

Weil Classes and Decomposable Abelian Fourfolds

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Received May 11, 2022, in final form December 06, 2022; Published online December 13, 2022

<https://doi.org/10.3842/SIGMA.2022.097>

Abstract. We determine which codimension two Hodge classes on $J \times J$, where J is a general abelian surface, deform to Hodge classes on a family of abelian fourfolds of Weil type. If a Hodge class deforms, there is in general a unique such family. We show how to determine the imaginary quadratic field acting on the fourfolds of Weil type in this family as well as their polarization. There are Hodge classes that may deform to more than one family. We relate these to Markman's Cayley classes.

Key words: abelian varieties; Hodge classes

2020 Mathematics Subject Classification: 14C30; 14C25; 14K20

1 Introduction

The Hodge conjecture asserts that for any smooth complex projective variety X and any non-negative integer p the vector space $B^p(X) := H^{2p}(X, \mathbf{Q}) \cap H^{p,p}(X)$ of Hodge classes is spanned by the classes of algebraic cycles.

From a study of the Mumford Tate groups of abelian fourfolds, Moonen and Zarhin [11, 12], obtained the result that if the Hodge conjecture holds for abelian fourfolds of Weil type, then it holds for all abelian fourfolds. An abelian fourfold is of Weil type if it has an imaginary quadratic field in its endomorphism algebra and each non-rational element in that field has two eigenvalues on $H^{1,0}$ with the same multiplicity. Such fourfolds are parametrized by four-dimensional subvarieties in the moduli space of all abelian fourfolds with a fixed polarization.

The first results on the Hodge conjecture for fourfolds of Weil type were obtained by Chad Schoen [15, 16]. The paper [17] provides an alternative construction of algebraic cycles representing 'exceptional' Hodge classes in one of the cases considered in [15] and might have much wider applications. Starting from a certain reducible surface $S \subset J \times J$, where J is the Jacobian of a general genus two curve, he shows that S deforms in a specific four-dimensional family of abelian fourfolds of Weil type. This verifies the Hodge conjecture for these abelian fourfolds. The general deformation of S , now called a Schoen surface, was studied in [1, 14] and a rather explicit description of these surfaces can be found there.

In this paper we consider more generally the question of which Hodge classes in $B^2(J \times J)$ can be deformed to an abelian fourfold of Weil type, and which is the imaginary quadratic field acting on such a fourfold. We provide answers to these questions in Section 4. In particular, such a Hodge class must be contained in a determinantal cubic fourfold $Z \subset \mathbf{P}B^2(J \times J) \cong \mathbf{P}^5$ which is the secant variety of a Veronese surface. This Veronese surface is the singular locus of Z . A Hodge class in Z , but not in $\text{Sing}(Z)$, will deform in at most one family. A Hodge class in $\text{Sing}(Z)$ may deform to infinitely many families with distinct quadratic fields. This rather curious fact is also the reason we restrict ourselves to fourfolds in the last sections. In case $\dim J = n > 2$ the deformation of a Hodge class to a family of Weil type, if it exists, seems to be unique. However, a good description of the subvariety of $\mathbf{P}B^n(J \times J)$ of classes that admit such deformations is much harder to find.

In his recent paper [10], E. Markman proved the Hodge conjecture for the abelian fourfolds of Weil type, with a polarization having trivial discriminant, using methods from deformation theory of hyperkähler manifolds. He also uses specific Hodge classes, called Cayley classes, that deform in Weil type families for any imaginary quadratic field. We briefly discuss the relation with our results in Section 5.

In Section 6, we discuss the Hodge classes defined by certain algebraic cycles from the papers [17] and [4]. In Section 7, we show that the discriminants of the polarized abelian fourfolds of Weil type we consider are trivial.

2 Hodge classes on abelian fourfolds of Weil type

2.1 Abelian varieties of Weil type

Let A be an abelian variety and let $K = \mathbf{Q}(\sqrt{-d})$, with $d \in \mathbf{Z}_{>0}$, be an imaginary quadratic field. An abelian variety of Weil type (with field K) is a pair (A, K) , where A is an abelian variety and $K \hookrightarrow \text{End}_{\mathbf{Q}}(A)$ is a subalgebra, such that for all $k \in K$ the endomorphism of T_0A defined by the differential of $k = a + b\sqrt{-d} \in K$ has eigenvalues $k = a + b\sqrt{-d}$ and $\bar{k} = a - b\sqrt{-d}$ with the same multiplicity. Equivalently, the eigenvalues of any $k \in K$ on $H^{1,0}$ have the same multiplicity. In particular, if (A, K) is of Weil type, then $\dim A$ is even.

Given an abelian variety of Weil type (A, K) , there exists a polarization $\omega_K \in B^1(A)$ on A such that for all $k \in K$ one has

$$k^* \omega_K = \text{Nm}(k) \omega_K, \quad \text{Nm}(k) = k \bar{k},$$

where $\text{Nm}(k)$ is the norm of $k \in K$ (see [3, Lemma 5.2.1]). We call such a 2-form a polarization of Weil type and (A, K, ω_K) is called a polarized abelian variety of Weil type.

2.2 The Weil classes

For a general (in particular A is not decomposable) abelian variety of Weil type (A, K) with dimension $2n$, the spaces of Hodge classes have dimensions [18] (see also [3, Theorem 6.12]):

$$\dim B^p(A) = 1, \quad p \neq n, \quad \dim B^n(A) = 3.$$

Since $\dim B^1(A) = 1$, any $\omega \in B^1(A)$, $\omega \neq 0$, defines (up to sign) a polarization on A which will be of Weil type. Writing $\omega_K := \omega$, the p -fold exterior product of this class with itself, denoted by $\omega_K^p \in \wedge^p B^1(A) \subset B^p(A)$, is non-zero for any $p \leq 2n$. Using the action of the multiplicative group $K^\times := K - \{0\}$ on $H^{2n}(A, \mathbf{Q})$, there is a natural two-dimensional subspace W_K in $B^n(A)$ which is a complement of $\mathbf{Q}\omega_K^n$:

$$B^n(A) = \mathbf{Q}\omega_K^n \oplus W_K, \quad \dim W_K = 2.$$

To define W_K we use that for all $k \in K^\times$ the eigenvalues of k^* on $H^1(A, \mathbf{Q})$ are k, \bar{k} , each with multiplicity $2n$. Let

$$H^1(A, K) = H^1(A, \mathbf{Q}) \otimes_{\mathbf{Q}} K = V_+ \oplus V_-, \quad k^* = \text{diag}(k, \bar{k})$$

be the direct sum decomposition into $2n$ -dimensional, conjugate, eigenspaces V_\pm of K^\times . The eigenvalues of k^* on $H^{1,0}(A)$ (and on $H^{0,1}(A)$) are k, \bar{k} each with multiplicity n , so V_+, V_- both have Hodge numbers $h^{1,0} = h^{0,1} = n$. Since $H^{2n}(A, \mathbf{Q}) = \wedge^{2n} H^1(A, \mathbf{Q})$, we get

$$H^{2n}(A, K) = \bigoplus_{m=0}^{2n} (\wedge^{2n-m} V_+) \otimes (\wedge^m V_-),$$

and the eigenvalues of k^* on the m -th summand are $k^{2n-m}\bar{k}^m$. The one-dimensional subspaces $\wedge^{2n}V_+$ and $\wedge^{2n}V_-$ are of Hodge type (n, n) and their direct sum is defined over \mathbf{Q} since they are conjugate. That is, there is two-dimensional subspace $W_K \subset H^{2n}(A, \mathbf{Q})$ such that

$$W_K \otimes_{\mathbf{Q}} K = \wedge^{2n}V_+ \oplus \wedge^{2n}V_-, \quad W_K \subset H^{2n}(A, \mathbf{Q}) \cap H^{n,n}(A) = B^n(A).$$

We give another definition of W_K . Notice that if $k \in K^\times$, and $k^{2n} = a + b\sqrt{-d} \in K$, with $a, b \in \mathbf{Q}$, then k^{2n}, \bar{k}^{2n} are the roots of the polynomial $P(T) := T^2 - 2aT + a^2 + db^2 \in \mathbf{Q}[T]$. Let W_K be the kernel of the linear map $P(k^*) \in \text{End}(H^{2n}(A, \mathbf{Q}))$, for any general $k \in K^\times$ (i.e., such that $k^{2n}, k^{2n-1}\bar{k}, \dots, \bar{k}^{2n}$ are distinct). Then $W_K \otimes_{\mathbf{Q}} K$ is the span of the eigenspaces of k^* with eigenvalues k^{2n}, \bar{k}^{2n} . Hence $W_K \otimes_{\mathbf{Q}} K = \wedge^{2n}V_+ \oplus \wedge^{2n}V_-$ as before, so $\dim_{\mathbf{Q}} W_K = 2$ and the classes in W_K are of Hodge type (n, n) .

Definition 2.1. Let (A, K, ω_K) be a $2n$ -dimensional abelian variety of Weil type. The two-dimensional subspace $W_K = W_{K,A}$ of $B^n(A)$ is called the space of Weil classes of (A, K) .

This subspace is characterized by $W_K \otimes_{\mathbf{Q}} K = W_{K,+} \oplus W_{K,-}$ where $W_{K,+}, W_{K,-}$ are the (one-dimensional) eigenspaces of the K^\times action on $H^{2n}(A, K)$, with eigenvalues k^{2n}, \bar{k}^{2n} respectively, for all $k \in K^\times$ where $K^\times := K - \{0\}$ be the multiplicative group of K .

The space of Hodge classes of (A, K, ω_K) is the three-dimensional subspace $B_{K,A}^n$ of the space of Hodge classes $B^n(A)$ defined as

$$B_{K,A}^n := \mathbf{Q}\omega_K^n \oplus W_K.$$

2.3 Families of abelian varieties of Weil type

Let (A, K, ω_K) be a polarized abelian variety of Weil type of dimension $2n$. Then A is a complex torus, $A = \mathbf{C}^{2n}/\Lambda$. Intrinsically, $\Lambda = \pi_1(A) = H_1(A, \mathbf{Z})$ and $\mathbf{C}^{2n} = \Lambda \otimes_{\mathbf{Z}} \mathbf{R} = H_1(A, \mathbf{R})$, with a certain complex structure $I \in \text{End}(H_1(A, \mathbf{R}))$, so $I^2 = -\text{Id}$. This complex structure induces one on the dual vector space $H^1(A, \mathbf{R})$ which has eigenspaces $H^{1,0}(A), H^{0,1}(A)$. These again determine the Hodge decomposition on all $H^k(A)$, since $H^k(A, \mathbf{Z}) = \wedge^k H^1(A, \mathbf{Z})$. The embedding $K \hookrightarrow \text{End}_{\mathbf{Q}}(A)$ determines (and is determined by) an embedding $K \hookrightarrow \text{End}_{\mathbf{Q}}(\Lambda \otimes_{\mathbf{Z}} \mathbf{Q})$ and the polarization $\omega_K \in H^2(A, \mathbf{Q})$ is an alternating 2-form on $\Lambda \otimes_{\mathbf{Z}} \mathbf{Q}$.

To deform (A, K, ω_K) to other abelian varieties of Weil type, one changes the complex structure I in such a way that K still acts by \mathbf{C} -linear maps on $\Lambda \otimes_{\mathbf{Z}} \mathbf{R}$ and such that ω_K still satisfies the Riemann conditions for a polarization (cf. [3, Section 5]). In this way one obtains an n^2 -dimensional family of abelian varieties of Weil type parametrized by a bounded Hermitian domain.

For each (A', K, ω_K) in this family there is a natural identification $H^k(A', \mathbf{Q}) = H^k(A, \mathbf{Q})$ since only the complex structure varies. As the K -action is fixed, both ω_K (and thus also ω_K^n) and the space of Weil cycles is fixed ($W_{K,A'} = W_{K,A}$) as is the spaces of Hodge classes of (A', K, ω_K) ($B_{K,A'}^n = B_{K,A}^n = \mathbf{Q}\omega_K^n \oplus W_{K,A}$) under the identification $H^{2n}(A', \mathbf{Q}) = H^{2n}(A, \mathbf{Q})$.

In this paper we consider, for a decomposable abelian $2n$ -fold $J \times J$, with J general, and all imaginary quadratic subfields $K \subset \text{End}_{\mathbf{Q}}(J^2)$, the corresponding three-dimensional spaces $B_{K,J^2}^n \subset B^n(J^2)$. In case $n = 2$, we will see that the intersection between these subspaces is non-trivial.

3 Weil classes for decomposable abelian varieties

3.1 General polarized abelian varieties

We fix a general abelian variety J of dimension n . In particular J has endomorphism algebra $\text{End}_{\mathbf{Q}}(J) = \mathbf{Q}$ and $\dim B^1(J) = 1$ (so J has a unique polarization up to scalar multiple).

Let $\omega_J \in B^1(J)$ be the class of an ample divisor, we denote by (J, ω_J) the corresponding polarized abelian variety. We choose a basis of 1-forms dt_j , $j = 1, \dots, 2n$, of $H^1(J, \mathbf{Q})$ such that

$$\omega_J = dt_1 \wedge dt_2 + \dots + dt_{2n-1} \wedge dt_{2n}.$$

Then we obtain a basis of $H^1(J^2, \mathbf{Q}) \cong H^1(J, \mathbf{Q})^{\oplus 2}$ given by the $dx_j = \pi_1^* dt_j$, $dy_j := \pi_2^* dt_j$, $j = 1, \dots, 2n$, where the $\pi_i: J^2 \rightarrow J$, $i = 1, 2$, are the projections on the factors.

The following proposition determines the dimensions, as well as bases, for the spaces of Hodge classes for the product variety $J \times J$.

Proposition 3.1. *For a general abelian n -fold J and for $1 \leq p \leq n$ we have*

$$\dim B^1(J^2) = 3, \quad B^p(J^2) \cong \text{Sym}^p B^1(J^2), \quad \text{so} \quad \dim B^p(J^2) = \binom{p+2}{2}.$$

A basis of $B^1(J \times J)$ is given by the following three 2-forms

$$\begin{aligned} \omega_1 &= dx_1 \wedge dx_2 + \dots + dx_{2n-1} \wedge dx_{2n}, \\ \omega_2 &= dy_1 \wedge dy_2 + \dots + dy_{2n-1} \wedge dy_{2n}, \\ \omega_\sigma &= dx_1 \wedge dy_2 - dx_2 \wedge dy_1 + \dots + dx_{2n-1} \wedge dy_{2n} - dx_{2n} \wedge dy_{2n-1}. \end{aligned}$$

A basis of $B^2(J \times J)$ is thus given by the following six 4-forms (where we write $\omega\theta$ for $\omega \wedge \theta$):

$$\omega_1^2, \quad \omega_1\omega_2, \quad \omega_1\omega_\sigma, \quad \omega_2^2, \quad \omega_2\omega_\sigma, \quad \omega_\sigma^2 \quad (\in B^2(J^2)).$$

Proof. Since J is general, the Hodge group, also known as the special Mumford–Tate group, of $V := H^1(J, \mathbf{Q})$ is $\text{Sp}(V) \cong \text{Sp}_{2n}$, the symplectic group of ω_J . The spaces of Hodge classes $B^p(J^2)$ are obtained as the invariants (for the diagonal action):

$$B^p(J^2) = (\wedge^{2p}(V^{\oplus 2}))^{\text{Sp}(V)} = \bigoplus_{k=0}^{2p} ((\wedge^k V) \otimes (\wedge^{2p-k} V))^{\text{Sp}(V)}.$$

Interchanging $\wedge^k V$ and $\wedge^{2p-k} V$ shows that we only need to find the invariants for $k \leq p$. The symplectic form ω_J induces a duality $\wedge^k V \cong (\wedge^k V)^*$ and thus

$$((\wedge^k V) \otimes (\wedge^{2p-k} V))^{\text{Sp}(V)} \cong \text{Hom}_{\text{Sp}(V)}(\wedge^k V, \wedge^{2p-k} V).$$

For each k , with $0 \leq k \leq n$, the $\text{Sp}(V)$ -representation $\wedge^k V$ is a direct sum of $k/2 + 1$, $(k+1)/2$, with k even and k odd respectively, irreducible $\text{Sp}(V)$ -representations, each with multiplicity 1. These representation are denoted by $V^{(k-2i)}$, $0 \leq i \leq k/2$ in [2, Theorem 17.5] and $\wedge^k V \cong V^{(k)} \oplus \wedge^{k-2} V$. In this way one obtains the Lefschetz decomposition [6, p. 122] of $H^k(J, \mathbf{Q}) = \wedge^k V$.

Notice that the number of these representations depends only on k and not on n . We denote by d_k this number of irreducible subrepresentations of $\wedge^k V$, so $d_k = 1, 1, 2, 2, 3, 3, 4, \dots$ for $k = 0, 1, 2, 3, 4, 5, 6, \dots$

If $k \leq p$, $\wedge^k V$ is isomorphic to the $\text{Sp}(V)$ -subrepresentation $\omega_J^{p-k} \wedge (\wedge^k V) \subset \wedge^{2p-k} V$. By Schur's lemma, the dimension of $\text{Hom}_{\text{Sp}(V)}(\wedge^k V, \wedge^{2p-k} V)$ is then equal to the number of irreducible $\text{Sp}(V)$ -representations in $\wedge^k V$. In particular $\dim B^1(J^2) = 1 + 1 + 1 = 3$ and $\dim B^2(J^2) = 1 + 1 + 2 + 1 + 1 = 6$. More generally, for $0 \leq k \leq p$,

$$\dim B^p(J^2) = \sum_{k=0}^{2p} d_k = 2 \sum_{k=0}^{p-1} d_k + d_p = \dim B^{p-1}(J^2) + d_{p-1} + d_p.$$

Since $d_{p-1} + d_p = p + 1$ and $\dim B^0(J^2) = 1$, we find $\dim B^p(J^2) = \binom{p+2}{2}$.

According to Hazama [8, Proposition 1.6] the ring of Hodge classes $B^*(J^2) = \bigoplus B^p(J^2)$ is generated by $B^1(J^2)$, so the cup product $\text{Sym}^p B^1(J^2) \rightarrow B^p(J^2)$ is surjective. Comparing dimensions we get $\text{Sym}^p B^1(J^2) \cong B^p(J^2)$.

Notice that $\omega_j = \pi_j^* \omega_J$, hence they are in $B^1(J^2)$. Using the addition map

$$\sigma: J \times J \longrightarrow J, \quad (x, y) \longmapsto t = x + y,$$

one finds $\omega_\sigma \in B^1(J^2)$ from

$$\sigma^* \omega_J = \omega_1 + \omega_2 + \omega_\sigma \quad (\in B^1(J^2)).$$

Since $\dim B^1(J^2) = 3$, these three 2-forms are a basis of $B^1(J^2)$. The six 4-forms are obviously a basis of $\text{Sym}^2 B^1(J^2) = B^2(J^2)$. ■

3.2 The GL_2 -action on the Hodge cycles

As $\text{End}_{\mathbf{Q}}(J) = \mathbf{Q}$, we have $\text{End}_{\mathbf{Q}}(J^2) = M_2(\mathbf{Q})$ (we write $(x, y) \in J^2$ as a column vector). We will first consider the induced action of GL_2 on the Hodge cycles $B^p(J^2)$. Since for pull-backs, $(gh)^* = h^*g^*$ where $g, h \in M_2(\mathbf{Q})$, to get a representation of GL_2 on the cohomology we define $g \cdot \omega := ({}^t g)^* \omega$ for $\omega \in H^*(J^2, \mathbf{Q})$. We denote the standard two-dimensional representation of GL_2 by V_1 . Then we have $H^1(J^2, \mathbf{Q}) \cong H^1(J, \mathbf{Q}) \otimes V_1$ and the GL_2 action is on the second factor.

Recall that SL_2 has a unique irreducible representation of dimension m for any $m \geq 1$. There is an isomorphism $\text{Sym}^2 V_1 \cong B^1(J^2)$ which simplifies some of the arguments in the rest of the paper.

Proposition 3.2. *The vector spaces $B^p(J^2)$ are GL_2 -representations in which the center \mathbf{Q}^\times of GL_2 acts by scalar multiplication by t^{2p} on $B^p(J^2)$. As representation for the subgroup SL_2 , the space $B^1(J^2)$ is the irreducible three-dimensional representation of SL_2 . Moreover, if e_+, e_- are the standard basis of V_1 , then there is an isomorphism of GL_2 -representations*

$$\text{Sym}^2 V_1 \xrightarrow{\cong} B^1(J^2), \quad \begin{cases} e_+^2 \longmapsto \omega_1, \\ e_+ e_- \longmapsto \frac{1}{2} \omega_\sigma, \\ e_-^2 \longmapsto \omega_2. \end{cases}$$

The space $B^2(J^2) = \text{Sym}^2 B^1(J^2)$ is the direct sum of the two irreducible SL_2 -subrepresentations V_0, V_4 of $B^2(J^2)$ of dimension 1 and 5:

$$V_0 := \langle \Omega := 4\omega_1\omega_2 - \omega_\sigma^2 \rangle, \quad V_4 = \langle \omega_1^2, \omega_1\omega_\sigma, 2\omega_1\omega_2 + \omega_\sigma^2, \omega_2\omega_\sigma, \omega_2^2 \rangle.$$

Proof. Since $t \in \mathbf{Q}^\times$ acts multiplication by t on $H^1(J^2, \mathbf{Q})$ and $H^k(J^2, \mathbf{Q}) = \wedge^k H^1(J^2, \mathbf{Q})$, it acts as t^{2p} on $B^p(J^2)$.

The diagonal matrix $\text{diag}(t, t^{-1}) \in \text{SL}_2$ acts as $\text{diag}(t^2, 1, t^{-2})$ on the basis $\omega_1, \omega_\sigma, \omega_2$ of $B^1(J^2)$ which proves that $B^1(J^2)$ is the three-dimensional irreducible representation of SL_2 . This representation is also isomorphic to $\text{Sym}^2 V_1$. Using the eigenvectors of the diagonal matrices we see that, up to scalar multiples, the isomorphism is as in the proposition. We fix the isomorphism by imposing that $e_+^2 \mapsto \omega_1$. We define

$$S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (\in \text{SL}_2).$$

Then $Se_+ = e_-$ and thus $Se_+^2 = e_-^2$, therefore e_-^2 maps to $S\omega_1 = \omega_2$. Finally we have $Te_+e_- = e_+(e_+ + e_-) = e_+^2 + e_+e_-$ and we know that e_+e_- maps to $a\omega_\sigma$ for some constant a . Thus T maps $a\omega_\sigma$ to $\omega_1 + a\omega_\sigma$. An explicit computation shows that $({}^t T)^* \omega_\sigma = 2\omega_1 + \omega_\sigma$, hence $a = 1/2$.

From the basis of $B^2(J^2)$ given in Proposition 3.1 one sees that $B^2(J^2) = \text{Sym}^2 B^1(J^2)$ and thus $B^2(J^2)$ is the sum of the irreducible five and one-dimensional representations of SL_2 . With standard Lie algebra computations one finds the explicit decomposition. ■

3.3 Homomorphisms $K \hookrightarrow \text{End}_{\mathbf{Q}}(J^2)$

For any imaginary quadratic field K , there are embeddings $K \hookrightarrow \text{End}_{\mathbf{Q}}(J^2) \cong M_2(\mathbf{Q})$, here an embedding is an injective homomorphism of rings. For example,

$$\phi: K \hookrightarrow M_2(\mathbf{Q}), \quad \phi(a + b\sqrt{-d}) = \begin{pmatrix} a & -db \\ b & a \end{pmatrix}.$$

Such an embedding gives a homomorphism (of algebraic groups defined over \mathbf{Q}) $K^\times \rightarrow \text{GL}_2$. The image of K^\times is a diagonalizable two-dimensional abelian subgroup of GL_2 , so it is a Cartan subgroup of GL_2 .

In particular, K has two eigenvectors $v_\pm \in V_1 \otimes_{\mathbf{Q}} K$ on which it acts as $k \cdot v_+ = kv_+$ and $k \cdot v_- = \bar{k}v_-$. Conversely, given a basis of conjugated vectors v_\pm of $V_1 \otimes_{\mathbf{Q}} K$, one finds an embedding $K \hookrightarrow M_2(\mathbf{Q})$. Such a pair of vectors can be chosen as $v_\pm = (\alpha_\pm, 1) \in K^2$ with $\bar{\alpha}_+ = \alpha_-$. We can thus parametrize all embeddings of K by $\alpha_+ \in K - \mathbf{Q}$.

3.4 Cartan subalgebras

We identify the Lie algebra of GL_2 with M_2 . Then a subfield $K \subset M_2(\mathbf{Q})$ is a Cartan subalgebra of M_2 , that is a diagonalizable two-dimensional subalgebra. More generally, we will consider Cartan subalgebras, still denoted by K , in $M_2(\mathbf{C})$. Such a Cartan subalgebra determines, and is determined by, a basis (of eigenvectors) of $V_1 \otimes_{\mathbf{Q}} \mathbf{C}$. Only rather special bases will correspond to imaginary quadratic subfields of $M_2(\mathbf{Q})$, but it is still convenient to consider all of them as we do from now on. The complexification of the space of Hodge cycles $B^p(J^2)$ is denoted simply by B^p :

$$B^p := B^p(J \times J) \otimes_{\mathbf{Q}} \mathbf{C}.$$

We refer to the eigenspaces and eigenvalues of the multiplicative group K^\times of K as the eigenspaces and eigenvalues of K .

The following lemma collects some explicit computations on the differential forms that are eigenvectors of a Cartan subalgebra $K \hookrightarrow M_2(\mathbf{C})$, where K is determined by a pair of independent eigenvectors $v_\pm \in \mathbf{C}^2$.

Lemma 3.3. *Let $v_\pm = (\alpha_\pm, 1) \in V_1 \otimes_{\mathbf{Q}} \mathbf{C} = \mathbf{C}^2$ and let $K \hookrightarrow M_2(\mathbf{C})$ be the Cartan subalgebra with these eigenvectors.*

1. *A basis of eigenvectors of K in B^1 is*

$$\omega_\pm := \alpha_\pm^2 \omega_1 + \alpha_\pm \omega_\sigma + \omega_2, \quad \text{and} \quad \omega_K := 2\alpha_+ \alpha_- \omega_1 + (\alpha_+ + \alpha_-) \omega_\sigma + 2\omega_2.$$

2. *Let K be an imaginary quadratic field. Then the eigenvalues of K on ω_+ , ω_- , ω_K are k^2 , \bar{k}^2 , $k\bar{k}$ respectively. Moreover, (J^2, K, ω_K) is an abelian variety of Weil type. The space of Weil cycles $W_K \subset H^{2n}(J^2, \mathbf{Q})$ of (J^2, K, ω_K) is spanned (over K) by ω_+^n and ω_-^n :*

$$W_K \otimes_{\mathbf{Q}} K = K\omega_+^n \oplus K\omega_-^n,$$

and $B_{K, J^2}^n = \mathbf{Q}\omega_K^n \oplus W_K$ (cf. Definition 2.1).

Proof. Writing $v_\pm = \alpha_\pm e_+ + e_-$ we get $v_\pm^2 = \alpha_\pm^2 e_+^2 + 2\alpha_\pm e_+ e_- + e_-^2$, which are eigenvectors of the K^\times action on $\text{Sym}^2 V_1$. The isomorphism $\text{Sym}^2 V_1 \cong B^1(J^2)$ maps these to the 2-forms in this lemma. Similar for $2v_+ v_-$ which is a third eigenvector in $\text{Sym}^2 V_1$.

Let $k \in K^\times$, it has eigenvalues k, \bar{k} on v_+, v_- , respectively, hence the eigenvalues of k on $B^1(J^2) \cong \text{Sym}^2 V_1$ are $k^2, \bar{k}^2, \text{Nm}(k)$. The K^\times eigenspace in $B^1(J^2)$ with eigenvalue $\text{Nm}(k)$ is

thus one-dimensional and therefore ω_K is unique up to scalar multiple, hence it is (up to sign) an ample class (see [4, Lemma 5.2]). Restricting ω_K to the first factor of J^2 we conclude that ω_K is in fact ample. Thus $\omega_K^{2n} \neq 0$. So (J^2, K, ω_K) is an abelian variety of Weil type.

Since $k^*\omega_+ = k^2\omega_+$ we find $k^*\omega_+^n = k^{2n}\omega_+^n$ and similarly $k^*\omega_-^n = \bar{k}^{2n}\omega_-^n$. As $\text{Sym}^n B^1(J^2) \cong B^n(J^2)$, the forms ω_n^\pm are non-zero. Definition 2.1 then shows that W_K is spanned by ω_+^n and ω_-^n . ■

By Lemma 3.3(2), any embedding $K \hookrightarrow \text{End}_{\mathbf{Q}}(J^2)$ of an imaginary quadratic field defines a polarized abelian variety, of dimension $2n$, of Weil type (J^2, K, ω_K) , which is unique up to multiplication of ω_K by a positive rational number. In particular, the embedding determines an n^2 -dimensional family of abelian fourfolds of Weil type as in Section 2.3 by changing the complex structure on $H_1(J^2, \mathbf{R})$. For any member A in this family there is a natural identification of the three-dimensional vector space of Hodge classes $B_{K,A}^n \subset H^{2n}(A, \mathbf{Q})$ of (A, K, ω_K) with B_{K,J^2}^n .

More generally, we define analogous complex subspaces in the B^p (the complexification of the $B^p(J^2)$) for any Cartan subalgebra.

Definition 3.4. Let $K \hookrightarrow M_2(\mathbf{C})$ be a Cartan subalgebra with eigenvectors $v_\pm = (\alpha_\pm, 1) \in V_1 \otimes_{\mathbf{Q}} \mathbf{C} = \mathbf{C}^2$. With the notation from Lemma 3.3(1) we define a two-dimensional subspace of $B^n = B^n(J^2) \otimes_{\mathbf{Q}} \mathbf{C}$ by

$$W_K := \mathbf{C}\omega_+^n \oplus \mathbf{C}\omega_-^n \quad (\subset B^n),$$

and we define a three-dimensional subspace B_K^n of Hodge classes of J^2 by

$$B_K^n := \mathbf{C}\omega_K^n \oplus W_K \quad (\subset B^n).$$

3.5 The invariant conic in B^1

There is a curious (if well-known) geometrical relation between the three points in $\mathbf{P}B^1 \cong \mathbf{P}^2$, where $B^1 := B^1(J^2) \otimes_{\mathbf{Q}} \mathbf{C}$ determined by the K^\times -eigenvectors $\omega_K, \omega_+, \omega_- \in B^1$.

The Veronese image of $\mathbf{P}V_1$, where V_1 is the standard representation of GL_2 , is a conic C_2 in $\mathbf{P}(\text{Sym}^2 V_1) \cong \mathbf{P}B^1$. This conic is thus invariant under the action of GL_2 on $\mathbf{P}B^1$. It contains the points $\omega_\pm = v_\pm^2$ where v_\pm are eigenvectors of K^\times acting on V_1 . From the Lemma 3.3 one then finds:

$$C_2 = \{[a\omega_1 + b\omega_\sigma + c\omega_2] \in \mathbf{P}B^1 : b^2 - ac = 0\}.$$

Let $p \in \mathbf{P}B^1$ be the point of intersection of the tangent lines to C_2 in the two points defined ω_\pm . Since these two points are fixed by the action of K^\times on B^1 , also the point p is fixed by K^\times . Since there are only three fixed points in B^1 by Lemma 3.3 it follows that p must be the point defined by ω_K .

Conversely, given ω_K , one finds ω_\pm (up to permutation) as the points of intersection of the (two) lines through p which are tangent to C_2 with C_2 itself.

3.6 The invariant conic and Pfaffians

This invariant conic also has another description in terms of the two-forms in $B^1 = B^1(J^2) \otimes_{\mathbf{Q}} \mathbf{C}$. The Pfaffian $\text{Pfaff}(\theta) \in \mathbf{C}$ of a two-form $\theta \in B^1$ is defined by

$$\wedge^{2n} \theta = ((2n)!) \text{Pfaff}(\theta) dx_1 \wedge \cdots \wedge dy_{2n} \quad (\in H^{4n}(J^2, \mathbf{C}) \cong \mathbf{C}).$$

Since the Pfaffian is essentially invariant under GL_2 , its zero locus is a union of GL_2 -orbits in B^1 . There are only two $\text{GL}_2(\mathbf{C})$ -orbits on $\mathbf{P}B^1$ and, since $\text{Pfaff}(\omega_\sigma) \neq 0$ but $\text{Pfaff}(\omega_1) = 0$, for $\theta \in B^1$ we must have $\text{Pfaff}(\theta) = e(b^2 - ac)^n$ for some non-zero constant e .

4 The Hodge classes of (J^2, K) when $n = \dim J = 2$

4.1 The Hodge classes B_K^2

For a general abelian surface J , an embedding of an imaginary quadratic field $K \hookrightarrow \text{End}_{\mathbf{Q}}(J^2) = M_2(\mathbf{Q})$ determines an abelian fourfold of Weil type (J^2, K, ω_K) . The space of Hodge classes of (J^2, K, ω_K) is the three-dimensional subspace of $B^2(J^2) \subset H^4(J^2, \mathbf{Q})$, see Definition 2.1,

$$B_{K, J^2}^2 := \mathbf{Q}\omega_K^2 \oplus W_{K, J^2}, \quad \text{with } W_{K, J^2} \otimes_{\mathbf{Q}} K = K\omega_+^2 \oplus K\omega_-^2.$$

As observed in Section 2.3, (J^2, K, ω_K) determines a four-dimensional family of abelian varieties of Weil type by changing the complex structure on $H_1(J^2, \mathbf{R})$. For any A in this family the three-dimensional space of Hodge classes $B_{K, A}^2$ of (A, K, ω_K) is the same as B_{K, J^2}^2 .

To study the subspaces of Hodge classes B_K^2 as $K \subset M_2(\mathbf{Q})$ varies over the imaginary quadratic subfields, it is convenient to complexify and projectivize. In that way, as K varies over the Cartan subalgebras of $M_2(\mathbf{C})$, we obtain a two-dimensional family of complex projective planes $\mathbf{P}B_K^2$, with B_K^2 as in Definition 3.4, in the complex projective space $\mathbf{P}B^2 = \mathbf{P}^5$.

First of all we show that any two $\mathbf{P}B_K^2$'s have a non-empty intersection.

Proposition 4.1. *Given two distinct Cartan subalgebras K_1, K_2 of $M_2(\mathbf{C})$, the intersection of their spaces of Hodge classes $B_{K_1}^2 \cap B_{K_2}^2 \subset B^2$ is one-dimensional and thus $\mathbf{P}B_{K_1}^2 \cap \mathbf{P}B_{K_2}^2$ is a point.*

Let K be a Cartan subalgebra, then the union of the points of intersection $\mathbf{P}B_K^2 \cap \mathbf{P}B_{K'}^2$, where K' runs over the Cartan subalgebra's distinct from K , is a conic, denoted by C_K , in $\mathbf{P}B_K^2$.

Proof. The action of $\text{GL}_2(\mathbf{C})$ on the Cartan subalgebras by conjugation is transitive, since any element in $\text{GL}_2(\mathbf{C})$ that maps the eigenvectors of K_1 to those of K_2 will conjugate K_1 to K_2 . So we may assume that K is the Cartan subalgebra of diagonal matrices. The corresponding eigenvectors in B^1 are $\theta_+ := \omega_1$, $\theta_- := \omega_2$ and $\theta_K := \omega_\sigma$ (where we wrote θ rather than ω), so they are the standard basis of B^1 . Therefore B_K^2 is spanned by $\omega_1^2, \omega_\sigma^2, \omega_2^2$ and an element in B^2 lies in B_K^2 iff in the standard basis of B^2 (given in Proposition 3.1) the coefficients of $\omega_1\omega_\sigma, \omega_2\omega_\sigma, \omega_1\omega_2$ are zero.

For the Cartan subalgebra $K', K' \neq K$, determined by $v_\pm := (\alpha_\pm, \beta_\pm) \in \mathbf{C}^2$, the eigenforms $\omega_\pm, \omega_{K'}$ are given in Lemma 3.3 (we assumed there that $\beta_\pm = 1$ but it is easy to homogenize the expressions). Computing their squares, one finds that the 3×3 matrix of their coefficients of $\omega_1\omega_\sigma, \omega_2\omega_\sigma, \omega_1\omega_2$ has rank two. It is then easy to find the intersection:

$$B_K^2 \cap B_{K'}^2 = \langle 2\alpha_-^2\beta_+^2\omega_1^2 - \alpha_+\alpha_-\beta_+\beta_-\omega_\sigma^2 + 2\alpha_+^2\beta_-^2\omega_2^2 \rangle.$$

In particular, the corresponding point $a\omega_1^2 + b\omega_\sigma^2 + c\omega_2^2 \in \mathbf{P}B_K^2$ lies on the conic C_K defined by $4b^2 = ac$. (This conic is not the one defined by the condition $\theta^2 = 0$, in fact $(a\omega_1^2 + b\omega_\sigma^2 + c\omega_2^2)^2 = 0$ in $H^8(J^2)$ iff $3b^2 = -ac$.) \blacksquare

4.2 The union Z of the planes of Hodge classes $\mathbf{P}B_K^2$

Using the basis for $B^2(J^2)$ given in Proposition 3.1, any element $\theta \in B^2$ in its complexification is a linear combination

$$\theta = x_0\omega_1^2 + x_1\omega_1\omega_2 + x_2\omega_2^2 + x_3\omega_1\omega_\sigma + x_4\omega_2\omega_\sigma + x_5\omega_\sigma^2 \quad (\in B^2),$$

with $(x_0, \dots, x_5) \in \mathbf{C}^6$. The following theorem identifies the union Z of all the projective planes $\mathbf{P}B_K^2$ as K varies over the Cartan subalgebras of $M_2(\mathbf{C})$. For dimension reasons we expect all these projective planes to be contained in a hypersurface. It should be noticed that the surface $S = \text{Sing}(Z)$ is *not* the Veronese surface of $\mathbf{P}B^1$.

Theorem 4.2. *Let Z be the Zariski closure in $\mathbf{P}B^2 \cong \mathbf{P}^5$ of the union $\cup_K \mathbf{P}B_K^2$, where the union is over all diagonalizable 2-dimensional Cartan subalgebras $K \hookrightarrow M_2(\mathbf{C})$:*

$$Z := \overline{\cup_K \mathbf{P}B_K^2} \quad (\subset \mathbf{P}B^2 \cong \mathbf{P}^5).$$

Then Z is a cubic fourfold in $\mathbf{P}B^2$ defined by the determinant of a symmetric matrix M_G :

$$Z = (G = 0), \quad G = \det M_G, \quad M_G := \begin{pmatrix} 2x_0 & -x_3 & x_1 - 4x_5 \\ -x_3 & x_1 & x_4 \\ x_1 - 4x_5 & x_4 & 2x_2 \end{pmatrix}.$$

The singular locus of Z is a Veronese surface $S (\cong \mathbf{P}^2)$ and Z is the secant variety of S . A parametrization of S is given by

$$\begin{aligned} \nu': \mathbf{P}^2 &\xrightarrow{\cong} S := \text{Sing}(Z), \\ (s : t : u) &\longmapsto (x_0 : \cdots : x_5) := (s^2 : 2u^2 : t^2 : -2us : 2ut : \frac{1}{2}(-st + u^2)). \end{aligned}$$

The intersection of the space of Hodge classes B_K^2 of (J^2, K) with S is the conic C_K from Proposition 4.1:

$$\mathbf{P}B_K^2 \cap S = C_K.$$

Proof. For a general Cartan subalgebra K we may assume that its eigenvectors are of the form $v_{\pm} = (\alpha_{\pm}, 1) \in \mathbf{C}^2$. These eigenvectors determine the classes $\omega_{\pm}, \omega_K \in B^1$ as in Lemma 3.3. Any element in B_K^2 can then be written as

$$\omega(\alpha_+, \alpha_-, r, s, t) := r\omega_+^2 + s\omega_K^2 + t\omega_-^2 \quad (\in B_K^2 \subset B^2),$$

for a certain $(r, s, t) \in \mathbf{C}^3$. So the problem is to determine the image of \mathbf{C}^5 , with coordinates $\alpha_+, \alpha_-, r, s, t$, in $\mathbf{P}B^2$ under this map. We used a computer to find the polynomial G defining Z and the singular locus of Z . (The ‘difference’ between the Veronese surface of $\mathbf{P}B_1$ in $\mathbf{P}B^2$, which is $\{\omega^2 \in \mathbf{P}B^2 : \omega \in B_1\}$ and which lies in Z , and S is only in the component along V_0 in the decomposition $B^2(J^2) = V_4 \oplus V_0$ in Proposition 3.2.) Choosing K to be the diagonal Cartan subalgebra as in the proof of Proposition 4.1, the plane $\mathbf{P}B_K^2$ is defined by $x_1 = x_3 = x_4 = 0$. The 2×2 minors of M_G , which define S , restrict to multiples of $4x_5^2 - x_0x_2$, which is thus the equation of C_K . Using the SL_2 -action, the same result then holds for any K .

More intrinsically, the second Veronese map $\nu_2: \mathbf{P}B^1 \rightarrow \mathbf{P}B^2$, with $\nu_2(\omega) = \omega^2$, maps the invariant conic $C_2 \subset \mathbf{P}B^1$ to a quartic rational normal curve $C_4 = \nu_2(C_2)$ which is the intersection of the Veronese surface $\nu_2(\mathbf{P}B^1)$ with a hyperplane $H \subset \mathbf{P}B^2$ (this hyperplane is the projectivization of the subrepresentation V_4). The plane $\mathbf{P}B_K^2$ is spanned by $\omega_K^2, \omega_+^2, \omega_-^2$ where $\omega_+, \omega_- \in C_2$. Hence $\mathbf{P}B_K^2$ intersects H in the line spanned by ω_+^2, ω_-^2 . Thus $Z \cap H$ is the secant variety of C_4 , which is a cubic (determinantal) threefold (cf. [7, Proposition 9.7]). Therefore Z is a cubic fourfold.

Since the planes $\mathbf{P}B_K^2$ intersect in points, the variety Z is singular and $\dim \text{Sing}(Z) \geq 2$. Any secant line of $\text{Sing}(Z)$ intersects Z with multiplicity at least four, hence it lies in Z . Thus Z is the secant variety of $\text{Sing}(Z)$. The intersection of $\text{Sing}(Z)$ with H is the curve C_4 , hence $\text{Sing}(Z)$ is a surface of degree four in \mathbf{P}^5 . Since Z contains the Veronese surface of $\mathbf{P}B^1$, it cannot be the secant variety of a rational normal scroll and thus it is the secant variety of a Veronese surface by [6, Proposition, p. 525]. \blacksquare

Remark 4.3. For a Cartan subalgebra K , the conic $C_K = S \cap \mathbf{P}B_K^2$ is a conic in the Veronese surface $S = \nu'(\mathbf{P}^2)$ and thus $C_K = \nu'(L_K)$ for a line $L_K \subset \mathbf{P}^2$. If K' is another Cartan subalgebra, then the lines L_K and $L_{K'}$ intersect in a point $p \in \mathbf{P}^2$. Then $\nu'(p) \in C_K \cap C_{K'} \subset \mathbf{P}B_K^2 \cap \mathbf{P}B_{K'}^2$. Thus we find another proof, besides Proposition 4.1, of the fact that any two planes B_K^2 intersect, see also Section 5.

4.3 The rational curve C_4 in the two Veronese surfaces

In $\mathbf{P}B^2$ there are two ‘natural’ Veronese surfaces. One is the image of $\nu_2: \mathbf{P}B^1 \rightarrow \mathbf{P}\mathrm{Sym}^2(B^1) = \mathbf{P}B^2$ and the other is the singular locus S of Z which is the image of a Veronese map $\nu': \mathbf{P}^2 \rightarrow \mathbf{P}B^2$. The intersection of these two surfaces turns out to be a smooth rational curve which we denote by C_4 . Given a general point $c \in \cup_K \mathbf{P}B_K^2 \subset Z$, this curve allows us to determine the Cartan algebra K such that $c \in B_K^2$, see Section 4.4. We apply this in Section 6.

Lemma 4.4. *The intersection of the two Veronese surfaces $\nu_2(\mathbf{P}B^1)$ and $\nu'(\mathbf{P}^2)$ is a smooth rational normal curve C_4 of degree four*

$$C_4 = \nu_2(\mathbf{P}B^1) \cap \nu'(\mathbf{P}^2).$$

The curve C_4 is the image under ν_2 and ν' of the following conics:

$$C_4 = \nu_2(C_2) = \nu'(\{(s:t:u) \in \mathbf{P}^2: u^2 = st\}),$$

where $C_2 \subset \mathbf{P}B^1$ is the invariant conic. The intersection of C_4 and $\mathbf{P}B_K^2$, for a Cartan subalgebra $K \subset M_2(\mathbf{C})$, consists of the two eigenvectors of K^\times in $\mathbf{P}B_K^2$:

$$C_4 \cap \mathbf{P}B_K^2 = \{\omega_+^2, \omega_-^2\}.$$

Proof. The invariant conic C_2 can be parametrised by $\omega = (xe_+ + ye_-)^2 = x^2\omega_1 + xy\omega_\sigma + y^2\omega_2 \in \mathbf{P}B^1$ with $(x, y) \in \mathbf{C}^2$. Then $\nu_2(C_2)$ is a rational normal quartic curve parametrised by the $\omega^2 \in \mathbf{P}B^2$. Similarly, the conic defined by $u^2 = st$ in \mathbf{P}^2 can be parametrised by $(s:t:u) = (x^2:y^2:xy)$. One verifies that its image under ν' is $\nu_2(C_2)$, hence we found a curve $C_4 \subset \nu_2(\mathbf{P}B^1) \cap \nu'(\mathbf{P}^2)$. A computer verified that there are no other points in the intersection.

From Theorem 4.2, we know that the intersection of $\mathbf{P}B_K^2$ with the singular locus $S = \nu'(\mathbf{P}^2)$ of Z is the conic $C_K \subset \mathbf{P}B_K^2$ defined in Proposition 4.1. The group K^\times fixes B_K^2 and has three orbits on $C_4 = \nu_2(C_2) = \nu_4(\mathbf{P}V_1)$, the fourth Veronese map, they are the eigenvectors $\omega_\pm^2 = \nu_4(v_\pm)$ and their complement. The eigenvectors are in $C_4 \cap \mathbf{P}B_K^2$, but since C_4 spans a \mathbf{P}^4 , the third orbit cannot intersect $\mathbf{P}B_K^2$. ■

Remark 4.5. We discuss another aspect of the relation between the Veronese surfaces $\nu_2(\mathbf{P}B^1)$, with $\nu_2(\theta) := \theta^2 \in \mathbf{P}B^2$, and $S = \nu'(\mathbf{P}^2) = \mathrm{Sing}(Z)$. For a Cartan subalgebra K , the plane of Hodge classes $\mathbf{P}B_K^2$ is the span of the three points ω_+^2 , ω_-^2 and ω_K^2 in $\nu_2(\mathbf{P}B^1)$. The closure of the union of these planes is the cubic fourfold Z which turned out to have another Veronese surface S as singular locus. We showed that Z is the secant variety of S and that the intersection of $\nu_2(\mathbf{P}B^1)$ and S is the quartic normal curve $C_4 = \nu_4(\mathbf{P}V_1)$.

One can recover the Veronese surface $\nu_2(\mathbf{P}B^1)$ from the secant variety Z of S and the curve C_4 as follows: the varieties $\mathbf{P}B^2 \supset Z \supset S$ parametrize the conics, those of rank at most two and those of rank one in a \mathbf{P}^2 . The duality map $g: \mathbf{P}B^2 \rightarrow \mathbf{P}^5$ is the birational map defined by duality of conics. It sends the symmetric 3×3 matrix defining the conic to its adjoint and the base locus of g is S . The image of Z is a Veronese surface $S^\vee \subset \mathbf{P}^5$. The image of a rank 2 conic, so the union of two distinct lines in \mathbf{P}^2 , is the intersection point of these two lines in \mathbf{P}^2 . The fiber of $g|_{Z-S}$ over p consists of all (unordered) pairs of distinct lines which intersect in p , so it is the complement of the diagonal in $\mathrm{Sym}^2 \mathbf{P}^1 \cong \mathbf{P}^2$. The closure of this fiber is a plane \mathbf{P}_p^2 in $\mathbf{P}B^2$ which intersects S along a conic C_p that parametrizes the double lines that contain p .

Recall that also the plane $\mathbf{P}B_K^2$ intersects S along a conic C_K . As a plane intersecting S along a conic is the span of $\nu'(L)$ for a line L in \mathbf{P}^2 , there is only a two-dimensional family of such planes and thus the general \mathbf{P}_p^2 is a $\mathbf{P}B_K^2$.

To recover $\nu_2(\mathbf{P}B^1)$ we recall that $\mathbf{P}B_K^2$ intersects C_4 in the two points ω_\pm^2 . By the proof of Proposition 4.1, C_K is the conic in $\mathbf{P}B_K^2 = \mathbf{P}\{\omega_+^2 + b\omega_K^2 + c\omega_-^2\}$ defined by $4b^2 = ac$. Thus the

tangent lines to C_K in the two points ω_{\pm}^2 intersect in ω_K^2 . As any point in $\nu_2(\mathbf{P}B^1)$, not on C_4 , is a ω_K^2 for a suitable K , we recover this Veronese surface from S and C_4 as follows: in \mathbf{P}_p^2 we consider the conic $C_p = S \cap \mathbf{P}_p^2$ and the two points $p_+, p_- \in C_p \cap C_4$. Let $q_p \in \mathbf{P}_p^2$ be the point of intersection of the tangent lines to C_p in p_{\pm} . The image of the map $S^{\vee} \rightarrow Z \subset \mathbf{P}^5$, $p \mapsto q_p$ is the Veronese surface $\nu_2(\mathbf{P}B^1)$. This map is a section of the duality map.

4.4 Finding K

Given a codimension two Hodge class $c \in \cup \mathbf{P}B_K^2 \subset Z \subset \mathbf{P}B^2$, we want to determine a Cartan subalgebra K such that $c \in \mathbf{P}B_K^2$. This K will be unique if c is general. The case where K is not unique is discussed in the next section and occurs only if $c \in S$, the singular locus of Z (see Proposition 4.1 and Theorem 4.2).

As in Remark 4.5, a point $c \in Z - S$ corresponds to a reducible conic $L \cup M$ in a \mathbf{P}^2 , which is the dual of the domain of ν' , whereas a point $q \in S = \nu'(\mathbf{P}^2)$ corresponds to a double line in this \mathbf{P}^2 . The pencil $sc + tq$ lies in Z iff all conics in it are reducible, so iff the double line which corresponds to q passes through the point of intersection of L and M . There are two lines passing through this point that are tangent to the conic whose (double) tangent lines parametrize C_4 . Thus for a given $c \in Z - S$ we find two points q on C_4 such that the lines $\langle c, q \rangle$ are contained in Z .

From Lemma 4.4, we know that there are exactly two points $\omega_{\pm}^2 \in C_4$ in any $\mathbf{P}B_K^2$ and these points in fact determine K . In particular the two lines spanned by c and ω_{\pm}^2 lie in $\mathbf{P}B_K^2$ and thus lie in Z . We therefore consider the following procedure.

Let $q(x, y)$, for $(x, y) \in \mathbf{C}^2$, be the parametrization of C_4 given by the composition $\mathbf{P}V_1 \rightarrow \mathbf{P}\text{Sym}^2(V_1) \cong \mathbf{P}B^1 \xrightarrow{\nu_2} \mathbf{P}B^2$:

$$q: \mathbf{P}(V_1 \otimes_{\mathbf{Q}} \mathbf{C}) \xrightarrow{\cong} C_4, \quad q(xe_+ + ye_-) = (x^2\omega_1 + xy\omega_{\sigma} + y^2\omega_2)^2.$$

The line spanned by c and $q(x, y)$ is parametrized by $sc + tq(x, y)$. As $C_4 \subset S$, the singular locus of Z , and Z is defined by the cubic polynomial $G = 0$, we must have

$$G(sc + tq(x, y)) = f(x, y)^2 s^2 t,$$

for some polynomial $f \in \mathbf{C}[x, y]$, homogeneous of degree 2. In case $f(x, y) = 0$, the line $sc + tq(x, y)$ lies in Z . Thus we conclude that for these (x, y) the points $q(x, y) = (xe_+ + ye_-)^4$ are the points ω_{\pm}^2 . Hence K is determined by the two eigenvectors $xe_+ + ye_- \in V_1 \otimes_{\mathbf{Q}} \mathbf{C}$ where the points $(x : y)$ are the zeroes of f .

The next theorem summarizes the main results we obtained.

Theorem 4.6. *A Hodge class in $Z - \text{Sing}(Z)$ deforms in at most one four-dimensional family of abelian fourfolds of Weil type, whereas a Hodge class in $\text{Sing}(Z)$ may deform in infinitely many four-dimensional families of Weil type with distinct imaginary quadratic fields.*

Proof. For $c \in \cup \mathbf{P}B_K^2 \subset Z$, $c \notin S = \text{Sing}(Z)$, there is a unique Cartan subalgebra K such that $c \in \mathbf{P}B_K^2$, see Section 4.4. If K is the complexification of an imaginary quadratic field embedded in $M_2(\mathbf{Q})$, then c deforms in a family of Weil type with that field. If $c \in S$ then c lies in infinitely many $\mathbf{P}B_K^2$, see also Section 5. \blacksquare

5 The Cayley classes

5.1 The Hodge classes in $\text{Sing}(Z)$

Combining Theorem 4.2 and Remark 4.3 we conclude that any point $c \in S = \text{Sing}(Z)$ lies in $\mathbf{P}B_K^2$ for a one-dimensional family of Cartan subalgebras K . Such classes, which we will call Cayley classes following [10], are thus of special interest.

For a fixed Cayley class $c \in S$ we want to find all Cartan subalgebras K such that $c \in B_K^2$. For this it is convenient to use the parametrization $\nu': \mathbf{P}^2 \rightarrow S$ given in Theorem 4.2. The plane of Hodge classes $\mathbf{P}B_K^2$ intersects S along a conic C_K (see Proposition 4.1) and the inverse image $L_K := \nu'^{-1}(C_K)$ is a line in \mathbf{P}^2 which is easily computed:

$$C_K = \nu'(L_K), \quad L_K: s - \alpha_+ \alpha_- t + (\alpha_+ + \alpha_-)u = 0 \quad (\subset \mathbf{P}^2),$$

where $v_{\pm} = (\alpha_{\pm}, 1)$ are the eigenvectors of K . So if $p \in \mathbf{P}^2$ is such that $c = \nu'(p)$ then $c \in B_K^2$ iff $p \in L_K$.

It is not always the case that a Cayley class in the \mathbf{Q} -vector space $B^2(J^2)$ underlying B^2 lies in a B_K^2 , where K is an imaginary quadratic field (rather than a Cartan subalgebra of $M_2(\mathbf{C})$). Taking for example the point $p = (n : 1 : 0) \in \mathbf{P}^2$ we find that it lies in L_K iff $\alpha_+ \alpha_- = n$ (cf. Proposition 5.2), but for an imaginary quadratic field we must have $\alpha_+ = \bar{\alpha}_- \in K$ and then $\alpha_+ \alpha_- > 0$. So for $n < 0$ the Cayley class $\nu'(p)$ does not deform to a Hodge class in any family of Weil type.

Example 5.1. We consider a Hodge class $C_{\phi} \in \mathbf{P}B^2$ which lies in S :

$$C_{\phi} = \omega_{\sigma}^2 + 4\omega_1\omega_2 = (0 : 1 : 0 : 0 : 0 : 4) = \nu'((0 : 0 : 1)).$$

The corresponding Cartan subalgebras are thus the $K \subset M_2(\mathbf{C})$, determined by $v_{\pm} = (\alpha_{\pm}, 1)$, such that the point $p = (s : t : u) = (0 : 0 : 1)$ lies on the line L_K , equivalently, $\alpha_+ + \alpha_- = 0$.

In particular, for any $d > 0$, the Cartan algebra defined by the embedding $\phi: K = \mathbf{Q}(\sqrt{-d}) \hookrightarrow M_2(\mathbf{Q})$ given in Section 3.3, which is determined by the eigenvectors $v_{\pm} = (\alpha_{\pm}, 1)$ with $\alpha_{\pm} = \pm\sqrt{-d}$, has this Cayley class in B_K^2 . Thus there are infinitely many, four-dimensional, families of abelian varieties of Weil type with the property that C_{ϕ} lies in the space of their Hodge classes.

5.2 Markman's Cayley class

Markman in [10] proves the Hodge conjecture for all abelian fourfolds of Weil type with trivial discriminant. For a fixed integer $n \geq 2$ he considers a five-dimensional family of four-dimensional complex tori T with $H^1(T, \mathbf{Z}) = V$, where $V := H^1(X, \mathbf{Z}) \oplus H^1(\hat{X}, \mathbf{Z})$ for an abelian surface X and \hat{X} is the dual abelian surface. The complex structure on T is determined by the subspace $H^{1,0}(T)$ in the complexification of V .

He shows that a certain class $C_n \in \wedge^4 V = H^4(T, \mathbf{Z})$ remains of Hodge type $(2, 2)$ for all members of the family. Moreover C_n is shown to be the second Chern class of a complex vector bundle \mathcal{E}_T over T using deformation theory in the hyperkähler context. The general torus in the family is not an abelian variety, but for each imaginary quadratic field K there is a four-dimensional subfamily of abelian fourfolds of Weil type, with trivial discriminant, with this field. Thus C_n is algebraic, being a Chern class. Using the K^{\times} -action on B_K^2 one finds that all classes in B_K^2 are algebraic for any of these fourfolds of Weil type.

We will now assume that $X = J$, and that ω_J defines a principal polarization on J . Then we may identify J^2 with $X \times \hat{X}$ and $H^1(J^2, \mathbf{Z})$ with V . The four-dimensional families of Weil type will in general not contain the three-dimensional subfamilies of tori T with $H^{1,0}(T) = H^{1,0}(J) \times H^{1,0}(J)$, where J varies over the principally polarized abelian surfaces. They will often intersect this family in codimension one and for the corresponding J 's one has $\text{End}_{\mathbf{Q}}(J) \neq \mathbf{Q}$, see Remark 5.3. The same is true for the similar families considered by O'Grady [13, Theorem 5.1].

The following proposition identifies the cases in which the four-dimensional families do contain all such decomposable abelian fourfolds of Weil type and their deformations. It shows that Markman's Cayley classes C_n are indeed Cayley classes as defined above. Moreover, it identifies

the corresponding imaginary quadratic fields and their embeddings into $\text{End}(J^2)$. The proposition shows that there must exist an $\alpha_+ \in K$ with conjugate $\alpha_- := \overline{\alpha_+}$ and $\text{Nm}(\alpha_+) = n$, which is quite restrictive (for example if $p \equiv 3 \pmod{4}$ is a prime number then there is no $\alpha \in \mathbf{Q}(\sqrt{-1})$ with $\alpha\bar{\alpha} = p$). This reflects the fact that the Néron Severi group of J must have rank two in the other cases.

Proposition 5.2. *The Cayley class in [10, Proposition 11.2] can also be written as*

$$C_n = n(-n\omega_\sigma^2 + 2n^2\omega_1^2 + 2\omega_2^2)$$

and $C_n \in \text{Sing}(Z)$. For any Cartan subalgebra $K \hookrightarrow M_2(\mathbf{Q})$ with eigenvectors $v_\pm = (\alpha_\pm, 1)$ such that $\alpha_+\alpha_- = n$ we have $C_n \in B_K^2$. In fact

$$(\alpha_+ - \alpha_-)^2 C_n = n(-\alpha_+\alpha_-\omega_K^2 + 2\alpha_-^2\omega_+^2 + 2\alpha_+^2\omega_-^2) \quad (\in B_K^2).$$

Proof. The Cayley class from [10, Proposition 11.2] is

$$C_n := n(-nc_1(\mathcal{P})^2 + 4n^2\pi_1^*[pt_X] + 4\pi_2^*[pt_{\hat{X}}]),$$

where \mathcal{P} is the normalized Poincaré bundle over $X \times \hat{X}$. Under the isomorphism $J^2 \rightarrow X \times \hat{X}$ we have $\omega_1^2 = 2\pi_1^*[pt_X]$ and $\omega_2^2 = 2\pi_2^*[pt_{\hat{X}}]$. The isomorphism $J = X \rightarrow \hat{X}$ induces the map on lattices $\Lambda = H_1(J, \mathbf{Z}) \rightarrow \hat{\Lambda} = H_1(\hat{X}, \mathbf{Z})$, $\lambda' \mapsto \omega_J(-, \lambda')$. The Chern class $c_1(\mathcal{P})$ is the 2-form on $\Lambda \times \hat{\Lambda}$ defined by $((\lambda, l), (\mu, m)) := m(\lambda) - l(\mu)$. One finds that $c_1(\mathcal{P})$ pulls back to ω_σ and that the first expression for C_n holds.

To find all Cartan subalgebras $K \subset M_2(\mathbf{Q})$ for which C_n is a Hodge class in B_K^2 , it is convenient to identify the point in \mathbf{P}^2 which maps to C_n under the Veronese map ν' :

$$C_n = n(-n\omega_\sigma^2 + 2n^2\omega_1^2 + 2\omega_2^2) = (2n^2 : 0 : 2 : 0 : 0 : -n) = \nu'((n : 1 : 0)) \in S = \text{Sing}(Z).$$

Thus $C_n \in B_K^2$ iff $(s : t : u) = (n : 1 : 0) \in L_K$, which is equivalent to $\alpha_+\alpha_- = n$. The last equality is easy to verify. \blacksquare

Remark 5.3. We discuss the fact that in Proposition 5.2 one doesn't find all imaginary quadratic fields. With the notation from [10, Section 12], the five-dimensional family is parametrized by Ω_{w^\perp} and $w \in S^+$ determines the class C_n . The four-dimensional subfamilies are parametrized by $\Omega_{\{w,h\}^\perp}$ and consist of the four dimensional tori \mathcal{T}_ℓ with $\ell \in \Omega_{\{w,h\}^\perp}$, in particular $(h, \ell) = 0$ for a class $h \in w^\perp \cap S^+$ with $(h, h)_{S^+} < 0$ (cf. [10, formula (12.6), Corollary 12.9]). The imaginary quadratic field $K \subset \text{End}_{\mathbf{Q}}(\mathcal{T}_\ell)$ is generated by an endomorphism $\Theta'_h \in \text{End}(\mathcal{T}_\ell)$ [10, Lemma 12.5]. One can find such h for which not all periods $\ell \in \Omega_{w^\perp}$ in the three-dimensional family parametrizing the abelian fourfolds $\mathcal{T}_\ell \cong J \times J$ will satisfy $(h, \ell) = 0$. For such an h and any $\ell \in \Omega_{\{w,h\}^\perp}$ such that $\mathcal{T}_\ell \cong J \times J$, the endomorphism Θ'_h of $J \times J$ cannot be in $M_2(\mathbf{Q})$, since it does not deform to all selfproducts. Therefore $\text{End}_{\mathbf{Q}}(J) \neq \mathbf{Q}$. From this one can deduce that the Picard number of J cannot be one.

6 Applications to algebraic cycles

6.1 Jacobians of genus two curves

We consider two cases, [17] and [4], where explicit Weil classes on decomposable fourfolds $A = J \times J$ are given in order to illustrate our results.

In both cases, the abelian surface J is the Jacobian of a general genus two curve C . After fixing a base point on C , the Abel–Jacobi image of $C \hookrightarrow J$, which we denote by C again, is a divisor

which defines a principal polarization. Using the notation from Section 3.1, $\omega_J \in H^2(J, \mathbf{Q})$ is the class of $C \subset J$. We now consider some codimension two classes defined by surfaces in J^2 . Recall that translations act trivially on the cohomology of abelian varieties.

The diagonal $S_1 := \Delta = \{(x, x) : x \in J \times J\}$ is a codimension two subvariety. It is the inverse image of the point $0 \in J$ along the difference map:

$$S_1 := \Delta := \delta^{-1}0, \quad \delta: J \times J \longrightarrow J, \quad (x, y) \longmapsto x - y.$$

Its class $[S_1]$ is thus the pull-back along δ of the class of a point on J . Since the intersection of C with a general translate consists of two points (in fact $\omega_J^2 = 2dx_1 \wedge \cdots \wedge dy_2$), we find $2[S_1] = \delta^*(\omega_J^2)$ and hence

$$[S_1] = [\Delta] = \frac{1}{2}\delta^*(\omega_J^2) = \frac{1}{2}(\omega_1 + \omega_2 - \omega_\sigma)^2 \quad (\in B^2(J^2)).$$

The surface $S_2 := C \times C \subset J \times J$ is the intersection of the divisors $C \times J$ and $J \times C$, these divisors have classes ω_1, ω_2 respectively and thus

$$[S_2] = [C \times C] = \omega_1\omega_2 \quad (\in B^2(J^2)).$$

6.2 The class of the Schoen surface

As in [17], we consider the 2-cycle $S := S_1 + S_2$ in $J \times J$ with S_1, S_2 as in Section 6.1. We will use our results to show that its class in $H^4(J^2, \mathbf{Q})$ lies in a unique B_K^2 . We also determine K as well as the type of the (essentially unique) polarization ω_K of Weil type. The class of S is given by

$$2[S] = 2([S_1] + [S_2]) = \omega_1^2 + 4\omega_1\omega_2 + \omega_2^2 - 2\omega_1\omega_\sigma - 2\omega_2\omega_\sigma + \omega_\sigma^2.$$

It is easy to check that $[S]$ lies on the cubic fourfold $Z = \overline{\cup B_K^2}$. To find the subalgebra $K \subset M_2(\mathbf{Q})$ such that $[S] \in B_K^2$ we consider the intersection of the line in \mathbf{P}^5 spanned by $c := 2[S]$ and a general point $q(x, y) = (xe_+ + ye_-)^4 \in C_4$ as in Section 4.4. A computation shows that

$$G(sc + tq) = 8(x^2 + xy + y^2)^2 s^2 t, \quad \text{hence} \quad x^2 + xy + y^2 = (x - \alpha_+ y)(x - \alpha_- y),$$

with $\alpha_\pm = (-1 \pm \sqrt{-3})/2$. The Cartan subalgebra K is determined by the eigenvectors $v_\pm = (\alpha_\pm, 1)$ and thus the field is $K = \mathbf{Q}(\sqrt{-3})$. The eigenvectors also determine the embedding $K \hookrightarrow M_2(\mathbf{Q})$:

$$K = \mathbf{Q}(\sqrt{-3}) \hookrightarrow M_2(\mathbf{Q}), \quad \frac{-1 + \sqrt{-3}}{2} \longmapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

With this embedding of K , the Hodge class $[S]$ lies in B_K^2 and will therefore deform to a (unique) family of abelian fourfolds of Weil type with this field K .

The polarization ω_K is (up to scalar multiple) equal to ω_K and we choose (cf. Lemma 3.3)

$$\omega_K = 2\omega_1 - \omega_\sigma + 2\omega_2, \quad \text{Pfaff}(\omega_K) = (1 - 2 \cdot 2)^2 = 9.$$

Notice that in [17, proof of Lemma 10.4] one finds an ample line bundle \mathcal{L} on a general deformation of Weil type which has $h^0(\mathcal{L}) = \text{Pfaff}(c_1(\mathcal{L})) = 9$, our computation is consistent with this. A further computation shows that the elementary divisors of the alternating form defined by ω_K on \mathbf{Z}^8 are $(1, 1, 3, 3)$.

In [17], Schoen proved that actually the surface $S_1 \cup S_2$ in J^2 deforms in a family of abelian varieties of Weil type with field $\mathbf{Q}(\sqrt{-3})$. These deformations were further studied in [1] and [14].

6.3 The field $\mathbf{Q}(i)$

In [4] one finds a four-dimensional family of principally polarized abelian fourfolds of Weil type with field $K = \mathbf{Q}(\sqrt{-1})$. There is a rational dominant map from such a fourfold A (in general with 16 base points) $A \rightarrow Q$, where $Q \subset \mathbf{P}^5$ is a smooth four-dimensional quadric. The pull-back to A of a plane $\mathbf{P}^2 \subset Q$ is shown to be an exceptional cycle c (i.e., its class lies in $B^2(A)$ but is not a scalar multiple of ω_K^2). The class of this cycle was not completely determined in [4], but it can be easily done now.

In the proof of Theorem 3.7 of [4] it is shown that, for some $a, b \in \mathbf{Z}$, the class c of the pull-back of a certain plane to $J \times J$ is

$$c = [T] + a[S_1] + b[S_{-1}], \quad [T] = [(2C) \times (2C)] = 4[S_2],$$

$$S_{-1} := \{(x, y) \in J \times J : x + y = 0\}.$$

The class of S_{-1} is obviously $(1, -1)^*[S_1]$, hence

$$[S_{-1}] = \frac{1}{2}(\omega_1 + \omega_2 + \omega_\sigma)^2.$$

To determine a, b we recall from [4] that the family contains fourfolds $A = J \times J$, with J a general abelian surface as before, the polarization ω_K is the product polarization $\omega_1 + \omega_2$ and the embedding of the field is

$$K = \mathbf{Q}(\sqrt{-1}) \hookrightarrow M_2(\mathbf{Q}), \quad \sqrt{-1} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This embedding has the eigenvectors $v_\pm = (\mp i, 1)$. From Lemma 3.3 one finds that B_K^2 is spanned by

$$\omega_K^2 = 4(\omega_1 + \omega_2)^2, \quad \omega_+^2 = (\omega_1 + i\omega_\sigma - \omega_2)^2, \quad \omega_-^2 = (\omega_1 - i\omega_\sigma - \omega_2)^2.$$

It is now easy to check that

$$[T] + a[S_1] + b[S_{-1}] \in B_K^2 \iff a = b = 1,$$

thus we determined the class of the exceptional cycle: $c = [T] + [S_1] + [S_{-1}]$.

7 The discriminant of the polarization

7.1 The discriminant

Let (A, K, E) be a polarized abelian variety of Weil type with imaginary quadratic field $K = \mathbf{Q}(\sqrt{-d})$ and $\dim A = n$. The \mathbf{Q} -bilinear alternating form E on $H_1(A, \mathbf{Q})$ satisfies $E(kx, ky) = \text{Nm}(k)E(x, y)$ for $x, y \in H_1(A, \mathbf{Q})$ and $k \in K$, where $\text{Nm}(k) = k\bar{k} \in \mathbf{Q}$ is the norm of k . Then $H_1(A, \mathbf{Q})$ is a K -vector space and E defines a K -valued Hermitian form H on this K -vector space by:

$$H: H_1(A, \mathbf{Q}) \times H_1(A, \mathbf{Q}) \longrightarrow K, \quad H(x, y) := E(x, (\sqrt{-d})_* y) + \sqrt{-d}E(x, y).$$

If $\Psi \in M_n(K)$ is the Hermitian matrix defining H w.r.t. some K -basis of $H_1(A, \mathbf{Q})$ then $\det(\Psi) \in \mathbf{Q}^\times = \mathbf{Q} - \{0\}$ and the class of $\det(\Psi) \in \mathbf{Q}^\times / \text{Nm}(K^\times)$ is independent of the choice of the basis, it is called the discriminant of (A, K, E) .

7.2 Trivial discriminant for selfproducts

Let $A = J \times J$, for an abelian surface J and let $K \subset \text{End}_{\mathbf{Q}}(A) = M_2(\mathbf{Q})$ be an imaginary quadratic field. Let $\omega_K \in H^2(A, \mathbf{Q})$ be the unique (up to scalar multiple) K -invariant polarization as in Lemma 3.3 on the abelian fourfold of Weil type (A, K) . We show in Proposition 7.1 that the discriminant of (A, K, ω_K) is trivial, and we briefly discuss more general decomposable abelian fourfolds of Weil type in Remark 7.3.

Proposition 7.1. *Let J be a general abelian surface, let $A = J \times J$, and let (A, K, ω_K) be an abelian fourfold of Weil type. Then the discriminant of (A, K, ω_K) in $\mathbf{Q}^\times / \text{Nm}(K^\times)$ is trivial.*

Proof. Recall the basis dt_i , $i = 1, \dots, 4$, of $H^1(J, \mathbf{Q})$ from Section 3.1. Let f_1, \dots, f_4 be the dual basis of $H_1(J, \mathbf{Q})$, so that $dt_i(f_j) = \delta_{ij}$, where δ_{kl} is Kronecker's delta. Then $H_1(A, \mathbf{Q}) = H_1(J, \mathbf{Q})^{\oplus 2}$ and, since $\text{End}_{\mathbf{Q}}(J) = \mathbf{Q}$, the vectors $(f_1, 0), \dots, (f_4, 0)$ are a K -basis of $H_1(A, \mathbf{Q})$.

In view of the definitions given in Proposition 3.1, for $i, j = 1, \dots, 4$, we find the following table:

$$\begin{aligned} \omega_1((f_i, 0), (f_j, 0)) &= \omega_J(f_i, f_j), & \omega_1((f_i, 0), (0, f_j)) &= 0, \\ \omega_1((0, f_i), (0, f_j)) &= 0, \\ \omega_2((f_i, 0), (f_j, 0)) &= 0, & \omega_2((f_i, 0), (0, f_j)) &= 0, \\ \omega_2((0, f_i), (0, f_j)) &= \omega_J(f_i, f_j), \\ \omega_\sigma((f_i, 0), (f_j, 0)) &= 0, & \omega_\sigma((f_i, 0), (0, f_j)) &= \omega_J(f_i, f_j), \\ \omega_\sigma((0, f_i), (0, f_j)) &= 0, \end{aligned}$$

here one computes (with σ as in the proof of Proposition 3.1):

$$\omega_\sigma((f_i, 0), (0, f_j)) = \sigma^* \omega_J((f_i, 0), (0, f_j))$$

since $\sigma^* \omega_J = \omega_1 + \omega_2 + \omega_\sigma$ and ω_1, ω_2 are zero on $((f_i, 0), (0, f_j))$, and finally $\sigma^* \omega_J((f_i, 0), (0, f_j)) = \omega_J(\sigma(f_i, 0), \sigma(0, f_j)) = \omega_J(f_i, f_j)$.

The action of $K \subset M_2(\mathbf{Q})$ on $H_1(A, \mathbf{Q}) = H_1(J, \mathbf{Q}) \otimes V_1$ is determined by $a, b \in \mathbf{Q}$ such that $\sqrt{-d}_*(f_i, 0) = (af_i, bf_i)$, for $i = 1, \dots, 4$. The polarization is a linear combination

$$\omega_K = a_1 \omega_1 + a_\sigma \omega_\sigma + a_2 \omega_2, \quad a_1, a_\sigma, a_2 \in \mathbf{Q}.$$

The Hermitian form H is determined by its matrix $\Psi = (\Psi_{ij})$ on the K -basis of $H_1(A, \mathbf{Q})$ given above:

$$\Psi_{ij} := \omega_K((f_i, 0), (af_j, bf_j)) + \sqrt{-d} \omega_K((f_i, 0), (f_j, 0)).$$

From the table we see that

$$\omega_K((f_i, 0), (f_j, 0)) = a_1 \omega_J(f_i, f_j), \quad \omega_K((f_i, 0), (0, f_j)) = a_\sigma \omega_J((f_i, 0), (0, f_j))$$

and thus

$$\Psi_{ij} = (a_1(a + \sqrt{-d}) + ba_\sigma) \omega_J(f_i, f_j).$$

Since H is a Hermitian form, we have $\Psi_{ij} = \overline{\Psi_{ji}}$ and using $\omega_J(f_i, f_j) = -\omega_J(f_j, f_i)$ we get $aa_1 + ba_\sigma = 0$. Therefore $\Psi_{ij} = (a_1 \sqrt{-d}) \omega_J(f_i, f_j)$. Since the f_i are a symplectic basis of $H_1(J, \mathbf{Q})$ we get $\det(\omega_J(f_i, f_j)) = 1$. Thus $\det(\Psi) = a_1^4 d^2 = \text{Nm}(a_1^2 d)$ is the norm of an element in $\mathbf{Q} \subset K$, so the discriminant of (A, K, ω_K) is trivial. \blacksquare

7.3 General discriminants

Whereas J^2 , with $\text{End}_{\mathbf{Q}}(J) = \mathbf{Q}$, only has structures of Weil type with trivial discriminant, for example the self product E^4 of an elliptic curve E with CM by K admits polarizations with any discriminant. In fact, if $\text{End}_{\mathbf{Q}}(E) = K$, then the embedding $K \hookrightarrow \text{End}_{\mathbf{Q}}(E^4) = M_4(K)$ given by $k \mapsto (k, k, \bar{k}, \bar{k})$ defines an abelian fourfold of Weil type. Let ω_E be a polarization on E (it is unique up to scalar multiple). Let f be a K -basis of $H_1(E, \mathbf{Q}) \cong K$ and let $b := \omega_E(f, \sqrt{-d}f) \in \mathbf{Q}$.

For any $a \in \mathbf{Q}_{>0}$ the polarization $\omega_a \in \wedge^2 H^1(E^4, \mathbf{Q})$ on E^4 defined by

$$\omega_a((x_1, \dots, x_4), (y_1, \dots, y_4)) = \omega_E(x_1, y_1) + \dots + \omega_E(x_3, y_3) + a\omega_E(x_4, y_4),$$

where $x_i, y_i \in H_1(E, \mathbf{Q})$, satisfies $\omega_a(k \cdot x, k \cdot y) = k^2 \bar{k}^2 \omega_a(x, y)$. The matrix of the associated Hermitian form on the K -basis $(f, 0, 0, 0), \dots, (0, 0, 0, f)$ of $H_1(E^4, \mathbf{Q})$ is given by $\text{diag}(b, b, -b, -b)$ and thus it has determinant ab^4 . Hence the discriminant of the polarized abelian variety of Weil type (E^4, ω_a, K) is a . The dimension of $B^2(E^4)$ is however much larger than 6 and the study of the limits of spaces of Weil classes is thus more complicated.

Remark 7.2. The Hodge conjecture for abelian fourfolds of Weil type with fields $\mathbf{Q}(\sqrt{-3})$, $\mathbf{Q}(\sqrt{-1})$ and any discriminant has been proven in [15], [16] and [9], respectively. The method for both results is the same: one verifies the Hodge conjecture for a family of abelian six-folds of Weil type with the same field and trivial discriminant. Then one specializes to a product $J \times B$, where B is a general abelian fourfold of Weil type and the discriminant of B can be arbitrary. A variant of this method, using abelian varieties with quaternionic multiplication, was proposed in [5] but thus far has not had any applications.

Acknowledgements

Discussions with E. Markman, K.G. O'Grady, F. Russo and C. Schoen were very helpful. I thank the referees for their comments and suggestions.

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