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Effective generation for foliated surfaces: Results and applications

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Abstract. We explore the birational structure and invariants of a foliated surface $(X, \mathcal{F})$ in terms of the adjoint divisor $K_{\mathcal{F}} + \epsilon K_X$, $0 < \epsilon \ll 1$. We then establish a bound on the automorphism group of an adjoint general type foliated surface $(X, \mathcal{F})$, provide a bound on the degree of a general curve invariant by an algebraically integrable foliation on a surface and prove that the set of ϵ-adjoint canonical models of foliations of general type and with fixed volume form a bounded family.

1. Introduction

A central challenge in the study of the birational geometry of varieties is to understand the behavior of the pluricanonical maps

$$\phi_{|mK_X|}\colon X \dashrightarrow \mathbb{P}H^0(X, mK_X)$$

as a function of $m \in \mathbb{N}$, for those varieties which admit non-trivial pluricanonical forms.

In this paper, we are interested in studying this question for rank one foliations on surfaces $(X, \mathcal{F})$. Already for surface foliations, the problem of understanding the maps induced by the sections of $mK_{\mathcal{F}}$ appears to be quite challenging. To remedy this issue, we instead consider a slight perturbation of this problem: namely, we aim to understand the behavior of the maps

$$\phi_{|mK_{\mathcal{F}}+nK_X|}\colon X \dashrightarrow \mathbb{P}H^0(X, mK_{\mathcal{F}} + nK_X), \quad m \gg n > 0.$$

Along the way, we prove several new results on the birational structure of foliations in terms of the adjoint divisors $K_{\mathcal{F}} + \epsilon K_X$, $0 < \epsilon \ll 1$. Considering adjoint divisors of this form is a natural approach to the study of foliated varieties as it allows us to apply classical results on the positivity of the canonical bundle of varieties that may not hold if one just considers the canonical bundle of the foliation, cf. [23].

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Among other applications, these new structural results provide bounds on the automorphism groups of foliations (Theorem 1.6), bounds on the degree of curves invariant by an algebraically integrable foliation (Theorem 1.7). Finally, we also prove a boundedness results for generalizations of surface foliations of general type, which constitutes important progress towards the construction of a moduli space of this class of foliations (Theorem 1.3).

1.1. Adjoint MMP. Our first main result is the proof of the existence and termination of the MMP for divisors of the form $K_{\mathcal{F}} + \epsilon K_X$ for $0 < \epsilon \ll 1$. While the existence (and termination) of the MMP for K_X is classical and the MMP for $K_{\mathcal{F}}$ is well known [20], it is not a priori clear that the sum of these results automatically implies that one can run a $(K_{\mathcal{F}} + \epsilon K_X)$-MMP. Moreover, even assuming that it was possible to run such MMP, it is not a priori clear how to bound the singularities of the foliation and variety on the minimal model.

Theorem 1.1 (= Theorem 3.1). *Let X be a smooth projective surface and $\mathcal{F}$ a rank one foliation with canonical singularities. Then, for any $0 < \epsilon < \frac{1}{5}$, there exists a birational morphism $\varphi\colon X \to Y$ such that either*

(1) *$K_{\mathcal{G}} + \epsilon K_Y$ is nef, where $\mathcal{G} = \varphi_* \mathcal{F}$; or*

(2) *there exists a morphism $f\colon Y \to Z$ such that $\rho(Y/Z) = 1$ and $-(K_{\mathcal{G}} + \epsilon K_Y)$ is f-ample.*

Moreover, Y has klt singularities and $\mathcal{G}$ has log canonical singularities.

We also prove the existence of ϵ-adjoint canonical models.

Theorem 1.2 (= Corollary 3.4). *Notation as in Theorem* 1.1. *Suppose in addition that $K_{\mathcal{F}}$ is big. Then there exists a birational morphism*

$$p\colon (Y, \mathcal{G}) \to (Y_{\text{can}}, \mathcal{G}_{\text{can}})$$

such that

(1) *Y_{can} is projective;*

(2) *$K_{\mathcal{G}_{\text{can}}} + \epsilon K_{Y_{\text{can}}}$ is an ample $\mathbb{Q}$-Cartier divisor; and*

(3) *Y_{can} has klt singularities and $\mathcal{G}_{\text{can}}$ has log canonical singularities.*

This ϵ-adjoint canonical model must be contrasted with McQuillan's notion of a canonical model of a foliation where the underlying space is, a priori, only an algebraic space, and its projectivity is not known. Our notion of an ϵ-adjoint canonical model should also be compared with the minimal partial du Val resolution of the canonical model of a surface foliation, see [7].

We are also able to provide a precise statement on the singularities of the underlying variety which arise in this MMP, see Corollary 3.3. This control on the singularities which arise in the course of the MMP is one of the key advantages of working with adjoint foliated divisors rather than simply with the canonical divisor of a foliation.

1.2. Boundedness and effective birationality. Our next main result is a boundedness result for ϵ-adjoint canonical models of foliations of general type.

Theorem 1.3 (= Theorem 6.1). *There exists a universal real constant $\tau > 0$ such that the following statement holds.*

Fix positive real numbers $C, 0 < \epsilon < \tau$. The set of foliated pairs

$$\begin{aligned}\mathcal{M}_{2,\epsilon,C} := \{(X, \mathcal{F}) \mid\ & X \text{ is a projective klt surface},\ \mathcal{F} \text{ is rank one},\\ & (X, \mathcal{F}) \text{ is an } \epsilon\text{-adjoint canonical foliated pair},\ K_{\mathcal{F}} \text{ is big},\\ & K_{\mathcal{F}} + \epsilon K_X \text{ is ample, and } (K_{\mathcal{F}} + \epsilon K_X)^2 \leq C\}\end{aligned}$$

forms a bounded family.

We refer to Section 2.7 for the precise definition of ϵ-adjoint canonical, but we remark here that it is a natural assumption on the singularities of $(X, \mathcal{F})$. Analogous results for a more restricted class of foliated pairs have recently appeared also in [8].

The key technical ingredient in the above statement is the following effective birationality statement which follows from our results on the MMP and some new results of Birkar on adjoint linear series [2].

Theorem 1.4 (= Corollary 4.8). *Let $\tau > 0$ be the constant whose existence is established in Theorem* 1.3. *Then, for all $0 < \epsilon < \tau$, there exists a positive integer $M = M(\epsilon)$ such that the following statement holds.*

Let X be a smooth projective surface and let $\mathcal{F}$ be a rank one foliation on X with canonical singularities. Suppose that $K_{\mathcal{F}}$ is big. Then

(1) *$K_{\mathcal{F}} + \epsilon K_X$ is big; and*

(2) *$|M(K_{\mathcal{F}} + \epsilon K_X)|$ defines a birational map.*

In fact, we are able to prove versions of Theorems 1.1, 1.2, 1.3 and 1.4 which allow for the presence of a boundary divisor.

Theorem 1.4 also supplies a partial answer to [23, Problem 6.8]. To provide a complete answer to this problem would require an exact value on the universal constant τ in Theorem 1.4.

Problem 1. Determine an effective upper bound for τ.

We also remark that, as examples in [17] show, there does not exist a universal M such that $|MK_{\mathcal{F}}|$ defines a birational map, so to get an effective birationality statement, the small perturbation by K_X is necessary.

1.3. Numerical invariants of surface foliations. We are able to provide several applications of the above results to the study of numerical invariants of foliated surfaces, automorphism groups of foliations and to the study of curves invariant by foliations.

Given a big divisor $\mathbb{Q}$-Cartier divisor D, the volume $\mathrm{vol}(D)$ is defined to be

$$\mathrm{vol}(D) := \limsup_{m\to\infty} \frac{h^0(mD)}{m^n/n!}.$$

By [16, Corollary 2.2.45], we may uniquely extend the volume to a function on $\mathbb{R}$-Cartier divisors. The volume is a fundamental invariant in birational geometry and, in analogy with the classical MMP, cf. [14], we expect the set

$$\begin{aligned}\mathcal{V}_n := \{\mathrm{vol}(X, K_{\mathcal{F}}) \mid\ & \mathcal{F} \text{ is a rank one foliation of general type with canonical}\\ & \text{singularities and } X \text{ is a klt projective variety of dimension } n\}\end{aligned}$$

to be highly structured for each fixed dimension n. In particular, we expect $\mathcal{V}_n$ to be bounded away from 0. This is a challenging problem, already for $n = 2$, but by perturbing $K_{\mathcal{F}}$ slightly, we can verify a related prediction.

Theorem 1.5 (= Theorem 5.2). *Let $\tau > 0$ be the constant whose existence is established in Theorem* 1.3*. Then, for all $0 < \epsilon < \tau$, there exists $0 < v(\epsilon)$ such that the following statement holds.*

If X is a smooth projective surface, $\mathcal{F}$ is a rank one foliation with canonical singularities and $K_{\mathcal{F}}$ is big, then $\mathrm{vol}(K_{\mathcal{F}} + \epsilon K_X) \geq v(\epsilon)$.

In fact, we are able to prove the above statement allowing for a boundary divisor. As a direct consequence of the above volume bound, we get another bound on the automorphism group of a foliation of general type.

Theorem 1.6 (= Theorem 5.3). *Let $\tau > 0$ be the constant whose existence is established in Theorem* 1.3*. Then, for all $0 < \epsilon < \tau$, there exists $0 < C = C(\epsilon)$ such that the following statement holds.*

If X is a smooth projective surface, $\mathcal{F}$ is a rank one foliation with canonical singularities and $K_{\mathcal{F}}$ is big, then

$$\#\,\mathrm{Bir}(X, \mathcal{F}) \leq C \cdot \mathrm{vol}(K_{\mathcal{F}} + \epsilon K_X).$$

In analogy with the classical situation, it would be nice to find a bound which depends only on $\mathrm{vol}(K_{\mathcal{F}})$. See also [9] for similar results in this direction.

We were also able to provide a refinement of the bound that is initially proven in [23], cf. also [7, 13].

Theorem 1.7 (= Theorem 5.1). *Let $\tau > 0$ be the constant whose existence is established in Theorem* 1.3*. Then, for all rational numbers $0 < \epsilon < \tau$, there exists $0 < C = C(\epsilon)$ such that the following statement holds.*

Let X be a smooth projective surface and let $\mathcal{F}$ be a rank one foliation on X. Assume that

(1) *$K_{\mathcal{F}}$ is big,*

(2) *$(X, \mathcal{F})$ is ϵ-adjoint canonical,*

(3) *$\mathcal{F}$ admits a meromorphic first integral, and*

(4) *the closure of a general leaf, L, has geometric genus g.*

Then, for any nef divisor H,

$$H \cdot L \leq gCH \cdot (K_{\mathcal{F}} + \epsilon K_X).$$

2. Preliminaries

Throughout, we work over an algebraically closed field k of characteristic 0. We refer to [4] for basic notions regarding foliations, and we refer to [15] for basic notions regarding the minimal model program. We will assume throughout that all of our foliations are of rank one.

2.1. ACC/DCC sets. Given a subset $I \subset \mathbb{R}$, we say that I satisfies the *ascending chain condition* (resp. *descending chain condition*), in short, ACC (resp. DCC), provided any increasing (resp. decreasing) sequence $x_n \in I$ is eventually constant.

Given a subset $I \subset [0, 1]$ and an $\mathbb{R}$-Weil divisor Δ on a normal variety, we write $\Delta \in I$ to indicate that all the coefficients of Δ are in I.

We will denote with $\mathcal{S}$ the following subset of $\mathbb{R}$:

$$\mathcal{S} := \left\{ \frac{n-1}{n} \;\middle|\; n \in \mathbb{N}_{>0} \right\} \cup \{1\}.$$

2.2. Pairs and triples. Let X be a normal variety and let $\mathcal{F}$ be a foliation on X. Let D be a $\mathbb{R}$-divisor on X. We may uniquely decompose $D = D_{\mathrm{inv}} + D_{n\text{-inv}}$ where the support of D_{inv} is $\mathcal{F}$-invariant and no component of the support of $D_{n\text{-inv}}$ is $\mathcal{F}$-invariant.

By a *(log) pair* (X, Δ), we mean the datum of a variety X and an effective $\mathbb{R}$-divisor Δ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier.

By a *foliated (log) pair* $(\mathcal{F}, \Delta)$ on a variety X, we mean the datum of a foliation $\mathcal{F}$ on X and an effective $\mathbb{R}$-divisor Δ such that $K_{\mathcal{F}} + \Delta$ is $\mathbb{R}$-Cartier. When we assume that X is projective, we shall say that the foliated pair is projective.

By a *foliated triple* $(X, \mathcal{F}, \Delta)$, we mean the datum of a variety X, a foliation $\mathcal{F}$ on X and an effective $\mathbb{R}$-divisor Δ such that both $K_X + \Delta$ and $K_{\mathcal{F}} + \Delta_{n\text{-inv}}$ are $\mathbb{R}$-Cartier. If $\Delta = 0$, then we will just write $(X, \mathcal{F})$ in place of $(X, \mathcal{F}, \Delta)$. When we assume that X is projective, we shall say that the foliated triple is projective.

2.3. Transform of a foliation under a rational map. Let X be a normal variety and let $\mathcal{F}$ be a foliation on X and let $\phi\colon X' \dashrightarrow X$ be a dominant rational map. Following [10, § 3.2], we may define the *pulled back* foliation, denoted $\phi^{-1}\mathcal{F}$, on X'. In the case where ϕ is birational and $\mathcal{G}$ is a foliation on X', we will denote by $\phi_*\mathcal{G}$ the pullback of $\mathcal{G}$ along the birational map ϕ^{-1} and refer to it as the *transform* of $\mathcal{G}$ by ϕ.

2.4. Foliation singularities. We are typically interested only in the case when $\Delta \geq 0$, although it simplifies some computations to allow Δ to have negative coefficients.

Given a birational morphism $\pi\colon \widetilde{X} \to X$ and a foliated pair $(\mathcal{F}, \Delta)$ on X, let $\widetilde{\mathcal{F}}$ be the pulled back foliation on $\widetilde{X}$. We may write

$$K_{\widetilde{\mathcal{F}}} = \pi^*(K_{\mathcal{F}} + \Delta) + \sum a(E, \mathcal{F}, \Delta)E,$$

where the sum runs over all the prime divisors of $\widetilde{X}$ and

$$\pi_* \sum a(E, \mathcal{F}, \Delta)E = -\Delta.$$

The rational number $a(E, \mathcal{F}, \Delta)$ denotes the *discrepancy* of $(\mathcal{F}, \Delta)$ with respect to E.

Definition 2.1. Let X be a normal variety and let $(\mathcal{F}, \Delta)$ be a foliated pair on X. We say that $(\mathcal{F}, \Delta)$ is *terminal* (resp. *canonical*, *klt*, *log canonical*) if $a(E, \mathcal{F}, \Delta) > 0$ (resp. ≥ 0, $> -\iota(E)$ and $\lfloor \Delta \rfloor = 0$, $\geq -\iota(E)$) for any birational morphism $\pi\colon \widetilde{X} \to X$ and for any prime divisor E on $\widetilde{X}$, where

$$\iota(E) := \begin{cases} 1 & \text{if } E \text{ is not } \mathcal{F}\text{-invariant}, \\ 0 & \text{if } E \text{ is } \mathcal{F}\text{-invariant}. \end{cases}$$

Moreover, we say that the foliated pair $(\mathcal{F}, \Delta)$ is *log terminal* if $a(E, \mathcal{F}, \Delta) > -\iota(E)$ for any birational morphism $\pi\colon \widetilde{X} \to X$ and for any π-exceptional prime divisor E on $\widetilde{X}$.

We shall say that a foliated pair $(\mathcal{F}, \Delta)$ on a normal variety X is *strictly log canonical* at a point $x \in X$ if there exists a geometric valuation E centered at x such that $\iota(E) = 1$ and $a(\mathcal{F}, \Delta, E) = -1$. In particular, a strictly log canonical foliated pair is not canonical.

In the next sections, we will also work with the class of F-dlt foliated pairs $(\mathcal{F}, \Delta)$ (in short, F-dlt pairs). We refer the reader to [6, Definition 3.6] for the definition of this class of foliated pairs.

Remark 2.2. Elsewhere in the literature, $\iota(D)$ is denoted by $\epsilon(D)$. However, in this paper, we will frequently use ϵ to denote a small positive real number, and so, to avoid confusion, we have adopted this new notation.

Remark 2.3. The quantities $\iota(E)$ and $a(E, \mathcal{F}, \Delta)$ are independent of π. If $\Delta = 0$, we will write $a(E, \mathcal{F})$ for $a(E, \mathcal{F}, \Delta)$.

In the case where $\mathcal{F} = T_X$, no exceptional divisor is invariant, i.e., $\iota(E) = 1$, and so this definition recovers the usual definitions of (log) terminal, (log) canonical, see [15]. In this case, we will write $a(E, X, \Delta)$ for $a(E, T_X, \Delta)$.

Definition 2.4. Given a pair (X, Δ) and $\eta \geq 0$, we say that (X, Δ) has η-lc singularities provided, for all birational morphisms $\pi\colon X' \to X$ and π-exceptional divisors E, we have

$$a(E, X, \Delta) \geq -(1 - \eta).$$

We say a foliated triple $(X, \mathcal{F}, D)$, where X is a surface and $\mathcal{F}$ has rank one, is *foliated log smooth* provided (X, D) has simple normal crossings and $\mathcal{F}$ has reduced singularities and each component of D which is not invariant is everywhere transverse to $\mathcal{F}$. We recall that these conditions entail that each component of D is disjoint from the singularities of $\mathcal{F}$. By [24], it is known that every surface foliated triple $(X, \mathcal{F}, D)$ admits a resolution $\pi\colon X' \to X$ such that

$$(X', \pi^{-1}\mathcal{F}, \pi_*^{-1}D + E)$$

is foliated log smooth, where $E = \operatorname{exc}(\pi)$. We call such a resolution a *foliated log resolution*.

For a choice of $\epsilon > 0$, we define the ϵ-adjoint log canonical divisor of a triple $(X, \mathcal{F}, \Delta)$ to be

$$K_{(X,\mathcal{F},\Delta),\epsilon} := (K_{\mathcal{F}} + \Delta_{n\text{-inv}}) + \epsilon(K_X + \Delta).$$

We say a triple $(X, \mathcal{F}, \Delta)$ is adjoint general type (pseudo-effective, etc.) if, for all $0 < \epsilon \ll 1$ sufficiently small, we have $K_{(X,\mathcal{F},\Delta),\epsilon}$ is big (pseudo-effective, etc.).

Let $P \in X$ be a germ of a normal variety and let $\mathfrak{m}$ be the maximal ideal of P. Let ∂ be a vector field on X which leaves P invariant. Since $\partial(\mathfrak{m}) \subset \mathfrak{m}$, we get an induced linear map $\partial_0\colon \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$, which we call the linear part of ∂ at P.

We recall the following characterization found in [20, Fact I.1.8].

Proposition 2.5. *Let $\mathcal{F}$ be a germ of a rank one foliation on a normal variety $P \in X$, and suppose that $K_{\mathcal{F}}$ is Cartier and that P is $\mathcal{F}$-invariant. Let ∂ be a vector field generating $T_{\mathcal{F}}$ near P. Then $\mathcal{F}$ is log canonical at P if and only if ∂_0 is non-nilpotent.*

2.5. Basic definitions of the MMP. We recall some of the main definitions commonly used in the Minimal Model Program. Let X be a normal projective variety. We denote by $N_1(X)$ the $\mathbb{R}$-vector space spanned by 1-cycles on X modulo numerical equivalence (e.g. see [15, Definition 1.16]). We denote by $\mathrm{NE}(X) \subset N_1(X)$ the subset of effective 1-cycles

$$\Big[\sum_{i=1}^{k} a_i C_i\Big],$$

where $a_1, \dots, a_k$ are positive real numbers and $C_1, \dots, C_k$ are curves in X, and we denote by $\overline{\mathrm{NE}(X)}$ its closure (e.g. see [15, Definition 1.17]). A *ray* is a 1-dimensional subcone R of $\overline{\mathrm{NE}(X)}$ and it is called *extremal* if for any $u, v \in \overline{\mathrm{NE}(X)}$ such that $u + v \in R$, we have that $u, v \in R$. If D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, then the extremal ray R is said to be D*-negative* if $D \cdot C < 0$ for any curve C such that $[C] \in R$. A projective birational morphism $f: X \to Y$ between normal projective varieties is said to be an *extremal contraction* if the relative Picard number $\rho(X/Y)$ is equal to one. The extremal contraction is called a *divisorial contraction* if its exceptional locus is a divisor. Given an extremal ray $R \subset \overline{\mathrm{NE}}(X)$, an extremal contraction $f: X \to Y$ is said to be *associated* to R if the locus of R coincides with the exceptional locus of f.

2.6. Recollection on the foliated MMP. We summarize some basic results on the existence of the MMP for surface foliations, as well as extending some well-known results to the case of pairs $(\mathcal{F}, \Delta)$ with log canonical singularities.

Lemma 2.6. *Let $P \in X$ be a germ of a surface singularity and let $\mathcal{F}$ be a rank one foliation on X which is strictly log canonical at P. Let $\mu: Y \to X$ be any foliated log resolution which is an isomorphism over $X \setminus P$. Then there is exactly one μ-exceptional divisor which is transverse to $\mu^{-1}\mathcal{F}$.*

Proof. Let $\nu: X' \to X$ be a foliated log resolution of $\mathcal{F}$. Observe that ν will extract every μ-exceptional divisor which is transverse to $\mu^{-1}\mathcal{F}$. We may write

$$\nu^* K_{\mathcal{F}} + F = K_{\mathcal{G}} + \sum_{i=1}^{k} E_i,$$

where $\mathcal{G} = \nu^{-1}\mathcal{F}$, where the E_i are the non-$\mathcal{G}$ invariant exceptional divisors and where $F \geq 0$. By [25, Corollary 2.26], we may run a $K_{\mathcal{G}}$-MMP over X, call it $\phi: X' \to X''$, set $\mathcal{H} = \phi_* \mathcal{G}$ and let $\rho: X'' \to X$ be the induced map. Only curves tangent to $\mathcal{G}$ will be contracted by this MMP, and so no component of $\sum_{i=1}^{k} E_i$ will be contracted.

By foliation adjunction, we see that

$$(K_{\mathcal{H}} + E_i') \cdot E_i' \geq 0, \quad \text{where } E_i' = \phi_* E_i,$$

see [25, Proposition 3.4] (note that in the notation of [25, Proposition 3.4] the restricted foliation $\mathcal{H}_{E_i'}$ is the foliation by points on E_i' and so $K_{\mathcal{H}_{E_i'}} = 0$). In particular, $K_{\mathcal{H}} + \sum E_i'$ is nef over X. By the negativity lemma [15, Lemma 3.39], we have $\phi_* F = 0$, and so $K_{\mathcal{H}} + \sum E_i'$ is numerically trivial over X. Since $K_{\mathcal{H}}$ is nef over X, it likewise follows that

$$-\sum E_i' = K_{\mathcal{H}} - \Big(K_{\mathcal{H}} + \sum E_i'\Big)$$

is nef over X. Since ρ has connected fibers, $\mathrm{exc}(\rho) = \sum E_i'$, and so $\sum E_i'$ is connected. This together with the inequalities

$$0 \leq (K_{\mathcal{H}} + E_i') \cdot E_i' \leq \left(K_{\mathcal{H}} + \sum_{i=1}^{k} E_i' \right) \cdot E_i' = 0$$

implies that $k = 1$, as required. □

Lemma 2.7. *Let X be a normal projective surface with a rank one foliation $\mathcal{F}$ and $\Delta \geq 0$ such that $(\mathcal{F}, \Delta)$ is log canonical. Suppose that $K_{\mathcal{F}}$ is $\mathbb{Q}$-Cartier. Let $R \subset \overline{\mathrm{NE}(X)}$ be a $(K_{\mathcal{F}} + \Delta)$-negative extremal ray and let $C \subset X$ be an $\mathcal{F}$-invariant curve such that*

(1) *$[C] \in R$; and*

(2) *C contains a strictly log canonical singularity of $\mathcal{F}$.*

Then X is covered by curves spanning R and $\rho(X) = 1$. In particular, $K_{\mathcal{F}} + \Delta$ is not pseudo-effective.

Proof. Since $(\mathcal{F}, \Delta)$ is log canonical by [25, Remark 2.12], we know that no component of Δ is $\mathcal{F}$-invariant. In particular, $\Delta \cdot C \geq 0$. So it follows that $K_{\mathcal{F}} \cdot C < 0$.

Let $P \in C$ be a strictly log canonical singularity of $\mathcal{F}$. If $n\colon \overline{C} \to C$ is the normalization, then [5, Proposition 2.16] implies that we may write $n^* K_{\mathcal{F}} = K_{\overline{C}} + \Theta$, where $\Theta \geq 0$ and $\lfloor \Theta \rfloor$ is supported exactly on the preimage of the non-terminal points of $\mathcal{F}$ contained in C. In particular, since $K_C + \Theta < 0$, it follows that, for all other $Q \in C$, $Q \neq P$, $\mathcal{F}$ is terminal at Q.

To see that C moves, we may freely replace X by a smaller open neighborhood of C so that $\mathcal{F}$ is strictly log canonical at only P. Let $f\colon X' \to X$ be an F-dlt modification, which exists by [6, Theorem 1.4]. Thus, $K_{\mathcal{F}'} + E = f^* K_{\mathcal{F}}$, where $\mathcal{F}' = f^{-1}\mathcal{F}$ and E is the unique irreducible f-exceptional divisor which is not $\mathcal{F}'$-invariant, see Lemma 2.6, which implies that $K_{\mathcal{F}'} \cdot C' < 0$, where C' is the strict transform of C. Let us observe that $(\mathcal{F}', E)$ is F-dlt, in particular log canonical, in a neighborhood of E. But, since $\mathcal{F}'$ is non-dicritical, then for any divisor F centered over a point in a neighborhood of E, $a(F, \mathcal{F}', E) \geq \iota(F) = 0$, which in turn implies that $\mathcal{F}'$ is terminal in a neighborhood of E. Hence, $\mathcal{F}'$ is terminal at $C' \cap E$. As $(K_{\mathcal{F}'} + E) \cdot C' < 0$, then, by adjunction [25, Lemma 8.9], $\mathcal{F}'$ is terminal at each point of C'. By Reeb stability [5, Proposition 3.3], C' moves in family covering X', and hence C moves in a family covering X.

Finally, we claim that $C^2 > 0$. This follows because

$$(C')^2 = 0, \quad C' \cdot E > 0 \quad \text{and} \quad f^*C = C' + aE + F,$$

where $a > 0$ and $F \geq 0$ is f-exceptional. Thus, C is a big divisor, and so C, and hence R, is contained in the interior of $\overline{\mathrm{NE}}(X)$. As R is also an extremal ray of $\overline{\mathrm{NE}}(X)$, then $\rho = 1$. □

Theorem 2.8. *Let X be a projective klt surface, let $\Delta \geq 0$ and let $\mathcal{F}$ be a rank one foliation such that $(\mathcal{F}, \Delta)$ and (X, Δ) are log canonical.*

Let $R \subset \overline{\mathrm{NE}}(X)$ be a $(K_{\mathcal{F}} + \Delta)$-negative extremal ray. Then there exists an $\mathcal{F}$-invariant curve C such that $R = \mathbb{R}_{>0}[C] \subset \overline{\mathrm{NE}}(X)$.

Moreover, there exists a contraction $c_R\colon X \to Y$ contracting exactly those curves in X whose numerical classes are contained in R and such that the following conditions holds:

(1) *if $c_R\colon X \to Y$ is birational, then c_R contracts only $\mathcal{F}$-invariant curves;*

(2) *if $c_R\colon X \to Y$ is a fiber type contraction, then R is $(K_X + \Delta)$-negative;*

(3) *if there is a strictly log canonical singularity of $\mathcal{F}$ on C, then $\rho(X) = 1$, $-(K_{\mathcal{F}} + \Delta)$ and $-(K_X + \Delta)$ are ample.*

Moreover, in all cases, the relative Picard number of the contraction is 1.

In particular, we may run a $(K_{\mathcal{F}} + \Delta)$-MMP.

Proof. By the cone theorem for surface foliations [25, Theorem 6.3 and Remark 6.4], a $(K_{\mathcal{F}} + \Delta)$-negative extremal ray $R \subset \overline{\mathrm{NE}}(X)$ is spanned by the class of a curve C which is $\mathcal{F}$-invariant.

Let us consider the case where $C^2 \geq 0$. By [25, Theorem 6.3], there exists a nef Cartier divisor H_R such that $H_R \cdot R = 0$ and H_R is positive on every other extremal ray. Since $C^2 \geq 0$, it follows that H_R cannot be big, i.e., $H_R^2 = 0$. Arguing as in the proof of [25, Theorem 6.3], using [25, Corollary 2.28], it follows that X is covered by a family of rational curves tangent to $\mathcal{F}$ whose numerical class spans R.

Claim 1. *Let Σ be a general choice of such a curve; then $(K_X + \Delta) \cdot \Sigma < 0$.*

Proof of Claim 1. Let $p\colon Y \to X$ be an F-dlt modification, which exists by [6, Theorem 1.4]. If $\mathcal{G} = p^{-1}\mathcal{F}$ and $\Delta' = p_*^{-1}\Delta$, then

$$K_{\mathcal{G}} + \Delta' + \sum_i \iota(E_i)E_i = p^*(K_{\mathcal{F}} + \Delta),$$

where E_i are the p-exceptional divisors. By construction, $\mathcal{G}$ is non-dicritical; hence, it is induced by a fibration $Y \to B$ such that $\Sigma' = p_*^{-1}\Sigma$ is a fiber of $Y \to C$. It follows that $K_Y \cdot \Sigma' = K_{\mathcal{G}} \cdot \Sigma' = -2$. We may write

$$K_Y + \Delta' + \sum_i a_i E_i = p^*(K_X + \Delta),$$

where $a_i \leq 1$ since (X, Δ) is log canonical. If $\iota(E_i) = 0$, then $E_i \cap \Sigma' = \emptyset$ since Σ' is general, and so $E_i \cdot \Sigma' = 0$. Hence,

$$\begin{aligned} \Big(\sum_i \iota(E_i)E_i\Big) \cdot \Sigma' &\geq \Big(\sum_i a_i E_i\Big) \cdot \Sigma' \quad \text{and} \\ 0 > (K_{\mathcal{F}} + \Delta) \cdot \Sigma &= \Big(K_{\mathcal{G}} + \Delta' + \sum_i \iota(E_i)E_i\Big) \cdot \Sigma' \\ &\geq \Big(K_Y + \Delta' + \sum_i a_i E_i\Big) \cdot \Sigma' = (K_X + \Delta) \cdot \Sigma. \end{aligned}$$ □

In view of Claim 1, then the existence of c_R is immediate since c_R can be constructed as the contraction of a $(K_X + (1-\epsilon)\Delta)$-negative extremal ray $0 < \epsilon \ll 1$, see [15, Theorem 3.7]. Since c_R is a fiber type contraction only if $C^2 \geq 0$, then item (2) above also follows at once.

Item (3) follows from Lemma 2.7 and (2).

Now consider the case where $C^2 < 0$. In this case, the contraction c_R, if it exists, will be birational and it will only contract C, which proves (1).

From item (3), we know that $\mathcal{F}$ has canonical singularities in a neighborhood of C. We may apply [20, Section III.1-2] to contract C; strictly speaking, in [20] an entire chain of rational curves is contracted, but the arguments provided work equally well to contract a single $K_{\mathcal{F}}$-negative curve. For the reader's convenience, we will supply an alternate proof of the existence of this contraction. By [6, Theorem 11.3], $\mathcal{F}$ has non-dicritical singularities in a neighborhood of C, and so, by [25, Lemma 8.14][1)], this implies that $(X, \Delta + C)$ is a log canonical pair.

Claim 2. $(K_X + \Delta + C) \cdot C < 0$.

Proof of Claim 2. Let $p: Y \to X$ be an F-dlt modification, which can be performed by [6, Theorem 1.4]. Let C' be the strict transform of C and let $\Delta' = p_*^{-1}\Delta$. We write $(K_Y + \Delta' + C' + \sum_i E_i) = p^*(K_X + \Delta + C) + \sum_i b_i E_i$, where the E_i are the p-exceptional prime divisors and $b_i \geq 0$. Since $\mathcal{F}$ is non-dicritical, then for all i, E_i is invariant; thus, $K_{\mathcal{G}} + \Delta' = p^*(K_{\mathcal{F}} + \Delta)$. Since $\mathcal{G}$ is F-dlt, [6, Lemma 3.12] implies that $\mathcal{G}$ is terminal at the singular points of X. We may then apply [25, Lemma 8.9] to conclude that

$$(K_{\mathcal{G}} + \Delta') \cdot C' \geq \Big(K_Y + \Delta' + C' + \sum_i E_i\Big) \cdot C' \geq (K_X + \Delta + C) \cdot C.$$

Again, here, the cited results are stated for threefolds but apply equally well to surfaces as explained above. □

We conclude by observing that $(X, (1-\epsilon)(\Delta + C))$ is klt for all $0 < \epsilon$ and

$$(K_X + (1-\epsilon)(\Delta + C)) \cdot C < 0 \quad \text{for } 0 < \epsilon \ll 1,$$

and so we may contract C by a $(K_X + (1-\epsilon)(\Delta + C))$-negative extremal contraction, see [15, Theorem 3.7].

Since all our contractions are $(K_X + \Theta)$-negative contractions for a klt pair (X, Θ), [15, Corollary 3.17] implies that they are of relative Picard number one.

Finally, it is a standard argument to show that the existence of divisorial contractions as explained above implies the existence of the $(K_{\mathcal{F}} + \Delta)$-MMP. □

Remark 2.9. Let notation be as in Theorem 2.8. The above proof shows that if $\sum_i C_i$ is any collection of reduced $\mathcal{F}$-invariant curves such that $\mathcal{F}$ has canonical singularities in a neighborhood of $\sum_i C_i$, then each step of the $K_{\mathcal{F}}$-MMP is also a step of the $(K_X + \Delta + \sum C_i)$-MMP. In particular, if $(X, \Delta + \sum C_i)$ is dlt and $\phi: X \to X'$ is a run of the $(K_{\mathcal{F}} + \Delta)$-MMP, then $(X', \phi_*(\Delta + \sum C_i))$ is again dlt.

Lemma 2.10. *Let X be a normal surface, let D be a reduced Weil divisor and let $\mathcal{F}$ be a rank one foliation on X such that*

(1) *$K_{\mathcal{F}}$ is Cartier; and*

(2) *every component of D is $\mathcal{F}$-invariant.*

Then there exists a log resolution of (X, D) which only extracts divisors E of foliation discrepancy at most $-\iota(E)$.

1) Let us observe that both [6, Theorem 11.3] and [25, Lemma 8.14] are stated for threefolds. We can deduce the analogous statement for rank one foliation on surfaces by applying the results to the threefold $X \times B$, where B is a smooth curve, and to the foliation $\pi^{-1}\mathcal{F}$, where $\pi: X \times B \to X$ is the projection.

Proof. The problem is local about any point $P \in X$, so we may freely assume (X, D) is not log smooth at P and that $\mathcal{F}$ is generated by a vector field ∂. Since (X, D) is not log smooth at P, either X or D is singular at P, and so, by [5, Lemma 2.6], P is invariant under ∂. By [3, Lemma 1.1.3], if $b\colon \widetilde{X} \to X$ is the blow up in P, then ∂ lifts to a vector field $\widetilde{\partial}$ on $\widetilde{X}$, which moreover leaves $b^{-1}(P)$ invariant.

A log resolution of (X, D) may be achieved by repeatedly blowing up centers where (X, D) is not log smooth, so by applying the above observation and arguing by induction on the number of blow ups in a log resolution, we may produce a log resolution $\pi\colon X' \to X$ and a lift ∂' of ∂ which leaves $\pi^{-1}(P)$ invariant.

Because ∂' leaves $\pi^{-1}(P)$ invariant, we see that if F is a π-exceptional divisor with $\iota(F) = 1$, then ∂' vanishes along F. Our first claim then follows by observing that the foliation discrepancy along a divisor F is exactly $-a$, where a is the order of vanishing of ∂' along F. □

Lemma 2.11. *Let $P \in X$ be a germ of a normal surface and let $\mathcal{F}$ be a rank one foliation on X. Suppose that $\mathcal{F}$ is strictly log canonical at P. Then $\mathcal{F}$ is Gorenstein at P.*

Proof. Let $\sigma\colon P' \in X' \to P \in X$ be the index one cover associated to $K_{\mathcal{F}}$ with Galois group $G \cong \mathbb{Z}/m\mathbb{Z}$, let $\mathcal{F}' = \sigma^{-1}\mathcal{F}$ and let ∂ be a vector field defining $\mathcal{F}'$. By Lemma 2.20, we have $\mathcal{F}'$ is strictly log canonical at P'.

Let $\mathfrak{m}$ denote the maximal ideal of X' at P'. Since $\mathcal{F}'$ is strictly log canonical by [21, Fact III.i.3] (up to renormalizing ∂ by a constant in $\mathbb{C}$), we may write the linear part of ∂ as

$$\partial_0 = \sum n_i x_i \frac{\partial}{\partial x_i},$$

where $n_1, \ldots, n_k$ are positive integers and $x_1, \ldots, x_k \in \mathfrak{m}$ give a basis of $\mathfrak{m}/\mathfrak{m}^2$.

Let $g \in G$. On one hand, by assumption, we may write $g_*\partial = \zeta\partial$, where ζ is a primitive m-th root of unity. On the other hand, from the equality $g^*(g_*\partial(x)) = \partial(g^*x)$ for any $x \in \mathfrak{m}$, we see that linear parts of $g_*\partial$ and ∂ have the same eigenvalues. It follows that

$$\{n_1, \ldots, n_k\} = \{\zeta n_1, \ldots, \zeta n_k\},$$

and so $\zeta = 1$, i.e., $K_{\mathcal{F}}$ was Gorenstein to begin with. □

2.7. (ϵ, δ)-adjoint log canonical foliated singularities. We wish to measure singularities of triples $(X, \mathcal{F}, \Delta)$ in terms of how $K_{(X,\mathcal{F},\Delta),\epsilon}$ changes under birational transformations. This idea was initially considered in [23], see Remark 2.13 below, and the approach here is a natural extension of the ideas introduced there.

Definition 2.12. Let $(X, \mathcal{F}, \Delta)$ be a foliated triple. Fix $\epsilon > 0$ and $\delta \geq 0$.

We say that $(X, \mathcal{F}, \Delta)$ is (ϵ, δ)*-adjoint log canonical* (resp. (ϵ, δ)*-adjoint klt*) provided that, for any birational morphism $\pi\colon X' \to X$, if we write

$$(K_{\mathcal{F}'} + \Delta'_{n\text{-inv}}) + \epsilon(K_{X'} + \Delta') = \pi^*\big((K_{\mathcal{F}} + \Delta_{n\text{-inv}}) + \epsilon(K_X + \Delta)\big) + E,$$

where

- $E = \sum a_i E_i$ is π-exceptional, and
- $\Delta' := \pi_*^{-1}\Delta$,

then for all i,

$$a_i \geq (\iota(E_i) + \epsilon)(-1 + \delta)$$
$$(\text{resp., } \lfloor \Delta \rfloor = 0 \text{ and } a_i > (\iota(E_i) + \epsilon)(-1 + \delta)).$$

When $\delta = 1$, we will refer to (ϵ, δ)-adjoint log canonical as ϵ-adjoint canonical. When $\delta = 0$, we will refer to (ϵ, δ)-adjoint log canonical as ϵ-adjoint log canonical.

Remark 2.13. In [23], the notion of ϵ-*canonical* was defined whereby singularities were measured by considering how the adjoint series $K_{\mathcal{F}} + \epsilon N^*_{\mathcal{F}}$ transforms under blow ups.

By re-writing

$$K_{\mathcal{F}} + \epsilon N^*_{\mathcal{F}} = (1 - \epsilon) K_{\mathcal{F}} + \epsilon K_X,$$

then it is immediate to see that if a foliated pair $(X, \mathcal{F})$ is $\frac{\epsilon}{1+\epsilon}$-canonical in the sense of [23], then it is also $(\epsilon, 1)$-adjoint log canonical in the sense of Definition 2.12. For various computations we need to perform, working with $K_{\mathcal{F}} + \epsilon K_X$ seemed preferable to us, hence the slight change in the definition.

The following three lemmata are immediate consequences of the definition of (ϵ, δ)-adjoint log canonical singularities.

Lemma 2.14. *Let $(X, \mathcal{F}, \Delta)$ be a foliated triple and $\epsilon > 0$.*

(1) *Let $0 \leq \delta' < \delta \in [0, 1]$. If $(X, \mathcal{F}, \Delta)$ is (ϵ, δ)-adjoint log canonical, then it also (ϵ, δ')-adjoint log canonical.*

(2) *Assume that $\Delta = 0$ and that $(X, \mathcal{F})$ is ϵ-adjoint canonical.*
   (a) *If $\mathcal{F}$ has canonical singularities, $(X, \mathcal{F})$ is ϵ'-adjoint canonical for all $0 < \epsilon' \leq \epsilon$.*
   (b) *If X has canonical singularities, $(X, \mathcal{F})$ is ϵ''-adjoint canonical for all $\epsilon'' \geq \epsilon$.*

Proof. Follows from a direct computation. □

Lemma 2.15. *Let X be a smooth surface and let $0 \leq \Delta$ be a $\mathbb{Q}$-divisor with snc support such that $\lfloor \Delta \rfloor = 0$. Let $\mathcal{F}$ be a rank one foliation such that $(\mathcal{F}, \Delta_{n\text{-inv}})$ is canonical. Let $\delta > 0$ be such that all coefficients of Δ are at most $1 - \delta$.*

Then $(X, \mathcal{F}, \Delta)$ is (ϵ, δ)-adjoint log canonical for all $\epsilon > 0$.

Proof. Follows from a direct computation. □

Lemma 2.16. *Let $p\colon X \to Y$ be a birational morphism between surfaces, let $(\mathcal{F}, \Delta)$ be a foliated pair on X, where $\mathcal{F}$ is of rank one, and let $\mathcal{G} := p_*\mathcal{F}$. Fix $\epsilon > 0$ and $\delta \geq 0$. Assume that*

(1) *the coefficients of Δ are at most $1 - \delta$;*
(2) *$-K_{(X,\mathcal{F},\Delta),\epsilon}$ is p-nef; and*
(3) *$(X, \mathcal{F}, \Delta)$ is (ϵ, δ)-adjoint log canonical.*

Then $(Y, \mathcal{G}, p_*\Delta)$ is (ϵ, δ)-adjoint log canonical.

Proof. This is a direct consequence of the negativity lemma [15, Lemma 3.38]. □

Our goal for the last part of this subsection is to prove the following generalization of [23, Proposition 4.9].

Proposition 2.17. *Let $I \subset [0,1]$ be a subset satisfying the DCC. Then there exists a positive real number $E = E(I)$ such that the following statement holds.*

Let $0 \in X$ be a germ of a klt surface singularity, let $\mathcal{F}$ be a rank one foliation on X and let $\Delta \geq 0$ be a divisor with $\Delta \in I$ so that $(X, \mathcal{F}, \Delta)$ is ϵ-adjoint log canonical for some $0 < \epsilon < E(I)$. Then $(\mathcal{F}, \Delta_{n\text{-inv}})$ is log canonical.

We start by proving three ancillary lemmata that will be used in the proof.

Lemma 2.18. *Let ∂ be a germ of a vector field on $P \in \mathbb{C}^2$, and suppose that ∂ is singular at P and the linear part of ∂ at P is equal to 0. Let $\pi: X \to \mathbb{C}^2$ be the blow up at P with exceptional divisor E and let $\mathcal{F}$ be the foliation generated by ∂. Then*

$$K_{\pi^{-1}\mathcal{F}} = \pi^* K_{\mathcal{F}} - bE,$$

where $b \geq \iota(E) + 1$.

Proof. Using notation as in [4, Chapter 1, § 2], let ω be a one form with an isolated zero at P and $\partial(\omega) = 0$. Let $a(P)$ denote the order of vanishing of ω at P and let $l(P)$ denote the order of vanishing of $\pi^*\omega$ along E.

A direct computation shows that $l(P) = a(P)$ when E is invariant and $l(P) = a(P) + 1$ when E is not invariant. Another straightforward calculation shows that the discrepancy of our blow up is $-(l(P) - 1) = -(a(P) + \iota(E) - 1)$, and by assumption, $a(P) \geq 2$, from which our claim follows. □

Lemma 2.19. *For $0 < \epsilon < \frac{1}{5}$, the following holds.*

Let $\mathcal{F}$ be a rank one foliation on $P \in \mathbb{C}^2$, and suppose that $(\mathbb{C}^2, \mathcal{F})$ is ϵ-adjoint log canonical at P. Then $\mathcal{F}$ is log canonical at P.

Proof. Suppose that $\mathcal{F}$ is not log canonical at P.

Following the proof of [23, Proposition 4.9] (see also the proof of [4, Theorem 1.1]), we may find a sequence of at most 3 blow ups $b_i: (X_i, \mathcal{F}_i) \to (X_{i-1}, \mathcal{F}_{i-1})$ such that

(1) $(X_0, \mathcal{F}_0) := (\mathbb{C}^2, \mathcal{F})$;

(2) b_i is a blow up in the singular locus of $\mathcal{F}_{i-1}$; and

(3) on the last blow up, call it b_n, we blow up a foliation singularity whose linear part is equal to 0.

Let $\pi: X_n \to \mathbb{C}^2$ denote the composition of these blow ups and let E be the exceptional divisor of b_n. By Lemma 2.18, $a(E, \mathcal{F}) \leq -(\iota(E) + 1)$, and a direct computation shows that $a(E, X) \leq 4$. Thus,

$$a(E, \mathcal{F}) + \epsilon a(E, X) \leq -(\iota(E) + 1) + 4\epsilon.$$

For all $\epsilon < \frac{1}{5}$, we have $-(\iota(E) + 1) + 4\epsilon < -(\iota(E) + \epsilon)$, which in turn implies that

$$a(E, \mathcal{F}) + \epsilon a(E, X) < -(\iota(E) + \epsilon),$$

contradicting the assumption that $(X, \mathcal{F})$ is ϵ-adjoint log canonical. □

We remark that if $\epsilon > 0$, then an ϵ-adjoint canonical singularity is not necessarily a canonical foliation singularity, i.e., it could be strictly log canonical. Moreover, if we fix $\epsilon > 0$, then a terminal singularity of $\mathcal{F}$ is not necessarily $(\epsilon, 1)$-adjoint log canonical. Consider for instance the quotient of $(\mathbb{C}^2, \langle \frac{\partial}{\partial x} \rangle)$ by the action of $\mathbb{Z}/m\mathbb{Z}$ given by $(x, y) \mapsto (\xi x, \xi^b y)$, where $(m, b) = 1$. This will always give a terminal foliation singularity; however, it is not ϵ-adjoint canonical for $\epsilon > \frac{1}{m-2}$.

Lemma 2.20. *Let X be a normal variety and let $\mathcal{F}$ be a rank one foliation on X. Let $\sigma\colon X' \to X$ be a finite morphism, étale in codimension 1, and set $\mathcal{F}' := \sigma^{-1}\mathcal{F}$. Write $\Delta' = \sigma^*\Delta$, and note that $\Delta'_{n\text{-inv}} = \sigma^*\Delta_{n\text{-inv}}$. Fix $\epsilon > 0$ and $\delta \geq 0$.*

(1) *Suppose that $(X, \mathcal{F}, \Delta)$ is (ϵ, δ)-adjoint log canonical. Then $(X', \mathcal{F}', \Delta')$ is (ϵ, δ)-adjoint log canonical.*

(2) *Suppose that $(\mathcal{F}', \Delta'_{n\text{-inv}})$ is (log) canonical. Then $(\mathcal{F}, \Delta_{n\text{-inv}})$ is (log) canonical.*

Proof. The proof of item (1) and the log canonical case of item (2) is essentially identical to the proof of [15, Proposition 5.20], making use of the statement of foliated Riemann–Hurwitz found in [10, Lemma 3.4] which gives us the following adjoint Riemann–Hurwitz formula for a finite morphism $\sigma\colon Y \to X$ for all $\epsilon \geq 0$:

$$K_{(Y,\mathcal{G},\sigma^*\Delta),\epsilon} = \sigma^* K_{(X,\mathcal{F},\Delta),\epsilon} + \sum_{D \in \mathrm{Div}(Y)} (r_D - 1)(\iota(D) + \epsilon)D,$$

where $\mathcal{G} = \sigma^{-1}\mathcal{F}$ and r_D is the ramification index of σ along D.

We now explain the proof of item (2) in the canonical case. Let $f\colon W \to X$ be a birational morphism, and consider the following diagram:

$$\begin{tikzcd} W' \arrow[r, "\tau"] \arrow[d, "g"] & W \arrow[d, "f"] \\ X' \arrow[r, "\sigma"] & X, \end{tikzcd}$$

where W' is the normalization of the main component of $W \times_X X'$. Set $\mathcal{H} := f^{-1}\mathcal{F}$ and $\mathcal{H}' := g^{-1}\mathcal{F}'$. Let $E \subset W$ be an exceptional divisor, let E' be a component of $\tau^{-1}(E)$ and let r be the ramification index of τ along E'. By assumption, $\mathcal{F}'$ has canonical singularities, and so, by [21, Corollary III.i.4], every g-exceptional divisor is $\mathcal{H}'$-invariant. Thus, every f-exceptional divisor is $\mathcal{H}$-invariant. Doing a calculation analogous to the one in [15, Proposition 5.20] and making use of the foliated Riemann–Hurwitz formula [10, Lemma 3.4] and noting that $\iota(E') = 0$ gives us

$$a(E', \mathcal{F}', \Delta') = ra(E, \mathcal{F}, \Delta).$$

Since $a(E', \mathcal{F}', \Delta') \geq 0$, it follows that $a(E, \mathcal{F}, \Delta) \geq 0$ as required. □

Proof of 2.17. Since $0 \in X$ is a klt singularity, it is a quotient singularity, and so, by Lemma 2.20, we may freely reduce to the case where $X = \mathbb{C}^2$. We may also assume, without loss of generality, that $\Delta = \Delta_{n\text{-inv}}$, and by taking $\epsilon < \frac{1}{5}$ and applying Lemma 2.19, we may assume that $\mathcal{F}$ is log canonical at 0.

We now proceed by arguing in cases, based on whether or not $\mathcal{F}$ is singular at 0.

Case 1. We assume that $\mathcal{F}$ is singular at 0. Since $\mathcal{F}$ is log canonical at 0, it suffices to show that, for ϵ sufficiently small, if $(X, \mathcal{F}, \Delta)$ is ϵ-adjoint log canonical, then Δ is disjoint from 0.

So suppose that 0 is in the support of Δ, let $\pi: X \to \mathbb{C}^2$ be the blow up at 0 with exceptional divisor C, let i_0 be the smallest strictly positive element of I, let $\mathcal{G} = \pi^{-1}\mathcal{F}$ and let $\Delta' = \pi_*^{-1}\Delta$.

Since $\mathcal{F}$ is log canonical and singular at 0, we have that the blow up at 0 has foliation discrepancy equal to $-\iota(C)$, and so $K_{\mathcal{G}} + \Delta' = \pi^*(K_{\mathcal{F}} + \Delta) + aC$, where $a \le -\iota(C) - i_0$, and $K_X + \Delta' = \pi^*(K_{\mathbb{C}^2} + \Delta) + bC$, where $b \le 1 - i_0$. It follows that

$$K_{(X,\mathcal{G},\Delta'),\epsilon} = \pi^*(K_{(\mathbb{C}^2,\mathcal{F},\Delta),\epsilon}) + (a + \epsilon b)C,$$

where $a + \epsilon b \le -\iota(C) - i_0 + \epsilon(1 - i_0)$.

Now $(\mathbb{C}^2, \mathcal{F}, \Delta)$ will fail to be ϵ-adjoint log canonical if we have the inequality

$$-\iota(C) - i_0 + \epsilon(1 - i_0) < -(\iota(C) + \epsilon) \quad \text{or equivalently} \quad \frac{\iota(C) + i_0 - \epsilon(1 - i_0)}{\iota(C) + \epsilon} > 1.$$

However, this inequality will hold for all ϵ smaller than some constant depending only on i_0.

Case 2. We assume that $\mathcal{F}$ is smooth at 0. Let $0 \in L$ be a germ of a leaf through 0, and observe that the discrepancies of $(\mathcal{F}, \Delta)$ are exactly the log discrepancies of $(X, \Delta + L)$, see [25, Lemma 8.14] and its proof (again, we remark that the cited lemma is proven for threefolds, but holds for surfaces as well). Thus, to show that $(\mathcal{F}, \Delta)$ is log canonical, it suffices to show that $(X, \Delta + L)$ is log canonical.

We now claim that $(X, \mathcal{F}, \Delta)$ being ϵ-adjoint log canonical implies that $(X, \Delta + \frac{1}{1+\epsilon}L)$ is log canonical. Indeed, let $\pi: Y \to X$ be any birational morphism, and let Δ_Y and L_Y denote the strict transforms of Δ and L respectively, and let $C = \sum C_i$ be the sum of the π-exceptional divisors with coefficient equal to 1. Note that all the C_i are invariant. Thus, by definition of ϵ-adjoint log canonical and the above observation, we may write

$$(K_Y + \Delta_Y + L_Y + C) + \epsilon(K_Y + \Delta_Y) = \pi^*\big((K_X + \Delta + L) + \epsilon(K_X + \Delta)\big) + \sum a_i C_i,$$

where $a_i \ge -\epsilon$. Dividing the above equality by $1 + \epsilon$ gives

$$K_Y + \Delta_Y + \frac{1}{1+\epsilon}(L_Y + C) = \pi^*\Big(K_X + \Delta + \frac{1}{1+\epsilon}L\Big) + \sum \frac{a_i}{1+\epsilon}C_i.$$

Note that we always have the inequality $\frac{1-a_i}{1+\epsilon} \le 1$ for all i, so it follows that $(X, \Delta + \frac{1}{1+\epsilon}L)$ is log canonical.

Let λ be the log canonical threshold of (X, Δ) with respect to L. We have just shown $\lambda \ge \frac{1}{1+\epsilon}$. By the ACC for log canonical thresholds [14, Theorem 1.1], we see that there is a fixed λ_0 depending only on I so that if $\lambda \ge \lambda_0$, then $\lambda = 1$, in which case $(X, \Delta + L)$ is log canonical, which implies that $(\mathcal{F}, \Delta)$ is log canonical.

Choose $E(I)$ so that $\epsilon < E(I)$ implies

(1) $\frac{1}{1+\epsilon} \ge \lambda_0$;

(2) $\epsilon < \frac{1}{5}$; and

(3) $\frac{\iota(C)+i_0-\epsilon(1-i_0)}{\iota(C)+\epsilon} > 1$.

We therefore see that if $\epsilon < E(I)$, then $(\mathcal{F}, \Delta)$ is log canonical. □

2.8. A general boundedness result.

Lemma 2.21. *Fix positive real numbers η, θ. Let X be a projective η-lc variety of dimension n and let N be an $\mathbb{R}$-divisor on X such that*

(1) *N is nef and big;*

(2) *$N - K_X$ is pseudo-effective; and*

(3) *$N = P + E$ with P integral and pseudo-effective, and $E \geq 0$ is effective and all its non-zero coefficients are at least θ.*

Then there exists an $m = m(\dim(X), \eta, \theta)$ such that, for any $m' \geq m$, $|\lfloor m'N \rfloor|$ defines a birational map.

Proof. This is [2, Theorem 4.2] □

2.9. Boundedness and foliations. We make note of a simple result on the boundedness of foliations in families.

We recall that a bounded family of proper normal surfaces is a proper and flat morphism $f : \mathcal{X} \to T$ of finite type varieties such that any fiber of f is a normal surface. When T is smooth (but not necessarily connected), then each connected component of $\mathcal{X}$ is normal.

Let X be a normal variety. By a Weil divisorial sheaf $\mathcal{K}$, we mean a sheaf of the form $\mathcal{O}_X(K)$, where K is a Weil divisor on X.

When discussing boundedness for foliated surfaces, we will use the following standard technical result about families of such pairs.

Lemma 2.22. *Let $f : \mathcal{X} \to T$ be a bounded family of normal surfaces and let $\mathcal{K}$ be a Weil divisorial sheaf on $\mathcal{X}$.*

Assume that the following hold:

(1) *T is smooth;*

(2) *$f : \mathcal{X} \to T$ is flat; and*

(3) *for all $t \in T$, the restriction $\mathcal{K}|_{X_t}$ is reflexive.*

Then there exists a bounded family of normal surfaces $f' : \mathcal{X}' \to T'$, with T' smooth, and a rank one foliation $\mathcal{F}$ on $\mathcal{X}'$ which is tangent to f' satisfying the following condition.

For all $t \in T$ and for any foliation $\mathcal{G}_t$ on $\mathcal{X}_t$ of canonical divisor $\mathcal{K}|_{\mathcal{X}_t}$, there exists $t' \in T'$ such that

$$\mathcal{X}_t \simeq \mathcal{X}'_{t'}, \quad \mathcal{F}|_{\mathcal{X}_{t'}} \simeq \mathcal{G}_t.$$

The lemma follows from hypotheses (1)–(3) and from applying classical results on flatness and base change.

The above result justifies the following definition a bounded family of foliated triples.

Definition 2.23. A bounded family of d-dimensional foliated triples is the datum of a foliated triple $(\mathcal{Y}, \mathcal{G}, \Gamma)$, where $\mathcal{G}$ has rank one, together with a projective morphism $f : \mathcal{Y} \to T$ to a variety of finite type T such that

(1) every fiber of f is normal and Γ does not contain any fiber of f;

(2) the singular locus of $\mathcal{Y}$ intersects every fiber in codimension at least 2;

(3) $\mathcal{G} \subseteq T_{\mathcal{Y}/T}$; and

(4) for all $t \in T$, $(\mathcal{Y}_t, \mathcal{G}_t, \Gamma_t)$ is a projective foliated triple with $\dim \mathcal{Y}_t = d$, where

$$\mathcal{G}_t := (\mathcal{G}|_{Y_t})^{**} \quad \text{and} \quad \Gamma_t := \Gamma|_{\mathcal{Y}_t}.$$

We will often use the streamlined notation $f\colon (\mathcal{Y}, \mathcal{G}, \Gamma) \to T$ to denote bounded families of triples. Given one such bounded family of triples and $t \in T$, we will denote by $(\mathcal{Y}_t, \mathcal{G}_t, \Gamma_t)$ the projective foliated triple induced on the fiber over the point t.

We will also say that a collection $\mathfrak{D}$ of projective d-dimensional foliated triples is bounded (or forms a bounded family) if there exists a bounded family of foliations $f\colon (\mathcal{Y}, \mathcal{G}, \Gamma) \to T$ such that any triple $(X, \mathcal{F}, \Delta) \in \mathfrak{D}$ appears as one of the fibers of family given by f.

3. Adjoint MMP

Let X be a smooth projective surface and $\mathcal{F}$ a foliation with reduced singularities. Under this assumption, it is known that we may run either one of a K_X-MMP or a $K_{\mathcal{F}}$-MMP starting at X or at $(X, \mathcal{F})$ respectively. The goal of this section is to show that we may also run a $K_{(X,\mathcal{F},\Delta),\epsilon}$-MMP for $\epsilon > 0$ sufficiently small, see Theorem 3.1. We will show, moreover, that the singularities that arise in the run of such MMP are relatively mild, cf. Corollary 3.3.

3.1. Running the $K_{(X,\mathcal{F},\Delta),\epsilon}$-MMP. We start by proving the existence and termination of the $K_{(X,\mathcal{F},\Delta),\epsilon}$-MMP.

Theorem 3.1. *Let $(X, \mathcal{F}, \Delta)$ be a projective foliated triple, where X is a surface, (X, Δ) is dlt and $\mathcal{F}$ is rank one. Fix $\delta \geq 0$. Suppose that $\Delta \in I \cap [0, 1-\delta]$, where $I \subset [0, 1]$ is a DCC set. Fix $0 < \epsilon < E(I)$, where $E(I)$ is as in Proposition* 2.17. *Suppose that $(X, \mathcal{F}, \Delta)$ is (ϵ, δ)-adjoint log canonical.*

Then we may run a $K_{(X,\mathcal{F},\Delta),\epsilon}$-MMP $\rho\colon X \to Y$. Moreover, setting

$$\mathcal{G} := \rho_*\mathcal{F} \quad \textit{and} \quad \Gamma := \rho_*\Delta,$$

the following properties hold:

(1) *$(Y, \mathcal{G}, \Gamma)$ is (ϵ, δ)-adjoint log canonical;*

(2) *$(\mathcal{G}, \Gamma_{n\text{-inv}})$ has log canonical singularities and (Y, Γ) is dlt;*

(3) *if $K_{(X,\mathcal{F},\Delta),\epsilon}$ is pseudo-effective, then $K_{(Y,\mathcal{G},\Gamma),\epsilon}$ is nef;*

(4) *if $K_{(X,\mathcal{F},\Delta),\epsilon}$ is not pseudo-effective, Y admits a fibration $f\colon Y \to Z$ with $\rho(X/Y) = 1$ and such that $-K_{(Y,\mathcal{G},\Gamma),\epsilon}$ is f-ample.*

Proof. First, let us note that Lemma 2.14 (1) implies that $(X, \mathcal{F}, \Delta)$ is ϵ-adjoint log canonical; thus, by our choice of ϵ, Proposition 2.17 implies that $(\mathcal{F}, \Delta_{n\text{-inv}})$ has log canonical singularities. We now explain how to run the $K_{(X,\mathcal{F},\Delta),\epsilon}$-MMP on X.

If $K_{(X,\mathcal{F},\Delta),\epsilon}$ is nef, then we immediately stop and define ρ to be the identity on X; in this case, properties (1)–(4) are straightforwardly satisfied. Thus, we may and will assume that

$K_{(X,\mathcal{F},\Delta),\epsilon}$ is not nef; hence, there exists an extremal ray

$$R_0 \subset \overline{\mathrm{NE}(X)} \quad \text{such that} \quad K_{(X,\mathcal{F},\Delta),\epsilon} \cdot R_0 < 0.$$

Since, by definition, $K_{(X,\mathcal{F},\Delta),\epsilon} = (K_{\mathcal{F}} + \Delta_{n\text{-inv}}) + \epsilon(K_X + \Delta)$, then

$$\text{either} \quad (K_{\mathcal{F}} + \Delta_{n\text{-inv}}) \cdot R_0 < 0, \quad \text{or} \quad (K_X + \Delta) \cdot R_0 < 0.$$

We shall consider the two cases separately.

(i) $(K_{\mathcal{F}} + \Delta_{n\text{-inv}}) \cdot R_0 < 0$: then R_0 is spanned by the class of an $\mathcal{F}$-invariant rational curve C_0, [25, Theorem 6.3]; moreover, Theorem 2.8 shows that R_0 can be contracted.

(ii) $(K_X + \Delta) \cdot R_0 < 0$: then R_0 can be contracted by the classical version of the Cone and Contraction Theorem, e.g., [15, Theorem 3.7].

We denote the contraction constructed in (i)/(ii) by $\rho_1 \colon X \to X_1$. If both

$$(K_{\mathcal{F}} + \Delta_{n\text{-inv}}) \cdot R_0 < 0 \quad \text{and} \quad (K_X + \Delta) \cdot R_0 < 0,$$

then the contraction ρ_1 is independent of the choice of (i)/(ii) because the contracted curves coincide.

If $\dim X_1 < \dim X$, then we stop and take ρ to be the identity again. In this case, then $K_{(X,\mathcal{F},\Delta),\epsilon}$ is not pseudo-effective since R_0 is contained in the cone of movable curves of X, as the fibers of ρ_1 move and their classes all lie in R_0.

If $\dim X_1 = \dim X$, then ρ_1 is birational and we set

$$\mathcal{F}_1 := \rho_{1*}\mathcal{F}, \quad \Delta_1 := \rho_{1*}\Delta.$$

Lemma 2.16 implies that $(X_1, \mathcal{F}_1, \Delta_1)$ is (ϵ, δ)-adjoint log canonical.

Claim. *If ρ_1 is birational, then (X_1, Δ_1) is dlt and $(\mathcal{F}_1, \Delta_{1,n\text{-inv}})$ is log canonical.*

Proof of the claim. We first deal with the singularities of (X_1, Δ_1). If ρ_1 is obtained via (ii), then the conclusion follows at once from the negativity lemma since (X, Δ) is dlt and ρ_1 is a $(K_X + \Delta)$-negative contraction. Hence, we can assume that ρ_1 is a $(K_{\mathcal{F}} + \Delta_{n\text{-inv}})$-negative birational contraction. Denoting by C_0 the rational invariant curve contracted by ρ_1, then $\mathcal{F}$ is canonical in a neighborhood of C_0 by Lemma 2.7, and the conclusion then follows from Remark 2.9 since ρ_1 is a $(K_X + \Delta + C_0)$-negative contraction and the pair $(X, \Delta + C_0)$ is dlt. To show that $(\mathcal{F}_1, \Delta_{1,n\text{-inv}})$ is log canonical, it suffices to apply Lemma 2.17 since $(X_1, \mathcal{F}_1, \Delta_1)$ is (ϵ, δ)-adjoint log canonical. □

We may then substitute $(X, \mathcal{F}, \Delta)$ with $(X_1, \mathcal{F}_1, \Delta_1)$ and repeat the above process. We therefore obtain a sequence of contractions

$$X =: X_0 \xrightarrow{\rho_1} X_1 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_i} X_i \xrightarrow{\rho_{i+1}} \cdots . \tag{3.1}$$

We set, inductively, $\mathcal{F}_j := \rho_{j*}\mathcal{F}_{j-1}$ and $\Delta_j := \rho_{j*}\Delta_{j-1}$. At each step, ρ_j contracts an extremal ray $R_{j-1} \subset \overline{\mathrm{NE}(X_{j-1})}$ having negative intersection with $K_{(X_{j-1},\mathcal{F}_{j-1},\Delta_{j-1}),\epsilon}$. Since X is a projective surface, X_j is projective as well for all j. Since the Picard number of X_j decreases at each step, the above sequence of contractions cannot be infinite. Hence, there exists $n \in \mathbb{N}$ such that

(a) either $K_{(X_n,\mathcal{F}_n,\Delta_n),\epsilon}$ is nef,

(b) or there exists a $K_{(X_n,\mathcal{F}_n,\Delta_n),\epsilon}$-negative extremal ray $R_n \subset \overline{\mathrm{NE}(X_n)}$ whose contraction induces a fibration $X_n \to X_{n+1}$ with $\dim X_n > \dim X_{n+1}$. Thus, $K_{(X_n,\mathcal{F}_n,\Delta_n),\epsilon}$ is not pseudo-effective since R_n is spanned by curves that move in X_n.

In both cases (a), (b), we set $Y := X_n$ and $\rho = \rho_{n-1} \circ \rho_{n-2} \circ \cdots \circ \rho_2 \circ \rho_1$, and in case (b), we set $Z := X_{n+1}$. We are then ready to prove the four statements of the theorem.

(1) The conclusion follows inductively from Lemma 2.16 since, at each step of (3.1), ρ_j contracts the $K_{(X_{j-1},\mathcal{F}_{j-1},\Delta_{j-1}),\epsilon}$-negative extremal ray R_{j-1}.

(2) As $(Y, \mathcal{G}, \Delta)$ is (ϵ, δ)-adjoint log canonical by (1), Lemma 2.14 (1) implies that it is also ϵ-adjoint log canonical; thus, by our choice of ϵ, Proposition 2.17 implies that $(\mathcal{G}, \Gamma_{n\text{-inv}})$ has log canonical singularities. The conclusion on the singularities of (Y, Γ) follows applying inductively the same proof as that of the claim.

(3) If $K_{(X,\mathcal{F},\Delta),\epsilon}$ is pseudo-effective, then $K_{(X_j,\mathcal{F}_j,\Delta_j),\epsilon}$ is also pseudo-effective for all $j = 1, \ldots, n$ since $K_{(X_j,\mathcal{F}_j,\Delta_j),\epsilon}$ is the pushforward of $K_{(X_{j-1},\mathcal{F}_{j-1},\Delta_{j-1}),\epsilon}$. Hence, the sequence of contractions in (3.1) must conclude with case (a) above.

(4) If $K_{(X,\mathcal{F},\Delta),\epsilon}$ is not pseudo-effective, then also $K_{(X_j,\mathcal{F}_j,\Delta_j),\epsilon}$ is not pseudo-effective for all $j = 1, \ldots, n$ by the negativity lemma. Hence, the sequence of contractions in (3.1) must terminate with (b) above. By [15, Corollary 3.17] and Theorem 2.8, we then have $\rho(X_n/X_{n+1}) = 1$. □

Exactly as in the classical case, the proof of Theorem 3.1 can be adapted to yield a proof of the following relative version of the statement. The interested reader can find a detailed explanation of how to reduce from the MMP on a projective variety to the relative case in [15, §§ 3.6–3.7].

Corollary 3.2. *Let $(X, \mathcal{F}, \Delta)$ be a foliated triple, where X is a surface, (X, Δ) is dlt and $\mathcal{F}$ is rank one, and let $\pi : X \to S$ be a projective morphism. Fix $\delta \geq 0$. Suppose that $\Delta \in I \cap [0, 1-\delta]$, where $I \subset [0, 1]$ is a DCC set. Fix $0 < \epsilon < E(I)$, where $E(I)$ is as in Proposition* 2.17. *Suppose that $(X, \mathcal{F}, \Delta)$ is (ϵ, δ)-adjoint log canonical.*

Then we may run a $K_{(X,\mathcal{F},\Delta),\epsilon}$-MMP relative over S,

$$\begin{tikzcd} X \arrow[rr, "\rho"] \arrow[dr] & & Y. \arrow[dl] \\ & S & \end{tikzcd}$$

Moreover, setting $\mathcal{G} := \rho_* \mathcal{F}$, $\Gamma := \rho_* \Delta$, the following properties hold:

(1) *$(Y, \mathcal{G}, \Gamma)$ is (ϵ, δ)-adjoint log canonical.*

(2) *$(\mathcal{G}, \Gamma_{n\text{-inv}})$ has log canonical singularities and (Y, Γ) is dlt.*

(3) *If $K_{(X,\mathcal{F},\Delta),\epsilon}$ is pseudo-effective over S, then $K_{(Y,\mathcal{G},\Gamma),\epsilon}$ is nef over S.*

(4) *If $K_{(X,\mathcal{F},\Delta),\epsilon}$ is not pseudo-effective over S, then Y/S admits a fibration $f : Y \to Z$ over S with $\rho(X/Y) = 1$ and such that $-K_{(Y,\mathcal{G},\Gamma),\epsilon}$ is f-ample.*

When running the adjoint MMP, we have precise control of the singularities of the underlying surface, as the following corollary shows.

Corollary 3.3. *Fix a positive real number δ and a DCC set $I \subset [0,1]$. Fix a positive real number $\epsilon < E(I)$, where $E(I)$ is as in Proposition* 2.17.

Let $(X, \mathcal{F}, \Delta)$ be a projective log smooth foliated triple, where X is a surface and $\mathcal{F}$ is rank one. Assume that $\Delta \in I \cap [0, 1-\delta]$. Let $\rho\colon (X, \mathcal{F}, \Delta) \to (Y, \mathcal{G}, \Theta := \rho_*\Delta)$ be a (finite) sequence of steps of the $K_{(X,\mathcal{F},\Delta),\epsilon}$-MMP. Then (Y, Θ) has η-lc singularities, where $\eta = \frac{\epsilon\delta}{1+\epsilon}$.

Proof. We argue by induction on the number n of steps of the $K_{(X,\Delta,\mathcal{F}),\epsilon}$-MMP that compose ρ,

$$X_0 := X \xrightarrow{\rho_1} X_1 \xrightarrow{\rho_2} \cdots \xrightarrow{\rho_{n-1}} X_{n-1} \xrightarrow{\rho_n} X_n = Y, \quad \rho = \rho_n \circ \cdots \circ \rho_1.$$

We denote inductively, $\mathcal{F}_i := \rho_{i,*}\mathcal{F}_{i-1}$, $\mathcal{F}_0 := \mathcal{F}$, $\Delta_i := \rho_{i,*}\Delta_{i-1}$, $\Delta_0 := \Delta$. Since $(X, \Delta, \mathcal{F})$ is log smooth and $\Delta \in [0, 1-\delta]$, then (X, Δ) is δ-lc. As for $\delta > 0$, $\delta > \eta = \frac{\epsilon\delta}{1+\epsilon}$, the case $n = 0$ is trivially settled. Hence, we will assume that $n > 0$. Moreover, the above observation implies that it suffices to show that (Y, Θ) is η-lc at all $P \in Y$ at which ρ^{-1} is not an isomorphism, i.e., $P \in \rho(\mathrm{exc}(\rho))$. We fix one such point, and we distinguish two cases based on whether or not P is a point at which $(\mathcal{G}, \Theta_{n\text{-inv}})$ is terminal.

Case 1. P is terminal for $(\mathcal{G}, \Theta_{n\text{-inv}})$. Let L be the unique germ of a leaf through P and let E be the union of the curves E_i contracted by ρ to P with the reduced structure. The E_i are all $\mathcal{F}$-invariant since $\mathcal{G}$ is non-dicritical. Hence, $\rho_*^{-1}\Theta_{n\text{-inv}} = \Delta_{n\text{-inv}}$. Moreover, near every point of $E \cap \mathrm{sing}(\mathcal{F})$, we know that $\mathcal{F}$ admits a holomorphic first integral. Thus, by [5, Lemma 2.16], $K_X + \rho_*^{-1}(\Theta_{n\text{-inv}} + L) + E$ is ρ-numerically equivalent to $K_{\mathcal{F}} + \Delta_{n\text{-inv}}$ over a sufficiently small neighborhood U_P of $P \in Y$. Moreover, $\Theta_{n\text{-inv}} + L \geq \Theta$, as L is the unique leaf of $\mathcal{F}$ through P. Writing $\Delta = \rho_*^{-1}\Theta + \sum a_i E_i$, then $a_i \leq 1-\delta$ by assumption and $K_{(X,\mathcal{F},\Delta),\epsilon}$ is ρ-numerically equivalent over U_P to

$$\begin{aligned}&\big(K_X + \rho_*^{-1}(\Theta_{n\text{-inv}} + L) + E\big) + \epsilon\Big(K_X + \rho_*^{-1}\Theta + \sum a_i E_i\Big)\\ &\qquad = (1+\epsilon)\Big(K_X + \frac{1}{1+\epsilon}\big(\rho_*^{-1}(\Theta_{n\text{-inv}} + L) + \epsilon\rho_*^{-1}\Theta\big) + \sum \frac{1+\epsilon a_i}{1+\epsilon} E_i\Big).\end{aligned}$$

Since, for all i,

$$\big(\rho_*^{-1}(\Theta_{n\text{-inv}} + L) + \epsilon\rho_*^{-1}\Theta\big) \cdot E_i \geq (1+\epsilon)(\rho_*^{-1}\Theta) \cdot E_i$$

and the support of L is not contained in E, it follows that, over U_P, ρ coincides with a run of the $(K_X + \rho_*^{-1}\Theta + \sum \frac{1+\epsilon a_i}{1+\epsilon} E_i)$-MMP. As $(X, \rho_*^{-1}\Theta + \sum E_i)$ is log smooth by assumption, $(X, \rho_*^{-1}\Theta)$ is δ-lc by assumption, and $\frac{1+\epsilon a_i}{1+\epsilon} \leq \frac{1+\epsilon(1-\delta)}{1+\epsilon}$, then

$$K_X + \rho_*^{-1}\Theta + \sum \frac{1+\epsilon a_i}{1+\epsilon} E_i \leq \eta(K_X + \rho_*^{-1}\Theta) + (1-\eta)\Big(K_X + \rho_*^{-1}\Theta + \sum E_i\Big)$$

for $\eta := \frac{\epsilon\delta}{1+\epsilon}$. Thus, the pair (Y, Θ) is η-lc since it is the outcome of a run of the MMP over U_P for the η-lc pair $(X, \rho_*^{-1}\Theta + \sum \frac{1+\epsilon a_i}{1+\epsilon} E_i)$.

Case 2. P is not terminal for $(\mathcal{G}, \Theta_{n\text{-inv}})$. By the inductive hypothesis, we can assume that (X_{n-1}, Δ_{n-1}) is η-lc. Moreover, we can assume that the step $\rho_n\colon X_{n-1} \to Y$ is the contraction of a $K_{(X_{n-1},\mathcal{F}_{n-1},\Delta_{n-1}),\epsilon}$-negative rational curve C_{n-1} with $\rho_n(C_{n-1}) = P$. If ρ_n is a $(K_{X_{n-1}} + \Delta_{n-1})$-negative contraction, then the proof terminates. Hence, we can assume that ρ_n is a $(K_{\mathcal{F}_{n-1}} + \Delta_{n-1})$-negative contraction. This leads to an immediate following contradiction, by the next claim.

Claim. *If ρ_n is a $(K_{\mathcal{F}_{n-1}} + \Delta_{n-1})$-negative contraction, $(\mathcal{G}, \Theta_{n\text{-inv}})$ is terminal at P.*

Proof of the claim. By Lemma 2.7, $(\mathcal{F}_{n-1}, \Delta_{n-1})$ is canonical at any point $Q \in X_{n-1}$ such that $\rho_n(Q) = P$ since ρ_n is birational. The negativity lemma readily shows that the terminality of $(\mathcal{G}, \Theta_{n\text{-inv}})$ at P follows from the hypothesis of the claim, cf. [15, Lemma 3.38]. □

This concludes the proof. □

We can also show that (ϵ, δ)-adjoint log canonical models exist in the projective category.

Corollary 3.4. *Set up as in Theorem* 3.1. *Suppose in addition that (X, Δ) is klt and that $K_{(X,\mathcal{F},\Delta),\epsilon}$ is big. Let $(Y, \mathcal{G}, \Gamma)$ be the output of a run of the $K_{(X,\mathcal{F},\Delta),\epsilon}$-MMP starting on X. Then there exists a birational contraction*

$$p: (Y, \mathcal{G}, \Gamma) \to (Y_{\mathrm{can}}, \mathcal{G}_{\mathrm{can}}, \Gamma_{\mathrm{can}})$$

such that

(1) Y_{can} *is projective;*

(2) $K_{(Y_{\mathrm{can}},\mathcal{G}_{\mathrm{can}},\Gamma_{\mathrm{can}}),\epsilon}$ *is an ample $\mathbb{Q}$-Cartier divisor;*

(3) $(Y_{\mathrm{can}}, \mathcal{G}_{\mathrm{can}}, \Gamma_{\mathrm{can}})$ *has (ϵ, δ)-adjoint log canonical singularities;*

(4) Y_{can} *has η-lc singularities where $\eta = \frac{\epsilon\delta}{1+\epsilon}$.*

Moreover, Y_{can} is uniquely determined.

Proof. First, note that $(Y, \mathcal{G}, \Gamma)$ has (ϵ, δ)-adjoint log canonical singularities.

Let $C \subset Y$ be a curve with $K_{(Y,\mathcal{G},\Gamma),\epsilon} \cdot C = 0$. Since $K_{(Y,\mathcal{G},\Gamma),\epsilon}$ is nef and big by construction, then $C^2 < 0$ by the Hodge Index Theorem. There are three possibilities at this point:

(i) $(K_{\mathcal{G}} + \Gamma_{n\text{-inv}}) \cdot C < 0$,

(ii) $(K_Y + \Gamma) \cdot C < 0$, or

(iii) $(K_{\mathcal{G}} + \Gamma_{n\text{-inv}}) \cdot C = (K_Y + \Gamma) \cdot C = 0$.

In case (i), $(\mathcal{G}, \Gamma_{n\text{-inv}})$ has canonical singularities in a neighborhood of C, as otherwise C would move, cf. the proof of Theorem 3.1; thus, we may contract C by a $(K_{\mathcal{G}} + \Gamma_{n\text{-inv}})$-negative contraction. In case (ii), we may contract C by a $(K_Y + \Gamma)$-negative contraction. In case (iii), we perform a $(K_Y + \Gamma + tC)$-negative contraction for some $t > 0$ sufficiently small, which does not constitute a problem since we are assuming that (X, Δ) is klt to start with. In any case, we obtain a morphism $p': Y \to Y'$ which contracts C to a point. Then $(Y', p'_*\mathcal{G}, p'_*\Gamma)$ is still (ϵ, δ)-adjoint log canonical, and the argument in Corollary 3.3 works equally well here to show that Y' has η-lc singularities.

Repeating this process, we will eventually terminate in model $(Y_0, \mathcal{G}_0, \Gamma_0)$ such that $K_{(Y_0,\mathcal{G}_0,\Gamma_0),\epsilon} \cdot C > 0$ for all curves C, and hence $K_{(Y_0,\mathcal{G}_0,\Gamma_0),\epsilon}$ is ample. That immediately implies the final part of the statement of the corollary, as $(p \circ \rho)^* K_{(Y_0,\mathcal{G}_0,\Gamma_0),\epsilon}$ realizes the positive part of the Zariski decomposition of $K_{(X,\mathcal{F},\Delta),\epsilon}$ which is uniquely determined. □

Remark 3.5. Analogously to Corollary 3.2, also Corollary 3.4 has a relative version with respect to a projective morphism $\pi: X \to S$.

Following the notation of Corollary 3.2, we assume in addition that (X, Δ) is klt and that $K_{(X,\mathcal{F},\Delta),\epsilon}$ is big over S. Denoting $(Y, \mathcal{G}, \Gamma)$ be the output of a run of the $K_{(X,\mathcal{F},\Delta),\epsilon}$-MMP over S, starting from X, there exists a birational contraction $p\colon (Y, \mathcal{G}, \Gamma) \to (Y_{\text{can}}, \mathcal{G}_{\text{can}}, \Gamma_{\text{can}})$ such that

(1) Y_{can} is projective over S;

(2) $K_{(Y_{\text{can}},\mathcal{G}_{\text{can}},\Gamma_{\text{can}}),\epsilon}$ is an ample $\mathbb{Q}$-Cartier divisor over S;

(3) $(Y_{\text{can}}, \mathcal{G}_{\text{can}}, \Gamma_{\text{can}})$ has (ϵ, δ)-adjoint log canonical singularities;

(4) Y_{can} has η-lc singularities where $\eta = \frac{\epsilon\delta}{1+\epsilon}$.

Moreover, Y_{can}/S is uniquely determined.

The last corollary motivates the following definition of an ample model.

Definition 3.6. We say that $(Y_{\text{can}}, \mathcal{G}_{\text{can}}, \Gamma_{\text{can}})$ is the (ϵ, δ)-*adjoint log canonical model* of $(X, \mathcal{F}, \Delta)$ (or the ϵ-*adjoint canonical model* when $\Delta = 0$).

Corollary 3.4 marks a notable difference with the theory of the MMP for the canonical divisor of general type foliations with canonical singularities. We recall that the canonical model of a surface foliation, in the sense of [20], is not necessarily projective, owing to the presence of cusp type singularities arising from the contraction of elliptic Gorenstein leaves (e.g.l.s) to points. We emphasize that the ϵ-adjoint canonical model, by contrast, is always projective and does not contract any e.g.l.s to points.

4. A bound on the pseudo-effective threshold in the big case

The goal of this section is to prove the following.

Theorem 4.1. *Fix a DCC set $I \subset [0, 1]$. There exists a positive real number $\tau = \tau(I)$ satisfying the following property.*

If $(X, \mathcal{F}, \Delta)$ is a projective log smooth triple such that X is a surface, $\mathcal{F}$ is rank one, $K_{\mathcal{F}} + \Delta_{n\text{-inv}}$ is big and $\Delta \in I$, then $K_{(X,\mathcal{F},\Delta),\tau}$ is big.

4.1. Outline of the proof of Theorem 4.1. We divide the outline of the proof into steps.

Step 0. As a preliminary step, we show that we may freely assume that Δ has coefficients in some fixed finite subset $J \subset I$ (Proposition 4.6).

For ease of exposition, we will assume that $\Delta = 0$ for the remainder of the proof sketch.

Step 1. Fix some $0 < \tau < E(I)$, where $E(I)$ is the constant given by Lemma 2.17. If $K_{(X,\mathcal{F}),\tau}$ is pseudo-effective, then we are done. Thus, we may assume that $K_{(X,\mathcal{F}),\tau}$ is not pseudo-effective.

Step 2. Assume for the moment that there exists a bounded family $\mathfrak{M}$ such that, for every $(X, \mathcal{F})$ with $K_{(X,\mathcal{F}),\tau}$ not pseudo-effective, there exist $(Z, \mathcal{G}) \in \mathfrak{M}$ and a birational morphism $(X, \mathcal{F}) \to (Z, \mathcal{G})$. Assume moreover that Z is smooth at all the strictly log canonical singularities of $\mathcal{G}$.

By boundedness, we may then find a τ' (independent of $(Z, \mathcal{G})$) such that $K_{(Z,\mathcal{G}),\tau'}$ is pseudo-effective.

In general, $(Z, \mathcal{G})$ may not be τ'-adjoint canonical, and so we cannot lift sections to $(X, \mathcal{F})$. However, if $(Z, \mathcal{G})$ is not τ'-adjoint canonical using some computations from [23], see Lemma 4.2, we show that the eigenvalues of the singular points of $\mathcal{G}$ belong to a finite set. In particular, it follows that we may resolve the log canonical singularities of $\mathcal{G}$ in a bounded way. Repeating the argument and using Noetherian induction allows us to conclude. This is achieved in Proposition 4.3.

Step 3. We may run an adjoint MMP which terminates in a foliated pair $(Z, \mathcal{G})$, where Z is a Fano surface with η-lc singularities, where $\eta > 0$ depends only on τ and J. In particular, $(Z, \mathcal{G})$ belongs to a bounded family. If we can arrange it so that Z is smooth at all strictly log canonical singularities of $\mathcal{G}$, then we are done by Step 2; this is done in Proposition 4.5.

Step 4. Finally, we show that we may modify our family $(Z, \mathcal{G})$ so that the condition that Z is smooth at all strictly log canonical singularities of $\mathcal{G}$ holds (this is done in Lemma 4.4).

4.2. Proof of Theorem 4.1.

Lemma 4.2. *Fix positive real numbers ϵ', δ. Then there exists a finite set $\Lambda \subset \mathbb{N} \times \mathbb{N}$ depending only on ϵ' such that the following holds.*

Let $(0 \in \mathbb{C}^2, \mathcal{F})$ be a germ of a rank one foliation such that $\mathcal{F}$ has log canonical singularities, but is not (ϵ', δ)-adjoint log canonical. Then $T_{\mathcal{F}}$ is generated by a vector field of the form $px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y}$, where $(p, q) \in \Lambda$.

Proof. Observe that $\mathcal{F}$ is strictly log canonical at 0, and so $T_{\mathcal{F}}$ is generated by a vector field of the form $px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y}$, where p, q are positive co-prime integers. Let $[u_1, \dots, u_k]$ be the continued fraction representation of p/q.

Next, observe that $\mathcal{F}$ is not ϵ'-adjoint canonical. It follows by combining [23, Corollary 4.10] – keeping in mind the slight difference of notations, Remark 2.13 – and [23, Lemma 4.7] that

$$\frac{1}{\epsilon'} \geq \sum_{i=1}^{k} u_i,$$

which implies that k and u_i are bounded in terms of ϵ'. Hence, p/q can take on only finitely many values, depending only on ϵ'. □

Proposition 4.3. *Fix a finite subset $J \subset [0, 1)$ and a real number $\epsilon > 0$. Let $\mathfrak{D}$ be a collection of projective foliated log smooth triples $(X, \mathcal{F}, \Delta)$ such that X is a surface, $\mathcal{F}$ is rank one, $\Delta = \Delta_{n\text{-inv}}$, $\Delta \in J$ and $K_{\mathcal{F}} + \Delta_{n\text{-inv}}$ is big. Suppose that there exists a bounded family of 2-dimensional foliated triples*

$$f : (\mathcal{Y}, \mathcal{G}, \Gamma) \to T \tag{4.1}$$

such that, for all $(X, \mathcal{F}, \Delta) \in \mathfrak{D}$, there exists $t \in T$ such that

(1) *there exists a contraction $\pi_t : X \to \mathcal{Y}_t$ and $\mathcal{G}_t = \pi_{t,*}\mathcal{F}$, $\Gamma_t = \pi_{t,*}\Delta$;*

(2) *$K_{(X,\mathcal{F},\Delta),\epsilon} = \pi_t^* K_{(\mathcal{Y}_t,\mathcal{G}_t,\Gamma_t),\epsilon} + E_\epsilon$, where $E_\epsilon \geq 0$;*

(3) *$\mathcal{Y}_t$ is klt, $(\mathcal{G}_t, \Gamma_t)$ is log canonical and $K_{\mathcal{G}_t} + \Gamma_t$ is big;*

(4) *$(\mathcal{Y}_t, \Gamma_t)$ is log smooth at all strictly log canonical points of $(\mathcal{G}_t, \Gamma_t)$.*

Then there exists a positive real number $\tau_0 = \tau_0(J, \epsilon)$ such that, for all $0 \leq \tau' < \tau_0$ and all $(X, \mathcal{F}, \Delta) \in \mathfrak{D}$, $K_{(X,\mathcal{F},\Delta),\tau'}$ is big.

Proof. By boundedness of the family in (4.1) and by (3), there exists $\tilde{\tau}_0 \leq \epsilon$ such that, for all $0 \leq s \leq \tilde{\tau}_0$ and all $t \in T$, $K_{(\mathcal{Y}_t,\mathcal{G}_t,\Gamma_t),s}$ is big. For a foliated triple $(X, \mathcal{F}, \Delta) \in \mathfrak{D}$, a point $t \in T$ as in the assumption of the proposition and $\lambda \in \mathbb{R}_{\geq 0}$, we define the effective divisors $E_{\lambda,t}, F_{\lambda,t}$ by

$$E_{\lambda,t} - F_{\lambda,t} := K_{(X,\mathcal{F},\Delta),\lambda} - \pi^* K_{(\mathcal{Y}_t,\mathcal{G}_t,\Gamma_t),\lambda},$$

where we assume that no prime divisor on X appears in the support of both $E_{\lambda,t}, F_{\lambda,t}$. Assumption (2) implies that $F_{\epsilon,t} = 0$; moreover, if $F_{\tilde{\tau}_0,t} = 0$, then $K_{(X,\mathcal{F},\Delta),\tilde{\tau}_0}$ is big.

Claim 1. *Fix $(X, \mathcal{F}, \Delta) \in \mathfrak{D}$ and $t \in T$ as in the statement of the proposition. Assume that $F_{\tilde{\tau}_0,t} \neq 0$. Then there exists a strictly log canonical singularity of $\mathcal{G}_t$ at which $(\mathcal{Y}_t, \mathcal{G}_t, \Gamma_t)$ is not $\tilde{\tau}_0$-adjoint canonical.*

Proof. By item (2) and since $F_{\tilde{\tau}_0,t} \neq 0$, there must exist a prime divisor $C \subset X$ that is π-exceptional and such that, for $a_C := \mu_C \Delta$,

$$a(C, \mathcal{G}_t, \Gamma_t) + \epsilon a(C, \mathcal{Y}_t, \Gamma_t) \geq -(\iota(C) + \epsilon) a_C,$$
$$a(C, \mathcal{G}_t, \Gamma_t) + \tilde{\tau}_0 a(C, \mathcal{Y}_t, \Gamma_t) < -(\iota(C) + \tilde{\tau}_0) a_C.$$

Hence, $a(C, \mathcal{G}_t, \Gamma_t) < 0$ and $\iota(C) = 1$, C is a strictly log canonical place of $(\mathcal{G}_t, \Gamma_t)$, with center P on $\mathcal{Y}_t$, and $a(C, \mathcal{G}_t, \Gamma_t) = -1$, as otherwise $\mathcal{G}_t$ would be non-dicritical at P contradicting that $a(C, \mathcal{G}_t, \Gamma_t) < 0$. Moreover, $P \notin \operatorname{supp}(\Gamma_t)$ and $\mathcal{G}_t$ is not $\tilde{\tau}_0$-adjoint canonical at P. □

Consider the following subset $Z_0 \subset T$:

(4.2) $$Z_0 := \{t' \in T \mid \text{there exists a strictly log canonical center } P' \in \mathcal{Y}_{t'} \text{ for } \mathcal{G}_{t'} \text{ at which } (\mathcal{Y}_{t'}, \mathcal{G}_{t'}, \Gamma_{t'}) \text{ is not } \tilde{\tau}_0\text{-adjoint canonical.}\}$$

By Lemma 4.2 and item (4) in the hypotheses of the proposition, the eigenvalues of $\mathcal{G}_{t'}$ at a point $P' \in \mathcal{Y}_{t'}$ as in (4.2) belong to a finite set; thus, $Z_0 \subset T$ is a Zariski closed subset – with possible equality.

Given $(X, \mathcal{F}, \Delta) \in \mathfrak{D}$ such that, for the foliated triple $(\mathcal{Y}_t, \mathcal{G}_t, \Gamma_t)$ associated to it, we have $t \in T \setminus Z_0$, then $F_{\tilde{\tau}_0,t} = 0$ and, for all $0 \leq s \leq \tilde{\tau}_0$, $K_{(X,\mathcal{F},\Delta),s}$ is big.

Up to stratifying Z_0 into a finite disjoint union of locally closed set, we may and will assume that Z_0 is the union of finitely many (disjoint) smooth irreducible components. We denote by $\overline{f}\colon (\overline{\mathcal{Y}}, \overline{\mathcal{G}}, \overline{\Gamma}) \to Z_0$ the restriction of $(\mathcal{Y}, \mathcal{G}, \Gamma)$ to Z_0.

Claim 2. *There exists a surjective morphism $e\colon Z_0' \to Z_0$ finite onto its image and a bounded family of foliated triples $f_0\colon (\mathcal{Y}^0, \mathcal{G}^0, \Gamma^0) \to Z_0'$ such that*

(i) *for any $t' \in Z_0'$, there exists a morphism $\mu_{t'}\colon \mathcal{Y}^0_{t'} \to \mathcal{Y}_{e(t')}$ which is a foliated log resolution in a neighborhood of any point of $\mathcal{Y}_{e(t')}$ at which $(\mathcal{Y}_{e(t')}, \Gamma_{e(t')})$ is not $\tilde{\tau}_0$-adjoint canonical;*

(ii) *if* $(X, \mathcal{F}, \Delta)$ *corresponds to* $(\mathcal{Y}_t, \mathcal{G}_t, \Gamma_t)$, $t \in Z_0$ *as in the statement of the proposition, then we may assume there exist* $t' \in e^{-1}(t)$ *and a morphism* $\nu_{t'} \colon X \to \mathcal{Y}^0_{t'}$;

(iii) *with the notation of item* (ii), *then* $\Gamma^0_{t'} = \nu_{t'*}\Delta$, $(\mathcal{Y}^0_{t'}, \mathcal{G}^0_{t'}, \Gamma^0_{t'})$ *is* $\tilde{\tau}_0$*-adjoint canonical at all log canonical centers of* $(\mathcal{G}^0_{t'}, \Gamma^0_{t'})$, *and* $K_{\mathcal{G}^0_{t'}} + \Gamma^0_{t'}$ *is big.*

Moreover,

$$K_{(X,\mathcal{F},\Delta),\epsilon} = \nu_t^* K_{(\mathcal{Y}^0_t,\mathcal{G}^0_t,\Gamma^0_t),\epsilon} + \widetilde{E}_{t,\epsilon}, \tag{4.3}$$

with $\widetilde{E}_{t,\epsilon} \geq 0$.

Proof. Passing to a stratification into locally closed subsets and a finite cover of Z_0, we may assume that the Zariski closed set S' defined by

$$\begin{aligned} S' := \{s \in \overline{\mathcal{Y}} \mid (\overline{\mathcal{G}}_{\overline{f}(s)}, \overline{\Gamma}_{\overline{f}(s)}) &\text{ is strictly log canonical at } s \in \overline{\mathcal{Y}}_{\overline{f}(s)} \\ \text{and } (\overline{\mathcal{Y}}_{\overline{f}(s)}, \overline{\mathcal{G}}_{\overline{f}(s)}, \overline{\Gamma}_{\overline{f}(s)}) &\text{ is not } \tilde{\tau}_0\text{-adjoint canonical at } s\} \end{aligned}$$

is flat over Z_0, all fibers of $\overline{f}|_{S'} \colon S' \to Z_0$ are everywhere reduced and the ratio of the eigenvalues of $\overline{\mathcal{G}}_{\overline{f}(s)}$ at s is constant on the components of S'. The last claim is a simple consequence of Lemma 4.2. Analogous reasoning shows also that there exists an upper bound on the number of strictly log canonical singularities of $(\overline{\mathcal{G}}_t, \overline{\Gamma}_t)$ independent of t.

Each of the log canonical foliated surface singularities parametrized by S' admits a foliated log resolution by a bounded number of blow ups, and the bound on the number of blow ups depends only on $\tilde{\tau}_0$, as shown in Lemma 4.2 and its proof, since such singularities are not $\tilde{\tau}_0$-adjoint canonical. Moreover, as the ratio of the eigenvalues of $\overline{\mathcal{G}}_{\overline{f}(s)}$ at s is constant on the components of S', we can perform these blow ups in family and, thus, obtain a bounded family of 2-dimensional triples

$$\begin{tikzcd} (\mathcal{Y}', \mathcal{G}', (\mu')^{-1}_*\overline{\Gamma}) \arrow[rr, "\mu"] \arrow[dr, "f'"'] & & (\overline{\mathcal{Y}}, \overline{\mathcal{G}}, \overline{\Gamma}), \arrow[dl, "\overline{f}"] \\ & Z_0 & \end{tikzcd}$$

where μ is the partial resolution whose construction we just explained and $\mathcal{G}' := \mu^{-1}\mathcal{G}$. Moreover, possibly passing to a stratification of Z_0 into locally closed sets, and a finite covering of the irreducible components of the stratification, we may assume that, for any $t \in Z_0$, there exists a 1-1 correspondence between the irreducible components of the exceptional locus of μ and those of μ_t, over the irreducible component of Z_0 containing t; moreover, we can assume that if E' is a μ-exceptional prime divisor, then $\iota(E'_t) = \iota(E')$ for any t contained in the image of E'. Let us denote by $\{E_1, \ldots, E_r\}$ the μ-exceptional divisors that are not $\mathcal{G}'$-invariant. We define $W := J \cup \{0\}$. We define $Z'_0 := \bigsqcup_{(a_1,\ldots,a_r)\in W^r} Z_0$ and $e \colon Z'_0 \to Z_0$ to be the identity on each copy of Z_0 contained in Z'_0. Then we define

$$(\mathcal{Y}^0, \mathcal{G}^0, \Gamma^0) := \bigsqcup_{(a_1,\ldots,a_r)\in W^r} \left(\mathcal{Y}', \mathcal{G}', (\mu')^{-1}_*\overline{\Gamma} + \sum_{i=1}^r a_i E_i\right).$$

The morphism f' induces a morphism $f_0 \colon (\mathcal{Y}^0, \mathcal{G}^0, \Gamma^0) \to Z'_0$, which yields a bounded family of 2-dimensional foliated triples.

To prove items (ii)–(iii), let $r\colon X' \to X$ resolve the indeterminacies of the rational map $X \dashrightarrow \mathcal{Y}^0_t$ so that
$$(X', \mathcal{F}' := r^{-1}\mathcal{F},\ \Delta' := r_*^{-1}\Delta)$$
is foliated log smooth. We denote by $\nu'_t\colon X' \to Y^0_t$ the induced morphism. If $K_{(X',\mathcal{F}',\Delta'),t}$ is big, then $K_{(X,\mathcal{F},\Delta),t}$ is big. Moreover, for any $t > 0$, we may write
$$K_{(X',\mathcal{F}',\Delta'),t} = r^* K_{(X,\mathcal{F},\Delta),t} + G_t,$$
where $G_t \geq 0$; thus, we are free to replace $(X, \mathcal{F}, \Delta)$ by $(X', \mathcal{F}', \Delta')$ in the statement of the proposition. By construction, $\pi_{t*}\Delta = \Gamma_t$. Since $\Delta \in J$ and $\Delta' = r_*^{-1}\Delta$, we then have $(\nu'_{t*}\Delta' - \mu_{t*}^{-1}\Gamma_t) \in W$, and its support is contained only on μ_t-exceptional components that are not $\mathcal{G}'$-invariant, by the definition of triples in $\mathfrak{D}$; thus, there exists $(a_1, \dots, a_r) \in W^r$ such that $\nu'_{t*}\Delta' - (\mu')_{t*}^{-1}\Gamma_t = \sum_i a_i E_{i,t}$. This completes the proof of items (ii)–(iii).

We now prove (4.3). Away from $\nu_t^{-1}(\mathrm{exc}(\mu_t))$, this is clear; thus, let Σ be a connected component of $\mathrm{exc}(\mu_t)$ and let $P := \mu_t(\Sigma)$. Since P is a strictly log canonical singularity of $\mathcal{G}_t$, P is not contained in the support of Γ_t. Moreover, by Lemma 2.6, there is exactly one divisor contained Σ which is transverse to $\mathcal{G}^0_t$. Hence, in a neighborhood of Σ, Γ^0_t is supported on at most 1 curve. Since $(Y^0_t, \mathcal{G}^0_t, \Gamma^0_t)$ is foliated log smooth in a neighborhood of Σ, it follows that $(Y^0_t, \mathcal{G}^0_t, \Gamma^0_t)$ is in fact ϵ-adjoint canonical for all $\epsilon > 0$. □

By boundedness of Z'_0, there exists $\tilde{\tau}_1 < \tilde{\tau}_0$ such that, for all $t \in Z'_0$, $K_{(Y^0_t,\mathcal{G}^0_t,\Gamma^0_t),\tilde{\tau}_1}$ is pseudo-effective. Let $Z'_1 \subset Z'_0$ be the Zariski closed subset
$$\begin{aligned} Z_1 := \{t' \in Z'_0 \mid {} & \text{there exists a strictly log canonical center } P' \in \mathcal{Y}^0_{t'} \\ & \text{for } \mathcal{G}^0_{t'} \text{ at which } (\mathcal{Y}^0_{t'}, \mathcal{G}^0_{t'}, \Gamma^0_{t'}) \text{ is not } \tilde{\tau}_1\text{-adjoint canonical}\} \end{aligned}$$

We may then repeat the above argument with Z'_1, and we define $Z_1 := e(Z_1)$. Iterating this process, we produce a decreasing sequence of Zariski closed subsets $Z_i \subsetneq Z_{i-1}$ of T and a decreasing sequence of positive real numbers $0 < \tilde{\tau}_i < \tilde{\tau}_{i-1}$ such that if $(X, \mathcal{F}, \Delta) \in \mathfrak{D}$ and the corresponding point $t \in T$ given by the proposition satisfies $t \in Z_{i-1} \setminus Z_i$, then we have that $K_{(X,\mathcal{F},\Delta),\tilde{\tau}_i}$ is big; moreover, the foliated surface triples parametrized by points of Z_i admit a log canonical singularity which is not $\tilde{\tau}_{i-1}$-adjoint canonical.

This process must eventually terminate since, at each step of the process, we reduce the number of strictly log canonical singularities on a foliated surface triple appearing in the fibers of our family. Hence, we must eventually obtain that, for some $n \gg 0$, $Z_n = \emptyset$ and $K_{(X,\mathcal{F},\Delta),\tilde{\tau}_{n-1}}$ is big for all $(X, \mathcal{F}, \Delta) \in \mathfrak{D}$. Hence, we set $\tau_0 := \tilde{\tau}_{n-1}$. □

Lemma 4.4. *Let $h\colon (\mathcal{Z}, \mathcal{L}, \Xi) \to T$ be a bounded family of 2-dimensional projective foliated triples $(Z_t, \mathcal{L}_t, \Xi_t)$. Assume that, for all $t \in T$, $\Xi_{t,n\text{-inv}} = \Xi_t$ and $(\mathcal{L}_t, \Xi_t)$ is log canonical.*

Passing to a stratification of T into locally closed sets, and a finite covering of the irreducible components of the stratification, there exists a bounded family of 2-dimensional projective foliated triples $j\colon (\mathcal{Y}, \mathcal{G}, \Gamma) \to T$ and a birational morphism over T, $g\colon \mathcal{Y} \to \mathcal{Z}$, such that, for all $t \in T$,

(1) *$\mathcal{G}_t := g^{-1}\mathcal{L}_t$ and $\Gamma_t := g_*^{-1}\Xi_t$;*

(2) *(Y_t, Γ_t) is log smooth in a neighborhood of $g_t^{-1}(P)$, where P is a strictly log canonical point of $(\mathcal{L}_t, \Xi_t)$;*

(3) *g_t only extracts divisors of discrepancy (resp. foliation discrepancy) at most* 0 *(resp. equal to $-\iota(E)$);*

(4) *any foliated log resolution $\tau_t\colon \overline{Z}_t \to Z_t$ of $(Z_t, \mathcal{L}_t, \Xi_t)$ factors as*

$$\begin{tikzcd} \overline{Z}_t \arrow[r] \arrow[rr, bend right, "\tau_t"'] & Y_t \arrow[r, "g_t"] & Z_t. \end{tikzcd}$$

Proof. Fix $t \in T$ and let $P \in \mathcal{Z}_t$ be a point, where $(\mathcal{L}_t, \Xi_t)$ is strictly log canonical. Thus, $P \notin \mathrm{supp}(\Xi_t)$ and $\mathcal{L}_t$ is strictly log canonical at P. By Lemmata 2.10 and 2.11, there exists a resolution $g_t\colon Y \to \mathcal{Z}_t$ by blowing up $\mathcal{Z}_t$ in $\mathcal{L}_t$-invariant centers. These blow ups only extract divisors of foliation discrepancy equal to $-\iota(E)$; taking g_t to be a minimal log resolution of Z_t around P, g_t only extracts divisors of discrepancy at most 0. Thus, items (2)–(4) are satisfied.

Substituting T with a stratification and taking finite covers of components, the minimal resolutions $g_t\colon Y_t \to \mathcal{Z}_t$ fit together in family to form a bounded family $j\colon \mathcal{Y} \to T$ of resolutions $g\colon \mathcal{Y} \to Z$ such that $g|_{\mathcal{Y}_t} = g_t$. To conclude, it suffices to define $\mathcal{G} := g^{-1}\mathcal{L}$, $\Gamma := g_*^{-1}\Xi$. □

Proposition 4.5. *Let $J \subset [0,1)$ be a finite subset. Let $\mathfrak{D}_J$ be the set of all triples $(X, \mathcal{F}, \Delta)$ such that*

(1) *$(X, \mathcal{F}, \Delta)$ is a projective foliated log smooth triple, X is a surface, $\mathcal{F}$ is rank one,*

(2) *$\Delta_{n\text{-inv}} = \Delta \in J$, and*

(3) *$K_{\mathcal{F}} + \Delta_{n\text{-inv}}$ is big.*

Then there exists $\tau_0 = \tau_0(J) > 0$ such that, for all $0 \le \epsilon < \tau$ and any triple $(X, \mathcal{F}, \Delta) \in \mathfrak{D}_J$, $K_{(X,\mathcal{F},\Delta),\epsilon}$ is big.

Proof. Fix $\epsilon_0 := \min\{\min_{j\in J} \frac{j}{3}, E(J)\}$, cf. Lemma 2.17 for the definition of $E(J)$. Clearly, $\epsilon_0 < E(J)$.

Fix $(X, \mathcal{F}, \Delta) \in \mathfrak{D}_J$. As J is finite and $1 \notin J$, there exists $\delta = \delta(J) := 1 - \max J$ such that $J \subset [0, 1-\delta]$ and $(X, \mathcal{F}, \Delta)$ is (ϵ, δ)-adjoint log canonical for all $\epsilon \ge 0$, cf. Lemma 2.15. If $K_{(X,\mathcal{F},\Delta),\epsilon_0}$ is pseudo-effective, then there is nothing to show; hence, we may assume that $K_{(X,\mathcal{F},\Delta),\epsilon_0}$ is not pseudo-effective.

Let $\rho\colon X \to Z$ be a run of the $K_{(X,\mathcal{F},\Delta),\epsilon_0}$-MMP which exists and terminates by Theorem 3.1. We set $\Xi := \rho_{t*}\Delta$ and $\mathcal{L} := \rho_{t*}\mathcal{F}$. By Theorem 3.1, as $K_{(X,\mathcal{F},\Delta),\epsilon_0}$ is not pseudo-effective, Z is endowed with a Mori fiber space structure with respect to $K_{(Z,\mathcal{L},\Xi),\epsilon_0}$, i.e., there exists a contraction $\psi\colon Z \to B$ with $\dim Z > \dim B$, $\rho(Z/B) = 1$ and $-K_{(Z,\mathcal{L},\Xi),\epsilon_0}$ is ψ-ample. Moreover, the following properties hold.

(i) Z has η-lc singularities for $0 < \eta = \frac{\epsilon_0\delta}{1+\epsilon_0}$, see Corollary 3.3. Here we need that $1 \notin J$ to conclude that $\eta > 0$.

(ii) $K_Z + \Xi$ is not pseudo-effective: it is antiample over B.

(iii) $K_{(Z,\mathcal{L},\Xi),\epsilon_0}$ is antiample over B. Hence, it is not pseudo-effective on Z. The same holds for $K_{\mathcal{L}} + \epsilon_0 K_Z$.

(iv) $K_{(X,\mathcal{F},\Delta),\epsilon_0} = \rho^* K_{(Z,\mathcal{L},\Xi),\epsilon_0} + E$, where $E \geq 0$.

(v) $(\mathcal{L}, \Xi_{t,n\text{-inv}})$ is log canonical.

All of these claims are direct consequences of the negativity lemma and Theorem 3.1, Corollary 3.3. We can also show that the geometry of Z is rather restrictive.

Claim 1. *Z is a Fano surface and $\rho(Z) = 1$, i.e.,* $\dim B = 0$.

Proof of Claim 1. If $\dim B > 0$, then B is a curve. Let F be a general fiber of ψ: F is rational since ψ is a Mori fiber space. Then F is not $\mathcal{L}$-invariant for the foliation as $(K_{\mathcal{L}} + \Xi) \cdot F > 0$ by the bigness of $K_{\mathcal{L}} + \Xi$; thus, $(K_{\mathcal{L}} + \Xi) \cdot F \geq j_0$, where $j_0 = \min J$, and $(K_Z + \Xi) \cdot F \geq -2$. Since $\epsilon_0 < \frac{j_0}{2}$ by definition, then $K_{(Z,\mathcal{L},\Xi),\epsilon_0} \cdot F \geq 0$. On the other hand, by item (iii), $K_{(Z,\mathcal{L},\Xi),\epsilon_0} \cdot F < 0$, which leads to the sought contradiction. □

Let $\mathfrak{D}_{J,\epsilon_0,\mathrm{Fano}}$ be the set of foliated triples $(Z, \mathcal{L}, \Xi)$ that appear as final outcomes (i.e., Mori fiber spaces) in a run of the $K_{(X,\mathcal{F},\Delta),\epsilon_0}$-MMP for $(X, \mathcal{F}, \Delta) \in \mathfrak{D}_J$ with $K_{(X,\mathcal{F},\Delta),\epsilon_0}$ not pseudo-effective. Claim 1 readily implies the following conclusion for $\mathfrak{D}_{J,\epsilon_0,\mathrm{Fano}}$.

Claim 2. $\mathfrak{D}_{J,\epsilon_0,\mathrm{Fano}}$ *forms a bounded family.*

Proof of Claim 2. For any triple $(Z, \mathcal{L}, \Xi) \in \mathcal{D}_{J,\mathrm{Fano}}$, (i) above implies that Z is η-lc for some fixed $\eta > 0$. By [1, Theorem 6.9], η-lc Fano surfaces form a bounded family. Thus, for any triple $(Z, \mathcal{L}, \Xi) \in \mathcal{D}_{J,\mathrm{Fano}}$, there exists $t = t(\eta) \in \mathbb{N}_{>0}$ such that $-tK_Z$ is very ample. Furthermore, (ii)–(iii) imply that

$$0 < \Xi \cdot (-K_Z) \leq -K_Z^2, \quad -\Xi \leq K_{\mathcal{L}} \leq -\frac{1}{\epsilon_0} K_Z. \tag{4.4}$$

Hence, (Z, Ξ) belong to a bounded family as $\Xi \in J$ (and J is finite) and $\deg_{-tK_Z} \Xi \leq -tK_Z^2$. Moreover, thanks to the fact that $K_{\mathcal{L}}$ is Weil and by (4.4), then the triples $(Z, \Xi, \mathcal{O}_Z(K_{\mathcal{L}}))$ belong to a bounded family. Possibly stratifying T into a disjoint union of locally closed subsets (which does not alter boundedness), we may assume that items (1) and (2) of Lemma 2.22 are satisfied. By [12, Théorème 12.2.1 (v)], possibly after further stratification of the base, we may assume that item (3) holds as well. We may then apply Lemma 2.22 to conclude. □

Given $(Z, \mathcal{L}, \Xi) \in \mathcal{D}_{J,\epsilon_0,\mathrm{Fano}}$, by Lemma 4.4, we may find a partial resolution $g: Y \to Z$ together with a morphism $\pi: X \to Y$ such that

(a) (Y, Γ) is log smooth near all strictly log canonical singularities of $\mathcal{G}$, where $\Gamma := \pi_* \Delta$ and $\mathcal{G} := \pi_* \mathcal{F}$;

(b) g only extracts divisors E with $a(E, Z, \Xi) \leq 0$ and $a(E, \mathcal{L}, \Xi) = -\iota(E)$; in particular, there exists a g-exceptional divisor $F \geq 0$ so that $K_{(Y,\mathcal{G},\Gamma),\epsilon_0} + F = g^*(K_{(Z,\mathcal{L},\Xi),\epsilon_0})$;

(c) $(\mathcal{G}, \Gamma)$ is log canonical (this is a direct consequence of item (b)) and $K_{\mathcal{G}} + \Gamma$ is big;

(d) (Y, Γ) is log smooth at all strictly log canonical singularities of $\mathcal{G}$; and,

(e) if $(X, \mathcal{F}, \Delta) \in \mathcal{D}_J$, then Z is a Mori fiber space obtained from a run of a $K_{(X,\mathcal{F},\Delta),\epsilon_0}$-MMP and there exists a contraction $\pi: X \to Y$.

Furthermore, by Lemma 4.4, the collection $\mathcal{D}_{J,\epsilon_0,\text{Fano,res}}$ of all triples $(Y, \mathcal{G}, \Gamma)$ as above is bounded. Thus, there exists a projective family of foliated triples $f\colon (\mathcal{Y}, \mathcal{G}_\mathcal{Y}, \Gamma_\mathcal{Y}) \to T$ over a base of finite type T such that $\mathcal{G}_\mathcal{Y}$ is tangent to f, and for any triple $(Y, \mathcal{G}, \Gamma) \in \mathcal{D}_{J,\epsilon_0,\text{Fano,res}}$, there exist $t \in T$ and an isomorphism $\psi\colon Y \to \mathcal{Y}_t$ with $\psi_*\Gamma = \Gamma_t$ and $\psi_*\mathcal{G} = \mathcal{G}_{\mathcal{Y},t}$.

The conclusion of the proof then follows from the next claim.

Claim 3. *Taking $\epsilon = \epsilon_0$, we can apply Proposition* 4.3 *to $\mathfrak{D} = \mathcal{D}_J$ and to the family $f\colon (\mathcal{Y}, \mathcal{G}_\mathcal{Y}, \Gamma_\mathcal{Y}) \to T$.*

At this point, we can conclude by defining τ_0 to be the number that Proposition 4.3 produces in the set-up of Claim 3.

Proof of Claim 3. It suffices to show that all the hypotheses (1)–(4) of Proposition 4.3 are satisfied.

Item (1) is (e) above. Item (3) follows from (a) and (c) above. Item (4) is (d) above. Thus, we are left to show that item (2) holds.

Fix $(X, \mathcal{F}, \Delta) \in D_J$, and let $(Y, \mathcal{G}, \Gamma) \in \mathcal{D}_{J,\epsilon_0,\text{Fano,res}}$ be a triple obtained as the resolution of a Mori fiber space outcome of the $K_{(X,\mathcal{F},\Delta),\epsilon_0}$-MMP. We adopt the same notation as in the previous part of the proof.

Suppose first that every component of $\mathrm{exc}(g)$ is $\mathcal{G}$-invariant. In this case, by (b) above, $(K_Y + \Gamma) + F = g^*(K_Z + \Xi)$ and $(K_\mathcal{G} + \Gamma) = g^*(K_\mathcal{L} + \Xi)$, where $F \geq 0$. In particular, $K_{(X,\mathcal{F},\Delta),\epsilon_0} = \pi^* K_{(Y,\mathcal{G},\Gamma),\epsilon_0} + \widetilde{E}$, where $\widetilde{E} \geq 0$.

So suppose that some component of $\mathrm{exc}(g)$ is not $\mathcal{G}$-invariant. Let C be one such component and let $P = g(C)$. Note that P is a strictly log canonical singularity of $(\mathcal{L}, \Xi)$, and so P is not contained in the support of Ξ. By Lemma 2.6, C is the unique non-invariant g-exceptional divisor mapping to P. Thus, in a neighborhood of $g^{-1}(P)$, we see that Γ is supported on at most one divisor. Note that, in this neighborhood, the support of Γ must be smooth; indeed, we know that $(\mathcal{G}, C)$ is log canonical, which implies that C is necessarily smooth. It follows that $(\mathcal{G}, \Gamma)$ and (Y, Γ) have canonical singularities, and so $(Y, \mathcal{G}, \Gamma)$ is ϵ-adjoint canonical for all $\epsilon > 0$; in particular, it is ϵ_0-adjoint canonical. □

This completes the proof of the proposition. □

Proposition 4.6. *Let $I \subset [0, 1]$ be a DCC set. Then there exists a finite subset*

$$J \subset (I \cup \mathcal{S}) \setminus \{1\}$$

with the following property.

Let $(X, \mathcal{F}, \Delta)$ be a projective foliated log smooth triple such that X is a surface, $\mathcal{F}$ is rank one, $K_\mathcal{F} + \Delta$ is big and $\Delta \in I$. Then there exists $\Delta' \leq \Delta$ with $\Delta' \in J$ such that $K_\mathcal{F} + \Delta'$ is big.

Proof. If $\Delta = 0$, then there is nothing to prove since $K_\mathcal{F}$ is big in its own right. Hence, we may assume that $\Delta \neq 0$. If $K_\mathcal{F}$ is pseudo-effective, then the result is straightforwardly true. In fact, since I is a DCC set, then $I \setminus \{0\}$ admits a minimum; call it i_1. It then suffices to take $J = \{i_1\}$: indeed, writing $\Delta = \sum_k \mu_{D_k} D_k$ the decomposition into prime components, then for $\Delta' := \sum_k i_1 D_k$, $0 < \Delta' \leq \Delta$, $\Delta' \in J$ and $K_\mathcal{F} + \Delta'$ is big. Hence, we may assume that $K_\mathcal{F}$ is not pseudo-effective.

We first claim that there exists a finite set $J \subset (I \cup \mathcal{S}) \setminus \{1\}$ such that if (C, Θ) is a log canonical pair on a smooth curve with $K_C + \Theta$ ample and $\Theta \in I$, then there exists $\Theta' \leq \Theta$ with $\Theta' \in J$ so that $K_C + \Theta'$ is ample. Indeed, by [14, Theorem 1.3], there exists an m, depending only on I, so that the map induced by $|m(K_C + \Theta)| = |\lfloor m(K_C + \Theta)\rfloor|$ is birational. For all $1 \leq k \leq m$, we set

$$J_k := \begin{cases} \min\left\{(I \cup \mathcal{S}) \cap \left[\frac{k-1}{m}, \frac{k}{m}\right]\right\} & \text{if } (I \cup \mathcal{S}) \cap \left[\frac{k-1}{m}, \frac{k}{m}\right] \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

When $(I \cup \mathcal{S}) \cap [\frac{k-1}{m}, \frac{k}{m}] \neq \emptyset$, then $\min\{(I \cup \mathcal{S}) \cap [\frac{k-1}{m}, \frac{k}{m}]\}$ is well-defined since $I \cup \mathcal{S}$ satisfies the DCC. Then it suffices to set $J := \{J_k \mid 1 \leq k \leq m\}$ and observe that J satisfies all our required properties. When $I = \{1\}$, we cannot just work with I, but we really need to work with the set $(I \cup \mathcal{S}) \setminus \{1\}$ to guarantee that $J \subset [0, 1)$.

By [3], $\mathcal{F}$ is uniruled; moreover, since $\mathcal{F}$ has canonical singularities, then there exists a morphism $X \to B$ inducing $\mathcal{F}$. Let C be a general fiber of $X \to B$, and set $\Theta := \Delta|_C$. Thus, we may find $\Delta' \leq \Delta$ with $\Delta' \in J$ such that $\Delta'|_C = \Theta'$.

We claim that $K_{\mathcal{F}} + \Delta'$ is pseudo-effective, from which we may conclude. To see the claim, observe that if $K_{\mathcal{F}} + \Delta'$ is not pseudo-effective, then X is covered by $(K_{\mathcal{F}} + \Delta')$-negative rational curves tangent to the foliation, see Theorem 2.8. However, by construction, $K_{\mathcal{F}} + \Delta'$ is positive on a general rational curve tangent to $\mathcal{F}$. □

Corollary 4.7. *Fix a DCC set* $I \subset [0, 1]$. *Let* $(X, \mathcal{F}, \Delta)$ *be a projective foliated log smooth triple such that* X *is a surface,* $\mathcal{F}$ *is rank one,* $K_{\mathcal{F}} + \Delta_{n\text{-inv}}$ *is big and* $\Delta \in I$. *Then there exists a positive real number* $\tau = \tau(I)$ *such that, for all* $0 \leq \epsilon < \tau$, $K_{(X,\mathcal{F},\Delta),\epsilon}$ *is big.*

Proof. We may assume without loss of generality that $\Delta = \Delta_{n\text{-inv}}$. By Proposition 4.6, we may assume without loss of generality that there exist a finite subset $J \subset (I \cup \mathcal{S}) \setminus \{1\}$ and $\Delta' \leq \Delta$ such that $\Delta' \in J$ such that $K_{\mathcal{F}} + \Delta'$ is big. We may then conclude by Proposition 4.5, defining $\tau := \min(\tau_0(J), E(I))$, where $\tau_0(J)$ is the positive real number produced by Proposition 4.5 and $E = E(I)$ is the positive real number defined in Proposition 2.17. □

Corollary 4.8. *Fix a DCC set* $I \subset [0, 1]$. *Then there exists a positive real number* $\tau = \tau(I)$ *such that, for all* $0 < \epsilon < \tau$, *the following statement holds.*

Let $(X, \mathcal{F}, \Delta)$ *be a* ϵ*-adjoint log canonical projective foliated triple such that* X *is a surface,* $K_{\mathcal{F}} + \Delta_{n\text{-inv}}$ *is big and* $\Delta \in I$. *Then there exists an integer* $M = M(\epsilon)$ *for which* $|MK_{(X,\mathcal{F},\Delta),\epsilon}|$ *defines a birational map.*

Proof. We define $\tau := \tau(I)$, where $\tau(I)$ is the positive real number produced by Corollary 4.7.

Fix $0 < \epsilon < \tau$ and a projective foliated triple $(X, \mathcal{F}, \Delta)$ satisfying the hypotheses of the statement. As $\epsilon < E(I)$ by construction, cf. the proof of Corollary 4.7, Proposition 2.17 implies that $(\mathcal{F}, \Delta_{n\text{-inv}})$ is log canonical. Let $p\colon X' \to X$ be a foliated log resolution of $(X, \mathcal{F}, \Delta)$ and let $\mathcal{F}' := p^{-1}\mathcal{F}$, $\mathrm{exc}(p) := E$ and $\Gamma := p_*^{-1}\Delta + E$. Since

$$K_{\mathcal{F}'} + \Gamma_{n\text{-inv}} = p^*(K_{\mathcal{F}} + \Delta_{n\text{-inv}}) + F,$$

where $F \geq 0$, then $K_{\mathcal{F}'} + \Gamma_{n\text{-inv}}$ is big. Moreover, $K_{(X',\mathcal{F}',\Gamma),\epsilon} = p^*K_{(X,\mathcal{F},\Delta),\epsilon} + G$, where $G \geq 0$. Hence, for all $m \in \mathbb{N}$, $|mK_{(X',\mathcal{F}',\Gamma),\epsilon}| = |mK_{(X,\mathcal{F},\Delta),\epsilon}|$, and if, for some $m \in \mathbb{N}_{>0}$,

$|mK_{(X',\mathcal{F}',\Gamma),\epsilon}|$ defines a birational map, then the same holds for $|mK_{(X,\mathcal{F},\Delta),\epsilon}|$. Therefore, we are free to replace $(X,\mathcal{F},\Delta)$ by $(X',\mathcal{F}',\Gamma)$, and thus, we may freely assume that $(X,\mathcal{F},\Delta)$ is a foliated log smooth triple.

By Proposition 4.6, we may assume without loss of generality that there exist a finite subset $J \subset (I \cup \mathcal{S}) \setminus \{1\}$ and $\Delta' \leq \Delta$ with $\Delta' \in J$, and $K_{\mathcal{F}} + \Delta'$ is big. Corollary 4.7 in turn implies that $K_{(X,\mathcal{F},\Delta'),\epsilon}$ is big. We run a $K_{(X,\mathcal{F},\Delta'),\epsilon}$-MMP, $\rho\colon X \to Y$, which must terminate with the ample model Y for $K_{(X,\mathcal{F},\Delta'),\epsilon}$, see Corollary 3.4. By Corollary 3.3 and the fact that $J \subset [0, 1-\delta] \cap I$ for some $\delta > 0$, Y has η-lc singularities for $\eta := \frac{\epsilon\delta}{1+\epsilon} > 0$.

We may then apply Lemma 2.21 to

$$N := \frac{1}{\epsilon}\rho_* K_{(X,\mathcal{F},\Delta'),\epsilon}$$

to conclude since $K_{(X,\mathcal{F},\Delta),\epsilon} \geq K_{(X,\mathcal{F},\Delta'),\epsilon}$. □

5. Applications

5.1. Bounding degrees of curves invariant by foliations. The following is an improvement on a bound proven in [23, Theorem 5.4].

Theorem 5.1. *Let $\tau = \tau(\emptyset) > 0$ be the real constant defined within Corollary 4.7. Then, for all positive rational numbers $0 < \epsilon < \tau$, there exists a positive integer $C = C(\epsilon)$ such that the following statement holds.*

Let $(X,\mathcal{F})$ be a projective foliated pair such that

(1) *X is a surface,*

(2) *$K_{\mathcal{F}}$ is big,*

(3) *$(X,\mathcal{F})$ is ϵ-adjoint canonical,*

(4) *$\mathcal{F}$ admits a meromorphic first integral, and*

(5) *the closure of a general leaf, L, has geometric genus g.*

Then, for any nef divisor H on X,

$$H \cdot L \leq gC\big(H \cdot (K_{\mathcal{F}} + \epsilon K_X)\big).$$

Proof. Let $p\colon X' \to X$ be a foliated log resolution of $\mathcal{F}$. We define $\mathcal{F}' = p^{-1}\mathcal{F}$ and $L' = p_*^{-1}L$. As $\mathcal{F}$ possesses a meromorphic first integral and by (5), we can assume that $\mathcal{F}'$ is given by a fibration in curves of genus g.

Since $(X,\mathcal{F})$ is ϵ-adjoint canonical, $K_{\mathcal{F}'} + \epsilon K_{X'} = p^*(K_{\mathcal{F}} + \epsilon K_X) + E$, where $E \geq 0$ and is p-exceptional, and for all $m \in \mathbb{N}$,

$$H^0\big(X', m(K_{\mathcal{F}'} + \epsilon K_{X'})\big) = H^0\big(X, m(K_{\mathcal{F}} + \epsilon K_X)\big).$$

Moreover, by Corollary 4.8, there exists a positive integer $M = M(\epsilon)$, that is, M independent of X' and $\mathcal{F}'$, such that $|M(K_{\mathcal{F}} + \epsilon K_X)|$ defines a birational map. Thus,

$$h^0\big(X', lM(K_{\mathcal{F}'} + \epsilon K_{X'})\big) \geq \binom{l+2}{2}.$$

As $m(K_{\mathcal{F}'} + \epsilon K_{X'})|_{L'} = m(1+\epsilon)K_{L'}$, then for all $m > 1$,

$$h^0\big(L', m(K_{\mathcal{F}'} + \epsilon K_{X'})|_{L'}\big) \le m(1+\epsilon)(2g-2) - g + 1.$$

We therefore have inequalities

$$\begin{aligned} h^0\big(X', l(K_{\mathcal{F}'} + \epsilon K_{X'}) - L'\big) \ge h^0\big(X', l(K_{\mathcal{F}'} + \epsilon K_{X'})\big) \\ - h^0\big(L', l(K_{\mathcal{F}'} + \epsilon K_{X'})\big)\big|_{L'} > 1, \end{aligned}$$

where the latter inequality holds for $l \ge 4gM^2$. Therefore, there exists

$$0 \le D \sim l(K_{\mathcal{F}'} + \epsilon K_{X'}) - L'.$$

Since H is nef, then $0 \le D \cdot p^* H$. Hence, taking $C(\epsilon) := 4M^2$, we obtain the desired result. □

5.2. Lower bound on adjoint volumes.

Theorem 5.2. *Fix a DCC set $I \subset [0,1]$. Let $\tau = \tau(I) > 0$ be the real constant defined within Corollary 4.7. Then, for all $0 < \epsilon < \tau$, there exists $0 < v(\epsilon)$ such that the following statement holds.*

If $(X, \mathcal{F}, \Delta)$ is an ϵ-adjoint log canonical foliated projective triple, where $\Delta \in I$, X is a surface and $K_{\mathcal{F}} + \Delta$ is big, then

$$\operatorname{vol}(K_{(X,\mathcal{F},\Delta),\epsilon}) \ge v(\epsilon).$$

Proof. This is a direct consequence of Corollary 4.8. □

5.3. Upper bound on automorphism group of foliations.

Theorem 5.3. *Let $\tau = \tau(\mathcal{S}) > 0$ be the real constant defined within Corollary 4.7. Then, for all $0 < \epsilon < \tau$, there exists $C = C(\epsilon)$ such that the following holds.*

Let $(X, \mathcal{F})$ be a projective foliated pair such that

(1) *X is a surface,*

(2) *$K_{\mathcal{F}}$ is big,*

(3) *$(X, \mathcal{F})$ is ϵ-adjoint canonical.*

Then

$$\#\operatorname{Bir}(X, \mathcal{F}) \le C \cdot \operatorname{vol}(K_{(X,\mathcal{F}),\epsilon}).$$

Proof. By [22], we know that $\#\operatorname{Bir}(X, \mathcal{F}) < +\infty$. Possibly replacing $(X, \mathcal{F})$ by a higher model, we may freely assume that $(X, \mathcal{F})$ is log smooth,

$$\operatorname{Bir}(X, \mathcal{F}) = \operatorname{Aut}(X, \mathcal{F}) = G,$$

and that, if $Y = X/G$ and $\mathcal{G} = \mathcal{F}/G$, then $(Y, \mathcal{G}, \Delta)$ is log smooth, where

$$\Delta = \sum_{D \text{ prime}} \frac{r_D - 1}{r_D} D$$

and r_D is the ramification index of q over the prime divisor $D \subset Y$.

By Riemann–Hurwitz and foliated Riemann–Hurwitz [10, Lemma 3.4], then

$$K_{(X,\mathcal{F}),t} = q^* K_{(Y,\mathcal{G},\Delta),t} \quad \text{for all } t \geq 0.$$

Thus,

$$\#G \leq \frac{\operatorname{vol}(K_{(X,\mathcal{F}),t})}{\operatorname{vol}(K_{(Y,\mathcal{G},\Delta),t})}.$$

Set $\tau := \tau(\mathcal{S})$ and $C(\epsilon) := \frac{1}{v(\epsilon)}$ for $\epsilon < \tau$, where $\tau(\mathcal{S})$ and $v(\epsilon)$ are as in Theorem 5.2. This gives our desired bound. □

6. Boundedness of ample models

In this section, we fix τ to be $\tau(\emptyset)$, the real number whose existence has been shown in Corollary 4.7.

Theorem 6.1. *Fix positive real numbers C and ϵ, with $\epsilon < \tau$. The set $\mathcal{M}_{2,\epsilon,C}$ of foliated pairs $(X, \mathcal{F})$ such that*

(1) *X is a projective klt surface,*

(2) *$\mathcal{F}$ is a rank one foliation on X with $K_{\mathcal{F}}$ big,*

(3) *$(X, \mathcal{F})$ is an ϵ-adjoint canonical foliated pair,*

(4) *$K_{\mathcal{F}} + \epsilon K_X$ is ample, and*

(5) $\operatorname{vol}(X, K_{\mathcal{F}} + \epsilon K_X) \leq C$

forms a bounded family.

Proof. We shall divide the proof into several steps.

Step 1. Effective birational boundedness. By Corollary 4.8, we know that if $(X, \mathcal{F})$ is one of the pairs in $\mathcal{M}_{2,\epsilon,C}$, then there exists an integer $M = M(\epsilon)$ such that the morphism $|MK_{(X,\mathcal{F}),\epsilon}|$ is birational onto the image.

Let Y be the Zariski closure of the image of X under the induced map. As

$$\operatorname{vol}(X, K_{(X,\mathcal{F}),\epsilon}) \leq C,$$

then Y belongs to a bounded family. Let $H \in |\mathcal{O}_Y(1)|$ be a general member.

Step 2. Normalization and boundedness. Let $\nu\colon Y^\nu \to Y$ be the normalization of Y and let $H_\nu = \nu^* H$. Then also Y^ν belongs to a bounded family, and we can assume that there exists a positive integer $l = l(\epsilon, C)$ such that lH_ν is very ample by Matsusaka's big theorem for normal surfaces, see [19, § 3].

Step 3. Relatively ample model. Let $(X^{(1)}, \mathcal{F}^{(1)})$ be a foliated log resolution of $(X, \mathcal{F})$ with morphism $r\colon X^{(1)} \to X$. The ϵ-adjoint canonical condition for $(X, \mathcal{F})$ implies that

$$K_{(X^{(1)},\mathcal{F}^{(1)}),\epsilon} = r^*(K_{(X,\mathcal{F}),\epsilon}) + E, \quad E \geq 0 \text{ and } r\text{-exceptional, and}$$
$$H^0(X, mK_{(X,\mathcal{F}),\epsilon}) = H^0(X^{(1)}, mK_{(X^{(1)},\mathcal{F}^{(1)}),\epsilon}) \quad \text{for all } m \in \mathbb{N}.$$

We note that $K_{\mathcal{F}^{(1)}}$ is pseudo-effective. Indeed, if that was not the case, then $\mathcal{F}^{(1)}$ would be induced by a fibration in rational curves $\rho\colon X^{(1)} \to B$. Given Σ a general fiber of ρ, then

$$(K_{\mathcal{F}^{(1)}} + \epsilon K_{X^{(1)}}) \cdot \Sigma = -2(1+\epsilon),$$

a contradiction to the pseudo-effectivity of $K_{\mathcal{F}^{(1)}} + \epsilon K_{X^{(1)}}$.

By taking $X^{(1)}$ to be a sufficiently high model of X, we can assume that there exists a morphism $p\colon X^{(1)} \to Y$ which is a resolution of indeterminacies of the rational map $X \dashrightarrow Y$ constructed in Step 1. As $X^{(1)}$ is smooth, then it automatically factors through the normalization, thus inducing $p_1\colon X^{(1)} \to Y^{\nu}$. We will denote by $\mathcal{F}_{Y^{\nu}}$ the strict transform of the foliation on Y^{ν}. In view of this and by construction, cf. Step 1, then

$$|MK_{(X^{(1)},\mathcal{F}^{(1)}),\epsilon}| = p_1^*|H_\nu| + F, \quad F \geq 0.$$

In particular, for any $t \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned}\operatorname{vol}(X^{(1)}, K_{(X^{(1)},\mathcal{F}^{(1)}),\epsilon} + tp_1^*H_\nu) &\leq \operatorname{vol}\big(X^{(1)}, (1+tM)K_{(X^{(1)},\mathcal{F}^{(1)}),\epsilon}\big)\\ &\leq (1+tM)^2C.\end{aligned}$$

Let us run the $K_{(X^{(1)},\mathcal{F}^{(1)}),\epsilon}$-MMP relatively over Y^{ν}, which exists by Corollary 3.2,

$$\begin{tikzcd} X^{(1)} \arrow[rr, "s'"] \arrow[dr, "p_1"'] & & X^{(2)} \arrow[dl, "p_2"] \\ & Y^{\nu}. & \end{tikzcd}$$

By Remark 3.5, we can pass to the ample model (over Y^{ν}) for the $K_{(X^{(1)},\mathcal{F}^{(1)}),\epsilon}$-MMP, that is, we can assume that $X^{(2)}$ satisfies the following conditions:

(i) $X^{(2)}$ is η-lc for some $\eta = \eta(\epsilon) > 0$;

(ii) $(X^{(2)}, \mathcal{F}^{(2)})$ is ϵ-adjoint canonical, where $\mathcal{F}^{(2)} := s'_*\mathcal{F}^{(1)}$;

(iii) $|MK_{(X^{(2)},\mathcal{F}^{(2)}),\epsilon} - p_2^*H_\nu| \neq \emptyset$;

(iv) $K_{(X^{(2)},\mathcal{F}^{(2)}),\epsilon}$ is ample over Y^{ν};

(v) $H^0(X^{(2)}, mK_{(X^{(2)},\mathcal{F}^{(2)}),\epsilon}) = H^0(X^{(1)}, mK_{(X^{(1)},\mathcal{F}^{(1)}),\epsilon})$ for all $m \in \mathbb{N}$; moreover, for any $t \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned}\operatorname{vol}(X^{(2)}, K_{(X^{(2)},\mathcal{F}^{(2)}),\epsilon} + tp_2^*H_\nu) &\leq \operatorname{vol}\big(X^{(2)}, (1+tM)K_{(X^{(2)},\mathcal{F}^{(2)}),\epsilon}\big)\\ &\leq (1+tM)^2C.\end{aligned}$$

Item (iv) and the Cone Theorem for surface foliations [25, Theorem 6.3] imply that

$$K_{(X^{(2)},\mathcal{F}^{(2)}),\epsilon} + (1+\lceil\epsilon\rceil)7p_2^*H_\nu$$

is ample on $X^{(2)}$. Item (v) implies that

$$\operatorname{vol}\big(X^{(2)}, K_{(X^{(2)},\mathcal{F}^{(2)}),\epsilon} + (1+\lceil\epsilon\rceil)7p_2^*H_\nu\big) \leq \big(1+(1+\lceil\epsilon\rceil)7M\big)^2C.$$

Step 4. Boundedness of intersection numbers. As

$$K_{(X^{(2)},\mathcal{F}^{(2)}),\epsilon} + (1+\lceil\epsilon\rceil)7p_2^*H_\nu$$

is ample, then the same holds for

$$K_{(X^{(2)},\mathcal{F}^{(2)}),\epsilon} + \big((1+\lceil\epsilon\rceil)7+s\big)p_2^*H_\nu \quad \text{for any } s \in \mathbb{R}_{\geq 0}.$$

Denoting

$$f(s) = \operatorname{vol}\big(X^{(2)}, K_{(X^{(2)},\mathcal{F}^{(2)}),\epsilon} + \big((1+\lceil\epsilon\rceil)7+s\big)p_2^*H_\nu\big),$$

then for all $0 \leq s < 1$,

$$\begin{aligned}
(6.1)\quad \big(1+(2+\lceil\epsilon\rceil)7M\big)^2C \geq f(s)-f(0) &= \int_0^s f'(x)\,dx \\
&= \int_0^s \frac{d}{dt}\Big|_{t=x}\big[\operatorname{vol}\big(X^{(2)}, K_{(X^{(2)},\mathcal{F}^{(2)}),\epsilon} + \big((1+\lceil\epsilon\rceil)7+t\big)p_2^*H_\nu\big)\big]\,dx \\
&= \int_0^s \frac{d}{dt}\Big|_{t=x}\big[\big(K_{(X^{(2)},\mathcal{F}^{(2)}),\epsilon} + \big((1+\lceil\epsilon\rceil)7+t\big)p_2^*H_\nu\big)^2\big]\,dx \\
&= \int_0^s \big[2K_{(X^{(2)},\mathcal{F}^{(2)}),\epsilon}\cdot p_2^*H_\nu + 2\big((1+\lceil\epsilon\rceil)7+x\big)H_\nu^2\big]\,dx \\
&= 2\big((K_{\mathcal{F}^{(2)}}+\epsilon K_{X^{(2)}})\cdot p_2^*H_\nu\big)s + 2H_\nu^2\big((1+\lceil\epsilon\rceil)7\big)s + H_\nu^2s^2.
\end{aligned}$$

As Y^ν is bounded, lH_ν is very ample and $H_\nu^2 \leq C$, cf. Step 2, H_ν^2 can only take finitely many values in the positive integers. Likewise, by the push-pull formula, $K_{X^{(2)}}\cdot p_2^*H_\nu = K_{Y^\nu}\cdot H_\nu$ can only take finitely many integral values. The boundedness of $p_2^*H_\nu^2$ and of $K_{X^{(2)}}\cdot p_2^*H_\nu$, together with the last line of (6.1), implies that also $K_{\mathcal{F}^{(2)}}\cdot p_2^*H_\nu$ can only take finitely many values in the positive integers: the positivity of $K_{\mathcal{F}^{(2)}}\cdot p_2^*H_\nu$ follows from the pseudo-effectivity of $K_{\mathcal{F}^{(2)}}$ and since H_ν is big and nef. Hence, there exists a finite set

$$L_{t,\epsilon} = L_{t,\epsilon}(t,\epsilon,C) \subset \mathbb{N}_{>0} + \epsilon\mathbb{N}_{>0} + t\mathbb{N}_{>0}$$

such that

$$(K_{(X^{(2)},\mathcal{F}^{(2)}),\epsilon} + tp_2^*H_\nu)\cdot p_2^*H_\nu \in L_{t,\epsilon}.$$

Step 5. A new model. Starting with $X^{(2)}$, we now run the $K_{X^{(2)}}$-MMP over Y^ν and then pass to the canonical model over Y^ν, which exists by (i) in Step 3,

$$\begin{tikzcd}
X^{(1)} \arrow[r, "s'"] \arrow[dr, "p_1"'] & X^{(2)} \arrow[r, "s''"] \arrow[d, "p_2"] & X^{(3)}. \arrow[dl, "p_3"] \\
& Y^\nu &
\end{tikzcd}$$

Let $\mathcal{F}^{(3)} := s''_*\mathcal{F}^{(2)}$. By Step 4 and the push-pull formula, there exists a positive real number $C'_{t,\epsilon} = C'(t,\epsilon,C)$ such that

$$0 < (K_{(X^{(3)},\mathcal{F}^{(3)}),\epsilon} + tp_3^*H_\nu)\cdot p_3^*H_\nu \leq C'_{t,\epsilon}.$$

Hence, there exists a positive real number $C''_{t,\epsilon} = C''_{t,\epsilon}(t,\epsilon,C)$ such that

$$0 < (K_{(X^{(3)},\mathcal{F}^{(3)}),\epsilon} + tp_3^*H_\nu)^2 \leq \frac{\big((K_{(X^{(3)},\mathcal{F}^{(3)}),\epsilon} + tp_3^*H_\nu)\cdot p_3^*H_\nu\big)^2}{(p_3^*H_\nu)^2} \leq C''_{t,\epsilon},$$

where the second inequality holds by the Hodge index theorem, i.e., $(A^2)(B^2) \leq (A\cdot B)^2$, where A is nef divisor.

Step 6. Boundedness of $X^{(3)}$. Let us observe that $X^{(3)}$ is η-lc, by construction. We now show that $X^{(3)}$ belongs to a bounded family. By the classical version of the Cone Theorem, [15, Theorem 3.7], $K_{X^{(3)}} + 7p_3^* H_\nu$ is ample on $X^{(3)}$; furthermore,

$$\begin{aligned}(6.2)\quad & \operatorname{vol}(X^{(3)}, K_{X^{(3)}} + 7p_3^* H_\nu) \\ & \leq \frac{1}{\epsilon^2} \operatorname{vol}(X^{(3)}, K_{(X^{(3)},\mathcal{F}^{(3)}),\epsilon} + 7\epsilon p_3^* H_\nu) \quad [K_{\mathcal{F}^{(3)}} \text{ is pseudo-effective}] \\ & \leq \frac{C''_{7\epsilon,\epsilon}}{\epsilon^2}.\end{aligned}$$

By Bertini's theorem, taking a general (irreducible) divisor $H' \in |7lp_3^* H_\nu|$, where $l = l(\epsilon, C)$ is the natural number defined in Step 2, we can ensure that $(X^{(3)}, \frac{H'}{l})$ is η'-lc, $\eta' := \min(\eta, \frac{1}{l})$; thus, [11, Theorem 1.3] and (6.2) together imply that

$$\operatorname{vol}(X^{(3)}, K_{X^{(3)}} + 7p_3^* H_\nu) = \operatorname{vol}\Big(X^{(3)}, K_{X^{(3)}} + \frac{H'}{l}\Big)$$

belongs to a finite set; finally, by [18, Theorem 6], $(X^{(3)}, H')$ is bounded. Thus, for any $s \in \mathbb{R}_{>0}$,

$$\begin{aligned}&\operatorname{vol}\big(X^{(3)}, K_{(X^{(3)},\mathcal{F}^{(3)}),\epsilon} + 6p_3^* H_\nu + s(K_{X^{(3)}} + 7p_3^* H_\nu)\big) \\ &\quad \leq \operatorname{vol}\big(X^{(3)}, (1+s)K_{(X^{(3)},\mathcal{F}^{(3)}),\epsilon} + (6+7s)p_3^* H_\nu\big) \quad [K_{\mathcal{F}^{(3)}} \text{ is pseudo-effective}] \\ &\quad \leq (1+s)^2 \operatorname{vol}\Big(X^{(3)}, K_{(X^{(3)},\mathcal{F}^{(3)}),\epsilon} + \frac{6+7s}{1+s} p_3^* H_\nu\Big) \\ &\quad \leq (1+s)^2 C''_{\frac{6+7s}{1+s},\epsilon}.\end{aligned}$$

Step 7. Boundedness of $\mathcal{F}^{(3)}$. By Step 3, the divisor

$$K_{(X^{(2)},\mathcal{F}^{(2)}),\epsilon} + (1 + \lceil \epsilon \rceil) 7p_2^* H$$

is ample on $X^{(2)}$; hence its pushforward

$$K_{(X^{(3)},\mathcal{F}^{(3)}),\epsilon} + (1 + \lceil \epsilon \rceil) 7p_3^* H_\nu$$

is big and nef on $X^{(3)}$.

As in Step 4, we compute the derivative

$$\begin{aligned}&\frac{d}{ds} \operatorname{vol}\big(X^{(3)}, K_{(X^{(3)},\mathcal{F}^{(3)}),\epsilon} + (1 + \lceil \epsilon \rceil)7p_3^* H_\nu + s(K_{X^{(3)}} + 7p_3^* H_\nu)\big) \\ &\quad = \frac{d}{ds}\big(K_{(X^{(3)},\mathcal{F}^{(3)}),\epsilon} + (1 + \lceil \epsilon \rceil)7p_3^* H_\nu + s(K_{X^{(3)}} + 7p_3^* H_\nu)\big)^2 \\ &\quad = 2\big(K_{(X^{(3)},\mathcal{F}^{(3)}),\epsilon} + (1 + \lceil \epsilon \rceil)7p_3^* H_\nu\big) \cdot (K_{X^{(3)}} + 7p_3^* H_\nu) \\ &\qquad + 2s(K_{X^{(3)}} + 7p_3^* H_\nu)^2.\end{aligned}$$

By boundedness and Steps 3–4, there exists a positive real number $D = D(\epsilon, C)$ such that $K^2_{X^{(3)}}$, $K_{X^{(3)}} \cdot p_3^* H_\nu$, $K_{\mathcal{F}^{(3)}} \cdot p_3^* H_\nu$ all belong to the interval $[-D, D]$. Repeating the same argument about the derivative of the volume, as in Step 4, then we can show that there exists a positive real number $D' = D'(\epsilon, C)$ such that

$$0 < K_{\mathcal{F}^{(3)}} \cdot (K_{X^{(3)}} + 7p_3^* H_\nu) \leq D'.$$

Since, for any $\lambda \in \mathbb{N}_{>0}$,

$$\begin{aligned}&2K_{X^{(3)}} + K_{\mathcal{F}^{(3)}} + \big((2+\lceil\epsilon\rceil)7+\lambda\big)p_3^*H_\nu \\ &\quad = K_{X^{(3)}} + \underbrace{K_{(X^{(3)},\mathcal{F}^{(3)}),\epsilon} + (1+\lceil\epsilon\rceil)7p_3^*H_\nu}_{\text{big and nef}} + \underbrace{(1-\epsilon)K_{X^{(3)}} + (7+\lambda)p_3^*H_\nu}_{\text{ample}},\end{aligned}$$

Kawamata–Viehweg vanishing implies that, for $i = 1, 2$,

$$h^i\big(X^{(3)}, 2K_{X^{(3)}} + K_{\mathcal{F}^{(3)}} + \big((2+\lceil\epsilon\rceil)7+\lambda\big)p_3^*H_\nu\big) = 0.$$

As H_ν is Cartier, then

$$\chi\big(X^{(3)}, 2K_{X^{(3)}} + K_{\mathcal{F}^{(3)}} + \big((2+\lceil\epsilon\rceil)7+\lambda\big)p_3^*H_\nu\big)$$

is a degree 2 polynomial in λ, cf. [13, Theorem 1.6]. Thus, for at least one value of $\lambda \in \{0, 1, 2\}$,

$$\big|2K_{X^{(3)}} + K_{\mathcal{F}^{(3)}} + \big((2+\lceil\epsilon\rceil)7+\lambda\big)p_3^*H_\nu\big| \neq \emptyset.$$

We fix such value of λ. Let $\Gamma \in |2K_{X^{(3)}} + K_{\mathcal{F}^{(3)}} + ((2+\lceil\epsilon\rceil)7+\lambda)p_3^*H_\nu|$; then the argument from the start of this step shows that there exists a positive real number $D'' = D''(\epsilon, C)$ such that

$$\Gamma \cdot (K_{X^{(3)}} + 7p_3^*H_\nu) \leq D''.$$

Hence, the couple $(X^{(3)}, \operatorname{Supp}\Gamma)$ constructed in this step belongs to a bounded family. Moreover, as $X^{(3)}$ itself is bounded, then we can conclude that there is a bounded family parametrizing the pairs $(X^{(3)}, \mathcal{O}_X(\Gamma - 2K_{X^{(3)}} - ((2+\lceil\epsilon\rceil)7+\lambda)p_3^*H_\nu)$ and

$$O_X\big(\Gamma - 2K_{X^{(3)}} - \big((2+\lceil\epsilon\rceil)7+\lambda\big)p_3^*H_\nu\big) \simeq \mathcal{O}(K_{\mathcal{F}^{(3)}}).$$

The dependence on λ here does not constitute an issue since $\lambda \in \{0, 1, 2\}$; thus, up to working with a larger family, we can assume that all three possible values of λ are considered. Hence, we have reconstructed the pair $(X^{(3)}, \mathcal{O}(K_{\mathcal{F}^{(3)}}))$, where $\mathcal{F}^{(3)}$ here is considered as an abstract Weil divisorial sheaf on $X^{(3)}$.

Finally, we want to reconstruct the foliation in a family. Indeed, the properties we proved so far imply that there exist a projective morphism

$$\widetilde{f}_3\colon \widetilde{\mathcal{X}^{(3)}} \to \widetilde{T},$$

where $\widetilde{T}$ is a quasi-projective variety, and a Weil divisorial sheaf $\widetilde{\mathcal{K}}$ on $\widetilde{\mathcal{X}^{(3)}}$ such that, for any $(X, \mathcal{F})$ in $\mathcal{M}_{2,\epsilon,C}$, there exists $t \in T$ such that

$$\big(X^{(3)}, \mathcal{O}(K_{\mathcal{F}^{(3)}})\big) \cong \big(\widetilde{\mathcal{X}_t^{(3)}}, \mathcal{O}(\widetilde{\mathcal{K}}|_{\widetilde{\mathcal{X}_t^{(3)}}})\big).$$

Possibly stratifying $\widetilde{T}$ into a disjoint union of locally closed subsets (which does not alter boundedness), we may assume that items (1) and (2) of Lemma 2.22 are satisfied for the couple $(\widetilde{\mathcal{X}^{(3)}}, \widetilde{\mathcal{K}})$. By [12, Théorème 12.2.1 (v)], possibly after further stratification of the base, we may assume that item (3) holds as well for $(\widetilde{\mathcal{X}^{(3)}}, \widetilde{\mathcal{K}})$. We may then apply Lemma 2.22 and produce a foliated pair $(\mathcal{X}^{(3)}, \mathcal{F}_{\mathcal{X}^{(3)}})$ given by a normal variety $\mathcal{X}^{(3)}$, a rank 1 foliation $\mathcal{F}_{\mathcal{X}^{(3)}}$ on $\mathcal{X}^{(3)}$, and a projective morphism $f_3\colon \mathcal{X}^{(3)} \to T$ of varieties of finite type such that

- $f_3\colon (\mathcal{X}^{(3)}, \mathcal{F}_{\mathcal{X}^{(3)}}, 0) \to T$ is a bounded family of surface foliated pairs; and,
- for any $(X, \mathcal{F}) \in \mathcal{M}_{2,\epsilon,C}$, there exists $t \in T$ and an isomorphism $\psi_t\colon \mathcal{X}_t^{(3)} \to X^{(3)}$ identifying $\mathcal{F}^{(3)}$ with $\mathcal{F}_{\mathcal{X}^{(3)}}|_{\mathcal{X}_t^{(3)}}$, where $(X^{(3)}, \mathcal{F}^{(3)})$ is the model constructed in Step 5 – starting from $(X, \mathcal{F})$.

Step 8. Resolutions in families: in this step, we show that then there exists a bounded family $(f_4\colon \mathcal{X}^{(4)} \to T, \mathcal{F}_{\mathcal{X}^{(4)}})$ *of foliated pairs such that, for any pair* $(X, \mathcal{F}) \in \mathcal{M}_{2,\epsilon,C}$*, there exists* $t \in T$ *such that*

- $X_t^{(4)}$ *is smooth;*
- $(X_t^{(4)}, \mathcal{F}_t^{(4)})$ *is* ϵ*-adjoint canonical; and,*
- $(X_t^{(4)}, \mathcal{F}_t^{(4)})$ *is birational to* $(X, \mathcal{F})$*.*

Continuing with the notation from the end of the previous step, up to passing to a finite stratification of T into locally closed subsets, we can assume that T is smooth. By further stratifying the base T into locally closed subsets and taking resolutions in families, we can moreover construct a pair $(f_4\colon \mathcal{X}^{(4)} \to T, \mathcal{F}_{\mathcal{X}^{(4)}})$ such that this gives a family of foliations on smooth projective surfaces, in the sense of [23, Definition 7.1]. Applying [23, Proposition 7.4] and further stratifying T into locally closed subsets yields a bounded family satisfying the properties above. In fact, denoting for $t \in T$ the restriction of $\mathcal{F}_{\mathcal{X}^{(4)}}$ to $\mathcal{X}_t^{(4)}$ by $\mathcal{F}_{\mathcal{X}_t^{(4)}}$, then the following properties hold:

(I) for all $t \in T$, $(\mathcal{X}_t^{(4)}, \mathcal{F}_{\mathcal{X}_t^{(4)}})$ is ϵ-adjoint canonical and it is birational to $(\mathcal{X}_t^{(3)}, \mathcal{F}_t^{(3)})$;

(II) $\mathcal{X}^{(4)}, g, T$ are all smooth and T is finite type;

Step 9. Properties of the family $(f_4\colon \mathcal{X}^{(4)} \to T, \mathcal{F}_{\mathcal{X}^{(4)}})$. We define the set $W \subset T$ as

$$\begin{aligned} W := \{t \in T \mid (\mathcal{X}_t^{(4)}, \mathcal{F}_{\mathcal{X}_t^{(4)}}) &\text{ admits a birational morphism} \\ &(\mathcal{X}_t^{(4)}, \mathcal{F}_{\mathcal{X}_t^{(4)}}) \to (X, \mathcal{F}) \text{ to a foliated pair } (X, \mathcal{F}) \in \mathcal{M}_{2,\epsilon,C}\} \end{aligned}$$

Further stratifying T into locally closed subsets and possibly discarding some of the components thus obtained, we can assume that also the following properties hold:

(III) $W \cap \widetilde{T}$ is Zariski dense in each connected component $\widetilde{T}$ of T; and,

(IV) for all $t \in T$, $K_{(\mathcal{X}_t^{(4)}, \mathcal{F}_{\mathcal{X}_t^{(4)}}),\epsilon}$ is big. To prove that this property holds, let us recall that, for each $t \in W$, $|MK_{(\mathcal{X}_t^{(4)}, \mathcal{F}_{\mathcal{X}_t^{(4)}}),\epsilon}|$ induces a birational map: indeed, since $(\mathcal{X}_t^{(4)}, \mathcal{F}_{\mathcal{X}_t^{(4)}})$ is birational equivalent to an ϵ-adjoint canonical model $(X, \mathcal{F}) \in \mathcal{M}_{2,\epsilon,C}$, then the conclusion follows at once from Corollary 4.8. As W is Zariski dense in each connected component of T and T is finite type, by semicontinuity of cohomology groups in family, we can assume that there exists a Zariski dense open subset $W^\circ \subset W$ such that, for any $t \in W^\circ$, the natural restriction map

$$H^0\big(W^\circ, f_*\mathcal{O}_{\mathcal{X}^{(4)}}(MK_{(\mathcal{X}^{(4)}, \mathcal{F}_{\mathcal{X}^{(4)}}),\epsilon})\big) \to H^0\big(\mathcal{X}_t^{(4)}, \mathcal{O}_{\mathcal{X}_t^{(4)}}(MK_{(\mathcal{X}_t^{(4)}, \mathcal{F}_{\mathcal{X}_t^{(4)}}),\epsilon})\big)$$

is surjective. In particular, this implies that, for $t \in W^\circ$, the rational map over T given by $f_*\mathcal{O}_{\mathcal{X}^{(4)}}(NK_{(\mathcal{X}^{(4)}, \mathcal{F}_{\mathcal{X}^{(4)}}),\epsilon})$ is birational along $\mathcal{X}_t^{(4)}$, which then proves the claim, as this is an open condition on T.

Step 10. Conclusion. Boundedness of ample models. Let T_1 be an irreducible component of T and let $\eta_1 \in T_1$ be its generic point. Let $\mathcal{X}_{\eta_1}^{(4)}$ be the fiber over η_1 and $\mathcal{X}_{\bar{\eta}_1}^{(4)}$ the extension of scalars to the algebraic closure $\overline{k(T_1)}$ of $k(T_1)$. We denote by $\mathcal{F}_{\mathcal{X}_{\bar{\eta}_1}^{(4)}}$ the induced foliation on $\mathcal{X}_{\bar{\eta}_1}^{(4)}$. By Lemma 2.15, $(\mathcal{X}_{\bar{\eta}_1}^{(4)}, \mathcal{F}_{\mathcal{X}_{\bar{\eta}_1}^{(4)}})$ is ϵ-adjoint canonical and $K_{(\mathcal{X}_{\bar{\eta}_1}^{(4)}, \mathcal{F}_{\mathcal{X}_{\bar{\eta}_1}^{(4)}}),\epsilon}$ is big. Hence, by Theorem 3.1 and Corollary 3.4, we can run the $K_{(\mathcal{X}_{\bar{\eta}_1}^{(4)}, \mathcal{F}_{\mathcal{X}_{\bar{\eta}_1}^{(4)}}),\epsilon}$-MMP

$$\mathcal{X}_{\bar{\eta}_1}^{(4)} := \widehat{\mathcal{X}}_0 \xrightarrow{g_0} \widehat{\mathcal{X}}_1 \xrightarrow{g_1} \widehat{\mathcal{X}}_2 \xrightarrow{g_2} \dots \xrightarrow{g_{n-1}} \widehat{\mathcal{X}}_n \tag{6.3}$$

terminating with a projective model $(\widehat{\mathfrak{X}}_n, \widehat{\mathcal{F}})$ defined over $\overline{k(T_1)}$ on which $K_{(\widehat{\mathfrak{X}}_n, \mathcal{F}_{\widehat{\mathfrak{X}}_n}),\epsilon}$ is ample and it has ϵ-adjoint canonical singularities. As we have a finite number of objects involved in defining all varieties, morphisms and foliations involved in (6.3), there exists a finite field extension $k(T) \subset k'$ over which the $\widehat{\mathfrak{X}}_i$, the morphisms g_i and the strict transforms $\mathcal{F}_{\widehat{\mathfrak{X}}_i}$ of $\mathcal{F}_{\mathfrak{X}^{(4)}_{\bar{\eta}_1}}$ are defined. Furthermore, as a line bundle $\mathcal{L}$ is ample over k' if and only if its extension of scalars to $\overline{k(T_1)}$ is ample over $\overline{k(T_1)}$, then, by construction, $K_{(\widehat{\mathfrak{X}}_n, \mathcal{F}_{\widehat{\mathfrak{X}}_n}),\epsilon}$ is defined over k' and already ample on that field.

The discussion in the previous paragraph implies that there exists a dominant étale morphism $q: U \to T_1$, with $k(U) \simeq k'$ and $q(U) \subset T_1$ open such that there exist families of foliated surfaces $(\widetilde{\mathfrak{X}}_i, \widetilde{\mathcal{F}}_i)$ projective over U together with morphisms $\widetilde{g}_i: \widetilde{\mathfrak{X}}_i \to \widetilde{\mathfrak{X}}_{i+1}$ and a commutative diagram of morphisms over U,

$$\begin{tikzcd}
(\widetilde{\mathfrak{X}}_0, \widetilde{\mathcal{F}}_0) \arrow[r, "\widetilde{g}_0"] \arrow[drr] & (\widetilde{\mathfrak{X}}_1, \widetilde{\mathcal{F}}_1) \arrow[r, "\widetilde{g}_1"] \arrow[dr] & (\widetilde{\mathfrak{X}}_2, \widetilde{\mathcal{F}}_2) \arrow[r, "\widetilde{g}_2"] \arrow[d] & \dots \arrow[r, "\widetilde{g}_{n-1}"] & (\widetilde{\mathfrak{X}}_n, \widetilde{\mathcal{F}}_n) \arrow[dll] \\
& & U & &
\end{tikzcd}$$

Up to possibly shrinking U, we can assume, by construction and the previous observations, that

- for all i, the $\widetilde{g}_i$ are all birational morphisms over U and $-(K_{(\widetilde{\mathfrak{X}}_i, \widetilde{\mathcal{F}}_i),\epsilon})$ is $\widetilde{g}_i$-nef;
- for all i, all fibers of $(\widetilde{\mathfrak{X}}_i, \widetilde{\mathcal{F}}_i) \to U$ are ϵ-adjoint canonical, by the negativity lemma applied fiber-wise;
- $K_{(\widetilde{\mathfrak{X}}_n, \widetilde{\mathcal{F}}_n),\epsilon}$ is ample when restricted to any fiber of $\widetilde{\mathfrak{X}}_n \to U$.

It follows that the fibers of $(\widetilde{\mathfrak{X}}_n, \widetilde{\mathcal{F}}_n) \to U$ are ϵ-adjoint canonical models and they are birational to the corresponding fiber of $(\widetilde{\mathfrak{X}}_0, \widetilde{\mathcal{F}}_0)$ over the same point on U. By uniqueness of ample models of ϵ-adjoint canonical foliated pairs, cf. Corollary 3.4, they all yield elements of $\mathcal{M}_{2,\epsilon,C}$. By Noetherian induction, it suffices to repeat the procedure of this step a finite number of times, first on each connected component $T_1 \setminus q(U)$ and then on all other connected components to T, to show that every element in $\mathcal{M}_{2,\epsilon,C}$ must appear as a fiber in the (finitely many) families of surface foliations produced by this process. □

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