_{\mathcal{E}}$ such that modulo $p\mathrm{Hdg}^{-1}$ the induced Hodge filtration on $\mathrm{H}'_{\mathcal{E}}$ admits a canonical splitting.

Definition 1.3.6. Define $\mathrm{H}^{\sharp}_{\mathcal{E}} := \mathrm{Hdg}^{\frac{1}{p-1}} \mathrm{H}'_{\mathcal{E}}$. We call this the modified de Rham sheaf.

Since $\mathrm{Hdg}^{\frac{1}{p-1}}$ is an invertible ideal, $\mathrm{H}^{\sharp}_{\mathcal{E}}$ is a locally free sheaf of rank 2 equipped with a Hodge filtration as follows.

$$0 \to \Omega_{\mathcal{E}} \to \mathrm{H}^{\sharp}_{\mathcal{E}} \to \mathrm{Hdg}^{\frac{p}{p-1}} \omega^{\vee}_{\mathcal{E}} \to 0$$

Moreover, this filtration splits upon pulling back via the closed immersion $i\colon \mathfrak{IG}_{n,r,I}/(p\mathrm{Hdg}^{-1}) \hookrightarrow \mathfrak{IG}_{n,r,I}$, and this splitting is induced by the splitting of $\mathrm{H}'_{\mathcal{E}}$.

Remark 1.3.1. We note that as modules $\mathrm{H}^{\sharp}_{\mathcal{E}}$ (by our definition) is the same as that defined in [AI21]. However we have the added information about the splitting of $\mathrm{H}^{\sharp}_{\mathcal{E}}$ in our case.

1.3.2.2 Functoriality

Consider the morphism $\tilde{F}\colon \mathfrak{IG}_{n,r+1,I} \to \mathfrak{IG}_{n,r,I}$ defined generically on the universal elliptic curve by $\mathcal{E} \mapsto \mathcal{E}' := \mathcal{E}/H_1$. Let $\lambda\colon \mathcal{E} \to \mathcal{E}'$ be the projection and let $\lambda^{\vee}\colon \mathcal{E}' \to \mathcal{E}$ be its dual. Then $\lambda^{\vee}$ induces an isomorphism $H_n(\mathcal{E}') \xrightarrow{\sim} H_n(\mathcal{E})$ over the generic fibre. This then induces an isomorphism $\Omega_{\mathcal{E}} \xrightarrow{\sim} \Omega_{\mathcal{E}'}$ sending the marked section of $\Omega_{\mathcal{E}}$ to the marked section of $\Omega_{\mathcal{E}'}$. We note that $\mathrm{Hdg}_{\mathcal{E}'} = \mathrm{Hdg}^{p}_{\mathcal{E}}$ since λ reduces to the relative Frobenius modulo $p\mathrm{Hdg}^{-1}$.

Lemma 1.3.5. *There exists an r large enough such that the induced map $(\lambda^{\vee})^*\colon \mathrm{H}_{\mathcal{E}} \to \mathrm{H}_{\mathcal{E}'}$ restricts to a well-defined map $(\lambda^{\vee})^*\colon \mathrm{H}^{\sharp}_{\mathcal{E}} \to \mathrm{H}^{\sharp}_{\mathcal{E}'}$ that sends marked sections to marked sections. Moreover, let $\mathcal{Q} \subset \mathrm{H}^{\sharp}_{\mathcal{E}}/p\mathrm{Hdg}_{\mathcal{E}}^{-(p+1)}$ be the kernel of the marked splitting, and let $\mathcal{Q}' \subset \mathrm{H}^{\sharp}_{\mathcal{E}'}/p\mathrm{Hdg}_{\mathcal{E}}^{-(p+1)}$ be the same for $\mathcal{E}'$. Then $(\lambda^{\vee})^*$ sends $\mathcal{Q}$ to $\mathcal{Q}'$.*

Proof. Since $(\lambda^{\vee})^*$ maps $\Omega_{\mathcal{E}}$ isomorphically onto $\Omega_{\mathcal{E}'}$, it is enough to show that the induced map $\mathrm{H}^{\sharp}_{\mathcal{E}}/\Omega_{\mathcal{E}} \to \mathrm{H}_{\mathcal{E}'}/\Omega_{\mathcal{E}'}$ factors through the inclusion $\mathrm{H}^{\sharp}_{\mathcal{E}'}/\Omega_{\mathcal{E}'} \hookrightarrow \mathrm{H}_{\mathcal{E}'}/\Omega_{\mathcal{E}'}$. We have a diagram as follows.

$$\begin{tikzcd}
 & \mathrm{H}^{\sharp}_{\mathcal{E}'}/\Omega_{\mathcal{E}'} \arrow[r, "\simeq", "i"'] \arrow[d, hook] & \mathrm{Hdg}_{\mathcal{E}}^{\frac{p^2}{p-1}} \omega^{\vee}_{\mathcal{E}'} \arrow[d, "\mathrm{Hdg}_{\mathcal{E}}^{\frac{p^2}{p-1}}"] \\
\mathrm{H}^{\sharp}_{\mathcal{E}}/\Omega_{\mathcal{E}} \arrow[r, "(\lambda^{\vee})^*"] \arrow[dr, "\simeq"'] & \mathrm{H}_{\mathcal{E}'}/\Omega_{\mathcal{E}'} \arrow[r, "\pi"] & \mathrm{H}_{\mathcal{E}'}/\omega_{\mathcal{E}'} \simeq \omega^{\vee}_{\mathcal{E}'} \\
 & \mathrm{Hdg}_{\mathcal{E}}^{\frac{p}{p-1}} \omega^{\vee}_{\mathcal{E}} \arrow[ur, "p\mathrm{Hdg}_{\mathcal{E}}^{-1+\frac{p}{p-1}}"'] &
\end{tikzcd}$$

The description of all the arrows above should be clear except perhaps the lower right diagonal arrow $\mathrm{Hdg}_{\mathcal{E}}^{\frac{p}{p-1}} \omega^{\vee}_{\mathcal{E}} \to \mathrm{H}_{\mathcal{E}'}/\omega_{\mathcal{E}'} \simeq \omega^{\vee}_{\mathcal{E}'}$. Let us explain what this map looks like. Firstly, abstractly this map is the restriction of the map $(\lambda^*)^{\vee}\colon \omega^{\vee}_{\mathcal{E}} \to \omega^{\vee}_{\mathcal{E}'}$ to the submodule $\mathrm{Hdg}_{\mathcal{E}}^{\frac{p}{p-1}} \omega^{\vee}_{\mathcal{E}}$. For a choice of local basis e of

$\omega_{\mathcal{E}}$, $\tilde{F}^*e$ is a local basis of $\omega_{\mathcal{E}'}$. With respect to the dual basis $e^\vee$ and $(\tilde{F}^*e)^\vee$ of $\omega_{\mathcal{E}}^\vee$ and $\omega_{\mathcal{E}'}^\vee$, the map $(\lambda^*)^\vee$ is given by sending $e^\vee \mapsto p\mathrm{Hdg}^{-1}(\tilde{F}^*e)^\vee$. Hence with respect to the basis $\mathrm{Hdg}_{\mathcal{E}}^{\frac{p}{p-1}}e^\vee$ and $(\tilde{F}^*e)^\vee$ of $\mathrm{Hdg}_{\mathcal{E}}^{\frac{p}{p-1}}\omega_{\mathcal{E}}^\vee$ and $\omega_{\mathcal{E}'}^\vee$ respectively, the $\mathcal{O}_{\mathfrak{IG}_{n,r+1,I}}$-linear map $(\lambda^*)^\vee$ can be described as multiplication by $p\mathrm{Hdg}^{-1+\frac{p}{p-1}}$.

Let $\mathrm{Hdg} = \mathrm{Hdg}_{\mathcal{E}}$ in this proof. Since multiplication by $p\mathrm{Hdg}^{-1+\frac{p}{p-1}}$ is injective, we see that the image of $\mathrm{H}_{\mathcal{E}}^\sharp/\Omega_{\mathcal{E}}$ under $(\lambda^\vee)^*$ does not intersect $\omega_{\mathcal{E}'}/\Omega_{\mathcal{E}'}$. We first show that $\pi\circ(\lambda^\vee)^*$ factors through $\mathrm{Hdg}^{\frac{p^2}{p-1}}\omega_{\mathcal{E}'}^\vee$. For this it is enough to show that $p\mathrm{Hdg}^{-1+\frac{p}{p-1}} \subset \mathrm{Hdg}^{\frac{p^2}{p-1}}$, i.e. $p\mathrm{Hdg}^{-(p+1)} \subset \mathcal{O}_{\mathfrak{IG}_{n,r+1,I}}$ which can be ensured by choosing large enough r. This proves that the map $(\lambda^\vee)^* - i^{-1}\circ\pi\circ(\lambda^\vee)^*$ factors through $\omega_{\mathcal{E}'}/\Omega_{\mathcal{E}'}$ which is a torsion module killed by $\mathrm{Hdg}^{\frac{p}{p-1}}$. Now since $\mathrm{H}_{\mathcal{E}}^\sharp = \Omega_{\mathcal{E}} + \mathrm{Hdg}^{\frac{p}{p-1}}\mathrm{H}_{\mathcal{E}}$, the difference $(\lambda^\vee)^* - i^{-1}\circ\pi\circ(\lambda^\vee)^* = 0$. This proves the first claim.

The second claim follows from a local computation. Choose an open affine $\mathrm{Spf}\, R = U \subset \mathfrak{X}_I$ such that $\mathrm{H}_{\mathcal{E}}$ admits a basis $\{e, f\}$ over U with e a basis of $\omega_{\mathcal{E}}$. Let $\{\bar{e}, \bar{f}\}$ be their image over $\mathrm{Spf}\, R/p$. The unit root subspace is generated by a vector $\bar{v} = \bar{C}\bar{e} + \mathrm{Ha}\bar{f}$ for some $\bar{C} \in R/p$. Let $\varphi\colon R/p \to R/p$ be the Frobenius. With respect to the basis $\{\bar{e}, \bar{f}\}$ of $\mathrm{H}_{\bar{\mathcal{E}}}$ and $\{\bar{e}^{(p)} := \varphi^*(\bar{e}), \bar{f}^{(p)} := \varphi^*(\bar{f})\}$ of $\mathrm{H}_{\bar{\mathcal{E}}^{(p)}}$, the matrix of Verschiebung $V\colon \mathrm{H}_{\bar{\mathcal{E}}} \to \mathrm{H}_{\bar{\mathcal{E}}^{(p)}}$ can be written as

$$V = \begin{pmatrix} \mathrm{Ha} & \bar{B} \\ 0 & 0 \end{pmatrix}.$$

Since Verschiebung kills the unit root subspace, $V(\bar{C}\bar{e} + \mathrm{Ha}\bar{f}) = (\bar{C}\mathrm{Ha} + \bar{B}\mathrm{Ha})\bar{e} = 0$. This shows that $\bar{B} = -\bar{C}$. Let $\mathrm{Spf}\, R_{n,r+1} \subset \mathfrak{IG}_{n,r+1,I}$ (resp. $\mathrm{Spf}\, R_{n,r} \subset \mathfrak{IG}_{n,r,I}$) be the inverse image of U in $\mathfrak{IG}_{n,r+1,I}$ (resp. $\mathfrak{IG}_{n,r,I}$). Then as discussed in §1.2, $\mathrm{H}'_{\mathcal{E}}$ over $\mathfrak{IG}_{n,r+1,I}$ is generated by the pullback of e and a lift $v = Ce + \mathrm{Hdg} f$ of $\bar{v}$. For notational simplicity we will write these sections as $\{e, v\}$ still. Similarly, $\mathrm{H}'_{\mathcal{E}'}$ is generated by pulling back via $\tilde{F}\colon \mathrm{Spf}\, R_{n,r+1} \to \mathrm{Spf}\, R_{n,r}$ the pullbacks of e and v to $\mathrm{Spf}\, R_{n,r}$. We will write these as $\{\tilde{F}^*e, \tilde{F}^*v\}$. Then with respect to $\{e, v\}$ and $\{\tilde{F}^*e, \tilde{F}^*v\}$ the matrix of $(\lambda^\vee)^*$ can be described as

$$\begin{aligned}(\lambda^\vee)^* &= \begin{pmatrix} 1 & -\tilde{F}^*(C\mathrm{Hdg}^{-1}) \\ 0 & \tilde{F}^*(\mathrm{Hdg}^{-1}) \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Hdg} & B \\ 0 & p\mathrm{Hdg}^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & C \\ 0 & \mathrm{Hdg} \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{Hdg} & C\mathrm{Hdg} + B\mathrm{Hdg} - p\tilde{F}^*(C)\mathrm{Hdg}^{-p} \\ 0 & p\mathrm{Hdg}^{-p} \end{pmatrix}.\end{aligned}$$

Here B is a lift of $\bar{B}$ modulo $p\mathrm{Hdg}^{-1}$. Therefore with respect to the basis $\{\mathrm{Hdg}^{\frac{1}{p-1}}e, \mathrm{Hdg}^{\frac{1}{p-1}}v\}$ and $\{\mathrm{Hdg}^{\frac{p}{p-1}}\tilde{F}^*e, \mathrm{Hdg}^{\frac{p}{p-1}}\tilde{F}^*v\}$, the matrix of $(\lambda^\vee)^*\colon \mathrm{H}_{\mathcal{E}}^\sharp \to \mathrm{H}_{\mathcal{E}'}^\sharp$ is written as

$$(\lambda^\vee)^* = \begin{pmatrix} 1 & C + B - p\tilde{F}^*(C)\mathrm{Hdg}^{-(p+1)} \\ 0 & p\mathrm{Hdg}^{-(p+1)} \end{pmatrix}.$$

This proves the second claim of the lemma. □

Definition 1.3.7. Let $\mathrm{H}^\sharp_{\mathcal{E}}, \Omega_{\mathcal{E}}, s$ be the modified de Rham sheaf, modified modular sheaf and its marked section respectively. Let $\mathcal{Q} \subset \mathrm{H}^\sharp_{\mathcal{E}}/p\mathrm{Hdg}^{\frac{-p^2}{p-1}}$ be the kernel of the marked splitting $\psi\colon \mathrm{H}^\sharp_{\mathcal{E}}/p\mathrm{Hdg}^{\frac{-p^2}{p-1}} \to \Omega_{\mathcal{E}}/p\mathrm{Hdg}^{\frac{-p^2}{p-1}}$. We call the marked splitting ψ as the modified unit root splitting and $\mathcal{Q}$ the modified unit root subspace.

We remark that by Lemma 1.3.5 the modified unit root subspace is functorial for $\lambda^\vee : \mathcal{E}' \to \mathcal{E}$.

Recall the notation $\beta_n = p^n\mathrm{Hdg}^{\frac{-p^n}{p-1}}$. Let $\eta := p\mathrm{Hdg}^{\frac{-p^2}{p-1}}$.

Associated to the data of $(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})$ we can define a formal vector bundle with marked section and marked splitting $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})$ as the functor on $\mathbf{FSch}^{(\alpha)}_{\mathfrak{IG}_{n,r,I}}$ whose points are

$$\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})(t\colon T \to \mathfrak{IG}_{n,r,I}) := \Big\{ f \in \mathrm{Hom}_{\mathcal{O}_T}(t^*\mathrm{H}^\sharp_{\mathcal{E}}, \mathcal{O}_T) \,|\, (f \bmod \beta_n)(t^*s) = 1, \\ (f \bmod \eta)(t^*\mathcal{Q}) = 0 \Big\}$$

A slight modification of the proof of Lemma 1.3.3 shows that $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})$ is representable by a formal scheme which is obtained as an open in a formal admissible blow-up of $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s)$.

Let $\rho\colon \mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q}) \to \mathfrak{IG}_{n,r,I}$ be the projection.

1.3.2.3 Formal group action on $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})$

The formal scheme $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})$ carries an action of the formal group $\mathfrak{T} = 1 + \beta_n\mathbb{G}_a$ over $\mathfrak{IG}_{n,r,I}$ that is compatible with the action of $\mathfrak{T}$ on $\mathbb{V}_0(\Omega_{\mathcal{E}}, s)$, i.e. the natural map $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q}) \to \mathbb{V}_0(\Omega_{\mathcal{E}}, s)$ that we get by composing the projection $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q}) \to \mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s)$ with the natural map $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s) \to \mathbb{V}_0(\Omega_{\mathcal{E}}, s)$ of Lemma 1.3.4, commutes with the action of $\mathfrak{T}$.

This action is defined on points as follows. Let $(\gamma, f) \in \mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})(R)$ be an R-valued point. Here $\gamma\colon \mathrm{Spf}\, R \to \mathfrak{IG}_{n,r,I}$ is a morphism of formal schemes, and $f \in \mathrm{Hom}_R(\gamma^*\mathrm{H}^\sharp_{\mathcal{E}}, R)$. By definition, $(f \bmod \beta_n)(\gamma^*s) = 1$ and $(f \bmod \eta)(\gamma^*\mathcal{Q}) = 0$. For any point $t \in 1 + \beta_n R$, let $t * (\gamma, f) := (\gamma, tf)$. Then clearly (γ, tf) defines a point of $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})(R)$.

We now give a description of this action in terms of local coordinates. Choose an open $\mathrm{Spf}\, R \subset \mathfrak{IG}_{n,r,I}$ such that we have a basis $\{X, Y\}$ of $\mathrm{H}^\sharp_{\mathcal{E}|\mathrm{Spf}\, R}$ with $X \in \Omega_{\mathcal{E}|\mathrm{Spf}\, R}$ being a lift of s and Y being a lift of a generator of $\mathcal{Q}_{|\mathrm{Spf}\, R/\eta}$. Then we have the following Cartesian diagram.

$$\begin{tikzcd}
\mathfrak{IG}_{n,r,I} & \mathbb{V}(\mathrm{H}^\sharp_{\mathcal{E}}) \arrow[l] & \mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s) \arrow[l] & \mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q}) \arrow[l] \\
\mathrm{Spf}\, R \arrow[u, hook] & \mathrm{Spf}\, R\langle X, Y\rangle \arrow[l] \arrow[u, hook] & \mathrm{Spf}\, R\langle Z, Y\rangle \arrow[l, "X\mapsto 1+\beta_n Z"'] \arrow[u, hook] & \mathrm{Spf}\, R\langle Z, W\rangle \arrow[l, "Y\mapsto \eta W"'] \arrow[u, hook]
\end{tikzcd}$$

In terms of these local coordinates, $t \in \mathfrak{T}(R)$ acts via $t * Z = \frac{t-1}{\beta_n} + tZ$, and $t * W = tW$.

Remark 1.3.2. In the following subsection we will define the p-adic interpolation sheaves $\mathfrak{w}'_{k,I}$ and $\mathbb{W}'_{k,I}$ over $\mathfrak{IG}_{n,r,I}$. There is an action of the group $\mathbb{Z}_p^\times$ on $\mathfrak{IG}_{n,r,I}$ over $\mathfrak{X}_{r,I}$. The $\mathfrak{T}$ action on $\mathbb{V}_0(\Omega_{\mathcal{E}}, s)$ and on $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})$ can be extended to an action of $\mathfrak{T}^{\mathrm{ext}} := \mathbb{Z}_p^\times(1+\beta_n\mathbb{G}_a)$ on the aforementioned formal schemes over $\mathfrak{X}_{r,I}$. For the case of $\mathbb{V}_0(\Omega_{\mathcal{E}}, s)$, this has already been discussed in [AI21, §3.2]. For $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})$ it is similar to the action of $\mathfrak{T}^{\mathrm{ext}}$ on $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s)$ as discussed in loc. cit. §3.3.1. It is necessary to consider this action to define p-adic interpolation of modular and de Rham sheaves over $\mathfrak{X}_{r,I}$ rather than over $\mathfrak{IG}_{n,r,I}$. Namely, the authors in loc. cit. define $\mathfrak{w}^0_{k,I}$ as a line bundle over $\mathfrak{X}_{r,I}$ by taking the k-invariants for the action of $\mathbb{Z}_p^\times$ on the sheaf $\mathfrak{w}'_{k,I}$ over $\mathfrak{IG}_{n,r,I}$. Similarly one should define a sheaf $\mathbb{W}^0_{k,I}$ over $\mathfrak{X}_{r,I}$ by taking the k-invariants of $\mathbb{W}'_{k,I}$ for the action of $\mathbb{Z}_p^\times$. Using this one can show that the construction of $\mathfrak{w}^0_{k,I}$ and $\mathbb{W}^0_{k,I}$ does not depend on the choice of n and is functorial with respect to r and I. But for the purpose of this chapter, which is to prove convergence of p-adic iteration of the Gauss-Manin connection on the generic fibre, it is not necessary to construct the p-adic interpolation sheaves over $\mathfrak{X}_{r,I}$, and it will be sufficient to work over $\mathfrak{IG}_{n,r,I}$. The efficacy of using $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})$ instead of $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s)$ will already be evident in constructing the Gauss-Manin connection over $\mathfrak{IG}_{n,r,I}$. Hence we ignore the residual action of $\mathbb{Z}_p^\times$. However we will discuss these issues in the next chapter when we deal with Hilbert modular forms.

1.3.3 p-adic interpolation of $\Omega_{\mathcal{E}}$ and $\mathrm{H}^\sharp_{\mathcal{E}}$

Let n, r, I be as fixed in the beginning of the section. Recall we have projections $\rho\colon \mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q}) \to \mathfrak{IG}_{n,r,I}$ and $\nu\colon \mathbb{V}_0(\Omega_{\mathcal{E}}, s) \to \mathfrak{IG}_{n,r,I}$.

Definition 1.3.8. For $k = k^0_I\colon \mathbb{Z}_p^\times \to \Lambda^0_I$ the universal character over $\mathfrak{W}^0_I$, define

$$\mathfrak{w}'_{k,I} := \nu_*\mathcal{O}_{\mathbb{V}_0(\Omega_{\mathcal{E}},s)}[k].$$

These are all functions $f \in \nu_*\mathcal{O}_{\mathbb{V}_0(\Omega_{\mathcal{E}},s)}$ such that $t * f = k(t)f$ for all $t \in \mathfrak{T}$.

Recall the description of $\mathbb{V}_0(\Omega_{\mathcal{E}}, s)$ in local coordinates as described in §1.3.2.1. Over an open Spf $R \subset \mathfrak{IG}_{n,r,I}$ that trivializes $\Omega_{\mathcal{E}}$, $\mathbb{V}_0(\Omega_{\mathcal{E}}, s)_{|\mathrm{Spf}\, R} \simeq \mathrm{Spf}\, R\langle Z\rangle$. The action of $\mathfrak{T}$ on Z is given by $t * Z = \frac{t-1}{\beta_n} + tZ$. Note that since k is analytic on $1+\beta_n\mathbb{G}_a$, $k(1+\beta_n Z)$ is an element of $R\langle Z\rangle$.

Proposition 1.3.1. *For Spf $R \subset \mathfrak{IG}_{n,r,I}$ as above, $\mathfrak{w}'_{k,I|\mathrm{Spf}\, R} = R\cdot k(1+\beta_n Z)$. In particular, $\mathfrak{w}'_{k,I}$ is a line bundle.*

Proof. This is [AI21, Lemma 3.9]. □

Definition 1.3.9. For $k = k^0_I$, define

$$\mathbb{W}'_{k,I} := \rho_*\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}},s,\mathcal{Q})}[k].$$

These are all functions $f \in \rho_*\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}},s,\mathcal{Q})}$ satisfying $t * f = k(t)f$ for all $t \in \mathfrak{T}$.

Recall from §1.3.2.3 the description of $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})$ in local coordinates. Choose an open $\mathrm{Spf}\, R \subset \mathfrak{IG}_{n,r,I}$ such that we have a basis $\{X, Y\}$ of $\mathrm{H}^\sharp_{\mathcal{E}|\mathrm{Spf}\, R}$ with $X \in \Omega_{\mathcal{E}|\mathrm{Spf}\, R}$ being a lift of s and Y being a lift of a generator of $\mathcal{Q}_{|\mathrm{Spf}\, R/\eta}$. Then $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})_{|\mathrm{Spf}\, R} \simeq \mathrm{Spf}\, R\langle Z, W\rangle$ with the projection $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})_{|\mathrm{Spf}\, R} \to \mathbb{V}(\mathrm{H}^\sharp_{\mathcal{E}})_{|\mathrm{Spf}\, R} \simeq \mathrm{Spf}\, R\langle X, Y\rangle$ being given by $X \mapsto 1 + \beta_n Z$ and $Y \mapsto \eta W$.

Proposition 1.3.2. *For Spf $R \subset \mathfrak{IG}_{n,r,I}$ as above, $\mathbb{W}'_{k,I}{}_{|\mathit{Spf}\, R} = R\langle \frac{W}{1+\beta_n Z}\rangle \cdot k(1 + \beta_n Z)$. In particular, $\mathbb{W}'_{k,I}$ is the p-adic completion of a direct limit of locally free sheaves.*

Proof. Let $\tilde{\mathbb{W}}(R) := R\langle \frac{W}{1+\beta_n Z}\rangle \cdot k(1 + \beta_n Z)$. Certainly $\tilde{\mathbb{W}}(R) \subset \mathbb{W}'_{k,I}{}_{|\mathrm{Spf}\, R}$. In order to prove the converse it will be sufficient to prove $R\langle Z, W\rangle^{\mathfrak{T}(R)} = R\langle \frac{W}{1+\beta_n Z}\rangle$, since $k(1+\beta_n Z) \in R\langle Z, W\rangle^\times$. Let $V = \frac{W}{1+\beta_n Z}$. We note first that the inclusion $R\langle Z, V\rangle \to R\langle Z, W\rangle$ is an isomorphism of topological rings. Suppose $f \in R\langle Z, V\rangle^{\mathfrak{T}(R)}$. Write $f = \sum_{i\geq 0} b_i(Z)V^i$ with $b_i(Z) \in R\langle Z\rangle$. Then since $t * f = f$ for all $t \in \mathfrak{T}(R)$, we have $b_i(Z) = b_i(t * Z)$ for all i. Write $t = 1 + \beta_n a$ for $a \in R$. Then we have $b_i(Z) = b_i(a + tZ)$ for any $a \in R$. Letting $Z = 0$, $b_i(0) = b_i(a)$ for any $a \in R$. By the Weierstrass preparation theorem, this implies that $b_i \in R$. Thus $f = \sum_i b_i V^i \in R\langle V\rangle$. This proves the proposition. □

Lemma 1.3.6. *Let $f_0\colon \mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q}) \to \mathbb{V}_0(\Omega_{\mathcal{E}}, s)$ be the projection. There is an increasing filtration $\{\mathrm{Fil}_i\}_{i\geq 0}$ on $f_{0*}\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}},s,\mathcal{Q})}$ with $\mathrm{Fil}_0(f_{0*}\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}},s,\mathcal{Q})}) = \mathcal{O}_{\mathbb{V}_0(\Omega_{\mathcal{E}},s)}$. On local coordinates, if $\mathbb{V}_0(\Omega_{\mathcal{E}}, s)_{|\mathit{Spf}\, R} =$ Spf $R\langle Z\rangle$ and $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})_{|\mathit{Spf}\, R} =$ Spf $R\langle Z, W\rangle$, then*

$$\mathrm{Fil}_n(f_{0*}\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}},s,\mathcal{Q})})_{|\mathit{Spf}\, R} = \sum_{i=0}^{n} R\langle Z\rangle W^i.$$

Proof. We only need to show that the description in local coordinate glues. This is obvious for Fil_0 by definition. Suppose $\{X, Y\}$ and $\{X', Y'\}$ be two bases of $\mathrm{H}^\sharp_{\mathcal{E}|\mathrm{Spf}\, R}$ with X, X' being lifts of s and Y, Y' being lifts of a generator of $\mathcal{Q}_{|\mathrm{Spf}\, R/\eta}$. Then these choices give two local coordinate description of $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})_{|\mathrm{Spf}\, R}$, viz. $R\langle Z, W\rangle \simeq \mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})_{|\mathrm{Spf}\, R} \simeq R\langle Z', W'\rangle$ with $Z = \frac{X-1}{\beta_n}$ and $W = \frac{Y}{\eta}$ and similarly for Z', W'. Since every filtration contains Fil_0, we can assume without loss of generality that $X = X'$. Then $Y' = uY + a\eta X$ for some $u \in R^\times$ and $a \in R$. The isomorphism $R\langle Z, W'\rangle \xrightarrow{\sim} R\langle Z, W\rangle$ is then given by $W' \mapsto uW + a(1 + \beta_n Z)$. Clearly this isomorphism respects the filtration on both sides. □

Theorem 1.3.1. *The $\mathfrak{T}$ action on $\rho_*\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}},s,\mathcal{Q})}$ is compatible with the filtration on $\rho_*\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}},s,\mathcal{Q})}$ induced from the one defined in Lemma 1.3.6 via pushforward. Then letting $\mathrm{Fil}_i\mathbb{W}'_{k,I} = \big(\mathrm{Fil}_i(\rho_*\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}},s,\mathcal{Q})})\big)[k]$, we have an increasing filtration of $\mathbb{W}'_{k,I}$ by locally free sheaves.*

Moreover, $\mathbb{W}'_{k,I} = \widehat{\varinjlim_i \mathrm{Fil}_i\mathbb{W}'_{k,I}}$.

We have $\mathrm{Fil}_0\mathbb{W}'_{k,I} = \mathfrak{w}'_{k,I}$, and $\mathrm{Gr}_i\mathbb{W}'_{k,I} \simeq \mathfrak{w}'_{k,I}\hat{\otimes}(\mathrm{Hdg}/\eta)^i\omega_{\mathcal{E}}^{-2i}$.

If $[0,1] \subset I$, then for any classical weight $m \in \mathbb{N}$ which is a point of $\mathcal{W}_I^0$, specializing at this point gives an isomorphism of sheaves

$$\mathrm{Sym}^m \, \mathrm{H}_{\mathcal{E}}[1/p] \simeq \mathrm{Fil}_m \mathbb{W}'_{k\mapsto m,I}[1/p].$$

over the generic fibre $\mathcal{IG}_{n,r,I} \times_{\mathcal{W}_I^0}^{k\mapsto m} \mathrm{Spf}\, \mathbb{Q}_p$, that preserves filtration on both sides considering the Hodge filtration on the left.

Proof. All of the statements follows immediately from the local description of $\mathbb{W}'_{k,I}$ as in Proposition 1.3.2. For instance $\mathrm{Fil}_n \mathbb{W}'_{k,I|\mathrm{Spf}\, R} = \sum_{i=0}^n R \cdot k(1+\beta_n Z) \cdot V^i$ with $V = \frac{W}{1+\beta_n Z}$ as in the proof of the Proposition. The last claim follows by observing that $\mathrm{H}_{\mathcal{E}}[1/p] = \mathrm{H}_{\mathcal{E}}^\sharp[1/p]$ and then using the local description of $\mathrm{Fil}_i \mathbb{W}'_{k,I}$. □

1.4 p-adic iteration of the Gauss-Manin connection

In this section we will define iteration of Gauss-Manin connection for analytic weights.

1.4.1 Gauss-Manin connection on $\mathrm{H}_{\mathcal{E}}^\sharp$

Let $\mathcal{IG}'_{n,r,I} \to \mathcal{IG}_{n,r,I}$ be the morphism classifying isomorphisms $(\mathbb{Z}/p^n\mathbb{Z})^2 \simeq \mathcal{E}[p^n]$. Let $\mathfrak{IG}'_{n,r,I}$ be the normalization of $\mathfrak{IG}_{n,r,I}$ in $\mathcal{IG}'_{n,r,I}$. Then [AI21, Proposition A.3] shows that the Gauss-Manin connection $\nabla \colon \mathrm{H}_{\mathcal{E}} \to \mathrm{H}_{\mathcal{E}} \hat{\otimes} \Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda_I^0}$ induces a connection $\nabla \colon \mathrm{H}_{\mathcal{E}}^\sharp \to \mathrm{H}_{\mathcal{E}}^\sharp \hat{\otimes} \Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda_I^0}$ over $\mathfrak{IG}'_{n,r,I}$ such that the marked section is horizontal for ∇ modulo β_n. In this section we will show that the modified unit root subspace $\mathcal{Q}$ is also horizontal for ∇ mod η. This will be necessary to apply Grothendieck's formalism of connections in our setting by using Lemma 1.3.4.

Lemma 1.4.1. *The Gauss-Manin connection $\nabla \colon \mathrm{H}_{\mathcal{E}} \to \mathrm{H}_{\mathcal{E}} \hat{\otimes} \Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda_I^0}$ restricts to a connection $\nabla \colon \mathrm{H}_{\mathcal{E}}^\sharp \to \mathrm{H}_{\mathcal{E}}^\sharp \hat{\otimes} \Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda_I^0}$ such that $(\nabla$ mod $\beta_n)(s) = 0$ and $(\nabla$ mod $\eta)(\mathcal{Q}) \subset \mathcal{Q} \otimes \Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda_I^0}/(\eta)$. In particular, if $t \in \mathcal{Q}$ is a local generator of $\mathcal{Q}$ that has been obtained as a pullback under the projection $\mathfrak{IG}'_{n,r,I} \to \mathfrak{IG}_{n,r,I}$ of a local generator of $\mathcal{Q}$ over $\mathfrak{IG}_{n,r,I}$, then $(\nabla$ mod $\eta)(t) = 0$.*

Proof. Let $\mathrm{Spf}\, R \subset \mathfrak{IG}_{n,r,I}$ be an open that trivializes $\mathrm{H}_{\mathcal{E}}$ and which is the inverse image of an open affine $\mathrm{Spf}\, R_0$ in $\mathfrak{X}_{r,I}$. Also assume that $\mathrm{Hdg}^{\frac{1}{p-1}}$ is trivialized over $\mathrm{Spf}\, R$. Let $\{\omega, \zeta\}$ be a basis of $\mathrm{H}_{\mathcal{E}}$ over $\mathrm{Spf}\, R$ which is a pullback of a basis from $\mathrm{Spf}\, R_0$. So hoping to not cause much confusion, we will often pretend that $\{\omega, \zeta\}$ lives over $\mathfrak{X}_{r,I}$. Then as shown in the proof of Lemma 1.3.5, there exists $C \in R_0$ and a choice of a generator of $\mathrm{Hdg}^{\frac{1}{p-1}}$ (to be also denoted by $\mathrm{Hdg}^{\frac{1}{p-1}}$ by an abuse of notation) such that $e := \mathrm{Hdg}^{\frac{1}{p-1}} \omega$ is a lift of the marked section s and $f := Ce + \mathrm{Hdg}^{\frac{p}{p-1}} \zeta$ is a lift of a generator of the modified unit root subspace $\mathcal{Q}$. Suppose

$$\nabla(\omega) = \omega \otimes m\theta + \zeta \otimes \theta, \qquad \nabla(\zeta) = \omega \otimes q\theta + \zeta \otimes r\theta.$$

Here $\theta = KS(\omega, \zeta)$ is the Kodaira-Spencer element associated to the basis $\{\omega, \zeta\}$. Since $\{\omega, \zeta\}$ is a basis over $\mathrm{Spf}\, R_0$, θ is a basis of $\Omega^1_{\mathfrak{X}_{r,I}/\Lambda_I^0}$.

It is shown in [AI21, Proposition A.3] that the restriction of ∇ to $\Omega_{\mathcal{E}}$ factors through $p^n\mathrm{H}_{\mathcal{E}}\hat{\otimes}\Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_I}$. Let Spf $R' \subset \mathfrak{IG}'_{n,r,I}$ be the inverse image of Spf R. Then over Spf R' we have

$$\nabla(e) = \omega \otimes \left(m\mathrm{Hdg}\theta + \mathrm{dHdg}^{\frac{1}{p-1}}\right) + \zeta \otimes \mathrm{Hdg}^{\frac{1}{p-1}}\theta.$$

This implies by the above fact that $\theta \in p^n\mathrm{Hdg}^{\frac{-1}{p-1}}\Omega^1_{R'/\Lambda^0_I}$, or in other words the image of $\Omega^1_{R/\Lambda^0_I} \to \Omega^1_{R'/\Lambda^0_I}$ lies in $p^n\mathrm{Hdg}^{\frac{-1}{p-1}}\Omega^1_{R'/\Lambda^0_I}$. This in fact shows why $\nabla\colon \mathrm{H}_{\mathcal{E}} \to \mathrm{H}_{\mathcal{E}}\hat{\otimes}\Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_I}$ restricts to a connection $\nabla\colon \mathrm{H}^\sharp_{\mathcal{E}} \to \mathrm{H}^\sharp_{\mathcal{E}}\hat{\otimes}\Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_I}$. Rewriting the connection on $\mathrm{H}^\sharp_{\mathcal{E}}$ in terms of the local basis we have the following equations.

$$\begin{aligned}\nabla(e) = \nabla(\mathrm{Hdg}^{\frac{1}{p-1}}\omega) &= \mathrm{Hdg}^{\frac{1}{p-1}}\omega \otimes \left(m\theta + \frac{\mathrm{dHdg}}{(p-1)\mathrm{Hdg}} - \frac{C\theta}{\mathrm{Hdg}}\right) + (Ce + \mathrm{Hdg}^{\frac{p}{p-1}}\zeta) \otimes \frac{\theta}{\mathrm{Hdg}} \\ &= e \otimes \left(m\theta + \frac{\mathrm{dHdg}}{(p-1)\mathrm{Hdg}} - \frac{C\theta}{\mathrm{Hdg}}\right) + f \otimes \frac{\theta}{\mathrm{Hdg}}.\end{aligned} \tag{1.2}$$

$$\begin{aligned}\nabla(f) &= \nabla(Ce + \mathrm{Hdg}^{\frac{p}{p-1}}\zeta) \\ &= e \otimes \left(\mathrm{d}C + q\mathrm{Hdg}\theta + mC\theta - \frac{C^2\theta}{\mathrm{Hdg}} - \frac{C\mathrm{dHdg}}{\mathrm{Hdg}}\right) + f \otimes \left(\frac{C\theta}{\mathrm{Hdg}} + \frac{p\mathrm{dHdg}}{(p-1)\mathrm{Hdg}}\right).\end{aligned} \tag{1.3}$$

Since both C and Hdg are defined over $\mathfrak{X}_{r,I}$, this shows that $\nabla(f) = 0 \bmod \eta$. This proves the lemma. □

1.4.2 Gauss-Manin connection on $\rho_*\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}},s,\mathcal{Q})}$

We make the observation that $\Omega^1_{(\mathfrak{IG}'_{n,r,I}/(\beta_n))/\Lambda^0_I} \simeq \Omega^1_{\mathfrak{IG}_{n,r,I}/\Lambda^0_I}/(\beta_n)$ and similarly $\Omega^1_{(\mathfrak{IG}'_{n,r,I}/(\eta))/\Lambda^0_I} \simeq \Omega^1_{\mathfrak{IG}_{n,r,I}/\Lambda^0_I}/(\eta)$. This follows from noting that $\mathrm{d}\beta_n \in \beta_n\Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_I}$ and $\mathrm{d}\eta \in \eta\Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_I}$.

Let $\mathcal{P}^{(1)}_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_I}$ be the first infinitesimal neighbourhood of the diagonal embedding $\Delta\colon \mathfrak{IG}'_{n,r,I} \hookrightarrow \mathfrak{IG}'_{n,r,I} \times_{\mathfrak{W}^0_I} \mathfrak{IG}'_{n,r,I}$. Let $p_i\colon \mathcal{P}^{(1)}_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_I} \to \mathfrak{IG}'_{n,r,I}$ be the two projections for $i = 1, 2$. Grothendieck's formalism of connection as discussed for example in [BO15, §2], allows us to view the connection ∇ on $\mathrm{H}^\sharp_{\mathcal{E}}$ as an isomorphism $\epsilon^\sharp\colon p_2^*\mathrm{H}^\sharp_{\mathcal{E}} \xrightarrow{\sim} p_1^*\mathrm{H}^\sharp_{\mathcal{E}}$ of locally free sheaves over $\mathcal{P}^{(1)}_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_I}$. This $\epsilon^\sharp$ is characterised by the properties that $\Delta^*\epsilon^\sharp = \mathrm{id}$, $\nabla(x) = \epsilon^\sharp(1 \otimes x) - x \otimes 1$ and it satisfies a suitable cocycle condition with respect to the three possible pullbacks of $\epsilon^\sharp$ to $\mathfrak{IG}'_{n,r,I} \times_{\mathfrak{W}^0_I} \mathfrak{IG}'_{n,r,I} \times_{\mathfrak{W}^0_I} \mathfrak{IG}'_{n,r,I}$.

The observation made in the previous paragraph implies that $p_1^*\beta_n = p_2^*\beta_n$ and $p_1^*\eta = p_2^*\eta$ as ideal sheaves on $\mathcal{P}^{(1)}_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_I}$. Thus without any confusion we will write them as β_n and η respectively. Then Lemma 1.4.1 implies that $\epsilon^\sharp\colon p_2^*\mathrm{H}^\sharp_{\mathcal{E}} \xrightarrow{\sim} p_1^*\mathrm{H}^\sharp_{\mathcal{E}}$ is an isomorphism of locally free sheaves satisfying

$$(\epsilon^\sharp \bmod \beta_n)(p_2^*s) = p_1^*s, \qquad (\epsilon^\sharp \bmod \eta)(p_2^*\mathcal{Q}) = p_1^*\mathcal{Q}.$$

Therefore we have an isomorphism

$$\epsilon_0\colon \Big(p_1^*\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q}) \simeq \mathbb{V}_0(p_1^*\mathrm{H}^\sharp_{\mathcal{E}}, p_1^*s, p_1^*\mathcal{Q})\Big) \xrightarrow{\sim} \Big(\mathbb{V}_0(p_2^*\mathrm{H}^\sharp_{\mathcal{E}}, p_2^*s, p_2^*\mathcal{Q}) \simeq p_2^*\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})\Big)$$

by Lemma 1.3.4, such that $\Delta^*\epsilon_0 = \mathrm{id}$. Moreover we have a commutative diagram as follows.

$$\begin{array}{ccc} p_1^*\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q}) & \xrightarrow{\epsilon_0} & p_2^*\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q}) \\ \downarrow & & \downarrow \\ p_1^*\mathbb{V}(\mathrm{H}^\sharp_{\mathcal{E}}) & \xrightarrow{\epsilon} & p_2^*\mathbb{V}(\mathrm{H}^\sharp_{\mathcal{E}}) \end{array}$$

Letting $\rho\colon \mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q}) \to \mathfrak{IG}'_{n,r,I}$, and $g\colon \mathbb{V}(\mathrm{H}^\sharp_{\mathcal{E}}) \to \mathfrak{IG}'_{n,r,I}$, we have isomorphism of sheaves over $\mathcal{P}^{(1)}_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_I}$ as follows.

$$\begin{array}{ccc} p_2^*\rho_*\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})} & \xrightarrow{\epsilon_0^\sharp} & p_1^*\rho_*\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})} \\ \uparrow & & \uparrow \\ p_2^*g_*\mathcal{O}_{\mathbb{V}(\mathrm{H}^\sharp_{\mathcal{E}})} & \xrightarrow{\epsilon^\sharp} & p_1^*g_*\mathcal{O}_{\mathbb{V}(\mathrm{H}^\sharp_{\mathcal{E}})} \end{array}$$

Here $\epsilon^\sharp$ is induced by the $\epsilon^\sharp$ on $\mathrm{H}^\sharp_{\mathcal{E}} \subset g_*\mathcal{O}_{\mathbb{V}(\mathrm{H}^\sharp_{\mathcal{E}})}$. Hence in particular $\epsilon^\sharp$ on $g_*\mathcal{O}_{\mathbb{V}(\mathrm{H}^\sharp_{\mathcal{E}})}$ satisfies the cocycle condition and so does $\epsilon_0^\sharp$. Thus $\epsilon_0^\sharp$ corresponds to a connection on $\rho_*\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})}$. We will study this connection in local coordinates to define the connection on $\mathbb{W}'_{k,I}$.

1.4.3 Gauss-Manin connection on $\mathbb{W}'_{k,I}$

Let Spf $R' \subset \mathfrak{IG}'_{n,r,I}$ be an open affine as in the proof of Lemma 1.4.1, and let $\{e, f\}$ be a basis of $\mathrm{H}^\sharp_{\mathcal{E}}$ as in that proof. Let $I(\Delta) := \ker(R'\hat{\otimes}_{\Lambda^0_I} R' \xrightarrow{\text{mult}} R')$. Let $R^{(1)} = R'\hat{\otimes}_{\Lambda^0_I} R'/I(\Delta)^2$. Then with respect to the basis $\{1 \otimes e, 1 \otimes f\}$ and $\{e \otimes 1, f \otimes 1\}$ we can write the matrix of $\epsilon^\sharp\colon p_2^*\mathrm{H}^\sharp_{\mathcal{E}} \to p_1^*\mathrm{H}^\sharp_{\mathcal{E}}$ as

$$\epsilon^\sharp = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since $\Delta^*\epsilon^\sharp = \mathrm{id}$, we see that $a, d \in 1 + I(\Delta)$ and $b, c \in I(\Delta)$. Moreover, letting $a_0 = a - 1$ and $d_0 = d - 1$, we have $a_0^2 = d_0^2 = b^2 = c^2 = 0$. Using Equation (1.2) we see that $c \cdot \mathrm{Hdg} = \theta$.

Recall, since k is analytic on $1 + \beta_n\mathbb{G}_a$, there exists a unique element $u_k \in p^{1-n}\Lambda^0_I$ such that $k(t) = \exp(u_k \log(t))$ for any $t \in 1 + \beta_n\mathbb{G}_a$.

Theorem 1.4.1. *There is an integrable connection on* $\mathbb{W}'_{k,I}$

$$\nabla_k\colon \mathbb{W}'_{k,I} \to \mathbb{W}'_{k,I}\hat{\otimes}\Omega^1_{\mathfrak{IG}_{n,r,I}}[1/\alpha]$$

which satisfies Griffiths' transversality for the filtration Fil_i *on* $\mathbb{W}'_{k,I}$ *defined on Theorem 1.3.1. Moreover the induced linear map on the graded pieces*

$$\mathrm{Gr}_m(\nabla_k)\colon \mathrm{Gr}_m\mathbb{W}'_{k,I} \to \mathrm{Gr}_{m+1}\mathbb{W}'_{k,I}\hat{\otimes}\Omega^1_{\mathfrak{IG}_{n,r,I}/\Lambda^0_I}[1/\alpha]$$

is an isomorphism times $(u_k - m)$*.*

Proof. By the previous section, we have an isomorphism $\epsilon_0^\sharp\colon p_2^*\rho_*\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}},s,\mathcal{Q})} \to p_1^*\rho_*\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}},s,\mathcal{Q})}$ that induces a connection on $\rho_*\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}},s,\mathcal{Q})}$. Its restriction to $\mathbb{W}'_{k,I}$ will give ∇_k.

Writing $\rho_*\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}},s,\mathcal{Q})|\mathrm{Spf}\,R'} = R'\langle Z, W\rangle$, we have $\mathbb{W}'_{k,I|\mathrm{Spf}\,R'} = R'\langle V\rangle \cdot k(1+\beta_n Z)$ with $V = \frac{W}{1+\beta_n Z}$. Thus to describe the connection on $\mathbb{W}'_{k,I}$ it will be enough to describe $\epsilon_0^\sharp$ on V and $(1+\beta_n Z)$. We have

$$\epsilon_0^\sharp(V) = \eta^{-1}(b+\eta dV)(a+\eta cV)^{-1}, \quad \epsilon_0^\sharp(1+\beta_n Z) = (a+\eta cV)(1+\beta_n Z).$$

From this one can deduce the following formula for $\nabla_k(x) = \epsilon_0^\sharp(1\otimes x) - x\otimes 1$.

$$\begin{aligned}\nabla_k\left(V^m\cdot k(1+\beta_n Z)\right) = \big(&mV^{m-1}\otimes b\eta^{-1} + (u_k-m)V^m\otimes a_0 \\ &+ mV^m\otimes d_0 + \eta(u_k-m)V^{m+1}\otimes c\big)\left(k(1+\beta_n Z)\otimes 1\right)\end{aligned} \tag{1.4}$$

The first part of the theorem follows by noting that the natural map $\Omega^1_{\mathfrak{IG}_{n,r,I}/\Lambda^0_I} \to \Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_I}$ is an isomorphism on the generic fibre, since $\mathcal{IG}'_{n,r,I} \to \mathcal{IG}_{n,r,I}$ is étale. The second statement follows because $c\cdot\mathrm{Hdg} = \theta$ is the Kodaira-Spencer element, which is a generator of $\Omega^1_{\mathfrak{IG}_{n,r,I}/\Lambda^0_I}[1/\alpha]$ since $\mathcal{IG}_{n,r,I}\to\mathcal{X}_{r,I}$ is étale. □

1.4.4 The U operator

Consider the morphisms $p_1, p_2\colon \mathfrak{IG}_{n,r+1,I} \to \mathfrak{IG}_{n,r,I}$ defined generically by sending $\mathcal{E}\mapsto\mathcal{E}$ and $\mathcal{E}\mapsto\mathcal{E}/H_1$ respectively. Here p_2 is the same as $\tilde{F}$ in §1.3.2.2. Let $\lambda\colon\mathcal{E}\to\mathcal{E}' = \mathcal{E}/H_1$ be the projection and let $\lambda^\vee\colon\mathcal{E}'\to\mathcal{E}$ be its dual. The map p_2 is finite flat of degree p on the generic fibre and so induces a well-defined trace map $\mathrm{Tr}\colon p_{2*}\mathcal{O}_{\mathfrak{IG}_{n,r+1,I}} \to \mathcal{O}_{\mathfrak{IG}_{n,r,I}}$.

Proposition 1.4.1. *There is a morphism* $\mathcal{U}\colon p_{2*}p_1^*\mathbb{W}'_{k,I} \to p_{2*}p_2^*\mathbb{W}'_{k,I}$ *induced by the isogeny* $\lambda^\vee$ *that commutes with the Gauss-Manin connection and preserves the filtration on both sides.*

Proof. The morphism $\lambda^\vee$ induces a map $p_1^*\mathrm{H}^\sharp_{\mathcal{E}} \to p_2^*\mathrm{H}^\sharp_{\mathcal{E}}$ as described in Lemma 1.3.5, which sends the marked section to the marked section and the modified unit root subspace to the modified unit root subspace. Then by Lemma 1.3.4 we get a morphism $p_2^*\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}},s,\mathcal{Q}) \to p_1^*\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}},s,\mathcal{Q})$. The required map $\mathcal{U}$ is obtained by taking k-invariants for the $\mathfrak{T}$-action on the induced map on the structure sheaves. □

Definition 1.4.1. The U operator is defined on the global sections of $\mathbb{W}'_{k,I}$ as the composition

$$U\colon H^0(\mathfrak{IG}_{n,r,I},\mathbb{W}'_{k,I}) \xrightarrow{\mathcal{U}\circ p_1^*} H^0(\mathfrak{IG}_{n,r,I}, p_{2*}p_2^*\mathbb{W}'_{k,I}) \xrightarrow{\frac{1}{p}\mathrm{Tr}} H^0(\mathfrak{IG}_{n,r,I},\mathbb{W}'_{k,I})[1/p].$$

1.4.5 p-adic iteration of ∇

In this section we will iterate the Gauss-Manin connection defined above for analytic weights. Since ∇_k maps $\mathbb{W}'_{k,I}[1/p] \to \mathbb{W}'_{k+2,I}[1/p]$, and we are interested in iterating the connection, for simplicity of notation we will write ∇ instead of ∇_k. The strategy is the same as in [AI21]. We first study the rate of convergence of ∇ on the ordinary locus. Here it will be evident that based on our definition of $\mathbb{W}'_{k,I}$, and in particular due to working with $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q})$ instead of $\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s)$, ∇ will converge faster than in loc. cit. This will simplify computations to a great extent. In particular, one can completely avoid the extremely complicated computations of [AI21, Proposition 3.41]. Then we will define the iteration of ∇ on a p-depleted overconvergent form by first defining the iteration on the ordinary locus, and then using a trick similar to [AI21, Proposition 4.11] to prove overconvergence of the resulting modular form.

1.4.5.1 ∇_k on q-expansions

We are going to study the effect of ∇ on q-expansions. Let $E = \mathrm{Tate}(q^N)$ be the Tate curve over $R = \Lambda_I^0((q))$. Fix the canonical basis $(\omega_{\mathrm{can}}, \eta_{\mathrm{can}} = \nabla(\partial)(\omega_{\mathrm{can}}))$ of $\mathrm{H}^\sharp_{\mathcal{E}} = H^1_{\mathrm{dR}}(E/R)$. Here $\partial = q\frac{\mathrm{d}}{\mathrm{d}q}$ is Serre's differential operator. With respect to this basis, the matrix of ∇ is given by

$$\nabla = \begin{pmatrix} 0 & 0 \\ \mathrm{d}q/q & 0 \end{pmatrix}.$$

Let $\mathbb{W}'_{k,I}(q)$ be the pullback of $\mathbb{W}'_{k,I}$ to Spf R via the structure morphism defining E. Then $\mathbb{W}'_{k,I}(q) = R\langle V\rangle \cdot k(1+p^nZ)$. Then it follows from Equation (1.4) that for any $a \in R$,

$$\nabla(aV^h(1+p^nZ)^k) = \partial(a)V^h(1+p^nZ)^{k+2} + p(u_k - h)V^{h+1}(1+p^nZ)^{k+2}. \tag{1.5}$$

Let $V_{k,h} = V^h(1+p^nZ)^k$.

Lemma 1.4.2. *Let $g(q) \in R$ and $N \geq 1$. Then we can write*

$$\nabla^N\left(g(q)V_{k,h}\right) = \sum_{j=0}^{N} p^j a_{N,k,h,j}\partial^{N-j}(g(q))V_{k+2N,h+j}.$$

We have $a_{N,k,h,0} = 1$ and for $j \geq 1$,

$$a_{N,k,h,j} = \binom{N}{j}\prod_{i=1}^{j-1}(u_k - h + N - 1 - i).$$

Proof. We prove the formula for $a_{N,k,h,j}$ by induction on N. For $N = 1$ the statement follows from Equation (1.5). Assume the statement is true for $N = n$. For $j = 0$ or $j = n+1$, the statement is also clear. So we assume that $0 < j < n+1$. In this case Equation (1.5) again gives

$$
\begin{aligned}
&a_{n+1,k,h,j} = a_{n,k,h,j} + (u_k - h + n + n - j + 1)a_{n,k,h,j-1} \\
&= \left[\frac{n!}{(n-j)!(j-1)!}\prod_{i=1}^{j-2}(u_k - h + n - 1 - i)\right]\left[\frac{(u_k - h + n - j)}{j} + \frac{(u_k - h + n + n - j + 1)}{n - j + 1}\right] \\
&= \left[\frac{n!}{(n-j)!(j-1)!}\prod_{i=1}^{j-2}(u_k - h + n - 1 - i)\right](n+1)(u_k - h + n) \\
&= \binom{n+1}{j}\prod_{i=1}^{j-1}(u_k - h + n + 1 - 1 - i).
\end{aligned}
$$

This proves the formula. $\square$

Let $\mathbb{W}'(k) := \sum_{n\geq 0}\mathbb{W}'_{k+2n,I}$ and $\mathbb{W}' := \rho_*\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}},s,\mathcal{Q})}$ where we recall $\rho\colon \mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{E}}, s, \mathcal{Q}) \to \mathfrak{IG}_{n,r,I}$ is the projection.

Corollary 1.4.1. *For any $g \in H^0(\mathfrak{IG}^{ord}_{n,I}, \mathbb{W}'_k)^{U=0}$, and $k = \exp(u_k \log(t))$ for all $t \in 1 + p^n\mathbb{Z}_p$, with $u_k \in \Lambda^0_I$, we have*

$$(\nabla^{p-1} - id)(g) \in pH^0(\mathfrak{IG}^{ord}_{n,I}, \mathbb{W}') \cap H^0(\mathfrak{IG}^{ord}_{n,I}, \mathbb{W}'(k)).$$

Proof. By the q-expansion principle, it will be enough to prove this for $g = g(q)V_{k,h}$ for some $g(q)$ $p-$depleted, i.e. $g(q) = \sum_{p\nmid n} a_n q^n$. By Lemma 1.4.2 it is enough to show that $\partial^{p-1}(g(q))V_{k+2(p-1),h} - g(q)V_{k,h} \in pH^0(\mathfrak{IG}^{\mathrm{ord}}_{n,I}, \mathbb{W}')$. But this follows from the obvious congruences $(1+p^nZ)^{p-1} \equiv 1 \bmod p$ and $\partial^{p-1}(g(q)) \equiv g(q) \bmod p$ for p-depleted $g(q)$. $\square$

Remark 1.4.1. Compare Corollary 1.4.1 with [AI21, Proposition 4.10]. There the authors prove a similar result, viz. for any $g \in H^0(\mathfrak{IG}^{\mathrm{ord}}_{n,I}, \mathbb{W}'_k)^{U=0}$, and k as above,

$$(\nabla^{p-1} - \mathrm{id})^p(g) \in pH^0(\mathfrak{IG}^{\mathrm{ord}}_{n,I}, \mathbb{W}') \cap H^0(\mathfrak{IG}^{\mathrm{ord}}_{n,I}, \mathbb{W}'(k)).$$

So one can see that our techniques yield faster convergence for p-adic iteration of ∇. Moreover the proof of their result relies on the proof of Proposition 3.41 of loc. cit. which is extremely complicated. We can avoid those computations entirely using our method.

1.4.5.2 Iteration of ∇

Before going further we need two preparatory lemmas.

Lemma 1.4.3. *Let $g_n\colon \mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,I}$ be the projection. Then the kernel and cokernel of the map $g_n^*\Omega^1_{\mathfrak{X}_{r,I}} \to \Omega^1_{\mathfrak{IG}_{n,r,I}}$ is killed by a power of Hdg. Let $h_n\colon \mathfrak{IG}'_{n,r,I} \to \mathfrak{IG}_{n,r,I}$ be the projection. Then the kernel of the map $h_n^*\Omega^1_{\mathfrak{IG}_{n,r,I}} \to \Omega^1_{\mathfrak{IG}'_{n,r,I}}$ is killed by a power of Hdg.*

Proof. The map $g_n^*\Omega^1_{\mathfrak{X}_{r,I}} \to \Omega^1_{\mathfrak{IG}_{n,r,I}}$ is an isomorphism over the ordinary locus. In particular the coherent sheaves kernel and cokernel are killed by tensoring with $\mathcal{O}_{\mathfrak{IG}_{n,r,I}}\langle\frac{1}{\mathrm{Hdg}}\rangle$. Since the completion $\mathcal{O}_{\mathfrak{IG}_{n,r,I}}[\frac{1}{\mathrm{Hdg}}] \to \mathcal{O}_{\mathfrak{IG}_{n,r,I}}\langle\frac{1}{\mathrm{Hdg}}\rangle$ is faithfully flat, the claim follows.

For the second statement, we note that over the ordinary locus, $\mathfrak{IG}'_{n,r,I} \to \mathfrak{IG}_{n,r,I}$ is a torsor for a group which is an extension of $\mathbb{Z}/p^n\mathbb{Z}^\times$ by μ_{p^n}. Indeed we can form an intermediate $\mathbb{Z}/p^n\mathbb{Z}^\times$-torsor $\mathfrak{IG}''^{,\mathrm{ord}}_{n,,I}$ over $\mathfrak{IG}^{\mathrm{ord}}_{n,I}$ that classifies trivializations of both the connected and étale parts of the p^n torsion of elliptic curves. Then $\mathfrak{IG}'^{,\mathrm{ord}}_{n,,I} \to \mathfrak{IG}''^{,\mathrm{ord}}_{n,,I}$ is the μ_{p^n}-torsor classifying splittings of the extension $0 \to \mu_{p^n} \to \mathcal{E}[p^n] \to \mathbb{Z}/p^n\mathbb{Z} \to 0$ away from the cusps. We only need to show the injectivity on Kahler differentials for this μ_{p^n}-torsor $\mathfrak{IG}'^{,\mathrm{ord}}_{n,,I} \to \mathfrak{IG}''^{,\mathrm{ord}}_{n,,I}$. Since $\mathfrak{IG}''^{,\mathrm{ord}}_{n,,I}$ is smooth, this reduces to checking injectivity on Kahler differentials for the map $\mu_{p^n} \times \mathbb{A}^1 \to \mathbb{A}^1$, where it is obvious. This proves that the map $h_n^*\Omega^1_{\mathfrak{IG}_{n,r,I}} \to \Omega^1_{\mathfrak{IG}'_{n,r,I}}$ is injective over the ordinary locus. Then the same argument as the first part completes the proof of the claim. $\square$

Lemma 1.4.4. *Let $C_n = p^{a+r+1}(n-1)+1$, where we recall $I = [p^a, p^b]$ (this has to do with the fact that $p/\mathrm{Hdg}^{p^{a+r+1}} \in \mathcal{O}_{\mathfrak{X}_{r,I}}$). Then the kernel of the restiction map $\mathcal{O}_{\mathfrak{IG}_{n,r,I}}/(\alpha^j) \xrightarrow{\phi_1} \mathcal{O}_{\mathfrak{IG}^{ord}_{n,I}}/(\alpha^j)$ is killed by* $\mathrm{Hdg}^{j(p^{r+1})+C_n}$.

Proof. It is clear from the local coordinates of $\mathfrak{X}_{r,I}$ that the kernel of $\mathcal{O}_{\mathfrak{X}_{r,I}}/(\alpha^j) \xrightarrow{\phi_0} \mathcal{O}_{\mathfrak{X}^{\mathrm{ord}}_I}/(\alpha^j)$ is killed by $\mathrm{Hdg}^{j(p^{r+1})}$. The trace map $\mathrm{Tr}\colon g_{n*}\mathcal{O}_{\mathfrak{IG}_{n,r,I}} \to \mathcal{O}_{\mathfrak{X}_{r,I}}$ then gives a commutative diagram as follows.

$$\begin{tikzcd}
0 \arrow[r] & \ker\phi_1 \arrow[r] \arrow[d] & \mathcal{O}_{\mathfrak{IG}_{n,r,I}}/(\alpha^j) \arrow[r] \arrow[d, "\mathrm{Tr}"] & \mathcal{O}_{\mathfrak{IG}^{\mathrm{ord}}_{n,I}}/(\alpha^j) \arrow[d, "\mathrm{Tr}"] \\
0 \arrow[r] & \ker\phi_0 \arrow[r] & \mathcal{O}_{\mathfrak{X}_{r,I}}/(\alpha^j) \arrow[r] & \mathcal{O}_{\mathfrak{X}^{\mathrm{ord}}_{\alpha,I}}/(\alpha^j)
\end{tikzcd}$$

Suppose $x \in \ker\phi_1$. Then $\mathrm{Tr}(x) \in \ker\phi_0$ and hence $\mathrm{Tr}(\mathrm{Hdg}^{j(p^{r+1})}x) = 0$. In other words, for any lift $\tilde{x} \in \mathcal{O}_{\mathfrak{IG}_{n,r,I}}$ of x, $\mathrm{Tr}(\mathrm{Hdg}^{j(p^{r+1})}\tilde{x}) \in \alpha^j\mathcal{O}_{\mathfrak{X}_{r,I}}$. Let $\mathfrak{D}^{-1} := \{y \in \mathrm{Frac}(\mathcal{O}_{\mathfrak{IG}_{n,r,I}}) \mid \mathrm{Tr}(yz) \in \mathcal{O}_{\mathfrak{X}_{r,I}} \text{ for all } z \in \mathcal{O}_{\mathfrak{IG}_{n,r,I}}\}$. Then $\mathrm{Hdg}^{j(p^{r+1})}\tilde{x} \in \alpha^j\mathfrak{D}^{-1}$ as $\ker\phi_1$ is an ideal. By using normality of the rings involved, the result follows by localizing at height 1 primes and noting that $\mathfrak{D}^{-1}$ is the usual inverse different in such extension of DVR's. $\square$

Remark 1.4.2. We note that the above Lemma is a weaker version of [AI21, Lemma 3.4], where the authors have a better estimate of the constant C_n (precisely $\frac{p^n-p}{p-1}$). Also in our proof we rely on the fact that $p \neq 0$ in Λ^0_I, i.e. we are away from the point at "∞" in the weight space, which they don't need to assume. But as we shall see, this difference will not be much of a problem for our application, which is to prove the convergence of p-adic iteration of ∇.

Assumption 1.1. Let $k\colon \mathbb{Z}_p^\times \to (\Lambda^0_I)^\times$ be a weight such that there exists $u_k \in \Lambda^0_I$ satisfying $k(t) = \exp(u_k\log(t))$ for all $t \in 1+p\mathbb{Z}_p$. In particular, we assume that k is a point in $\mathcal{W}^0_{[0,1]}$ and hence we can take $\alpha = p$.

By Lemma 1.4.3 and the explicit local description of ∇ in Equation (1.4), we see that there exists an integer $D>0$ such that $\nabla(\mathbb{W}'_k) \subset \frac{1}{p\mathrm{Hdg}^D}\mathbb{W}'_{k+2}$. In particular, for any $N \geq 1$

$$(\nabla^{p-1}-\mathrm{id})^N(\mathbb{W}'_{k,I}) \subset \frac{1}{p^{(p-1)N}\mathrm{Hdg}^{D(p-1)N}}\mathbb{W}'(k).$$

Lemma 1.4.5. *There exists an integer ℓ depending on r, n and p, and an integer $C>0$, such that for any $g \in H^0(\mathfrak{IG}_{n,r,I}, \mathbb{W}'_{k,I})^{U=0}$, and every positive integer N, we have*

$$\left(\nabla^{p-1}-id\right)^N(g) \in \left(\frac{p}{\mathrm{Hdg}^C}\right)^N H^0(\mathfrak{IG}_{n,\ell,I}, \mathbb{W}') \cap H^0(\mathfrak{IG}_{n,\ell,I}, \mathbb{W}'(k)).$$

Proof. By Corollary 1.4.1 we see that

$$p^{(p-1)N}\left(\nabla^{p-1}-\mathrm{id}\right)^N(g)_{|\mathfrak{IG}^{\mathrm{ord}}_{n,I}} \in p^{pN}H^0(\mathfrak{IG}^{\mathrm{ord}}_{n,I}, \mathbb{W}') \cap H^0(\mathfrak{IG}^{\mathrm{ord}}_{n,I}, \mathbb{W}'(k)).$$

Locally on $\mathfrak{IG}_{n,r,I}$, we then have that

$$p^{(p-1)N}\mathrm{Hdg}^{D(p-1)N}\left(\nabla^{p-1}-\mathrm{id}\right)^N(g) \in \ker\left(\mathbb{W}'/(p^{pN}) \to \mathbb{W}'_{\mathrm{ord}}/(p^{pN})\right).$$

Here $\mathbb{W}'_{\mathrm{ord}} = \mathbb{W}'_{|\mathfrak{IG}^{\mathrm{ord}}_{n,I}}$. Recall that $\mathbb{W}'/(p^j)$ is a polynomial algebra over $\mathcal{O}_{\mathfrak{IG}_{n,r,I}}/(p^j)$ for any j. Thus by Lemma 1.4.4 this kernel is killed by $\mathrm{Hdg}^{pN(p^{r+1})+C_n}$. This implies that

$$p^{(p-1)N}\mathrm{Hdg}^{N(p(p^{r+1})+D(p-1)+C_n}\left(\nabla^{p-1}-\mathrm{id}\right)^N(g) \in p^{pN}H^0(\mathfrak{IG}_{n,r,I}, \mathbb{W}')\cap H^0(\mathfrak{IG}_{n,r,I}, \mathbb{W}'(k)).$$

In particular, choosing $C \gg 0$, such that $CN \geq N(p^{r+2}+D(p-1))+C_n$ for all $N \geq 0$, we see that

$$\mathrm{Hdg}^{CN}\left(\nabla^{p-1}-\mathrm{id}\right)^N(g) \in p^N H^0(\mathfrak{IG}_{n,r,I}, \mathbb{W}') \cap H^0(\mathfrak{IG}_{n,r,I}, \mathbb{W}'(k)).$$

Choosing $\ell \geq r$ such that $p/\mathrm{Hdg}^C \in \mathcal{O}_{\mathfrak{X}_{\ell,I}}$, we get that

$$\left(\nabla^{p-1}-\mathrm{id}\right)^N(g) \in \left(\frac{p}{\mathrm{Hdg}^C}\right)^N H^0(\mathfrak{IG}_{n,\ell,I}, \mathbb{W}') \cap H^0(\mathfrak{IG}_{n,\ell,I}, \mathbb{W}'(k)).$$

□

Proposition 1.4.2. *Let $s\colon \mathbb{Z}_p^\times \to (\Lambda_I^0)^\times$ be a weight such that $s = \exp(u_s\log(t))$ for any $t \in 1+p^n\mathbb{Z}_p$ and $u_s \in \Lambda_I^0$. Then for any prime $p \geq 3$, there exists an integer ℓ depending on r, n and p, such that for every $g \in H^0(\mathfrak{IG}_{n,r,I}, \mathbb{W}'_{k,I})^{U=0}$, the sequences*

$$A(g,s)_m := \sum_{j=1}^m (-1)^{j-1}\frac{(\nabla^{p-1}-id)^j(g)}{j}$$

and if we write $H_{i,m}$ for the set of tuples $(j_1, \cdots, j_i)$ of i positive integers with $j_1 + \cdots + j_i \leq m$,

$$B(g,s)_m := \sum_{i=0}^{m} \frac{u_s^i}{i!(p-1)^i} \left(\sum_{(j_1,\ldots,j_i)\in H_{i,m}} \left(\prod_{a=1}^{i} \frac{(-1)^{j_a}-1}{j_a} \right) (\nabla^{p-1} - id)^{j_1+\cdots+j_i} \right) (g)$$

converge in $H^0(\mathfrak{IG}_{n,\ell,I}, \mathbb{W}')$ for all $m \geq 0$. Moreover, if we denote the limits

$$\log\left(\nabla^{p-1}\right)(g) := \lim_{m\to\infty} A(g,s)_m$$

and

$$\nabla^s(g) = \exp(\frac{u_s}{p-1}\log(\nabla^{p-1}))(g) := \lim_{m\to\infty} B(g,s)_m$$

then $\nabla^s(g) \in H^0(\mathfrak{IG}_{n,\ell,I}, \mathbb{W}'_{k+2s,I})$. The same results hold for $p=2$ if $u_s \in 4\Lambda_I^0$.

Proof. The convergence of $A(g,s)_m$ is clear from Lemma 1.4.5. We prove convergence for $B(g,s)_m$. Let's first deal with the case $p \geq 3$. Let

$$X := \frac{(\nabla^{p-1} - \mathrm{id})^{j_1+\cdots+j_i}(g)}{i!\prod j_a}.$$

Then by Lemma 1.4.5, $X \in (p/\mathrm{Hdg}^C)^{\sum j_a - v_p(i!) - \sum v_p(j_a)} H^0(\mathfrak{IG}_{n,\ell,I}, \mathbb{W}')$. Now $v_p(i!) \leq \frac{i-1}{p-1} \leq \frac{i}{p-1}$. Hence $v_p(j_a) \leq \frac{j_a-1}{p-1}$ too. Using these inequalities,

$$\sum_{a=1}^{i} j_a - v_p(i!) - \sum_{a=1}^{i} v_p(j_a) \geq \sum_{a=1}^{i} \left(j_a - \frac{1}{p-1} - v_p(j_a) \right) \geq \sum_{a=1}^{i} j_a \left(1 - \frac{1}{p-1} \right).$$

This proves convergence in this case. For the case $p=2$ we note that the terms $\frac{(\nabla - \mathrm{id})^{j_1+\cdots+j_i}(g)}{\prod j_a}$ do not have poles and the term $u_s^i/i!$ is divisible by 2^i, which gives convergence in this case. Finally $\nabla^s(g) \in H^0(\mathfrak{IG}_{n,\ell,I}, \mathbb{W}'_{k+2s,I})$ as can be seen from the fact that $t * \nabla(g) = t^{k+2}\nabla(g)$ for any $t \in \mathbb{Z}_p^\times$. $\square$

Chapter 2

Overconvergent Modular and de Rham Sheaves for $\operatorname{Res}_{L/\mathbb{Q}} \mathbf{GL}_{2,L}$

2.1 The setup

Notation

Let L be a totally real number field of degree $[L : \mathbb{Q}] = g$. Denote by $\mathfrak{d}$ the different ideal of $\mathcal{O}_L$. Fix an integer $N \geq 4$. Let $p \nmid N$ be a prime which is unramified in L. Suppose p splits as $p = \mathfrak{P}_1 \cdots \mathfrak{P}_r$ and let their inertia degree be $f(\mathfrak{P}_i|p) = f_i$. Fix a finite unramified Galois extension K of $\mathbb{Q}_p$ where L is split and let $\Sigma := \{\sigma \colon L \to K\}$ be the set of embeddings of L in K. Let $q = p$ if $p \neq 2$ and $q = 4$ otherwise.

2.1.1 The weight space

Let $\mathbb{T} = \operatorname{Res}_{\mathcal{O}_L/\mathbb{Z}} \mathbb{G}_m$. Then $\mathbb{T}(\mathbb{Z}_p) = (\mathcal{O}_L \otimes \mathbb{Z}_p)^\times$. Denote by $\Lambda := \mathcal{O}_K[\![\mathbb{T}(\mathbb{Z}_p)]\!]$ the base change of the Iwasawa algebra $\mathbb{Z}_p[\![\mathbb{T}(\mathbb{Z}_p)]\!]$ to $\mathcal{O}_K$. Then $\Lambda = \mathcal{O}_K[\Delta][\![\mathbb{T}(\mathbb{Z}_p)_{\text{tf}}]\!] \simeq \mathcal{O}_K[\Delta][\![T_1, \dots, T_g]\!]$, where $\Delta \subset \mathbb{T}(\mathbb{Z}_p)$ is the torsion subgroup and we choose an isomorphism $\mathbb{T}(\mathbb{Z}_p)_{\text{tf}} \simeq \mathbb{Z}_p^g$ of the torsion free part with $\mathbb{Z}_p^g$. Under this isomorphism, the standard basis elements e_i of $\mathbb{Z}_p^g$ are sent to $1 + T_i$. Let $\Lambda^0 = \mathcal{O}_K[\![\mathbb{T}(\mathbb{Z}_p)_{\text{tf}}]\!]$ be the quotient of Λ that sends $\Delta \mapsto 1$, and let $\mathfrak{m} = (p, T_1, \dots, T_g)$ be its maximal ideal.

Let $\mathfrak{W} = \operatorname{Spf} \Lambda$ and $\mathfrak{W}^0 = \operatorname{Spf} \Lambda^0$. Let $\mathcal{W} = \operatorname{Spa}(\Lambda, \Lambda)^{\text{an}}$ be the analytic adic space associated to $\mathfrak{W}$ and similarly define $\mathcal{W}^0 := \operatorname{Spa}(\Lambda^0, \Lambda^0)^{\text{an}}$. $\mathcal{W}$ satisfies the following universal property: for any complete Huber pair (R, R^+) over $(\mathbb{Q}_p, \mathbb{Z}_p)$,

$$\operatorname{Hom}_{\operatorname{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)} \left(\operatorname{Spa}(R, R^+), \mathcal{W} \right) = \operatorname{Hom}_{\text{gp-sch}} \left(\mathbb{T}(\mathbb{Z}_p)_{R^+}, \mathbb{G}_{m\,R^+} \right) = \operatorname{Hom}^{\text{cont}}_{\mathbb{Z}} \left(\mathbb{T}(\mathbb{Z}_p), (R^+)^\times \right).$$

$\mathcal{W}^0$ satisfies a similar universal property with respect to $\mathbb{T}(\mathbb{Z}_p)_{\text{tf}}$, i.e.

$$\operatorname{Hom}_{\operatorname{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)} \left(\operatorname{Spa}(R, R^+), \mathcal{W}^0 \right) = \operatorname{Hom}^{\text{cont}}_{\mathbb{Z}} \left(\mathbb{T}(\mathbb{Z}_p)_{\text{tf}}, (R^+)^\times \right).$$

Let $\widetilde{\mathfrak{W}^0}$ be the admissible blow-up of $\mathfrak{W}^0$ along $\mathfrak{m}$. For $\alpha \in \mathfrak{m} \setminus \mathfrak{m}^2$, let $\mathfrak{W}^0_\alpha$ be the open in $\widetilde{\mathfrak{W}^0}$ where $\mathfrak{m}$ is generated by α. Suppose $\mathfrak{W}^0_\alpha = \mathrm{Spf}\, B^0_\alpha$. Let $\mathcal{W}^0_\alpha$ be the rational open in $\mathcal{W}^0$ obtained by taking the generic fibre of $\mathfrak{W}^0_\alpha$. This is given by the affinoid adic $\mathrm{Spa}(B^0_\alpha[1/\alpha], B^0_\alpha)$ and is thus the adic spectrum of a Tate ring. The $\mathfrak{W}^0_\alpha$ cover $\widetilde{\mathfrak{W}^0}$ for varying α and hence their adic generic fibre cover $\mathcal{W}^0$. In particular, the map induced by $\widetilde{\mathfrak{W}^0} \to \mathfrak{W}^0$ on the associated analytic adic spaces is an isomorphism.

The natural inclusion $\Lambda^0 \to \Lambda$ is finite flat and the induced finite flat morphism $\mathcal{W} \to \mathcal{W}^0$ realizes $\mathcal{W}$ as a disjoint union of copies of $\mathcal{W}^0$ indexed by Δ. For $\alpha \in \mathfrak{m} \setminus \mathfrak{m}^2$ we let $\mathcal{W}_\alpha$ be the inverse image of $\mathcal{W}^0_\alpha$ under this morphism.

We remark that all classical weights can be realized as points in $\mathcal{W}_p$.

For $I = [p^a, p^b]$ with $a \in \mathbb{N} \cup \{-\infty\}$ and $b \in \mathbb{N} \cup \{\infty\}$, let $\mathcal{W}^0_{\alpha,I} \subset \mathcal{W}^0_\alpha$ be the rational open subset defined as follows:

$$\mathcal{W}^0_{\alpha,I} := \{x \in \mathcal{W}^0_\alpha \,:\, |p(x)| \leq |\alpha(x)|^{p^a} \neq 0,\ |\alpha(x)|^{p^b} \leq |p(x)| \neq 0\}.$$

Let $\Lambda^0_{\alpha,I} := \Gamma(\mathcal{W}^0_{\alpha,I}, \mathcal{O}^+_{\mathcal{W}^0_{\alpha,I}})$ and $\Lambda_{\alpha,I} := \Gamma(\mathcal{W}_{\alpha,I}, \mathcal{O}^+_{\mathcal{W}_{\alpha,I}})$. Let $\mathfrak{W}^0_{\alpha,I} := \mathrm{Spf}\, \Lambda^0_{\alpha,I}$ and $\mathfrak{W}_{\alpha,I} := \mathrm{Spf}\, \Lambda_{\alpha,I}$.

For α varying in $\mathfrak{m} \setminus \mathfrak{m}^2$, the different $\mathfrak{W}^0_{\alpha,I}$ glue together to form a formal scheme $\mathfrak{W}^0_I$ with adic generic fibre $\mathcal{W}^0_I$. $\mathcal{W}^0_I$ can be described as follows.

$$\mathcal{W}^0_I = \{x \in \mathcal{W}^0 \,:\, (\exists \alpha \in \mathfrak{m}, |p|_x \leq |\alpha^{p^a}|_x \neq 0) \wedge (\forall \alpha \in \mathfrak{m}, |\alpha^{p^b}|_x \leq |p|_x \neq 0)\}$$

Then $\mathcal{W}^0_{[0,1]} = \mathcal{W}^0_p$ and $\mathcal{W}^0_{[1,\infty]} = \mathcal{W}^0$. At the level of formal models, $\mathfrak{W}^0_{[0,1]} = \mathfrak{W}^0_p$ and $\mathfrak{W}^0_{[1,\infty]} = \widetilde{\mathfrak{W}^0}$.

We fix one such α. For the purpose of defining the p-adic iteration of the Gauss-Manin connection, which is the technical heart of this work (§2.4.4) we need to assume $\alpha = p$. For everything before that section we can choose any α. But we should also mention that the construction of the main objects of this work, i.e. the interpolation sheaves of modular forms and de Rahm classes of varying weight k, all take place over a weight space where $p \neq 0$. In particular we do not study the perfect overconvergent modular forms of [AIP18] or [AIP16b].

Analyticity of the universal character:
Let $k^{\mathrm{un}} \colon \mathbb{T}(\mathbb{Z}_p) \to \Lambda$ be the universal character. Denote by $k^0 \colon \mathbb{T}(\mathbb{Z}_p) \to \Lambda \to \Lambda^0$ the character obtained by composing the universal character with the projection onto the component of the trivial character, and let $k^0_{\alpha,I} \colon \mathbb{T}(\mathbb{Z}_p) \to \Lambda^0 \to \Lambda^0_{\alpha,I}$ be its restriction to $\mathcal{W}^0_{\alpha,I}$.

Lemma 2.1.1. *For $I \subset [0, q^{-1}p^n]$, the restriction of $k^0_{\alpha,I}$ to $1 + qp^{n-1}(\mathcal{O}_L \otimes \mathbb{Z}_p)$ is analytic. Thus it extends to a character*

$$k^0_{\alpha,I} \colon \mathcal{W}^0_{\alpha,I} \times \mathbb{T}(\mathbb{Z}_p)(1 + qp^{n-1}\,\mathrm{Res}_{\mathcal{O}_L/\mathbb{Z}}\, \mathbb{G}_a^+) \to \mathbb{G}_m^+$$

which restricts to a character

$$k^0_{\alpha,I} \colon \mathcal{W}^0_{\alpha,I} \times (1 + qp^{n+m-1}\,\mathrm{Res}_{\mathcal{O}_L/\mathbb{Z}}\, \mathbb{G}_a^+) \to 1 + qp^m \mathbb{G}_a^+.$$

Proof. This is an adaptation of the proof of [AIP18, Proposition 2.1]. See also [AIP16b, Proposition 2.8]. □

Since $k^0_{\alpha,I}$ is analytic, and $1+qp^{n-1}\operatorname{Res}_{\mathcal{O}_L/\mathbb{Z}}\mathbb{G}_a \simeq \prod_{\sigma\in\Sigma} 1+qp^{n-1}\mathbb{G}_a$, the universal character splits into components $k^0_{\alpha,I} = \prod_{\sigma\in\Sigma} k_\sigma$.

Remark 2.1.1. Note that the analyticity of the character does not depend on α. So we can glue the different $k^0_{\alpha,I}$ together to obtain a character $k^0_I\colon \mathcal{W}^0_I \times \mathbb{T}(\mathbb{Z}_p)(1+qp^{n-1}\operatorname{Res}_{\mathcal{O}_L/\mathbb{Z}_p}\mathbb{G}_a^+) \to \mathbb{G}_m^+$.

2.1.1.1 The weight space for the group $\operatorname{Res}_{L/\mathbb{Q}}\mathbf{GL}_{2,L}$

In the literature there are two distinct notions of Hilbert modular forms. The first one is realized as sections of a modular sheaf on moduli of abelian varieties with real multiplication and additional data. The associated weight space is the one we defined above. There is also the notion of arithmetic Hilbert modular forms which are sections of automorphic line bundles on the Shimura variety associated to the group $G := \operatorname{Res}_{L/\mathbb{Q}}\mathbf{GL}_{2,L}$. It is necessary to consider these modular forms for arithmetic applications. The relation between these two different notions will be clarified in the next section. Here we will define the weight space associated to the group G.

Let $\Lambda^G := \mathcal{O}_K[\![\mathbb{T}(\mathbb{Z}_p)\times\mathbb{Z}_p^\times]\!]$, and let $\mathfrak{W}^G := \operatorname{Spf}\Lambda^G$. Let $\mathcal{W}^G := \operatorname{Spa}(\Lambda^G,\Lambda^G)^{\mathrm{an}}$ be the associated analytic adic space. There is a natural morphism $\mathfrak{W}^G \to \mathfrak{W}$ given by the map $\Lambda \to \Lambda^G$ induced by $(\mathcal{O}_L\otimes\mathbb{Z}_p)^\times \to (\mathcal{O}_L\otimes\mathbb{Z}_p)^\times\times\mathbb{Z}_p^\times$ sending $t\mapsto(t^2,\mathrm{N}_{L/\mathbb{Q}}(t))$. This induces a map on the corresponding analytic adic spaces. On classical points this map can be described as sending $(v,w)\in\mathcal{W}^G(\mathbb{C}_p)\mapsto v^2\cdot(w\circ\mathrm{N}_{L/\mathbb{Q}})$, where $v\colon(\mathcal{O}_L\times\mathbb{Z}_p)^\times\to\mathbb{C}_p^\times$ is a continuous character and so is $w\colon\mathbb{Z}_p^\times\to\mathbb{C}_p^\times$. Denote the universal character by $k_G^{\mathrm{un}}\colon(\mathcal{O}_L\times\mathbb{Z}_p)^\times\times\mathbb{Z}_p^\times\to(\Lambda^G)^\times$.

2.1.2 The Hilbert modular variety

Let $\mathfrak{c}$ be a fractional ideal of L and let $\mathfrak{c}^+$ be the cone of totally positive elements. Let $M(\mu_N,\mathfrak{c})$ be the moduli scheme over $\mathbb{Z}_p$ classifying tuples $(A_{/S},\iota,\lambda,\psi)$ consisting of (1) an abelian scheme $A\to S$ for any $\mathbb{Z}_p$-scheme S, (2) an embedding $\iota\colon\mathcal{O}_L\hookrightarrow\operatorname{End}_S(A)$, (3) if $P\subset\operatorname{Hom}_{\mathcal{O}_L}(A,A^\vee)$ is the étale sheaf of symmetric $\mathcal{O}_L$-linear homomorphisms from A to its dual $A^\vee$, and P^+ is the cone of polarizations, then an isomorphism $\lambda\colon(P,P^+)\simeq(\mathfrak{c},\mathfrak{c}^+)$ of étale sheaves of invertible $\mathcal{O}_L$-modules with a notion of positivity such that the induced map $A\otimes\mathfrak{c}\xrightarrow{\sim}A^\vee$ is an isomorphism (the Deligne-Pappas condition), and (4) a closed immersion $\psi\colon\mu_N\otimes\mathfrak{d}^{-1}\hookrightarrow A[N]$ compatible with $\mathcal{O}_L$-action.

Let $[\mathfrak{c}]\in\mathrm{Cl}^+(L)$ be the class of $\mathfrak{c}$ in the strict class group of L. Any two representatives of a class are related via multiplication by a totally positive unit, and hence the corresponding moduli problems are isomorphic. Using this, we henceforth fix $\mathfrak{c}$ coprime to p, since that will ensure upon a choice of a generator of $\mathfrak{c}\otimes\mathbb{Z}_p$ as an $\mathcal{O}_L\otimes\mathbb{Z}_p$-module, the p-divisible groups of $\mathfrak{c}$-polarized abelian varieties satisfying the Deligne-Pappas condition are principally polarized.

Let $\bar{M}(\mu_N,\mathfrak{c})$ be a toroidal compactification of $M(\mu_N,\mathfrak{c})$ and let $M^*(\mu_N,\mathfrak{c})$ be the minimal compactification. There is a semi-abelian scheme $\pi\colon\mathcal{A}\to\bar{M}(\mu_N,\mathfrak{c})$ which restricts to the universal abelian scheme over $M(\mu_N,\mathfrak{c})$ and degenerates to a torus at the cusps. Let $\omega_{\mathcal{A}}$ be the canonical extension of the sheaf of invariant differentials of the universal abelian scheme to the cusps. There is a largest open subscheme $M^R(\mu_N, \mathfrak{c}) \subset \bar{M}(\mu_N, \mathfrak{c})$, called the Rapoport locus where $\omega_{\mathcal{A}}$ is an invertible $\mathcal{O}_L \otimes \mathcal{O}_{M^R(\mu_N, \mathfrak{c})}$-module. Since we assume p is unramified in L, the complement of the Rapoport locus is empty and $\bar{M}(\mu_N, \mathfrak{c})$ is projective and smooth over $\operatorname{Spec} \mathbb{Z}_p$ [DP94]. The boundary $D := \bar{M}(\mu_N, \mathfrak{c}) \setminus M(\mu_N, \mathfrak{c})$ is a relative normal crossings divisor. The minimal compactification $M^*(\mu_N, \mathfrak{c})$ is normal and projective.

Henceforth denote $\bar{M}(\mu_N, \mathfrak{c})$ by X. Let $\mathfrak{X}$ be its p-adic completion. Let $\mathrm{H}_{\mathcal{A}}$ be the canonical extension of the relative de Rham sheaf of the universal abelian scheme to the cusps. It is a locally free $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{X}}$-module of rank 2 and it is endowed with an integrable connection $\nabla \colon \mathrm{H}_{\mathcal{A}} \to \mathrm{H}_{\mathcal{A}} \otimes \Omega^1_{\mathfrak{X}/\mathbb{Z}_p}(\log(D))$ called the Gauss-Manin connection. It also fits in an exact sequence

$$0 \to \omega_{\mathcal{A}} \to \mathrm{H}_{\mathcal{A}} \to \omega^{\vee}_{\mathcal{A}^{\vee}} \to 0 \tag{2.1}$$

which defines the Hodge filtration. The maps in the sequence are $\mathcal{O}_L$-linear. We also use the principal polarization of the p-divisible group of $\mathcal{A}$ to henceforth identify $\omega_{\mathcal{A}^{\vee}}$ and $\omega_{\mathcal{A}}$.

Fix $\alpha \in \mathfrak{m} \setminus \mathfrak{m}^2$ and I as above. Let $\mathfrak{X}_{\alpha,I} := \mathfrak{X} \times_{\operatorname{Spf} \mathbb{Z}_p} \mathfrak{W}^0_{\alpha,I}$. Since L splits in K, $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{X}_{\alpha,I}} \simeq \prod_{\sigma \in \Sigma} \mathcal{O}_{\mathfrak{X}_{\alpha,I}}$. Later it will be useful to collect the σ's in r different groups according to the valuation they induce on $\mathcal{O}_L$ (recall $p = \mathfrak{P}_1 \cdots \mathfrak{P}_r$ in L). Hence over $\mathfrak{X}_{\alpha,I}$, the exact sequence (2.1) splits into g exact sequences

$$0 \to \omega_{\mathcal{A}}(\sigma) \to \mathrm{H}_{\mathcal{A}}(\sigma) \to \omega^{\vee}_{\mathcal{A}}(\sigma) \to 0$$

where each $\omega_{\mathcal{A}}(\sigma)$ is an invertible $\mathcal{O}_{\mathfrak{X}_{\alpha,I}}$-module and $\mathrm{H}_{\mathcal{A}}(\sigma)$ is a locally free $\mathcal{O}_{\mathfrak{X}_{\alpha,I}}$-module of rank 2.

Let $\xi \colon \mathfrak{X}_{\alpha,I}/(\alpha) \hookrightarrow \mathfrak{X}_{\alpha,I}$ be the closed subscheme defined by $\alpha = 0$. The Hasse invariant is a section $\mathrm{Ha} \in (\Lambda^g(\xi^* \omega_{\mathcal{A}}))^{\otimes(p-1)}$. Define the Hasse ideal to be $\underline{\mathrm{Ha}} := \mathrm{Ha} \cdot (\Lambda^g(\xi^*\omega_{\mathcal{A}}))^{\otimes(1-p)}$.

Theorem 2.1.1. *The Hasse invariant vanishes with multiplicity one along the irreducible components of its divisor.*

Proof. [AG05, Corollary 8.18]. □

With the notation above, for each $r \geq 1$, consider the inverse image of $\underline{\mathrm{Ha}}^{p^{r+1}}$ under the map $\mathcal{O}_{\mathfrak{X}_{\alpha,I}} \to \mathcal{O}_{\mathfrak{X}_{\alpha,I}/(\alpha)}$ and call this ideal Hdg_r. We call a local lift of a generator of $\underline{\mathrm{Ha}}$ as Hdg. Then locally Hdg_r is the ideal $(\alpha, \mathrm{Hdg}^{p^{r+1}})$.

Recall a classical weight is an element of $\mathbb{Z}[\Sigma]$. The Hilbert modular sheaf of a classical weight $k = \sum n_\sigma \sigma$ is defined as

$$\omega^k_{\mathcal{A}} := \hat{\bigotimes_{\sigma}} \omega_{\mathcal{A}}(\sigma)^{n_\sigma}.$$

Note that $\omega^k_{\mathcal{A}}$ is obtained as the image of $\omega_{\mathcal{A}}$ under the map induced by change of the structural group

$$H^1(\mathfrak{X}_{\alpha,I}, (\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{X}_{\alpha,I}})^{\times}) \xrightarrow{k} H^1(\mathfrak{X}_{\alpha,I}, \mathcal{O}^{\times}_{\mathfrak{X}_{\alpha,I}}).$$

2.1.2.1 The Shimura variety associated to G

Let $\mathrm{Sh}_K(G)$ be the Shimura variety associated to the group $G = \mathrm{Res}_{L/\mathbb{Q}}\,\mathbf{GL}_{2,L}$ and level subgroup $K = K_1(N)$,

$$K_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\hat{\mathcal{O}}_L) \mid a \equiv 1, c \equiv 0 \bmod N \right\}$$

whose complex points are $\mathrm{Sh}_K(G)(\mathbb{C}) = G(\mathbb{Q})\backslash(\mathfrak{h}^{\pm})^{\Sigma} \times G(\mathbb{A}^{\infty})/K$. Here $\mathfrak{h}$ is the Poincaré upper-half plane endowed with the usual action of $\mathbf{GL}_2(\mathbb{R})$ via Möbius transformation. The Shimura variety $\mathrm{Sh}_K(G)$ is defined over its reflex field $\mathbb{Q}$. For a weight $(v, w) \in \mathbb{Z}[\Sigma] \times \mathbb{Z}$, one can define the automorphic line bundle $\underline{\omega}^{(v,w)}$ on $\mathrm{Sh}_K(G)(\mathbb{C})$ as follows. Let $k = 2v + wt_L$, where $t_L = \sum_{\tau\in\Sigma} \tau$ is the generator of parallel weights. Consider the compact dual $\mathbb{P}^1_{\mathbb{C}}$ of $\mathfrak{h}$, and let ω be the dual of the tautological quotient bundle on $\mathbb{P}^1_{\mathbb{C}}$. There is a natural action on ω of $\mathbf{GL}_2(\mathbb{C})$ for which the projection $p\colon \omega \to \mathbb{P}^1_{\mathbb{C}}$ is equivariant. Consider the line bundle

$$\mathscr{L} := \bigotimes_{\tau\in\Sigma} \mathrm{pr}_{\tau}^{*}\left(\omega^{\otimes k_\tau} \otimes \det{}^{\frac{w+k_\tau}{2}}\right)$$

on $\mathbb{P}^1_{\mathbb{C}}$. Let $Z_s = \ker(\mathrm{Res}_{L/\mathbb{Q}}\,\mathbb{G}_m \xrightarrow{\mathrm{N}_{L/\mathbb{Q}}} \mathbb{G}_m)$. The action of $G(\mathbb{C}) = \mathbf{GL}_2(\mathbb{C})^{\Sigma}$ on $\mathscr{L}$ factors through the quotient of G by Z_s. Hence the pullback of $\mathscr{L}$ to $\mathfrak{h}^{\Sigma} \times G(\mathbb{A}^{\infty})$ via the Borel embedding descends to a line bundle on $\mathrm{Sh}_K(G)(\mathbb{C})$ and it has a canonical model over the Galois closure of L [Mil90]. We define this line bundle to be $\underline{\omega}^{(v,w)}$.

The determinant map $\det\colon G \to \mathrm{Res}_{L/\mathbb{Q}}\,\mathbb{G}_m$ gives a bijection between the set of geometrically connected components of $\mathrm{Sh}_K(G)$ and the strict class group $\mathrm{Cl}^+(L)$ [TX16, §2.3]. For any fractional ideal $\mathfrak{c}$ coprime to p, let $\mathrm{Sh}^{\mathfrak{c}}_K(G)$ be the connected component of $\mathrm{Sh}_K(G)$ corresponding to the class of $\mathfrak{c}$. This space is related to the moduli of $\mathfrak{c}$-polarized abelian varieties in the following manner.

The moduli scheme $M(\mu_N, \mathfrak{c})$ is defined over $\mathbb{Z}[1/N]$. Consider the action of $\mathcal{O}_L^{\times,+}$ on $M(\mu_N, \mathfrak{c})$ defined on points by $\epsilon \cdot (A, \iota, \lambda, \psi) = (A, \iota, \epsilon\lambda, \psi)$. Notice that for $\epsilon = \eta^2$, with $\eta \in U_N := 1 + N\mathcal{O}_L$, the isomorphism $\eta\colon A \to A$ induces an isomorphism $\epsilon \cdot (A, \iota, \lambda, \psi) = \eta^*(A, \iota, \lambda, \psi) \simeq (A, \iota, \lambda, \psi)$. Thus the action of $\mathcal{O}_L^{\times,+}$ factors through the finite quotient $\Gamma := \mathcal{O}_L^{\times,+}/U_N^2$. We have then the following proposition.

Proposition 2.1.1. *There exists an isomorphism between the quotient $M(\mu_N, \mathfrak{c})(\mathbb{C})/\Gamma$ and $\mathrm{Sh}^{\mathfrak{c}}_K(G)(\mathbb{C})$. In other words $\mathrm{Sh}^{\mathfrak{c}}_K(G)(\mathbb{C})$ is a coarse moduli space over $\mathbb{C}$ of $\mathfrak{c}$-polarized abelian varieties with real multiplication by $\mathcal{O}_L$. Moreover the quotient map $p\colon \bar{M}(\mu_N, \mathfrak{c}) \to \bar{M}(\mu_N, \mathfrak{c})/\Gamma$ is finite étale with Galois group Γ.*

Proof. For the first part see [TX16, Proposition 2.4]. For the étaleness of the quotient see [AIP16b, Lemma 8.1]. □

Using this one can define integral models of the automorphic sheaves $\underline{\omega}^{(v,w)}$. Let L^{Gal} be the Galois closure of L. Let R be an $L^{\mathrm{Gal}}_{(p)}$-algebra. For (v, w) a classical weight for the group G with $k = 2v + wt_L$, consider the sheaf $\omega^k_{\mathcal{A}} = \otimes_\tau \omega_{\mathcal{A}}(\tau)^{k_\tau}$ on $\bar{M}(\mu_N, \mathfrak{c})_R$. Define an action of $\mathcal{O}_L^{\times,+}$ on $\omega^k_{\mathcal{A}}$ as follows. For a section f of $\omega_{\mathcal{A}}^k$, define $\epsilon \cdot f$ as the section whose evaluation at points $(A_{/R}, \iota, \lambda, \psi, \omega)$ for an $\mathcal{O}_L \otimes R$-generator ω of $\omega_{\mathcal{A}}$ satisfies

$$(\epsilon \cdot f)(A, \iota, \lambda, \psi, \omega) = v(\epsilon^{-1}) f(A, \iota, \epsilon\lambda, \psi, \omega).$$

We can then check that the action of U_N^2 is trivial on $\omega_{\mathcal{A}}^k$. Indeed, for $\eta \in U_N$ and $\epsilon = \eta^2$ we have

$$f(A, \iota, \lambda, \psi, \omega) = f(A, \iota, \epsilon\lambda, \psi, \eta\omega) = k(\eta)^{-1} f(A, \iota, \epsilon\lambda, \psi, \omega) = v(\epsilon)^{-1} f(A, \iota, \epsilon\lambda, \psi, \omega)$$

which proves the claim. We remark that this action only depends on k and not the pair (v, w).

Definition 2.1.1. Define the sheaf of Hilbert modular forms for the group G of tame level μ_N, $\mathfrak{c}$-polarization and weight (v, w) with coefficients in R to be

$$\underline{\omega}_R^{(v,w)} := (p_* \omega_{\mathcal{A},R}^k)^\Gamma$$

where $p \colon \bar{M}(\mu_N, \mathfrak{c})_R \to \bar{M}(\mu_N, \mathfrak{c})_R / \Gamma$ is the quotient map. Alternatively we will sometimes call them arithmetic Hilbert modular forms.

Let $\mathrm{M}(\mu_N, \mathfrak{c}, k; R) := \Gamma(\bar{M}(\mu_N, \mathfrak{c})_R, \omega_{\mathcal{A}}^k)$ be the space of geometric Hilbert modular forms. Let $\mathrm{M}^G(\mu_N, \mathfrak{c}, (v, w); R) = \Gamma(\bar{M}(\mu_N, \mathfrak{c})_R / \Gamma, \underline{\omega}_R^{(v,w)})$ be the space of arithmetic Hilbert modular forms. If $\#\Gamma \in R^\times$, then $\mathrm{M}^G(\mu_N, \mathfrak{c}, (v, w); R) = \mathrm{M}(\mu_N, \mathfrak{c}, k = 2v + wt_L; R)^\Gamma$ can be realized as the image of the projector:

$$e := \frac{1}{\#\Gamma} \sum_{\epsilon \in \Gamma} \epsilon.$$

Let $x \in L^{\times,+}$ be coprime to p. Then we have an isomorphism $L_{(x\mathfrak{c},\mathfrak{c})} \colon \mathrm{M}^G(\mu_N, \mathfrak{c}, (v, w); R) \xrightarrow{\sim} \mathrm{M}^G(\mu_N, x\mathfrak{c}, (v, w); R)$ given by

$$L_{(x\mathfrak{c},\mathfrak{c})}(f)(A, \iota, \lambda, \psi, \omega) := v(x) f(A, \iota, x^{-1}\lambda, \psi, \omega).$$

Moreover this isomorphism only depends on the principla ideal (x). Let $\mathrm{Frac}(L)^{(p)}$ be the group of fractional ideals which are coprime to p and let $\mathrm{Princ}(L)^{+,(p)}$ be the group of positive elements which are coprime to p. Then $\mathrm{Frac}(L)^{(p)} / \mathrm{Princ}(L)^{+,(p)} \simeq \mathrm{Cl}^+(L)$.

Definition 2.1.2. Define the R-module of Hilbert modular forms for G of tame level N and weight (v, w) to be

$$\mathrm{M}^G(\mu_N, (v, w); R) := \left(\bigoplus_{\mathfrak{c} \in \mathrm{Frac}(L)^{(p)}} \mathrm{M}^G(\mu_N, \mathfrak{c}, (v, w); R) \right) \Big/ \Big(L_{(x\mathfrak{c},\mathfrak{c})}(f) - f \Big)_{x \in \mathrm{Princ}(L)^{+,(p)}}.$$

Upon choosing representatives $\mathfrak{c}_1, \ldots, c_{h_L^+}$ of $\mathrm{Cl}^+(L)$ in $\mathrm{Frac}(L)^{(p)}$ we have a non-canonical isomorphism

$$\mathrm{M}^G(\mu_N, (v, w); R) \simeq \bigoplus_{i=1}^{h_L^+} \mathrm{M}^G(\mu_N, \mathfrak{c}_i, (v, w); R)$$

which shows that $\mathrm{M}^G(\mu_N, (v, w); R)$ is a finite R-module.

Remark 2.1.2. For $(v,w),(v',w')$ satisfying $2v+wt_L = 2v'+w't_L$, the autormorphic sheaves $\underline{\omega}^{(v,w)}$ and $\underline{\omega}^{(v',w')}$ on $\mathrm{Sh}_K(G)(\mathbb{C})$ are related by $\underline{\omega}^{(v',w')} = \underline{\omega}^{(v,w)} \otimes \det^{\frac{w'-w}{2}}$, where the determinant factor should be thought of as a Tate twist. In particular the underlying function spaces of automorphic forms of weight (v,w) and weight (v',w') are isomorphic. In the definition of Hilbert modular forms for G, we have ignored this discrepancy coming from the determinant.

Remark 2.1.3. For $\mathfrak{c} = \mathfrak{d}^{-1}$, the moduli scheme $M(\mu_N, \mathfrak{d}^{-1})$ is an integral model for the Shimura variety associated to the group $G^* := G \times_{\mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m} \mathbb{G}_m$ where the arrow $G \to \mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m$ is the determinant and $\mathbb{G}_m \to \mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m$ is the diagonal embedding [Rap78].

2.1.3 The partial Igusa tower

Fix an $I = [p^a, p^b]$ and $\alpha \in \mathfrak{m} \setminus \mathfrak{m}^2$. We now work over $\mathfrak{X}_{\alpha,I}$. Recall the ideal Hdg_r was given locally by $(\alpha, \mathrm{Hdg}^{p^{r+1}})$ for a local lift Hdg of the Hasse invariant. Let $g_r \colon \mathfrak{X}_{r,\alpha,I} \to \mathfrak{X}_{\alpha,I}$ be the open subscheme of the blow-up of $\mathfrak{X}_{\alpha,I}$ with respect to the ideal Hdg_r, where the inverse image ideal is generated by $\mathrm{Hdg}^{p^{r+1}}$. For any integer n with $1 \leq n \leq r$ if $I = [0,1]$ and $1 \leq n \leq a+r$ let $\lambda = \mathrm{Hdg}^{\frac{p^n-1}{p-1}}$. Note that $\frac{p}{\lambda} \in \mathcal{O}_{\mathfrak{X}_{r,\alpha,I}}$.

Proposition 2.1.2. *For I, r, n, α as above the semiabelian scheme $\mathcal{A} \to \mathfrak{X}_{r,\alpha,I}$ has a canonical subgroup H_n of order p^n [AIP18, Appendice A]. This is a finite, locally free subgroup scheme that satisfies the following properties:*

1. *H_n lifts* $\ker F^n$ *modulo $\frac{p}{\lambda}$.*
2. *For any α-adically complete admissible $\Lambda^0_{\alpha,I}$-algbera R, together with a morphism $f \colon$ Spf $R \to \mathfrak{X}_{r,\alpha,I}$,*
$$H_n(R) = \{s \in \mathcal{A}[p^n](R) \mid s \bmod \frac{p}{\lambda} \in \ker F^n\}.$$
3. *Suppose $L_n = \mathcal{A}[p^n]/H_n$. Then ω_{L_n} is killed by λ, and we have $\omega_{L_n} \simeq \omega_{\mathcal{A}}/\lambda\omega_{\mathcal{A}}$.*
4. *$\mathcal{A}^\vee[p^n]/H_n(\mathcal{A}^\vee) \simeq H_n^\vee$ through the Weil pairing and it is étale over the adic generic fibre $\mathcal{X}_{r,\alpha,I}$ of $\mathfrak{X}_{r,\alpha,I}$.*

Proof. [AIP18, Appendice A]. ☐

Definition 2.1.3. For every r, n as above define $\mathcal{IG}_{n,r,I} \to \mathcal{X}_{r,\alpha,I}$ to be the adic space classifying isomorphisms $\mathcal{O}_L/p^n\mathcal{O}_L \xrightarrow{\sim} H_n^\vee$ of the group scheme $H_n^\vee \to \mathcal{X}_{r,\alpha,I}$. Define $\mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,\alpha,I}$ to be the normalization of $\mathfrak{X}_{r,\alpha,I}$ in $\mathcal{IG}_{n,r,I}$.

Proposition 2.1.3. *$\mathcal{IG}_{n,r,I} \to \mathcal{X}_{r,\alpha,I}$ is an étale, Galois morphism with Galois group $(\mathcal{O}_L/p^n\mathcal{O}_L)^\times$. The morphism $\mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,\alpha,I}$ is finite and is endowed with an action of $(\mathcal{O}_L/p^n\mathcal{O}_L)^\times$ induced by the action on the generic fibre.*

Proof. Similar to Proposition 1.1.5. ☐

(Note we suppressed the index α in our notation for the partial Igusa tower to avoid clumsiness.)

2.2 Splitting of de Rham sheaf

In this section we redo the theory developed for elliptic curves in §1.2 more generally for abelian schemes. In the following we put an overline on the names of objects (abelian schemes, sheaves etc.) to denote they are obtained by base change along the closed immersion $i\colon X_{\mathbb{F}_p} \hookrightarrow \mathfrak{X}$. Denote by $\bar{\pi}\colon \bar{\mathcal{A}} \to X_{\mathbb{F}_p}$ the base change of π along i. Let $\bar{\omega}_{\mathcal{A}} := i^*\omega_{\mathcal{A}} = \bar{\omega}_{\mathcal{A}}$. The Verschiebung $V\colon \bar{\mathcal{A}}^{(p)} \to \bar{\mathcal{A}}$ induces a map on the Lie algebra $HW\colon \omega^\vee_{\bar{\mathcal{A}}^{(p)}} \to \omega^\vee_{\bar{\mathcal{A}}}$, whose determinant is the Hasse invariant $\mathrm{Ha} \in (\Lambda^g\bar{\omega}_{\mathcal{A}})^{\otimes(p-1)}$. Let $\underline{\mathrm{Ha}} := \mathrm{Ha}\cdot(\Lambda^g\bar{\omega}_{\mathcal{A}})^{\otimes(1-p)}$ be the ideal generated by the values of Ha. This is an invertible ideal with zeroes of order 1 along each of the prime divisors that appear in $\mathrm{Div}(\underline{\mathrm{Ha}})$ (Theorem 2.1.1). Denote by $\bar{\mathrm{H}}_{\mathcal{A}}$ the pullback of $\mathrm{H}_{\mathcal{A}}$ along i. Let $j\colon X^{\mathrm{ord}}_{\mathbb{F}_p} \hookrightarrow X_{\mathbb{F}_p}$ be the ordinary locus, which is the open subscheme of $X_{\mathbb{F}_p}$ where $\underline{\mathrm{Ha}} = \mathcal{O}_{X_{\mathbb{F}_p}}$.

Let $\varphi\colon X_{\mathbb{F}_p} \to X_{\mathbb{F}_p}$ be the Frobenius. The Frobenius induces a φ-linear endomorphism of $\bar{\mathrm{H}}_{\mathcal{A}}$.

Proposition 2.2.1. *Over the ordinary locus $X^{ord}_{\mathbb{F}_p}$ we have the unit root splitting which is a canonical splitting $\psi_{Frob}\colon j^*\bar{\mathrm{H}}_{\mathcal{A}} \to j^*\bar{\omega}_{\mathcal{A}}$ of the Hodge filtration on $\bar{\mathrm{H}}_{\mathcal{A}}$, that respects the Frobenius action. The kernel of ψ_{Frob} is called the unit root subspace. It is characterized by the property that it is stable under the Frobenius action and Frobenius acts invertibly on it.*

Proof. Suppose $\operatorname{Spec} R \subset X_{\mathbb{F}_p}$ is a local chart for which $\bar{\omega}_{\mathcal{A}}, \bar{\mathrm{H}}_{\mathcal{A}}$ are trivial and choose a basis compatible with the Hodge filtration. With respect to such a basis we can write the matrix of the Frobenius action on $\bar{\mathrm{H}}_{\mathcal{A}}$ in $g\times g$ blocks as follows.

$$\mathrm{Frob} = \begin{pmatrix} 0 & C \\ 0 & HW. \end{pmatrix}$$

Here we abuse notation to write HW for the matrix corresponding to the φ-linear map induced on $\bar{\omega}^\vee_{\mathcal{A}}$ by $\mathrm{F}_{\mathrm{abs}}\colon \bar{\mathcal{A}} \to \bar{\mathcal{A}}$. For the base change of $j_*j^*\bar{\mathrm{H}}_{\mathcal{A}}(R) = \bar{\mathrm{H}}_{\mathcal{A}}(R)[1/\mathrm{Ha}]$ given by the matrix

$$P = \begin{pmatrix} \mathrm{Id} & C\cdot HW^{-1} \\ 0 & \mathrm{Id} \end{pmatrix},$$

the matrix of Frobenius becomes

$$P^{-1}\mathrm{Frob}P = \begin{pmatrix} 0 & 0 \\ 0 & HW \end{pmatrix}$$

Note that P is only defined over the ordinary locus. Hence we have a splitting $\psi_{\mathrm{Frob}}\colon j^*\bar{\mathrm{H}}_{\mathcal{A}} \to j^*\bar{\omega}_{\mathcal{A}}$ of the Hodge filtration over the ordinary locus that respects the Frobenius action. The kernel of this splitting is *uniquely* characterized by the fact that it is stable under the Frobenius action and Frobenius acts invertibly on it. $\square$

Consider the map $\psi\colon \bar{\mathrm{H}}_{\mathcal{A}} \to j_*j^*\bar{\mathrm{H}}_{\mathcal{A}} \xrightarrow{\psi_{\mathrm{Frob}}} j_*j^*\bar{\omega}_{\mathcal{A}}$. Then let $\bar{\mathrm{H}}'_{\mathcal{A}} := \psi^{-1}\bar{\omega}_{\mathcal{A}}$. The inclusion $\bar{\omega}_{\mathcal{A}} \to \bar{\mathrm{H}}'_{\mathcal{A}}$ admits a retraction given by the map ψ. As a subsheaf of $\bar{\mathrm{H}}_{\mathcal{A}}$ containing $\bar{\omega}_{\mathcal{A}}$, $\bar{\mathrm{H}}'_{\mathcal{A}}$ is equipped with the induced Hodge filtration. In the following lemma we describe the 1st graded piece of this Hodge filtration.

Lemma 2.2.1. *The sheaf $\bar{\mathrm{H}}'_{\mathcal{A}}$ sits in the following split exact sequence.*

$$0 \to \bar{\omega}_{\mathcal{A}} \to \bar{\mathrm{H}}'_{\mathcal{A}} \to HW(\omega^{\vee}_{\bar{\mathcal{A}}^{(p)}}) \to 0. \tag{2.2}$$

Proof. Choose a local chart $\operatorname{Spec} R$ as above. Explicitly, suppose $e_1, \dots, e_g, f_1, \dots, f_g$ form an R-basis of $\bar{\mathrm{H}}'_{\mathcal{A}}$ such that $e_1, \dots, e_g$ span $\bar{\omega}_{\mathcal{A}}$ and the images of $f_1, \dots, f_g$ span $\bar{\omega}^{\vee}_{\mathcal{A}}$. Also assume that the matrix of Frobenius with respect to this basis is given as above. Then $\psi(e_i) = e_i$ and $\psi(f_i) = (\mathrm{Id} - P)(f_i)$ for all i. Let $V = \sum_i Rf_i$. By an abuse of notation we will denote by C the linear map it defines $C\colon V \to \bar{\omega}_{\mathcal{A}}$. Then, denoting by Ha a generator of $\underline{\mathrm{Ha}}$ over $\operatorname{Spec} R$, we have

$$\begin{aligned}
\psi^{-1}\bar{\omega}_{\mathcal{A}} \cap V &= \{f \in V \mid C \cdot HW^{-1}(f) \in \bar{\omega}_{\mathcal{A}}\} \\
&= \{f \in V \mid C \cdot \operatorname{adj}(HW)(f) \in \underline{\mathrm{Ha}} \cdot \bar{\omega}_{\mathcal{A}}\} \\
&= \{f \in V \mid \operatorname{adj}(HW)(f) \in C^{-1}(\underline{\mathrm{Ha}} \cdot \bar{\omega}_{\mathcal{A}})\} \\
&= \frac{1}{\mathrm{Ha}} \cdot HW\big(C^{-1}(\underline{\mathrm{Ha}} \cdot \bar{\omega}_{\mathcal{A}})\big) \cap V \\
&= \frac{1}{\mathrm{Ha}} \cdot HW\big(C^{-1}(\underline{\mathrm{Ha}} \cdot \bar{\omega}_{\mathcal{A}}) \cap \operatorname{adj}(HW)(V)\big).
\end{aligned}$$

Now suppose $f \in C^{-1}(\underline{\mathrm{Ha}} \cdot \bar{\omega}_{\mathcal{A}}) \cap \operatorname{adj}(HW)(V)$. Then, $f \pmod{\underline{\mathrm{Ha}}}$ lies in the kernel of $C \otimes 1\colon V \otimes R/\underline{\mathrm{Ha}} \to \bar{\omega}_{\mathcal{A}} \otimes R/\underline{\mathrm{Ha}}$ as well as in the kernel of $HW \otimes 1$. Thus in particular, denoting by $\operatorname{Spec} k(y) \to Y_{\mathbb{F}_p}$ a generic point of a prime divisor of $\underline{\mathrm{Ha}}$, f lies in the kernel of Frobenius acting on $H^1_{\mathrm{dR}}(\bar{\mathcal{A}}_{k(y)}/k(y))$. But the image of Frobenius has rank g, and hence its kernel is precisely $\omega_{\bar{A}_{k(y)}}$. Thus $\ker(C \otimes 1) \cap \ker(HW \otimes 1) = 0$. Hence $f \in \underline{\mathrm{Ha}} \cdot V$. This implies that $\psi^{-1}\bar{\omega}_{\mathcal{A}} \cap V = \frac{1}{\mathrm{Ha}} \cdot HW(\underline{\mathrm{Ha}} \cdot V) = HW(V)$. This proves the lemma. □

Corollary 2.2.1. *The sheaf $\bar{\mathrm{H}}'_{\mathcal{A}}$ is stable under the $\mathcal{O}_L$ action. It is a locally free $\mathcal{O}_L \otimes \mathcal{O}_{X_{\mathbb{F}_p}}$-module of rank 2 and $\mathrm{H}'_{\mathcal{A}} = \bar{\omega}_{\mathcal{A}} \oplus$ Frob$(\bar{\mathrm{H}}_{\mathcal{A}})$.*

Proof. The subsheaf $\mathrm{Frob}(\bar{\mathrm{H}}_{\mathcal{A}})$ is the image of the map $F^*_{\mathcal{A}}\colon \mathrm{H}_{\bar{\mathcal{A}}^{(p)}} \to \bar{\mathrm{H}}_{\mathcal{A}}$ induced by the relative Frobenius. It is killed by the unit root splitting and maps surjectively onto $HW(\omega^{\vee}_{\bar{\mathcal{A}}^{(p)}})$. Hence $\bar{\mathrm{H}}'_{\mathcal{A}} = \bar{\omega}_{\mathcal{A}} \oplus \mathrm{Frob}(\bar{\mathrm{H}}_{\mathcal{A}})$. Since the relative Frobenius commutes with the $\mathcal{O}_L$-action, $HW(\omega^{\vee}_{\bar{\mathcal{A}}^{(p)}})$ is stable under the $\mathcal{O}_L$-action and hence so is $\bar{\mathrm{H}}'_{\mathcal{A}}$. Moreover $HW\colon \omega^{\vee}_{\bar{\mathcal{A}}^{(p)}} \to \bar{\omega}_{\mathcal{A}}$ is an $\mathcal{O}_L \otimes \mathcal{O}_{X_{\mathbb{F}_p}}$-linear map of invertible $\mathcal{O}_L \otimes \mathcal{O}_{X_{\mathbb{F}_p}}$-modules such that $\mathrm{N}_{\mathcal{O}_L \otimes \mathcal{O}_{X_{\mathbb{F}_p}}/\mathcal{O}_{X_{\mathbb{F}_p}}} HW = \det HW = \mathrm{Ha}$ is a non-zero divisor. Hence $HW(\omega^{\vee}_{\bar{\mathcal{A}}^{(p)}})$ is an invertible $\mathcal{O}_L \otimes \mathcal{O}_{X_{\mathbb{F}_p}}$-module. □

In the following we will construct a locally free $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{X}_{r,\alpha,I}}$ subsheaf $\mathrm{H}'_{\mathcal{A}} \subset \mathrm{H}_{\mathcal{A}}$ of rank 2, together with the induced Hodge filtration such that its reduction modulo a small power of p will give us the split exact sequence (2.2).

Let $i\colon \mathfrak{X}_{\alpha,I}/(p) \hookrightarrow \mathfrak{X}_{\alpha,I}$ be the base change of $X_{\mathbb{F}_p} \hookrightarrow \mathfrak{X}$ to $\mathfrak{X}_{\alpha,I}$. Let $i_0\colon \mathfrak{X}_{r,\alpha,I}/(p\mathrm{Hdg}^{-1}) \hookrightarrow \mathfrak{X}_{r,\alpha,I}$

be the closed subscheme defined by the ideal $\frac{p}{\mathrm{Hdg}}$. Thus we have a commutative diagram as follows.

$$\begin{tikzcd} \mathfrak{X}_{r,\alpha,I}/(p\mathrm{Hdg}^{-1}) \arrow[r, "i_0"] \arrow[d, "q"] & \mathfrak{X}_{r,\alpha,I} \arrow[d, "g_r"] \\ \mathfrak{X}_{\alpha,I}/(p) \arrow[r, "i"] & \mathfrak{X}_{\alpha,I} \end{tikzcd}$$

Let $\bar{\omega}^{\vee}_{\mathcal{A},0} := HW(\omega_{\bar{\mathcal{A}}^{(p)}})$ where we now denote by $\bar{\mathcal{A}}$ the pullback of $\mathcal{A}$ along $i\colon \mathfrak{X}_{\alpha,I}/(p) \hookrightarrow \mathfrak{X}_{\alpha,I}$. Let $\tilde{\omega}^{\vee}_{\mathcal{A}} := (i^{\sharp})^{-1}(i_*\bar{\omega}^{\vee}_{\mathcal{A},0})$ where $i^{\sharp}\colon \omega^{\vee}_{\mathcal{A}} \to i_*i^*\omega^{\vee}_{\mathcal{A}}$ is the unit of the adjunction. Note that $\tilde{\omega}^{\vee}_{\mathcal{A}}$ is stable under the $\mathcal{O}_L$-action.

Lemma 2.2.2. $\omega^{\vee}_{\mathcal{A},0} := \mathrm{im}\,(g_r^*\tilde{\omega}^{\vee}_{\mathcal{A}} \to g_r^*\omega^{\vee}_{\mathcal{A}})$ *is a locally free* $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{X}_{r,\alpha,I}}$*-module of rank* 1.

Proof. Choose a local chart $\mathrm{Spf}\, R = U \subset \mathfrak{X}_{\alpha,I}$, such that $\omega^{\vee}_{\mathcal{A}}$ is free as an $\mathcal{O}_L \otimes R$-module over $\mathrm{Spf}\, R$, and $\underline{\mathrm{Ha}}$ is free over $\mathrm{Spec}\, R/(p)$. Let v be an $\mathcal{O}_L \otimes R$ basis of $\omega^{\vee}_{\mathcal{A}|U}$ and let $(\mathcal{O}_L \otimes R)\cdot v = \oplus_{i=1}^{g} Rv_i$ be the decomposition induced by the splitting of $\mathcal{O}_L$ in R. Let $\bar{v}$ be the image of v in $\bar{\omega}^{\vee}_{\mathcal{A}} := i^*\omega^{\vee}_{\mathcal{A}}$. Let $\bar{w} = HW(\bar{v})$ and pick a lift w of $\bar{w}$. Let $w = (w_i)_i$ be its components. Then $\tilde{\omega}^{\vee}_{\mathcal{A}} = \sum_{i=1}^{g} \mathcal{O}_U w_i + p\omega^{\vee}_{\mathcal{A}}$. Consider the $\mathcal{O}_L$-linear map $\widetilde{HW}$ which sends $v \mapsto w$ and which reduces to $HW \bmod p$. Then $\det \widetilde{HW}$ is a lift of $\mathrm{Ha} = \det HW$. Keeping with previous notation, we will call this lift Hdg. Consider now $V = g_r^{-1}(U) = \mathrm{Spf}\, R\langle \frac{\alpha}{\mathrm{Hdg}^{p^{r+1}}}\rangle$. Using $\frac{p}{\mathrm{Hdg}} \in \mathcal{O}_{\mathfrak{X}_{r,I}}$, we see that over $\mathrm{Spf}\, R\langle \frac{\alpha}{\mathrm{Hdg}^{p^{r+1}}}\rangle$, $p\cdot g_r^*\omega^{\vee}_{\mathcal{A}} \subset \sum_{i=1}^{n} \mathcal{O}_V g_r^*(w_i)$ as submodules of $g_r^*\omega^{\vee}_{\mathcal{A}}$. Thus $\omega^{\vee}_{\mathcal{A},0} = \sum_{i=1}^{n} \mathcal{O}_V g_r^*(w_i)$. The sum is direct because Hdg is a non-zero divisor in $\mathcal{O}_{\mathfrak{X}_{r,I}}$. It is clearly stable under the $\mathcal{O}_L$-action. Since locally HW can be seen as a non-zero divisor in $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{X}_{\alpha,I}/(p)}$, the same is true of the lift $\widetilde{HW}$, and since $\omega^{\vee}_{\mathcal{A},0} = \widetilde{HW}\cdot\omega^{\vee}_{\mathcal{A}}$, the lemma follows. ☐

Following the proof of Lemma 2.2.2, let $\widetilde{HW}$ be the $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{X}_{r,\alpha,I}}$-ideal sheaf defined by $\omega^{\vee}_{\mathcal{A},0} = \widetilde{HW}\cdot \omega^{\vee}_{\mathcal{A}}$.

Definition 2.2.1. Define $\mathrm{H}'_{\mathcal{A}}$ to be the inverse image of $\omega^{\vee}_{\mathcal{A},0}$ in $\mathrm{H}_{\mathcal{A}}$ under the projection coming from the Hodge filtration.

We have the following commutative diagram of sheaves over $\mathfrak{X}_{r,\alpha,I}$.

$$\begin{tikzcd} 0 \arrow[r] & \omega_{\mathcal{A}} \arrow[r] \arrow[d, equal] & \mathrm{H}'_{\mathcal{A}} \arrow[r] \arrow[d] & \omega^{\vee}_{\mathcal{A},0} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \omega_{\mathcal{A}} \arrow[r] & \mathrm{H}_{\mathcal{A}} \arrow[r] & \omega^{\vee}_{\mathcal{A}} \arrow[r] & 0 \end{tikzcd}$$

Proposition 2.2.2. *There is an isomorphism of* $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{X}_{r,\alpha,I}/(p\mathrm{Hdg}^{-1})}$*-modules* $i_0^*\omega^{\vee}_{\mathcal{A},0} \xrightarrow{\sim} q^*\bar{\omega}^{\vee}_{\mathcal{A},0}$*, that commutes with the induced maps to* $\bar{\omega}^{\vee}_{\mathcal{A}}$*. (We abuse notation to denote* $q^*\bar{\omega}^{\vee}_{\mathcal{A}}$ *by* $\bar{\omega}^{\vee}_{\mathcal{A}}$*.)*

$$\begin{tikzcd} i_0^*\omega^{\vee}_{\mathcal{A},0} \arrow[r, "\sim"] \arrow[d] & q^*\bar{\omega}^{\vee}_{\mathcal{A},0} \arrow[d] \\ \bar{\omega}^{\vee}_{\mathcal{A}} \arrow[r, equal] & \bar{\omega}^{\vee}_{\mathcal{A}} \end{tikzcd}$$

Proof. In fact we prove the following. The natural surjective map $i^*\tilde{\omega}_{\mathcal{A}}^\vee \to \bar{\omega}_{\mathcal{A},0}^\vee$ induces by pullback a surjective map $q^*i^*\tilde{\omega}_{\mathcal{A}}^\vee = i_0^*g_r^*\tilde{\omega}_{\mathcal{A}}^\vee \to q^*\bar{\omega}_{\mathcal{A},0}^\vee$ that commutes with the induced maps to $\bar{\omega}_{\mathcal{A}}^\vee$. We will show that this map factors naturally as $i_0^*g_r^*\tilde{\omega}_{\mathcal{A}}^\vee \to i_0^*\omega_{\mathcal{A},0}^\vee \to q^*\bar{\omega}_{\mathcal{A},0}^\vee$, and the last two sheaves being both locally free of rank g, the last arrow is an isomorphism. Also since $i_0^*g_r^*\tilde{\omega}_{\mathcal{A}}^\vee \to q^*\bar{\omega}_{\mathcal{A},0}^\vee$ is $\mathcal{O}_L$-linear and $i_0^*g_r^*\tilde{\omega}_{\mathcal{A}}^\vee \to i_0^*\omega_{\mathcal{A},0}^\vee$ is surjective $\mathcal{O}_L$-linear, the induced isomorphism is $\mathcal{O}_L$-linear too. This will be the isomorphism claimed in the proposition.

We use the notation of the proof of Lemma 2.2.2, except that to avoid clumsiness we write v_i (resp. w_i) instead of $g_r^*(v_i)$ (resp. $g_r^*(w_i)$). Since $\tilde{\omega}_{\mathcal{A}}^\vee$ is generated by the w_i and pv_i for $i = 1, \ldots, g$, there is a surjective map $\mathcal{O}_V^{2g} \to g_r^*\tilde{\omega}_{\mathcal{A}}^\vee$ that sends $e_i \mapsto w_i$ for $1 \le i \le g$, and $e_j \mapsto pv_j$ for $g+1 \le j \le 2g$. Using the basis $\bar{w}_i$ for $\bar{\omega}_{\mathcal{A},0}^\vee$, we have a surjective map $M\colon \mathcal{O}_{i_0^{-1}V}^{2g} \to \mathcal{O}_{i_0^{-1}V}^{g}$ given by $e_i \mapsto e_i$ for $1 \le i \le g$, and $e_j \mapsto 0$ for $g+1 \le j \le 2g$ and which induces the map $i_0^*g_r^*\tilde{\omega}_{\mathcal{A}}^\vee \to q^*\bar{\omega}_{\mathcal{A},0}^\vee$. On the other hand, the images of $g_r^*(w_i)$ form a basis for $\omega_{\mathcal{A},0}^\vee$. With respect to this basis, $pv_i = \frac{p}{\mathrm{Hdg}}\mathrm{adj}(\widetilde{HW})(e_i)$. Thus the surjective map $g_r^*\tilde{\omega}_{\mathcal{A}}^\vee \to \omega_{\mathcal{A},0}^\vee$ is induced by the map $N\colon \mathcal{O}_V^{2g} \to \mathcal{O}_V^{g}$ that sends $e_i \mapsto e_i$ for $1 \le i \le g$, and $e_{i+g} \mapsto \frac{p}{\mathrm{Hdg}}\mathrm{adj}(\widetilde{HW})(e_i)$ for $1 \le i \le g$. Suppose $(a_1, \ldots, a_{2g}) \in \ker N$. Since $N\big(\sum_{i=g+1}^{2g} \mathcal{O}_V e_i\big) \subset \frac{p}{\mathrm{Hdg}} \cdot \mathcal{O}_V^g$, we see that $a_i \in \frac{p}{\mathrm{Hdg}}$ for $1 \le i \le g$. Thus M kills the kernel of the pullback of N to $\mathfrak{X}_{r,\alpha,I}/(p\mathrm{Hdg}^{-1})$, $i_0^*N\colon \mathcal{O}_{i_0^{-1}V}^{2g} \to \mathcal{O}_{i_0^{-1}V}^{g}$. This proves the proposition. □

Proposition 2.2.3. *The pullback of the exact sequence*

$$0 \to \omega_{\mathcal{A}} \to \mathrm{H}'_{\mathcal{A}} \to \omega_{\mathcal{A},0}^\vee \to 0$$

along $i_0\colon \mathfrak{X}_{r,\alpha,I}/(p\mathrm{Hdg}^{-1}) \hookrightarrow \mathfrak{X}_{r,\alpha,I}$ admits a canonical splitting induced by the splitting of (2.2) which moreover commutes with the splitting induced by the decomposition $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{X}_{r,\alpha,I}} \simeq \prod_{\sigma\in\Sigma} \mathcal{O}_{\mathfrak{X}_{r,\alpha,I}}$.

Proof. This is immediate from Proposition 2.2.2 and the $\mathcal{O}_L$-linearity of the splitting. □

2.3 p-adic interpolation of modular and de Rham sheaves

Henceforth fix n a positive integer. Fix $I = [p^a, p^b]$ such that $k_{\alpha,I}^0$ is analytic on $1 + p^{n-1}(\mathcal{O}_L \otimes \mathbb{Z}_p)$ and r such that H_n is defined on $\mathfrak{X}_{r,\alpha,I}$. Depending on the two cases $I = [0,1]$ (i.e. $\alpha = p$) or $I = [p^a, p^b]$ for $a, b \in \mathbb{N}$, these conditions are satisfied if

1. $I = [0,1]$, $r \ge 2$ if $p \ne 2$ and $2 \le n \le r$, or $r \ge 4$ if $p = 2$ and $4 \le n \le r$,
2. $I = [p^a, p^b]$ with $a, b \in \mathbb{N}$, $r \ge 1$ and $r + a \ge b + 2$ if $p \ne 2$ and $b + 2 \le n \le a + r$, or $r \ge 2$ and $r + a \ge b + 4$ if $p = 2$, and $b + 4 \le n \le r + a$.

In this section we construct overconvergent modular and de Rham sheaves, denoted $\mathfrak{w}_{k,\alpha,I}$ and $\mathbb{W}_{k,\alpha,I}$ on the Hilbert modular scheme $\mathfrak{X}_{r,\alpha,I} \times_{\mathfrak{W}_{\alpha,I}^0} \mathfrak{W}_{\alpha,I}$ for the universal weight $k = k^{\mathrm{un}}\colon (\mathcal{O}_L \otimes \mathbb{Z}_p)^\times \to \Lambda_{\alpha,I}^\times$, as we did in §1.3 for the case of elliptic curves. The modular sheaf interpolates $\omega_{\mathcal{A}}^k$ for classical weights k. The techniques are essentially similar to the elliptic case in the following sense. By passing to a partial Igusa tower depending on n, we construct a modified modular sheaf $\Omega_{\mathcal{A}}$ and a modified de Rham sheaf $\mathrm{H}^\sharp_{\mathcal{A}}$ together with a modified unit root splitting. We decompose the universal character into its components induced by the splitting of $\mathcal{O}_L$ in $\mathcal{O}_K$. We decompose $\Omega_{\mathcal{A}}$ and $\mathrm{H}^\sharp_{\mathcal{A}}$ likewise. On each component we carry out the construction of interpolation for the corresponding component of the universal weight following the technique developed in §1.3. Finally we define $\mathfrak{w}_{k,\alpha,I}$ and $\mathbb{W}_{k,\alpha,I}$ by taking the tensor product of these individual components.

The construction of $\mathfrak{w}_{k,\alpha,I}$ appears in the joint work of Andreatta, Iovita, Pilloni and Stevens [AIS14], [AIP16a], [AIP16b]. Our construction is similar to [AIP16b]. However using the theory of vector bundles with marked sections we make it more explicit by actually constructing sections that generate $\mathfrak{w}_{k,\alpha,I}$ locally. This is inspired by the work of Andreatta-Iovita [AI21]. At the end of the section we compare our construction of $\mathfrak{w}_{k,\alpha,I}$ with the construction in [AIP16b] and show why they are isomorphic. The definition of $\mathbb{W}_{k,\alpha,I}$ is new, but as we will see it is inspired by loc. cit. and the improved technique of using modified unit root subspaces as discussed in Chapter 1. The main theorem of this section is the following.

Theorem. *For n, r, α, I as above and $k = k^{un}$ the universal weight on $\mathfrak{W}_{\alpha,I}$, there are formal sheaves $\mathfrak{w}_{k,\alpha,I}$ and $\mathbb{W}_{k,\alpha,I}$ on $\bar{\mathfrak{M}}_{r,\alpha,I} := \mathfrak{X}_{r,\alpha,I} \times_{\mathfrak{W}^0_{\alpha,I}} \mathfrak{W}_{\alpha,I}$. For $\alpha = p$ and $I = [0,1]$, viewing a classical weight κ as a point of $\mathcal{W}_{p,[0,1]}(\mathbb{C}_p) = \mathcal{W}_p(\mathbb{C}_p)$, the restriction of the sheaf $\mathfrak{w}_{k,\alpha,I}[1/p]_{|k\mapsto\kappa}$ on the associated analytic adic space $\mathcal{X}_{r,p} \times_{\mathcal{W}^0_p} \mathcal{W}_{p|\kappa}$ gives the sheaf $\omega^\kappa_{\mathcal{A}}$ of classical Hilbert modular forms of weight κ. The sheaf $\mathbb{W}_{k,\alpha,I}$ is equipped with a filtration by coherent $\mathcal{O}_{\bar{\mathfrak{M}}_{r,\alpha,I}}$-modules $\{\mathrm{Fil}_i\mathbb{W}_{k,\alpha,I}\}_{i\geq 0}$, and $\mathbb{W}_{k,\alpha,I}$ is the α-adic completion of $\varinjlim_i \mathrm{Fil}_i\mathbb{W}_{k,\alpha,I}$. Moreover $\mathrm{Fil}_0\mathbb{W}_{k,\alpha,I} = \mathfrak{w}_{k,\alpha,I}$.*

Remark 2.3.1. Note that we cannot relate the sheaf $\mathbb{W}_{k,\alpha,I}$ with symmetric powers of $\mathrm{H}_{\mathcal{A}}$ at classical points in an obvious way like we did for elliptic curves.

2.3.1 The sheaves $\Omega_{\mathcal{A}}$ and $\mathrm{H}^\sharp_{\mathcal{A}}$

The trivialization of $H_n^\vee$ on $\mathcal{IG}_{n,r,I}$ induces an equality of groups $H_n^\vee(\mathfrak{IG}_{n,r,I}) = H_n^\vee(\mathcal{IG}_{n,r,I}) \simeq \mathcal{O}_L/p^n\mathcal{O}_L$. Let P^{univ} be the image of $1 \in \mathcal{O}_L/p^n\mathcal{O}_L$ in $H_n^\vee(\mathfrak{IG}_{n,r,I})$. We have a map of $\mathcal{O}_L \otimes \mathfrak{IG}_{n,r,I}$-modules,

$$\begin{array}{ccccc} & & \omega_{\mathcal{A}} & & \\ & & \downarrow & & \\ H_n^\vee(\mathcal{IG}_{n,r,I}) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathfrak{IG}_{n,r,I}} = H_n^\vee(\mathfrak{IG}_{n,r,I}) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathfrak{IG}_{n,r,I}} & \xrightarrow{\mathrm{dlog}\otimes 1} & \omega_{H_n} & \xrightarrow{\sim} & \omega_{\mathcal{A}}/p^n\mathrm{Hdg}^{-\frac{p^n-1}{p-1}}\omega_{\mathcal{A}}. \end{array} \tag{2.3}$$

Definition 2.3.1. Define the sheaf $\Omega_{\mathcal{A}}$ to be the inverse image under the map $\omega_A \to \omega_{H_n}$ of the image of $\mathrm{dlog}\otimes 1$. We call this the modified modular sheaf.

Proposition 2.3.1. *The sheaf $\Omega_{\mathcal{A}}$ is a locally free $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{n,r,I}}$ sheaf of rank 1. The cokernel of $\Omega_{\mathcal{A}} \subset \omega_{\mathcal{A}}$ is killed by $\mathrm{Hdg}^{\frac{1}{p-1}}$. Moreover dlog induces an isomorphism*

$$\mathrm{dlog}\otimes 1\colon H_n^\vee(\mathfrak{IG}_{n,r,I}) \otimes \mathcal{O}_{\mathfrak{IG}_{n,r,I}}/p^n\mathrm{Hdg}^{-\frac{p^n}{p-1}} \xrightarrow{\sim} \Omega_A \otimes \mathcal{O}_{\mathfrak{IG}_{n,r,I}}/p^n\mathrm{Hdg}^{-\frac{p^n}{p-1}}.$$

Proof. [AIP16b, Proposition 4.1]. □

Since $\Omega_{\mathcal{A}} \subset \omega_{\mathcal{A}}$ is an invertible $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{n,r,I}}$- module, there exists an invertible ideal $\underline{\xi} \subset \mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{n,r,I}}$, such that $\Omega_{\mathcal{A}} = \underline{\xi}\omega_{\mathcal{A}}$.

Definition 2.3.2. Define the sheaf $\mathrm{H}^{\sharp}_{\mathcal{A}} := \underline{\xi}\mathrm{H}'_{\mathcal{A}}$. We call this the modified de Rham sheaf.

Corollary 2.3.1. *We have a short exact sequence of locally free $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{n,r,I}}$-modules*

$$0 \to \Omega_{\mathcal{A}} \to \mathrm{H}^{\sharp}_{\mathcal{A}} \to \underline{\xi}\omega^{\vee}_{\mathcal{A},0} \to 0 \tag{2.4}$$

which splits upon pulling back via $i_n \colon \mathfrak{IG}_{n,r,I}/(p\mathrm{Hdg}^{-1}) \hookrightarrow \mathfrak{IG}_{n,r,I}$.

Proof. This is immediate from Proposition 2.2.3. □

Letting $s = \mathrm{dlog}(P^{\mathrm{univ}}) \in \Omega_{\mathcal{A}}/p^n\mathrm{Hdg}^{\frac{-p^n}{p-1}}\Omega_{\mathcal{A}}$, and taking its σ-components s_σ for $\sigma \in \Sigma$, we get a vector bundle with marked sections $(\Omega_{\mathcal{A}}, \{s_\sigma\}_{\sigma\in\Sigma})$. Associated to this pair, one can consider the geometric vector bundle with marked sections $\mathbb{V}_0(\Omega_{\mathcal{A}}, \{s_\sigma\}_{\sigma\in\Sigma})$ in the sense of Definition 1.3.3. We will show that the points of this geometric vector bundle have a natural interpretation as $\mathcal{O}_L$-linear functions on $\Omega_{\mathcal{A}}$ that evaluate to 1 on s. But before that we need to study functoriality of the sheaf $\mathrm{H}^{\sharp}_{\mathcal{A}}$ with respect to the U correspondence. Note that we are not interested in $\mathrm{H}^{\sharp}_{\mathcal{A}}$ solely for its structure as a vector bundle with marked sections, but we are also interested in the splitting modulo some small power of p. We have already seen above that such a splitting exists modulo $p\mathrm{Hdg}^{-1}$. In studying functoriality of $\mathrm{H}^{\sharp}_{\mathcal{A}}$ for the U correspondence, we will pin down the small power of p for which the splitting is functorial too.

Let $\beta_n := p\mathrm{Hdg}^{-\frac{p^n}{p-1}}$.

2.3.1.1 Functoriality

Consider the projection $\lambda\colon \mathcal{A} \to \mathcal{A}' := \mathcal{A}/H_1$. Let $\lambda'\colon \mathcal{A}'^{\vee} \to \mathcal{A}^{\vee}$ be the isogeny such that $\lambda' \circ \lambda = [p]$. Then λ' maps $H_n(\mathcal{A}')$ to $H_n(\mathcal{A})$ and induces an isomorphism of canonical subgroups $H_n(\mathcal{A}') \simeq H_n(\mathcal{A})$ on the generic fibres. The generic trivialization of $H_n^{\vee}(\mathcal{A}')$ induced by this isomorphism defines a map $\tilde{F}\colon \mathfrak{IG}_{1,r,I} \to \mathfrak{IG}_{1,r-1,I}$ that sends $\mathcal{A} \mapsto \mathcal{A}'$ together with this trivialization of the generic fibre of the dual of the canonical subgroup. Note that a priori if $\mathcal{A}$ is $\mathfrak{c}$-polarized then $\mathcal{A}'$ is $p\mathfrak{c}$-polarized. But since multiplication by p induces a canonical isomorphism $M(\mu_N, \mathfrak{c}) \to M(\mu_N, p\mathfrak{c})$, we indeed get a map $\tilde{F}$ as above. By abuse of notation we also denote by $\tilde{F}$ the map $\mathfrak{X}_{r,\alpha,I} \to \mathfrak{X}_{r-1,\alpha,I}$ induced by sending $\mathcal{A} \mapsto \mathcal{A}'$. The following diagram is commutative with h and h' being the usual projections.

$$\begin{tikzcd} \mathfrak{IG}_{1,r,I} \arrow[r, "\tilde{F}"] \arrow[d, "h"] & \mathfrak{IG}_{1,r-1,I} \arrow[d, "h'"] \\ \mathfrak{X}_{r,\alpha,I} \arrow[r, "\tilde{F}"] & \mathfrak{X}_{r-1,\alpha,I} \end{tikzcd}$$

The functoriality of the dlog map provides the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \Omega_{\mathcal{A}} \arrow[r] \arrow[d, "\simeq"] & \omega_{\mathcal{A}} \arrow[d, "(\lambda')^*"] \\ 0 \arrow[r] & \Omega_{\mathcal{A}'} \arrow[r] & \omega_{\mathcal{A}'} \end{tikzcd}$$

The map $(\lambda')^*$ is the adjoint of a lift of the Hasse-Witt map $HW\colon \omega^\vee_{\mathcal{A}'/(p\mathrm{Hdg}^{-1})} \to \omega^\vee_{\mathcal{A}/(p\mathrm{Hdg}^{-1})}$ modulo $p\mathrm{Hdg}^{-1}$ because λ' is a lift of the Verschiebung. As a map between invertible $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{1,r,I}}$-modules, $(\lambda')^*$ corresponds to multiplication by an invertible $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{1,r,I}}$ ideal $\widetilde{HW}$. Since $\Omega_{\mathcal{A}} = \underline{\xi}\omega_{\mathcal{A}}$, we have the following relation between $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{1,r,I}}$ ideals.

Lemma 2.3.1. $\tilde{F}^*\underline{\xi} = \underline{\xi}\widetilde{HW}$.

Proof. Follows from the discussion above. □

Definition 2.3.3. Define the σ-components $\widetilde{HW}(\sigma)$ of $\widetilde{HW}$ as the partial Hasse ideals.

Corollary 2.3.2. $\prod_\sigma \widetilde{HW}(\sigma) = \mathrm{Hdg}$.

Proof. Immediate as the determinant of $\widetilde{HW}$ is Hdg. □

We will prove a result relating the σ-components of $\underline{\xi}$ and $\widetilde{HW}$. For that we need to choose a numbering of Σ. Recall p splits as $p = \mathfrak{P}_1 \cdots \mathfrak{P}_r$ in $\mathcal{O}_L$ with their inertia degree $f(\mathfrak{P}_i|p) = f_i$. Choose a bijection $\Xi_i\colon \{i_1, \ldots, i_{f_i}\} \simeq D(\mathfrak{P}_i)$ of the decomposition groups for each i.

Corollary 2.3.3. *$(\tilde{F}^*\omega_{\mathcal{A}})(i_j) = \tilde{F}^*(\omega_{\mathcal{A}}(i_{j-1}))$. In particular, $\tilde{F}^*(\underline{\xi}(i_{j-1})) = \underline{\xi}(i_j)\widetilde{HW}(i_j)$.*

Proof. Recall that for any $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{1,r,I}}$-module $\mathcal{F}$, $\mathcal{F}(i_j)$ is the component on which $\mathcal{O}_L$ acts via $\Xi_i(i_j)$. The claim follows immediately by noting that modulo $p\mathrm{Hdg}^{-1}$, $\tilde{F}$ induces a morphism such that the following diagram commutes with φ being the Frobenius.

$$\begin{tikzcd}
\mathfrak{IG}_{1,r,I}/(p\mathrm{Hdg}^{-1}) \arrow[r, "\tilde{F}"] \arrow[d] & \mathfrak{IG}_{1,r-1,I}/(p\mathrm{Hdg}^{-1}) \arrow[d] \\
\mathfrak{X}_{\alpha,I}/(p) \arrow[r, "\varphi"] & \mathfrak{X}_{\alpha,I}/(p)
\end{tikzcd}$$

□

Lemma 2.3.2. *$\mathfrak{X}_{r,\alpha,I}$ and $\mathfrak{IG}_{1,r,I}$ are normal schemes.*

Proof. For the case of $\mathfrak{X}_{r,\alpha,I}$ see [AIP16b, Corollary 3.8]. The map $\mathcal{IG}_{1,r,I} \to \mathcal{X}_{r,\alpha,I}$ is finite étale. Hence $\mathcal{IG}_{1,r,I}$ is normal. Thus $\mathfrak{IG}_{1,r,I}$ is normal being the normalization of $\mathfrak{X}_{r,\alpha,I}$ in $\mathcal{IG}_{1,r,I}$. □

In the next lemma and the remark following it, we record a result that seemed interesting to us, even though it is not used for any argument further ahead.

Lemma 2.3.3. *For all i_j, $\tilde{F}^*(\underline{\xi}(i_j))$ is a p-th power at all height 1 localizations.*

Proof. Since this is an equality of ideals in a normal scheme it is enough to check the statement locally at height 1 primes. So choose an affine open U in $\mathfrak{X}_{\alpha,I}$ such that the pullback of $\omega_{\mathcal{A}}$ to $U/(p)$ is trivial as an $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{X}_{\alpha,I}}/(p)$-module. Let Spf R_{r-1} and Spf R_r be its inverse image in $\mathfrak{X}_{r-1,\alpha,I}$ and $\mathfrak{X}_{r,\alpha,I}$ respectively. Let S_{r-1} and S_r be their respective inverse images in $\mathfrak{IG}_{1,r-1,I}$ and $\mathfrak{IG}_{1,r,I}$. So $\tilde{F}$ induces a commutative diagram as follows.

$$\begin{array}{ccc} S_{r-1} & \xrightarrow{\tilde{F}^*} & S_r \\ {\scriptstyle h'}\uparrow & & \uparrow{\scriptstyle h} \\ R_{r-1} & \xrightarrow{\tilde{F}^*} & R_r \end{array}$$

Pick a height 1 prime $\mathfrak{p} \in$ Spf S_r that contains a local generator of $\underline{\xi}(i_j)$. Since $\prod_{i,j} \underline{\xi}(i_j)^{p-1} = \mathrm{Hdg}$ [AIP18, Proposition A.3], $\mathfrak{p} \in V(\mathrm{Hdg})$. Let $\mathfrak{q} = h^{-1}\mathfrak{p}$. Then $\mathfrak{q}$ is a height 1 prime containing Hdg. In particular $\mathfrak{q}R_{rq}$ is generated by Hdg as Hdg has simple zeroes along degree 1 divisors in $\mathfrak{X}_{\alpha,I}/(p)$ (Theorem 2.1.1). Let $\mathfrak{p}' = (\tilde{F}^*)^{-1}\mathfrak{p}$ and $\mathfrak{q}' = (\tilde{F}^*)^{-1}\mathfrak{q}$. Localizing at the primes gives a diagram as follows.

$$\begin{array}{ccc} S_{r-1\mathfrak{p}'} & \xrightarrow{\tilde{F}^*} & S_{r\mathfrak{p}} \\ {\scriptstyle h'}\uparrow & & \uparrow{\scriptstyle h} \\ R_{r-1\mathfrak{q}'} & \xrightarrow{\tilde{F}^*} & R_{r\mathfrak{q}} \end{array}$$

All the rings are DVR. The bottom arrow has ramification index p. h and h' are tamely ramified. This forces the upper arrow to be ramified of index p. This proves the lemma. $\square$

Remark 2.3.2. It seems that in fact $\tilde{F}^*(\underline{\xi}(i_j)) = \underline{\xi}(i_j)^p$. This would imply that $\underline{\xi}(i_{j-1})^p = \underline{\xi}(i_j)\widetilde{HW}(i_j)$ which reflects the fact that the partial Hasse invariant of degree i_j is of weight $(p, -1)$ concentrated at degree (i_{j-1}, i_j) [Gor01, Theorem 2.1]. Moreover this shows a posteriori that $\prod_{i,j} \underline{\xi}(i_j)^{p-1} = \mathrm{Hdg}$. But as of now we are not able to prove this.

Proposition 2.3.2. *There exists an r large enough such that the map $(\lambda')^*\colon \mathrm{H}_{\mathcal{A}} \to \mathrm{H}_{\mathcal{A}'}$ restricts to a well-defined map $(\lambda')^*\colon \mathrm{H}^\sharp_{\mathcal{A}} \to \mathrm{H}^\sharp_{\mathcal{A}'}$ that sends marked sections to marked sections. Moreover, let $\mathcal{Q} \subset \mathrm{H}^\sharp_{\mathcal{A}}/p\mathrm{Hdg}(\mathcal{A})^{-(p+1)}$ be the kernel of the marked splitting, and let $\mathcal{Q}' \subset \mathrm{H}^\sharp_{\mathcal{A}'}/p\mathrm{Hdg}(\mathcal{A})^{-(p+1)}$ be the same for $\mathcal{A}'$. Then $(\lambda')^*$ sends $\mathcal{Q}$ to $\mathcal{Q}'$.*

Proof. Since $(\lambda')^*$ maps $\Omega_{\mathcal{A}}$ isomorphically onto $\Omega_{\mathcal{A}'}$ sending the marked section to the marked section, it is enough to show that the induced map $\mathrm{H}^\sharp_{\mathcal{A}}/\Omega_{\mathcal{A}} \to \mathrm{H}_{\mathcal{A}'}/\Omega_{\mathcal{A}'}$ factors through the inclusion $\mathrm{H}^\sharp_{\mathcal{A}'}/\Omega_{\mathcal{A}'} \hookrightarrow \mathrm{H}_{\mathcal{A}'}/\Omega_{\mathcal{A}'}$. Choosing suitable local generators ξ of $\underline{\xi}(\mathcal{A})$ and $\widetilde{HW}$ of $\widetilde{HW}(\mathcal{A})$ respectively as $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{n,r,I}}$-modules, we have a diagram as follows.

$$\begin{tikzcd}
 & \mathrm{H}^{\sharp}_{\mathcal{A}'}/\Omega_{\mathcal{A}'} \arrow[r, "\simeq", "i"'] \arrow[d, hook] & \tilde{F}^*(\underline{\xi}(\mathcal{A})\widetilde{HW}(\mathcal{A}))\cdot\omega^{\vee}_{\mathcal{A}'} \arrow[d, "\tilde{F}^*(\xi\widetilde{HW})"] \\
\mathrm{H}^{\sharp}_{\mathcal{A}}/\Omega_{\mathcal{A}} \arrow[r, "(\lambda')^*"] \arrow[dr, "\simeq"'] & \mathrm{H}_{\mathcal{A}'}/\Omega_{\mathcal{A}'} \arrow[r, "\pi"] & \mathrm{H}_{\mathcal{A}'}/\omega_{\mathcal{A}'} \simeq \omega^{\vee}_{\mathcal{A}'} \\
 & \underline{\xi}(\mathcal{A})\widetilde{HW}(\mathcal{A})\cdot\omega^{\vee}_{\mathcal{A}} \arrow[ur, "p\widetilde{HW}^{-1}\xi\widetilde{HW}"'] &
\end{tikzcd}$$

Here the bottom right diagonal arrow is the map induced on the Lie algebra by $\lambda\colon \mathcal{A} \to \mathcal{A}'$. Choosing basis of $\omega^{\vee}_{\mathcal{A}}$ and of $\omega^{\vee}_{\mathcal{A}'}$, this $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{n,r,I}}$-linear map is multiplication by $p\widetilde{HW}^{-1}$ upto a unit. Hence we get a description of the arrow as in the diagram.

Let $\mathrm{Hdg} = \mathrm{Hdg}(\mathcal{A})$ in this proof. Since multiplication by $p\xi$ is injective, we see that the image of $\mathrm{H}^{\sharp}_{\mathcal{A}}/\Omega_{\mathcal{A}}$ under $(\lambda')^*$ does not intersect $\omega_{\mathcal{A}'}/\Omega_{\mathcal{A}'}$. We first show that $\pi \circ (\lambda')^*$ factors through the submodule $\tilde{F}^*(\underline{\xi}(\mathcal{A})\widetilde{HW}(\mathcal{A}))\omega^{\vee}_{\mathcal{A}'} \subset \omega^{\vee}_{\mathcal{A}'}$. For this it is enough to show that $p\underline{\xi}(\mathcal{A}) \subset \tilde{F}^*(\underline{\xi}(\mathcal{A})\widetilde{HW}(\mathcal{A}))$. Using Lemma 2.3.1 this reduces to showing $p/(\widetilde{HW}\tilde{F}^*(\widetilde{HW})) \subset \mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{n,r,I}}$ which can be ensured by choosing large enough r since $\det \widetilde{HW} = \mathrm{Hdg}$. This proves that the map $(\lambda')^* - i^{-1} \circ \pi \circ (\lambda')^*$ factors through $\omega_{\mathcal{A}'}/\Omega_{\mathcal{A}'}$ which is a torsion module killed by $\tilde{F}^*(\underline{\xi}(\mathcal{A})) = \underline{\xi}(\mathcal{A})\widetilde{HW}(\mathcal{A})$ (Lemma 2.3.1). Now since $\mathrm{H}^{\sharp}_{\mathcal{A}} = \Omega_{\mathcal{A}} + \underline{\xi}(\mathcal{A})\widetilde{HW}(\mathcal{A}) \cdot \mathrm{H}_{\mathcal{A}}$, the difference $(\lambda')^* - i^{-1} \circ \pi \circ (\lambda')^* = 0$. This proves the first claim.

The second claim follows from a local computation. Choose an open affine $\mathrm{Spf}\, R = U \subset \mathfrak{X}_{\alpha,I}$ such that $\mathrm{H}_{\mathcal{A}}$ admits an $\mathcal{O}_L \otimes R$ basis $\{e, f\}$ over U with e a basis of $\omega_{\mathcal{A}}$. Let $\{\bar{e}, \bar{f}\}$ be their image over $\mathrm{Spf}\, R/p$. The unit root subspace is generated by a vector $\bar{v} = \bar{C}\bar{e} + HW\bar{f}$ for some $\bar{C} \in (\mathcal{O}_L \otimes R)/p$. Let $\varphi\colon R/p \to R/p$ be the Frobenius. With respect to the basis $\{\bar{e}, \bar{f}\}$ of $\mathrm{H}_{\bar{\mathcal{A}}}$ and $\{\bar{e}^{(p)} := \varphi^*(\bar{e}), \bar{f}^{(p)} := \varphi^*(\bar{f})\}$ of $\mathrm{H}_{\bar{\mathcal{A}}^{(p)}}$, the matrix of Verschiebung $V\colon \mathrm{H}_{\bar{\mathcal{A}}} \to \mathrm{H}_{\bar{\mathcal{A}}^{(p)}}$ can be written as

$$V = \begin{pmatrix} HW & \bar{B} \\ 0 & 0 \end{pmatrix}.$$

Since Verschiebung kills the unit root subspace, $V(\bar{C}\bar{e} + HW\bar{f}) = (\bar{C}HW + \bar{B}HW)\bar{e} = 0$. This shows that $\bar{B} = -\bar{C}$. Let $\mathrm{Spf}\, R_{n,r} \subset \mathfrak{IG}_{n,r,I}$ (resp. $\mathrm{Spf}\, R_{n,r-1} \subset \mathfrak{IG}_{n,r-1,I}$) be the inverse image of U in $\mathfrak{IG}_{n,r,I}$ (resp. $\mathfrak{IG}_{n,r-1,I}$). Then as discussed in §2.2, $\mathrm{H}'_{\mathcal{A}}$ over $\mathfrak{IG}_{n,r,I}$ is generated by the pullback of e and a lift $v = Ce + \widetilde{HW}f$ of $\bar{v}$. For notational simplicity we will write these sections as $\{e, v\}$ still. Similarly, $\mathrm{H}'_{\mathcal{A}'}$ is generated by pulling back via $\tilde{F}\colon \mathrm{Spf}\, R_{n,r} \to \mathrm{Spf}\, R_{n,r-1}$ the pullbacks of e and v to $\mathrm{Spf}\, R_{n,r-1}$. We will write these as $\{\tilde{F}^*e, \tilde{F}^*v\}$. Then with respect to $\{e, v\}$ and $\{\tilde{F}^*e, \tilde{F}^*v\}$ the matrix of $(\lambda')^*$ can be described as

$$(\lambda^\vee)^* = \begin{pmatrix} 1 & -\tilde{F}^*(C\widetilde{HW}^{-1}) \\ 0 & \tilde{F}^*(\widetilde{HW}^{-1}) \end{pmatrix} \cdot \begin{pmatrix} \widetilde{HW} & B \\ 0 & p\widetilde{HW}^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & C \\ 0 & \widetilde{HW} \end{pmatrix}$$
$$= \begin{pmatrix} \widetilde{HW} & C\widetilde{HW} + B\widetilde{HW} - p\tilde{F}^*(C\widetilde{HW}^{-1}) \\ 0 & p\tilde{F}^*(\widetilde{HW}^{-1}) \end{pmatrix}.$$

Here B is a lift of $\bar{B}$ modulo $p\mathrm{Hdg}^{-1}$. Therefore with respect to the basis $\{\xi e, \xi v\}$ and $\{\tilde{F}^*(\xi e), \tilde{F}^*(\xi v)\}$, the matrix of $(\lambda')^* \colon \mathrm{H}^\sharp_{\mathcal{A}} \to \mathrm{H}^\sharp_{\mathcal{A}'}$ is written as

$$(\lambda')^* = \begin{pmatrix} 1 & C + B - p\tilde{F}^*(C\widetilde{HW}^{-1})\widetilde{HW}^{-1} \\ 0 & p\widetilde{HW}^{-1}\tilde{F}^*(\widetilde{HW}^{-1}) \end{pmatrix}.$$

Since $\det \widetilde{HW} = \mathrm{Hdg}$, this proves the second claim of the lemma. □

Definition 2.3.4. Let $\mathrm{H}^\sharp_{\mathcal{A}}, \Omega_{\mathcal{A}}, s$ be the modified de Rham sheaf, the modified modular sheaf and the marked section respectively. Let $\mathcal{Q} \subset \mathrm{H}^\sharp_{\mathcal{A}}/p\mathrm{Hdg}^{-p^2}$ be the kernel of the splitting $\psi \colon \mathrm{H}^\sharp_{\mathcal{A}}/p\mathrm{Hdg}^{-p^2} \to \Omega_{\mathcal{A}}/p\mathrm{Hdg}^{-p^2}$. We call this marked splitting the modified unit root splitting and $\mathcal{Q}$ the modified unit root subspace.

We remark that by Proposition 2.3.2 the modified unit root subspace is functorial for $\lambda' \colon \mathcal{A}' \to \mathcal{A}$.

Recall the notation $\beta_n = p^n \mathrm{Hdg}^{\frac{-p^n}{p-1}}$. Let $\eta := p\mathrm{Hdg}^{-p^2}$.

2.3.2 Formal $\mathcal{O}_L$-vector bundles with marked sections and marked splitting

In this section we define the relevant vector bundles with marked sections and marked splitting enriched with an action of $\mathcal{O}_L$. Although we focus on the sheaves relevant for our purpose, i.e. $\Omega_{\mathcal{A}}$ and $\mathrm{H}^\sharp_{\mathcal{A}}$, the theory can be developed more generally.

Recall $s = \mathrm{dlog}(P_n^{\mathrm{univ}}) \in \Omega_{\mathcal{A}}/\beta_n\Omega_{\mathcal{A}}$ is the image of the universal generator of $H_n^\vee(\mathfrak{IG}_{n,r,I})$ under the dlog map. Let s_σ be its σ-component under the splitting $\Omega_{\mathcal{A}} = \prod_{\sigma\in\Sigma} \Omega_{\mathcal{A}}(\sigma)$. Following the VBMS formalism explained in §1.3.1 we define the following formal $\mathcal{O}_L$-vector bundles with marked sections.

Definition 2.3.5. Define $\mathbb{V}^{\mathcal{O}_L}(\Omega_{\mathcal{A}})$ as the functor that associates to any α-admissible formal scheme $\mathfrak{Z} \xrightarrow{\gamma} \mathfrak{IG}_{n,r,I}$ the following set:

$$\mathbb{V}^{\mathcal{O}_L}(\Omega_{\mathcal{A}})(\mathfrak{Z} \xrightarrow{\gamma} \mathfrak{IG}_{n,r,I}) := \mathrm{Hom}_{\mathcal{O}_L\otimes\mathcal{O}_{\mathfrak{Z}}}(\gamma^*\Omega_{\mathcal{A}}, \mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{Z}}).$$

Similarly, define $\mathbb{V}^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}})$ as the functor

$$\mathbb{V}^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}})(\mathfrak{Z} \xrightarrow{\gamma} \mathfrak{IG}_{n,r,I}) := \mathrm{Hom}_{\mathcal{O}_L\otimes\mathcal{O}_{\mathfrak{Z}}}(\gamma^*\mathrm{H}^\sharp_{\mathcal{A}}, \mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{Z}}).$$

Definition 2.3.6. Define $\mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s)$ as the functor that associates to any α-admissible formal scheme $\mathfrak{Z} \xrightarrow{\gamma} \mathfrak{IG}_{n,r,I}$, the following set:

$$\mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s)(\mathfrak{Z} \xrightarrow{\gamma} \mathfrak{IG}_{n,r,I}) := \left\{ h \in \mathrm{Hom}_{\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{Z}}} \left(\gamma^* \Omega_{\mathcal{A}}, \mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{Z}}\right) \mid (h \bmod \gamma^* \beta_n)(\gamma^* s) = 1 \right\}.$$

Similarly, define $\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s)$ as the functor

$$\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s)(\mathfrak{Z} \xrightarrow{\gamma} \mathfrak{IG}_{n,r,I}) := \left\{ h \in \mathrm{Hom}_{\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{Z}}} \left(\gamma^* \mathrm{H}^\sharp_{\mathcal{A}}, \mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{Z}}\right) \mid (h \bmod \gamma^* \beta_n)(\gamma^* s) = 1 \right\}.$$

Proposition 2.3.3. *1. We have natural isomorphisms of functors*

$$\mathbb{V}^{\mathcal{O}_L}(\Omega_{\mathcal{A}}) \simeq \mathbb{V}(\Omega_{\mathcal{A}}) = \prod_{\sigma \in \Sigma} \mathbb{V}\big(\Omega_{\mathcal{A}}(\sigma)\big), \qquad \mathbb{V}^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}) \simeq \mathbb{V}(\mathrm{H}^\sharp_{\mathcal{A}}) = \prod_{\sigma \in \Sigma} \mathbb{V}\big(\mathrm{H}^\sharp_{\mathcal{A}}(\sigma)\big).$$

where $\mathbb{V}(\mathcal{E})$ for any locally finite free sheaf $\mathcal{E}$ is defined as in Definition 1.3.2.

2. We have natural isomorphisms of functors

$$\mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s) \simeq \prod_{\sigma \in \Sigma} \mathbb{V}_0\big(\Omega_{\mathcal{A}}(\sigma), s_\sigma\big), \qquad \mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s) \simeq \prod_{\sigma \in \Sigma} \mathbb{V}_0\big(\mathrm{H}^\sharp_{\mathcal{A}}(\sigma), s_\sigma\big).$$

where $\mathbb{V}_0(\mathcal{E}, s) \subset \mathbb{V}(\mathcal{E})$ for any locally finite free sheaf $\mathcal{E}$ with a marked section s is defined as in Definition 1.3.3.

Proof. We note that since p is unramified in L, the natural map $\mathcal{O}_L \otimes \mathcal{O}_K \to \mathcal{O}_K^\Sigma$ is an isomorphism. The claims of the proposition then follow immediately from the definitions. $\square$

Taking into account the modified unit root subspace $\mathcal{Q} \subset \mathrm{H}^\sharp_{\mathcal{A}}/p\mathrm{Hdg}^{-p^2}$, we define a geometric $\mathcal{O}_L$-vector bundle with marked section and marked splitting as follows.

Definition 2.3.7. 1. For any $\sigma \in \Sigma$, define $\mathbb{V}_0\big(\mathrm{H}^\sharp_{\mathcal{A}}(\sigma), s_\sigma, \mathcal{Q}(\sigma)\big)$ as the functor that associates to any α-admissible formal scheme $\mathfrak{Z} \xrightarrow{\gamma} \mathfrak{IG}_{n,r,I}$ the following set:

$$\mathbb{V}_0\big(\mathrm{H}^\sharp_{\mathcal{A}}(\sigma), s_\sigma, \mathcal{Q}(\sigma)\big)(\mathfrak{Z}) := \left\{ h \in \mathbb{V}_0\big(\mathrm{H}^\sharp_{\mathcal{A}}(\sigma), s_\sigma\big)(\mathfrak{Z}) \mid (h \bmod \gamma^* \eta)(\gamma^* \mathcal{Q}(\sigma)) = 0 \right\}.$$

2. Define $\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q})$ as the functor that associates to any α-admissible formal scheme $\mathfrak{Z} \xrightarrow{\gamma} \mathfrak{IG}_{n,r,I}$ the following set:

$$\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q})(\mathfrak{Z}) := \left\{ h \in \mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s)(\mathfrak{Z}) \mid (h \bmod \gamma^* \eta)(\gamma^* \mathcal{Q}) = 0 \right\}.$$

Proposition 2.3.4. *We have a natural isomorphism of functors*

$$\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q}) \simeq \prod_{\sigma \in \Sigma} \mathbb{V}_0\big(\mathrm{H}^\sharp_{\mathcal{A}}(\sigma), s_\sigma, \mathcal{Q}(\sigma)\big).$$

Proof. Clear from the definitions. □

Proposition 2.3.5. *The functors* $\mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s)$, $\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s)$ *and* $\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q})$ *are representable.*

Proof. Follows immediately from the discussion in §1.3. □

2.3.2.1 Formal group action on formal $\mathcal{O}_L$-vector bundles

The vector bundles $\mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s)$, $\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s)$ and $\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q})$ carry an action of the formal group $\mathfrak{T} := 1 + \beta_n \operatorname{Res}_{\mathcal{O}_L/\mathbb{Z}} \mathbb{G}_a$ over $\mathfrak{IG}_{n,r,I}$. This action realizes $\mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s)$ as a $\mathfrak{T}$-torsor over $\mathfrak{IG}_{n,r,I}$. Moreover, there is a natural action of $\mathbb{T}(\mathbb{Z}_p) = (\mathcal{O}_L \otimes \mathbb{Z}_p)^\times$ on the aforementioned formal vector bundles over $\mathfrak{X}_{r,\alpha,I}$ which we will describe too. Together we get an action of $\mathfrak{T}^{\text{ext}} := \mathbb{Z}_p^\times(1+\beta_n \operatorname{Res}_{\mathcal{O}_L/\mathbb{Z}} \mathbb{G}_a)$ on these vector bundles over $\mathfrak{X}_{r,\alpha,I}$.

1. $\mathfrak{T}^{\text{ext}}$-action on $\mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s)$: Let $(\rho, h) \in \mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s)(R)$ be a R-valued point of $\mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s)$. Here $\rho\colon \operatorname{Spf} R \to \mathfrak{IG}_{n,r,I}$ is a morphism of formal schemes and $h \in \operatorname{Hom}_{\mathcal{O}_L\otimes R}(\rho^*\Omega_{\mathcal{A}}, \mathcal{O}_L \otimes R)$. Then $\lambda \in \mathfrak{T}(R)$ acts as $\lambda * (\rho, h) := (\rho, \lambda h)$. Let $\lambda \in \mathbb{T}(\mathbb{Z}_p)$. Denote its class in $\mathcal{O}_L \otimes \mathbb{Z}/p^n\mathbb{Z}$ by $\bar{\lambda}$. Then λ induces an isomorphism $[\lambda]\colon \mathfrak{IG}_{n,r,I} \xrightarrow{\sim} \mathfrak{IG}_{n,r,I}$ that induces the map
$$\begin{aligned} H_n^\vee(R) &\to H_n^\vee(R) \\ P &\mapsto \bar{\lambda}^{-1} P \end{aligned}$$
on the R-valued points of $\mathfrak{IG}_{n,r,I}$. There is a natural isomorphism $\gamma_\lambda\colon [\lambda]^*\Omega_{\mathcal{A}} \xrightarrow{\sim} \Omega_{\mathcal{A}}$ such that $(\gamma_\lambda \bmod \beta_n)([\lambda]^*s) = \bar{\lambda}^{-1}s$. Then we define the $\mathbb{T}(\mathbb{Z}_p)$-action as $\lambda * (\rho, h) := ([\lambda]\circ\rho, \lambda h \circ \gamma_\lambda)$.
2. $\mathfrak{T}^{\text{ext}}$-action on $\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s)$: Let $(\rho, h) \in \mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s)(R)$, with $\rho\colon \operatorname{Spf} R \to \mathfrak{IG}_{n,r,I}$ a morphism of formal schemes and $h \in \operatorname{Hom}_{\mathcal{O}_L\otimes R}(\rho^*\mathrm{H}^\sharp_{\mathcal{A}}, \mathcal{O}_L \otimes R)$. Then $\lambda \in \mathfrak{T}(R)$ acts as $\lambda * (\rho, h) := (\rho, \lambda h)$. If $\lambda \in \mathbb{T}(\mathbb{Z}_p)$, then as before we have an isomorphism $[\lambda]\colon \mathfrak{IG}_{n,r,I} \xrightarrow{\sim} \mathfrak{IG}_{n,r,I}$. This gives a natural isomorphism $\gamma_\lambda\colon [\lambda]^*\mathrm{H}^\sharp_{\mathcal{A}} \xrightarrow{\sim} \mathrm{H}^\sharp_{\mathcal{A}}$ such that $(\gamma_\lambda \bmod \beta_n)([\lambda]^*s) = \bar{\lambda}^{-1}s$. Then we define the $\mathbb{T}(\mathbb{Z}_p)$-action as $\lambda * (\rho, h) := ([\lambda] \circ \rho, \lambda h \circ \gamma_\lambda)$.
3. $\mathfrak{T}^{\text{ext}}$-action on $\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q})$: This is defined by restricting the action defined on $\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s)$.

Lemma 2.3.4. *The formal group* $\mathfrak{T}$ *decomposes as* $\mathfrak{T} = \prod_{\sigma\in\Sigma}(1 + \beta_n\mathbb{G}_a)$ *over* $\mathfrak{IG}_{n,r,I}$. *The action of* $\mathfrak{T}$ *on* $\mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s)$ *and* $\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s)$ *is compatible with the splitting of* $\mathfrak{T}$ *and the vector bundles. That is to say, if* $\lambda = (\lambda_\sigma) \in \prod_\sigma(1+\beta_n\mathbb{G}_a)(R)$, *and* $(\rho, h) \in \mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s)(R)$, *with* $h = (h_\sigma) \in \prod_\sigma \operatorname{Hom}_R(\rho^*\Omega_{\mathcal{A}}(\sigma), R)$, *then* $\lambda * (\rho, h) = \prod_\sigma(\rho, \lambda_\sigma h_\sigma)$. *Similarly for* $\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s)$.

Proof. This is clear. □

$\mathfrak{T}$-action in terms of local coordinates:
Based on Lemma 2.3.4, the action of $\mathfrak{T}$ on $\mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s)$ and on $\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q})$ can be described on local coordinates in the following manner.

Let Spf $R \hookrightarrow \mathfrak{IG}_{n,r,I}$ be an open subscheme such that $\Omega_{\mathcal{A}}$ and $\mathrm{H}^{\sharp}_{\mathcal{A}}$ are trivialized as $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{n,r,I}}$-modules, and such that $\mathcal{Q}$ is trivial too over Spf R/η. Let $X \in \Omega_{\mathcal{A}}(R)$ be a lift of s and $Y \in \mathrm{H}^{\sharp}_{\mathcal{A}}(R)$ be a lift of a local $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{n,r,I}}$ generator t of $\mathcal{Q}$. Let X_σ, Y_σ be their σ-components. The formal schemes $\mathbb{V}_0(\Omega_{\mathcal{A}}(\sigma), s_\sigma)$ and $\mathbb{V}_0(\mathrm{H}^{\sharp}_{\mathcal{A}}(\sigma), s_\sigma, \mathcal{Q}(\sigma))$ are realized as admissible blow-ups of $\mathbb{V}(\Omega_{\mathcal{A}}(\sigma))$ and $\mathbb{V}(\mathrm{H}^{\sharp}_{\mathcal{A}}(\sigma))$ respectively. Then as elaborated in §1.3 the blow-ups are described by the following diagrams.

$$\begin{tikzcd}
\mathfrak{IG}_{n,r,I} & \mathbb{V}(\Omega_{\mathcal{A}}(\sigma)) \arrow[l] & \mathbb{V}_0(\Omega_{\mathcal{A}}(\sigma), s_\sigma) \arrow[l] \\
\mathrm{Spf}\, R \arrow[u, hook] & \mathrm{Spf}\, R\langle X_\sigma \rangle \arrow[l] \arrow[u, hook] & \mathrm{Spf}\, R\langle Z_\sigma \rangle \arrow[l, "X_\sigma \mapsto 1+\beta_n Z_\sigma"'] \arrow[u, hook]
\end{tikzcd} \tag{2.5}$$

$$\begin{tikzcd}
\mathfrak{IG}_{n,r,I} & \mathbb{V}(\mathrm{H}^{\sharp}_{\mathcal{A}}(\sigma)) \arrow[l] & \mathbb{V}_0(\mathrm{H}^{\sharp}_{\mathcal{A}}(\sigma), s_\sigma) \arrow[l] & \mathbb{V}_0(\mathrm{H}^{\sharp}_{\mathcal{A}}, s_\sigma, \mathcal{Q}(\sigma)) \arrow[l] \\
\mathrm{Spf}\, R \arrow[u, hook] & \mathrm{Spf}\, R\langle X_\sigma, Y_\sigma \rangle \arrow[l] \arrow[u, hook] & \mathrm{Spf}\, R\langle Z_\sigma, Y_\sigma \rangle \arrow[l, "X_\sigma \mapsto 1+\beta_n Z_\sigma"'] \arrow[u, hook] & \mathrm{Spf}\, R\langle Z_\sigma, W_\sigma \rangle \arrow[l, "Y_\sigma \mapsto \alpha W_\sigma"'] \arrow[u, hook]
\end{tikzcd} \tag{2.6}$$

Let $\lambda \in \mathfrak{T}(R)$ and let $\lambda = (\lambda_\sigma)_\sigma$ be its decomposition into coordinates. The action of λ_σ on Z_σ is such that $\lambda_\sigma * (1 + \beta_n Z_\sigma) = \lambda_\sigma(1 + \beta_n Z_\sigma)$. In other words,

$$\lambda_\sigma * Z_\sigma = \frac{\lambda_\sigma - 1}{\beta_n} + \lambda_\sigma Z_\sigma.$$

Similarly, λ_σ acts on W_σ via

$$\lambda_\sigma * W_\sigma = \lambda_\sigma W_\sigma.$$

2.3.3 p-adic interpolation of $\Omega_{\mathcal{A}}$ and $\mathrm{H}^{\sharp}_{\mathcal{A}}$

Let n, r, α, I be as fixed in the beginning of the section. Denote by $\rho' \colon \mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^{\sharp}_{\mathcal{A}}, s, \mathcal{Q}) \to \mathfrak{IG}_{n,r,I}$ and $\nu' \colon \mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s) \to \mathfrak{IG}_{n,r,I}$ the projections.

Definition 2.3.8. 1. For $k = k^0_{\alpha,I} \colon (\mathcal{O}_L \otimes \mathbb{Z}_p)^\times \to \Lambda^0_{\alpha,I}$ the universal character, define

$$\mathfrak{w}'_{k,\alpha,I} := \nu'_* \mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s)}[k].$$

The sections of this sheaf by definition are the functions $f \in \nu'_* \mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s)}$ that transform via $\lambda * f = k(\lambda) f$ under the $\mathfrak{T}$-action.

2. For $k = k^0_{\alpha,I}$ define

$$\mathbb{W}'_{k,\alpha,I} := \rho'_* \mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp, s, \mathcal{Q})}[k].$$

Let $k_\sigma \colon 1 + \beta_n \mathbb{G}_a \to \mathbb{G}_m$ be the restriction of $k^0_{\alpha,I}$ to the σ-component of $\mathfrak{T} = \prod_\sigma (1 + \beta_n \mathbb{G}_a)$.

Proposition 2.3.6. *Let $\nu'_\sigma\colon \mathbb{V}_0(\Omega_{\mathcal{A}}(\sigma), s_\sigma) \to \mathfrak{IG}_{n,r,I}$ be the projection for each σ. Then*

$$\mathfrak{w}'_{k,\alpha,I} = \hat{\otimes}_\sigma (\nu'_\sigma)_* \mathcal{O}_{\mathbb{V}_0(\Omega_{\mathcal{A}}(\sigma), s_\sigma)}[k_\sigma].$$

In particular, $\mathfrak{w}'_{k,\alpha,I}$ is a line bundle on $\mathfrak{IG}_{n,r,I}$.

Proof. Denote $\hat{\otimes}_\sigma (\nu'_\sigma)_* \mathcal{O}_{\mathbb{V}_0(\Omega_{\mathcal{A}}(\sigma), s_\sigma)}[k_\sigma]$ by $\tilde{w}$. Then indeed $\tilde{\omega} \subset \mathfrak{w}'_{k,\alpha,I}$ as follows from Lemma 2.3.4.

Take a Zariski open $\operatorname{Spf} R \subset \mathfrak{IG}_{n,r,I}$ that trivializes $\Omega_{\mathcal{A}}$. Then (2.5) shows that $(\nu')^{-1}(\operatorname{Spf} R) \simeq \operatorname{Spf} R\langle \{Z_\sigma\}_{\sigma \in \Sigma}\rangle$.

The description of the $\mathfrak{T}$-action on $\mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s)$ in terms of local coordinates imply by [AI21, Lemma 3.9]

$$\tilde{\omega}(\operatorname{Spf} R) = R \cdot \prod_\sigma k_\sigma(1 + \beta_n Z_\sigma).$$

Denote $\prod_\sigma k_\sigma(1+\beta_n Z_\sigma)$ by $k(1+\beta_n Z)$.

If $f \in \mathfrak{w}'_{k,\alpha,I}(\operatorname{Spf} R)$, then $f/k(1+\beta_n Z) \in R\langle\{Z_\sigma\}_\sigma\rangle^{\mathfrak{T}(R)}$. Then the problem reduces to showing that the $\mathfrak{T}$ invariant functions are simply R.

For the one variable case, this follows from an application of the Weierstrass preparation theorem. If $1+\beta_n R$ acts on $R\langle Z\rangle$ via $t * Z = \frac{t-1}{\beta_n} + tZ$, then take $f \in R\langle Z\rangle$ invariant under the action of $1+\beta_n R$. Suppose $f = \sum a_n Z^n$. Then for any $a \in R$, $\sum a_n Z^n = (1+\beta_n a) * (\sum a_n Z^n) = \sum a_n (a + (1+\beta_n a)Z)^n$. Letting $Z = 0$, we see that $a_0 = \sum a_n a^n$ for any $a \in R$, which shows that $a_n = 0$ for all $n > 0$.

For the general case the result follows by induction on the number of variables. Choose a bijection $\Sigma \simeq \{1, \dots, g\}$. Suppose $f = \sum a_n Z_g^n \in R\langle Z_1, \dots, Z_g\rangle^{\mathfrak{T}(R)}$, with $a_n \in R\langle Z_1, \dots, Z_{g-1}\rangle$ for all n. Then for any element $\lambda = (\lambda_i) \in \mathfrak{T}(R)$, such that $\lambda_g = 1$, $f = \lambda * f = \sum (\lambda * a_n) Z_g^n$. This shows that $\lambda * a_n = a_n$ for all n, and then by induction $a_n \in R$. Finally $a_n = 0$ for all $n > 0$ by the same argument as above. □

Remark 2.3.3. The isomorphism classes of $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{n,r,I}}$ line bundles can be naturally identified with elements of $H^1(\mathfrak{IG}_{n,r,I}, \operatorname{Res}_{\mathcal{O}_L/\mathbb{Z}} \mathbb{G}_m)$. The subgroup $H^1(\mathfrak{IG}_{n,r,I}, 1+\beta_n \operatorname{Res}_{\mathcal{O}_L/\mathbb{Z}} \mathbb{G}_a)$ classifies precisely $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{n,r,I}}$ line bundles $\mathscr{L}$ with a marked section $s \in \mathscr{L}/\beta_n \mathscr{L}$. Thus the isomorphism class of $(\Omega_{\mathcal{A}}, \operatorname{dlog}(P_n^{\mathrm{univ}}))$ defines an element of $H^1(\mathfrak{IG}_{n,r,I}, \mathfrak{T})$. Then $\mathfrak{w}'_{k,\alpha,I}$ defined as above is nothing but its image under the map induced by extension of structural group $H^1(\mathfrak{IG}_{n,r,I}, \mathfrak{T}) \xrightarrow{k} H^1(\mathfrak{IG}_{n,r,I}, \mathbb{G}_m)$.

Next we give a local description of $\mathbb{W}'_{k,\alpha,I}$.

Proposition 2.3.7. *Let $\rho'_\sigma\colon \mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{A}}(\sigma), s_\sigma, \mathcal{Q}(\sigma)) \to \mathfrak{IG}_{n,r,I}$ be the projection for each σ. Then*

$$\mathbb{W}'_{k,\alpha,I} = \hat{\otimes}_\sigma (\rho'_\sigma)_* \mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{A}}(\sigma), s_\sigma, \mathcal{Q}(\sigma))}[k_\sigma].$$

Proof. Let $\tilde{\mathbb{W}} = \hat{\otimes}_\sigma (\rho'_\sigma)_* \mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{A}}(\sigma), s_\sigma, \mathcal{Q}(\sigma))}[k_\sigma]$. Then clearly $\tilde{\mathbb{W}} \subset \mathbb{W}'_{k,\alpha,I}$.

Take a Zariski open $\operatorname{Spf} R \subset \mathfrak{IG}_{n,r,I}$ that trivializes $\mathrm{H}^\sharp_{\mathcal{A}}$ compatibly with a trivialization of $\Omega_{\mathcal{A}}$ and of $\mathcal{Q}$ modulo η. Choosing coordinates as in the local description (2.6), we see that $(\rho')^{-1}(\operatorname{Spf} R) \simeq \operatorname{Spf} R\langle\{Z_\sigma, W_\sigma\}_{\sigma\in\Sigma}\rangle$.

The description of $\mathfrak{T}$-action on $\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q})$ in local coordinates, together with Proposition 1.3.2 shows that

$$\tilde{\mathbb{W}}(\operatorname{Spf} R) = R\langle\{\frac{W_\sigma}{1+\beta_n Z_\sigma}\}_{\sigma\in\Sigma}\rangle \cdot k(1+\beta_n Z)$$

where we recall from the previous Proposition that $k(1+\beta_n Z) = \prod_\sigma k_\sigma(1+\beta_n Z_\sigma)$. Since $k(1+\beta_n Z)$ is a unit, in order to prove the reverse inclusion, it will be sufficient to show that $R\langle\{Z_\sigma, W_\sigma\}_\sigma\rangle^{\mathfrak{T}(R)} = R\langle\{\frac{W_\sigma}{1+\beta_n Z_\sigma}\}_{\sigma\in\Sigma}\rangle$. We prove this by induction on the cardinality of Σ.

For the one variable case, this has been proved in Proposition 1.3.2. For the general case, choose a bijection $\Sigma \simeq \{1,\dots,g\}$. Let $V_g := \frac{W_g}{1+\beta_n Z_g}$. The inclusion $R\langle\{Z_i, W_i\}_{i=1}^{g-1}\rangle\langle Z_g, V_g\rangle \to R\langle\{Z_i, W_i\}_{i=1}^{g}\rangle$ is an isomorphism of topological rings. Let $f \in R\langle\{Z_i, W_i\}_{i=1}^{g}\rangle^{\mathfrak{T}(R)}$. Write $f = \sum_{n\geq 0} A_n(Z_g)V_g^n$ for $A_n(Z_g) \in R\langle\{Z_i, W_i\}_{i=1}^{g-1}\rangle\langle Z_g\rangle$. For any $\lambda \in \mathfrak{T}(R)$ with $\lambda_i = 1$ for all $i \neq g$, we have $A_n(Z_g) = A_n(\lambda_g * Z_g)$ for all n. Thus for $\lambda_g = 1+\beta_n a$ for $a \in R$, we have $A_n(Z_g) = A_n(a+\lambda_g Z_g)$. Putting $Z_g = 0$, we have $A_n(0) = A_n(a)$ for any $a \in R$. The Weierstrass preparation theorem then implies that $A_n(Z_g) = A_n(0) \in R\langle\{Z_i, W_i\}_{i=1}^{g-1}\rangle$. Thus $f = \sum A_n V_g^n$ with $A_n \in R\langle\{Z_i, W_i\}_{i=1}^{g-1}\rangle$. The induction hypothesis then implies that $A_n \in R\langle\{\frac{W_i}{1+\beta_n Z_i}\}_{i=1}^{g-1}\rangle$. This proves the claim. □

Corollary 2.3.4. *Let $\operatorname{Spf} R \subset \mathfrak{IG}_{n,r,I}$ be a Zariski open subset where $\mathrm{H}^\sharp_{\mathcal{A}}, \Omega_{\mathcal{A}}$ and $\mathcal{Q}$ are trivialized. Then with the notation of (2.6),*

$$\mathbb{W}'_{k,\alpha,I}(\operatorname{Spf} R) \simeq R\left\langle\{V_\sigma\}_{\sigma\in\Sigma}\right\rangle \cdot k(1+\beta_n Z), \quad V_\sigma := \frac{W_\sigma}{1+\beta_n Z_\sigma}.$$

Let $\nu\colon \mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s) \xrightarrow{\nu'} \mathfrak{IG}_{n,r,I} \xrightarrow{h_n} \mathfrak{X}_{r,\alpha,I}$ and $\rho\colon \mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q}) \xrightarrow{\rho'} \mathfrak{IG}_{n,r,I} \xrightarrow{h_n} \mathfrak{X}_{r,\alpha,I}$ be the projections.

Definition 2.3.9.

1. For $k = k^0_{\alpha,I}$ define $\mathfrak{w}^0_{k,\alpha,I} := \left(\nu_*\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}},s)}\right)[k]$. This by definition is the sheaf of sections $f \in \nu_*\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}},s)}$ that transform via k for the action of $\mathfrak{T}^{\mathrm{ext}}$. This is the interpolation sheaf of Hilbert modular forms for the universal weight k.
2. For $k = k^0_{\alpha,I}$ define $\mathbb{W}^0_{k,\alpha,I} := \left(\rho_*\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}},s,\mathcal{Q})}\right)[k]$. This by definition is the sheaf of sections $f \in \rho_*\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}},s,\mathcal{Q})}$ that transform via k for the action of $\mathfrak{T}^{\mathrm{ext}}$. This is the interpolation sheaf of de Rham classes for the universal weight k.

Lemma 2.3.5. $\mathfrak{w}^0_{k,\alpha,I} = (h_n)_*\mathfrak{w}'_{k,\alpha,I}[k]$ *and* $\mathbb{W}^0_{k,\alpha,I} = (h_n)_*\mathbb{W}'_{k,\alpha,I}[k]$ *for the residual action of* $(\mathcal{O}_L \otimes \mathbb{Z}_p)^\times$

Proof. This is clear. □

Remark 2.3.4. Note that the universal weight $k = k^0_{\alpha,I}$ kills the torsion group $\Delta \subset (\mathcal{O}_L \otimes \mathbb{Z}_p)^\times$. We will later take care of the torsion part of the character and define the interpolation sheaves $\mathfrak{w}_{k,\alpha,I}$ and $\mathbb{W}_{k,\alpha,I}$ for the univeral weight $k_{\alpha,I} \colon (\mathcal{O}_L \otimes \mathbb{Z}_p)^\times \to \Lambda^\times_{\alpha,I}$, as promised in the beginning of the section by tensoring $\mathfrak{w}^0_{k,\alpha,I}$ and $\mathbb{W}^0_{k,\alpha,I}$ respectively with an appropriate coherent sheaf on $\mathfrak{X}_{r,\alpha,I} \times_{\mathfrak{W}^0_{\alpha,I}} \mathfrak{W}_{\alpha,I}$. In particular, $\mathfrak{w}^0_{k,\alpha,I}$ and $\mathbb{W}^0_{k,\alpha,I}$ will be the restriction of $\mathfrak{w}_{k,\alpha,I}$ and $\mathbb{W}_{k,\alpha,I}$ to the connected component of the trivial character.

Filtration on $\mathbb{W}^0_k$: An important result of this work is the following.

Theorem. *The sheaf $\mathbb{W}^0_{k,\alpha,I}$ comes equipped with a natural Hodge filtration induced by the Hodge filtration on $\mathrm{H}_{\mathcal{A}}$ such that the Gauss-Manin connection ∇ on $\mathbb{W}^0_{k,\alpha,I}$ satisfies Griffiths' transversality with respect to the Hodge filtration.*

We define the Hodge filtration on $\mathbb{W}^0_{k,\alpha,I}$ later in Lemma 2.4.1 and prove Griffiths' transversality for ∇ in Theorem 2.4.1. The way we show that $\mathbb{W}^0_{k,\alpha,I}$ is equipped with a Hodge filtration is by producing a filtration locally on coordinates, and then proving that it glues. But before we prove these results, we will introduce a finer filtration on $\mathbb{W}^0_{k,\alpha,I}$ that will eventually help us to prove that the Hodge filtration is well-defined.

Choose a bijection $\Sigma \simeq \{1, \dots, g\}$. Consider the lexicographic order on $\mathbb{N}^g$:
$(a_1, \dots, a_g) > (b_1, \dots, b_g)$ if and only if
(1) $\sum a_i > \sum b_i$, or
(2) if $\sum a_i = \sum b_i$, then for the first index where $a_i \neq b_i$, $a_i > b_i$.
Since this defines a well ordering on $\mathbb{N}^g$ we get an order preserving bijection $\Xi \colon \mathbb{N} \simeq \mathbb{N}^g$. This allows us to define a natural filtration on $\mathbb{W}^0_k$.

Lemma 2.3.6. *Let $f_0 \colon \mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q}) \to \mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s)$ be the projection. There is an increasing filtration $\{\mathrm{Fil}_i\}_{i \geq 0}$ on $f_{0*}\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q})}$ with $\mathrm{Fil}_0(f_{0*}\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q})}) = \mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s)}$. On local coordinates as in (2.6),*

$$\mathrm{Fil}_i\left(f_{0*}\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q})}(\mathit{Spf}\, R)\right) = \sum_{j \leq i} R\langle\{Z_1, \dots, Z_g\}\rangle \cdot W^{\Xi(j)}.$$

Proof. We follow the proof of Lemma 1.3.6. We need to show that the local description glues. For Fil_0 this is obvious by definition. For two different choice of $\mathcal{O}_L \otimes R$-basis of $\mathrm{H}^\sharp_{\mathcal{A}|\mathrm{Spf}\, R}$, say X, Y and X', Y', with X, X' being lifts of the marked section and Y, Y' being lifts of some generator of $\mathcal{Q}$, we get two different local coordinate description of $\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q})_{|\mathrm{Spf}\, R}$. Since all the filtered pieces contain Fil_0, we may assume $X = X'$. Then the components of Y and Y' are related by $Y_i' = u_i Y_i + a_i \eta X_i$ for all i, with $u_i, a_i \in R$. This implies that the isomorphism $R\langle\{Z_i, W_i'\}_{i=1}^g\rangle \xrightarrow{\sim} R\langle\{Z_i, W_i\}_{i=1}^g\rangle$ is given by sending $W_i' \mapsto u_i W_i + a_i(1 + \beta_n Z_i)$. Clearly, this isomorphism respects the filtration given by the lexicographic ordering. $\square$

Recall $\rho' \colon \mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q}) \to \mathfrak{IG}_{n,r,I}$ was the projection.

Lemma 2.3.7. *The filtration on* $\rho'_*\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}},s,\mathcal{Q})}$ *induced by taking direct image of the filtration defined in Lemma 2.3.6 is stable for the action of* $\mathfrak{T}$. *Therefore, we can define a filtration on* $\mathbb{W}'_{k,\alpha,I}$ *as* $\mathrm{Fil}_i\mathbb{W}'_{k,\alpha,I} = \mathrm{Fil}_i\left(\rho'_*\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}},s,\mathcal{Q})}\right)[k]$ *with the property that* $\mathrm{Fil}_0\mathbb{W}'_{k,\alpha,I} = \mathfrak{w}'_{k,\alpha,I}$ *and* $\mathrm{Gr}_i\mathbb{W}'_{k,\alpha,I} \simeq \mathfrak{w}'_{k,\alpha,I} \otimes \eta^{-\ell(i)}(HW\cdot\omega_{\mathcal{A}}^{-2})^{\Xi(i)}$, *where for* $\Xi(i) = (a_1,\ldots,a_g)$ *we let* $\ell(i) = \sum a_k$, *and* $HW^{\Xi(i)} = \prod HW(k)^{a_k}$.

Proof. This is clear from the description of the action on local coordinates. □

Here we collect a few results that will allow us to prove that $\mathfrak{w}^0_{k,\alpha,I}$ is a line bundle and the filtration on $\mathbb{W}'_{k,\alpha,I}$ descends to a filtration on $\mathbb{W}^0_{k,\alpha,I}$ which then can be realized as the completion of the colimit of its filtered pieces, which are locally free sheaves of finite rank.

Lemma 2.3.8. *Let* $\mathcal{O}^{\circ\circ}_{\mathfrak{IG}_{1,r,I}}$ *be the ideal of topologically nilpotent elements in* $\mathcal{O}_{\mathfrak{IG}_{1,r,I}}$. *With* I, r *as fixed in the beginning of the section, for any* $2 \le l \le a+r$,

$$k(1+p^{l-1}(\mathcal{O}_L\otimes\mathbb{Z}_p)) - 1 \subset \mathrm{Hdg}^{\frac{p^l-p}{p-1}}\mathcal{O}^{\circ\circ}_{\mathfrak{IG}_{1,r,I}}.$$

Proof. [AIP16b, Lemma 4.4]. □

Lemma 2.3.9. *The natural* $\mathfrak{T}^{ext}$*-equivariant map* $\mathcal{O}_{\mathfrak{IG}_{n,r,I}} \to \mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}},s)}$ *induces an isomorphism*

$$\mathcal{O}_{\mathfrak{IG}_{n,r,I}}/q\mathcal{O}_{\mathfrak{IG}_{n,r,I}} \xrightarrow{\sim} \mathfrak{w}'_k/q\mathfrak{w}'_k.$$

Proof. [AIP16b, Lemme 4.5]. □

Lemma 2.3.10. *Let* $h\colon \mathfrak{IG}_{n,r,I} \to \mathfrak{IG}_{n-1,r,I}$ *be the projection for any* n. h *is finite and the trace map* $\mathrm{Tr}_h\colon h_*\mathcal{O}_{\mathfrak{IG}_{n,r,I}} \to \mathcal{O}_{\mathfrak{IG}_{n-1,r,I}}$ *induced by the trace on the adic generic fibre satisfies*

$$\mathrm{Hdg}^{p^{n-1}}\mathcal{O}_{\mathfrak{IG}_{n-1,r,I}} \subset \mathrm{Tr}_h(h_*\mathcal{O}_{\mathfrak{IG}_{n,r,I}})$$

for any $2 \le n \le a+r$.

Proof. [AIP16b, Proposition 3.4]. □

Lemma 2.3.11. *Let Spf* $R \subset \mathfrak{X}_{r,\alpha,I}$ *be an open where* Hdg *is trivialized. Letting* $c_0 = 1$, *for every* $1 \le n \le a+r$, *there exists* $c_n \in \mathrm{Hdg}^{-\frac{p^n-p}{p-1}}\mathcal{O}_{\mathfrak{IG}_{n,r,I}}(\mathit{Spf}\ R)$ *satisfying* $\mathrm{Tr}_h(c_n) = c_{n-1}$.

Proof. Immediate from Lemma 2.3.10. □

Recall $\rho\colon \mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}},s,\mathcal{Q}) \to \mathfrak{X}_{r,\alpha,I}$ was the projection and $\mathbb{W}^0_{k,\alpha,I} = \rho_*\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}},s,\mathcal{Q})}[k]$.

Theorem 2.3.1. *The action of* $\mathfrak{T}^{ext}$ *on* $\rho_*\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}},s,\mathcal{Q})}$ *preserves the filtration* $\{\mathrm{Fil}_i\}_i$ *induced by taking direct image of the filtration defined in Lemma 2.3.6. Let* $\mathrm{Fil}_i\mathbb{W}^0_{k,\alpha,I} = \mathrm{Fil}_i\left(\rho_*\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}},s,\mathcal{Q})}\right)[k]$.

1. $\mathrm{Fil}_i\mathbb{W}^0_{k,\alpha,I}$ *is a finite locally free* $\mathcal{O}_{\mathfrak{X}_{r,\alpha,I}}$*-module.*

2. $\mathbb{W}^0_{k,\alpha,I}$ *is the* α*-adic completion of* $\varinjlim_i \mathrm{Fil}_i\mathbb{W}^0_{k,\alpha,I}$.

3. $\mathrm{Fil}_0\mathbb{W}^0_{k,\alpha,I} = \mathfrak{w}^0_{k,\alpha,I}$ *and* $\mathrm{Gr}_i\mathbb{W}^0_{k,\alpha,I} \simeq \mathfrak{w}^0_{k,\alpha,I} \otimes \eta^{-\ell(i)}(HW \cdot \omega_{\mathcal{A}}^{-2})^{\Xi(i)}$.

Proof. We already know the similar results for $\mathbb{W}'_{k,\alpha,I}$. Also it is clear that Fil_0 is preserved by the $\mathfrak{T}^{\mathrm{ext}}$-action and $\mathrm{Fil}_0\mathbb{W}^0_{k,\alpha,I} = \mathfrak{w}^0_{k,\alpha,I}$. By Lemma 2.3.5, $\mathbb{W}^0_{k,\alpha,I} = (h_n)_*\mathbb{W}'_{k,\alpha,I}[k]$ for the residual action of $(\mathcal{O}_L \otimes \mathbb{Z}_p)^\times$. The idea is to pick generators of $\mathrm{Gr}_i\mathbb{W}'_{k,\alpha,I}$ and modify them to produce generators of $\mathrm{Gr}_i\mathbb{W}^0_{k,\alpha,I}$. To that intent, recall we have a $\mathfrak{T}^{\mathrm{ext}}$ equivariant isomorphism $\mathcal{O}_{\mathfrak{IG}_{n,r,I}}/q \simeq \mathfrak{w}'_{k,\alpha,I}/q$ by Lemma 2.3.9. Let Spf $R \subset \mathfrak{X}_{r,\alpha,I}$ be a Zariski open that trivializes $\omega_{\mathcal{A}}$ and let ω be a $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{X}_{r,\alpha,I}}$ generator. Let V be the pullback of Spf R to $\mathfrak{IG}_{n,r,I}$. This gives a $\mathfrak{T}^{\mathrm{ext}}$ equivariant isomorphism

$$\mathcal{O}_{\mathfrak{IG}_{n,r,I}}(V)/q \otimes \eta^{-\ell(i)}(HW \cdot \omega_{\mathcal{A}}^{-2})^{\Xi(i)} \xrightarrow{\sim} \mathrm{Gr}_i\mathbb{W}'_k(V)/q = \mathfrak{w}'_k(V)/q \otimes \eta^{-\ell(i)}(HW \cdot \omega_{\mathcal{A}}^{-2})^{\Xi(i)}.$$

Let $\bar{s}_i$ be the image of the class of $\eta^{-\ell(i)}(HW \cdot \omega_{\mathcal{A}}^{-2})^{\Xi(i)}$. In particular $t * \bar{s}_i = \bar{s}_i$ for all $t \in (\mathcal{O}_L \otimes \mathbb{Z}_p)^\times$, since $\eta^{-\ell(i)}(HW \cdot \omega_{\mathcal{A}}^{-2})^{\Xi(i)}$ is defined over $\mathfrak{X}_{r,\alpha,I}$. Pick a lift s_i of $\bar{s}_i$ to $\mathrm{Fil}_i\mathbb{W}'_{k,\alpha,I}$.

Choose lifts $\tilde{\tau}$ of $\tau \in (\mathcal{O}_L/p^n\mathcal{O}_L)^\times$ in $(\mathcal{O}_L \otimes \mathbb{Z}_p)^\times$. With c_n as in Lemma 2.3.11, define

$$\tilde{s}_i := \sum_{\tau \in (\mathcal{O}_L/p^n\mathcal{O}_L)^\times} k(\tilde{\tau})^{-1}\tau(c_n s_i) \in \mathrm{Hdg}^{-\frac{p^n-p}{p-1}}\mathbb{W}'_{k,\alpha,I}(V).$$

We claim that $\tilde{s}_i \in \mathrm{Fil}_i\mathbb{W}'_{k,\alpha,I}(V)$ and its image generates $\mathrm{Gr}_i\mathbb{W}'_{k,\alpha,I}$. Moreover since $(\mathcal{O}_L \otimes \mathbb{Z}_p)^\times$ acts on $\tilde{s}_i$ via k, it descends to $\mathbb{W}^0_k$. To prove the claim we note that

$$\begin{aligned}
\tilde{s}_i - s_i &= \left(\sum_{\tau \in (\mathcal{O}_L/p^n\mathcal{O}_L)^\times} k(\tilde{\tau})^{-1}\tau(c_n s_i) \right) - s_i \\
&\in \sum_{\tau \in (\mathcal{O}_L/p^n\mathcal{O}_L)^\times} k(\tilde{\tau})^{-1}(\tau(c_n s_i) - \tau(c_n)s_i) + R^{\circ\circ}\mathrm{Fil}_i\mathbb{W}'_{k,\alpha,I} \\
&\subset \sum_{\tau \in (\mathcal{O}_L/p^n\mathcal{O}_L)^\times} k(\tilde{\tau})^{-1}\tau(c_n)(\tau(s_i) - s_i) + R^{\circ\circ}\mathrm{Fil}_i\mathbb{W}'_{k,\alpha,I} \\
&\subset R^{\circ\circ}\mathrm{Fil}_i\mathbb{W}'_{k,\alpha,I} + \mathrm{Fil}_{i-1}\mathbb{W}'_{k,\alpha,I}.
\end{aligned}$$

Here we used the fact that $\sum k(\tilde{\tau})^{-1}\tau(c_n) \in 1 + R^{\circ\circ}\mathcal{O}_{\mathfrak{IG}_{n,r,I}}$ whose proof we refer to [AIP18, Lemme 5.4]. The fact that $\tilde{s}_i$ generates $\mathrm{Gr}_i\mathbb{W}'_{k,\alpha,I}$ follows by noting that it does so modulo $R^{\circ\circ}$.

So we have produced a local basis of $\mathrm{Fil}_i\mathbb{W}'_{k,\alpha,I}$ for each i that descends to $\mathbb{W}^0_{k,\alpha,I}$. This proves that the filtration on $\rho_*\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}},s,\mathcal{Q})}$ is preserved by the $\mathfrak{T}^{\mathrm{ext}}$-action and so the filtration on $\mathbb{W}'_{k,\alpha,I}$ descend to a filtration on $\mathbb{W}^0_{k,\alpha,I}$. The rest of the claims in the theorem follow immediately. ☐

Let $\chi\colon \Delta \subset (\mathcal{O}_L \otimes \mathbb{Z}_p)^\times \xrightarrow{k^{\mathrm{un}}} \Lambda_{\alpha,I}^\times$ be the finite part of the universal character. For $p \neq 2$, $\Delta = (\mathcal{O}_L \otimes Z_p)^\times/1 + p(\mathcal{O}_L \otimes Z_p)$ and for $p = 2$, Δ is a quotient of $(\mathcal{O}_L \otimes Z_p)^\times/1 + 4(\mathcal{O}_L \otimes \mathbb{Z}_p)$. Using this we view χ as a character of $(\mathcal{O}_L/q\mathcal{O}_L)^\times$. Let $p\colon \bar{\mathfrak{M}}_{r,\alpha,I} := \mathfrak{X}_{r,\alpha,I} \times_{\mathfrak{W}^0_{\alpha,I}} \mathfrak{W}_{\alpha,I} \to \mathfrak{X}_{r,\alpha,I}$ be the projection induced by the finite flat base change $\mathfrak{W}_{\alpha,I} \to \mathfrak{W}^0_{\alpha,I}$.

Definition 2.3.10. For $i = 1$ if $p \neq 2$ and $i = 2$ if $p = 2$, define a coherent sheaf $\mathfrak{w}^\chi_{k,\alpha,I} := \left(p^*(f_i)_*\mathcal{O}_{\mathfrak{IG}_{i,r,I}}\right)[\chi^{-1}]$ for the action of $(\mathcal{O}_L/q\mathcal{O}_L)^\times$ on $f_i\colon \mathfrak{IG}_{i,r,I} \to \mathfrak{X}_{r,\alpha,I}$.

Definition 2.3.11.

1. Define the sheaf of overconvergent Hilbert modular forms of weight $k = k^{\mathrm{un}}$ to be $\mathfrak{w}_{k,\alpha,I} = p^*\mathfrak{w}^0_{k,\alpha,I} \otimes \mathfrak{w}^\chi_{k,\alpha,I}$.
2. Define the sheaf of overconvergent de Rham classes to be $\mathbb{W}_{k,\alpha,I} = p^*\mathbb{W}^0_{k,\alpha,I} \otimes \mathfrak{w}^\chi_{k,\alpha,I}$.

Proposition 2.3.8. *The sheaf $\mathbb{W}_{k,\alpha,I}$ is equipped with a filtration by coherent $\mathcal{O}_{\bar{\mathfrak{M}}_{r,\alpha,I}}$ modules $\mathrm{Fil}_i\mathbb{W}_{k,\alpha,I}$ and moreover $\mathbb{W}_{k,\alpha,I}$ is the α-adic completion of $\varinjlim_i \mathrm{Fil}_i\mathbb{W}_{k,\alpha,I}$. We have $\mathrm{Fil}_0\mathbb{W}_{k,\alpha,I} = \mathfrak{w}_{k,\alpha,I}$ and $\mathrm{Gr}_i\mathbb{W}_{k,\alpha,I} \simeq \mathfrak{w}_{k,\alpha,I} \otimes \eta^{-\ell(i)}(HW \cdot \omega_{\mathcal{A}}^{-2})^{\Xi(i)}$.*

Proof. Define $\mathrm{Fil}_i\mathbb{W}_{k,\alpha,I} := p^*\mathrm{Fil}_i\mathbb{W}^0_{k,\alpha,I} \otimes \mathfrak{w}^\chi_{k,\alpha,I}$. The rest of the claims follow immediately from Theorem 2.3.1. ☐

We remarked in the introduction to the section that the construction of $\mathfrak{w}_{k,\alpha,I}$ appears in the previous work of Andreatta-Iovita-Pilloni [AIP16b]. Here we compare our construction to theirs and show why we get isomorphic sheaves.

In [AIP16b] the authors consider a torsor $\mathfrak{F}_{n,r,I}$ over $\mathfrak{IG}_{n,r,I}$ for the group $\mathfrak{T}$. This torsor is defined on points Spf $R \xrightarrow{\gamma} \mathfrak{IG}_{n,r,I}$ for any normal admissible $\Lambda^0_{\alpha,I}$-algebra R as follows.

$$\mathfrak{F}_{n,r,I}(R) := \{\omega \in \omega_{\mathcal{A}} \,|\, \omega = \gamma^*(s) \in \gamma^*\Omega_{\mathcal{A}}/\beta_n\}.$$

The action of $\mathfrak{T}$ on $\mathfrak{F}_{n,r,I}$ is the obvious one, i.e. $\lambda \in \mathfrak{T}(R)$ acts via $\lambda * \omega = \lambda\omega$. Moreover there is an action of $\mathfrak{T}^{\mathrm{ext}}$ on $\mathfrak{F}_{n,r,I}$ over $\mathfrak{X}_{r,\alpha,I}$. This is given by first noting that any point of $\mathfrak{F}_{n,r,I}(R)$ can be seen as a pair (P,ω) where $P \in H_n^\vee(R)$ and $\omega \in \omega_{\mathcal{A}}(R), \omega = \mathrm{dlog}(P) \bmod \beta_n$. Then $\lambda \in (\mathcal{O}_L \otimes \mathbb{Z}_p)^\times$ acts via $\lambda * (P,\omega) = (\bar{\lambda}P, \lambda\omega)$, where $\bar{\lambda}$ is the class of λ in $(\mathcal{O}_L/p^n\mathcal{O}_L)^\times$. Then they define the sheaf of Hilbert modular forms for universal weight $k = k^0_{\alpha,I}$ as follows. Let $\nu'\colon \mathfrak{F}_{n,r,I} \to \mathfrak{X}_{r,\alpha,I}$ be the projection.

Definition 2.3.12. The sheaf of Hilbert modular forms for weight $k = k^0_{\alpha,I}$ is defined as $\mathfrak{w}^{0,\mathrm{old}}_{k,\alpha,I} := \nu'_*\mathcal{O}_{\mathfrak{F}_{n,r,I}}[k^{-1}]$ for the action of $\mathfrak{T}^{\mathrm{ext}}$.

We now show why the sheaf $\mathfrak{w}^0_{k,\alpha,I}$ is naturally isomorphic to $\mathfrak{w}^{0,\mathrm{old}}_{k,\alpha,I}$.

Proposition 2.3.9. *There is an isomorphism of formal schemes over $\mathfrak{X}_{r,\alpha,I}$, $a\colon \mathfrak{F}_{n,r,I} \xrightarrow{\sim} \mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s)$. This isomorphism interacts with the $\mathfrak{T}^{ext}$ action in the following manner: for any point $x \in \mathfrak{F}_{n,r,I}(R)$, $\lambda * a(x) = a(\lambda^{-1} * x)$.*

Proof. Define a by sending a point $(P,\omega) \mapsto (P,\omega^\vee)$. Then it is easy to check the rest of the claims. ☐

Proposition 2.3.10. *There is a natural isomorphism $\mathfrak{w}^0_{k,\alpha,I} \simeq \mathfrak{w}^{0,old}_{k,\alpha,I}$.*

Proof. Let $\nu\colon \mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}}, s) \to \mathfrak{X}_{r,\alpha,I}$ be the projection. Then the isomorphism induced by a,

$$\nu_*\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\Omega_{\mathcal{A}},s)} \simeq \nu'_*\mathcal{O}_{\mathfrak{F}_{n,r,I}}$$

induces an isomorphism $\mathfrak{w}^0_{k,\alpha,I} \simeq \mathfrak{w}^{0,\text{old}}_{k,\alpha,I}$ due to Proposition 2.3.9. ☐

Here we recall an important result about the surjectivity of the specialization map for cusp forms.

Let $\mathfrak{X}^*_{r,\alpha,I} \to \mathfrak{W}^0_{\alpha,I}$ be the blow-up spaces constructed exactly as $\mathfrak{X}_{r,\alpha,I}$ but now starting from the minimal compactification $M^*(\mu_N, \mathfrak{c})$. These are formal models for overconvergent neighbourhoods of $M^*(\mu_N, \mathfrak{c})$. Let $\mathfrak{M}^*_{r,\alpha,I} := \mathfrak{X}^*_{r,\alpha,I} \times_{\mathfrak{W}^0_{\alpha,I}} \mathfrak{W}_{\alpha,I}$. There is a natural map $f\colon \bar{\mathfrak{M}}_{r,\alpha,I} \to \mathfrak{M}^*_{r,\alpha,I}$ induced by the projection $\bar{M}(\mu_N, \mathfrak{c}) \to M^*(\mu_N, \mathfrak{c})$. Let $\bar{\mathcal{M}}_{r,\alpha,I}, \mathcal{M}^*_{r,\alpha,I}$ be their adic generic fibre. Recall that D was the boundary divisor of $\bar{M}(\mu_N, \mathfrak{c})$. By an abuse of notation we denote by D its inverse image in $\bar{\mathfrak{M}}_{r,\alpha,I}$.

Theorem 2.3.2. *We have $R^i f_*\mathfrak{w}_{k,\alpha,I}(-D) = 0$ for all $i > 0$. Let $g\colon \bar{\mathcal{M}}_{r,\alpha,I} \to \mathcal{W}_{\alpha,I}$ be the projection to the weight space. Then for any weight $\kappa \in \mathcal{W}_{\alpha,I}$,*

$$\kappa^* g_*\mathfrak{w}_{k,\alpha,I}(-D)[1/\alpha] = H^0(\bar{\mathcal{M}}_{r,\alpha,I}, \kappa^*\mathfrak{w}_{k,\alpha,I}(-D)[1/\alpha])$$

is the space of r-overconvergent Hilbert cuspforms of tame level μ_N, $\mathfrak{c}$-polarization and weight κ.

Proof. The first part follows from [AIP16a, Corollary 3.20]. For the second part we remark that the projection g factors through $\bar{\mathcal{M}}_{r,\alpha,I} \xrightarrow{f} \mathcal{M}^*_{r,\alpha,I} \to \mathcal{W}_{\alpha,I}$, and $\mathcal{M}^*_{r,\alpha,I}$ is affinoid. Then the claim follows from the first part. ☐

2.3.4 Overconvergent arithmetic Hilbert modular forms

Recall that we defined the notion of arithmetic Hilbert modular forms in §2.1.2.1. These were Hilbert modular forms associated to the group $G = \mathrm{Res}_{L/\mathbb{Q}}\,\mathbf{GL}_{2,L}$. With $\Gamma = \mathcal{O}_L^{\times,+}/U_N^2$, we saw that the quotient $\bar{M}(\mu_N, \mathfrak{c}) \to \bar{M}(\mu_N, \mathfrak{c})/\Gamma$ is finite étale. Given a classical weight $(v, w) \in \mathfrak{W}^G$ the sheaf of arithmetic Hilbert modular forms of tame level μ_N, $\mathfrak{c}$-polarization and weight (v, w) with coefficients in R was defined to be $\underline{\omega}_R^{(v,w)} := (p_*\omega^k_{\mathcal{A},R})^\Gamma$. The definition of overconvergent arithmetic Hilbert modular forms is given in a similar manner. We follow [AIP16b].

Let $\bar{\mathfrak{M}}^G_{r,\alpha,I} := (\bar{\mathfrak{M}}_{r,\alpha,I} \times_{\mathfrak{W}} \mathfrak{W}^G)/\Gamma$. By Proposition 2.1.1 the quotient map $p\colon \bar{\mathfrak{M}}_{r,\alpha,I} \times_{\mathfrak{W}} \mathfrak{W}^G \to \bar{\mathfrak{M}}^G_{r,\alpha,I}$ is finite étale. Consider the pullback of $\mathfrak{w}_{k,\alpha,I}$ along $f\colon \bar{\mathfrak{M}}_{r,\alpha,I} \times_{\mathfrak{W}} \mathfrak{W}^G \to \bar{\mathfrak{M}}_{r,\alpha,I}$. The action of Γ on $\bar{\mathfrak{M}}_{r,\alpha,I} \times_{\mathfrak{W}} \mathfrak{W}^G$ can be lifted to an action on $f^*\mathfrak{w}_{k,\alpha,I}$ as follows. Let $(v_{\alpha,I}, w_{\alpha,I})\colon (\mathcal{O}_L \otimes \mathbb{Z}_p)^\times \times \mathbb{Z}_p^\times \to (\Lambda_{\alpha,I}\hat{\otimes}_\Lambda \Lambda^G)^\times =: (\Lambda^G_{\alpha,I})^\times$ be the universal character. Any $\epsilon \in \mathcal{O}_L^{\times,+}$ induces an isomorphism $[\epsilon]^* f^*\mathfrak{w}_{k,\alpha,I} \to f^*\mathfrak{w}_{k,\alpha,I}$. We define the action on $f^*\mathfrak{w}_{k,\alpha,I}$ by multiplying this isomorphism by $v_{\alpha,I}(\epsilon^{-1})$. That is, viewing a section g of $f^*\mathfrak{w}_{k,\alpha,I}$ by Koecher's principle as a rule that associates to any tuple $(A, \iota, \lambda, \psi, \omega)$ a value $g(A, \iota, \lambda, \psi, \omega) \in \Lambda^G_{\alpha,I}$, the action of ϵ is given by

$$(\epsilon \cdot g)(A, \iota, \lambda, \psi, \omega) = v_{\alpha,I}(\epsilon^{-1})g(A, \iota, \epsilon\lambda, \psi, \omega).$$

Definition 2.3.13. The sheaf of r-overconvergent arithmetic Hilbert modular forms of tame level μ_N, $\mathfrak{c}$-polarization and weight $k^G_{\alpha,I} := (v_{\alpha,I}, w_{\alpha,I})$ is defined to be

$$\mathfrak{w}^{G,\mathfrak{c}}_{k,\alpha,I} := (p_* f^* \mathfrak{w}_{k,\alpha,I})^{\Gamma}.$$

Similarly one defines the sheaf of r-overconvergent arithmetic Hilbert cuspforms of tame level μ_N, $\mathfrak{c}$-polarization and weight $k^G_{\alpha,I}$ to be

$$\mathfrak{w}^{G,\mathfrak{c}}_{k,\alpha,I}(-D) := (p_* f^* \mathfrak{w}_{k,\alpha,I}(-D))^{\Gamma}.$$

We have a surjectivity result about the specialization map of arithmetic Hilbert cuspforms analogous to Theorem 2.3.2.

Let $\bar{\mathcal{M}}^G_{r,\alpha,I}$ be the adic generic fibre of $\bar{\mathfrak{M}}^G_{r,\alpha,I}$. Let $g\colon \bar{\mathcal{M}}^G_{r,\alpha,I} \to \mathcal{W}^G_{\alpha,I}$ be the projection to the weight space, where $\mathcal{W}^G_{\alpha,I} = \mathrm{Spa}(\Lambda^G_{\alpha,I}[1/\alpha], \Lambda^G_{\alpha,I})$.

Theorem 2.3.3. *For any weight $k^G \in \mathcal{W}^G_{\alpha,I}$, $(k^G)^* \mathfrak{w}^{G,\mathfrak{c}}_{k,\alpha,I}(-D)[1/\alpha]$ is the space of r-overconvergent arithmetic Hilbert cuspforms of tame level μ_N, $\mathfrak{c}$-polarization and weight k^G.*

Proof. We need to show that $R^i g_* \mathfrak{w}^{G,\mathfrak{c}}_{k,\alpha,I}(-D)[1/\alpha] = 0$ for all $i > 0$. This follows from Theorem 2.3.2 by noting that over the generic fibre applying the invariant functor $(\cdot)^{\Gamma}$ to a Čech resolution of $p_* f^* \mathfrak{w}_{k,\alpha,I}(-D)$ is exact since $(\cdot)^{\Gamma}$ is obtained by the application of the projector $e = \frac{1}{\#\Gamma}\sum_{\epsilon\in\Gamma} \epsilon$. $\square$

Let $g_{\mathfrak{c}}$ denote the projection to the weight space $\mathfrak{W}^G_{\alpha,I}$ from the formal model $\bar{\mathfrak{M}}^G_{r,\alpha,I}$ corresponding to the moduli of abelian schemes with $\mathfrak{c}$-polarization. For $x \in L^{\times,+}$ coprime to p, consider the isomorphism $L_{(x\mathfrak{c},\mathfrak{c})}\colon g_{\mathfrak{c}*}\mathfrak{w}^{G,\mathfrak{c}}_{k,\alpha,I} \to g_{x\mathfrak{c}*}\mathfrak{w}^{G,x\mathfrak{c}}_{k,\alpha,I}$ given by

$$L_{(x\mathfrak{c},\mathfrak{c})}(f)(A,\iota,\lambda,\psi,\omega) = v(x) f(A,\iota,x^{-1}\lambda,\psi,\omega).$$

The isomorphism depends only on the principal ideal (x) and preserves cuspidality.

Definition 2.3.14. Define the sheaf of r-overconvergent arithmetic Hilbert modular forms of tame level μ_N and weight $k^G_{\alpha,I}$ to be

$$\mathfrak{w}^G_{k,\alpha,I} := \left(\bigoplus_{\mathfrak{c}\in \mathrm{Frac}(L)^{(p)}} g_{\mathfrak{c}*}\mathfrak{w}^{G,\mathfrak{c}}_{k,\alpha,I} \right) \Big/ \Big(L_{(x\mathfrak{c},\mathfrak{c})}(f) - f \Big)_{x\in \mathrm{Princ}(L)^{+,(p)}}.$$

One defines similarly the subsheaf of r-overconvergent arithmetic Hilbert cuspforms, which we denote by $\mathfrak{w}^G_{k,\alpha,I}(-D)$. (Note the D is the notation does not have anything to do any boundary divisor, but we choose this notation to stay consistent with our previous notation when the polarization module was fixed and we were working over a fixed toroidal compactification.)

2.4 p-adic iteration of the Gauss-Manin connection

In this section we will define iteration of the Gauss-Manin connection for analytic weights. For simplicity of notation we will ignore the log poles at the cusps. The first main result of this section is the following.

Theorem. *There is a filtration* $\{\mathrm{Fil}_i\}_{i\geq 0}$ *on* $\mathbb{W}^0_{k,\alpha,I}$ *such that the graded pieces over the generic fibre are* $\mathrm{Gr}_n\mathbb{W}^0_{k,\alpha,I}[1/\alpha] \simeq \mathfrak{w}^0_{k,\alpha,I} \otimes \mathrm{Sym}^n\, \omega_{\mathcal{A}}^{-\otimes 2}[1/\alpha]$*. There is a connection*

$$\nabla_k \colon \mathbb{W}^0_{k,\alpha,I}[1/\alpha] \to \mathbb{W}^0_{k,\alpha,I}\hat{\otimes}\Omega^1_{\mathfrak{X}_{r,\alpha,I}/\Lambda^0_{\alpha,I}}[1/\alpha]$$

induced by the Gauss-Manin connection on $\mathrm{H}_{\mathcal{A}}$*, which satisfies Griffiths' transversality with respect to the filtration* Fil_i *above. Moreover, it also induces a connection*

$$\mathbb{W}_{k,\alpha,I}[1/\alpha] \to \mathbb{W}_{k,\alpha,I}\hat{\otimes}\Omega^1_{\mathfrak{M}_{r,\alpha,I}/\Lambda_{\alpha,I}}[1/\alpha]$$

that satisfies Griffiths' transversality with respect to the filtration defined by tensoring Fil_i *with* $\mathfrak{w}^{\chi}_{k,\alpha,I}$*.*

2.4.1 The Gauss-Manin connection on $\mathrm{H}^\sharp_{\mathcal{A}}$

Let $\mathcal{IG}'_{n,r,I} \to \mathcal{IG}_{n,r,I}$ be the analytic adic space classifying trivializations $(\mathcal{O}_L/p^n\mathcal{O}_L)^2 \xrightarrow{\sim} \mathcal{A}[p^n]^\vee$ compatible with the trivializations $\mathcal{O}_L/p^n\mathcal{O}_L \xrightarrow{\sim} H_n^\vee$. Let $\mathfrak{IG}'_{n,r,I} \to \mathfrak{IG}_{n,r,I}$ be the normalization.

Proposition 2.4.1. *The Gauss-Manin connection on* $\mathrm{H}_{\mathcal{A}}$ *over* $\mathfrak{IG}'_{n,r,I}$ *restricts to a connection*

$$\nabla\colon \mathrm{H}^\sharp_{\mathcal{A}} \to \mathrm{H}^\sharp_{\mathcal{A}}\hat{\otimes}\Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_{\alpha,I}}$$

such that $(\nabla \text{ mod } \beta_n)(s) = 0$ *and* $(\nabla \text{ mod } \eta)(\mathcal{Q}) \subset \mathcal{Q} \otimes \Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_{\alpha,I}}/(\eta)$*.*

Proof. We have a commutative diagram as follows coming from the functoriality of the dlog map.

$$\begin{array}{ccc}
(\mathcal{O}_L/p^n\mathcal{O}_L)^2 & \xrightarrow{\mathrm{dlog}} & \omega_{\mu^2_{p^n}} \otimes \mathfrak{d}^{-1} \\
\downarrow{\simeq} & & \downarrow \\
\mathcal{A}[p^n]^\vee(\mathfrak{IG}'_{n,r,I}) & \xrightarrow{\mathrm{dlog}} & \omega_{\mathcal{A}}/p^n
\end{array}$$

The bottom arrow $\mathrm{dlog}_{\mathcal{A}[p^n]^\vee}$ composed with the projection on to $\omega_{\mathcal{A}}/\beta_n \simeq \omega_{H_n}/\beta_n$ factors through $\mathcal{A}[p^n]^\vee \to H_n^\vee \xrightarrow{\mathrm{dlog}} \omega_{H_n}$. The connection on $\mathrm{H}_{\mathcal{A}}$ modulo p^n is the connection $\bar{\nabla}$ on the invariant differentials of the universal vector extension of $\mathcal{A}[p^n]^\vee$. Since μ_{p^n} is isotrivial, the functoriality of the Gauss-Manin connection and the commutativity of the diagram shows that $\bar{\nabla}(\mathrm{dlog}(P_n^{\mathrm{univ}})) = 0$. This shows that $\nabla(\Omega_{\mathcal{A}}) = 0 \bmod \beta_n\mathrm{H}_{\mathcal{A}}\hat{\otimes}\Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_{\alpha,I}}$. In particular $\nabla(\Omega_{\mathcal{A}}) \subset \mathrm{H}^\sharp_{\mathcal{A}}\hat{\otimes}\Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_{\alpha,I}}$.

On the other hand since $\mathrm{H}^\sharp_{\mathcal{A}} = \Omega_{\mathcal{A}} + \underline{\xi}\widetilde{HW} \cdot \mathrm{H}_{\mathcal{A}}$ we are left to show that ∇ maps $\underline{\xi}\widetilde{HW} \cdot \mathrm{H}_{\mathcal{A}}$ to $\mathrm{H}^\sharp_{\mathcal{A}}$. Since the Gauss-Manin connection is functorial, it commutes with the splitting of $\mathrm{H}_{\mathcal{A}}$ into the g different components. Recalling the notation from Corollary 2.3.3 we need to show that

$$\nabla\left(\underline{\xi}(i_j)\widetilde{HW}(i_j)\cdot \mathrm{H}_{\mathcal{A}}(i_j)\right) \subset \mathrm{H}^\sharp_{\mathcal{A}}(i_j)\hat{\otimes}\Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_{\alpha,I}}.$$

To show this note that by Lemma 2.3.3, $\underline{\xi}(i_j)\widetilde{HW}(i_j) = \tilde{F}^*(\underline{\xi}(i_{j-1}))$ is a p-th power at all height 1 localizations. Thus ∇ maps it to $\mathrm{H}^\sharp_{\mathcal{A}} + p\mathrm{H}_{\mathcal{A}}$. Since $p \in \underline{\xi}(i_j)\widetilde{HW}(i_j)$ for all i_j we conclude. This proves the first two claims of the proposition. Now we show that $(\nabla \bmod \eta)(\mathcal{Q}) \subset \mathcal{Q} \otimes \Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_{\alpha,I}}/(\eta)$.

Let Spf $R_0 \subset \mathfrak{X}_{r,\alpha,I}$ be an open such that $\mathrm{H}_{\mathcal{A}}$ is trivialized over Spf R as $\mathcal{O}_L \otimes R_0$-modules. Let Spf $R \subset \mathfrak{IG}_{n,r,I}$ and Spf $R' \subset \mathfrak{IG}'_{n,r,I}$ be its inverse image in $\mathfrak{IG}_{n,r,I}$ and $\mathfrak{IG}'_{n,r,I}$ respectively. Assume also that $\mathrm{H}^\sharp_{\mathcal{A}}$ is trivialized over Spf R. Pick $\mathcal{O}_L \otimes R_0$-basis ω, ζ of $\mathrm{H}_{\mathcal{A}}$ such that $e := \xi\omega$ is a lift of s for some local generator ξ of $\underline{\xi}$ and such that $f := Ce + (\xi HW)\zeta$ is a lift of a generator of $\mathcal{Q}$ for some local generator HW of $\widetilde{HW}$ and some $C \in \mathcal{O}_L \otimes R$ as in the proof of Proposition 2.3.2. Assume also that the image of ζ in $\omega^\vee_{\mathcal{A}}$ is the dual of ω.

Let $\omega_\sigma, \zeta_\sigma$ be the σ-components of ω and ζ. Let $\Theta_\sigma = KS(\omega_\sigma, \zeta_\sigma)$ be the Kodaira-Spencer class in $\Omega^1_{\mathfrak{X}_{r,\alpha,I}/\Lambda^0_{\alpha,I}}$ corresponding to the image of $\omega_\sigma^{\otimes 2}$ under the Kodaira-Spencer isomorphism

$$KS\colon \omega_{\mathcal{A}}^{\otimes 2} \xrightarrow{\sim} \Omega^1_{\mathfrak{X}_{r,\alpha,I}/\Lambda^0_{\alpha,I}}.$$

The Θ_σ thus form a basis for $\Omega^1_{\mathfrak{X}_{r,\alpha,I}/\Lambda^0_{\alpha,I}}$. Suppose that

$$\nabla(\omega_\sigma) = \sum_\tau \omega_\sigma \otimes \alpha^\sigma_\tau \Theta_\tau + \zeta_\sigma \otimes \Theta_\sigma \tag{2.7}$$

$$\nabla(\zeta_\sigma) = \sum_\tau \omega_\sigma \otimes \beta^\sigma_\tau \Theta_\tau + \sum_\tau \zeta_\sigma \otimes \gamma^\sigma_\tau \Theta_\tau \tag{2.8}$$

for $\alpha^\sigma_\tau, \beta^\sigma_\tau, \gamma^\sigma_\tau \in \mathcal{O}_{\mathfrak{X}_{r,\alpha,I}}$. Therefore we have

$$\begin{aligned}\nabla(e_\sigma) &= \nabla(\xi(\sigma)\cdot\omega_\sigma)\\ &= \xi(\sigma)\sum_\tau \omega_\sigma \otimes \alpha^\sigma_\tau\Theta_\tau + \xi(\sigma)\cdot\zeta_\sigma\otimes\Theta_\sigma + \xi(\sigma)\cdot\omega_\sigma\otimes \mathrm{dlog}\,\xi(\sigma)\\ &= \omega_\sigma \otimes \left(\xi(\sigma)\sum_\tau \alpha^\sigma_\tau\Theta_\tau + \mathrm{d}\xi(\sigma)\right) + \zeta_\sigma\otimes\xi(\sigma)\Theta_\sigma\\ &= e_\sigma \otimes \left(\sum_\tau \alpha^\sigma_\tau\Theta_\tau + \mathrm{dlog}\,\xi(\sigma) - \frac{C_\sigma\Theta_\sigma}{HW(\sigma)}\right) + f_\sigma \otimes \frac{\Theta_\sigma}{HW(\sigma)}\end{aligned} \tag{2.9}$$

$$\begin{aligned}\nabla(f_\sigma) &= \nabla(C_\sigma e_\sigma + \xi(\sigma)HW(\sigma)\cdot\zeta_\sigma)\\ &= \nabla(C_\sigma e_\sigma) + \xi(\sigma)HW(\sigma)\sum_\tau \omega_\sigma\otimes\beta^\sigma_\tau\Theta^\tau + \xi(\sigma)HW(\sigma)\sum_\tau \zeta_\sigma\otimes\gamma^\sigma_\tau\Theta_\tau\\ &+ \xi(\sigma)HW(\sigma)\cdot\zeta_\sigma\otimes\mathrm{dlog}(\xi(\sigma)HW(\sigma))\\ &= \nabla(C_\sigma e_\sigma) + e_\sigma\otimes\left(\sum_\tau HW(\sigma)\beta^\sigma_\tau\Theta_\tau - C_\sigma\sum_\tau\gamma^\sigma_\tau\Theta_\tau - \mathrm{dlog}(\xi(\sigma)HW(\sigma))\right)\\ &+ f_\sigma\otimes\left(\sum_\tau\gamma^\sigma_\tau\Theta_\tau + \mathrm{dlog}(\xi(\sigma)HW(\sigma))\right)\end{aligned} \tag{2.10}$$

Now the proof of the previous part of the proposition shows that over $\mathfrak{IG}'_{n,r,I}$, $\nabla(\Omega_{\mathcal{A}}) \subset \beta_n \mathrm{H}_{\mathcal{A}} \otimes \Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_{\alpha,I}}$. Then the third equality of (2.9) implies that $\Theta_\sigma \in \beta_n \Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_{\alpha,I}}$ for all σ. In other words the image of $\Omega^1_{\mathfrak{X}_{r,\alpha,I}/\Lambda^0_{\alpha,I}} \to \Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_{\alpha,I}}$ is contained in $\beta_n \Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_{\alpha,I}}$. Moreover the same is true of $\mathrm{d}\xi(\sigma)$. This implies together with the explicit formula of (2.10) that $\nabla(f_\sigma) \equiv 0 \bmod (\eta)$, proving the final part of the proposition. ☐

2.4.2 The Gauss-Manin connection on $\mathbb{W}^0_{k,\alpha,I}$

Recall that the universal character $k = k^0_{\alpha,I}$ is analytic on $1 + p^{n-1}(\mathcal{O}_L \otimes \mathbb{Z}_p)$. In particular there are $u_\sigma \in p^{1-n}\Lambda^0_{\alpha,I}$ such that $k_\sigma(t) = \exp(u_\sigma \log(t))$ for all $t \in 1 + \beta_n \mathbb{G}_a$. In this section we will define the connection on $\mathbb{W}^0_{k,\alpha,I}$. We would like to have Griffith's transversality for some filtration on $\mathbb{W}^0_{k,\alpha,I}$. But the filtration given in Theorem 2.3.1, i.e. the filtration given by lexicographic ordering for a choice of numbering of the set Σ, doesn't satisfy this. Thus as promised before, here we define the Hodge filtration on $\mathbb{W}^0_{k,\alpha,I}$.

Lemma 2.4.1. *Let $\rho' \colon \mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q}) \to \mathfrak{IG}_{n,r,I}$ and $h_n \colon \mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,\alpha,I}$ be the projections. Then the filtration on $\rho'_* \mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q})}$ defined on local coordinates Spf $R \subset \mathfrak{IG}_{n,r,I}$ by*

$$\mathrm{Fil}_i \left(\rho'_* \mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q})}(\mathit{Spf}\, R) \right) = \mathrm{Fil}_i R\langle \{Z_\sigma, W_\sigma\}_{\sigma \in \Sigma}\rangle := \bigoplus_{j=0}^{i} R\langle \{Z_\sigma\}_{\sigma\in\Sigma}\rangle \otimes_R \mathrm{Sym}^j\, R[W_\sigma]_{\sigma\in\Sigma}$$

is well-defined. Moreover, $(h_n)_* \mathrm{Fil}_i$ is stable under the action of $\mathfrak{T}^{ext}$ for all i. In particular it induces a filtration on $\mathbb{W}^0_{k,\alpha,I}$, by defining $\mathrm{Fil}_i \mathbb{W}^0_{k,\alpha,I} := \mathrm{Fil}_i \left(\rho_* \mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q})} \right)[k]$, where $\rho = h_n \circ \rho'$. This is defined to be the Hodge filtration. The graded pieces over the generic fibre are $\mathrm{Gr}_i \mathbb{W}^0_{k,\alpha,I}[1/\alpha] \simeq \mathfrak{w}^0_{k,\alpha,I} \otimes \mathrm{Sym}^i\, \omega_{\mathcal{A}}^{-\otimes 2}[1/\alpha] \simeq \mathfrak{w}^0_{k,\alpha,I} \otimes \mathrm{Sym}^i(\oplus_\sigma \omega_{\mathcal{A}}(-2\sigma))[1/\alpha]$.

Proof. We note that $\oplus_{j=0}^i \mathrm{Sym}^j\, R[W_\sigma]_{\sigma\in\Sigma}$ contains all polynomials in $\{W_\sigma\}_{\sigma\in\Sigma}$ of degree $\leq i$. In particular, choosing an ordering $\Xi \colon \Sigma \simeq \{1, \ldots, g\}$, there is a greatest element in $\mathrm{Sym}^i\, R[W_\sigma]_{\sigma\in\Sigma}$ corresponding to the multi-index $(i, 0, \ldots, 0)$. Let $i_0 := \Xi^{-1}(i, 0, \ldots, 0)$. Denoting by Fil'_i the lexicographic ordering of Theorem 2.3.1, we then have $\mathrm{Fil}_i = \mathrm{Fil}'_{i_0}$. The lemma follows. ☐

Convention: Henceforth, unless otherwise stated $\{\mathrm{Fil}_i\}_{i\geq 0}$ will denote the Hodge filtration of Lemma 2.4.1, and **not** the filtration induced by a lexicographic ordering on Σ. Also we will denote by Fil_i the filtration induced on $\mathbb{W}_{k,\alpha,I}$ by tensoring the above filtration with $\mathfrak{w}^\chi_{k,\alpha,I}$.

Let $\mathcal{P}^{(1)}_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_{\alpha,I}}$ be the first infinitesimal neighborhood of the closed subscheme of $\mathfrak{IG}'_{n,r,I} \times_{\mathfrak{W}^0_{\alpha,I}} \mathfrak{IG}'_{n,r,I}$ defined by the diagonal embedding $\Delta \colon \mathfrak{IG}'_{n,r,I} \hookrightarrow \mathfrak{IG}'_{n,r,I} \times_{\mathrm{Spf}\, \Lambda^0_{\alpha,I}} \mathfrak{IG}'_{n,r,I}$. Let p_1, p_2 be the first and second projections $\mathcal{P}^{(1)}_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_{\alpha,I}} \to \mathfrak{IG}'_{n,r,I}$. Then using Grothendieck's formalism of connections [BO15, §2], we get an $\mathcal{O}_L \otimes \mathcal{O}_{\mathcal{P}^{(1)}_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_{\alpha,I}}}$-linear isomorphism $\epsilon^\sharp \colon p_2^* \mathrm{H}^\sharp_{\mathcal{A}} \xrightarrow{\sim} p_1^* \mathrm{H}^\sharp_{\mathcal{A}}$ associated to the connection $\nabla \colon \mathrm{H}^\sharp_{\mathcal{A}} \to \mathrm{H}^\sharp_{\mathcal{A}} \otimes \Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_{\alpha,I}}$. This $\epsilon^\sharp$ is characterized by the properties that

$\Delta^*\epsilon^\sharp = \mathrm{id}$, $\nabla(x) = \epsilon^\sharp(1\otimes x) - x\otimes 1$ and it satisfies a suitable cocycle condition with respect to the three possible pullbacks of $\epsilon^\sharp$ to $\mathfrak{IG}'_{n,r,I}\times_{\mathfrak{W}^0_{\alpha,I}}\mathfrak{IG}'_{n,r,I}\times_{\mathfrak{W}^0_{\alpha,I}}\mathfrak{IG}'_{n,r,I}$.

Let $\tilde{\rho}\colon \vartheta^*\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q}) \to \mathfrak{IG}'_{n,r,I}$ be the pullback of $\rho'\colon \mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q}) \to \mathfrak{IG}_{n,r,I}$ to $\vartheta\colon \mathfrak{IG}'_{n,r,I}\to \mathfrak{IG}_{n,r,I}$. Sometimes we will drop the notation ϑ for simplicity.

Lemma 2.4.2. *The connection $\nabla\colon \mathrm{H}^\sharp_{\mathcal{A}} \to \mathrm{H}^\sharp_{\mathcal{A}}\hat{\otimes}\Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_{\alpha,I}}$ induces an isomorphism associated to a connection (in the sense of Grothendieck) on $\tilde{\rho}_*\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}},s,\mathcal{Q})}$.*

$$p_2^*\tilde{\rho}_*\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}},s,\mathcal{Q})} \xrightarrow[\epsilon_0^\sharp]{\simeq} p_1^*\tilde{\rho}_*\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}},s,\mathcal{Q})}$$

Proof. The isomorphism $\epsilon^\sharp\colon p_2^*\mathrm{H}^\sharp_{\mathcal{A}} \xrightarrow{\sim} p_1^*\mathrm{H}^\sharp_{\mathcal{A}}$ splits by $\mathcal{O}_L$-linearity into $\epsilon^\sharp_\sigma\colon p_2^*\mathrm{H}^\sharp_{\mathcal{A}}(\sigma) \xrightarrow{\sim} p_1^*\mathrm{H}^\sharp_{\mathcal{A}}(\sigma)$ for all $\sigma\in\Sigma$. Let $\tilde{\rho}_\sigma\colon \mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{A}}(\sigma), s_\sigma, \mathcal{Q}(\sigma)) \to \mathfrak{IG}'_{n,r,I}$ be the σ-component of $\tilde{\rho}$. Each $\epsilon^\sharp_\sigma$ induces a connection $\epsilon^\sharp_{\sigma,0}\colon p_2^*(\tilde{\rho}_\sigma)_*\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{A}}(\sigma),s_\sigma,\mathcal{Q}(\sigma))} \xrightarrow{\sim} p_1^*(\tilde{\rho}_\sigma)_*\mathcal{O}_{\mathbb{V}_0(\mathrm{H}^\sharp_{\mathcal{A}}(\sigma),s_\sigma,\mathcal{Q}(\sigma))}$ by §1.4.2. Then $\epsilon_0^\sharp$ is defined by the tensor product $\otimes\epsilon^\sharp_{\sigma,0}$. □

Lemma 2.4.3. *The action of $(\mathcal{O}_L\otimes\mathbb{Z}_p)^\times$ on $\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q})$ over $\mathfrak{X}_{r,\alpha,I}$ as defined in §2.3.2.1 can be lifted to an action on $\vartheta^*\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q})$ over $\mathfrak{X}_{r,\alpha,I}$ such that the induced action commutes with $\epsilon_0^\sharp$.*

Proof. The map $\mathcal{IG}'_{n,r,I}\to\mathcal{X}_{r,\alpha,I}$ is a torsor for the group

$$\begin{pmatrix} (\mathcal{O}_L/p^n\mathcal{O}_L)^\times & \mu_{p^n}\otimes\mathfrak{d}^{-1} \\ 0 & (\mathcal{O}_L/p^n\mathcal{O}_L)^\times \end{pmatrix}.$$

Then $(\mathcal{O}_L\otimes\mathbb{Z}_p)^\times$ acts on $\mathfrak{IG}'_{n,r,I}$ through the quotient

$$(\mathcal{O}_L\otimes\mathbb{Z}_p)^\times \to \begin{pmatrix} 1 & 0 \\ 0 & (\mathcal{O}_L/p^n\mathcal{O}_L)^\times \end{pmatrix}$$

and this action lifts the action on $\mathfrak{IG}_{n,r,I}$. For $\lambda\in(\mathcal{O}_L\otimes\mathbb{Z}_p)^\times$ we get an isomorphism $[\lambda]\colon \mathfrak{IG}'_{n,r,I}\to\mathfrak{IG}'_{n,r,I}$ over $\mathfrak{X}_{r,\alpha,I}$, that induces an isomorphism $\gamma_\lambda\colon \mathrm{H}^\sharp_{\mathcal{A}}\to\mathrm{H}^\sharp_{\mathcal{A}}$ sending the marked section $s\mapsto\bar{\lambda}^{-1}s$ and the marked subspace $\mathcal{Q}$ to itself. Since the connection on $\mathrm{H}^\sharp_{\mathcal{A}}$ is induced by the Gauss-Manin connection on $\mathrm{H}_{\mathcal{A}}$, by functoriality of the Gauss-Manin connection, ∇ commutes with γ_λ. The last claim follows by noticing that the action on $\vartheta^*\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q})$ is induced by the isomorphism $\mathrm{H}^\sharp_{\mathcal{A}}\xrightarrow{\gamma_\lambda}\mathrm{H}^\sharp_{\mathcal{A}}\xrightarrow{\times\lambda}\mathrm{H}^\sharp_{\mathcal{A}}$ which obviously commutes with ∇. □

Theorem 2.4.1. *There is an integrable connection on $\mathbb{W}^0_{k,\alpha,I}$,*

$$\nabla_k\colon \mathbb{W}^0_{k,\alpha,I}\to\mathbb{W}^0_{k,\alpha,I}\hat{\otimes}\Omega^1_{\mathfrak{X}_{r,\alpha,I}/\Lambda^0_{\alpha,I}}[1/\alpha]$$

for which the filtration on $\mathbb{W}^0_{k,\alpha,I}$ defined in Lemma 2.4.1 satisfies Griffith's transversality.

Proof. We use the notation of the proof of Proposition 2.4.1. Recall Spf $R' \subset \mathfrak{IG}'_{n,r,I}$ was an open that was the inverse image of an open Spf $R_0 \subset \mathfrak{X}_{r,\alpha,I}$ that trivializes $\mathrm{H}_{\mathcal{A}}$, and such that over Spf R', $\mathrm{H}^\sharp_{\mathcal{A}}$ is trivial with $\mathcal{O}_L \otimes R'$-basis $\{e, f\}$ adapted to the marked section s and modified unit root subspace $\mathcal{Q}$. Let $I(\Delta) = \ker\left(R'\hat{\otimes}_{\Lambda^0_{\alpha,I}} R' \xrightarrow{\text{mult}} R'\right)$, and let $R^{(1)} = R'\hat{\otimes}_{\Lambda^0_{\alpha,I}} R'/I(\Delta)^2$.

Then in terms of the basis $\{e, f\}$, $\epsilon^\sharp$ is given by a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2\left(\mathcal{O}_L \otimes R^{(1)}\right).$$

Decomposing into components, we get matrices

$$A_\sigma = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} \in \mathrm{GL}_2\left(R^{(1)}\right)$$

with respect to the basis e_σ, f_σ of $\mathrm{H}^\sharp_{\mathcal{A}}(\sigma)$ for each σ.

Since $\Delta^*(\epsilon^\sharp) = \mathrm{id}$, we have that $a_\sigma = 1 + a^0_\sigma$ and $d_\sigma = 1 + d^0_\sigma$ with $a^0_\sigma, d^0_\sigma, b_\sigma, c_\sigma \in I(\Delta)$ for all σ. Moreover, the squares of $a^0_\sigma, d^0_\sigma, b_\sigma, c_\sigma$ are all 0 in $R^{(1)}$.

Comparing the expression of ∇ in terms of a, b, c, d on the one hand and that in (2.9) and (2.10) on the other, we see that $c_\sigma HW(\sigma)$ is the Kodaira-Spencer class Θ_σ.

By Lemma 2.4.2 there is an isomorphism $\epsilon^\sharp_0\colon p_2^*\tilde{\rho}_*\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}},s,\mathcal{Q})} \xrightarrow{\sim} p_1^*\tilde{\rho}_*\mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}},s,\mathcal{Q})}$ induced by $\epsilon^\sharp\colon p_2^*\mathrm{H}^\sharp_{\mathcal{A}} \simeq p_1^*\mathrm{H}^\sharp_{\mathcal{A}}$. We show that $\epsilon^\sharp_0$ restricts to a connection on $\mathbb{W}'_{k,\alpha,I}$. We show this on local coordinates. So recalling the local description of $\mathbb{W}'_{k,\alpha,I}$ from Corollary 2.3.4, we have $\mathbb{W}'_{k,\alpha,I|\mathrm{Spf}\,R'} = R'\langle\{V_\sigma\}\rangle k(1+\beta_n Z)$, where we recall $V_\sigma = \frac{W_\sigma}{1+\beta_n Z_\sigma}$ and $k(1+\beta_n Z)$ was the notation for $\prod_\sigma k_\sigma(1+\beta_n Z_\sigma)$. Thus $\epsilon^\sharp_0$ is described by its action on V_σ and $1+\beta_n Z_\sigma$. We have

$$\epsilon^\sharp_0(V_\sigma) = \eta^{-1}(b_\sigma + \eta d_\sigma V_\sigma)(a_\sigma + \eta c_\sigma V_\sigma)^{-1} \quad \epsilon^\sharp_0(1+\beta_n Z_\sigma) = (a_\sigma + \eta c_\sigma V_\sigma)(1+\beta_n Z_\sigma).$$

From this one can deduce the following formula for $\nabla_k(x) = \epsilon^\sharp_0(1\otimes x) - x \otimes 1$.

$$\begin{aligned}\nabla_k\left(\prod_\sigma V_\sigma^{i_\sigma}\cdot k(1+\beta_n Z)\right) = \prod_\sigma V_\sigma^{i_\sigma}\Bigg(&\sum_\sigma i_\sigma V_\sigma^{-1}\otimes b_\sigma\eta^{-1} + \sum_\sigma (u_\sigma - i_\sigma)\otimes a^0_\sigma \\ &+ \sum_\sigma i_\sigma \otimes d^0_\sigma + \sum_\sigma (u_\sigma - i_\sigma)V_\sigma \otimes \eta c_\sigma\Bigg)\left(k(1+\beta_n Z)\otimes 1\right)\end{aligned} \tag{2.11}$$

This connection descends to $\mathfrak{IG}_{n,r,I}$ after inverting α, by the formula above. Then we descend ∇_k to $\mathfrak{X}_{r,\alpha,I}$ by taking k-invariants for the $(\mathcal{O}_L\otimes\mathbb{Z}_p)^\times$ action using Lemma 2.4.3 and noting $\mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,\alpha,I}$ is generically étale. □

Corollary 2.4.1. *The connection* $\nabla_k\colon \mathbb{W}^0_{k,\alpha,I} \to \mathbb{W}^0_{k,\alpha,I}\hat{\otimes}\Omega^1_{\mathfrak{X}_{r,\alpha,I}/\Lambda^0_{\alpha,I}}[1/\alpha]$ *induces a connection which we still denote by* ∇_k

$$\nabla_k\colon \mathbb{W}_{k,\alpha,I} \to \mathbb{W}_{k,\alpha,I}\hat{\otimes}\Omega^1_{\mathfrak{M}_{r,\alpha,I}/\Lambda_{\alpha,I}}[1/\alpha]$$

that satisfies Griffiths transverality with respect to the filtration on $\mathbb{W}_{k,\alpha,I}$ *defined by tensoring* Fil_i *of Lemma 2.4.1 with* $\mathfrak{w}^{\chi}_{k,\alpha,I}$.

Proof. Recall $\mathfrak{w}^{\chi}_{k,\alpha,I}$ was defined as $\left((f_i)_*\mathcal{O}_{\mathfrak{IG}_{i,r,I}} \otimes_{\Lambda^0_{\alpha,I}} \Lambda_{\alpha,I}\right)[\chi^{-1}]$ for $f_i\colon \mathfrak{IG}_{i,r,I} \to \mathfrak{X}_{r,\alpha,I}$ the projection with $i=1$ if $p\neq 2$ and $i=2$ if $p=2$. The universal derivation $(f_i)_*\mathcal{O}_{\mathfrak{IG}_{i,r,I}} \otimes_{\Lambda^0_{\alpha,I}} \Lambda_{\alpha,I} \to (f_i)_*\Omega^1_{\mathfrak{IG}_{i,r,I}/\Lambda^0_{\alpha,I}} \otimes \Lambda_{\alpha,I}$ commutes with the action of $(\mathcal{O}_L/q\mathcal{O}_L)^\times$ and thus induces a connection $\mathfrak{w}^{\chi}_{k,\alpha,I} \to \mathfrak{w}^{\chi}_{k,\alpha,I}\hat{\otimes}\Omega^1_{\mathfrak{M}_{r,\alpha,I}}[1/\alpha]$ by taking χ^{-1}-invariants and upon inverting α. This along with ∇_k defined on $\mathbb{W}^0_{k,\alpha,I}$ above induces the required connection on $\mathbb{W}_{k,\alpha,I}$. $\square$

The Kodaira-Spencer isomorphism $\omega_{\mathcal{A}}^{\otimes 2} \xrightarrow{\sim} \Omega^1_{\mathfrak{X}_{r,\alpha,I}/\Lambda^0_{\alpha,I}}$ induces a decomposition of $\Omega^1_{\mathfrak{X}_{r,\alpha,I}/\Lambda^0_{\alpha,I}}$ corresponding to the decomposition $\omega_{\mathcal{A}}^{\otimes 2} = \prod_\sigma \omega_{\mathcal{A}}^{2\sigma}$. Here the tensor product is taken as $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{X}_{\alpha,I}}$-modules. This induces an isomorphism $\Omega^1_{\mathfrak{X}_{r,\alpha,I}}[1/\alpha] \simeq \prod_\sigma \omega_{\mathcal{A}}^{2\sigma}[1/\alpha]$.

Definition 2.4.1. Define $\nabla_k(\sigma)\colon \mathbb{W}^0_{k,\alpha,I} \to \mathbb{W}^0_{k+2\sigma,\alpha,I}[1/\alpha]$ as the map obtained by composing ∇_k with the projection onto the σ component of $\Omega^1_{\mathfrak{X}_{r,\alpha,I}/\Lambda^0_{\alpha,I}}[1/\alpha] \simeq \prod_\sigma \omega_{\mathcal{A}}^{2\sigma}$, followed by the natural map to $\mathbb{W}^0_{k+2\sigma,\alpha,I}$.

$$\nabla_k(\sigma)\colon \mathbb{W}^0_{k,\alpha,I} \xrightarrow{\nabla_k} \mathbb{W}^0_{k,\alpha,I}\hat{\otimes}\Omega^1_{\mathfrak{X}_{r,\alpha,I}/\Lambda^0_{\alpha,I}}[1/\alpha] \to \mathbb{W}^0_{k,\alpha,I}\hat{\otimes}\omega_{\mathcal{A}}^{2\sigma}[1/\alpha] \to \mathbb{W}^0_{k+2\sigma,\alpha,I}[1/\alpha]$$

Similarly denote still by $\nabla_k(\sigma)\colon \mathbb{W}_{k,\alpha,I} \to \mathbb{W}_{k+2\sigma,\alpha,I}[1/\alpha]$ the map obtained by twisting $\nabla_k(\sigma)$ as above with the connection on $\mathfrak{w}^{\chi}_{k,\alpha,I}$ followed by the projection onto the σ-component under the Kodaira-Spencer isomorphism.

Corollary 2.4.2. *The* $\mathcal{O}_{\mathcal{X}_{r,\alpha,I}}$*-linear map induced by the connection* $\nabla_k(\sigma)$ *on the graded piece*

$$\nabla_k(\sigma)\colon \mathrm{Gr}_n\mathbb{W}^0_{k,\alpha,I}[1/\alpha] \to \mathrm{Gr}_{n+1}\mathbb{W}^0_{k+2\sigma,\alpha,I}[1/\alpha]$$

sends an element $\omega^{k-2\mathbf{i}} \in \mathfrak{w}^0_{k,\alpha,I} \otimes \mathrm{Sym}^n\,\omega_{\mathcal{A}}^{-\otimes 2}[1/\alpha]$ *to* $(u_\sigma - i_\sigma)\omega^{k-2\mathbf{i}}$ *for* $\mathbf{i} = (i_\tau)_{\tau\in\Sigma} \in \mathbb{N}^g$.

Proof. Follows from (2.11) above. $\square$

2.4.3 ∇_k on q-expansions

For simplicity of notation, in this section we drop α, I from the notation $\mathbb{W}^0_{k,\alpha,I}$ and simply write $\mathbb{W}^0_k$. Also since we want to iterate the connection, and the connection $\nabla_k(\sigma)$ maps $\mathbb{W}^0_k[1/\alpha]$ to $\mathbb{W}^0_{k+2\sigma}[1/\alpha]$, in our notation we forget the dependency of the connection on the weight k, and simply write $\nabla(\sigma)$. Later we will need to compose $\nabla(\sigma)$ and $\nabla(\tau)$ for $\sigma \neq \tau \in \Sigma$. Lemma 2.4.4 below will show us that the order of composition does not matter.

In this section we will study the effect of ∇_k on q-expansions. We begin by reviewing the definition of Tate objects for Hilbert-Blumenthal abelian varieties following [Kat78].

Fix fractional ideals $\mathfrak{a}, \mathfrak{b}$ such that $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$. Let S be a set of g linearly independent $\mathbb{Q}$-linear forms $\ell_i \colon L \to \mathbb{Q}$, such that $\ell_i(x) > 0$ for all $x \gg 0$, where by $x \gg 0$ we mean x is a totally positive element. We say an element is S-positive if $\ell_i(x) \geq 0$ for all i. Let $\mathfrak{a}\mathfrak{b}_S = \{x \in \mathfrak{a}\mathfrak{b} \mid x \text{ is } S\text{-positive}\}$ be the set of S-positive elements in $\mathfrak{a}\mathfrak{b}$. $\mathfrak{a}\mathfrak{b}_S$ is a finitely generated monoid.

Definition 2.4.2. Define $\mathbb{Z}[\![\mathfrak{a}\mathfrak{b}, S]\!]$ to be the ring of all formal series $\sum_{\beta \in \mathfrak{a}\mathfrak{b}_S} a_\beta q^\beta$. Define $\mathbb{Z}(\!(\mathfrak{a}\mathfrak{b}, S)\!) = \mathbb{Z}[\![\mathfrak{a}\mathfrak{b}, S]\!][1/q^\beta]$ for some $\beta \gg 0$.

We remark that inverting q^β for some $\beta \gg 0$ inverts q^γ for all $\gamma \gg 0$. So $\mathbb{Z}(\!(\mathfrak{a}\mathfrak{b}, S)\!)$ is well-defined. In particular, $\mathbb{Z}(\!(\mathfrak{a}\mathfrak{b}, S)\!)$ is the collection of all formal series $\sum_{\beta \in \mathfrak{a}\mathfrak{b}} a_\beta q^\beta$ such that for some integer $n \gg 0$, we have $\ell_i(\beta) \geq -n$ whenever $a_\beta \neq 0$.

Over the ring $\mathbb{Z}(\!(\mathfrak{a}\mathfrak{b}, S)\!)$, we have the g-dimensional algebraic torus $\mathbb{G}_m \otimes \mathfrak{d}^{-1}\mathfrak{a}^{-1}$ together with an $\mathcal{O}_L$-linear group homomorphism $\underline{q} \colon \mathfrak{b} \to \mathbb{G}_m \otimes \mathfrak{d}^{-1}\mathfrak{a}^{-1}$ defined as follows. To give such a group homomorphism is the same as giving an $\mathcal{O}_L$-linear group homomorphism $\mathfrak{a}\mathfrak{b} \to \mathbb{G}_m \otimes \mathfrak{d}^{-1}$. This is equivalent to giving a group homomorphism $\mathfrak{a}\mathfrak{b} \to \mathbb{G}_m$ which we define to be $\beta \mapsto q^\beta \in \mathbb{G}_m(\mathbb{Z}(\!(\mathfrak{a}\mathfrak{b}, S)\!))$. The rigid analytic quotient $\mathbb{G}_m \otimes \mathfrak{d}^{-1}\mathfrak{a}^{-1}/\underline{q}(\mathfrak{b})$ is algebraizable to a Hilbert-Blumenthal abelian variety denoted $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ over $\mathbb{Z}(\!(\mathfrak{a}\mathfrak{b}, S)\!)$ which carries a canonical $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$ polarization

$$\lambda_{\mathrm{can}} \colon \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)^\vee \xrightarrow{\sim} \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q) \otimes \mathfrak{a}\mathfrak{b}^{-1} \simeq \mathrm{Tate}_{\mathfrak{b},\mathfrak{a}}(q).$$

We quickly recall that there exists canonical isomorphisms as follows [Kat78, (1.1.17), (1.1.18)].

1. $\omega_{\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)} \simeq \mathfrak{a} \otimes \mathbb{Z}(\!(\mathfrak{a}\mathfrak{b}, S)\!)$; $\omega^\vee_{\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)} \simeq \mathfrak{d}^{-1}\mathfrak{a}^{-1} \otimes \mathbb{Z}(\!(\mathfrak{a}\mathfrak{b}, S)\!)$.
2. $\Omega^1_{\mathbb{Z}(\!(\mathfrak{a}\mathfrak{b},S)\!)} \simeq \mathfrak{a}\mathfrak{b} \otimes \mathbb{Z}(\!(\mathfrak{a}\mathfrak{b}, S)\!)$; $\mathrm{Der}\Big(\mathbb{Z}(\!(\mathfrak{a}\mathfrak{b}, S)\!), \mathbb{Z}(\!(\mathfrak{a}\mathfrak{b}, S)\!)\Big) \simeq \mathfrak{d}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1} \otimes \mathbb{Z}(\!(\mathfrak{a}\mathfrak{b}, S)\!)$.

We now base change to $\Lambda^0_{\alpha,I}$, so that $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ is defined over $R := \Lambda^0_{\alpha,I}(\!(\mathfrak{a}\mathfrak{b}, S)\!)$.

For simplicity assume $\mathfrak{a}, \mathfrak{b}$ are coprime to p. Everything that follows holds true with appropriate modifications in the general case by choosing an isomorphism $\mathcal{O}_L \otimes \mathbb{Z}_p \simeq \mathfrak{a}^{-1} \otimes \mathbb{Z}_p$ which amounts to choosing a $\Gamma_{00}(p^\infty)$-structure on $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ [Kat78, (1.1.15)]. When $\mathfrak{a}$ is coprime to p, we have the natural equality $\mathcal{O}_L \otimes \mathbb{Z}_p = \mathfrak{a}^{-1} \otimes \mathbb{Z}_p$ inside $L \otimes \mathbb{Q}_p$ which induces a canonical $\Gamma_{00}(p^\infty)$-structure on $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$.

For any $\sigma \in \Sigma$, let $e_\sigma \in \mathcal{O}_L \otimes R$ be the corresponding idempotent. Let $\omega_{\mathrm{can}}(\sigma)$ be the image of $e_\sigma \in \mathfrak{a} \otimes R = \mathcal{O}_L \otimes R$ under the canonical identification $\omega_{\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)} \simeq \mathfrak{a} \otimes R$. Let $\Theta_\sigma = KS(\omega_{\mathrm{can}}^{\otimes 2}(\sigma))$ be the corresponding Kodaira-Spencer class. Then Θ_σ is the image of $e_\sigma \in \mathcal{O}_L \otimes R$ under the identification $\Omega^1_{R/\Lambda^0_{\alpha,I}} \simeq \mathfrak{a}\mathfrak{b} \otimes R = \mathcal{O}_L \otimes R$. The homomorphism $e_\sigma^\vee \colon \Omega^1_{R/\Lambda^0_{\alpha,I}} \to R$ that is dual to e_σ, induces the derivation $\theta_\sigma \in \mathrm{Der}(R, R)$ defined as

$$\theta_\sigma\Big(\sum a_\beta q^\beta\Big) = \sum \sigma(\beta) a_\beta q^\beta.$$

Having recalled generalities about the Tate objects, we go back to computing the effect of ∇_k on q-expansions.

Let $\nabla(\omega_{\text{can}}(\sigma)) = \zeta_{\text{can}}(\sigma) \otimes \Theta_\sigma$. Then $\nabla(\zeta_{\text{can}}(\sigma)) = 0$. Let ω_{can} and ζ_{can} be the $\mathcal{O}_L \otimes R$-basis of $\mathrm{H}^\sharp_{\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)} = H^1_{\mathrm{dR}}(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)/R)$, whose σ-components are $\omega_{\text{can}}(\sigma)$ and $\zeta_{\text{can}}(\sigma)$ respectively. With respect to this basis the matrix of $\nabla = (\nabla_\sigma)_\sigma$ is thus given as follows.

$$\nabla_\sigma = \begin{pmatrix} 0 & 0 \\ \Theta_\sigma & 0 \end{pmatrix}$$

(Note $\nabla_\sigma \colon \mathrm{H}^\sharp_{\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)}(\sigma) \to \mathrm{H}^\sharp_{\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)}(\sigma)\hat{\otimes}\Omega^1_{R/\Lambda^0_{\alpha,I}}$ is just the σ component of ∇. In particular, it should not be confused with $\nabla_k(\sigma)$ of Definition 2.4.1).

Let $\mathbb{W}^0_k(q)$ be the pullback of $\mathbb{W}^0_k$ to Spf R along the structure morphism defining $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ together with the canonical $\Gamma_{00}(p^n)$-structure defined as above. Then we can write $\mathbb{W}^0_k(q) \simeq R\langle \{V_\sigma\}_\sigma \rangle \cdot k(1+p^nZ)$ as in Corollary 2.3.4. Then formula (2.11) gives us

$$\begin{aligned} \nabla_k(\sigma)\left(a\prod_\tau V_\tau^{i_\tau}\cdot k(1+p^nZ)\right) &= \theta_\sigma(a)\prod_\tau V_\tau^{i_\tau}\cdot(k+2\sigma)(1+p^nZ) \\ &\quad + p(u_\sigma - i_\sigma)V_\sigma\prod_\tau V_\tau^{i_\tau}\cdot(k+2\sigma)(1+p^nZ) \end{aligned} \tag{2.12}$$

for any $a \in R$.

Lemma 2.4.4. *For any $\sigma, \tau \in \Sigma$, the maps $\nabla_k(\sigma)$ and $\nabla_k(\tau)$ commute, i.e.*

$$\nabla_{k+2\sigma}(\tau) \circ \nabla_k(\sigma) = \nabla_{k+2\tau}(\sigma) \circ \nabla_k(\tau)$$

as maps $\mathbb{W}^0_k \to \mathbb{W}^0_{k+2\sigma+2\tau}[1/\alpha]$.

Proof. It is enough to check this on the ordinary locus $\mathfrak{X}^{\mathrm{ord}}_{\alpha,I}$, as the ordinary locus is dense in $\mathfrak{X}_{r,\alpha,I}$. On the ordinary locus the result follows by verifying on q-expansions using (2.12) and the q-expansion principle. (See also [Kat78, (2.1.14)]) ☐

Lemma 2.4.5. *Let $g(q) \in R$ and $N \geq 1$. Then we can write*

$$\nabla(\sigma)^N\Big(g(q)\prod_\tau V_\tau^{i_\tau}\cdot k(1+p^nZ)\Big) = \sum_{j=0}^N p^j a_{N,k,i_\sigma,j}\theta_\sigma^{N-j}(g(q))V_\sigma^j\prod_\tau V_\tau^{i_\tau}\cdot(k+2N\sigma)(1+p^nZ).$$

Here $a_{N,k,i_\sigma,0} = 1$ and for $j \geq 1$, we have

$$a_{N,k,i_\sigma,j} = \binom{N}{j}\prod_{i=1}^{j-1}(u_\sigma - i_\sigma + N - 1 - i).$$

Proof. This is the exact same computation as Lemma 1.4.2. ☐

Let $\mathbb{W}_k^0(\sigma) := \sum_n \mathbb{W}^0_{k+2n\sigma}$ and let $\mathbb{W}^0 = \rho_* \mathcal{O}_{\mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q})}$ where $\rho\colon \mathbb{V}_0^{\mathcal{O}_L}(\mathrm{H}^\sharp_{\mathcal{A}}, s, \mathcal{Q}) \to \mathfrak{X}_{r,\alpha,I}$ is the projection. Let $\mathbb{W}_k(\sigma) = \mathbb{W}^0_k(\sigma) \otimes \mathfrak{w}^{\chi}_{k,\alpha,I}$ and let $\mathbb{W} = \mathbb{W}^0 \otimes \mathfrak{F}$. Here $\mathfrak{F} := (f_i)_* \mathcal{O}_{\mathfrak{IG}_{i,r,I}} \otimes_{\Lambda^0_{\alpha,I}} \Lambda_{\alpha,I}$, where $f_i\colon \mathfrak{IG}_{i,r,I} \to \mathfrak{X}_{r,\alpha,I}$ is the projection with $i=1$ for $p \neq 2$ and $i=2$ otherwise. We have defined the U operator in the next chapter (Definition 3.1.1) which we will use now.

Corollary 2.4.3. *Let f_σ be the inertia degree for the embedding σ. Let $k = \chi \cdot k^0$ with $\chi = k|_\Delta$ the torsion part of the character and $k^0 = k\chi^{-1}$. Assume $k^0_\sigma(t) = \exp(u_\sigma \log t)$ for $t \in 1 + \beta_n \mathbb{G}_a$ and $u_\sigma \in \Lambda^0_{\alpha,I}$. Let $\bar{\mathfrak{M}}^{ord}_{\alpha,I} = \mathfrak{X}^{ord}_{\alpha,I} \times_{\mathfrak{W}^0_{\alpha,I}} \mathfrak{W}_{\alpha,I}$. For any $g \in H^0(\bar{\mathfrak{M}}^{ord}_{\alpha,I}, \mathbb{W}_k)^{U=0}$,*

$$(\nabla(\sigma)^{p^{f_\sigma}-1} - id)(g) \in pH^0\big(\bar{\mathfrak{M}}^{ord}_{\alpha,I}, \mathbb{W}\big) \cap H^0\big(\bar{\mathfrak{M}}^{ord}_{\alpha,I}, \mathbb{W}_k(\sigma)\big).$$

Proof. We recall that $\mathbb{W}_k = \mathbb{W}^0_{k^0} \otimes \mathfrak{w}^\chi_{k,\alpha,I}$ and the connection ∇ on $\mathbb{W}_k$ is defined by the composite of the connection on $\mathbb{W}^0_{k^0}$ as defined in Theorem 2.4.1 and the connection on $\mathfrak{w}^\chi_{k,\alpha,I}$ which is defined by the universal derivation on $\mathcal{O}_{\mathfrak{IG}_{i,r,I}}$ for i as above. Let $\Lambda_{\alpha,I}(\!(\mathfrak{ab}, S)\!) = \Lambda_{\alpha,I} \otimes_{\Lambda^0_{\alpha,I}} R$. The base change of $\mathrm{Spf}\, \Lambda_{\alpha,I}(\!(\mathfrak{ab}, S)\!)$ to $\mathfrak{IG}_{i,r,I} \to \mathfrak{X}_{r,\alpha,I}$ is just copies of $\mathrm{Spf}\, \Lambda_{\alpha,I}(\!(\mathfrak{ab}, S)\!)$ indexed by $(\mathcal{O}_L/q\mathcal{O}_L)^\times$. Hence the universal derivation on $\Lambda_{\alpha,I}(\!(\mathfrak{ab}, S)\!) \otimes_{\mathcal{O}_{\mathfrak{X}_{r,\alpha,I}}} \mathcal{O}_{\mathfrak{IG}_{i,r,I}}$ is determined by the universal derivation on $\Lambda_{\alpha,I}(\!(\mathfrak{ab}, S)\!)$. The q-expansion of any section $g \in H^0(\bar{\mathfrak{M}}_{r,\alpha,I}, \mathbb{W})^{U=0}$ at $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ corresponds to a tuple $(g_i)_{i \in (\mathcal{O}_L/q\mathcal{O}_L)^\times}$ with $g_i \in \Lambda_{\alpha,I}(\!(\mathfrak{ab}, S)\!)\langle \{V_\sigma\}_\sigma \rangle \cdot k^0(1+p^nZ)$. Moreover, for each i, $U(g_i) = 0$. By the q-expansion principle it will be enough to prove the corollary for $g = g(q) \prod_\tau V_\tau^{i_\tau} \cdot k^0(1 + p^nZ)$ for $g(q) \in \Lambda_{\alpha,I}(\!(\mathfrak{ab}, S)\!)$, such that $g(q)$ is p-depleted. By Lemma 2.4.5, it is enough to show $\theta_\sigma^{p^{f_\sigma}-1}(g(q)) \cdot (k + 2(p^{f_\sigma} - 1))(1 + p^nZ) \equiv g(q)k(1 + p^nZ) \bmod p$, which is clear. □

2.4.4 Iteration of ∇

In this section we will finally define the p-adic iteration of the Gauss-Manin connection. We begin with a preparatory lemma.

Lemma 2.4.6. *For $I = [p^a, p^b]$,*

1. $\Lambda^0_{\alpha,I} = \mathcal{O}_K[[T_1, \ldots, T_g]]\langle \frac{p}{\alpha}, \frac{T_1}{\alpha}, \ldots, \frac{T_g}{\alpha}, u, v\rangle / (\alpha^{p^a} v - p, uv - \alpha^{p^{b-a}})$ *if $b \neq \infty$.*
2. $\Lambda^0_{\alpha,I} = \mathcal{O}_K[[T_1, \ldots, T_g]]\langle \frac{p}{\alpha}, \frac{T_1}{\alpha}, \ldots, \frac{T_g}{\alpha}, u\rangle / (\alpha^{p^a} u - p)$ *if $b = \infty$.*

Let $U := \mathrm{Spf}\, A$ be a Zariski open in $\mathfrak{X}$ where $\omega_{\mathcal{A}}$ is trivial. Then

$$U \times_{\mathfrak{X}} \mathfrak{X}_{r,I} = \mathrm{Spf}\, A \hat{\otimes} \Lambda^0_{\alpha,I}\langle w\rangle / (w\mathrm{Hdg}^{p^{r+1}} - \alpha).$$

Proof. See [AIP16b, §3.4.1]. □

Lemma 2.4.7. *Let $C_1 = \#(\mathcal{O}_L/p\mathcal{O}_L)^\times$. Then the kernel of the restriction map $\mathcal{O}_{\mathfrak{IG}_{1,r,I}}/(\alpha^j) \xrightarrow{\phi_1} \mathcal{O}_{\mathfrak{IG}^{\mathrm{ord}}_{1,I}}/(\alpha^j)$ is killed by $\mathrm{Hdg}^{j(p^{r+1})+C}$. The kernel of the restriction map $\mathcal{O}_{\mathfrak{IG}_{n,r,I}}/(\alpha^j) \xrightarrow{\phi_n} \mathcal{O}_{\mathfrak{IG}^{ord}_{n,I}}/(\alpha^j)$ is killed by $\mathrm{Hdg}^{j(p^{r+1})+C_n}$ where $C_n = C_1 + \frac{p^n - p}{p-1}$.*

Proof. The formulas of Lemma 2.4.6 show that the kernel of $\mathcal{O}_{\mathfrak{X}_{r,I}}/(\alpha^j) \xrightarrow{\phi_0} \mathcal{O}_{\mathfrak{X}^{\mathrm{ord}}_{\alpha,I}}/(\alpha^j)$ is killed by $\mathrm{Hdg}^{j(p^{r+1})}$. The trace map $\mathrm{Tr}\colon \mathcal{O}_{\mathfrak{IG}_{1,r,I}} \to \mathcal{O}_{\mathfrak{X}_{r,I}}$ then gives a commutative diagram as follows.

$$\begin{tikzcd}
0 \arrow[r] & \ker\phi_1 \arrow[r] \arrow[d] & \mathcal{O}_{\mathfrak{IG}_{1,r,I}}/(\alpha^j) \arrow[r] \arrow[d, "\mathrm{Tr}"] & \mathcal{O}_{\mathfrak{IG}^{\mathrm{ord}}_{1,I}}/(\alpha^j) \arrow[d, "\mathrm{Tr}"] \\
0 \arrow[r] & \ker\phi_0 \arrow[r] & \mathcal{O}_{\mathfrak{X}_{r,I}}/(\alpha^j) \arrow[r] & \mathcal{O}_{\mathfrak{X}^{\mathrm{ord}}_{\alpha,I}}/(\alpha^j)
\end{tikzcd}$$

Suppose $x \in \ker\phi_1$. Then $\mathrm{Tr}(x) \in \ker\phi_0$ and hence $\mathrm{Tr}(\mathrm{Hdg}^{j(p^{r+1})}x) = 0$. In other words, for any lift $\tilde{x} \in \mathcal{O}_{\mathfrak{IG}_{1,r,I}}$ of x, $\mathrm{Tr}(\mathrm{Hdg}^{j(p^{r+1})}\tilde{x}) \in \alpha^j\mathcal{O}_{\mathfrak{X}_{r,I}}$. Let $\mathfrak{D}^{-1} := \{y \in \mathrm{Frac}(\mathcal{O}_{\mathfrak{IG}_{1,r,I}}) \mid \mathrm{Tr}(yz) \in \mathcal{O}_{\mathfrak{X}_{r,I}} \text{ for all } z \in \mathcal{O}_{\mathfrak{IG}_{1,r,I}}\}$. Then $\mathrm{Hdg}^{j(p^{r+1})}\tilde{x} \in \alpha^j\mathfrak{D}^{-1}$ as $\ker\phi_1$ is an ideal. By using normality of the rings involved, the first claim then follows by localizing at height 1 primes and noting that $\mathfrak{D}^{-1}$ is the usual inverse different in such tamely ramified extensions of DVR's. For the second part, we note that $\mathfrak{IG}_{n,r,I}$ is the normalization of $\mathfrak{Y} := \mathfrak{IG}_{1,r,I} \times_{H_1^\vee} H_n^\vee$ where $\mathfrak{IG}_{1,r,I} \to H_1^\vee$ is the universal generator of $H_1^\vee$. The faithfully flat extension $H_n^\vee \to H_1^\vee$ has different $\mathfrak{D}(H_n^\vee/H_1^\vee)$ that contains $\mathrm{Hdg}^{\frac{p^n-p}{p-1}}$ [AIP18, Proposition 3.5]. By flatness the kernel of $\mathcal{O}_{\mathfrak{Y}}/(\alpha^j) \to \mathcal{O}_{\mathfrak{Y}^{\mathrm{ord}}}/(\alpha^j)$ is killed by $\mathrm{Hdg}^{j(p^{r+1})+C_1}$. Since $\mathfrak{IG}_{n,r,I}$ is the normalization of $\mathfrak{Y}$, it is finite and in particular $\mathfrak{IG}_{n,r,I} \subset \mathfrak{D}^{-1}(\mathfrak{Y}/\mathfrak{IG}_{1,r,I})$. Thus $\mathrm{Hdg}^{\frac{p^n-p}{p-1}}\mathcal{O}_{\mathfrak{IG}_{n,r,I}} \subset \mathcal{O}_{\mathfrak{Y}}$, which proves the second claim. □

Lemma 2.4.8. *Let $g_n\colon \mathfrak{IG}_{n,r,I} \to \mathfrak{X}_{r,\alpha,I}$ be the projection. The kernel and cokernel of $g_n^*\Omega^1_{\mathfrak{X}_{r,\alpha,I}/\Lambda^0_{\alpha,I}} \to \Omega^1_{\mathfrak{IG}_{n,r,I}/\Lambda^0_{\alpha,I}}$ is killed by a power of* Hdg. *Let $\vartheta\colon \mathfrak{IG}'_{n,r,I} \to \mathfrak{IG}_{n,r,I}$ be the projection. The kernel of $\vartheta^*\Omega^1_{\mathfrak{IG}_{n,r,I}/\Lambda^0_{\alpha,I}} \to \Omega^1_{\mathfrak{IG}'_{n,r,I}/\Lambda^0_{\alpha,I}}$ is killed by a power of* Hdg.

Proof. This is similar to Lemma 1.4.3. For the second part one views $\mathfrak{IG}'^{,\mathrm{ord}}_{n,I}$ as a torsor over $\mathfrak{X}^{\mathrm{ord}}_{\alpha,I}$ for the group $\begin{pmatrix} (\mathcal{O}_L/p^n\mathcal{O}_L)^\times & \mu_{p^n}\otimes\mathfrak{d}^{-1} \\ 0 & (\mathcal{O}_L/p^n\mathcal{O}_L)^\times \end{pmatrix}$, and argues as before using smoothness of $\mathfrak{IG}^{\mathrm{ord}}_{n,I}$. □

Assumption 2.1. Let $k\colon \mathbb{T}(\mathbb{Z}_p) \to (\Lambda_{\alpha,I})^\times$ be a weight such that $k = \chi k^0$ where $\chi = k|_\Delta$ is the finite part of the character and $k^0 = k\chi^{-1}$. Assume that for all $\sigma \in \Sigma$, there exists $u_\sigma \in \Lambda^0_{\alpha,I}$, such that k^0 factors as

$$k^0\colon (\mathcal{O}_L \otimes \mathbb{Z}_p)^\times \to (\mathcal{O}_L \otimes \mathcal{O}_K)^\times \simeq \prod_\sigma \mathcal{O}_K^\times \xrightarrow{(k^0_\sigma)_\sigma} (\Lambda^0_{\alpha,I})^\times$$

with $k^0_\sigma(t) = \exp(u_\sigma \log t)$ for all $t \in \mathcal{O}_K^\times$.

Let $s\colon \mathbb{T}(\mathbb{Z}_p) \to (\Lambda^0_{\alpha,I})^\times$ be a weight such that for all $\sigma \in \Sigma$, there exists $v_\sigma \in \Lambda^0_{\alpha,I}$ such that s factors as

$$s\colon (\mathcal{O}_L \otimes \mathbb{Z}_p)^\times \to (\mathcal{O}_L \otimes \mathcal{O}_K)^\times \simeq \prod_\sigma \mathcal{O}_K^\times \xrightarrow{(s_\sigma)_\sigma} (\Lambda^0_{\alpha,I})^\times.$$

with $s_\sigma(t) = \exp(v_\sigma \log t)$ for all $t \in \mathcal{O}_K^\times$.

In particular, we can take $I = [0,1]$ and $\alpha = p$.

Note that the explicit description of the Gauss-Manin connection in (2.11) together with Lemma 2.4.8 implies that there exists an integer D such that $\nabla_k(\sigma)(\mathbb{W}_k) \subset \frac{1}{p\mathrm{Hdg}^D}\mathbb{W}_{k+2\sigma}$ for all σ. Let $\nabla(\sigma)\colon \mathbb{W} \to \frac{1}{p\mathrm{Hdg}^D}\mathbb{W}$ be the map defined by $\nabla(\sigma)|_{\mathbb{W}_k} = \nabla_k(\sigma)$. In particular, for all $N \geq 1$,

$$(\nabla(\sigma)^{p^{f_\sigma}-1} - \mathrm{id})^N(\mathbb{W}_k) \subset \frac{1}{(p\mathrm{Hdg}^D)^{(p^{f_\sigma}-1)N}}\mathbb{W}_k(\sigma).$$

Lemma 2.4.9. *There exists an integer ℓ depending on r, n and p, and an integer $C > 0$, such that for any $g \in H^0(\bar{\mathfrak{M}}_{r,p,[0,1]}, \mathbb{W}_k)^{U=0}$, and every positive integer N, we have*

$$\left(\nabla(\sigma)^{p^{f_\sigma}-1} - \mathrm{id}\right)^N(g) \in \left(\frac{p}{\mathrm{Hdg}^C}\right)^N H^0(\bar{\mathfrak{M}}_{\ell,p,[0,1]}, \mathbb{W}) \cap H^0(\bar{\mathfrak{M}}_{\ell,p,[0,1]}, \mathbb{W}_k(\sigma)).$$

Proof. By Corollary 2.4.3 we see that

$$\left(\nabla(\sigma)^{p^{f_\sigma}-1} - \mathrm{id}\right)^N(g)|_{\bar{\mathfrak{M}}^{\mathrm{ord}}_{p,[0,1]}} \in p^N H^0(\bar{\mathfrak{M}}^{\mathrm{ord}}_{p,[0,1]}, \mathbb{W}) \cap H^0(\bar{\mathfrak{M}}^{\mathrm{ord}}_{p,[0,1]}, \mathbb{W}_k(\sigma)).$$

Locally on $\bar{\mathfrak{M}}_{r,p,[0,1]}$, we then have that

$$(p\mathrm{Hdg}^D)^{(p^{f_\sigma}-1)N}\left(\nabla(\sigma)^{p^{f_\sigma}-1} - \mathrm{id}\right)^N(g) \in \ker\left(\mathbb{W}/(p^{p^{f_\sigma}N}) \to \mathbb{W}^{\mathrm{ord}}/(p^{p^{f_\sigma}N})\right).$$

Here $\mathbb{W}^{\mathrm{ord}} = \mathbb{W}|_{\bar{\mathfrak{M}}^{\mathrm{ord}}_{p,[0,1]}}$. By Corollary 2.3.4 $\mathbb{W}'/(p^j)$ is a polynomial algebra over $\mathcal{O}_{\mathfrak{IG}_{n,r,I}}/(p^j)$ for any j. Since $\mathbb{W} = \mathbb{W}^0 \otimes \mathfrak{F}$, we first deal with $\mathbb{W}^0$. Here we see by Lemma 2.4.7 the kernel of $\mathbb{W}^0/(p^{p^{f_\sigma}N}) \to \mathbb{W}^{0,\mathrm{ord}}/(p^{p^{f_\sigma}N})$ is killed by $\mathrm{Hdg}^{p^{f_\sigma}N(p^{r+1})+C_n}$. By the same lemma $\ker\left(\mathfrak{F}/(p^{p^{f_\sigma}N}) \to \mathfrak{F}^{\mathrm{ord}}/(p^{p^{f_\sigma}N})\right)$ is killed by $\mathrm{Hdg}^{p^{f_\sigma}N(p^{r+1})+C_2}$. Therefore

$$p^{(p^{f_\sigma}-1)N}\mathrm{Hdg}^{N(2p^{f_\sigma}(p^{r+1})+D(p^{f_\sigma}-1)+C_n+C_2}\left(\nabla(\sigma)^{p^{f_\sigma}-1} - \mathrm{id}\right)^N(g) \in p^{p^{f_\sigma}N}H^0(\bar{\mathfrak{M}}_{r,\alpha,I}, \mathbb{W}).$$

In particular, choosing $C \gg 0$, such that $CN \geq N(2p^{f_\sigma}(p^{r+1}) + D(p^{f_\sigma}-1) + C_n + C_2$ for all $N > 0$, we see that

$$\mathrm{Hdg}^{CN}\left(\nabla(\sigma)^{p^{f_\sigma}-1} - \mathrm{id}\right)^N(g) \in p^N H^0(\bar{\mathfrak{M}}_{r,\alpha,I}, \mathbb{W}).$$

Choosing $\ell \geq r$ such that $p/\mathrm{Hdg}^C \in \mathcal{O}_{\mathfrak{X}_{\ell,I}}$, we get that

$$\left(\nabla(\sigma)^{p^{f_\sigma}-1} - \mathrm{id}\right)^N(g) \in \left(\frac{p}{\mathrm{Hdg}^C}\right)^N H^0(\bar{\mathfrak{M}}_{\ell,\alpha,I}, \mathbb{W}).$$

□

Proposition 2.4.2. *Let k, s be as in Assumption 2.1. Then for any prime $p \geq 3$, there exists an integer ℓ depending on r, n and p such that for every $g \in H^0(\bar{\mathfrak{M}}_{r,\alpha,I}, \mathbb{W}_k)^{U=0}$ the sequences in m*

$$A(g, s_\sigma)_m := \sum_{j=1}^{m} (-1)^{j-1} \frac{(\nabla(\sigma)^{p^{f_\sigma}-1} - id)^j(g)}{j}$$

and if we write $H_{i,m}$ for the set of tuples $(j_1, \dots, j_i)$ of i positive integers with $j_1 + \cdots + j_i \leq m$,

$$B(g, s_\sigma)_m := \sum_{i=0}^{m} \frac{v_\sigma^i}{i!(p^{f_\sigma}-1)^i} \left(\sum_{(j_1,\dots,j_i) \in H_{i,m}} \Big(\prod_{a=1}^{i} \frac{(-1)^{j_a-1}}{j_a} \Big) (\nabla(\sigma)^{p^{f_\sigma}-1} - id)^{j_1+\cdots+j_i} \right)(g)$$

converge in $H^0(\bar{\mathfrak{M}}_{\ell,\alpha,I}, \mathbb{W})$. Moreover, if we denote the limits

$$\log\Big(\nabla(\sigma)^{p-1}\Big)(g) := \lim_{m\to\infty} A(g, s_\sigma)_m$$

and

$$\nabla(\sigma)^{s_\sigma}(g) = \exp\left(\frac{v_\sigma}{p^{f_\sigma}-1} \log\left(\nabla(\sigma)^{p^{f_\sigma}-1}\right)\right)(g) := \lim_{m\to\infty} B(g, s_\sigma)_m$$

then $\nabla(\sigma)^{s_\sigma}(g) \in H^0(\bar{\mathfrak{M}}_{\ell,\alpha,I}, \mathbb{W}_{k+2s_\sigma})$. The same results hold for $p = 2$ if $v_\sigma \in 4\Lambda^0_{\alpha,I}$.

Proof. The convergence of $A(g, s_\sigma)_m$ is clear from Lemma 2.4.9. We prove convergence for $B(g, s_\sigma)_m$. Let's first deal with the case $p \geq 3$. Let

$$X := \frac{(\nabla(\sigma)^{p^{f_\sigma}-1} - \mathrm{id})^{j_1+\cdots+j_i}(g)}{i! \prod j_a}.$$

Then by Lemma 2.4.9, $X \in (p/\mathrm{Hdg}^C)^{\sum j_a - v_p(i!) - \sum v_p(j_a)} H^0(\bar{\mathfrak{M}}_{\ell,\alpha,I}, \mathbb{W})$. Now $v_p(i!) \leq \frac{i-1}{p-1} \leq \frac{i}{p-1}$. Hence $v_p(j_a) \leq \frac{j_a-1}{p-1}$ too. Using these inequalities,

$$\sum_{a=1}^{i} j_a - v_p(i!) - \sum_{a=1}^{i} v_p(j_a) \geq \sum_{a=1}^{i} \Big(j_a - \frac{1}{p-1} - v_p(j_a) \Big) \geq \sum_{a=1}^{i} j_a \left(1 - \frac{1}{p-1}\right).$$

This proves convergence in this case. For the case $p = 2$ we note that the terms $\frac{(\nabla(\sigma)^{2^{f_\sigma}-1} - \mathrm{id})^{j_1+\cdots+j_i}(g)}{\prod j_a}$ do not have poles and the term $v_\sigma^i/i!$ is divisible by 2^i, which gives convergence in this case. Finally, $\nabla(\sigma)^{s_\sigma}(g) \in H^0(\bar{\mathfrak{M}}_{\ell,\alpha,I}, \mathbb{W}_{k+2s_\sigma})$ as can be seen from its expansion as a power series and the fact that $t * \nabla(\sigma)(g) = (k + 2\sigma)(t)\nabla(\sigma)(g)$. □

Thus given any $g \in H^0(\bar{\mathfrak{M}}_{r,\alpha,I}, \mathbb{W}_k)^{U=0}$, there exists a large enough ℓ depending on r, n and p such that one can consider $\prod_\sigma \nabla(\sigma)^{s_\sigma}(g)$ as an element of $H^0(\bar{\mathfrak{M}}_{\ell,\alpha,I}, \mathbb{W}_{k+2s})$. Here $\prod_\sigma \nabla(\sigma)^{s_\sigma}$ means the composition of the different $\nabla(\sigma)^{s_\sigma}$'s in any order. Note the order of composition does not matter since they mutually commute by Lemma 2.4.4. Thus we fix such an ℓ and define the following.

Definition 2.4.3. For $s\colon \mathbb{T}(\mathbb{Z}_p) \to (\Lambda^0_{\alpha,I})^\times$ as in Proposition 2.4.2 and k as in Assumption 2.1, define $\nabla^s(g)$ for $g \in H^0(\bar{\mathfrak{M}}_{r,p,[0,1]}, \mathbb{W}_k)^{U=0}$ to be $\prod_\sigma \nabla(\sigma)^{s_\sigma}(g) \in H^0(\bar{\mathfrak{M}}_{\ell,p,[0,1]}, \mathbb{W}_{k+2s})$ for some ℓ for which the expression makes sense by Proposition 2.4.2.

Chapter 3

Hecke operators and overconvergent projection

In this chapter we always assume that $\alpha = p$ and $I = [0,1]$.

3.1 The U and V operators

Let $p_1 \colon \mathfrak{X}_{r+1,\alpha,I} \to \mathfrak{X}_{r,\alpha,I}$ and $p_2 \colon \mathfrak{X}_{r+1,\alpha,I} \to \mathfrak{X}_{r,\alpha,I}$ be the two maps defined on the generic fibre by sending $\mathcal{A} \mapsto \mathcal{A}$ and $\mathcal{A} \mapsto \mathcal{A}/H_1$ respectively. The map p_2 is the one denoted $\tilde{F}$ in §2.3.1.1. The quotient $\lambda \colon \mathcal{A} \to \mathcal{A}/H_1 =: \mathcal{A}'$ induces via the unique isogeny $\lambda' \colon \mathcal{A}' \to \mathcal{A}$ such that $\lambda' \circ \lambda = [p]$ an isomorphism $H_n(\mathcal{A}') \simeq H_n(\mathcal{A})$ on the generic fibre. Hence they induce an isomorphism on the duals of the canonical subgroups, and the functoriality of the dlog map gives an isomorphism $p_1^*\Omega_{\mathcal{A}} \simeq p_2^*\Omega_{\mathcal{A}}$ of $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{IG}_{n,r+1,I}}$ invertible modules. This isomorphism extends to a map $p_1^*\mathrm{H}^\sharp_{\mathcal{A}} \to p_2^*\mathrm{H}^\sharp_{\mathcal{A}}$ that respects the marked section and marked splitting by Proposition 2.3.2. The map p_2 is finite flat of rank p^g on generic fibres and so induces a trace map $\mathrm{Tr} \colon p_{2*}\mathcal{O}_{\mathfrak{X}_{r+1,\alpha,I}} \to \mathcal{O}_{\mathfrak{X}_{r,\alpha,I}}$ on the formal models by normality. The maps p_1, p_2 have obvious lifts to maps $p_1 \colon \mathfrak{IG}_{n,r+1,I} \to \mathfrak{IG}_{n,r,I}$ and $p_2 \colon \mathfrak{IG}_{n,r+1,I} \to \mathfrak{IG}_{n,r,I}$.

Lemma 3.1.1. *There is a morphism $\mathcal{U} \colon p_{2*}p_1^*\mathbb{W}^0_{k,\alpha,I} \to p_{2*}p_2^*\mathbb{W}^0_{k,\alpha,I}$ of $\mathcal{O}_{\mathfrak{X}_{r,\alpha,I}}$-modules induced by the isogeny λ' which is an isomorphism on the modular sheaf $\mathfrak{w}^0_{k,\alpha,I}$ and which preserves the filtration and commutes with the Gauss-Manin connection. We also have a morphism $\mathcal{U} \colon p_{2*}p_1^*\mathbb{W}_{k,\alpha,I} \to p_{2*}p_2^*\mathbb{W}_{k,\alpha,I}$ of $\mathcal{O}_{\bar{\mathfrak{M}}_{r,\alpha,I}}$-modules satisfying the same properties as above. Moreover, the induced map on the m-graded pieces is 0 modulo $(p/\mathrm{Hdg}^{p+1})^m$.*

Proof. The first claim follows simply from the definition of $\mathbb{W}^0_{k,\alpha,I}$ and the morphism $p_1^*\mathrm{H}^\sharp_{\mathcal{A}} \to p_2^*\mathrm{H}^\sharp_{\mathcal{A}}$ defined above. Considering the $\mathcal{U}$-correspondence on $\mathfrak{w}^\chi_{k,\alpha,I}$ we get the required map for $\mathbb{W}_{k,\alpha,I}$. For the last claim we observe that by Proposition 2.3.2, the induced map $\mathrm{H}^\sharp_{\mathcal{A}}/\Omega_{\mathcal{A}} \to \mathrm{H}^\sharp_{\mathcal{A}'}/\Omega_{\mathcal{A}'}$ is multiplication by $p/HW(\sigma)^{p+1}$ on the σ-component. The claim then follows from the local description of $\mathbb{W}^0_{k,\alpha,I}$. $\square$

Definition 3.1.1. Define the U operator as the composition

$$U\colon H^0(\bar{\mathfrak{M}}_{r,\alpha,I},\mathbb{W}_{k,\alpha,I}) \xrightarrow{p_{2*}\circ\mathcal{U}\circ p_1^*} H^0(\bar{\mathfrak{M}}_{r,\alpha,I},p_{2*}p_2^*\mathbb{W}_{k,\alpha,I}) \xrightarrow{\frac{1}{p^g}\mathrm{Tr}} H^0(\bar{\mathfrak{M}}_{r,\alpha,I},\mathbb{W}_{k,\alpha,I})[1/p].$$

Corollary 3.1.1. *For any $g\in H^0(\bar{\mathfrak{M}}_{r,\alpha,I},\mathfrak{w}_{k,\alpha,I})$ with q-expansion $g=\sum_{\mathfrak{ab}} a_\beta q^\beta$ at a cusp $\Lambda^0_{\alpha,I}((\mathfrak{ab},S))$, $U(g)=\sum_{\mathfrak{ab}} a_{p\beta}q^\beta$ on q-expansions.*

As noted above the isomorphism $(\lambda')^*\colon p_1^*\Omega_{\mathcal{A}} \xrightarrow{\sim} p_2^*\Omega_{\mathcal{A}}$ induces an isomorphism $\mathcal{U}\colon p_1^*\mathfrak{w}_{k,\alpha,I} \xrightarrow{\sim} p_2^*\mathfrak{w}_{k,\alpha,I}$.

Definition 3.1.2. Define the V operator as the map

$$V\colon H^0(\bar{\mathfrak{M}}_{r,\alpha,I},\mathfrak{w}_{k,\alpha,I}) \xrightarrow{\mathcal{U}^{-1}\circ p_2^*} H^0(\bar{\mathfrak{M}}_{r+1,\alpha,I},\mathfrak{w}_{k,\alpha,I}).$$

Corollary 3.1.2. *For any $g\in H^0(\bar{\mathfrak{M}}_{r,\alpha,I},\mathfrak{w}_{k,\alpha,I})$ with q-expansion $g=\sum_{\mathfrak{ab}} a_\beta q^\beta$ at a cusp $\Lambda^0_{\alpha,I}((\mathfrak{ab},S))$, $V(g)=\sum_{\mathfrak{ab}} a_\beta q^{p\beta}$ on q-expansions.*

Corollary 3.1.3. *1. $U\circ V = id$ on $H^0(\bar{\mathfrak{M}}_{r,\alpha,I},\mathfrak{w}_{k,\alpha,I})$.*

2. For $g\in H^0(\bar{\mathfrak{M}}_{r,\alpha,I},\mathfrak{w}_{k,\alpha,I})$, if we denote by $g^{[p]} := (id - V\circ U)(g)$ the p-depletion of g, then $U(g^{[p]})=0$. Moreover if $g=\sum_{\mathfrak{ab}} a_\beta q^\beta$, then $g^{[p]}=\sum_{p\nmid\beta} a_\beta q^\beta$.

Proposition 3.1.1. *For every non-negative rational h, the $\Lambda_{\alpha,I}[1/\alpha]$-Banach module $H^0(\bar{\mathcal{M}}_{r,\alpha,I},\mathbb{W}_{k,\alpha,I})$ admits a slope h decomposition which restricts to a slope h decomposition on $H^0(\bar{\mathcal{M}}_{r,\alpha,I},\mathrm{Fil}_n\mathbb{W}_{k,\alpha,I})$ for all $n\in\mathbb{N}$. Moreover, the inclusion $H^0(\bar{\mathcal{M}}_{r,\alpha,I},\mathrm{Fil}_n\mathbb{W}_{k,\alpha,I})^{\le h}\subset H^0(\bar{\mathcal{M}}_{r,\alpha,I},\mathbb{W}_{k,\alpha,I})^{\le h}$ is an isomorphism for n large enough (depending on h).*

Proof. The operator U is compact on the coherent sheaf $\mathrm{Fil}_n\mathbb{W}_{k,\alpha,I}$, and so by the usual formalism of slope decomposition we have locally on the weight space a slope h decomposition

$$H^0(\bar{\mathcal{M}}_{r,\alpha,I},\mathrm{Fil}_n\mathbb{W}_{k,\alpha,I}) = H^0(\bar{\mathcal{M}}_{r,\alpha,I},\mathrm{Fil}_n\mathbb{W}_{k,\alpha,I})^{\le h}\oplus H^0(\bar{\mathcal{M}}_{r,\alpha,I},\mathrm{Fil}_n\mathbb{W}_{k,\alpha,I})^{>h}.$$

By Lemma 3.1.1, the U operator on $H^0(\bar{\mathcal{M}}_{r,\alpha,I},\mathbb{W}_{k,\alpha,I}/\mathrm{Fil}_n\mathbb{W}_{k,\alpha,I})$ is divisible by p^{h+1} for n large enough. It follows that $H^0(\bar{\mathcal{M}}_{r,\alpha,I},\mathbb{W}_{k,\alpha,I}/\mathrm{Fil}_n\mathbb{W}_{k,\alpha,I})$ also admits a slope h decomposition and that $H^0(\bar{\mathcal{M}}_{r,\alpha,I},\mathbb{W}_{k,\alpha,I}/\mathrm{Fil}_n\mathbb{W}_{k,\alpha,I})^{\le h}=0$. □

3.2 Hecke operators

In order to define Hecke operators we work over the non-compactified moduli scheme $M(\mu_N,\mathfrak{c})$ instead of the toroidal compactification to avoid problems of finding toroidal compactifications stable under the correspondences. Instead we will use Koecher principle to extend these operators to the cusps.

Let $\mathfrak{M}^{\mathfrak{c}}_{r,\alpha,I}\subset\bar{\mathfrak{M}}_{r,\alpha,I}$ be the inverse image of $M(\mu_N,\mathfrak{c})\subset\bar{M}(\mu_N,\mathfrak{c})$ under the projection $\bar{\mathfrak{M}}_{r,\alpha,I}\to\bar{M}(\mu_N,\mathfrak{c})$, and let $\mathcal{M}^{\mathfrak{c}}_{r,\alpha,I}$ be its generic fibre. Let $\ell\subset\mathcal{O}_L$ be an ideal. If ℓ divides pN, assume it is a prime. Let $\mathcal{Y} \subset \mathcal{M}^{\mathfrak{c}}_{r,\alpha,I} \times \mathcal{M}^{\ell\mathfrak{c}}_{r,\alpha,I}$ be the subspace classifying pairs $(A, \iota, \lambda, \psi)$ and $(A', \iota', \lambda', \psi')$, together with an isogeny $\pi_\ell \colon A \to A'$ compatible with $\iota, \iota', \lambda, \lambda', \psi, \psi'$, such that $\ker \pi_\ell$ is étale locally isomorphic to $\mathcal{O}_L/\ell\mathcal{O}_L$, $\ker \pi_\ell \cap \mathrm{Im}\psi = \{0\}$ and $\ker \pi_\ell \cap H_1 = \{0\}$ where H_1 is the canonical subgroup of level 1 of A. Let $p_1 \colon \mathcal{Y} \to \mathcal{M}^{\mathfrak{c}}_{r,\alpha,I}$ and $p_2 \colon \mathcal{Y} \to \mathcal{M}^{\ell\mathfrak{c}}_{r,\alpha,I}$ be the two projections defining the Hecke correspondence. Let $\mathfrak{Y} \subset \mathfrak{M}^{\mathfrak{c}}_{r,\alpha,I} \times \mathfrak{M}^{\ell\mathfrak{c}}_{r,\alpha,I}$ be the normalization of the Hecke correspondence. We note that if ℓ is prime to pN, then p_1 is finite étale of degree $q_\ell + 1 := \mathrm{N}_{L/\mathbb{Q}}(\ell) + 1$, and otherwise it is finite flat of degree $q_\ell := \mathrm{N}_{L/\mathbb{Q}}(\ell)$.

Lemma 3.2.1. *The universal isogeny π_ℓ defines an isomorphism $\pi_\ell^* \colon p_2^*\mathfrak{w}_{k,\alpha,I}[1/p] \xrightarrow{\sim} p_1^*\mathfrak{w}_{k,\alpha,I}[1/p]$ of invertible sheaves over the generic fibre.*

Proof. [AIP16b, Corollary 8.6]. □

We now define the Hecke operator T_ℓ. By abuse of notation let g be the structural maps for both $\bar{\mathcal{M}}^{\mathfrak{c}}_{r,\alpha,I}$ and $\bar{\mathcal{M}}^{\ell\mathfrak{c}}_{r,\alpha,I}$ to the weight space $\mathcal{W}_p$ and let g_0 be their restriction to the opens $\mathcal{M}^{\mathfrak{c}}_{r,\alpha,I}$ and $\mathcal{M}^{\ell\mathfrak{c}}_{r,\alpha,I}$ respectively. Then Koecher's principle tells us that $g_{0*}\mathfrak{w}_{k,\alpha,I} \simeq g_*\mathfrak{w}_{k,\alpha,I}$ [AIP16b, Proposition 8.4].

Definition 3.2.1. Define the Hecke operator T_ℓ for ℓ as above as the map from an invertible sheaf over $\bar{\mathcal{M}}^{\ell\mathfrak{c}}_{r,\alpha,I}$ to an invertible sheaf over $\bar{\mathcal{M}}^{\mathfrak{c}}_{r,\alpha,I}$

$$g_*\mathfrak{w}_{k,\alpha,I}[1/p] \to g_*(p_{1*}p_2^*\mathfrak{w}_{k,\alpha,I}[1/p]) \xrightarrow{\pi_\ell^*} g_*(p_{1*}p_1^*\mathfrak{w}_{k,\alpha,I}[1/p]) \xrightarrow{\frac{1}{q_\ell}\mathrm{Tr}} g_*\mathfrak{w}_{k,\alpha,I}[1/p].$$

For ℓ an ideal prime to pN, define a map $S_\ell \colon \mathfrak{M}^{\mathfrak{c}}_{r,\alpha,I} \to \mathfrak{M}^{\mathfrak{c}\ell^2}_{r,\alpha,I}$ as the normalization of the map induced on generic fibres by sending $A \mapsto A \otimes \ell^{-1}$ together with the induced real multiplication, polarization and level structure. As before it is easy to see that there is a morphism $\pi_\ell^* \colon S_\ell^*\mathfrak{w}^{\mathfrak{c}\ell^2}_{k,\alpha,I}[1/p] \to \mathfrak{w}^{\mathfrak{c}}_{k,\alpha,I}[1/p]$. The Hecke operator S_ℓ is defined as

$$S_\ell \colon g_*\mathfrak{w}^{\mathfrak{c}\ell^2}_{k,\alpha,I}[1/p] \xrightarrow{S_\ell^*} g_*S_\ell^*\mathfrak{w}^{\mathfrak{c}\ell^2}_{k,\alpha,I}[1/p] \xrightarrow{\frac{1}{q_\ell^2}\pi_\ell^*} g_*\mathfrak{w}^{\mathfrak{c}}_{k,\alpha,I}[1/p].$$

The Hecke operators induce maps on the sheaf of arithmetic Hilbert modular forms, which we still denote by $T_\ell \colon \mathfrak{w}^G_{k,\alpha,I}[1/p] \to \mathfrak{w}^G_{k,\alpha,I}[1/p]$ for $\ell \nmid pN$ and similarly for S_ℓ. For $\ell = \mathfrak{P}_i$, where we recall $\mathfrak{P}_i$ is a prime above p, choose an $x_i \in L^{\times,+}$ such that $v_{\mathfrak{P}_i}(x_i) = 1$ and $v_{\mathfrak{P}_j}(x_i) = 0$ for $j \neq i$. Then multiplication by x_i induces a positive isomorphism $M(\mu_N, \mathfrak{P}_i x_i^{-1}\mathfrak{c}) \xrightarrow{\sim} M(\mu_N, \mathfrak{P}_i\mathfrak{c})$. We let $U_{\mathfrak{P}_i} \colon \mathfrak{w}^{G,\mathfrak{c}}_{k,\alpha,I}[1/p] \to \mathfrak{w}^{G,\mathfrak{P}_i x_i^{-1}\mathfrak{c}}_{k,\alpha,I}[1/p]$ be the induced map. Then the T_ℓ and $U_{\mathfrak{P}_i}$ define operators on $\mathfrak{w}^G_{k,\alpha,I}[1/p]$. We warn that although $\prod_i \mathfrak{P}_i = (p)$, $\prod_i U_{\mathfrak{P}_i} = v_{\mathrm{un}}(p\prod x_i^{-1})U$ where we recall $k^G_{\mathrm{un}} = (v_{\mathrm{un}}, w_{\mathrm{un}})$ is the universal weight on $\mathcal{W}^G_p$.

For $a \in (\mathcal{O}_L/N)^\times$, define the operator $T(a,1)$ as the map induced by the action of $(\mathcal{O}_L/N)^\times$ on the level structure.

Definition 3.2.2. Define the Hecke algebra $\mathcal{H}$ as the Λ^G_p-subalgebra of $\mathrm{End}(\mathfrak{w}^G_{k,\alpha,I}(-D)[1/p])$ generated by the T_ℓ, S_ℓ for $\ell \nmid pN$, U_ℓ for $\ell | pN$ and $T(a,1)$ for $a \in (\mathcal{O}_L/N)^\times$.

3.2.1 Adelic q-expansion

Next we turn our attention to classical complex Hilbert modular forms. Let $\mathfrak{N}$ be an ideal deep enough such that the moduli of $\mathfrak{c}$-polarized abelian varieties with $\mu_{\mathfrak{N}}$-level structure is representable [Hid04, §4.1.2]. Consider the following compact open subgroups.

$$K_1(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\hat{\mathcal{O}}_F) \mid a \equiv 1 \bmod \mathfrak{N}\hat{\mathcal{O}}_F \right\}$$

$$K_{11}(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\hat{\mathcal{O}}_F) \mid a \equiv d \equiv 1 \bmod \mathfrak{N}\hat{\mathcal{O}}_F \right\}.$$

Let us briefly discuss the Shimura varieties associated to the group G at these levels.

In §2.1.2.1 we already showed that $\mathrm{Sh}_{K_1(N}(G)(\mathbb{C})$ is a disjoint union

$$\mathrm{Sh}_{K_1(N)}(G) = \bigsqcup_{[\mathfrak{c}]\in \mathrm{Cl}_L^+} \mathrm{Sh}^{\mathfrak{c}}_{K_1(N)}(G)(\mathbb{C})$$

where each connected component $\mathrm{Sh}^{\mathfrak{c}}_{K_1(N)}(G)(\mathbb{C})$ is the étale quotient of the moduli scheme $M(\mu_N, \mathfrak{c})$ by the action of $\Gamma = \mathcal{O}_L^{\times,+}/U_N^2$. The same result is true for N replaced by any $\mathfrak{N}$ deep enough.

Let $\mathrm{Cl}_L^+(\mathfrak{N})$ be the strict ray class group modulo N. Then [Hid04, §4.1.3] shows that $\mathrm{Sh}_{K_{11}(\mathfrak{N})}(G)(\mathbb{C})$ is a disjoint union $\mathrm{Sh}_{K_{11}(\mathfrak{N})}(G)(\mathbb{C}) = \bigsqcup_{[\mathfrak{c}]\in \mathrm{Cl}_L^+(\mathfrak{N})} \mathrm{Sh}^{\mathfrak{c}}_{K_{11}(\mathfrak{N})}(G)(\mathbb{C})$ of connected components indexed by $\mathrm{Cl}_L^+(\mathfrak{N})$. Here each connected component $\mathrm{Sh}^{\mathfrak{c}}_{K_{11}(\mathfrak{N})}(G)(\mathbb{C})$ is an étale quotient of $M(\mu_{\mathfrak{N}}, \mathfrak{c})$ by $(\det K_{11}(\mathfrak{N}) \cap \mathcal{O}_L^{\times,+})/(K_{11}(\mathfrak{N}) \cap \mathcal{O}_L^{\times})^2$. Therefore the projection $\mathrm{Sh}^{\mathfrak{c}}_{K_{11}(\mathfrak{N})}(G)(\mathbb{C}) \to \mathrm{Sh}^{\mathfrak{c}}_{K_1(\mathfrak{N})}(G)(\mathbb{C})$ is an étale Galois quotient under the action of the group $\mathcal{O}_L^{\times,+}/(\det K_{11}(\mathfrak{N}) \cap \mathcal{O}_L^{\times,+})$.

The $\mathfrak{c}$-polarized Hilbert modular forms of weight (v, n) and level $K_{11}(\mathfrak{N})$ can then be realized as the sections $H^0(M(\mu_{\mathfrak{N}}, \mathfrak{c}), \omega_{\mathcal{A}}^k)^{\Gamma'}$ for $k = 2v + nt_L$, and $\Gamma' = (\det K_{11}(\mathfrak{N}) \cap \mathcal{O}_L^{\times,+})/(K_{11}(\mathfrak{N}) \cap \mathcal{O}_L^{\times})^2$.

Viewed as an automorphic form, a Hilbert modular cuspform of weight (v, n) and level $K_{11}(\mathfrak{N})$ is a function $f\colon G(\mathbb{A}) \to \mathbb{C}$ satisfying a bunch of properties that we list below. First note that choosing a square root $i \in \mathbb{C}$ of -1, we have an identification $\mathbb{H}^g \simeq G(\mathbb{R})^+/C_\infty^+$, where by $G(\mathbb{R})^+$ we mean the connected component of $1 \in G(\mathbb{R})$, and C_∞^+ is the stabilizer of $\mathbf{i} = (i, \dots, i) \in \mathbb{H}$ for the action of $G(\mathbb{R})^+$ via Möbius transformations. Then the cuspform f satisfies:

1. $f(axu) = f(x)j(u_\infty, \mathbf{i})^{-1}$ for $a \in G(\mathbb{Q})$, $u \in K_{11}(\mathfrak{N})C_\infty^+$, and the automorphy factor is $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z) = (ad - bc)^{-v}(cz + d)^k$ for $k = 2v + nt_L$, $z \in \mathbb{H}^g$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbb{R})$.
2. For every finite adelic point $x \in G(\mathbb{A}^\infty)$, the well-defined function $f_x\colon \mathbb{H}^g \to \mathbb{C}$ defined as $f_x(z) = f(xu_\infty)j(u_\infty, \mathbf{i})$ is holomorphic, where we choose $u_\infty \in G(\mathbb{R})^+$ such that $u_\infty \mathbf{i} = z$.
3. For all adelic points $x \in G(\mathbb{A})$ and for all additive measures on $F\backslash\mathbb{A}_F$, we have

$$\int_{F\backslash\mathbb{A}_F} f\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} x\right) da = 0.$$

Let $a_1, \ldots, a_{h^+(\mathfrak{N})} \in \mathbb{A}_L^{\infty,\times}$ be representatives of $\mathrm{Cl}_L^+(\mathfrak{N})$ and assume that a_i is coprime to $p\mathfrak{N}$ for all i. Let a_i be the ideal generated by a_i. Then we have the following decomposition.

$$G(\mathbb{A}) = \bigsqcup_{i=1}^{h^+(\mathfrak{N})} G(\mathbb{Q}) t_i K_{11}(\mathfrak{N}) G(\mathbb{R})^+, \quad t_i = \begin{pmatrix} a_i^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Fix a finite extension L_0 of L^{Gal} such that for any ideal $\mathfrak{a} \subset \mathcal{O}_L$, and any embedding $\tau \in \Sigma$, $\mathfrak{a}^\tau \mathcal{O}_{L_0}$ is principal. Choose a generator $\{\mathfrak{q}^\tau\} \in \mathcal{O}_{L_0}$ of $\mathfrak{q}^\tau \mathcal{O}_{L_0}$ for each prime ideal $\mathfrak{q}$, and extend it to all fractional ideals multiplicatively. Fix an idele $\mathrm{d} \in \mathbb{A}_F^{\infty,\times}$ whose ideal is the different $\mathfrak{d}$ of $L/\mathbb{Q}$.

Given a Hilbert cuspform f of weight (v, n) and level $K_{11}(\mathfrak{N})$, the discussion above implies that f corresponds to a tuple $(f_1, \ldots, f_{h^+(\mathfrak{N})})$ where f_i is an $\mathrm{a}_i^{-1}\mathfrak{d}$-polarized Hilbert modular form of weight (v, n) and $\mu_{\mathfrak{N}}$-level structure. Then the holomorphic function $f_i \colon \mathbb{H}^g \to \mathbb{C}$ has a Fourier expansion

$$f_i(z) = y_\infty^{-v} f\left(\begin{pmatrix} y_\infty & x_\infty \\ 0 & 1 \end{pmatrix} t_i\right) = \sum_{\xi \in (\mathrm{a}_i \mathfrak{d}^{-1})_+} a(\xi, f_i) e_L(\xi z).$$

Here $z = x_\infty + \mathbf{i} y_\infty$ and $e_L(\xi z) = \exp(2\pi i \sum_{\tau \in \Sigma} \tau(\xi) z_\tau)$. Every idele $y \in \mathbb{A}_{L,+}^\times = \mathbb{A}_L^{\infty,\times} L_{\infty,+}^\times$ can be written as $y = \xi a_i^{-1} \mathrm{d} u$ for $\xi \in L_+^\times$ and $u \in \det K_{11}(\mathfrak{N}) L_{\infty,+}^\times$. Define two functions $c(\cdot, f) \colon \mathbb{A}_{L,+}^\times \to \mathbb{C}$ and $c_p(\cdot, f) \colon \mathbb{A}_{L,+}^\times \to \bar{\mathbb{Q}}_p$ as follows.

$$c(y, f) := a(\xi, f_i) \{y^{v - t_L}\} \xi^{t_L - v} |a_i|_{\mathbb{A}_L} \qquad c_p(\xi, f) := a(\xi, f_i) y_p^{v - t_L} \xi^{t_L - v} \mathcal{N}_L(a_i)^{-1}$$

if $y \in \hat{\mathcal{O}}_L L_{\infty,+}^\times$ and 0 otherwise. Here $\mathcal{N}_L$ is defined by $y \mapsto y_p^{-t_L} |y^\infty|_{\mathbb{A}_L}^{-1}$. The function $c_p(\cdot, f)$ makes sense only if the coefficients $a(\xi, f_i)$ are algebraic for all i. Moreover, for our choice of the a_i as being coprime to p, we have

$$c_p(y, f) = c(y, f) \{y^{t_L - v}\} y_p^{v - t_L}.$$

Theorem 3.2.1. *Consider the map $e_L \colon \mathbb{C}^\Sigma \to \mathbb{C}^\times$ defined by $e_L(z) = \exp(2\pi i \sum_{\tau \in \Sigma} z_\tau)$ and the unique additive character of the ideles $\chi_L \colon \mathbb{A}_L/L \to \mathbb{C}^\times$ which satisfies $\chi_L(x_\infty) = e_L(x_\infty)$. Each Hilbert cuspform of weight (v, n) has an adelic q-expansion of the form*

$$f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = |y|_{\mathbb{A}_L} \sum_{\xi \in L_+} c(\xi y \mathrm{d}, f) \{(\xi y \mathrm{d})^{t_L - v}\} (\xi y_\infty)^{v - t_L} e_L(\mathbf{i} \xi y_\infty) \chi_L(\xi x)$$

for $y \in \mathbb{A}_{L,+}^\times$, $x \in \mathbb{A}_L^\times$, where $c(\cdot, f) \colon \mathbb{A}_{L,+}^\times \to \mathbb{C}$ vanishes outside $\hat{\mathcal{O}}_L L_{\infty,+} \times$ and depends only on the coset $y^\infty \det K_{11}(\mathfrak{N})$. The adelic q-expansion agrees with the Fourier expansions of the f_i in the following sense.

$$f\left(\begin{pmatrix} y_\infty & x_\infty \\ 0 & 1 \end{pmatrix} t_i\right) = y_\infty^v \sum_{\xi \in (\mathrm{a}_i \mathfrak{d}^{-1})_+} a(\xi, f_i) e_L(\xi z).$$

Proof. [Hid91, Theorem 1.1]. ☐

In [Hid91], Hida uses adelic q-expansions to define Hecke operators on the space of cuspforms. These are defined in the following way. Let ϖ be an uniformizer of the localization of $\mathcal{O}_L$ at a prime $\mathfrak{q}$. Then define the double coset operators

$$T_0(\varpi) = \{\varpi^{v-t_L}\}\left[K_{11}(\mathfrak{N})\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix}K_{11}(\mathfrak{N})\right] \quad \text{if } \mathfrak{q}\nmid\mathfrak{N}$$

$$U_0(\varpi) = \{\varpi^{v-t_L}\}\left[K_{11}(\mathfrak{N})\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix}K_{11}(\mathfrak{N})\right] \quad \text{if } \mathfrak{q}|\mathfrak{N}$$

and for $a \in \mathcal{O}_{L,\mathfrak{N}}^{\times} = \prod_{\mathfrak{q}|\mathfrak{N}} \mathcal{O}_{L,\mathfrak{q}}^{\times}$, the double coset operator

$$T(a,1) = \left[K_{11}(\mathfrak{N})\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}K_{11}(\mathfrak{N})\right].$$

If the prime $\mathfrak{q}$ is coprime to the level, then $T_0(\varpi)$ is independent of the choice of ϖ and we simply denote it as $T_0(\mathfrak{q})$. For any finite adelic point $z \in \mathbb{A}_L^{\infty,\times}$, we define the diamond operator associated to it by $f_{|\langle z\rangle}(x) = f(xz)$ where z acts through the embedding of $\mathbb{A}_L^{\infty,\times}$ into the center of $G(\mathbb{A}^\infty)$.

Remark 3.2.1. The Hecke operators defined by Hida as above should match with the ones defined earlier in the section using Hecke correspondences. But unfortunately we cannot confirm this yet.

The Hecke algebra $\mathcal{H}_0(A)$ is defined to be the A-subalgebra of $\mathrm{End}_{\mathbb{C}}(S(K_{11}(\mathfrak{N}),(v,n),\mathbb{C})$ generated by the $T_0(\mathfrak{q})$ for $\mathfrak{q}\nmid\mathfrak{N}$, $U_0(\varpi)$ for primes dividing $\mathfrak{N}$, $T(a,1)$ for $a\in\mathcal{O}_{L,\mathfrak{N}}^{\times}$ and the diamond operators. Here $S(K_{11}(\mathfrak{N},(v,n),\mathbb{C})$ is the space of cuspforms of weight (v,n) and level $K_{11}(\mathfrak{N})$ with coefficients in $\mathbb{C}$.

Theorem 3.2.2. *For any finite extension F/L^{Gal} and any $\mathcal{O}_{L_0}$ subalgebra A of F, there is a natural isomorphism $S(K_{11}(\mathfrak{N},(v,n),F) \simeq S(K_{11}(\mathfrak{N},(v,n),A)\otimes_A F$. Moreover, if A is an integrally closed domain containing $\mathcal{O}_{L_0}$, finite flat over $\mathcal{O}_L$, then $S(K_{11}(\mathfrak{N}),(v,n),A)$ is stable for the action of $\mathcal{H}_0(A)$, and the pairing of A-modules*

$$\begin{aligned}\mathcal{H}_0(A)\times S(K_{11},(v,n),A) &\to A\\ (h,f) &\mapsto c(1,f_{|h})\end{aligned}$$

is a perfect pairing.

Proof. [Hid91, Theorem 2.2]. □

3.3 Overconvergent projection

In this section we will define the overconvergent projection in families upon studying the cohomology of the complex of $\mathcal{O}_{\bar{\mathcal{M}}_{r,\alpha,I}}$ sheaves obtained by the connection ∇ which is described as follows. In particular in this section all sheaves are considered over the analytic adic spaces. Hence to simplify notation we will still write $\mathbb{W}_{k,\alpha,I}$ but it is to be understood that this is a sheaf over $\bar{\mathcal{M}}_{r,\alpha,I}$. Consider the complex

$$\mathbb{W}_{k,\alpha,I} \xrightarrow{\nabla} \mathbb{W}_{k,\alpha,I}\hat{\otimes}\Omega^1_{\bar{\mathcal{M}}_{r,\alpha,I}/\Lambda_{\alpha,I}} \cdots \xrightarrow{\nabla} \mathbb{W}_{k,\alpha,I}\hat{\otimes}\Omega^g_{\bar{\mathcal{M}}_{r,\alpha,I}/\Lambda_{\alpha,I}}. \tag{3.1}$$

Denote by $\mathbb{W}^\bullet_{k,\alpha,I}$ the complex obtained by tensoring the above complex by $\mathcal{O}_{\bar{\mathcal{M}}_{r,\alpha,I}}(-D)$ where we recall D is the boundary divisor. By Griffiths' transversality we obtain a complex corresponding to the filtration on $\mathbb{W}_{k,\alpha,I}$ as follows.

$$\mathrm{Fil}_n\mathbb{W}_{k,\alpha,I} \xrightarrow{\nabla} \mathrm{Fil}_{n+1}\mathbb{W}_{k,\alpha,I}\hat{\otimes}\Omega^1_{\bar{\mathcal{M}}_{r,\alpha,I}/\Lambda_{\alpha,I}} \cdots \xrightarrow{\nabla} \mathrm{Fil}_{n+g}\mathbb{W}_{k,\alpha,I}\hat{\otimes}\Omega^g_{\bar{\mathcal{M}}_{r,\alpha,I}/\Lambda_{\alpha,I}}. \tag{3.2}$$

Denote by $\mathrm{Fil}^\bullet_n\mathbb{W}_{k,\alpha,I}$ the complex obtained by tensoring the above complex with $\mathcal{O}_{\bar{\mathcal{M}}_{r,\alpha,I}}(-D)$. By taking quotient of the first complex by the second we obtain a third complex $(\mathbb{W}_{k,\alpha,I}/\mathrm{Fil}_n\mathbb{W}_{k,\alpha,I})^\bullet$ which sits in a short exact sequence of complexes on $\bar{\mathcal{M}}_{r,\alpha,I}$ that gives a long exact sequence of hypercohomology groups.

$$\begin{aligned} 0 &\to H^0_{\mathrm{dR}}(\bar{\mathcal{M}}_{r,\alpha,I}, \mathrm{Fil}^\bullet_n\mathbb{W}_{k,\alpha,I}) \to H^0_{\mathrm{dR}}(\bar{\mathcal{M}}_{r,\alpha,I}, \mathbb{W}^\bullet_{k,\alpha,I}) \to H^0_{\mathrm{dR}}(\bar{\mathcal{M}}_{r,\alpha,I}, (\mathbb{W}_{k,\alpha,I}/\mathrm{Fil}_n\mathbb{W}_{k,\alpha,I})^\bullet) \\ &\to H^1_{\mathrm{dR}}(\bar{\mathcal{M}}_{r,\alpha,I}, \mathrm{Fil}^\bullet_n\mathbb{W}_{k,\alpha,I}) \to H^1_{\mathrm{dR}}(\bar{\mathcal{M}}_{r,\alpha,I}, \mathbb{W}^\bullet_{k,\alpha,I}) \to \cdots \end{aligned} \tag{3.3}$$

Lemma 3.3.1. *The cohomology of the de Rham complex of coherent sheaves* $\mathrm{Fil}^\bullet_n\mathbb{W}_{k,\alpha,I}$ *can be computed using global sections.*

Proof. We recall that if $f\colon \bar{\mathcal{M}}_{r,\alpha,I} \to \mathcal{M}^*_{r,\alpha,I}$ is the projection to the minimal compactification, which is an affinoid adic space, $R^if_*\mathfrak{w}_{k,\alpha,I}(-D) = 0$. We note that in order to prove the lemma, it will be enough to prove that each sheaf in the complex $\mathrm{Fil}^\bullet_n\mathbb{W}_{k,\alpha,I}$ is acyclic for the direct image functor f_*. Because then an injective resolution of $\mathrm{Fil}^\bullet_n\mathbb{W}_{k,\alpha,I}$ will give an acyclic resolution of $f_*\mathrm{Fil}^\bullet_n\mathbb{W}_{k,\alpha,I}$, and since $\Gamma(\mathcal{M}^*_{r,\alpha,I},\cdot)$ is exact, the lemma will follow.

We first show that $\mathrm{Fil}_n\mathbb{W}_{k,\alpha,I}(-D)$ is acyclic for f_*. By Lemma 2.4.1, the sheaf $\mathrm{Fil}_n\mathbb{W}_{k,\alpha,I}(-D)$ is equipped with a finite filtration such that the graded pieces are finite direct sums of sheaves of cuspforms. Thus the graded pieces of the filtration are acyclic for f_* by what we just recalled above. Then a simple spectral sequence argument proves that $\mathrm{Fil}_n\mathbb{W}_{k,\alpha,I}(-D)$ is f_*-acyclic. Moreover, using the Kodaira-Spencer isomorphism, this same proof shows that $\mathrm{Fil}_{n+i}\mathbb{W}_{k,\alpha,I}(-D)\hat{\otimes}\Omega^i_{\bar{\mathcal{M}}_{r,\alpha,I}/\Lambda_{\alpha,I}}$ is also f_*-acyclic for all $0 \le i \le g$. □

Lemma 3.3.2. *Let* $t_L = \sum \sigma$ *be the generator of the parallel weights. There exists an exact sequence*

$$0 \to H^0(\bar{\mathcal{M}}_{r,\alpha,I}, \mathfrak{w}_{k+2t_L}(-D)) \xrightarrow{i} H^g_{dR}(\bar{\mathcal{M}}_{r,\alpha,I}, \mathrm{Fil}^\bullet_n\mathbb{W}_{k,\alpha,I}) \to \operatorname{coker} i \to 0$$

where i *is* U*-equivariant and* $\operatorname{coker} i$ *is killed by* $\prod_\sigma \prod_{i=0}^{n+g-1}(u_\sigma - i)$.

Proof. We first note that

$$H^g_{\mathrm{dR}}(\bar{\mathcal{M}}_{r,\alpha,I}, \mathrm{Fil}^\bullet_n \mathbb{W}_k) = \frac{H^0(\bar{\mathcal{M}}_{r,\alpha,I}, \mathrm{Fil}_{n+g}\mathbb{W}_{k+2t_L}(-D))}{\nabla H^0(\bar{\mathcal{M}}_{r,\alpha,I}, \mathrm{Fil}_{n+g-1}\mathbb{W}_k(-D)\hat{\otimes}\Omega^{g-1}_{\bar{\mathcal{M}}_{r,\alpha,I}/\Lambda_{\alpha,I}})}$$

The U-equivariant inclusion of $H^0(\bar{\mathcal{M}}_{r,\alpha,I}, \mathfrak{w}_{k+2t_L}(-D))$ inside $H^0(\bar{\mathcal{M}}_{r,\alpha,I}, \mathrm{Fil}_{n+g}\mathbb{W}_{k+2t_L}(-D))$ induces a map $H^0(\bar{\mathcal{M}}_{r,\alpha,I}, \mathfrak{w}_{k+2t_L}(-D)) \to H^g_{\mathrm{dR}}(\bar{\mathcal{M}}_{r,\alpha,I}, \mathrm{Fil}^\bullet_n \mathbb{W}_k)$ which is an inclusion as can be seen from the local description below. We are left to understand the cokernel of the inclusion, and in particular to show that the cokernel is killed by $\prod_\sigma \prod_{i=0}^{n+g-1}(u_\sigma - i)$.

The proof relies on the local description of the connection (2.11). Choosing a numbering $\sigma\colon \{1,\dots,g\} \simeq \Sigma$, we first write the sheaf $\mathrm{Fil}_{n+g-1}\mathbb{W}_k\hat{\otimes}\Omega^{g-1}_{\bar{\mathcal{M}}_{r,\alpha,I}/\Lambda_{\alpha,I}}$ on local coordinates as

$$\mathrm{Fil}_{n+g-1}\mathbb{W}_k\hat{\otimes}\Omega^{g-1}_{\bar{\mathcal{M}}_{r,\alpha,I}/\Lambda_{\alpha,I}} = \bigoplus_i \mathfrak{w}_k^{\le n+g-1}[V_1,\dots,V_g]\mathrm{d}\hat{X}_i.$$

Here $\mathrm{d}\hat{X}_i$ corresponds via Kodaira-Spencer to a generator of $\omega_{\mathcal{A}}^{2(t_L-\sigma_i)}$ and the superscript $\le n+g-1$ denotes we take the polynomials in V_j's of degree at most $n+g-1$.

The map $\nabla\colon \mathrm{Fil}_{n+g-1}\mathbb{W}_k(-D)\hat{\otimes}\Omega^{g-1}_{\bar{\mathcal{M}}_{r,\alpha,I}/\Lambda_{\alpha,I}} \to \mathrm{Fil}_{n+g}\mathbb{W}_{k+2t_L}(-D)$ can be described as the twist by $\mathcal{O}_{\bar{\mathcal{M}}_{r,\alpha,I}}(-D)$ of a map

$$\bigoplus_i \mathfrak{w}_k^{\le n+g-1}[V_1,\dots,V_g]\mathrm{d}\hat{X}_i \xrightarrow{\nabla} \mathfrak{w}_{k+2t_L}^{\le n+g}[V_1,\dots,V_g].$$

that can be described using formula (2.11). In particular, the image of ∇ consists of polynomials in V_i of positive total degree and hence the map

$$\mathfrak{w}_{k+2t_l}(-D) \to \frac{\mathrm{Fil}_{n+g}\mathbb{W}_{k+2t_L}(-D)}{\nabla\mathrm{Fil}_{n+g-1}\mathbb{W}_k(-D)\hat{\otimes}\Omega^{g-1}_{\bar{\mathcal{M}}_{r,\alpha,I}/\Lambda_{\alpha,I}}}$$

is injective. Thus taking global sections we get $H^0(\bar{\mathcal{M}}_{r,\alpha,I}, \mathfrak{w}_{k+2t_L}(-D)) \hookrightarrow H^g_{\mathrm{dR}}(\bar{\mathcal{M}}_{r,\alpha,I}, \mathrm{Fil}^\bullet_n \mathbb{W}_k)$.

We prove the claim about the annihilator of $\operatorname{coker} i$ by induction on n, the base case being $n = 1-g$. For $n = 1-g$ we have $\mathrm{Fil}_1\mathbb{W}_{k+2t_L}/(\nabla\mathfrak{w}_k + \mathfrak{w}_{k+2t_L}) \simeq \oplus_i \mathfrak{w}_{k+2t_L}V_i/u_i\mathfrak{w}_{k+2t_L}V_i$. This proves the base case. We have a diagram as follows with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Fil}_{n+g-2}\mathbb{W}_k\hat{\otimes}\Omega^{g-1} & \longrightarrow & \mathrm{Fil}_{n+g-1}\mathbb{W}_k\hat{\otimes}\Omega^{g-1} & \longrightarrow & \mathrm{Gr}_{n+g-1}\mathbb{W}_k\hat{\otimes}\Omega^{g-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\nabla} & & \downarrow{\scriptstyle\nabla} & & \downarrow{\scriptstyle\nabla} & & \\
0 & \longrightarrow & \mathrm{Fil}_{n+g-1}\mathbb{W}_{k+2t_L} & \longrightarrow & \mathrm{Fil}_{n+g}\mathbb{W}_{k+2t_L} & \longrightarrow & \mathrm{Gr}_{n+g}\mathbb{W}_{k+2t_L} & \longrightarrow & 0
\end{array}$$

To complete the induction, we need to understand the connection on the graded pieces. Letting $k(1+\beta_n Z)$ be a local generator of $\mathfrak{w}_k$, the map ∇ on the graded pieces can be described as follows.

$$\begin{aligned}
\oplus\mathfrak{w}_k^{n+g-1}[v_1,\dots,V_g]\mathrm{d}\hat{X}_i &\xrightarrow{\nabla} \mathfrak{w}_{k+2t_L}^{n+g}[V_1,\dots,V_g] \\
k(1+\beta_n Z)\prod_i V_i^{n_i}\mathrm{d}\hat{X}_j &\mapsto (u_j - n_j)(k+2t_L)(1+\beta_n Z)V_j\prod_i V_i^{n_i}
\end{aligned}$$

This shows that the cokernel of $\nabla\colon \mathrm{Gr}_{n+g-1}\mathbb{W}_k\hat{\otimes}\Omega^{g-1}_{\bar{\mathcal{M}}_{r,\alpha,I}/\Lambda_{\alpha,I}} \to \mathrm{Gr}_{n+g}\mathbb{W}_{k+2t_L}$ is annihilated by $\prod_\sigma\prod_{i=0}^{n+g-1}(u_\sigma - i)$. The lemma then follows by applying Snake lemma to the diagram above and by the induction hypothesis. □

Lemma 3.3.3. *For $h \geq 0$, there exists $n \geq 0$ such that the map*

$$H^i_{dR}(\bar{\mathcal{M}}_{r,\alpha,I}, \mathrm{Fil}^\bullet_n\mathbb{W}_{k,\alpha,I})^{\leq h} \to H^i_{dR}(\bar{\mathcal{M}}_{r,\alpha,I}, \mathbb{W}^\bullet_{k,\alpha,I})^{\leq h}$$

is an isomorphism.

Proof. Arguing as in Proposition 3.1.1, the sheaf $H^i_{\mathrm{dR}}(\bar{\mathcal{M}}_{r,\alpha,I}, (\mathbb{W}_k/\mathrm{Fil}_n\mathbb{W}_k)^\bullet)$ admits slope decomposition locally on the weight space. Moreover, by the same proposition $H^i_{\mathrm{dR}}(\bar{\mathcal{M}}_{r,\alpha,I}, (\mathbb{W}_k/\mathrm{Fil}_n\mathbb{W}_k)^\bullet)^{\leq h} = 0$ for large enough n. The lemma then follows from the long exact sequence (3.3). □

Definition 3.3.1. For $h \geq 0$, let n be as in the above lemma. Let $\lambda = \prod_\sigma\prod_{i=0}^{n+g-1}(u_\sigma - i)$. For the finite slope $h \geq 0$, define the overconvergent projection in families to be the map induced by the isomorphisms as follows.

$$\begin{aligned} H^\dagger\colon H^g_{\mathrm{dR}}(\bar{\mathcal{M}}_{r,\alpha,I}, \mathbb{W}^\bullet_k)^{\leq h} \otimes \Lambda_{\alpha,I}[\lambda^{-1}] &\xrightarrow{\sim} H^g_{\mathrm{dR}}(\bar{\mathcal{M}}_{r,\alpha,I}, \mathrm{Fil}^\bullet_n\mathbb{W}_k)^{\leq h} \otimes \Lambda_{\alpha,I}[\lambda^{-1}] \\ &\xrightarrow{\sim} H^0(\bar{\mathcal{M}}_{r,\alpha,I}, \mathfrak{w}_{k+2t_L}(-D))^{\leq h} \otimes \Lambda_{\alpha,I}[\lambda^{-1}] \end{aligned}$$

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