# BOUNDEDNESS OF ELLIPTIC CALABI–YAU VARIETIES WITH A RATIONAL SECTION

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Abstract. We show that for each fixed dimension $d \geq 2$, the set of $d$-dimensional klt elliptic varieties with numerically trivial canonical bundle is bounded up to isomorphism in codimension one, provided that the torsion index of the canonical class is bounded and the elliptic fibration admits a rational section. This case builds on an analogous boundedness result for the set of rationally connected log Calabi-Yau pairs with bounded torsion index. In dimension 3, we prove the more general statement that the set of $\epsilon$-lc pairs $(X, B)$ with $-(K_X + B)$ nef and rationally connected $X$ is bounded up to isomorphism in codimension one.

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## 1. Introduction

Throughout this paper, we work over an algebraically closed field of characteristic 0, for instance, the field of complex numbers $\mathbb{C}$.

The central task in algebraic geometry is the classification of projective varieties. There are two possible approaches to this end: either by identifying two distinct varieties that are isomorphic or by saying that two distinct varieties are birational equivalent (or simply, birational) if they both possess isomorphic dense open sets.

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Birational equivalence, preserving many numerical and geometrical quantities, is a sufficiently coarse equivalence relation in the category of algebraic varieties; at the same time, it is more flexible than the classification by isomorphism type: we can modify a given variety as long as a dense open set is left untouched, thus constructing a new variety birational to the original one. The Minimal Model Program predicts that, up to a special class of birational transformations, each projective variety decomposes into iterated fibrations with general fibers of 3 basic types:

- *Fano varieties*: mildly singular varieties with ample anti-canonical bundle;
- *K-trivial varieties*: mildly singular varieties with torsion canonical bundle;
- *log canonical models*: mildly singular varieties with ample canonical bundle.

As these classes of varieties constitute the fundamental building blocks in the birational classification of algebraic varieties, the task of understanding their possible algebraic and topological structures is a central one.

The second part of the classification scheme aims to construct compact *moduli spaces* that parametrize all isomorphism classes for the above 3 classes of varieties. For this purpose, one big question is whether or not in any given dimension there are just finitely many families of varieties for each of the 3 classes – perhaps after fixing certain geometric invariants. This property goes under the name of *boundedness*, see § 2.4. This is a central question as these 3 classes of varieties are building blocks for many constructions in geometry and theoretical physics.

Showing boundedness for a certain class of varieties is a rather difficult task: in fact, one needs to embed all varieties in the chosen class in a fixed projective space, while, at the same time, controlling the volume of these embeddings. There have been some recent extraordinary developments on the study of boundedness for two of the building blocks introduced above: Hacon–M$^{c}$Kernan–Xu proved that log canonical models with fixed volume are bounded in fixed dimension, see [HMX18]; this result also implies the existence of moduli spaces of log canonical models, thanks to work of Kollár–Shepherd-Barron, Alexeev, and others, see [Kol13]. Kollár-Miyaoka-Mori showed in [KMM92] that smooth Fano varieties form a bounded family; recently, Birkar proved the boundedness of $d$-dimensional $\epsilon$-klt Fano varieties for fixed $\epsilon > 0$ and $d \in \mathbb{N}$, thus, proving the BAB Conjecture, see [Bir16a]; moreover, he has also generalized this result to the case of log Fano fibrations, [Bir18], where also some ideas and techniques from [DCS16] are used.

K-trivial varieties, in turn, are not bounded in the category of algebraic varieties: for example, it is well-known that there are infinitely many algebraic families of projective K3 surfaces, or of abelian varieties in each dimension. Nonetheless, both K3 surfaces and abelian varieties of fixed dimension all fit into a unique topological family, once we consider also the non-algebraic ones. In dimension higher than 2, the situation is even more varied. In these contexts the situation can be remedied by fixing a polarization with bounded volume – a classic approach in the study of moduli of algebraic varieties that do not possess an obvious polarization, cf. [Vie95]. More recently, [Bir20], Birkar has shown that polarised Calabi–Yau varieties and log Calabi–Yau pairs are well-behaved from the point of view of the Minimal Model Program and boundedness questions, much like in the case of Fano varieties or canonical models, despite the lack of uniqueness for the choice of polarization.

**Boundedness for elliptic Calabi–Yau varieties.** A now classical result due to Beauville and Bogomolov, [Bea83], shows that, up to an étale cover, every smooth variety with numerically trivial canonical class can be decomposed into a product of abelian, hyperkähler, and Calabi–Yau manifolds. A smooth projective variety $Y$ is Calabi–Yau if it is simply connected, $K_Y \sim 0$ and $H^i(Y, \mathcal{O}_Y) = 0$ for $0 < i < \dim Y$.

While we know that in any fixed dimension $n \geq 2$, there exists infinitely many algebraic families of abelian or algebraic hyperkähler manifolds, Calabi–Yau varieties constitute a large class of K-trivial varieties for which boundedness is still a hard unresolved question.

**Question 1.1.** *Fix $n \in \mathbb{N}$. Are $n$-dimensional Calabi–Yau manifolds bounded?*

The only known affirmative answer to the above question is due to Gross, [Gro94], who proved that boundedness holds for Calabi–Yau threefolds carrying an elliptic fibration, up to birational equivalence. This is in opposition to the two-dimensional case: in fact, there exist infinitely many algebraic families of elliptic K3 surfaces. Starting from dimension three, theoretical physicists have formulated the expectation, based on the known examples, that Calabi–Yau manifolds with sufficiently large Picard number may be modified birationally to obtain a model that is endowed with a fibration to a lower dimensional variety. In view of this, then, proving boundedness results for elliptically fibered Calabi–Yau manifolds would provide an important step towards answering Question 1.1, as it would show that the Picard number can be bounded from above.

It is not hard to show that if we require the presence of a section, already in the case of elliptic K3 surfaces, it is possible to construct polarizations with bounded volume, thus proving boundedness for this class of K3 surfaces. The main goal of this paper is to prove the following generalization to arbitrary dimension of this phenomenon.

**Theorem 1.2.** *Fix a positive integer $d$. Then the set of projective varieties $Y$ such that*

*(1) $Y$ is a Calabi–Yau manifold of dimension $d$ and*
*(2) $Y \to X$ is an elliptic fibration with a rational section $X \dashrightarrow Y$*

*is bounded up to flops.*

Theorem 1.2 is an important foundational result: it implies that there are just finitely many families of elliptic Calabi-Yau varieties carrying a rational section, up to birational operations that modify the variety in a subset of codimension at least 2. Such birational operations do not modify excessively the geometry of the variety; for example, the Hodge diamond is left unaltered, as shown in [Bat98]. In turn, our result implies that, in any fixed dimension, there are finitely many possible different Hodge diamonds of elliptic Calabi-Yau manifolds admitting a rational section. This is relevant to applications to physics, where F-theory constructions are based on a choice of a Calabi-Yau manifold $Y$ with an elliptic fibration over a base space $X$. In this context, physical properties of the model can be translated into the geometry of the elliptically fibered Calabi-Yau and the uniform boundedness of some of their Hodge numbers is a property that has long been sought after in the field, see, for example, [TW15].

When studying elliptic Calabi–Yau varieties in general, those admitting a rational section play a central role. Weierstrass models and Jacobian fibrations always carry a rational section and their boundedness is a fundamental step towards the proof of boundedness of elliptic Calabi–Yau threefolds in Gross' work. Thus, Theorem 1.2 represents the realization of this first fundamental step in the generalization of [Gro94] to higher dimension. Theorem 1.2 combined with a careful analysis of the Tate-Shafarevich group for elliptic fibrations should eventually produce a proof of the boundedness of general elliptic Calabi-Yau manifolds.

A similar result to Theorem 1.2 holds for singular K-trivial varieties as long as we bound the torsion index of the canonical divisor.

**Theorem 1.3.** *Fix positive integers $d, l$. Then the set of varieties $Y$ such that*

*(1) $Y$ is klt and projective of dimension $d$,*
*(2) $lK_Y \sim 0$*
*(3) $Y \to X$ is an elliptic fibration with a rational section $X \dashrightarrow Y$, and*
*(4) $X$ is rationally connected*

*is bounded up to flops.*

As in Theorem 1.3 we dropped the assumption on the simple connectedness of $Y$, there is a price to pay in terms of the assumptions we make: namely, we have to assume that the base $X$ of the fibration is rationally connected. Considering elliptic K-trivial varieties with a rationally connected base is a rather natural restriction: indeed, it is simple to see that when $Y$ is Calabi–Yau then the base of an elliptic fibration is always rationally connected, cf. Theorem 5.1. More generally, imposing only that $K_Y$ be numerically trivial, it may happen that the base $X$ of the elliptic fibration $Y \to X$ is not rationally connected; nonetheless, such assumption implies that the elliptic fibration is isotrivial, i.e., the isomorphism type of the fibers does not change, and, moreover, that the fibers over points of codimension one on the base all have semi log-canonical singularities. Furthermore, $K_X \sim_{\mathbb{Q}} 0$ and either $X$ has canonical singularities so that the Kodaira dimension of $X$ is 0, or, else, $X$ has strictly klt singularities and it is uniruled. When $X$ is rationally connected, a slightly weaker result than the one in Theorem 1.3 was proven by the second and third named authors, [DCS16], for dimension up to 5; they showed that the conclusion of the theorem holds for those elliptic Calabi–Yau manifolds with a section under some extra assumptions on the birational structure of the base $X$ of the fibration. Hence, already in dimension 4 and 5, Theorem 1.3 provides a considerable improvement of our current knowledge.

**Boundedness for rationally connected log Calabi–Yau pairs.** The heart of the proof of Theorem 1.2 relies on showing that the bases of elliptic Calabi–Yau manifolds form in turn a bounded family, up to isomorphism in codimension 1. In the elliptically fibered setting, the canonical bundle formula, see § 2.7, implies that the base $X$ carry a structure of log Calabi-Yau pair, that is, there exists an effective divisor $B$ on $X$ with coefficients in $(0, 1)$, $K_X + B$ is numerically trivial and the singularities of the pair $(X, B)$ are mild (klt), see § 2.1 for the type of singularities that are allowed. The study of the bases of elliptic fibrations is our motivation to study the boundedness of rationally connected log Calabi–Yau pairs. We prove the following general result that holds in any dimension.

**Theorem 1.4.** *Fix positive integers $d, l$. Then the set of pairs $(X, B)$ such that*

*(1) $(X, B)$ is a projective klt pair of dimension $d$,*
*(2) $l(K_X + B) \sim 0$, and*

*(3) $X$ is rationally connected*

*is log bounded up to flops.*

This result is not only central to the study of boundedness for elliptic Calabi–Yau varieties, but it maintains its own independent relevance: in view of the unavoidable appearance of singularities in the birational classification of varieties, it is often necessary to work not just with mildly singular varieties, but rather with pairs of a normal projective variety and an effective divisor with coefficients in $[0, 1]$. All conjectures and questions for Calabi–Yau manifolds discussed so far can be also formulated in the more general context of log Calabi–Yau pairs. Hence, the importance of boundedness for log Calabi–Yau pairs should be immediately clear, being the natural extension to the realm of pairs of the K-trivial case. As such, then, it is an even more intriguing and complicated problem.

In proving Theorem 1.4, we first show that, up to birational modifications, we can decompose $X$ into a tower of fibrations whose general fibers are Fano varieties with bounded singularities. This result allows us to control the torsion index of the successive log Calabi–Yau pairs that we can construct on the base of each of the fibrations that decompose $X$. At this point, we show inductively, using Birkar's recent results on boundedness for log Fano fibrations, [Bir18], that, at each step in this tower of fibrations, if we assume that the base of the fibration belongs to a bounded family, then the same holds also for the total space of the fibration.

We regard Theorem 1.4 as providing strong evidence for a more general conjecture, generalizing the BAB Conjecture: the conjecture predicts the boundedness of all rationally connected varieties of log Calabi–Yau type with bounded singularities.

**Conjecture 1.5.** *(cf.* [MP04, Conj. 3.9], [CDCH+18, Conj. 1.3]*) Fix a positive integer $d$ and positive real number $\epsilon$. Then the set of varieties $X$ such that*

*(1) $X$ is normal projective and rationally connected of dimension $d$*
*(2) $(X, B)$ is $\epsilon$-lc for some effective $\mathbb{R}$-divisor $B$, and*
*(3) $-(K_X + B)$ is nef,*

*is bounded.*

The 2-dimensional case of this conjecture was proved by Alexeev in [Ale94]. In dimension three, combining our techniques with the recent result of Jiang [Jia19], we are able to prove a slightly weaker version of the above conjecture, showing that rather than boundedness, we can obtain boundedness up to isomorphism in codimension 1.

**Theorem 1.6.** *Fix a positive real number $\epsilon$. The set of varieties $X$ such that*

*(1) $X$ is normal projective of dimension 3,*
*(2) $(X, B)$ is $\epsilon$-lc for some effective $\mathbb{R}$-divisor $B$,*
*(3) $-(K_X + B)$ is nef, and*
*(4) $X$ is rationally connected*

*is bounded up to flops.*

**Strategy of proof of Theorem 1.2-1.3.** The standard approach to bound a set of fibered varieties is to first bound the set of bases and general fibers, and then lift boundedness from the bases to the set of total spaces. Here we follow this very same strategy; the existence of a rational section makes bounding the set of bases of an elliptic fibration the most important and technical step in our proof. One of the main novelties here is that we are able to deal with the case where $X$ is rationally connected and $K_X \equiv 0$, an important missing piece in [DCS16]. Recall that examples of this type of fibration are known to exists already in dimension 3, see [Ogu93].

From the viewpoint of boundedness, the current techniques in the Minimal Model Program do not provide enough tools to approach rationally connected varieties with $K_X \equiv 0$. As predicted in Conjecture 1.5 for the case when $B = 0$, these varieties are expected to be bounded once their singularities are. In general, boundedness is expected to hold once the torsion index of the canonical divisor is uniformly bounded, cf. [CDCH+18, Jia19]; unfortunately, it is not easy to show that such an hypothesis is satisfied in higher dimension. We manage to show that those rationally connected K-trivial varieties that are bases of K-trivial elliptic fibrations with bounded torsion index satisfy boundedness of the torsion index in turn. Once uniform boundedness of the torsion index is settled, we can reduce the theorem to boundedness of Fano fibrations (or towers of such fibrations) which is one of the main results of [Bir18]. Here for simplicity, we state only the case where the total space is smooth.

**Theorem 1.7** (cf. Theorem 5.4)**.** *Fix a positive integer $d$. Then there exists a positive integer $m = m(d)$ such that if*

*(1) $Y$ is a Calabi–Yau manifold of dimension $d$ and*
*(2) $Y \to X$ is an elliptic fibration with $K_X \equiv 0$,*

*then $mK_X \sim 0$.*

Once boundedness of the bases is settled, the following step consists in showing that boundedness also holds for the total space of an elliptic fibration carrying a rational section, see Theorem 5.2. In this case, the idea is to use the Zariski closure $S$ of the rational section $X \dashrightarrow Y$ together with the pullback of a suitable very ample divisor $H$ from the base $X$ with bounded volume – a bound guaranteed by boundedness of $X$ – to arrive to a suitable birational model $Y'$ of $Y$ on which $K_Y \equiv 0$, and the strict transform of $S + H$ on $Y'$ has bounded singularities and bounded positive volume. Then by the results of [HMX14] and [Bir18], these conditions imply that $Y'$ is bounded and the divisors contracted in the birational map $Y \dashrightarrow Y'$ can be extracted to yield a bounded model $Y_1$ isomorphic in codimension 1 to $Y$. These techniques have been recently applied also to the study of the boundedness of $n$-dimensional minimal models of Kodaira dimension $n - 1$, see [FS20, Fil20a].

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## 2. Preliminaries

We adopt the standard notation and definitions from [KMM87] and [KM98], and we freely use those.

A set $I \subset \mathbb{R}$ is said to be a DCC (resp. ACC) set, if it does not contain any strictly decreasing (resp. increasing) sequence $\{i_k\}$ of elements of $I$.

2.1. **Pairs and singularities.** A *log pair* $(X, B)$ consists of a normal projective variety $X$ and an effective $\mathbb{R}$-divisor $B$ on $X$ such that $K_X + B$ is $\mathbb{R}$-Cartier. Given $f\colon Y \to X$ a log resolution of the log pair $(X, B)$, we write

$$K_Y + B' = f^*(K_X + B),$$

where $B'$ is the sum of the strict transform $f_*^{-1}B$ of $B$ on $Y$ and a divisor completely supported on the exceptional locus of $f$, $B' = f_*^{-1}B + \sum a_i E_i$. We will denote by $\mu_D B'$ the multiplicity of $B'$ along a prime divisor $D$ on $Y$. For a non-negative real number $\epsilon$, the log pair $(X, B)$ is called

(a) *$\epsilon$-kawamata log terminal* (*$\epsilon$-klt*, in short) if $\mu_D B' < 1 - \epsilon$ for all $D \subset Y$;
(b) *$\epsilon$-log canonical* (*$\epsilon$-lc*, in short) if $\mu_D B' \leq 1 - \epsilon$ for all $D \subset Y$;
(c) *terminal* if $\mu_D B' < 0$ for all $f$-exceptional $D \subset Y$ and all possible choices of $f$.
(d) *canonical* if $\mu_D B' \leq 0$ for all $f$-exceptional $D \subset Y$ and all possible choices of $f$.

Let us note that 0-klt (resp., 0-lc) is just klt (resp., lc) in the usual sense. Moreover, $\epsilon$-lc singularities only make sense if $\epsilon \in [0, 1]$, and $\epsilon$-klt singularities only make sense if $\epsilon \in [0, 1)$, Usually we write $X$ instead of $(X, 0)$, that is, when we consider the case $B = 0$.

The *log discrepancy* of a prime divisor $D$ on $Y$ is defined to be $a(D, X, B) := 1 - \mu_D B'$. It does not depend on the choice of the log resolution $f$.

2.2. **Generalised pairs.** For the definition of b-divisor and related notions, we refer the reader to [BZ16]. There, the authors introduced also the notion of generalised pairs. Let us recall that a b-$\mathbb{R}$-divisor $\mathbf{N}$ is said to descend to the divisor $N'$ on a model $X'$ if $\mathbf{N}$ equals the Cartier closure of its trace $\mathbf{N}_{X'}$ on $X'$ and $\mathbf{N}_{X'} = N'$.

**Definition 2.1.** *Let $Z$ be a variety. A generalised polarised pair over $Z$ is a tuple $(X' \to X, B, M')$ consisting of the following data:*

- *a normal variety $X \to Z$ projective over $Z$ equipped with a projective birational morphism $\phi : X' \to X$,*
- *an effective $\mathbb{R}$-Weil divisor $B$ on $X$,*
- *a b-$\mathbb{R}$-Cartier b-divisor $\mathbf{M}$ over $X$ which descends on $X'$ such that $M' := \mathbf{M}_{X'}$ is nef over $Z$, and*
- *$K_X + B + M$ is $\mathbb{R}$-Cartier, where $M := \phi_* M'$.*

When no confusion arises, we refer to the pair by saying that $(X, B + M)$ is a generalised pair with data $X' \to X \to Z$ and $M'$. We call $M'$ the nef part of the generalised pair.

Similarly to log pairs, we can define discrepancies and singularities for generalised pairs. Replacing $X'$ with a higher birational model, we can assume that $\phi$ is a log resolution of $(X, B)$. Then we can write

$$K_{X'} + B' + M' = \phi^*(K_X + B + M)$$

for some uniquely determined divisor $B'$. For a prime divisor $D$ on $X'$ the generalised log discrepancy $a(D, X, B+M)$ is defined to be $1-\mu_D B'$. We say $(X, B+M)$ is generalised lc (resp. generalised klt, generalised $\epsilon$-lc) if for each $D$ the generalised log discrepancy $a(D, X, B + M)$ is $\geq 0$ (resp.$> 0$, $\geq \epsilon$).

2.3. **Minimal model program.** Moreover, we will make use of the Minimal Model Program (MMP, in short) for (generalised) pairs with non-pseudo-effective log canonical class. In this case, the existence and termination of the MMP has been established in the pair setting by Birkar-Cascini-Hacon-M$^c$Kernan, [BCHM10, Cor. 1.3.3], and by Birkar-Zhang in the generalised setting, [BZ16, Lemma 4.4].

**Theorem 2.2.** [BZ16, BCHM10] *Let $(X, B+M)$ be a projective $\mathbb{Q}$-factorial generalised klt pair. Assume $K_X+B+M$ is not pseudo-effective. Then we may run a $(K_X+B+M)$-MMP $g\colon X \dashrightarrow Y$ that terminates with a Mori fibre space $f\colon Y \to T$.*

Let us recall the definition of Mori fibre space.

**Definition 2.3.** *Let $(X, B+M)$ be a generalised klt pair and let $f\colon X \to T$ be a projective morphism of normal varieties with $\dim(T) < \dim(X)$. Assume that $f_*\mathcal{O}_X = \mathcal{O}_T$. Then $f$ is a Mori fibre space if*

*(1) $X$ is $\mathbb{Q}$-factorial,*
*(2) $f$ is a primitive contraction, i.e. the relative Picard number $\rho(X/T) = 1$ and*
*(3) $-(K_X+B+M)$ is $f$-ample.*

2.4. **Boundedness of pairs.** We recall the different notions of boundedness for varieties and log pairs.

**Definition 2.4.** *A collection of projective varieties $\mathfrak{D}$ is said to be* bounded *(resp.,* birationally bounded*, or* bounded in codimension one*) if there exists $h\colon \mathcal{Z} \to S$ a projective morphism of schemes of finite type such that each $X \in \mathfrak{D}$ is isomorphic (resp., birational, or isomorphic in codimension one) to $\mathcal{Z}_s$ for some closed point $s \in S$.*

**Definition 2.5.** *A collection of projective log pairs $\mathfrak{D}$ is said to be* log birationally bounded *(resp.,* log bounded*, or* log bounded in codimension one*) if there is a quasi-projective variety $\mathcal{Z}$, which may possibly be reducible, a reduced divisor $\mathcal{E}$ on $\mathcal{Z}$, and a projective morphism $h\colon \mathcal{Z} \to S$, where $S$ is of finite type and $\mathcal{E}$ does not contain any fiber, such that for every $(X, B) \in \mathfrak{D}$, there is a closed point $s \in S$ and a birational map $f\colon \mathcal{Z}_s \dashrightarrow X$ (resp., isomorphic, or isomorphic in codimension one) such that $\mathcal{E}_s$ contains the support of $f_*^{-1}B$ and any $f$-exceptional divisor (resp., $\mathcal{E}_s$ coincides with the support of $f_*^{-1}B$, $\mathcal{E}_s$ coincides with the support of $f_*^{-1}B$).*
*A collection of projective log pairs $\mathfrak{D}$ is said to be* strongly log bounded *if there is a quasi-projective log pair $(\mathcal{Z}, \mathcal{B})$ and a projective morphism $h\colon \mathcal{Z} \to S$, where $S$ is of finite type, such that, $\mathcal{B}$ does not contain any fiber of $h$, and for every $(X, B) \in \mathfrak{D}$, there is a closed point $s \in S$ and an isomorphism $f\colon \mathcal{Z}_s \to X$ such that $f_*\mathcal{B}_s = B$.*

In the case of Calabi–Yau pairs, we will use an equivalent notion of log boundedness in codimension one that is more suitable for our proofs.

**Remark 2.6.** If $\mathfrak{D}$ is a collection bounded in codimension one (resp., log bounded in codimension one) of projective klt Calabi–Yau varieties (resp., klt log Calabi–Yau pairs), then by replacing $\mathcal{Z}$ with its normalization in the definition above, we can assume that the fibre $\mathcal{Z}_s$ isomorphic in codimension one to $X$ is normal projective, and $K_{\mathcal{Z}_s}$ is $\mathbb{Q}$-Cartier (resp., $K_{\mathcal{Z}_s} + f_*^{-1}B$ is $\mathbb{R}$-Cartier). We will refer to such a collection as to a collection *bounded modulo flops* (resp., *log bounded modulo flops*). The indication "modulo flops" comes from the fact that, if we assume that $X$ and $\mathcal{Z}_s$ are both $\mathbb{Q}$-factorial, then they are connected by flops by running a

$(K_X + B + \delta f_* H)$-MMP where $H$ is an ample divisor on $\mathcal{Z}_s$ and $\delta$ is a sufficiently small positive number, cf. [BCHM10, Kaw08].

If $\mathfrak{D}$ is a collection of klt log Calabi–Yau pairs which is log bounded modulo flops, $(X, B) \in \mathfrak{D}$, and $f \colon \mathcal{Z}_s \dashrightarrow X$ is an isomorphism in codimension one as in the definition above, then $(\mathcal{Z}_s, f_*^{-1} B)$ is again a klt log Calabi–Yau pair by the Negativity Lemma. Moreover, $(X, B)$ is $\epsilon$-lc if and only if $(\mathcal{Z}_s, f_*^{-1} B)$ is so. The same statement holds for a set $\mathfrak{D}$ of klt Calabi–Yau varieties which is bounded modulo flops.

To show that a given set of pairs is log birationally bounded, we will mainly use the following theorem which is a combination of results in [HMX13, HMX14].

**Theorem 2.7.** [HMX13, Theorem 3.1], [HMX14, Theorem 1.3] *Fix two positive integers $n$ and $V$ and a DCC set $I \subset [0, 1]$. Then the collection $\mathfrak{D}$ of log pairs $(X, B)$ satisfying*

*(1) $X$ is a projective variety of dimension $n$,*
*(2) $(X, B)$ is lc,*
*(3) the coefficients of $B$ belong to $I$, and*
*(4) $0 < \mathrm{Vol}(K_X + B) \leq V$,*

*is log birationally bounded.*

In certain special cases it is possible to deduce boundedness from log birational boundedness.

**Theorem 2.8.** [HMX14, Theorem 1.6] *Fix a positive integer $n$ and two positive real numbers $b$ and $\epsilon$. Let $\mathfrak{D}$ be a collection of log pairs $(X, B)$ such that:*

*(1) $X$ is a projective variety of dimension $n$,*
*(2) $K_X + B$ is ample,*
*(3) the coefficients of $B$ are at least $b$, and*
*(4) the log discrepancy of $(X, B)$ is greater than $\epsilon$.*

*If $\mathfrak{D}$ is log birationally bounded then $\mathfrak{D}$ is a log bounded set of log pairs.*

Theorem 2.8 can be strengthened when we further impose control on the singularities and coefficients of pairs. The following result is a straightforward consequence of [MST16, Theorem 6] and [Fil20b, Theorem 1.3].

**Theorem 2.9.** *Fix a natural number $d$, a positive rational number $C$, a positive real number $\epsilon$, and $I \subset \mathbb{Q} \cap [0, 1)$ a finite set. The collection $\mathfrak{D}$ of all pairs $(X, B)$ such that*

*(1) $(X, B)$ is $\epsilon$-klt, projective, of dimension $d$,*
*(2) the coefficients of $B$ belong to $I$,*
*(3) $K_X + B$ is big and nef, and*
*(4) $\mathrm{Vol}(K_X + B) \leq C$*

*is strongly log bounded.*

In the course of our treatment, we will need to use several times the following technical result which allows us to pass from bounded collections of log pairs to log bounded ones.

**Proposition 2.10.** *Fix a finite set $\mathfrak{R} \subset (0, 1)$ and a positive real number $\epsilon$. Let $\mathfrak{D}$ be a bounded collection of $\epsilon$-lc pairs. Assume that for any pair $(X, B) \in \mathfrak{D}$ the coefficients of $D$ belong to $\mathfrak{R}$. Then $\mathfrak{D}$ is strongly log bounded.*

The important point in the statement of the proposition is that there exists a proper morphism of quasi-projective varieties $h\colon \mathcal{Z} \to S$ with $S$ smooth, and a divisor $\mathcal{B}$ on $\mathcal{Z}$ such that for any $(X, B) \in \mathfrak{D}$ there exists $s \in S$ and an isomorphism $f_s\colon \mathcal{Z}_s \to X$ with $f_s^*(B) = \mathcal{B}_s$. In this context, both $\mathcal{Z}$ and $S$ are not necessarily irreducible, but, nonetheless, they are of finite type.

*Proof.* By definition of boundedness there exists a couple $(\mathcal{Z}, \mathcal{E})$ with $\mathcal{E}$ reduced divisor, and a projective morphism of quasi-projective varieties $h\colon \mathcal{Z} \to S$ such that for any pair $(X, B) \in \mathfrak{D}$ there exists $s \in S$ and isomorphism $f_s\colon \mathcal{Z}_s \to X$ such that $f_s$ maps $\mathcal{E}_s$ to the support of $B$. We will denote by $\mathcal{H}$ a relatively very ample Cartier divisor for $\mathcal{Z} \to S$.
Decomposing $S$ into a finite union of locally closed subsets and possibly discarding some components, we may assume that every fibre $\mathcal{Z}_s$ is a variety. Blowing up $\mathcal{Z}$ and decomposing $S$ into a finite union of locally closed subsets, we may assume that there exists a pair $(\mathcal{Z}', \mathcal{E}')$ that has simple normal crossings and such that

$$\begin{tikzcd} \mathcal{Z}' \arrow[rr, "g"] \arrow[dr, "h'"'] & & \mathcal{Z} \arrow[dl, "h"] \\ & S & \end{tikzcd}$$

and for any $s \in S$, $(\mathcal{Z}'_s, \mathcal{E}'_s)$ is a log resolution of $(\mathcal{Z}_s, \mathcal{E}_s)$, with $g_*\mathcal{E}' = \mathcal{E}$.
Decomposing $S$ into a finite union of locally closed subsets, we may assume that over an irreducible component of $S$, the fibres of $\mathcal{Z}' \to S$ are equidimensional log pairs, and that $(\mathcal{Z}', \mathcal{E}')$ has simple normal crossings over $S$. Passing to a finite cover of $S$, we may assume that every stratum of $(\mathcal{Z}', \mathcal{E}')$ has irreducible fibres over $S$. Decomposing $S$ into a finite union of locally closed subsets, we may assume that $S$ is smooth. In particular, over an irreducible component $\bar{S}$ of $S$, where all fibers of $\mathcal{Z}' \to S$ are $d$-dimensional, if $\mathcal{E}' = \sum_{i=1}^n \mathcal{E}'_i$ is the decomposition into its irreducible components, up to passing to an open set of $\bar{S}_1 \subset \bar{S}$, we can assume that for any $i$ either $\mathcal{E}'_i|_{\mathcal{Z}_s}$ is exceptional over $\mathcal{Z}_s$ for any $s \in \bar{S}_1$ or that $\mathcal{E}'_i|_{\mathcal{Z}_s}$ maps to a divisor on $\mathcal{Z}_s$ for any $s \in \bar{S}_1$. We will denote by $\mathcal{E}'_{i,s}$ the divisors $\mathcal{E}'_i|_{\mathcal{Z}'_s}$.
Let $\{\mathcal{E}'_1, \ldots, \mathcal{E}'_l\}$ be those components of $\mathcal{E}'$ that are exceptional over $\mathcal{Z}$ – equivalently, those components of $\mathcal{E}'$ such that $\mathcal{E}'_{i,s}$ is exceptional over $\mathcal{Z}_s$ for some (or any) $s \in \bar{S}_1$.
By [HMX18, Theorem 4.2], for any fixed $n$-tuple $(a_1, \ldots, a_n) \in (0,1)^n$ and any positive integers $t, m$, then

$$h^0(\mathcal{Z}'_s, \mathcal{O}_{\mathcal{Z}'_s}(m(K_{\mathcal{Z}'_s} + \sum_i a_i \mathcal{B}'_{i,s} + tg^*\mathcal{H}_s)))$$

is independent of $s \in \bar{S}_1$. Let us consider the set $\mathfrak{N}$ of $n$-tuples of the form

$$(1 - \frac{\epsilon}{2}, \ldots, 1 - \frac{\epsilon}{2}, b_{l+1}, b_{l+2}, \ldots, b_n),\ b_i \in \mathfrak{R}.$$

As $\mathfrak{R}$ is finite, $\mathfrak{N}$ is a finite set as well. Then, the MMP for

$$K_{\mathcal{Z}'} + \sum_{i=1}^{l} (1 - \frac{\epsilon}{2}) \mathcal{B}'_i + \sum_{j=1}^{n-l} b_{j+l} \mathcal{B}'_{j+l} + (2d+2) g^*\mathcal{H}$$

over $\bar{S}_1$ exists and it must terminate by [BZ16, Lemma 4.4] with a log canonical model as

$$K_{\mathcal{Z}'} + \sum_{i=1}^{l}(1-\frac{\epsilon}{2})\mathcal{B}'_i + \sum_{j=1}^{n-l} b_{j+l}\mathcal{B}'_{j+l} + (2d+2)g^*\mathcal{H}_s$$

is big over $S_1$. For an element $e \in \mathfrak{N}$, $e = (1-\frac{\epsilon}{2}, \dots, 1-\frac{\epsilon}{2}, b_{l+1}, b_{l+2}, \dots, b_n)$, such that

$$K_{\mathcal{Z}'} + \sum_{i=1}^{l}(1-\frac{\epsilon}{2})\mathcal{B}'_i + \sum_{j=1}^{n-l} b_{j+l}\mathcal{B}'_{j+l} + (2d+2)g^*\mathcal{H}_s$$

is pseudo-effective over $S_1$, we will indicate by $\mathcal{Z}_e \to S$ the corresponding log canonical model. Moreover, we will denote by $\mathfrak{C}_e$ the pushforward of $\mathcal{B}'$ on $\mathcal{Z}_e$.
Given a pair $(X, B) \in \mathfrak{D}$ corresponding to the fiber $\mathcal{Z}_s$ over a point $s \in \bar{S}_1$, we can write $B = \sum_{k=1}^{n-l} b_{k+l}\mathcal{B}'_{k+l}$, $b_{k+l} \in \mathfrak{N}$ and

$$K_{\mathcal{Z}'_s} + \sum_{i=1}^{l}(1-\frac{\epsilon}{2})\mathcal{B}'_{i,s} + \sum_{j=1}^{n-l} b_{j+l}\mathcal{B}'_{j+l,s} + (2d+2)g^*\mathcal{H}_s = g^*|_{\mathcal{Z}'_s}(K_X + B + (2d+2)\mathcal{H}_s) + F,$$

where $F$ is effective and its support coincides with the union of all the $\mathcal{B}'_{i,s}$, $1 \leq i \leq l$. Moreover, by the cone theorem $K_X + B + (2d+2)\mathcal{H}_s$ is ample on $X$. Hence, it follows from [HMX18, Theorems 4.2 and 6.2] that there exists an isomorphism $h_s \colon \mathcal{Z}_s \to X$ and $h_s^*(B) = \mathfrak{C}_e$. As the cardinality of $\mathfrak{N}$ is finite, the union of all the $\mathcal{Z}_e \to S_1$ is a bounded family.
The proof then concludes by repeating the same argument on $\bar{S} \setminus \bar{S}_1$. This iteration must terminate by noetherian induction in a finite number of steps. Repeating the same argument for the remaining (finitely many) components of $S$ yields the desired result. □

### 2.5. Boundedness of Fano fibrations.

To simplify the statements in this section, we recall the following definition introduced in [Bir18].

**Definition 2.11.** [Bir18, Definition 1.1] *Let $d, r$ be natural numbers and $\epsilon$ be a positive real number. A $(d, r, \epsilon)$-*Fano type (log Calabi-Yau) fibration *consists of a pair $(X, B)$ and a contraction $f \colon X \to Z$ such that we have the following:*

- *$(X, B)$ is a projective $\epsilon$-lc pair of dimension $d$,*
- *$K_X + B \sim_{\mathbb{R}} f^*L$ for some $\mathbb{R}$-divisor $L$,*
- *$-K_X$ is big over $Z$, i.e. $X$ is of Fano type over $Z$,*
- *$A$ is a very ample divisor on $Z$ with $A^{\dim Z} \leq r$, and*
- *$A - L$ is ample.*

Birkar has shown that $(d, r, \epsilon)$-Fano type fibrations are bounded. Let us recall the following two results which are going to be relevant later in our treatment.

**Theorem 2.12.** [Bir18, Theorem 1.2] *Let $d, r$ be natural numbers and $\epsilon$ be a positive real number. Consider the set of all $(d, r, \epsilon)$-Fano type fibrations $(X, B) \to Z$. Then the set of varieties $X$ is bounded.*

A more general log boundedness result holds if we impose a lower bound on the coefficients of the boundary of a log Calabi–Yau pair endowed with a $(d, r, \epsilon)$-Fano type fibration.

**Theorem 2.13.** [Bir18, Theorem 1.3] *Let $d, r$ be natural numbers and $\epsilon, \tau$ be positive real numbers. Consider the set of all $(d, r, \epsilon)$-Fano type fibrations $(X, B) \to Z$ such that $0 \leq \Delta \leq B$ and the coefficients of $\Delta$ are $\geq \tau$. Then the set of such pairs $(X, \Delta)$ is log bounded.*

Definition 2.11 has an analogue for generalised pairs, where rather than $(X, B)$ one considers a generalised $\epsilon$-lc pair $(X, B + M)$ such that $K_X + B + M \sim_{f,\mathbb{R}} 0$ and everything else is unchanged. In this context, there is an exact analogue of Theorem 2.13 for generalised pairs, see [Bir18, Theorem 2.2].

2.6. **Log bounded families of morphisms.** We extend the notion of boundedness to collection of log pairs endowed with morphisms.

**Definition 2.14.** *(i) A set $\mathfrak{F}$ of morphisms is a collection of 4-uples $(f, Y, B, X)$ where $f \colon Y \to X$ is a surjective morphism of projective varieties and the pair $(Y, B)$ is a log pair.*

*(ii) We say that a set $\mathfrak{F}$ of morphisms is* log bounded *(resp.,* strongly log bounded*) if there exist quasi-projective varieties $\mathcal{Y}, \mathcal{X}$, a reduced divisor $\mathcal{E}$ on $\mathcal{Y}$ (resp., a log pair $(\mathcal{Y}, \mathcal{E})$), and a commutative diagram of projective morphisms*

$$\begin{tikzcd} \mathcal{Y} \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{X} \arrow[dl] \\ & S & \end{tikzcd}$$

*such that*

- *$S$ is of finite type,*
- *for any 4-uple $(f, Y, B, X) \in \mathfrak{F}$, there is a closed point $s \in S$ and isomorphisms $p \colon \mathcal{X}_s \to X$, $q \colon \mathcal{Y}_s \to Y$ such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{Y}_s \arrow[r, "\phi|_{\mathcal{Y}_s}"] \arrow[d, "q"] & \mathcal{X}_s \arrow[d, "p"] \\ Y \arrow[r, "f"] & X, \end{tikzcd}$$

  *and*
- *$q_*\mathcal{E}_s$ coincides with the support of $B$ (resp., $q_*\mathcal{E}_s = B$).*

**Remark 2.15.** Let $\mathfrak{F}$ be a bounded (resp., log bounded) collection of morphisms. Then the sets

$$\mathfrak{T} := \{(Y, B) \mid \exists (f, Y, B, X) \in \mathfrak{F}\},$$
$$\mathfrak{B} := \{X \mid \exists (f, Y, B, X) \in \mathfrak{F}\}$$

are a bounded (resp., log bounded) set of pairs and a bounded set of varieties, respectively. This is an immediate consequence of the conditions of Definition 2.14.

**Lemma 2.16.** *Let $d$ be a positive integer, $\epsilon$ be a positive real number, and let $\mathcal{R} \subset \mathbb{Q} \cap (0, 1)$ be a finite set.*
*Let $\mathcal{D}$ be a log bounded set of pairs $(Y, B)$ such that*

*(1) $(Y, B)$ is projective, $\epsilon$-klt of dimension $d$,*

*(2) the coefficients of $B$ are in $\mathcal{R}$,*
*(3) $K_Y + B$ is semi-ample.*

*Then, the set $\mathfrak{F}$ of 4-uples $(f, Y, B, X)$ such that*

*(i) there exists a log pair $(Y, B) \in \mathcal{D}$,*
*(ii) $X = \operatorname{Proj}(\oplus_{i=0}^{\infty} H^0(Y, \mathcal{O}_Y(i(K_Y + B))))$, and*
*(iii) $f \colon Y \to X$ is the Iitaka fibration of $K_Y + B$*

*is a strongly log bounded set of morphisms.*

*Proof.* As $\mathcal{D}$ is a log bounded set of $\epsilon$-klt pairs and $\mathcal{R}$ is finite, we can assume that $\mathcal{D}$ is strongly log bounded by Proposition 2.10. Hence, there exists a scheme of finite type $S$, a log pair $\pi \colon (\mathcal{Y}, \mathcal{B}) \to S$ over $S$ such that for any log pair $(Y, B) \in \mathcal{D}$, there exists $s \in S$ and $(\mathcal{Y}_s, \mathcal{B}_s)$ is isomorphic to $(Y, B)$.
Decomposing $S$ into a finite union of locally closed subsets and possibly discarding some components, we may assume that every fibre $\mathcal{Y}_s$ is a variety and that $\mathcal{B}$ does not contain any fiber. Decomposing $S$ into a finite union of locally closed subsets and passing to a finite cover of $S$, we may assume that for any $s \in S$ there is a 1-1 correspondence between the irreducible components of $\mathcal{B}$ and those of $\mathcal{B}_s$. Decomposing $S$ into a finite union of locally closed subsets, we may assume that $S$ is smooth. By [Bir19, Lemma 2.25], we can assume that there exists $I = I(\mathfrak{D})$ such that $I(K_{\mathcal{Y}} + \mathcal{B})$ is Cartier along any fibre $\mathcal{Y}_s$. Up to shrinking $S$ and decomposing it into a finite union of locally closed subsets, we can also assume that the set

$$S' := \{s \in S \mid \exists (Y, B) \in \mathfrak{D} \text{ such that } (Y, B) \simeq (\mathcal{Y}_s, \mathcal{B}_s)\}$$

is Zariski dense in $S$. By [FM18, Proposition 2.10], up to shrinking $S$ and decomposing it into a finite union of locally closed subsets, we can assume that $\mathcal{Y}$ is $\mathbb{Q}$-factorial and $S'$ is still dense. In particular, this implies that for all $s \in S$, $(\mathcal{Y}_s, \mathcal{B}_s)$ is $\epsilon$-klt.
**Claim**. *Decomposing $S$ into a finite union of locally closed subsets, we may assume that the restriction map*

$$H^0(\mathcal{Y}, O_{\mathcal{Y}}(lI(K_{\mathcal{Y}} + \mathcal{B}))) \to H^0(\mathcal{Y}_s, lI(K_{\mathcal{Y}_s} + \mathcal{B}|_{\mathcal{Y}_s})), \ \forall l > 0 \tag{1}$$

*is surjective at any point $s \in S$, and for any connected component $\bar{S}$ of $S$*

$$h^0(\mathcal{Y}_s, \mathcal{O}_{\mathcal{Y}_s}(lI(K_{\mathcal{Y}_s} + \mathcal{B}|_{\mathcal{Y}_s}))) \text{ is independent of } s \in \bar{S}, \ \forall l > 0. \tag{2}$$

*Proof.* Decomposing $S$ into a finite union of locally closed subset, we can assume that there exists a log resolution $\psi \colon (\mathcal{Y}', \overline{\mathcal{B}}) \to \mathcal{Y}$ of $(\mathcal{Y}, \mathcal{B})$, where $\overline{\mathcal{B}} := \psi_*^{-1}\mathcal{B} + E$ and $E$ is the exceptional divisor of $\psi$, and for any $s \in S$, $(\mathcal{Y}'_s, \overline{\mathcal{B}}_s)$ is a log resolution of $(\mathcal{Y}_s, \mathcal{B}_s)$. In particular, for any $s \in S$,

$$H^0(\mathcal{Y}_s, \mathcal{O}_{\mathcal{Y}_s}(l(K_{\mathcal{Y}_s} + \mathcal{B}_s))) = H^0(\mathcal{Y}'_s, \mathcal{O}_{\mathcal{Y}'_s}(l(K_{\mathcal{Y}'_s} + \overline{\mathcal{B}}^{\epsilon}_s))), \ \overline{\mathcal{B}}^{\epsilon} := \psi_*^{-1}\mathcal{B} + (1-\epsilon)E,$$

where in the equation above we have used that the pairs we consider are $\epsilon$-klt. Hence, the properties in (1)-(2) follow from [HMX18, Theorems 1.4, 4.2], since for any $s \in S'$ the pair $(\mathcal{Y}_s, \mathcal{B}_s)$ has a good minimal model, by assumption (3) in the statement of the Lemma. □

If $s \in S$ is a point such that $mI(K_{\mathcal{Y}_s} + \mathcal{B}|_{\mathcal{Y}_s})$ is base point free, then it follows that the natural map

$$\pi^*\pi_* O_{\mathcal{Y}}(mI(K_{\mathcal{Y}} + \mathcal{B})) \to O_{\mathcal{Y}}(mI(K_{\mathcal{Y}} + \mathcal{B})) \tag{3}$$

is surjective along $\mathcal{Y}_s$, and hence along all fibers in a Zariski neighborhood of $s$ in $S$. Hence, up to decomposing $S$ into a finite union of locally closed subsets, we can assume that there exists a positive integer $m'$ such that $m'I(K_{\mathcal{Y}} + \mathcal{B})$ is relatively base point free over $S$. The relative Iitaka fibration for $K_{\mathcal{Y}} + \mathcal{B}$ over $S$

$$\begin{tikzcd} (\mathcal{Y}, \mathcal{B}) \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{X} \arrow[dl] \\ & S & \end{tikzcd}$$

provides the desired bounded family of triples: in fact, given $(Y, B) \in \mathfrak{D}$, if $s \in S$ is such that there exists an isomorphism $h\colon \mathcal{Y}_s \to Y$ and $h_*\mathcal{B}_s = B$, then (1)-(2) imply that there exists also an isomorphism $g\colon \mathcal{X}_s \to X$ and a commutative diagram

$$\begin{tikzcd} \mathcal{Y}_s \arrow[r, "\phi|_{\mathcal{Y}_s}"] \arrow[d, "h"] & \mathcal{X}_s \arrow[d, "g"] \\ Y \arrow[r, "f"] & X. \end{tikzcd}$$

□

Let us recall the notion of degenerateness for effective Weil divisors.

**Definition 2.17.** [Lai11, Definition 2.9] *Let $Y$ be a normal variety and let $f : Y \to X$ be a proper surjective morphism of normal varieties. An effective Weil $\mathbb{R}$-divisor $D$ on $Y$ is said to be*

*(i) $f$-exceptional if* $\operatorname{codim}(\operatorname{Supp}(f(D))) \geq 2$*;*
*(ii) of insufficient fibre type if* $\operatorname{codim}(\operatorname{Supp}(f(D))) = 1$ *and there exists a prime divisor $\Gamma$ on $Y$ such that $\Gamma \not\subseteq \operatorname{Supp}((D))$ and $\operatorname{Supp}(f(\Gamma)) \subset \operatorname{Supp}(f(D))$ has codimension one in $X$.*

In the notation of Definition 2.17, we shall say that the divisor $D$ is $f$-degenerate if is satisfies either one of the conditions stated the definition above. This type of divisors were already studied by Shokurov who referred to them as very exceptional, cf. [Sho03, Definition 3.2].

**Remark 2.18.** Let $f : Y \to X$ be a proper surjective morphism of normal varieties and let $\phi\colon Y' \dashrightarrow Y$ be a birational contraction over $X$, that is, there exists a proper surjective morphism $f'\colon Y' \to X$ and a commutative diagram

$$\begin{tikzcd} Y' \arrow[rr, dashed, "\phi"] \arrow[dr, "f'"'] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

Let us denote by $E'$ the divisorial part of the exceptional locus of $\phi$. If $D$ is $f$-degenerate for $f$ on $Y$, then $D'$ is $f'$-degenerate for $f'$ on $Y'$, where $D'$ is the strict transform of $D$ on $Y'$. Moreover, if $f$ is of relative dimension 1 and $D$ is $f$-degenerate for $f$ on $Y$, then $D' + E'$ is $f'$-degenerate for $f'$ on $Y'$.

**Lemma 2.19.** *Let $\mathfrak{F}$ be a log bounded set of morphisms. Then the set $\mathfrak{F}_{\deg}$ of morphism given by 4-uples $(f, Y, B + D, X)$ such that*

- *there exists $(f, Y, B, X) \in \mathfrak{F}$,*

- *$D$ is reduced and $\mathbb{Q}$-Cartier, and*
- *$D$ is $f$-degenerate*

*is a log bounded set of morphisms.*
*Moreover, if $\mathfrak{F}$ is strongly log bounded, then also $\mathfrak{F}_{\deg}$ is strongly log bounded as well.*

*Proof.* By the log boundedness of $\mathfrak{F}$, there exists a commutative diagram of projective morphisms

$$\begin{tikzcd} (\mathcal{Y},\mathcal{B}) \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{X} \arrow[dl] \\ & S & \end{tikzcd} \tag{4}$$

such that for any 4-uple $(f, Y, B, X) \in \mathfrak{F}$, there is a closed point $s \in S$ and isomorphisms $g\colon \mathcal{X}_s \to X$, $h\colon \mathcal{Y}_s \to Y$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{Y}_s \arrow[r, "\phi|_{\mathcal{Y}_s}"] \arrow[d, "h"] & \mathcal{X}_s \arrow[d, "g"] \\ Y \arrow[r, "f"] & X, \end{tikzcd}$$

and $h_*\mathcal{B}_s = \mathrm{Supp}(B)$. If $\mathfrak{F}$ is strongly log bounded, then the only difference is that we can take $\mathcal{B}$ such that $h_*\mathcal{B}_s = B$.
Decomposing $S$ into a finite union of locally closed subsets and passing to a finite cover of $S$, we may assume that for any $s \in S$ there is a 1-1 correspondence between the irreducible components of $\mathcal{B}$ and those of $\mathcal{B}_s$. Decomposing $S$ into a finite union of locally closed subsets, we may assume that $S$ is smooth.
On $\mathcal{X}$, we define the following two Zariski closed subsets

$$\mathcal{J}_{\mathcal{X}} := \{x \in \mathcal{X} \mid \dim \mathcal{Y}_x > \dim \mathcal{Y} - \dim \mathcal{X}\},$$
$$\mathcal{I}_{\mathcal{X}} := \overline{\{x \in \mathcal{X} \mid \mathcal{Y}_x \text{ is reducible}\}}^{\mathrm{Zar}},$$

where $\overline{\phantom{xx}}^{\mathrm{Zar}}$ denotes the Zariski closure of a set. Furthermore, we define

$$\mathcal{J}_{\mathcal{Y}} := \phi^{-1}(\mathcal{J}_{\mathcal{X}}),\ \mathcal{I}_{\mathcal{Y}} := \phi^{-1}(\mathcal{I}_{\mathcal{X}}),\ \mathcal{L}_{\mathcal{Y}} := \mathcal{J}_{\mathcal{Y}} \cup \mathcal{I}_{\mathcal{Y}},$$

and we consider the variety $\mathcal{J}_{\mathcal{Y}}$ (resp. $\mathcal{J}_{\mathcal{X}}$, $\mathcal{I}_{\mathcal{Y}}$, $\mathcal{I}_{\mathcal{X}}$, $\mathcal{L}_{\mathcal{Y}}$) as a scheme with the reduced scheme structure. We will denote by $\mathcal{J}_{\mathcal{Y},s}$ (resp. $\mathcal{I}_{\mathcal{Y},s}$, $\mathcal{L}_{\mathcal{Y},s}$) the schematic fibre of $\mathcal{J}_{\mathcal{Y}}$ (resp. $\mathcal{I}_{\mathcal{Y}}$, $\mathcal{L}_{\mathcal{Y}}$) at $s \in S$.
It follows from the definition of degeneracy that for any $s \in S$ if an effective reduced divisor $D$ on $\mathcal{Y}_s$ is degenerate with respect to $\phi|_{\mathcal{Y}_s}$ then $\mathrm{Supp}(D) \subset \mathrm{Supp}(\mathcal{L}_{\mathcal{Y},s})$. Decomposing $S$ into a finite union of locally closed subsets, we can assume that $\mathcal{L}_{\mathcal{Y}}$ is flat over $S$. Hence, if $\mathcal{H}$ is a very ample polarization on $\mathcal{Y}$ relatively over $S$ then there exists a positive constant $C = C(\mathfrak{F})$ such that for any $s \in S$

$$\deg_{\mathcal{H}_s} \mathcal{L}_{\mathcal{Y},s} \leq C, \tag{5}$$

where $\deg_{\mathcal{H}_s}$ indicates the degree with respect to the polarization $\mathcal{H}_s$. As $D \subset \mathcal{L}_{\mathcal{Y},s}$ is a reduced Weil divisor on $\mathcal{Y}_S$, then (5) implies that

$$D \cdot \mathcal{H}_s^{\dim \mathcal{Y}_s - 1} = \deg_{\mathcal{H}_s} D \leq deg_{\mathcal{H}_s} \mathcal{L}_{\mathcal{Y},s} \leq C.$$

In view of the theory of Chow varieties, see [Kol96, §1.3], then it follows that there exists a commutative diagram of projective morphisms

$$\begin{tikzcd}(\mathcal{Y}', \mathcal{B}' + \mathcal{D}') \arrow[rr, "\phi'"] \arrow[dr] & & \mathcal{X}' \arrow[dl] \\ & S' & \end{tikzcd} \tag{6}$$

such that for any 4-uple $(f, Y, B + D, X) \in \mathfrak{F}_{\deg}$, there is a closed point $s' \in S'$ and isomorphisms $g' \colon \mathcal{X}'_{s'} \to X$, $h' \colon \mathcal{Y}'_s \to Y$ such that the following diagram commutes

$$\begin{tikzcd}\mathcal{Y}'_s \arrow[r, "\phi'|_{\mathcal{Y}'_s}"] \arrow[d, "h'"] & \mathcal{X}'_s \arrow[d, "g'"] \\ Y \arrow[r, "f"] & X,\end{tikzcd}$$

and $h'_*\mathcal{B}'_s = \mathrm{Supp}(B)$, $h'_*\mathcal{D}'_s = D$. If $\mathfrak{F}$ is strongly log bounded, then the only difference is that we can take $\mathcal{B}'$ such that $h'_*\mathcal{B}'_s = B$. In particular, $h'_*\mathcal{B}'_s + \mathcal{D}'_s = B + D$, which implies that $\mathfrak{F}_{\deg}$ is strongly log bounded. □

Let us recall the following result that will be useful in dealing with degenerate divisors.

**Lemma 2.20.** *Let $(Y, B)$ be a klt pair. Let $f \colon Y \to X$ a projective contraction of normal $\mathbb{Q}$-factorial varieties. Let $D = D_1 + D_2$ be an effective Weil $\mathbb{R}$-divisor on $Y$ such that $D_1$ is $f$-exceptional and $D_2$ is of insufficient fibre type with respect to $f$. Assume that $K_Y + B \sim_{\mathbb{R},Y} 0$.*
*Then, there exists a rational contraction $Y \dashrightarrow \hat{Y}$ over $X$ which contracts exactly the support of $D$.*

*Proof.* Let $Y' \to Y$ be a small $\mathbb{Q}$-factorialization of $Y$. Fix $0 < a \ll 1$ such that the pair $(Y', B' + aD')$ is klt, where $B'$ (resp. $D'$) is the strict transform of $B$ (resp. $D$) on $Y'$. We can apply [Bir12, Theorem 1.8] to obtain the desired result. □

In Section 5, we will study boundedness of families of elliptic Calabi–Yau varieties with bounded bases. In order to prove their boundedness up to flops, we will have to show that we can contract degenerate divisors in log bounded families of morphisms. Using the above lemma, we can ensure that this result holds.

**Proposition 2.21.** *Let $d$ be a positive integer, $\epsilon$ be a positive real number, and let $\mathcal{R} \subset \mathbb{Q} \cap (0, 1)$ be a finite set.*
*Let $\mathcal{D}$ be a log bounded set of log pairs $(Y, B)$ such that*

*(1) $(Y, B)$ is $\mathbb{Q}$-factorial projective, $\epsilon$-klt of dimension $d$,*
*(2) the coefficients of $B$ are in $\mathcal{R}$,*
*(3) $Y$ is a good minimal model for $K_Y + B$,*

*Then, there exists a log bounded set $\mathfrak{F}_{\deg,\min}$ of 4-uples $(f', Y', B', X)$ such that*

*(i) there exists a log pair $(Y, B) \in \mathcal{D}$ such that*

$$X := \mathrm{Proj}(\oplus_{i=0}^{\infty} H^0(Y, \mathcal{O}_Y(i(K_Y + B))))$$

*and $f \colon Y \to X$ is the Iitaka fibration of $K_Y + B$,*

(ii) *there exists an effective divisor $D$ on $Y$ that is $f$-degenerate such that $f'\colon Y' \to X$ is a good minimal model for $D$ over $X$ and $B'$ is the strict transform of $B$ on $X'$.*

Under the assumptions of the proposition, as $K_Y + B \sim_{\mathbb{Q},f} 0$, then $K_Y + B + \lambda D \sim_{\mathbb{Q},f} \lambda D$, for all $\lambda \in \mathbb{R}$, and the existence of a good minimal model for $D$ over $X$ is implied by Lemma 2.20.

*Proof.* By Lemma 2.16, the set $\mathfrak{F}$ of 4-uples $(f, Y, B, X)$ such that

- $(Y, B) \in \mathfrak{D}$, and
- $f\colon Y \to X$ is the Iitaka fibration of $K_Y + B$,

is a strongly log bounded set of morphisms. By Lemma 2.19, any $f$-degenerate divisor $D$ is bounded on $Y$. Hence, the set $\mathfrak{F}_{\mathrm{deg}}$ of 4-uples $(f, Y, B + D, X)$ such that

- $(Y, B) \in \mathfrak{D}$,
- $f\colon Y \to X$ is the Iitaka fibration of $K_Y + B$, and
- $D$ is $f$-degenerate,

is a strongly log bounded set of morphisms and there is a commutative diagram

$$\begin{tikzcd}(\mathcal{Y}, \mathcal{B} + \mathcal{D}) \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{X} \arrow[dl] \\ & S & \end{tikzcd} \tag{7}$$

such that

- $\mathcal{X}$ is the Iitaka fibration of $K_{\mathcal{Y}} + \mathcal{B}$ over $S$,
- for any 4-tuple $(f, Y, B + D, X) \in \mathfrak{F}_{\mathrm{deg}}$ there is a closed point $s \in S$ and isomorphisms $g\colon \mathcal{X}_s \to X$, $h\colon \mathcal{Y}_s \to Y$ such that the following diagram commutes

$$\begin{tikzcd}\mathcal{Y}_s \arrow[r, "\phi|_{\mathcal{Y}_s}"] \arrow[d, "h"] & \mathcal{X}_s \arrow[d, "g"] \\ Y \arrow[r, "f"] & X,\end{tikzcd} \tag{8}$$

  and $h_*(\mathcal{B}_s + \mathcal{D}_s) = B + D$.

Decomposing $S$ into a finite union of locally closed subsets and possibly discarding some components, we may assume that every fibre $\mathcal{Y}_s$ is a variety and that $\mathrm{Supp}(\mathcal{B} + \mathcal{D})$ does not contain any fiber. Decomposing $S$ into a finite union of locally closed subsets and passing to a finite cover of $S$, we may assume that for any $s \in S$ and there is a 1-1 correspondence between the irreducible components of $\mathcal{B} + \mathcal{D}$ and those of $\mathcal{B}_s + \mathcal{D}_s$. Decomposing $S$ into a finite union of locally closed subsets, we may assume that $S$ is smooth; up to shrinking $S$, we can also assume that the set

$$S' := \{s \in S \mid \exists (f, Y, B + D, X) \in \mathfrak{F}_{\mathrm{deg}} \text{ such that } (Y, B + D) \simeq (\mathcal{Y}_s, \mathcal{B}_s + \mathcal{D}_s)\}$$

is Zariski dense in $S$. Applying [FM18, Proposition 2.10] to the pairs $(Y, B)$ in each of the 4-uples $(f, Y, B + D, X) \in \mathfrak{F}'$, we may assume that $\mathcal{Y}$ is $\mathbb{Q}$-factorial, after decomposing $S$ again into a finite union of locally closed subsets.

**Claim**. *Up to decomposing $S$ into a finite union of locally closed subsets and discarding those components that do not contain points of $S'$, we may assume that there exists a positive rational number $\delta$ such that $(Y, B+\delta D)$ is $\frac{1}{2}\epsilon$-klt for any 4-uple $(f, Y, B+D, X) \in \mathfrak{F}'$.*

*Proof of the Claim.* Given a point $s \in S'$, there exists a positive rational number $\delta'$ such that $(\mathcal{Y}_s, \mathcal{B}_s + \delta' \mathcal{D}_s)$ is $\frac{1}{2}\epsilon$-klt. As $\mathcal{Y}$ is $\mathbb{Q}$-factorial, then $(\mathcal{Y}, \mathcal{B}+\delta'\mathcal{D})$ is $\frac{1}{2}\epsilon$-klt in a neighborhood of $\mathcal{Y}_s$ in $\mathcal{Y}$, as being $\frac{1}{2}\epsilon$-klt is an open property. Thus, decomposing $S$ into a finite union of locally closed subsets we can find a uniform choice of $\delta$ such that $(\mathcal{Y}, \mathcal{B}+\delta\mathcal{D})$ is $\frac{1}{2}\epsilon$-klt, which proves the desired claim. By noetherian induction and up to discarding those components of $S$ not containing points of $S'$, we can still assume that $S'$ is dense in $S$ and repeat the same argument, which then concludes the proof of the claim. □

In view of the claim, we can run the $(K_{\mathcal{Y}} + \mathcal{B} + \delta\mathcal{D})$-MMP over $\mathcal{X}$ and this must terminate with a good minimal model by [HX13, Theorem 1.1], since $\mathcal{D}$ is vertical over $\mathcal{X}$ by construction: indeed, $\mathcal{D}|_{\mathcal{Y}_s}$ is vertical over $\mathcal{X}_s$ for any $s \in S'$. Let

$$
\begin{tikzcd}
(\mathcal{Y}, \mathcal{B}+\mathcal{D}) \arrow[rr, dashed] \arrow[dr, "\phi"'] & & (\mathcal{Y}', \mathcal{B}'+\mathcal{D}') \arrow[dl, "\phi'"] \\
& \mathcal{X} \arrow[d] & \\
& S &
\end{tikzcd} \tag{9}
$$

be a good minimal model for this MMP, and by [HX13, Corollary 2.9] the good minimal model in (9) is the outcome of an MMP with scaling of an ample divisor. By decomposing $S$ into a finite union of locally closed subsets and using Noetherian induction, we can assume that at each step of this MMP, the exceptional locus is horizontal over $S$ and that if a given step is a divisorial contraction (resp. a flip), then the same holds along each fibre over $S$. From this, we can conclude that for all $s \in S'$, $(\mathcal{Y}'_s, \mathcal{B}'_s) \to \mathcal{X}_s$ is a good relatively minimal model of $(\mathcal{Y}_s, \mathcal{D}_s)$ over $\mathcal{X}_s$. Hence, the conclusion follows from Lemma 2.20. □

2.7. **Canonical bundle formula.** As our main focus in this paper will be on Mori fibrations of log Calabi–Yau pairs, we collect in this subsection some results about lc-trivial fibrations which can be applied to our setting. For more details, we refer the reader to [Amb04, Amb05], and to [FG12, Theorem 3.1] for the case of real coefficients. For the definition of b-divisor and related notions, we refer the reader to [BZ16].

Given a log pair $(X, B)$ and a lc-trivial (resp. klt-trivial) fibration, that is, a contraction $f : X \to Z$ of normal quasi-projective varieties such that $(X, B)$ is log canonical (resp. klt) at the generic point of $Z$, and $K_X + B \sim_{\mathbb{R}} f^*L$, then we can define a divisor $B_Z$ on $Z$ by posing

$$
B_Z := \sum (1 - l_D) D, \tag{10}
$$

where the sum is taken over every prime divisor $D$ on $Z$, and $l_D$ is the log canonical threshold of $f^*D$ with respect to $(X, B)$ over the generic point of $D$. We can also define a divisor $M_Z$ on $Z$, by posing

$$M_Z := L - (K_Z + B_Z). \tag{11}$$

By these definitions, we get the following formula, usually referred to as the canonical bundle formula,

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z). \tag{12}$$

**Remark 2.22.** It is actually not hard to see that the following more refined version of (12) holds when $B_Z$ is a $\mathbb{Q}$-divisor,

$$r(K_X + B) \sim (f^* r(K_Z + B_Z + M_Z)),$$

where

$$r := \min\{m \in \mathbb{N}_{>0} \mid m(K_F + B|_F) \sim 0, \text{ for a general fibre } F\},$$

cf. [Flo14, Remark 2.10].

By working analogously on any higher model $Z' \to Z$ of $Z$, it is possible to define divisors $B_{Z'}$, $M_{Z'}$ and the collections of these divisors in turn define b-$\mathbb{R}$-divisors $\mathbf{B}$, $\mathbf{M}$ whose traces on $Z$ are $B_Z$, $M_Z$, respectively. The b-divisors $\mathbf{B}$, $\mathbf{M}$ are referred to as the boundary and the moduli part of $f$. In the rest of the paper, we will refer to $B_Z$ as the boundary divisor and to $M_Z$ as the moduli part of the fibration $f$ on $Z$.

When the singularities of the pair $(X, B)$ are mild, then one can describe more precisely the singularities of $(Z, B_Z + M_Z)$ and the positivity properties of $\mathbf{M}$.

**Theorem 2.23.** [FG12, Theorem 3.1] *Let $f\colon X \to Z$ be a contraction of normal varieties and let $(X, B)$ be a log canonical pair that is klt over the generic point of $Z$ and $K_X + B \sim_{f,\mathbb{R}} 0$.*
*Moreover, if $(X, B)$ is klt, then there exists a choice of an effective $\mathbb{R}$-divisor $M \sim_{\mathbb{R}} M_Z$ such that $(Z, B_Z + M)$ is klt.*

**Remark 2.24.** We will use the same notation as in the theorem above. When $B$ is a divisor with rational coefficients and $(X, B)$ is klt, the above result was first proved by Ambro, [Amb05], who also proved that there exists a sufficeintly high birational model $Z'$ of $Z$ on which $\mathbf{M}$ descends to $M_{Z'}$ which is a nef $\mathbb{Q}$-divisor. This is usually summarized by saying that the moduli part $\mathbf{M}$ is a b-nef b-$\mathbb{Q}$-divisor. Later, in [FG14], the same result was extended to the case of lc pairs.
Under such assumption, it is an immediate consequence that $(Z' \to Z, B_Z, M_{Z'})$ is a generalised pair in the sense of § 2.2.
If $B$ is an $\mathbb{R}$-divisor, it is not true in general that $\mathbb{M}$ is going to be a b-nef b-$\mathbb{R}$ divisor, but rather its trace on any model of $X$ will be pseudoeffective.
Nonetheless, by writing $K_X + B$ as a convex sum of rational klt log divisors

$$K_X + B = \sum_{i=1}^{s} r_i(K_X + B_i), \ B_i \ \mathbb{Q}\text{-divisor}, \ K_X + B_i \sim_{\mathbb{Q}} 0 \ \forall i = 1, \dots, s,$$

$$r_i > 0 \ \forall i = 1, \dots, s, \ \sum_{i=1}^{s} r_i = 1,$$

it is possible to write $K_X + B \sim_{\mathbb{R}} f^*(K_Z + C_Z + N)$, where $N$ is the trace on $Z$ of a b-nef b-$\mathbb{R}$-divisor $\mathbf{N}$ obtained as the convex sum of the moduli parts of the log divisors $K_X + B_i$, while $C_Z$ is the convex sum of the boundary parts of the log divisors $K_X + B_i$ effective divisor. On the other hand, this kind of operation is non-canonical and thus there is no way to enforce uniqueness of $C_Z$ and $\mathbf{N}$ cf. [HL19].

It is conjectured that when $(X, B)$ is klt and $B$ has rational coefficients, then the divisor $M_{Z'}$ should be semi-ample. Actually, the following much stronger statement, known as the effective semi-ampleness conjecture of the moduli part, is expected to hold; it first appeared in [PS09, Conjecture 7.13].

**Conjecture 2.25.** *Let $k, r$ be positive integers. There exists a positive integer $m = m(k, r)$ such that for any klt projective pair $(X, B)$, where $B$ has rational coefficients, and any klt-trivial fibration $f: X \to Z$ as in Theorem 2.23, where the relative dimension of $f$ is $k$ and $r$ is the positive integer defined in Remark 2.22, there exists a birational model $Z' \to Z$ where the multiple $mM_{Z'}$ of the moduli b-divisor on $Z'$ descends and is base point free.*

The effective semi-ampleness conjecture remains an hard unsolved problem, but for our purposes we only need the case where $M_Z$ is numerically trivial and the generic fibre of $f$ is smooth: this was settled in [Flo14].

**Theorem 2.26.** [Flo14, Theorem 1.3] *Fix a positive integer $b$. There exists an integer $m = m(b)$ such that for any klt-trivial fibration $f: (X, B) \to Z$ with*

- *$B$ a Weil $\mathbb{Q}$-divisor,*
- *$M_Z \equiv 0$, and*
- *$\dim H^{\dim E}(E, \mathbb{C}) = b$ for a non-singular model $E$ of the cover $E' \to F$ associated to the unique element of $|r(K_X + B)|_F|$ of a general fibre $F$ of $f$, where $r$ is the positive integer defined in Remark 2.22.*

*we have that $mM_Z \sim 0$.*

The statement of Theorem 2.26 is slightly different than that of the effective semi-ampleness conjecture: in fact, while in Theorem 2.26 the integer $m$ depends on the dimension $b$ of the middle cohomology of the resolution of a finite cover of the geometric fiber, in Conjecture 2.25 the integer $m$ depends on the relative dimension $k$ of $f$ and on the integer $r$. While it is clear from the statement of Theorem 2.26 that the integer $b$ is inherently related to the integer $r$, as $E$ is a resolution of the ramified cover of degree $r$ induced by $r(K_F + B|_F) \sim 0$, it is not evident that this result provides a full solution to Conjecture 2.25 when the moduli part is numerically trivial. Nonetheless, when the log pair induced on the general fiber of the klt-trivial fibration $f$ belongs to a bounded family, using Theorem 2.26 and standard techniques in the theory of bounded pairs, we can show that the positive integer $m$ that trivializes the moduli part $M_Z$ can be chosen to be uniformly bounded.

**Corollary 2.27.** *Fix a DCC set $\mathfrak{R} \subset (0, 1) \cap \mathbb{Q}$. Let $\mathfrak{D}$ be a bounded set of klt pairs. Assume that for any pair $(X, B) \in \mathfrak{D}$ the coefficients of $B$ are in $\mathfrak{R}$ and $K_X + B \sim_{\mathbb{Q}} 0$. Then there exists an integer number $m = m(\mathfrak{D}, \mathfrak{R})$ such that for any klt-trivial fibration $f: (X, B) \to Z$ with*

- *$M_Z \equiv 0$, and*
- *the pair $(F, B|_F) \in \mathfrak{D}$, where $F$ is the general fibre of $f$,*

*we can choose a divisor $M$ in the class $M_Z$ with $mM \sim 0$.*

*Proof.* As $\mathfrak{R}$ is DCC, by [HMX14, Theorem 1.5] there exists a finite subset $\mathfrak{R}_0 \subset \mathfrak{R}$ such that the coefficients of pairs in $\mathfrak{D}$ are in $\mathfrak{R}_0$. Moreover, [DCS16, Corollary

2.9] implies that there exists $\epsilon > 0$ such that all pairs in $\mathfrak{D}$ are $\epsilon$-klt. Hence, by Proposition 2.10 we can assume that $\mathfrak{D}$ is strongly log bounded. In particular, there exists a morphism of quasi-projective varieties $\mathcal{Z} \to S$ and a effective $\mathbb{Q}$-divisor $\mathcal{B}$ on $\mathcal{Z}$ such that for any pair $(X, B)$ in $\mathfrak{D}$ there exists $s \in S$ and an isomorphism $f_s\colon \mathcal{Z}_s \to X$ and $\mathcal{B}_s = f_s^* B$. Moreover, up to decomposing $S$ into a disjoint union of finitely many locally closed subsets, we can assume that on each component of $S$ there exists a positive integer $l$ such that $l(K_{\mathcal{Z}} + \mathcal{B}) \sim_S 0$. In fact, constructing a log resolution

$$\begin{tikzcd} \mathcal{Z}' \arrow[rr] \arrow[dr] & & \mathcal{Z} \arrow[dl] \\ & S & \end{tikzcd}$$

of $(\mathcal{Z}, \mathcal{B})$ which is log smooth over the base $S$ as in the proof of 2.10, and using [HMX18, Theorem 4.2] there must exist a positive integer $n$ such that $n(K_X + B) \sim 0$ for any pair in $\mathfrak{D}$. Up to stratifying $\mathfrak{D}$ in the Zariski topology, we can construct a bounded set of varieties $\mathfrak{C}$ whose elements are smooth projective varieties $E$ that are non-singular models of the cover $E' \to X$ associated to the unique element of $|n(K_X + B)|$. The statement of the Corollary then follows by noticing that since $\mathfrak{C}$ is bounded then there exists a natural number $b = b(\mathfrak{C})$ such that for any $E \in \mathfrak{C}$, $h^{\dim E}(E) \leq b$. The conclusion then follows from Theorem 2.26 and noetherian induction. □

Similarly to the classical case, we can define a canonical bundle formula for generalised pairs. Let $(X, B+M)$ be a projective generalised pair, and let $f\colon X \to Z$ be a contraction where $\dim Z > 0$. We shall assume that $(X, B + M)$ is generalised log canonical over the generic point of $Z$ and that $K_X + B + M \sim_{f,\mathbb{Q}} 0$. We shall also fix a divisor $L$ on $Z$ such that $K_X + B + M \sim_{\mathbb{Q}} f^* L$. Given any prime divisor $D$ on $Z$, let $l_D$ be the generalised log canonical threshold of $f^* D$ with respect to $(X, B + M)$ over the generic point of $D$. Then, we define $G_Z := \sum c_D D$, where $c_D := 1 - l_D$ and $N_Z := L - (K_Z + B_Z)$, so that

$$K_X + B + M \sim_{\mathbb{Q}} f^*(K_Z + G_Z + N_Z).$$

The divisor $G_Z$ and $N_Z$ are just the traces on $Z$ of Weil b-divisors $\mathbf{G}$, $\mathbf{N}$ and it was proven in [Fil18] that these two b-divisors induce a structure of generalised pair on the base of a relatively trivial fibration of a generalised pair.

2.8. **The different of a section of a fibration.** In this subsection, we introduce a result that will be used in the proof of Theorem 5.2.

Let us first recall the notion of *different*. Let $\Sigma$ be a reduced divisor and $B$ be an effective divisor on a normal quasi projective variety $Y$. We denote by $\nu\colon \Sigma^\nu \to \Sigma$ the normalization of $\Sigma$. Assuming that $\Sigma$ has no common components with $B$, if $(Y, \Sigma + B)$ is a log pair, then there exists a canonically defined effective $\mathbb{R}$-divisor $\mathrm{Diff}_{\Sigma^\nu}(B)$ on $\Sigma^\nu$, called the *different* of $B$, defined as follows – see [K+92, §16] for details. By taking hyperplane cuts, it is enough to consider the case where $Y$ is a surface. We can find a log resolution of singularities for $(Y, \Sigma)$, $r\colon \bar{Y} \to Y$, such that the strict transform $\tilde{\Sigma}$ of $\Sigma$ coincides with the normalisation $\Sigma^\nu$ of $\Sigma$. Hence, there exists an $\mathbb{R}$-divisor $B_{\bar{Y}}$ on $\bar{Y}$ such that

$$K_{\bar{Y}} + \Sigma^\nu + B_{\bar{Y}} = r^*(K_Y + \Sigma + B), \text{and } r_* B_{\bar{Y}} = B.$$

The different $\mathrm{Diff}_{\Sigma^\nu}(B)$ is then defined as the $\mathbb{R}$-divisor $(B_{\bar{Y}})|_{\Sigma^\nu}$. Moreover, the pair $(\Sigma^\nu, \mathrm{Diff}_{\Sigma^\nu}(B))$ is a log pair which satisfies $K_{\Sigma^\nu} + \mathrm{Diff}_{\Sigma^\nu}(B) = \nu^*((K_Y+\Sigma+B)|_\Sigma)$.

We will now consider a contraction of normal quasi-projective varieties $f\colon Y \to X$ of relative dimension one. We assume that there exists a rational section $s\colon X \dashrightarrow Y$ and denote by $S$ the Zariski closure of $s(X)$ in $Y$.

**Lemma 2.28.** *Let $g\colon T \to B$ be a surjective morphism of normal quasi-projective varieties with $\dim T - \dim B = 1$. Assume that $(T, 0)$ is terminal and that there exists a rational section $s\colon B \dashrightarrow T$. Let $S$ be the Zariski closure of $s(B)$ and let $\nu\colon S^\nu \to S$ be the normalization of $S$. Then, $\mathrm{Diff}_{S^\nu}(0)$ is exceptional over $B$.*

*Proof.* Since $s$ is a rational section, the finite part in the Stein factorization of $g$ is an isomorphism. Hence we can assume that $g$ is a contraction.
Since $T$ is terminal, then it is smooth in codimension two. In particular, $\omega_T(S)$ is locally free at each codimension one point of $S$. Let $P \in S$ be a codimension one point such that $g|_S(P) = Q \in B$ and $Q$ is codimension one. As $T$ is smooth at $P$, it suffices to show that $S$ is normal at $P$, since then $\nu$ would be an isomorphism locally around $P$ and the coefficient of $\mathrm{Diff}_{S^\nu}(0)$ at the codimension one point $\nu^{-1}(P)$ would be 0.
Considering the birational morphism $g|_S \circ \nu\colon S^\nu \to B$, the generic point $P'$ of the strict transform of the closure of $Q$ on $S^\nu$ is the unique codimension one point which is mapped to $Q$, since $B$ is normal quasi-projective. Hence, $g|_S^{-1}(Q) = P$ and, furthermore, $\nu^{-1}(P) = P'$, as $P, P'$, and $Q$ are generic points. Since $B$ is normal quasi-projective, the map $s$ is well defined at $Q$, and $s(Q) = P$ since $g|_S \circ s$ is the identity around $Q$ and $P$ is the only point that dominates $Q$. Hence, there exists a lift $s_\nu\colon B \dashrightarrow S^\nu$ which is well defined at $Q$ and such that $\nu \circ s_\nu = s$ and $s_\nu(Q) = P'$, by construction. But then, since $S^\nu$ is smooth at $P'$ and the map $g|_S \circ \nu \circ s_\nu$ is $\mathrm{Id}_B$ and it is a morphism around $P$, it follows that $s_\nu$ is an isomorphism locally around $Q$, which implies that its differential $ds_\nu : T_{S^\nu}|_{P'} \to T_B|_Q$ is an isomorphism of vector spaces. Since $ds_\nu$ factors as

$$T_{S^\nu}|_{P'} \xrightarrow{d\nu} T_T|_P \longrightarrow T_B|_Q$$

and it is an isomorphism, it follows that $d\nu$ is injective. Combined with the fact that $\nu^{-1}(P) = P'$, we obtain that $\nu$ is an isomorphism locally around $P'$ and $S$ is normal at $P$, which concludes the proof of the lemma. $\square$

## 3. Rationally connected log Calabi–Yau pairs

### 3.1. **Towers of Fano fibrations and boundedness.** 

We briefly recall some important results from [Bir18, DCS16].

Given a klt log Calabi-Yau pair $(X, B)$ with $B > 0$, any run of a $K_X$-MMP

$$X \dashrightarrow X' \to Z$$

terminates with a Mori fibre space structure $f'\colon X' \to Z$, see Theorem 2.2. By the canonical bundle formula, cf. § 2.7, $Z$ carries a structure of log Calabi–Yau pair $(Z, \Gamma)$ as implied by Theorem 2.23. If $K_Z \sim_{\mathbb{Q}} 0$, i.e. $\Gamma = 0$, then we say that the pair $(X, B)$ is of product-type, see [DCS16, Definition 2.23]. Otherwise, $\Gamma > 0$ and, assuming $K_Z$ is $\mathbb{Q}$-factorial, we can run a $K_Z$-MMP in turn and repeat the same analysis as above. The $\mathbb{Q}$-factoriality of $Z$ is not a very strong assumption, as, for example, it is readily implied by $X$ being $\mathbb{Q}$-factorial.

By iterating this procedure and using the so-called two-ray game, see [K+92, Chapter 5], the following description of Calabi-Yau pairs was given in [DCS16, Theorem 3.2].

**Theorem 3.1.** *Let $(X, B)$ be a klt Calabi-Yau pair with $B \neq 0$. Then there exists a birational contraction*

$$\pi \colon X \dashrightarrow X'$$

*to a $\mathbb{Q}$-factorial Calabi-Yau pair $(X', B' = \pi_* B)$, $B' \neq 0$ and a tower of morphisms*

$$X' = X_0 \xrightarrow{p_0} X_1 \xrightarrow{p_1} X_2 \xrightarrow{p_2} \dots \xrightarrow{p_{k-1}} X_k \tag{13}$$

*such that*

*(i) for any $1 \leq i < k$ there exists a boundary $B_i \neq 0$ on $X_i$ and $(X_i, B_i)$ is a klt Calabi-Yau pair,*
*(ii) for any $0 \leq i < k$ the morphism $p_i \colon X_i \to X_{i+1}$ is a Mori fibre space, with $\rho(X_i/X_{i+1}) = 1$, and*
*(iii) either $\dim X_k = 0$, or $\dim X_k > 0$ and $X_k$ is a klt variety with $K_{X_k} \equiv_{\mathbb{Q}} 0$.*

When $\dim X_k > 0$, $K_{X_k} \sim_{\mathbb{Q}} 0$, then we say that $(X, B)$ is of product type, see [DCS16, Definition 2.23].

Using Theorem 3.1, the strategy of [DCS16], and the techniques of [Bir19, Bir16a], Birkar has shown that log Calabi–Yau pairs with bounded singularities admitting a tower of fibration as in (13) are log bounded, provided we assume that the last element of the tower belongs to a bounded family.

**Theorem 3.2.** [Bir18, Theorem 1.4] *Let $d, r$ be natural numbers, $\epsilon, \tau$ be positive real numbers. Consider pairs $(X, B)$ and contractions $f \colon X \to Z$ such that*

- *$(X, B)$ is projective $\epsilon$-lc of dimension $d$,*
- *$K_X + B \sim_{\mathbb{Q}} f^* L$ for some $\mathbb{R}$-divisor $L$,*
- *the coefficients of $B$ are at least $\tau$,*
- *$f$ factors as a sequence of non-birational contractions*

$$X = X_1 \to X_2 \to \cdots \to X_k = Z, \tag{14}$$

- *for each $i$, $-K_{X_i}$ is ample over $X_{i+1}$,*
- *there is a very ample divisor $A$ on $Z$ with $A^{\dim Z} \leq r$, and*
- *$A - L$ is ample.*

*Then the set of such $(X, B)$ forms a log bounded family.*

Combining the two results above, Birkar also showed that boundedness holds for log Calabi–Yau pairs of non-product type.

**Theorem 3.3.** [Bir18, Theorem 1.5] *Let $d$ be a natural number and $\epsilon, \tau$ be positive real numbers. Consider pairs $(X, B)$ with the following properties:*

- *$(X, B)$ is projective $\epsilon$-lc of dimension $d$,*
- *$K_X + B \sim_{\mathbb{R}} 0$,*
- *$B \neq 0$ and its coefficients are $\geq \tau$, and*
- *$(X, B)$ is not of product type.*

*Then the set of such $(X, B)$ is log bounded up to isomorphism in codimension one.*

To obtain this result, it is necessary to bound the singularities of the pair together with the coefficients of the boundary $B$. While the former condition is unavoidable as already clear in the case of singular del Pezzo surfaces, the latter is a technical condition that it should be possible to waive, as predicted by Conjecture 1.5. Indeed, this is what we achieve in Theorem 1.4 at the expense of fixing the torsion index of $K_X+B$. A first step in this direction is represented by Theorem 3.5 below.

As an immediate corollary to Theorem 3.2, we get the following log boundedness result that will be useful in the next subsection.

**Corollary 3.4.** *Fix $d$ and $\Phi \subset [0,1)$ a DCC set of rational numbers. Then the set of klt log CY pairs $(X,B)$ such that*

- *$(X,B)$ is projective klt of dimension $d$,*
- *$K_X+B \sim_{\mathbb{Q}} 0$,*
- *the coefficients of $B$ are in $\Phi$,*
- *the constant map $X \to \{pt.\}$ factors as a sequence of non-birational contractions*

$$X = X_1 \to X_2 \to \cdots \to X_l = \{pt.\}, \tag{15}$$

*and,*

- *for each $i$, $-K_{X_i}$ is ample over $X_{i+1}$.*

*Then the set of such $(X,B)$ forms a log bounded family.*

*Proof.* As $\Phi$ is a DCC subset, then [HMX14, Theorem 1.1] implies that there exists $\epsilon = \epsilon(d,\Phi)$ such that all the pairs $(X,B)$ here considered are $\epsilon$-klt, see [CDCH+18, Lemma 3.12]. The result then follows by applying Theorem 3.2 with $Z$ equal to a point and $l=d$, cf. also [Bir18, §7]. □

3.2. **Boundedness with fixed torsion index.** As we have seen above, there is no boundedness result currently for the case of log Calabi–Yau pairs of product type. On the one hand, this is not surprising, as, for example, among such pairs there are those of the form $(S \times \mathbb{P}^1, p_2^*(0+\infty))$, where $S$ is a K3 surface and $p_2$ is the projection to the second factor: these log Calabi–Yau pairs cannot possibly be bounded, as K3 surfaces are not bounded. On the other hand, Conjecture 1.5 predicts that if the total space of the pair is rationally connected, then even in the product-type case we should expect boundedness. As already mentioned, our aim is to take care of those log Calabi–pairs that are of product-type, as those have yet to be fully understood as far as their boundedness goes. To this end, we prove that when the torsion index of $K_X+B$ is bounded on the total space of a product-type log Calabi–Yau pair $(X,B)$ and $X$ is endowed with a tower of Mori fibre spaces terminating with a K-trivial variety $Z$, then also the torsion index of $K_Z$ is bounded.

**Theorem 3.5.** *Fix $d,l$ positive integers. Consider pairs $(X,B)$ and contractions $f\colon X \to Z$ such that*

- *$(X,B)$, $B>0$ is a klt projective pair of dimension $d$,*
- *$l(K_X+B) \sim 0$,*
- *$f$ factors as a sequence of non-birational contraction*

$$X = X_1 \to X_2 \to \cdots \to X_k = Z, \tag{16}$$

- *for each $i$, $-K_{X_i}$ is ample over $X_{i+1}$, and*
- *$K_Z \equiv 0$*

*Then there exists $m = m(d, l)$ such that $mK_Z \sim 0$.*

*Proof.* As $l(K_X + B) \sim 0$, we can write the canonical bundle formula for $(X, B)$ and $f$, cf. Remark 2.22, as

$$l(K_X + B) \sim f^*l(K_Z + B_Z + M_Z).$$

As $K_Z \equiv 0$, it immediately follows from the definition of $B_Z, M_Z$ that $B_Z = 0 \equiv M_Z$. Since $f$ factors as in (16), the general fibre $(F, B|_F)$ is one of the pairs described in Corollary 3.4. Hence, Corollary 2.27 implies that there exists $m' = m'(d, \Phi)$ such that $m'M_Z \sim 0$. Thus,

$$lm'(K_X + B) \sim 0 \sim f^*lm'(K_Z + M_Z) \sim f^*lm'K_Z.$$

To conclude, it suffices to take $m := lm'$. □

3.3. **Birational transformations of fibered log Calabi–Yau pairs.** In this subsection we collect a few technical results on birational transformations of log Calabi–Yau pairs endowed with a fibration that will be useful in the paper.

**Proposition 3.6.** *Let $(Y, D)$ be a klt pair, $D$ a $\mathbb{Q}$-divisor, and let $f\colon Y \to Z$ be a projective contraction of normal varieties. Assume that $K_Y + D \sim_{f,\mathbb{Q}} 0$ and let $s\colon Z' \to Z$ be a small contraction. Then there exists a $\mathbb{Q}$-factorial klt pair $(Y', D')$ isomorphic to $(Y, D)$ in codimension 1 and a projective contraction of normal varieties $f'\colon Y' \to Z'$.*

*Proof.* Let

$$\begin{tikzcd} Y \arrow[d, "f"'] & \overline{Y} \arrow[l, "p"'] \arrow[d, "\overline{f}"] \\ Z & Z' \arrow[l, "s"'] \end{tikzcd} \tag{17}$$

be a smooth resolution of indeterminacies of the rational map $Y \dashrightarrow Z'$. As $(Y, D)$ is klt, it follows that for $0 < \delta \ll 1$, $K_{\overline{Y}} + \tilde{D} + (1 - \delta)E = p^*(K_Y + D) + F$, where $\tilde{D}$ is the strict transform of $D$, $E$ is the support of the $p$-exceptional divisor and $F$ is an effective and $p$-exceptional divisor whose support coincides with that of $E$. Hence $K_{\overline{Y}} + \tilde{D} + (1 - \delta)E \sim_{\overline{f},\mathbb{Q}} F$. As the diagram in (17) is a resolution of indeterminacies of $Y \dashrightarrow Z'$ and $s$ is a small contraction, it follows that the relative Kodaira dimension of $F$ over $Z'$ is 0. In particular the support of $F$ coincides with the divisorial part of the relative base locus of $K_{\overline{Y}} + \tilde{D} + (1 - \delta)E$ over $Z'$. By [HX13, Theorem 1.1] and [Bir12, Theorem 1.4] a $(K_{\overline{Y}} + \tilde{D} + (1 - \delta)E)$-good minimal model must exist, as the general fibre of $f$ is a good minimal model for the restriction of $K_{\overline{Y}} + \tilde{D} + (1 - \delta)E$ to a general fibre of $\overline{f}$. It follows that the relative $(K_{\overline{Y}} + \tilde{D} + (1 - \delta)E)$-MMP$/Z'$ contracts $F$ and hence it terminates with a projective contraction $f'\colon Y' \to Z'$ such that $K_{Y'} + D' \sim_{f',\mathbb{Q}} 0$, where $D'$ is the strict transform of $\tilde{D}$. As this run of the MMP contracts exactly $F$, which is the exceptional locus of $p$, it follows that $Y$ and $Y'$ are isomorphic in codimension 1. □

The next result shows that we can modify the base $Z$ of a fibration $f\colon Y \to Z$ by means of an isomorphism $Z \dashrightarrow Z'$ in codimension 1 and show that $Z'$ is also the base of an elliptic log Calabi–Yau pair. Contrary to the previous statement, here we assume that the base is $\mathbb{Q}$-factorial.

**Proposition 3.7.** *Let $(Y, D)$ be a klt pair and let $f\colon Y \to Z$ be a projective contraction of normal varieties. Assume that $K_Y + D \sim_{\mathbb{Q}} 0$ and $Z$ is $\mathbb{Q}$-factorial and let $t\colon Z \dashrightarrow Z'$ be a birational contraction of normal projective varieties. Then there exists a $\mathbb{Q}$-factorial klt pair $(Y', D')$ isomorphic to $(Y, D)$ in codimension 1 and a projective contraction of normal varieties $f'\colon Y' \to Z'$.*

A birational map $t\colon Z \dashrightarrow Z'$ is a birational contraction if $t$ is proper and $t^{-1}$ does not contract any divisors.

*Proof.* Let $H'$ be an ample divisor on $Z'$ and let $H$ be its pullback on $Z$. $H$ exists and it is well defined as $t$ is a birational contraction. If we consider the pair $(Y, D + \epsilon f^*H)$, $0 < \epsilon \ll 1$ then $f^*H$ is abundant, since the Kodaira and numerical dimension are invariant by pullback under contraction morphisms. By [GL13, Theorem 4.3], there exists a run of the $(K_Y + D + \epsilon f^*H)$-MMP, $Y \dashrightarrow Y''$ which terminates with a good minimal model $Y'' \to Z'$. Moreover, $Y''$ admits a structure of a log Calabi–Yau pair $(Y'', D'')$, where $D''$ is the strict transform of $D$ on $Y''$. Let $\{E_1, \dots, E_k\}$ be the divisors contracted by the birational contraction $Y \dashrightarrow Y''$. The log discrepancy of any $E_i$ with respect to $(Y, D)$ (or, equivalently, $(Y'', D'')$) is at most 1. Hence, by [BCHM10, Corollary 1.4.3], there exists a model $Y' \to Y''$ of $Y''$ on which the only extracted divisors are the $E_i$. This yields the desired model in the statement of the proposition. □

## 4. Rationally connected K-trivial varieties

In this section we show that the set of $d$-dimensional rationally connected klt projective varieties with torsion canonical bundle are bounded up to flops, if we bound the torsion index, i.e., if we assume that there exists a fixed integer $l$ such that $lK_X \sim 0$. When the dimension of $X$ is 3, this result was implicitly proven in [CDCH+18, Theorem 5.1].

4.1. **Partial resolutions of RC K-trivial varieties and towers structure.** Given a rationally connected klt projective variety $X$ with $K_X \sim_{\mathbb{Q}} 0$, by [BCHM10, Corollary 1.4.3], we can construct a partial resolution $\pi\colon X' \to X$ of $X$ such that

$$K_{X'} + D = \pi^* K_X, \ \lfloor D \rfloor = 0,$$

the divisorial part of the exceptional locus of $\pi$ coincides with the support of $D$, and the pair $(X', D)$ is canonical, $\mathbb{Q}$-factorial with $K_{X'} + D \sim_{\mathbb{Q}} 0$. The above conditions imply that $(X', 0)$ is canonical; thus, as $X'$ is rationally connected, $K_{X'}$ cannot be pseudo-effective and we obtain that

$$D > 0.$$

Moreover, [Fuj11, Theorem 2.3] implies that for any $0 < \epsilon \ll 1$, $X$ is a good minimal model for the $K_{X'} + (1+\epsilon)D$-MMP. If we assume that $lK_X \sim 0$, $l \in \mathbb{N}$, then also $l(K_{X'} + D) \sim 0$ and the coefficients of $D$ belong to the subset $\{\frac{1}{l}, \frac{2}{l}, \dots, \frac{l-1}{l}\}$. Under this assumption, the pair $(X', D)$ is a $\frac{1}{l}$-lc pair.

As $K_{X'}$ is not pseudo-effective, we can apply Theorem 3.1 and there is a crepant birational contraction $(X', D) \dashrightarrow (X'', D')$ to a $d$-dimensional log Calabi–Yau pair $(X'', D')$ such that $l(K_{X''} + D') \sim 0$ and $X''$ is equipped with a non-birational contraction $g\colon Y \to Z$ which can be factored into a sequence of Mori fibrations:

$$(X'', D') = (X_0, D_0) \longrightarrow X_1 \longrightarrow \dots \longrightarrow X_{s-1} \longrightarrow X_s = Z. \tag{18}$$

If the pair $(X_0, D_0)$ is not of product type then we know that $\dim Z = 0$ and $(X'', D')$ belongs to a log bounded family, by Theorem 3.4. If $\dim Z > 0$, then $K_Z \equiv 0$, $Z$ is $\mathbb{Q}$-factorial and rationally connected: Theorem 3.5 then implies that there exists $m = m(d, l)$ such that $mK_Z \sim 0$.

4.2. **Boundedness of RC connected K-trivial varieties with bounded torsion index.** The decomposition introduced in the previous section, suggest that the following result should hold inductively.

**Theorem 4.1.** *Fix positive integers $d, l$. Consider varieties $X$ such that*

- *$X$ is klt projective of dimension $d$,*
- *$X$ is rationally connected, and*
- *$lK_X \sim 0$.*

*Then the set of such $X$ is bounded up to flops.*

*Proof.* We prove the Theorem by induction on $d$. The case $d = 1$ is trivial.
We will use the notation from § 4.1 and prove the inductive step, assuming the theorem holds in dimension at most $d - 1$. For the reader's convenience, we divide the proof into different steps.

**Step 1**: We show that the pair $(X'', D')$ in (18) is log bounded up to flops.
It has already been discussed in § 4.1 that the conclusion holds when $X''$ is not of product type. Hence we are left to prove the case when in (18) $\dim Z > 0$ and $K_Z \sim_{\mathbb{Q}} 0$: we have already noticed that there exists $m = m(d, l)$ such that $mK_Z \sim 0$. By the inductive hypothesis, then it follows that $Z$ is bounded up to flops. Hence, there exists a klt variety $\overline{Z}$ isomorphic to $Z$ in codimension 1, $mK_{\overline{Z}} \sim 0$, and $\overline{Z}$ belongs to a bounded family. By Theorem 2.13, there exists a $\mathbb{Q}$-factorialization of $\overline{Z}$ that also belongs to a bounded family. In particular, replacing $\overline{Z}$ with a $\mathbb{Q}$-factorial model, we can assume that $\overline{Z}$ is $\mathbb{Q}$-factorial, as well. Hence, by repeatedly applying Proposition 3.7 we can assume that we have a tower of Mori fibrations analogous to that in (18)

$$(\overline{X}'', \overline{D}') = (\overline{X}_0, \overline{D}_0) \longrightarrow \overline{X}_1 \longrightarrow \ldots \longrightarrow \overline{X}_{s-1} \longrightarrow \overline{X}_s = \overline{Z},$$

where each $\overline{X}_i$ is $\mathbb{Q}$-factorial and isomorphic in codimension 1 to the corresponding $X_i$ in (18). Hence, as $\rho(X_i/X_{i+1}) = 1$ the same holds for $\rho(\overline{X}_i/\overline{X}_{i+1})$ which implies that $-K_{\overline{X}_i}$ is ample over $\overline{X}_{i+1}$. Finally, Theorem 3.2 implies that $(\overline{X}'', \overline{D}')$ is log bounded.

**Step 2**: We show that the pair $(X', D)$ is log bounded up to flops.
The pairs $(X', D), (X'', D')$, and $(\overline{X}'', \overline{D}')$ are all crepant birational. Let $\{E_1, \ldots, E_j\}$ be the divisors contracted by the birational contraction $X' \dashrightarrow X''$. As $X''$, $\overline{X}''$ are isomorphic in codimension 1, the $E_i$ are exceptional for the rational contraction $X' \dashrightarrow \overline{X}''$, too. The log discrepancy of the $E_i$ with respect to $(X', D), (X'', D')$, and $(\overline{X}'', \overline{D}')$ is the same and it is contained in $(0, 1]$. By [BCHM10, Corollary 1.4.3], there exists a $\mathbb{Q}$-factorial klt log Calabi–Yau pair $(\overline{X}', \overline{D})$ and a morphism $\overline{r} \colon \overline{X}' \to \overline{X}''$ extracting all and only the valuations corresponding to $\{E_1, \ldots, E_j\}$ and such that $K_{\overline{X}'} + \overline{D} = \overline{r}^*(K_{\overline{X}''} + \overline{D}')$. By construction, $(X', D)$ and $(\overline{X}', \overline{D})$ are isomorphic in codimension 1. Once again, Theorem 2.13 implies that also $(\overline{X}', \overline{D})$ is log bounded, thus terminating the proof of this part.

**Step 3**: We show that $X$ is bounded up to flops.
We know that the pair $(\overline{X}', \overline{D})$ is log bounded and it is isomorphic in codimension 1 to $(X', D)$. Moreover, the Kodaira dimension of $K_{\overline{X}'} + \lceil \overline{D} \rceil$ is 0 as $K_X \sim_{\mathbb{Q}} 0$; thus, a minimal model for the $(K_{\overline{X}'} + (1+\epsilon)\overline{D})$-MMP on $\overline{X}'$, for $\epsilon \ll 1$, will be a $\mathbb{Q}$-factorial variety $\overline{X}$ with $lK_{\overline{X}} \sim 0$. By construction, $X$ and $\overline{X}$ are isomorphic in codimension 1. If we can prove that $\overline{X}$ belongs to a bounded family then the theorem follows at once. But this is just a consequence of Proposition 4.2 below. $\square$

**Proposition 4.2.** *Fix a DCC set $\mathfrak{R} \subset (0,1) \cap \mathbb{Q}$. Let $\mathfrak{D}$ be a bounded set of klt pairs. For any pair $(X, B) \in \mathfrak{D}$ assume that:*

- *$B > 0$ and its coefficients belong to $\mathfrak{R}$,*
- *$K_X + B \sim_{\mathbb{Q}} 0$,*
- *the Kodaira dimension of $-K_X$ is zero, and*
- *there exists a good minimal model $X_1$ for $K_X + (1+\epsilon)B$, $0 < \epsilon \ll 1$.*

*Then the set $\mathfrak{D}_1 := \{X_1 \mid (X, B) \in \mathfrak{D},\ X_1$ is a good minimal model for $(K_X + (1+\epsilon)B),\ 0 < \epsilon \ll 1\}$ is bounded up to flops.*

In the hypotheses of Proposition 4.2 the minimal model $X_1$ is a Calabi–Yau variety, $K_{X_1} \sim_{\mathbb{Q}} 0$, which is klt but non-canonical. Moreover, under these assumptions, the minimal model $X_1$ is independent of the choice of $1 < \epsilon \ll 1$.

*Proof.* As $\mathfrak{R}$ is DCC, by [HMX14, Theorem 1.5] there exists a finite subset $\mathfrak{R}_0 \subset \mathfrak{R}$ such that the coefficients of pairs in $\mathfrak{D}$ are in $\mathfrak{R}_0$. Moreover, [DCS16, Corollary 2.9] implies that there exists $\epsilon > 0$ such that all pairs in $\mathfrak{D}$ are $\epsilon$-klt. By Proposition 2.10, there exists a pair $(\mathcal{X}, \mathcal{B})$ together with a projective morphism of quasi-projective varieties $\mathcal{X} \to S$ such that for any pair $(X, B) \in \mathfrak{D}$ there exists $s \in S$ and an isomorphism $h_s \colon \mathcal{X}_s \to X$ and $\mathcal{B}|_{\mathcal{Z}_s} = h_s^* B$. By taking the Zariski closure of the points $s \in S$ corresponding to pairs in $\mathfrak{D}$, and decomposing that into a disjoint union of finitely many locally closed subsets, we can assume that $S$ is smooth and the points on $S$ corresponding to pairs in $\mathfrak{D}$ are dense. By [HX15, Proposition 2.4] up to substituting $S$ with a Zariski dense open, we can assume that $(\mathcal{X}, \mathcal{B})$ is klt and $\mathbb{Q}$-factorial. Restricting to a Zariski open set of $S$ and/or decomposing $S$ into a disjoint union of finitely many locally closed subsets are operations that we are allowed to perform: by noetherian induction, these operations can be performed only finitely many times, hence they do not affect any boundedness argument.
By passing to a sufficiently high log resolution

$$\begin{tikzcd} \mathcal{Y} \arrow[rr, "\sigma"] \arrow[dr] & & \mathcal{X} \arrow[dl] \\ & S & \end{tikzcd}$$

of $(\mathcal{X}, \mathcal{B})$, there exist effective divisors $\mathcal{E}, \mathcal{F}$ on $\mathcal{Y}$ such that $K_{\mathcal{Y}} + \mathcal{E} = \sigma^*(K_{\mathcal{X}} + \mathcal{B}) + \mathcal{F}$, with $\sigma_* \mathcal{E} = \mathcal{B}$ and $\mathcal{F}$ is $\sigma$-exceptional. Up to decomposing $S$ in a disjoint union of finitely many locally closed subsets and possibly passing to a higher model $\mathcal{Y}$, we can assume that $(\mathcal{Y}, \mathcal{E})$ is log smooth over $S$. Moreover, we can also assume that the support of $\mathcal{E}$ contains all $\sigma$-exceptional divisors and that $\lfloor \mathcal{E} \rfloor = 0$.
Let us notice that for any sufficiently divisible $m \in \mathbb{N}$, and for $0 < \epsilon \ll 1,\ \epsilon \in \mathbb{Q}$,

$$h^0(\mathcal{Y}_s, \mathcal{O}_{\mathcal{Y}_s}(m(K_{\mathcal{Y}_s} + (1+\epsilon)\mathcal{E}|_{\mathcal{Y}_s}))) = h^0(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s}(m(K_{\mathcal{X}_s} + (1+\epsilon)\mathcal{B}|_{\mathcal{X}_s}))).$$

By [HMX18, Theorem 4.2],

$$h^0(\mathcal{Y}_s, \mathcal{O}_{\mathcal{Y}_s}(m(K_{\mathcal{Y}_s} + (1+\epsilon)\mathcal{E}|_{\mathcal{Y}_s})))$$

is a constant function of $s \in S$; in particular $\kappa(\mathcal{Y}_s, \mathcal{O}_{\mathcal{Y}_s}((K_{\mathcal{Y}_s} + (1+\epsilon)\mathcal{E}|_{\mathcal{Y}_s}))) = 0$, $\forall s \in S$. Thus, for any $s \in S$, $K_{\mathcal{Y}_s} + (1+\epsilon)\mathcal{E}|_{\mathcal{Y}_s}$ is not movable. Then [HMX18, Theorem 1.2] implies that for any $s \in S$ a good minimal model for $K_{\mathcal{Y}_s} + (1+\epsilon)\mathcal{E}|_{\mathcal{Y}_s}$ exists. If we run the $K_{\mathcal{Y}} + (1+\epsilon)\mathcal{E}$-MMP over $S$, [HMX18, Theorem 1.2] implies that a good minimal $\mathcal{Z} \to S$ model must exist and by [HMX18, Lemma 6.1] $\mathcal{Z}_s$ will yield a good minimal model for each pair $(\mathcal{Y}_s, (1+\epsilon)\mathcal{E}|_{\mathcal{Y}_s})$, thus concluding the proof. As $\mathcal{Z}_s$ is a good minimal model for $K_{\mathcal{Y}_s} + (1+\epsilon)\mathcal{E}|_{\mathcal{Y}_s}$ it follows from the definition of good minimal model that $X_1$ and $\mathcal{Z}_s$ are isomorphic in codimension one. □

The following immediate corollary of the Theorem 4.1 extends [CDCH+18, Cor. 5.2] to any dimension. It relates the boundedness of rationally connected Calabi–Yau manifolds to the boundedness of the torsion index of the canonical divisor inside the class group.

**Corollary 4.3.** *Fix a positive integer $d$. Let $\mathfrak{C}$ be the set of varieties $X$ satisfying the following hypotheses:*

*(1) $X$ is a klt projective variety of dimension $d$,*
*(2) $X$ is rationally connected, and*
*(3) $K_X \equiv 0$.*

*Then, $\mathfrak{C}$ is bounded up to flops if and only if there exists a positive integer $l = l(d)$ such that $lK_X \sim 0$ for any $X \in \mathfrak{C}$.*

*Proof.* If there exists $l = l(d)$ such that $lK_X \sim 0$ for any $X \in \mathfrak{C}$, then the boundedness up to flops of $\mathfrak{C}$ is a consequence of Theorem 4.1.
Let us assume that $\mathfrak{C}$ is bounded up to flops. Hence, there exists $h\colon \mathcal{Z} \to S$ a projective morphism of schemes of finite type such that for each $X \in \mathfrak{C}$ is isomoprhic in codimension one to $\mathcal{Z}_s$ for some closed point $s \in S$ and, moreover, $\mathcal{Z}_s$ is normal. As $lK_{\mathcal{Z}_s} \sim 0$ if and only if $lK_X \sim 0$, it suffices to show that there exists a positive integer $l$ such that $lK_{\mathcal{Z}_s} \sim 0$ for any $s \in S$. Moreover, by [Bir19, Lemma 2.25] we can assume that there exists $I = I(\mathfrak{C})$ such that $IK_{\mathcal{Z}_s}$ is Cartier for any $s \in S$.
Decomposing $S$ into a finite union of locally closed subsets and possibly discarding some components, we may assume that $S$ is smooth and that every fibre $\mathcal{Y}_s$ is a normal variety. Up to shrinking $S$, we can also assume that the set

$$S' := \{s \in S \mid \exists X \in \mathfrak{C} \text{ such that } X \text{ is isomorphic in codimension one to } \mathcal{Z}_s\}$$

is Zariski dense in $S$. Furthermore, by [HX13, Proposition 2.4], decomposing $S$ into a finite union of locally closed subsets and discarding those components that do not contain points of $S'$, we can assume that $K_{\mathcal{Z}}$ is $\mathbb{Q}$-Cartier.

**Claim**. *Up to decomposing $S$ into a finite union of locally closed subsets, we may assume that for any connected component $\bar{S}$ of $S$ $h^0(\mathcal{Z}_s, \mathcal{O}_{\mathcal{Z}_s}(mI(K_{\mathcal{Z}_s})))$ is independent of $s \in \bar{S}$, for all $m > 0$.*

*Proof.* Up to decomposing $S$ into a finite union of locally closed subset, we may assume that there exists a log resolution $\psi\colon (\mathcal{Z}', \mathcal{B}') \to \mathcal{Z}$ of $(\mathcal{Z}, 0)$, where $\mathcal{B}'$ is the exceptional divisor of $\psi$. Furthermore, we may also assume that for any $s \in S$, $(\mathcal{Y}'_s, \mathcal{B}'_s)$ is a log resolution of $\mathcal{Y}_s$. In particular, for any $s \in S$, for all $m > 0$

$$H^0(\mathcal{Y}_s, \mathcal{O}_{\mathcal{Y}_s}(m(K_{\mathcal{Y}_s} + \mathcal{B}_s))) = H^0(\mathcal{Y}'_s, \mathcal{O}_{\mathcal{Y}'_s}(mK_{\mathcal{Y}'_s})).$$

The conclusion then follows from [HMX18, Theorem 4.2]. □

At this point, we discard those connected components of $S$ that do not contain points of $S'$. By construction then, given a connected component $S_i$ of $S$, there is a point $s \in S' \cap S_i$; if $m_i$ is a positive integer such that $m_i K_{\mathcal{Z}_s} \sim 0$, the claim above implies that

$$h^0(\mathcal{Y}'_s, \mathcal{O}_{\mathcal{Y}'_s}(m_i K_{\mathcal{Y}'_s})) = 1, \text{ for all } s \in S_i. \tag{19}$$

We define $l$ to be the maximum of the positive integers $m_i$ just defined. This is well defined since $S$ has only finitely many connected components, being of finite type. As for any $s \in S'$, $K_{\mathcal{Z}_s} \sim_{\mathbb{Q}} 0$, then (19) implies that $lK_{\mathcal{Z}_s} \sim 0$. □

Moreover, exactly as in [CDCH+18, §5], we expect that there will be a tight connection be

## 5. Elliptic Calabi–Yau varieties with a rational section

In this section we will prove some results regarding the geometric structure of elliptic Calabi–Yau varieties and of the bases of such fibrations that will be needed in the proof of the main theorems of this paper.

5.1. **Birational geometry of bases of fibered Calabi–Yau varieties.** In this subsection we prove that the bases of a smooth Calabi–Yau endowed with a non-birational fibration are rationally connected.

Kollár and Larsen, [KL09, Theorem 3], proved that a simply connected smooth (or canonical) K-trivial variety $Y$ endowed with a dominant rational map $m\colon Y \dashrightarrow X$ to a non-uniruled variety is isomorphic to a product $Y \simeq Y_1 \times Y_2$, where the map $m$ restricted to $Y_2$ induces a dominant generically finite map $Y_2 \dashrightarrow X$. In the case of a fibered Calabi–Yau manifold, such a product cannot exist, as we now show. We remind the reader that in this paper a smooth Calabi–Yau variety is assumed to be simply connected.

**Corollary 5.1.** *Let $Y$ be a smooth projective Calabi–Yau variety. Assume that $Y$ is endowed with a morphism $f\colon Y \to X$ of relative dimension $0 < d < \dim Y$. Then $X$ is rationally connected.*

*Proof.* Let us assume that $X$ were not rationally connected. Considering the MRC fibration of $X$, $X \dashrightarrow W$, its image $W$ is a non-uniruled variety of positive dimension. Hence, by the results of Kollár and Larsen $Y$ would have to be isomorphic to a product $Y \simeq Y_1 \times Y_2$, with $\dim Y_1, \dim Y_2 > 0$. As $K_Y \sim 0$, Künneth formula for Hodge numbers implies that $h^0(K_{Y_1}) = 1 = h^0(K_{Y_2})$. As $h^0(K_{Y_1}) = h^{\dim Y_1, 0}(Y_1)$, pulling back from $Y_1$ to $Y$, we get that $h^{\dim Y_1, 0}(Y) \neq 0$, which contradicts $Y$ being Calabi-Yau, since $\dim Y_1 < \dim Y$. □

5.2. **Boundedness for elliptic Calabi–Yau varieties over bounded bases.** The main goal of this subsection is to prove a generalization of [DCS16, Theorem 1.3]. In what follows, by a rational section $X \dashrightarrow Y$ of a contraction $f \colon Y \to X$, we will mean a rational map $s \colon X \dashrightarrow Y$ such that $f \circ s$ is the identity on $X$. With this notation, we will denote by $S$ the Zariski closure of the image of $X$ via $s$ and by $S^\nu$ the normalization $\nu \colon S^\nu \to S$.

**Theorem 5.2.** *Fix positive integers $n, d, l$. Then the set of varieties $Y$ such that*

*(1) $Y$ is a klt variety of dimension $n$,*
*(2) $lK_Y \sim 0$*
*(3) $Y \to X$ is an elliptic fibration with a rational section $X \dashrightarrow Y$,*
*(4) there exists a very ample Cartier divisor $A$ on $X$ with $A^{n-1} \leq d$*

*is bounded up to flops.*

The existence of a very ample Cartier divisor $A$ with bounded volume implies that the set of bases of the elliptic fibrations is bounded.

*Proof.* For the reader's convenience, we divide the proof in several steps.

**Step** 0. *In this step we show that there exists a $\mathbb{Q}$-factorialization $X'$ of $X$ which belongs to a bounded family and we construct an elliptic fibration $Y' \to X'$ with $Y'$ $\mathbb{Q}$-factorial and isomorphic to $Y$ in codimension 1.*
As $X$ is the base of an elliptic fibration and $Y$ is klt, then $X$ supports an effective divisor $\Gamma$ such that $(X, \Gamma)$ is klt, see Theorem 2.23. By [HMX14, Theorem 1.1] and [PS09, Theorem 8.1], it is possible to choose $\Gamma$ so that its coefficients vary in a DCC set $I = I(n)$. Hence, by [Bir19, Lemma 2.48] it follows that there exists $\epsilon = \epsilon(n, I)$ such that $(X, \Gamma)$ is $\epsilon$-lc. As the DCC set defined $I$ above only depends from $n$ and $\epsilon$ is a function of $n, I$, then we conclude that $\epsilon$ only depends on $n$.
Hence, by [BCHM10, Corollary 1.4.3] $X$ admits a small $\mathbb{Q}$-factorialization $X' \to X$. $X'$ belongs to a bounded family, by Theorem 2.13. In particular, there exists an integer $d' = d'(n, d, l)$ and a very ample Cartier divisor $A'$ on $X'$ such that $A'^{n-1} \leq d'$. Applying Proposition 3.6 with $D = 0$, we see that there exists a $\mathbb{Q}$-factorial K-trivial klt variety $Y'$ isomorphic to $Y$ in codimension 1 and an elliptic fibration $f' \colon Y' \to X'$. It is immediate to see that the strict transform $S'$ of $S$ on $Y'$ is a rational section of $f'$.
We will denote by $A''$ the divisor $A'' := (4n+4)A'$.

**Step** 1. *In this step, we show that it suffices to prove that $Y'$ is bounded up to flops.*
As $Y$ and $Y'$ are K-trivial varieties that are klt and isomorphic in codimension 1, the claim is straightforward.

**Step** 2. *In this step, we show that there exists a log bounded set of morphisms $\mathfrak{F}$ and a 4-uple $(f_{Y''}, Y'', 0, X') \in \mathfrak{F}$ together with a birational contraction $Y'' \dashrightarrow Y'$ over $X'$ that is crepant with respect to $(Y', 0)$.*
The proof of Step 2 is divided into several steps (Step 2.1-2.6). We start by introducing some notation.
Let $(Y^t, \Delta^t)$ be a terminalization $t \colon Y^t \to Y'$ of $(Y', 0)$, $K_{Y^t} + \Delta^t = t^* K_{Y'} \sim_{\mathbb{Q}} 0$. We denote with $S^t$ the strict transform of $S'$ on $Y^t$. As $Y'$ is klt and $lK_{Y'} \sim 0$, then $\Delta^t \geq 0$ and its positive coefficients belong to the set $\{\frac{1}{l}, \frac{2}{l}, \dots, \frac{l-1}{l}\}$. It follows from

Definition 2.17 that the exceptional divisor $E$ of $t$ is degenerate for the composition $f' \circ t$, cf. Remark 2.18. The support of $E$ consists of the support of $\Delta$ together with divisors of log discrepancy one for $(Y, 0)$.

**Step 2.1** *In this step, we show that there exists a $\mathbb{Q}$-factorial projective variety $Z$ which is isomorphic in codimension 1 to $Y^t$, such that $(Z, 0)$ is terminal and the strict transform $S_Z$ of $S^t$ is big and nef over $X'$.*
Fix $0 < \epsilon \ll 1$ and run the $(K_{Y^t} + \Delta^t + \epsilon S^t)$-MMP over $X'$. As $(Y^t, \Delta^t + \epsilon S^t)$ is klt and $K_{Y^t} + \Delta^t + \epsilon S^t \sim_{\mathbb{R}} \epsilon S^t$ is relatively big over $X'$, this run of the MMP must terminate with a minimal model $f_Z \colon Z \to X'$ for $K_{Y^t} + \Delta^t + \epsilon S^t$. Denoting by $\Delta_Z$ (resp. $S_Z$) the strict transform on $Z$ of $\Delta^t$ (resp. $S^t$), it follows that $S_Z$ is nef and big over $X'$, since $K_{Y^t} + \Delta^t \sim_{\mathbb{Q}} 0$ and $S^t$ is horizontal over $X'$. Moreover, as $(Y^t, \Delta^t)$ is terminal and $K_{Y^t} + \Delta^t + \epsilon S^t \equiv \epsilon S^t$, it follows that every step of this MMP is a $(K_{Y^t} + \Delta^t)$-flop and $Z$ will also be terminal as no divisor is contracted. We denote by $\Delta_Z$ the strict transform of $\Delta^t$ on $Z$.

**Step 2.2**. *In this step, we show that we can run the $(K_Z + S_Z)$-MMP over $X'$ with scaling of an ample divisor and that terminates with a minimal model $f_{Z_1} \colon (Z_1, S_{Z_1}) \to X'$ and $(Z_1, S_{Z_1})$ is plt, where $S_{Z_1}$ denotes the strict transform of $S_Z$ on $Z_1$. We prove that each step of this MMP must be a $K_Z + \Delta_Z$-flop and that $(Z_1, \Delta_1)$ is also terminal, where $\Delta_1$ denotes the strict transform of $\Delta_Z$ on $Z_1$.*
As $S_Z$ is big and nef over $X'$, then it is relatively semiample, by the Base point free theorem [KM98, Theorem 3.3], as $\mu S_Z \sim_{\mathbb{R}} K_Z + \Delta_Z + \mu S_Z$. Thus, we can choose $\bar{S} \sim_{f_Z, \mathbb{Q}} S_Z$ such that $(Z, \bar{S})$ is terminal, by Bertini's theorem, since $(Z, 0)$ is terminal. Hence, we can run the $(K_Z + \bar{S})$-MMP over $X'$ with scaling and this must terminate with a $\mathbb{Q}$-factorial minimal model $f_{Z_1} \colon Z_1 \to X'$, since $\deg((K_Z + \bar{S})|_F) = 1$ along a general fibre $F$ of $f_Z$.
As the pair $(Z, \bar{S})$ is terminal and $Z_1$ is a minimal model for the $K_Z + \bar{S}$-MMP, then $(Z_1, 0)$ is terminal as well. It follows that $K_{Z_1} + S_{Z_1}$ is also big and nef over $X'$, as $S_Z \sim_{\mathbb{Q}} \bar{S}$. We will denote by $\nu_{S_{Z_1}} \colon S^{\nu}_{Z_1} \to S_{Z_1}$ the normalization of $S_{Z_1}$. By Lemma 2.28, as $S_{Z_1}$ is a rational section for $f_{Z_1}$ and $Z_1$ is terminal, it follows that $(f_{Z_1} \circ \nu_{S_{Z_1}})_* \operatorname{Diff}(0) = 0$. The negativity lemma then implies that

$$K_{S^{\nu}_{Z_1}} + \operatorname{Diff}(0) = (f_{Z_1} \circ \nu_{S_{Z_1}})^*(K_{X'}) - E, \tag{20}$$

where $E$ is an effective divisor exceptional over $X'$. Hence, the pair $(S^{\nu}_{Z_1}, \operatorname{Diff}(0))$ is klt, since $(X', 0)$ is. By inversion of adjunction, $S_{Z_1}$ is the only lc center of the pair $(Z_1, S_{Z_1})$: this automatically implies that $S_{Z_1}$ is normal.

**Step 2.3**. *Let $G_1 \in |f^*_{Z_1}(4n+4)A'|$ be a general member. In this step, we show that for $\lambda \in (0, 1)$, $(Z_1, \lambda(S_{Z_1} + G_1))$ is $(1-\lambda)$-klt and $K_{Z_1} + S_{Z_1} + G_1$ is big and nef.*
$(Z_1, S_{Z_1} + G_1)$ is dlt: in fact, $(Z_1, S_{Z_1})$ is plt, and $G_1$ is a general element of a base point free linear system; thus we can conclude by Bertini's Theorem for discrepancies. As discrepancies are linear functions of the divisorial part of a pair, and $(Z_1, 0)$ is terminal, the pair $(Z_1, \lambda(S_{Z_1} + G_1))$ is $(1-\lambda)$-klt. As $K_{Z_1} + S_{Z_1}$ is nef and big$/X'$ and $G_1 \sim f^*_{Z_1} A''$ with $A'' = (4n+4)A'$ Cartier and very ample, the cone theorem implies that $K_{Z_1} + S_{Z_1} + G_1$ is nef and big.

**Step 2.4**. *In this step, we show that there exists $C' = C'(n,d,l)$ such that $(K_{Z_1} + S_{Z_1} + (n-2)G_1)^n \leq C'$.*
Let us denote by
$$\alpha = G_1,\ \beta = K_{Z_1} + S_{Z_1} + (n-2)G_1.$$
Then,
$$(\alpha^{n-2}\beta^2)^{n-1} \geq (\alpha^{n-1}\beta)^{n-2}\beta^n,$$
see [Laz04, Theorem 1.6.3.(i)]. Hence, to prove that the statement of Step 2.4 holds it suffices to show that there exists positive constants $C_1 = C_1(n,d,l)$, $C_2 = C_2(n,d,l)$ such that
$$(\alpha^{n-2}\beta^2)^{n-1} \leq C_1 \text{ and } (\alpha^{n-1}\beta)^{n-2} \geq C_2.$$
To compute $C_2$, let us note that $0 < \alpha^{n-1}\beta$ since $0 \neq G_1^{n-1}$ is a movable class while $K_{Z_1} + S_{Z_1} + G_1$ is big and nef, cf. [BDPP13, Theorem 0.2]. Moreover,
$$\begin{aligned} & 0 < \alpha^{n-1}\beta = \\ & G_1^{n-1} \cdot (K_{Z_1} + S_{Z_1} + (n-2)G_1) = \\ & G_1^{n-1} \cdot (K_{Z_1} + S_{Z_1}) = && [\text{since } G_1^n = 0] \\ & G_1^{n-1} \cdot S_{Z_1} - G_1^{n-1} \cdot \Delta_{Z_1}. && [\text{since } K_{Z_1} \equiv -\Delta_{Z_1}] \end{aligned} \tag{21}$$
As $G_1$ is Cartier and $G_1 \sim f_{Z_1}^*(4n+4)A'$,
$$G_1^{n-1} \cdot S_{Z_1} = (4n+4)^{n-1}A'^{n-1} \in \mathbb{N}_{>0}, \tag{22}$$
where we used the fact that $S_{Z_1} \to X'$ is a birational map and $A'$ is very ample and Cartier on $X'$. Moreover, as the coefficients of $\Delta_{Z_1}$ belong to $\{\frac{1}{l}, \frac{2}{l}, \dots \frac{l-1}{l}\}$, we can write $\Delta_{Z_1}$ as the sum of its prime components as follows
$$\Delta_{Z_1} = \sum_{j=1}^{s} c_j D_j,\ c_j \in \{\frac{1}{l}, \frac{2}{l}, \dots \frac{l-1}{l}\},\ D_j \text{ prime divisor } \forall j = 1, \dots, s.$$
Hence, as $G_1$ is Cartier and nef, then
$$G_1^{n-1} \cdot \Delta_{Z_1} = \sum_{j=1}^{s} c_j (G_1^{n-1} \cdot D_j) \text{ and } G_1^{n-1} \cdot D_j \in \mathbb{N}. \tag{23}$$
By putting together (21)-(23), we obtain that
$$\begin{aligned} & 0 < \alpha^{n-1}\beta = \\ & G_1^{n-1} \cdot S_{Z_1} - G_1^{n-1} \cdot \Delta_{Z_1} \ \in \frac{1}{l}\mathbb{N}_{>0}. \end{aligned} \tag{24}$$
Hence, it suffices to take $C_2(n,d,l) := \frac{1}{l}$.
We now show the existence of the constant $C_1 = C_1(n,d,l)$ such that
$$(\alpha^{n-2}\beta^2)^{n-1} \leq C_1.$$
Let $\bar{A} \subset X'$ be a sufficiently general curve which is a complete intersection of $n-2$ elements of $|A''|$. In particular, $\bar{A}$ belongs to a bounded family.
Since $X'$ is normal, $\bar{A}$ is a smooth curve and by the adjunction formula
$$\begin{aligned} \deg_{\bar{A}} K_{\bar{A}} = & (K_{X'} + (n-2)A'') \cdot (A'')^{n-2} \\ \leq & (n-2)A''^{n-1} && [\text{since } K_{X'} \geq 0 \text{ and } A'' \text{ is ample}] \\ \leq & d'(n-2)(4n+4)^{n-1} && [A'' = (4n+4)A',\ \mathrm{Vol}(A') \leq d']. \end{aligned} \tag{25}$$

Let us define $T := Z_1 \times_{X'} \bar{A}$ and $S_T := S_{Z_1}|_T$. As $T$ is the complete intersection of $n-2$ sufficiently general divisors in a base point free linear system and $(Z_1, 0)$ is terminal, Bertini's theorem and adjunction imply that $T$ is smooth and $(T, S_T)$ is plt. Moreover, $T$ is an elliptic surface over $\bar{A}$, by construction admitting the section $S_T$. We will denote by $F$ the numerical class of a fiber of the morphism $f_T := f_{Z_1}|_T \colon T \to \bar{A}$. With this notation, then

$$\alpha^{n-2}\beta^2 = (K_{Z_1} + S_{Z_1} + (n-2)G_1)^2 \cdot T = ((K_{Z_1} + S_{Z_1} + (n-2)G_1)|_T)^2. \tag{26}$$

By the adjunction formula,

$$K_T =(K_{Z_1} + (n-2)G_1)|_T. \tag{27}$$

Hence,

$$\begin{aligned}
K_T =&(K_{Z_1} + (n-2)G_1) \cdot T && [(27)]\\
=&K_{Z_1} \cdot T + (n-2)(G_1 \cdot T)\\
=&K_{Z_1} \cdot G_1^{n-2} + (n-2)(G_1 \cdot T) && [T = f^*_{Z_1}\bar{A} \equiv f^*_{Z_1}A''^{n-2},\ G_1 \sim f^*_{Z_1}A''] \qquad (28)\\
\equiv& -\Delta_{Z_1} \cdot G_1^{n-2} + (n-2)(A''^{n-1})F && [K_{Z_1} + \Delta_{Z_1} \sim_{\mathbb{Q}} 0,\ G_1|_T \equiv (A''^{n-1})F]\\
\leq& d'(n-2)(4n+4)^{n-1}F && [-\Delta_{Z_1} \cdot G_1^{n-2} \leq 0 \text{ and}\\
& && A''^{n-1} \leq d'(4n+4)^{n-1}]
\end{aligned}$$

Thus, we can rewrite (26), as

$$\alpha^{n-2}\beta^2 = (K_T + S_T)^2 \leq (K_T + S_T) \cdot (d'(4n+4)^{n-1}(n-2)F + S_T).$$

As the general fibre of $T \to \bar{A}$ has genus 1 and $S_T$ is a section, $(K_T + S_T) \cdot F = 1$. On the other hand, adjunction formula implies that

$$(K_T + S_T) \cdot S_T = \deg_{S_T} K_{S_T} = \deg_{\bar{A}} K_{\bar{A}} \leq d'(n-2)(4n+4)^{n-1}$$

since $S_T \to \bar{A}$ is an isomorphism; the displayed inequality is just (25). Hence, taking $C_1 = 2d'(n-2)(4n+4)^{n-1}$ proves the claim.

**Step 2.5**. *In this step, we show that there exists a log bounded set of morphisms $\mathfrak{F}_1$ and a 4-uple $(f'_{Z_1}, Z'_1 + G'_1, \Delta'_1, X') \in \mathfrak{F}_1$ such that $(Z'_1, \Delta'_1 + G'_1)$ is $\mathbb{Q}$-factorial and isomorphic to $(Z_1, \Delta_1 + G_1)$ in codimension one.*

We know that $Z_1$ is terminal, $(Z_1, \Delta_1)$ is $\frac{1}{l}$-klt, while $(Z_1, S_{Z_1} + G_1)$ is dlt. Hence, $(Z_1, \frac{1}{2}(S_{Z_1} + G_1))$ is $\frac{1}{2}$-klt and $(Z_1, \frac{1}{2}(\Delta_1 + S_{Z_1} + G_1))$ is $\frac{1}{2l}$-klt.

By Step 2.4, we have that $\mathrm{vol}(K_{Z_1} + \frac{1}{2}(S_{Z_1} + G_1)) \leq C'$. Running the $(K_{Z_1} + \frac{1}{2}(S_{Z_1} + G_1))$-MMP, this terminates with a minimal model $Z_1 \dashrightarrow Z_2$. Since $G_1 \sim |f^*_{Z_1}(4n+4)A'|$ and $A'$ is Cartier and very ample on $X'$, this run of the MMP is over $X'$ and moreover $\mathrm{vol}(K_{Z_1} + \frac{1}{2}(S_{Z_1} + G_1)) > 0$. Hence, Theorem 2.9 implies that $(Z_2, \frac{1}{2}(S_{Z_2} + G_2))$ is bounded, where $S_{Z_2}$ (resp. $G_2$) denote the strict transform of $S_{Z_1}$ (resp. $G_1$) on $Z_2$.

Denoting with $\Delta_2$ the strict transform of $\Delta_1$ on $Z_2$, it follows that $(Z_2, \Delta_2)$ is a log Calabi–Yau pair such that $l(K_{Z_2} + \Delta_2) \sim 0$. As $Z_2$ is bounded, there exists a very ample Cartier divisor $H$ and a constant $C_2 = C_2(n, d, l)$ such that

$$|K_{Z_2} \cdot H^{n-1}| \leq C_2.$$

As $K_{Z_2} + \Delta_2 \sim_{\mathbb{Q}} 0$, it follows that also

$$\Delta_2 \cdot H^{n-1} \leq C_2,$$

so that $(Z_2, \Delta_2 + G_2)$ is bounded since the coefficients of $\Delta_2$ belong to the set $\{\frac{1}{l}, \frac{2}{l}, \dots, \frac{l-1}{l}\}$. The birational map $Z \dashrightarrow Z_2$ which is the composition of the birational contractions $Z \dashrightarrow Z_1$ (Step 2.2) and $Z_1 \dashrightarrow Z_2$ is in turn a birational contraction. All these maps are by construction maps over $X'$.
As $K_Z + \Delta_Z \sim_{\mathbb{Q}} 0$, it follows from Theorem 2.13 that there exists a $\mathbb{Q}$-factorial variety $Z'$ with a partial resolution over $X'$ that is isomorphic to $Z$ in codimension 1 and such that the pair $(Z', \Delta' + G')$ belongs to a bounded family, where we denote by $\Delta'$ the strict transform of $\Delta_Z$ and by $G'$ the pullback on $Z'$ of $G_2$. As $Z'$ is a partial resolution of $Z_2$ over $X'$, it follows that there exists a morphism $f_{Z'} \colon Z' \to X'$ which is the Iitaka fibration of the semiample divisor $K_{Z'} + \Delta' + \frac{1}{2}G'_1$. Thus Lemma 2.16 implies that the set $\mathfrak{F}_1$ of 4-uples $(f_{Z'}, Z', \Delta' + G', X')$ is a log bounded set of morphisms.

**Step 2.6**. *In this step, we construct the set $\mathfrak{F}$ whose existence is claimed in the statement of Step 2.*
By construction, $(Z', \Delta')$ is isomorphic in codimension one to $(Y^t, \Delta^t)$, by Step 2.1, as it is isomorphic in codimension one to $(Z, \Delta_Z)$. As $\Delta^t$ is exceptional over $Y'$, by Remark 2.18 $\Delta'$ is degenerate for $f_{Z'}$. Applying Proposition 2.21, it follows that the set

$$\begin{aligned}\mathfrak{F} :=& \{(f_{Y''}, Y'', 0, X') \mid \exists (f_{Z'}, Z', \Delta' + G', X') \in \mathfrak{F}' \text{ such that } f_{Y''} \colon Y'' \to X' \\ & \text{is good minimal model for } (Z', (1+\epsilon)\Delta' + G') \text{ over } X',\ 0 < \epsilon \ll 1\}\end{aligned}$$

is a log bounded set of morphisms.

**Step 3**. *In this step we show that $Y'$ is bounded up to flops.*
We have shown in Step 2 that there exists a log bounded set of morphism $\mathfrak{F}$ and a 4-uple $(f_{Y''}, Y'', 0, X') \in \mathfrak{F}$, together with a birational contraction $Y'' \dashrightarrow Y'$ over $X'$ which is a birational contraction is crepant with respect to $(Y', 0)$.
Let $D''$ be the exceptional divisor for the map $Y'' \dashrightarrow Y'$. Remark 2.18 implies that $D''$ is degenerate with respect to $f_{Y''}$. By Lemma 2.19 the set

$$\mathfrak{F}_{\rm deg} := \{(f_{Y''}, Y'', D, X') \mid (f_{Y''}, Y'', 0, X') \in \mathfrak{F} \text{ and } D \text{ is degenerate for } f_{Y''}\}$$

is a log bounded set of morphisms, and $(f_{Y''}, Y'', D'', X') \in \mathfrak{F}_{\rm deg}$ by construction. Applying Proposition 2.21 it follows that the set

$$\begin{aligned}\mathfrak{F}_{\rm deg} :=& \{(f_{Y'''}, Y''', 0, X') \mid \exists (f_{Y''}, Y'', D'', X') \in \mathfrak{F}_{\rm deg} \text{ such that} \\ & f_{Y'''} \colon Y''' \to X' \text{ is good minimal model for } (Y'', D'') \text{ over } X'\}\end{aligned}$$

is a log bounded set of morphisms. As $f_{Y'''} \colon Y''' \to X'$ is a relatively good minimal model for $(Y'', D'')$ over $X'$ and $D''$ is the exceptional divisor of the birational contraction $Y'' \dashrightarrow Y'$, it follows that $Y'$ is isomorphic in codimension one to $Y'''$. As $Y'''$ belongs to a bounded family, see Remark 2.15, the conclusion follows. □

5.3. **Finiteness of index for bases of elliptic Calabi–Yau varieties.** Given an elliptic fibration $f \colon Y \to X$ where $K_Y \sim_{f,\mathbb{Q}} 0$, using the canonical bundle formula, see § 2.7, we can write

$$K_Y \sim_{\mathbb{Q}} f^*(K_X + B_X + M_X).$$

**Remark 5.3.** Since $f$ is an elliptic fibration we know that the coefficients of $B_X$ vary in a DCC subset of $[0, 1)$, see [PS09, Example 3.1] for the precise determination of what coefficients can appear. Moreover, on a sufficiently high birational model of $X$, say, $r\colon X' \to X$, the moduli part $M_{X'}$ becomes semi-ample: more precisely, $|12M_{X'}|$ is base point free, cf. [PS09, Example 7.16]. Hence, we can choose $\overline{M}' \sim 12M_{X'}$ and define $\overline{M} := r_*\overline{M}'$ such that $(X, B_X + \frac{1}{12}\overline{M})$ is klt and $12(K_X + B_X + M_X) \sim 12(K_X + B_X + \frac{1}{12}\overline{M})$.

**Theorem 5.4.** *Fix positive integers $d, l$. Let $f\colon Y \to X$ be a contraction of normal projective varieties such that*

- *$Y$ is of dimension $d$,*
- *$(Y, 0)$ is klt,*
- *$lK_Y \sim 0$, and*
- *$f\colon Y \to X$ is an elliptic fibration.*

*Then there exists $m = m(d, l)$ such that exactly one of the following cases is realized:*

*(a) $mK_X \sim 0$; or*
*(b) there exists an effective divisor $B > 0$ on $X$, $(X, B)$ is a klt log Calabi–Yau pair and $m(K_X + B) \sim 0$.*

*Proof.* We use the same notation as in the paragraph before the theorem.
By Remark 2.22 and the hypotheses, we can write the canonical bundle formula in the following form

$$0 \sim lK_Y \sim f^*l(K_X + B_X + M_X).$$

As $12M_{X'} \sim \overline{M}$, then

$$0 \sim 12lK_Y \sim f^*12l(K_X + B_X + M_X) \sim f^*12l(K_X + B_X + \frac{1}{12}\overline{M}).$$

Hence it suffices to take $m = 12l$ to settle both case (a) and (b). In fact, case (b) follows at once, while in case (a), if $K_X \sim_{\mathbb{Q}} 0$, then $B = 0 \equiv M_X$. As $\overline{M}'$ is base point free and $r_*\overline{M}' \equiv 0$, it follows $\overline{M}' \sim 0 \sim \overline{M}$. Hence, in this case

$$0 \sim 12lK_Y \sim 12l(K_X + \frac{1}{12}\overline{M}) \sim 12lK_X.$$

□

Much in the same vein one can prove the following generalization of this result.

**Theorem 5.5.** *Fix positive integers $d, l$ and a bounded set $\mathfrak{D}$ of $K$-trivial varieties. Consider projective varieties $X$ and contractions $f\colon Y \to X$ such that*

- *$Y$ is a klt projective of dimension $d$,*
- *$lK_Y \sim 0$, and*
- *$f\colon Y \to X$ is a fibration and the general fiber $F$ belongs to $\mathfrak{D}$.*

*Then there exists $m = m(d, l, \mathfrak{D})$ such that exactly one of the following cases is realized:*

*(a) $X$ is projective klt and $mK_X \sim 0$; or*
*(b) there exists a generalised klt log Calabi–Yau pair $(X, B + M)$ and $m(K_X + B + M) \sim 0$.*

*Proof.* By Remark 2.22 and the hypotheses, we can write the canonical bundle formula in the following form

$$0 \sim lK_Y \sim f^*l(K_X + B_X + M_X). \tag{29}$$

**Case 1**: $K_X < 0$.
The existence of the klt log Calabi–Yau generalised pair is simply a consequence of the canonical bundle formula above (29) and Remark 2.24.

**Case 2**: $K_X \sim_{\mathbb{Q}} 0$.
In this case, as in the proof of Theorem 5.4, $lK_Y \sim f^*(l(K_X + M_X))$ and $M_X$ is a torsion divisor. Moreover, by Theorem 2.26, there exists a constant $t$ such that $tM_X \sim 0$. Hence,

$$0 \sim tlK_Y \sim f^* tlK_X.$$

□

## 6. Rationally connected generalised log Calabi–Yau pairs

6.1. **Singularities in Mori fibre spaces.** When studying the structure of a Mori fibre space $f\colon X \to Y$ it is natural to ask whether there is any way to control the singularities of the base $Y$ in terms of the singularities of the total space $X$.

A conjecture of Shokurov, later refined by Birkar, [Bir16b, Conjecture 1.2] predicts that one should expect an affirmative answer to the question above. We have seen in § 2.7 that when we have a generalised pair $(X, B + M)$ and a morphism $f\colon X \to Z$ such that $K_X + B + M \sim_{f,\mathbb{Q}} 0$, then we can define a generalised pair $(Z, G + N)$ providing a generalization of the standard form of the canonical bundle formula. Using the canonical bundle formula for generalised pairs, Birkar's conjecture can be expressed in a more general context.

**Conjecture 6.1.** [Bir18, Conjecture 2.4] *Let $d$ be a positive integer and $\epsilon$ be a positive real number. Then there exists a positive real number $\delta = \delta(d, \epsilon)$ such that if $(X, B + M)$ is a generalised pair with a contraction $f\colon X \to Z$ such that*

- *$(X, B + M)$ is $\epsilon$-lc of dimension $d$,*
- *$K_X + B + M \sim_{f,\mathbb{Q}} 0$, and*
- *$-K_X$ is big over $Z$,*

*then the generalised pair structure $(Z, G + N)$ induced by the canonical bundle formula is $\delta$-lc.*

Birkar proved in [Bir16b, Corollary 1.7] that Conjecture 6.1 holds when $\dim X = \dim Z + 1$ and the $M \equiv 0$ as a b-divisor. In the following we extend Birkar's result to the case of generalised pairs.

**Lemma 6.2.** *Fix a positive integer $d$ and a positive real number $\epsilon$. Let $(X, B+M)$ be an $\epsilon$-lc generalised pair and let $f\colon X \to T$ be a contraction of relative dimension 1 with $K_X+B+M \sim_{\mathbb{R}} 0$. Assume that $B+M$ is $f$-big. For any big Cartier divisor $H_T$ on $T$, there exists $\delta = \delta(d, \epsilon)$stisfying the following:*
*for any positive real number $\lambda \ll 1$, there exists an effective divisor $\Gamma_\lambda$ such that $\lambda H_T \sim_{\mathbb{R}} K_T + \Gamma_\lambda$ and $(T, \Gamma_\lambda)$ is $\delta$-lc.*

In particular, the lemma implies that if $T$ is $\mathbb{Q}$-Gorenstein then the pair $(T, 0)$ is automatically $\delta$-lc.

*Proof.* Let $X' \to X$ be a log resolution where $M$ descends. We will denote by $E$ the exceptional divisor for $X' \to X$ and by $B'$ the strict transform of $B$.
As $H_T$ is big on $T$, $H_T \sim_{\mathbb{R}} A_T + F_T$ the sum of an ample $A_T$ and an effective $\mathbb{R}$-divisors $F_T$ on $T$. We will denote by $H$ (respectively $H'$), $A$ (resp.$A'$), $F$ (resp. $F'$) the pullback of $H_T, A_T, F_T$ on $X$ (resp. $X'$).

It is easy to see that $\eta_1(B'+M')+\eta_2A'$ is big for any positive real numbers $\eta_1,\eta_2$. Hence, $\eta_1(B'+M')+\eta_2A' \sim_{\mathbb{R}} G'_{\eta_1,\eta_2}+A''_{\eta_1,\eta_2}$, where $G'_{\eta_1,\eta_2}$ is effective and $A''_{\eta_1,\eta_2}$ is ample. We fix a choice of positive real numbers $\eta_1,\eta_2$. Thus, for any positive constant $c$ and choosing $D'_{c\eta_1,c\eta_2}$ a sufficiently general effective divisor in the ample $\mathbb{R}$-linear system $|(1-c\eta_1)M'+cA''_{\eta_1,\eta_2}|_{\mathbb{R}}$, it follows that

$$
\begin{aligned}
&K_{X'}+B'+(1-\epsilon)E+M'+c\eta_2(A'+F') \sim_{\mathbb{R}}\\
&K_{X'}+(1-\epsilon)E+(1-c\eta_1)(B'+M')+c(\eta_1(B'+M')+\eta_2A')+c\eta_2F' \sim_{\mathbb{R}} \qquad (30)\\
&K_{X'}+(1-c\eta_1)B'+(1-\epsilon)E+D'_{c\eta_1,c\eta_2}+cG'_{\eta_1,\eta_2}+c\eta_2F'.
\end{aligned}
$$

For $c \ll 1$, the pair $(X',(1-c\eta_1)B'+(1-\epsilon)E+D'_{c\eta_1,c\eta_2}+cG'_{\eta_1,\eta_2}+c\eta_2F')$ is $\frac{\epsilon}{2}$-lc. By (30) and the fact that

$$K_{X'}+(1-c\eta_1)B'+(1-\epsilon)E+D'_{c\eta_1,c\eta_2}+cG'_{\eta_1,\eta_2}+c\eta_2F' \sim_{\mathbb{R},X} E',$$

with $E'$ effective and exceptional over $X$, also the induced pair $(X,(1-c\eta_1)B+D_{c\eta_1,c\eta_2}+cG_{\eta_1,\eta_2}+c\eta_2F)$ is $\frac{\epsilon}{2}$-lc, where $D_{c\eta_1,c\eta_2},G_{\eta_1,\eta_2},\eta_2F$ are the push-forward of $D'_{c\eta_1,c\eta_2},G'_{\eta_1,\eta_2},F'$ on $X$.
Let us fix one such choice of the constants $c$ and define $\lambda := c\eta_2$. Since

$$K_X+(1-c\eta_1)B+D_{c\eta_1,\lambda}+cG_{\eta_1,\lambda}+\lambda F \sim_{\mathbb{R}} \lambda H,$$

by [Bir16b, Corollary 1.7], there exist $0<\delta=\delta(d,\epsilon)$ and an effective divisor $\Gamma_\lambda$ on $T$ such that $\lambda H_T \sim_{\mathbb{R}} K_T+\Gamma_\lambda$ and $(T,\Gamma_\lambda)$ is $\delta$-lc. $\square$

In the Introduction we addressed the reasons that make Conjecture 1.5 a central problem in the study of boundedness. Also this conjecture can be extended to the more general setting of generalised pairs.

**Conjecture 6.3.** *Fix a positive integer $d$ and positive real number $\epsilon$. Then the set of varieties $X$ such that*

*(1) $(X,B+M)$ is an $\epsilon$-lc generalised pair of dimension $d$,*
*(2) $K_X+B+M \sim_{\mathbb{Q}} 0$, and*
*(3) $X$ is rationally connected*

*is bounded.*

We will denote by $6.3_{\leq d}$ the statement of Conjecture 6.3 for generalised pairs of dimension at most d. We can prove the following conditional step towards the solution of the conjecture above.

**Theorem 6.4.** *Assume Conjecture $6.3_{\leq d}$ and Conjecture 6.1 hold. Then the set of varieties $X$ such that*

*(1) $(X,B+M)$ is an $\epsilon$-lc generalised pair of dimension $d+1$,*
*(2) $K_X+B+M \sim_{\mathbb{Q}} 0$*
*(3) $K_X \not\equiv 0$, and*
*(4) $X$ is rationally connected*

*is bounded up to flops.*

*Proof.* By [Bir18, Theorem 2.2], it is enough to prove the Theorem when $X$ is endowed with a Mori fibre space structure $f\colon X \to Z$. The target variety $Z$ will also be rationally connected. The canonical bundle formula, see § 2.7, implies that there is a log Calabi–Yau structure $(Z,G+N)$ on $Z$. Moreover, by Conjecture 6.1 $(Z,G+N)$ is $\delta$-lc for some $\delta=\delta(d+1,\epsilon)$. Hence, by applying Conjecture 6.3, $Z$

belongs to a bounded family. A second application of [Bir18, Theorem 2.2] to the Fano fibration $f$ completes the proof. $\square$

## 7. Proofs of the theorems and corollaries

*Proof of 1.4.* By Theorems 3.1 and 2.13, it is enough to prove the conclusion holds for a pair $(X, B)$ satisfying all the conditions in the statement of the theorem and that is moreover endowed with a tower of Mori fibre spaces

$$(X, B) = (X_0, B_0) \longrightarrow X_1 \longrightarrow \ldots \longrightarrow X_{s-1} \longrightarrow X_s = Z. \tag{31}$$

If $(X, B)$ is not of product type, then $\dim Z = 0$ and the conclusion follows from Corollary 3.4. If $\dim Z > 0$ and $K_Z \sim_{\mathbb{Q}} 0$, then Theorem 3.5 implies that there exists $m = m(d, l)$ such that $mK_Z \sim 0$. Then, by Theorem 4.1, $Z$ is bounded up to flops. That is, there exists a klt variety $Z'$ isomorphic to $Z$ in codimension 1 which belongs to a bounded family. By Theorem 2.12, we can assume that $Z'$ is $\mathbb{Q}$-factorial. As, $Z' \dashrightarrow Z$ is an isomorphism in codimension 1 of projective varieties, it is also a birational contraction. Thus, we can apply Proposition 3.7 to the MFS $X_{s-1} \to Z$ in (31) and obtain a commutative diagram

$$\begin{tikzcd} X'_{s-1} \arrow[r, dashed] \arrow[d] & X_{s-1} \arrow[d] \\ Z' \arrow[r, dashed] & Z \end{tikzcd}$$

where the horizontal arrow $X'_{s-1} \dashrightarrow X_{s-1}$ is an isomorphism in codimension 1 of $\mathbb{Q}$-factorial projective varieties. As all horizontal arrows in the diagram are isomorphisms in codimension 1 of $\mathbb{Q}$-factorial projective varieties, it follows that $\rho(X_{s-1}/Z) = \rho(X'_{s-1}/Z') = 1$; hence, $X_{s-1} \to Z'$ is a Mori fibre space. As the map $X'_{s-1} \dashrightarrow X_{s-1}$ is in turn a birational contraction, we can apply Proposition 3.7 to the MFS $X_{s-2} \to X_{s-1}$ in (31) and obtain a commutative diagram

$$\begin{tikzcd} X'_{s-2} \arrow[r, dashed] \arrow[d] & X_{s-2} \arrow[d] \\ X'_{s-1} \arrow[r, dashed] & X_{s-1} \end{tikzcd}$$

where the horizontal arrow $X'_{s-2} \dashrightarrow X_{s-2}$ is an isomorphism in codimension 1 of $\mathbb{Q}$-factorial projective varieties.
By inductively applying Propositions 3.7, as we just did, we construct a pair $(X', B')$ isomorphic in codimension 1 to $(X, B)$ and endowed with a tower of Mori fibre spaces

$$\begin{tikzcd} (X', B') \arrow[r] \arrow[d, dashed] & X'_1 \arrow[r] \arrow[d, dashed] & \ldots \arrow[r] & X'_{s-1} \arrow[r] \arrow[d, dashed] & Z' \arrow[d, dashed] \\ (X, B) \arrow[r] & X_1 \arrow[r] & \ldots \arrow[r] & X_{s-1} \arrow[r] & Z, \end{tikzcd} \tag{32}$$

where all the vertical arrows are isomorphisms in codimension 1 and $\rho(X'_i/X'_{i+1}) = 1$, for $i = 0, \ldots, s-1$.
The conclusion then follows by applying Theorem 3.2 to the pair $(X', B')$ and the tower of morphisms in (32). $\square$

*Proof of 1.2.* Since $Y$ is a simply connected Calabi–Yau, by Corollary 5.1 $X$ is rationally connected. Thus, Theorem 5.4 implies that there is a choice of an effective divisor $B \geq 0$ (where we allow the possibility of $B = 0$), such that the pair $(X, B)$ is klt, log Calabi–Yau, and the torsion index of $K_X + B$ is bounded. By Theorem 1.4, the pair $(X, B)$ is log bounded up to flops. By Propositions 3.6 and 3.7, up to substituting $Y$ with an elliptic terminal Calabi–Yau $Y' \to X'$ with $Y'$ isomorphic to $Y$ in codimension 1, we can assume that the base of the elliptic fibration is actually bounded. The proof then follows immediately from Theorem 5.2. □

*Proof of 1.3.* As $Y \to X$ is an elliptic fibration, Theorems 5.4 implies that there is a choice of an effective divisor $B \geq 0$ (where we allow the possibility of $B = 0$), such that the pair $(X, B)$ is klt, log Calabi–Yau, and the torsion index of $K_X + B$ is bounded by a constant $m = m(l, d)$. Hence, as $X$ is rationally connected, the pair $(X, B)$ is log bounded up to flops, by Theorem 1.4. By Propositions 3.6 and 3.7, up to substituting $Y$ with an elliptic K-trivial variety $Y' \to X'$ with $Y'$ isomorphic to $Y$ in codimension 1, we can assume that the base of the elliptic fibration is actually bounded. The proof then follows immediately from Theorem 5.2, since $Y$ is $\frac{1}{l}$-lc. □

*Proof of 1.6.* The case $K_X \equiv 0$ follows from [Jia19, Theorem 1.6]. Hence, we can assume that $K_X \not\equiv 0$. Then, by Theorem 3.1 and [Bir18, Theorem 2.2], it suffices to prove that the conclusion holds for a $\epsilon$-lc generalised pair $(X, B + M)$ such that $K_X + B + M \sim_{\mathbb{R}} 0$ and that is moreover endowed with a tower of Mori fibre spaces

$$(X, B) = (X_0, B_0) \longrightarrow X_1 \longrightarrow \dots \longrightarrow X_{s-1} \longrightarrow X_s = Z, \tag{33}$$

and either $\dim Z = 0$ or $K_Z \equiv 0$. Let us notice that the proof of Theorem 3.1 applies also to the case of generalised pairs, cf. [DCS16, Theorem 3.1] and its proof. If $\dim X_1 = 0$ then $s = 1$ and the conclusion follows immediately from the proof of the BAB Conjecture. If $\dim X_1 = 1$, then $s = 2$, $\dim Z = 0$ and the conclusion follows immediately from Theorem 2.12. If $\dim X_1 = 2$, then either

(a) $\rho(X_1) = 1$ and $s = 2$, $\dim Z = 0$; or
(b) $\rho(X_1) = 2$ and $X_2 = \mathbb{P}^1$, and $s = 3$; or
(c) $s = 1$ and $K_{X_1} \equiv 0$.

Let us analyze these 3 cases separately.
*Case (a).* We start by proving the following claim.
**Claim 1**. $X_1$ is a $\delta$-lc Fano surface, for some positive $\delta = \delta(\epsilon)$.

*Proof of Claim 1.* If $M \sim_{X_1,\mathbb{R}} 0$ then $K_X + B \sim_{X_1,\mathbb{R}} 0$, hence [Bir16b, Corollary 1.7] implies that $X_1$ is $\delta$-lc. If $M$ is relatively ample over $X_1$ then letting $H_1$ be an ample Cartier divisor on $X_1$, Lemma 6.2 implies that there exist $0 < \delta = \delta(\epsilon)$ such that $X_1$ is $\delta$-lc. □

Claim 1 and the BAB conjecture for surfaces, [Ale94, Theorem 6.8], imply that $X_1$ belongs to a bounded family. Then, Theorem 2.12 implies the statement of the theorem.
*Case (c).* The same proof as that of Claim 1 shows that there exists $0 < \delta = \delta(\epsilon)$ such that $X_1$ is a $\delta$-lc surface. As $K_{X_1} \equiv 0$ and $X_1$ is rationally connected, [Ale94, Theorem 6.8] implies that $X_1$ belongs to a bounded family. Then, Theorem 2.12 implies the statement of the theorem.
*Case (b).* We start by proving the following claim.

**Claim 2**. There exists a birational contraction $X_1 \to Y$ onto surface $Y$ and $Y$ belongs to a bounded family.

*Proof of Claim 2.* If the contraction $X_1 \to Y$ exists, then considering the contraction $X \to Y$, the same proof as for Claim 1 shows that $Y$ is $\delta$-lc, for some $0 < \delta = \delta(\epsilon)$.
As $\rho(X_1) = 2$, the nef cone of $X_1$ is spanned by 2 extremal rays

$$\mathrm{Nef}(X_1) = \mathbb{R}_{\geq 0} F_1 + \mathbb{R}_{\geq 0} F_2,$$

where $F_1$ is the class of a fibre of the $\mathbb{P}^1$-bundle $X_1 \to X_2$; thus, $F_1^2 = 0$.
We distinguish several cases based on the sign of $F_2^2, K_{X_1} \cdot F_2$:

**$F_2^2 > 0$:** $F_2$ is big and nef. By Lemma 6.2, for $0 < \lambda \ll 1$ there exists a boundary $\Gamma_\lambda$ such that $(X_1, \Gamma_\lambda)$ is klt and $K_{X_1} + \Gamma_\lambda \sim_{\mathbb{R}} \lambda F_2$. Hence, the Rationality Theorem and the Contraction Theorem imply that $F_2$ is a semi-ample $\mathbb{Q}$-divisor. The Iitaka fibration $X_1 \to Y$ of $F_2$ is a birational contraction such that $\rho(Y) = 1$. By Lemma 6.2, then either $Y$ is Fano or $K_Y \equiv 0$. As $-K_Y$ is nef and $Y$ is rationally connected and $\delta$-lc, [Ale94, Theorem 6.8] implies that $Y$ belongs to a bounded family.

**$F_2^2 = 0, K_{X_1} \cdot F_2 < 0$:** Since $F_1^2 = 0 = F_2^2$, the nef cone of $X_1$ coincides with $\overline{NE}(X_1)$. Moreover, $K_{X_1} \cdot F_i < 0$, hence, $K_{X_1}$ is negative along $\overline{NE}(X_1) \setminus \{0\}$. Thus, $X_1$ is Fano and it suffices to take $Y = X_1$ and the identity map $X_1 \to Y$. Hence, [Ale94, Theorem 6.8] implies that $Y$ belongs to a bounded family, as it is a $\delta$-lc Fano.

**$F_2^2 = 0, K_{X_1} \cdot F_2 > 0$:** As $K_{X_1} \cdot F_1 < 0 < K_{X_1} \cdot F_2$, Hodge Index Theorem implies that $K_{X_1} \equiv_{\mathbb{R}} a_1 F_1 + a_2 F_2$ with $a_2 < 0 < a_1$.
Let $A$ be an ample Cartier divisor on $X_1$. Lemma 6.2 implies that for any $0 < \lambda \ll 1$ there exists an effective divisor $\Gamma_\lambda$ s.t. $\lambda A \sim_{\mathbb{R}} K_{X_1} + \Gamma_\lambda$. For $\lambda$ sufficiently small, then $\Gamma_\lambda \in \mathbb{R}_{<0} F_1 + \mathbb{R}_{>0} F_2$ which is impossible, as $F_1^2 = 0 = F_2^2$ implies that $\mathrm{Nef}(X_1) = \mathrm{Pseff}(X_1)$. Hence, this case cannot happen.

**$F_2^2 = 0, K_{X_1} \cdot F_2 = 0$:** In this case $K_{X_1} + tF_2 \sim_{\mathbb{R}} 0$ for some $t > 0$ by Hodge Index Theorem. Taking $X_1 = Y$ and $X_1 \to Y$ to be the identity, then $-K_Y$ is nef and $Y$ is rationally connected and $\delta$-lc. Hence, [Ale94, Theorem 6.8] implies that $Y$ belongs to a bounded family.

□

The statement of the theorem then follows from Claim 2 together with Theorem 2.12 and [Ale94, Theorem 6.8]. □

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