



Boundedness of Bergman projectors on homogeneous Siegel domains

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Received: 1 April 2022 / Accepted: 17 July 2022 / Published online: 21 September 2022
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Abstract

In this paper we study the boundedness of Bergman projectors on weighted Bergman spaces on homogeneous Siegel domains of Type II. As it appeared to be a natural approach in the special case of tube domains over irreducible symmetric cones, we study such boundedness on the scale of mixed-norm weighted Lebesgue spaces. The sharp range for the boundedness of such operators is essentially known only in the case of tube domains over Lorentz cones. In this paper we prove that the boundedness of such Bergman projectors is equivalent to various notions of atomic decomposition, duality, and characterization of boundary values of the mixed-norm weighted Bergman spaces, extending results mostly known only in the case of tube domains over irreducible symmetric cones. Some of our results are new even in the latter simpler context. We also study the simpler, but still quite interesting, case of the “positive” Bergman projectors, the integral operator in which the Bergman kernel is replaced by its modulus. We provide a useful characterization which was previously known for tube domains.

Keywords Bergman space · Bergman projection · Homogeneous Siegel domain · Atomic decomposition · Decoupling inequality

1 Introduction

In this paper we study the boundedness of the Bergman projectors $P_{s'}$ on the mixed-norm Lebesgue spaces $L_s^{p,q}(D)$ on a Siegel domain D of Type II, where $p, q \in [1, \infty]$ and $s, s' \in \mathbb{R}^r$. We refer to Sect. 2 for the relevant definitions and the main basic properties. Here we just mention that $r \in \mathbb{N}$ is the *rank* of the homogeneous cone Ω that appears in the definition of the underlying Siegel domain D . One of the main features of the present work is the generality of our setting, in which the analysis and geometric properties of Ω are crucial and, generally speaking, still quite elusive.

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We shall now try to describe our results to a broader audience than just that of the specialists and this will lead to a somewhat longer introduction. We hope that it will make it easier to understand the problems we address and that it may serve as a general reference.

The simplest case of the domains we consider is the upper half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. For $p, q \in (0, \infty]$ and $s \in \mathbb{R}$, consider the Lebesgue spaces

$$L_s^{p,q}(\mathbb{C}_+) = \left\{ f \text{ measurable in } \mathbb{C}_+ : \int_0^\infty \left(\int_{\mathbb{R}} |f(x+iy)|^p dx \right)^{q/p} y^{qs-1} dy < \infty \right\},$$

with the obvious modifications when $\max(p, q) = \infty$. The mixed-norm Bergman spaces are defined as

$$A_s^{p,q}(\mathbb{C}_+) := L_s^{p,q}(\mathbb{C}_+) \cap \operatorname{Hol}(\mathbb{C}_+),$$

where $\operatorname{Hol}(\mathbb{C}_+)$ denotes the space of holomorphic functions on $\mathbb{C}_+$. For the time being, we simply write $A_s^{p,q}$ in place of $A_s^{p,q}(\mathbb{C}_+)$. Then, $A_s^{p,q} = \{0\}$ if $s < 0$, and also if $s = 0$ and $q < \infty$. In addition, $A_0^{p,\infty}$ is the classical Hardy space $H^p(\mathbb{C}_+)$. In order to avoid trivialities, from now on we shall assume $s > 0$. In addition, for simplicity, in this introductory section we shall generally limit ourselves to the case $p, q \geq 1$. Then, the $A_s^{p,p}$ are the classical weighted Bergman spaces, the unweighted case corresponding to the choice $s = 1/p$. The spaces $A_s^{p,q}$ embed continuously into $\operatorname{Hol}(\mathbb{C}_+)$, so that $A_s^{p,q}$ is a closed subspace of $L_s^{p,q}(\mathbb{C}_+)$. In particular, the $A_s^{2,2}$ are reproducing kernel Hilbert spaces. Their reproducing kernel, that is, the weighted Bergman kernel, is the kernel function

$$K_s(z, w) = c_s(z - \bar{w})^{-1-2s}$$

where $c_s = s(2i)^{2s+1}/\pi$. The orthogonal projection of $L_s^{2,2}(\mathbb{C}_+)$ onto $A_s^{2,2}$ is the Bergman projector $\mathcal{P}_s$ given by

$$\mathcal{P}_s f(z) = \int_{\mathbb{C}_+} f(w) K_s(z, w) (\operatorname{Im} w)^{2s-1} d\mathcal{H}^2(w),$$

where $\mathcal{H}^2$ denotes the (suitably normalized) 2-dimensional Hausdorff measure on $\mathbb{C}_+$, that is, the Lebesgue measure. One of the main questions in the theory of holomorphic function spaces is the boundedness of projection operators. In this setting, it is known the operator $\mathcal{P}_s$ induces a continuous endomorphism of $L_s^{p,q}(\mathbb{C}_+)$ if and only if $2\tilde{s} > s$ see, e.g., [18, Proposition 5.20 and Corollary 5.27].¹

We also define an associated ‘positive’ operator by

$$\mathcal{P}_{s,+} f(z) = \int_{\mathbb{C}_+} f(w) |K_s(z, w)| (\operatorname{Im} w)^{2s-1} d\mathcal{H}^2(w).$$

Clearly, the boundedness of $\mathcal{P}_{s,+}$ implies the boundedness of $\mathcal{P}_s$ on the same spaces. In the case of $\mathbb{C}_+$, it turns out that $\mathcal{P}_s$ is bounded on $L_s^{p,q}(\mathbb{C}_+)$ if and only if $\mathcal{P}_{\tilde{s},+}$ is, but this equivalence fails in more general settings, as we shall soon discuss.

¹ When $q = 1$, the cited result only proves that $2\tilde{s} \geq s$. Nonetheless, if the assertion held for $2\tilde{s} = s$, then the argument below would show that the Hardy space $H^p(\mathbb{C}_+)$ is canonically antilinearly isomorphic to $(A_s^{p,1})'$; this latter fact is false, as one may see by inspection of the boundary values (cf., e.g., [18, Proposition 5.12]).

The boundedness of the Bergman projector $\mathcal{P}_{\tilde{s}}$ is tightly related to the characterization of the dual space $(A_s^{p,q})'$ as the space $A_{2\tilde{s}-s}^{p',q'}$, where p' denotes the conjugate exponent of p , that is, $p' = p/(p-1)$: on the one hand, it is readily seen that the continuity of a Bergman projector implies a characterization of the dual of $A_s^{p,q}$, at least when $p, q < \infty$, by standard arguments (cf., e.g., [2, Theorem 5.2]); on the other hand, more subtle arguments show that also the converse implication holds, even in greater generality² (cf. [4, Theorem 1.6] for the case $p = q$, and Corollary 4.7 below for the general case). As a matter of fact, even for $p, q \in (0, \infty)$, the sesquilinear form

$$A_s^{p,q} \times A_{\tilde{s}}^{p',q'} \ni (f, g) \rightarrow \int_{\mathbb{C}_+} f(z) \overline{g(z)} (\operatorname{Im} z)^{s+\tilde{s}+(1/p-1)_+-1} d\mathcal{H}^2(z)$$

is continuous and induces an antilinear isomorphism of $A_{\tilde{s}}^{p',q'}$ onto $(A_s^{p,q})'$ for every $s, \tilde{s} > 0$. Notice that we set $p' = \max(1, p)'$ for every $p \in (0, \infty]$.

The boundedness of Bergman projectors is also strictly related to the existence of suitable *atomic decompositions* for the Bergman spaces $A_s^{p,q}$. For a fixed $R > 0$, consider the family $(Q_{j,k}^{(R)})_{j,k \in \mathbb{Z}}$ of open boxes in $\mathbb{C}_+$ defined by

$$Q_{j,k}^{(R)} := (2^{kR}Rj, 2^{kR}R(j+1)) \times (2^{kR}, 2^{(k+1)R}),$$

Define

$$\ell^{p,q}(\mathbb{Z}) = \left\{ (\lambda_{j,k}) \in \mathbb{C}^{\mathbb{Z} \times \mathbb{Z}} : \|(\lambda_{j,k})_j\|_{\ell^p} \|_{\ell^q} < \infty \right\},$$

and consider the mapping

$$\mathcal{A} : \ell^{p,q}(\mathbb{Z}) \ni (\lambda_{j,k}) \rightarrow \sum_{j,k} \lambda_{j,k} K_{\tilde{s}}(\cdot, w_{j,k,R}) (\operatorname{Im} w_{j,k,R})^{2\tilde{s}+1-s-\frac{1}{p}} \in A_s^{p,q},$$

where the $w_{j,k,R} \in Q_{j,k}^{(R)}$ are fixed, but otherwise arbitrary. When $p, q \in (0, \infty]$, it is known that this mapping is continuous (and onto, for R small) if and only if $2\tilde{s} > s + \left(\frac{1}{p} - 1\right)_+$ (cf. [28] and also [18, Lemma 3.29 and Theorem 3.34]).³ Then, it turns out that (for $p, q \geq 1$) $\mathcal{A}$ is bounded (and onto, for R small) if and only if $\mathcal{P}_{\tilde{s}}$ is bounded on $A_s^{p,q}$.

Observe that the $A_{s/p}^{p,p}$ ($s > 0$, $p \in (0, \infty)$) are all weighted Bergman spaces corresponding to the same measure. In other words, $A_{s/p}^{p,p} = L^p((\operatorname{Im} z)^{s-1} d\mathcal{H}^2(z)) \cap \operatorname{Hol}(\mathbb{C}_+)$. For this reason, it is somewhat more natural to investigate the continuity of the Bergman projector $\mathcal{P}_{s/2}$ on $A_{s/p}^{p,p}$, rather than that of the other projectors $\mathcal{P}_{\tilde{s}}$. Nonetheless, even though the general problem for the operators $\mathcal{P}_{\tilde{s}}$ has been somewhat more rarely considered, the corresponding problem for atomic decomposition has usually been considered in the general case (or, at least, with uncomparable restrictions). Because of the deep connection between the continuity of Bergman projectors and atomic decomposition (cf., e.g., Corollary 4.7), it is therefore quite natural to consider all the Bergman projectors $\mathcal{P}_{\tilde{s}}$. Besides that, considering the general problem has also some technical advantages.

² Of course, the equivalence is trivial in this case, as both properties are (separately) known to be true in the same cases.

³ Again, when $q \leq 1$, the cited results only show that $2\tilde{s} \geq s + \left(\frac{1}{p} - 1\right)_+$, but a direct argument involving duality shows that the weak inequality cannot occur (cf. also Corollary 4.14 below).

Analogously, considering the general mixed-norm family of spaces $A_s^{p,q}$ instead of the ‘pure-norm’ $A_s^{p,p}$ has some technical advantages, as it both sheds light on some phenomena, such as the different behaviour of the operators $\mathcal{P}_{\bar{s}}$ and $\mathcal{P}_{\bar{s},+}$, since the continuity of the latter on $L_s^{p,q}$ does *not* depend on p (cf. Theorem 3.1), and allows to apply Fourier techniques to the spaces $L_s^{2,q}$ by means of the Plancherel formula.

We now discuss the characterization of the boundary values of functions in $A_s^{p,q}$, that is, of the limits f_0 of f_y for $y \rightarrow 0^+$ (in a suitable topology), where $f_y := f(\cdot + iy)$ for $y > 0$. It turns out that the space of boundary values of the functions in $A_s^{p,q}$ is the closed subspace of the homogeneous Besov space $\dot{B}_{p,q}^{-s}(\mathbb{R})$ whose elements have Fourier transform supported in $[0, \infty)$. The mapping

$$\mathcal{B} : A_s^{p,q} \ni f \mapsto f_0 \in \dot{B}_{p,q}^{-s}(\mathbb{R})$$

is an isomorphism onto its image.

Furthermore, we observe that the operators $f \mapsto f^{(k)}$ induce isomorphisms of $A_s^{p,q}$ onto $A_{s+k}^{p,q}$ for every $s > 0$, $p, q \in (0, \infty]$, and $k \in \mathbb{N}$. The continuity of the aforementioned operators is an easy consequence of Cauchy’s estimates and extends to more general spaces without difficulty. The continuity of the inverse operator then appears as a Hardy-type inequality (cf., also, [2, Sect. 1.6]), and its validity in more general contexts has been proved to be equivalent to the continuity of Bergman projectors, under suitable circumstances (cf. [11, Theorem 5.2], [4, Theorem 1.3], and [18, Corollaries 5.16 and 5.28]).

A very natural extension of the upper half-plane to several complex variables are the *tube domains over symmetric cones*. Let $\Omega \subseteq \mathbb{R}^m$ be an open convex cone not containing affine lines. The cone Ω is said to be homogeneous if the subgroup of $GL(m, \mathbb{R})$ that preserves Ω acts transitively on Ω itself. Then, Ω is said to be symmetric if it coincides with its dual Ω' , where

$$\Omega' := \left\{ \lambda \in \mathbb{R}^m : \forall h \in \overline{\Omega} \setminus \{0\} \langle h, \lambda \rangle > 0 \right\},$$

for a suitable scalar product on $\mathbb{R}^m$. The tube domain over the cone Ω is then

$$T_\Omega := \{z \in \mathbb{C}^m : \text{Im } z \in \Omega\}.$$

It is possible to define a natural polynomial function Δ on Ω such that the Bergman kernel for the unweighted Bergman space $A^{2,2}(T_\Omega)$ is given by $c\Delta^{-2m/r} \left(\frac{z-\bar{w}}{i} \right)$ for a suitable $c > 0$; cf. Sect. 2 for more information. A natural family of mixed-norm Lebesgue spaces may then be defined as

$$L_s^{p,q}(T_\Omega) = \left\{ f \text{ measurable in } T_\Omega : \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} |f(x+iy)|^p d\mathcal{H}^m(x) \right)^{q/p} \Delta^{qs-\frac{m}{r}}(y) d\mathcal{H}^m(y) < \infty \right\}.$$

Note that the invariant measure dy/y on $(0, \infty)$ is now replaced by the measure $\Delta^{-\frac{m}{r}}(y) d\mathcal{H}^m(y)$, which is invariant under the group of linear transformations that preserve Ω .

To the cone Ω is associated a positive integer r , which is called the *rank* of Ω , see Sect. 2. The only homogeneous cone of rank 1 is the half-line $\mathbb{R}_+^* := (0, \infty)$, and thus it is obvious that $\mathbb{C}_+ = T_{(0,\infty)}$ is the only tube domain over a symmetric cone in one

dimension. Apart from the reducible cone $(\mathbb{R}_+^*)^2$, the only (homogeneous, and actually) symmetric cones of rank 2 are, up to isomorphisms, the Lorentz cones

$$\Omega_L := \{h \in \mathbb{R}^m : h_1 > 0, h_1^2 - h_2^2 - \dots - h_m^2 > 0\}.$$

These cones constitute perhaps the simplest, yet typical, examples of symmetric cones. The first typical examples of (non-symmetric) homogeneous cones arise in rank 3.

The problem of determining the boundedness of the Bergman projectors on Bergman spaces on the tube domains over Lorentz cones was undertaken by D. Bekollé and A. Bonami [1], where they obtained the sharp range of boundedness of the unweighted positive projector $\mathcal{P}_+$.

It was later shown in [5], for tube domains over Lorentz cones, and then in [3], for tube domains over irreducible symmetric cones, that the Bergman projectors $\mathcal{P}_s$ may be bounded even when $\mathcal{P}_{s,+}$ is unbounded, thus showing that a finer analysis of the Bergman projectors must take cancellations into account. The projectors on this class of domains provide the only known examples of such phenomenon, making their analysis of great interest.

In order to exploit the cancellations of the weighted Bergman kernels, the authors of [5] considered the mixed-normed spaces $L_s^{2,q}(T_\Omega)$ and used the Fourier transform in the “horizontal” variable $x \in \mathbb{R}^m$. Then, interpolating with the range of boundedness of the operator $\mathcal{P}_{s,+}$, they obtained a larger range of boundedness for $\mathcal{P}_s$. More precise arguments, based on suitable decoupling-type inequalities on the relevant cone, were later developed in [3]. These methods, combined with the sharp ℓ^2 -decoupling inequality of Bourgain and Demeter [15], led to a precise characterization of the boundedness of Bergman projectors on the spaces $L_s^{p,q}$ (cf. [8]).

It seems worth remarking the connection between an extremely deep result in harmonic analysis such as the sharp ℓ^2 -decoupling inequality [15] and the sharp range of boundedness of Bergman projectors on the spaces $L_s^{p,q}$. In fact, as shown in [3, 18], certain decoupling-type inequalities adapted to the cone Ω , which reduce to the usual decoupling inequalities when Ω is a Lorentz cone and $D = T_\Omega$, are the main tool to prove the boundedness of the Bergman projectors on the spaces $L_s^{p,q}$. This approach extends the results which were previously obtained interpolating the sharp results that could be proved using Fourier techniques on $L_s^{2,q}$ with the results obtained by means of Schur’s lemma to prove the boundedness of $\mathcal{P}_{s',+}$ (cf. [2, 27]).

The aforementioned problems have also been considered for mixed norm weighted Bergman spaces on homogeneous Siegel domains of type II. In particular, in [27] it was shown that also in the case of homogeneous Siegel domains of type II, the Bergman projectors $\mathcal{P}_s$ may be bounded even when $\mathcal{P}_{s,+}$ is unbounded. We point out, however, that two non-equivalent classes of mixed norm weighted Bergman spaces on Siegel domains which are *not* of tube type have been considered, see Sect. 2 for more information.

One of the main themes of this paper is to extend the theory and results from the case of tube domains over homogeneous cones, that is, homogeneous Siegel domains of type I, to the case of homogeneous Siegel domains of type II. Perhaps the main feature, and difficulty, of such endeavour is the fact that the Euclidean Fourier transform on $\mathbb{R}^m$ needs to be replaced by the noncommutative Fourier transform on a (suitable) 2-step nilpotent Lie group.

In order to describe some of the aspects of the problems in this more general setting, we now briefly discuss the case of the Siegel upper half-space

$$\mathcal{U} = \{(\zeta, z) \in \mathbb{C}^n \times \mathbb{C} : \operatorname{Im} z > |\zeta|^2\}.$$

The domain $\mathcal{U}$ is the appropriate analogue of the upper half-plane $\mathbb{C}_+$ in this setting, since it is biholomorphically equivalent to the unit ball in $\mathbb{C}^{n+1}$. Notice that we can parametrize the boundary of $\mathcal{U}$ as the sets of points $\{(\zeta, x + i|\zeta|^2) : \zeta \in \mathbb{C}^n, x \in \mathbb{R}\}$ and then $\mathcal{U}$ as $\{(\zeta, x + i(y + |\zeta|^2)) : \zeta \in \mathbb{C}^n, x \in \mathbb{R}, y \in (0, \infty)\}$. We remark here that the boundary of $\mathcal{U}$ is then identified with the Heisenberg group $\mathbb{H}_n$ and that the non-commutative Fourier transforms on $\mathbb{H}_n$ plays a crucial role in this analysis.

Then, setting $f_y : (\zeta, x) \mapsto f(\zeta, x + i(y + |\zeta|^2))$, we define

$$L_s^{p,q}(\mathcal{U}) = \left\{ f \text{ measurable on } \mathcal{U} : \int_0^\infty \left(\int_{\mathbb{H}_n} |f_y(\zeta, x)|^p d\mathcal{H}^{2n+1}(\zeta, x) \right)^{q/p} y^{qs-1} dy < \infty \right\},$$

and $A_s^{p,q}(\mathcal{U}) := \operatorname{Hol}(\mathcal{U}) \cap L_s^{p,q}(\mathcal{U})$. Notice that we highlighted the L^p -integration on the translates of the boundary of $\mathcal{U}$, so that $A_0^{p,\infty}(\mathcal{U})$ is the Hardy space $H^p(\mathcal{U})$. In addition, $A_s^{p,q}(\mathcal{U}) = \{0\}$ unless $s > 0$ or $q = \infty$ and $s = 0$. Consequently, we shall again assume that $s > 0$, for simplicity. In this case, the weighted Bergman kernel, that is, the reproducing kernel of $A_s^{2,2}(\mathcal{U})$, is

$$K_s((\zeta, z), (\zeta', z')) = c'_s \left(\frac{z - \bar{w}}{2i} - \langle \zeta | \zeta' \rangle \right)^{-n-1-2s},$$

where $c'_s = \frac{(2s+n) \cdots (2s)}{4\pi^{n+1}}$ and $\langle \cdot | \cdot \rangle$ is the standard (hermitian) scalar product on $\mathbb{C}^n$. Then, the orthogonal projection of $L_s^{2,2}(\mathcal{U})$ onto $A_s^{2,2}(\mathcal{U})$ is given by

$$\mathcal{P}_s f(\zeta, z) = \int_{\mathcal{U}} f(\zeta', z') K_s((\zeta, z), (\zeta', z')) (\operatorname{Im} z' - |\zeta'|^2)^{2s-1} d\mathcal{H}^{2n+2}(\zeta', z').$$

In this setting it is known that the operator $\mathcal{P}_{\tilde{s}}$ induces a continuous endomorphism of $L_s^{p,q}(\mathcal{U})$ if and only if $2\tilde{s} > s$, see, e.g. [18, Proposition 5.20 and Corollary 5.27], with the same remarks as above. Also in this case the associated integral operator $\mathcal{P}_{\tilde{s},+}$ is bounded on $L_s^{p,q}(\mathcal{U})$ if and only if $\mathcal{P}_{\tilde{s}}$ is. Again, for $p, q \in (0, \infty)$, the sesquilinear form

$$(f, g) \mapsto \int_{\mathcal{U}} f(\zeta, z) \overline{g(\zeta, z)} (\operatorname{Im} z - |\zeta|^2)^{s+\tilde{s}+(1/p-1)_+(n+1)-1} d\mathcal{H}^{2n+2}(\zeta, z)$$

is continuous on $A_s^{p,q}(\mathcal{U}) \times A_{\tilde{s}}^{p',q'}(\mathcal{U})$ and induces an antilinear isomorphism of $A_s^{p',q'}(\mathcal{U})$ onto $A_s^{p,q}(\mathcal{U})'$.

Concerning atomic decomposition, the mapping

$$(\lambda_{j,k}) \mapsto \sum_{j,k} \lambda_{j,k} K_{\tilde{s}}(\cdot, (\zeta_{j,k,R}, z_{j,k,R})) (\operatorname{Im} z_{j,k,R} - |\zeta_{j,k,R}|^2)^{2\tilde{s}-s+(n+1)(1-1/p)}$$

maps $\ell^{p,q}$ continuously into (onto, for R small) $A_s^{p,q}(\mathcal{U})$ if and only if $2\tilde{s} > s + \left(\frac{1}{p} - 1\right)_+ (n+1)$ (cf. [18, Lemma 3.29 and Theorem 3.34], with the same remarks as above), where

$$(\zeta_{j,k,R}, z_{j,k,R}) = 2^{kR} (R^{1/2} j_1, R j_2 + i)$$

for every $j \in \mathbb{Z}[i]^n \times \mathbb{Z}$ and for every $k \in \mathbb{Z}$ (cf. the proof of [18, Lemma 2.55] and [21, Lemma 6.1]). One may also consider more general sequences, as in the case of $\mathbb{C}_+$.

Finally, the boundary values of $A_s^{p,q}(\mathcal{U})$, that is, the limits of $f_y := f(\cdot + (0, iy))$ as $y \rightarrow 0^+$ (in a suitable topology) belong to a closed subspace of a suitable homogeneous Besov space $\dot{B}_{p,q}^{-s}(\mathbb{H}_n)$ associated with a sub-Laplacian $\mathcal{L}$ which is homogeneous for the natural dilations defined by $R \cdot (\zeta, x) = (R\zeta, R^2x)$. Also in this case, the subspace is determined by a condition on the (non-commutative) Fourier transform on $\mathbb{H}_n$, cf. [16, Corollary 8.2] for more details. Notice that an analogous description of the space of boundary values of the Bergman spaces in terms of ‘classical’ Besov spaces is no longer possible when $r > 1$, that is, when Ω is not a half-line, cf. [16, Sect. 8].

Various relationships between the aforementioned properties have been studied at different levels of generality. As a non-exhaustive review of the literature, we mention the following works. In [28] the authors studied the mixed-normed Bergman spaces in $\mathbb{C}_+$ and provided a characterization of the dual as a consequence of atomic decomposition. In [11] some connections between the boundedness of Bergman projectors, generalized Hardy’s inequalities, and the characterization of the boundary values were highlighted. In [2, 9] the atomic decomposition is seen as a consequence of the characterization of the dual, hence of the continuity of Bergman projectors, while in [5] the continuity of the Bergman projectors is obtained as a consequence of the characterization of boundary values. In [4] generalized Hardy’s inequality, the continuity of Bergman projectors, and the characterization of the duals were proved to be equivalent in some situations. Finally, in [18] the above properties are proved to be equivalent (in complete generality, but) in a suitable weak sense. We also mention [25] where the authors provide a necessary and a sufficient condition for the boundedness of the positive Bergman projectors on Siegel domains of Type II, and [19, 20] for some related recent works on Bergman spaces on homogeneous Siegel domains of Type II.

We shall now outline the new results and the organization of the paper. In Sect. 2 we provide some preliminary material on homogeneous Siegel domains of Type II. In Sect. 3, we extend some previous results (cf. [2, Theorem 4.10]) on the continuity of $\mathcal{P}_{s,+}$. In particular, we shall prove that the boundedness of $\mathcal{P}_{s,+}$ on $L_s^{p,q}$ does *not* depend on $p \in [1, \infty]$. In Sect. 4, we present a rather general and complete treatment of the equivalence between the various problems presented above (except for what concerns generalized Hardy’s inequalities) for $p, q \in [1, \infty]$. Our main results are summarized in Corollary 4.7, where we prove the equivalence between the following properties:

- (1) the continuity of $P_{\bar{s}}$ on $L_s^{p,q}$;
- (2) the continuity (and surjectivity, for fine ‘lattices’) of the atomic decomposition mappings in $A_s^{p,q}$ associated with the kernel $K_{\bar{s}}$;
- (3) the identification of $A_s^{p,q}$ and $A_{2\bar{s}-s}^{p',q'}$ with the duals of one another with respect to the natural sesquilinear pairing;
- (4) the identification of the spaces of boundary values of $A_s^{p,q}$ and $A_{2\bar{s}-s}^{p',q'}$ with suitable Besov spaces of analytic type.

We shall also indicate some results for the general case $p, q \in (0, \infty]$, even though the lack of duality arguments does not allow a complete treatment in this more general situation. In order to do that, we shall also prove the equivalence of the validity of each one

of the properties (1) to (4) on $A_s^{p,q}$ and on a suitable closed subspace of $A_s^{p,q}$ (which is different from $A_s^{p,q}$ only when $\max(p, q) = \infty$, and differs from $A_s^{p,q}$ because of some vanishing conditions at infinity).

In Sect. 5, we shall consider how complex interpolation interacts with the preceding properties, thus extending some results of [6]. In particular, we shall prove that the set of $(p, q, s, \bar{s})$ for which the (equivalent) properties (1)–(4) hold is convex.

In Sect. 6 we shall prove some transference results between weighted Bergman spaces on a Siegel domain of type II and the corresponding tube domain, thus extending some results of [8]. In particular, we shall prove that, under suitable assumptions, if $\mathcal{P}_{T_{\Omega, \bar{s}}}$ is bounded on $L_s^{p,p}(T_{\Omega})$, where T_{Ω} denotes the tube domain associated with the Siegel domain D , then $\mathcal{P}_{D, \bar{s}}$ is bounded on $L_s^{p,p}(D)$. Conversely, if $\mathcal{P}_{D, \bar{s}}$ is bounded on $L_s^{p,q}(D)$, then $\mathcal{P}_{T_{\Omega, \bar{s}+b/2}}$ is bounded on $L_{s-b/p}^{p,q}(D)$ for a suitable b depending only on D . Notice that one implication allows to deal with the case $p \neq q$, while the other one does not: this is related to our definition of $L_s^{p,q}(D)$. The other definition of $L_s^{p,q}(D)$ available in the literature would lead to a similar result where these restrictions are swapped (cf. [8]).

Finally, Sect. 7 is devoted to the proof of Proposition 2.5, which extends [3, Proposition 4.34] (cf., also, [8]) to the present setting.

We conclude this introduction by pointing out that one of our contributions lies in the treatment of the more delicate cases in which the spaces $A_s^{p,q}$ are *not* reflexive (namely, when $\max(p, q, p', q') = \infty$). In other situations, we simplify the arguments present in the literature and employ the Besov spaces and the associated extension operator developed in [18] to work in full generality. In this case, the passage from tube domains to general Siegel domains requires a deeper analysis of the Fourier transform associated with the canonical (non-commutative) group structure on the Šilov boundary of D , cf. [16, 18] for a more detailed treatment of this subject (cf., also, Remark 2.6). Finally, also the passage from the case of symmetric domains to the case of homogeneous ones sometimes requires to develop rather different proofs, as in Sect. 7. Also the proof of the fundamental so-called ‘Korányi’s lemma’ requires the development of further techniques in the case of homogeneous domains, cf. [9].

2 Preliminaries

2.1 Homogeneous Siegel domains

We fix a complex Hilbert space E of dimension n , a real Hilbert space of dimension m , an open convex cone $\Omega \subseteq F$ not containing affine lines, and an Ω -positive non-degenerate hermitian form $\Phi : E \times E \rightarrow F_{\mathbb{C}}$, where $F_{\mathbb{C}}$ denotes the complexification of F (i.e., $\Phi(\zeta, \zeta) \in \overline{\Omega} \setminus \{0\}$ for every non-zero $\zeta \in E$). We shall assume that there is a triangular subgroup⁴ T_+ of $GL(F)$ which acts simply transitively on Ω , and that for every $t \in T_+$ there is $g \in GL(E)$ such that $t\Phi = \Phi(g \times g)$. Observe that T_+ is uniquely determined up to the conjugation by a linear automorphism of F which preserves Ω (cf. [31, 32]). We define $\Phi(\zeta) := \Phi(\zeta, \zeta)$ for every $\zeta \in E$,

$$\rho : E \times F_{\mathbb{C}} \ni (\zeta, z) \mapsto \operatorname{Im} z - \Phi(\zeta) \in F$$

⁴ This means that there is a basis of F with respect to which all elements of T_+ are represented by an upper triangular matrix. Equivalently, all eigenvalues of every element of T_+ are real (cf. [31]).

and $D := \rho^{-1}(\Omega)$, so that D is a homogeneous Siegel domain of type II.

We denote by $\Omega' := \left\{ \lambda \in F' : \forall h \in \overline{\Omega} \setminus \{0\} \langle \lambda, h \rangle > 0 \right\}$ the dual cone of Ω , so that T_+ acts simply transitively on Ω' by transposition (cf. [32, Theorem 1]). We denote by $t \cdot h$ and $\lambda \cdot t$ the actions of $t \in T_+$ on $h \in \Omega$ and $\lambda \in \Omega'$. We use the same notation for the extensions of the actions of T_+ to $F_{\mathbb{C}}$ and $F'_{\mathbb{C}}$.

Then, there are $r \in \mathbb{N}$ and a homomorphism Δ of T_+ onto $(\mathbb{R}_+^*)^r$ with kernel $[T_+, T_+]$ (cf. [18, § 2.1]), so that the characters of T_+ are of the form $\Delta^s = \prod_{j=1}^r \Delta_j^{s_j}$, $s \in \mathbb{C}^r$. Once we fix two base points $e_{\Omega} \in \Omega$ and $e_{\Omega'} \in \Omega'$, we may then define ‘generalized power functions’ Δ_{Ω}^s and $\Delta_{\Omega'}^s$ on Ω and Ω' , respectively, so that

$$\Delta_{\Omega}^s(t \cdot e_{\Omega}) = \Delta^s(t) = \Delta_{\Omega'}^s(e_{\Omega'} \cdot t)$$

for every $t \in T_+$. Then, Δ_{Ω}^s and $\Delta_{\Omega'}^s$ extend to holomorphic functions on $\Omega + iF$ and $\Omega' + iF'$, respectively, for every $s \in \mathbb{C}^r$ (cf. [18, Corollary 2.25]). In order to simplify the notation, we shall introduce two orderings on $\mathbb{R}^r$:

$$s \leq s' \iff \forall j = 1, \dots, r \ s_j \leq s'_j \iff s' - s \in \mathbb{R}_+^r$$

and

$$s \leq s' \iff s = s' \vee \forall j = 1, \dots, r \ s_j < s'_j \iff s' - s \in \{0\} \cup (\mathbb{R}_+^*)^r,$$

which coincide only when $r = 1$.

We shall then assume that Δ is chosen so that Δ_{Ω}^s (or, equivalently, $\Delta_{\Omega'}^s$) is bounded on the bounded subsets of Ω if and only if $s \geq 0$, that is, $s \in \mathbb{R}_+^r$ (cf. [18, Lemma 2.34]). We shall further assume that $\Delta^s(t_r) = r^{\sum_j s_j}$ for every $r > 0$ and for every $s \in \mathbb{C}^r$, where t_r is the unique element of T_+ such that $t_r \cdot x = rx$ for every $x \in F$ (notice that T_+ necessarily contains elements which act as homotheties). Notice that this does not determine Δ completely; nonetheless, if Δ' is another homomorphism with the same properties as Δ , then there is a permutation σ on $\{1, \dots, r\}$ such that $\Delta'_j = \Delta_{\sigma(j)}$ for every $j = 1, \dots, r$. This allows us to use the results of [18], where specific choices of T_+ and Δ are made, with only minor modifications. Besides that, we hope that this axiomatic approach may help comparing the following formulae and results with those appearing in the literature, where a number of different conventions appear.

Observe that there are $\mathbf{d} < 0$, $\mathbf{b} \leq 0$, and $\mathbf{m}, \mathbf{m}' \geq 0$ such that the following hold:

- the measures

$$\nu_{\Omega} := \Delta_{\Omega}^{\mathbf{d}} \cdot \mathcal{H}^m, \quad \nu_{\Omega'} := \Delta_{\Omega'}^{\mathbf{d}} \cdot \mathcal{H}^m, \quad \text{and} \quad \nu_D := (\Delta_{\Omega}^{\mathbf{b}+2\mathbf{d}} \circ \rho) \cdot \mathcal{H}^{2n+2m}, \quad (1)$$

where $\mathcal{H}^k$ denotes the k -th dimensional Hausdorff measure (that is, Lebesgue measure), are invariant under all linear automorphisms of F and F' , and under all biholomorphisms of D , respectively (cf. [18, Propositions 2.19 and 2.44] and [22, Proposition I.3.1]);

- the measures $\Delta_{\Omega}^s \cdot \nu_{\Omega}$ and $\Delta_{\Omega'}^s \cdot \nu_{\Omega'}$ induce Radon measures on F and F' , respectively, if and only if $\operatorname{Re} s > \mathbf{m}$ and $\operatorname{Re} s > \mathbf{m}'$, respectively, in which case

$$\mathcal{L}(\Delta_{\Omega}^s \cdot \nu_{\Omega}) = \Gamma_{\Omega}(s) \Delta_{\Omega}^{-s} \quad \text{and} \quad \mathcal{L}(\Delta_{\Omega'}^s \cdot \nu_{\Omega'}) = \Gamma_{\Omega'}(s) \Delta_{\Omega'}^{-s},$$

for suitable $\Gamma_{\Omega}(s), \Gamma_{\Omega'}(s) \in \mathbb{C}$, where $\mathcal{L}$ denotes the Laplace transform (cf. [18, Proposition 2.19]).

Then, $\mathbf{d} = -(\mathbf{1}_r + \mathbf{m}/2 + \mathbf{m}'/2)$ (cf. [18, Definition 2.8, Lemma 2.18, and Proposition 2.19]).

In addition, there are unique holomorphic families $(I_\Omega^s)_{s \in \mathbb{C}^r}$ and $(I_{\Omega'}^s)_{s \in \mathbb{C}^r}$ of tempered distributions on F and F' , respectively, such that $\mathcal{L}I_\Omega^s = \Delta_{\Omega'}^{-s}$ and $\mathcal{L}I_{\Omega'}^s = \Delta_\Omega^{-s}$ on Ω' and Ω , respectively (cf. [18, Lemma 2.26 and Proposition 2.28]). In particular, $I_\Omega^s = \frac{1}{\Gamma_\Omega(s)} \Delta_\Omega^s \cdot \nu_\Omega$ for $s > \mathbf{m}$, while $I_{\Omega'}^s = \frac{1}{\Gamma_{\Omega'}(s)} \Delta_{\Omega'}^s \cdot \nu_{\Omega'}$ for $s > \mathbf{m}'$ (cf. [18, Proposition 2.28]). We denote by $\mathbb{N}_\Omega$ and $\mathbb{N}_{\Omega'}$ the sets of $\mathbf{s} \in \mathbb{R}^r$ such that $\Delta_\Omega^{\mathbf{s}}$ and $\Delta_{\Omega'}^{\mathbf{s}}$ are polynomials, respectively; equivalently, $I_{\Omega'}^{-\mathbf{s}}$ and $I_\Omega^{-\mathbf{s}}$ are supported in $\{0\}$, respectively.

Let us now briefly describe an example.

Example 2.1 Fix $r \in \mathbb{N}^*$ and $k \in \mathbb{N}$. Let E be the space of $k \times r$ matrices over $\mathbb{C}$, and let F be the space of self-adjoint $r \times r$ matrices over $\mathbb{C}$, so that $n = kr$ and $m = r^2$. Let Ω be the cone of non-degenerate positive self-adjoint $r \times r$ matrices over $\mathbb{C}$, and define

$$\Phi : E \times E \ni (\zeta, \zeta') \mapsto \zeta'^* \zeta \in F_{\mathbb{C}},$$

so that $\Phi(\zeta, \zeta) = \zeta^* \zeta \in \overline{\Omega}$ for every $\zeta \in E$.⁵ In this case, one may choose T_+ to be the group of upper triangular $r \times r$ matrices over $\mathbb{C}$ with strictly positive elements on the diagonal, endowed with the action

$$T_+ \times F \ni (t, x) \mapsto txt^* \in F.$$

Then, one may define $\Delta : T_+ \ni (t_{j,k}) \mapsto (t_{j,j}^2) \in (\mathbb{R}_+^*)^r$. With this choice,

$$\Delta_\Omega^{\mathbf{s}}(x) = (\det(x_{j,k})_{j,k=1,\dots,r})^{s_1-s_2} \cdots (\det(x_{j,k})_{j,k=r-1,r})^{s_{r-1}-s_r} (x_{r,r})^{s_r}$$

for every $\mathbf{s} \in \mathbb{C}^r$ and for every $x \in \Omega$.

The resulting Siegel domain D is then homogeneous, since

$$t \cdot \Phi(\zeta, \zeta') = \Phi(\zeta t^*, \zeta' t^*)$$

for every $t \in T_+$ and for every $\zeta, \zeta' \in E$. In this case, we have $\mathbf{m} = (2(r-j))_{j=1,\dots,r}$, $\mathbf{m}' = (2(j-1))_{j=1,\dots,r}$, $\mathbf{d} = -r\mathbf{1}_r$, and $\mathbf{b} = -k\mathbf{1}_r$. Cf. [18, Examples 1.3, 2.6, and 2.14].

2.2 Fourier analysis on the Šilov boundary

We endow $E \times F_{\mathbb{C}}$ with the product

$$(\zeta, z) \cdot (\zeta', z') := (\zeta + \zeta', z + z' + 2i\Phi(\zeta', \zeta))$$

for every $(\zeta, z), (\zeta', z') \in E \times F_{\mathbb{C}}$, so that $E \times F_{\mathbb{C}}$ becomes a 2-step nilpotent Lie group, and the Šilov boundary $bD = \rho^{-1}(0)$ a closed subgroup such that $bD \cdot D \subseteq D$. We identify bD with $\mathcal{N} := E \times F$ by means of the mapping $(\zeta, x) \mapsto (\zeta, x + i\Phi(\zeta))$, so that $\mathcal{N}$ inherits the product

$$(\zeta, x)(\zeta', x') = (\zeta + \zeta', x + x' + 2\text{Im } \Phi(\zeta, \zeta')),$$

for every $(\zeta, x), (\zeta', x') \in \mathcal{N}$. In addition, $\mathcal{N}$ inherits the structure of a CR manifold (cf. [10]), with holomorphic tangent bundle generated by the left-invariant vector field Z_v which induce the Wirtinger derivatives $\partial_{E,v} := \frac{1}{2}(\partial_v - i\partial_{iv})$ at $(0, 0)$, for $v \in E$. A distribution u

⁵ Here we identify $F_{\mathbb{C}}$ with the space of $r \times r$ matrices over $\mathbb{C}$, since $F \cap (iF) = \{0\}$.

on $\mathcal{N}$ is then CR if $\overline{Z_v}u = 0$ for every $v \in E$. We endow $\mathcal{N}$ with the dilations defined by $R \cdot (\zeta, x) := (R^{1/2}\zeta, Rx)$ for every $R > 0$ and for every $(\zeta, x) \in \mathcal{N}$. Even though these dilations may seem unnatural, they are well adapted to the Fourier transform we are about to describe, as well as the functional calculi we shall need in the sequel.

We now recall some basic facts on Fourier analysis on $\mathcal{N}$. We limit ourselves to describing the Plancherel formula for CR elements of $L^2(\mathcal{N})$. Define

$$\Lambda_+ := \{\lambda \in F' : \forall \zeta \in E \setminus \{0\} \langle \lambda, \Phi(\zeta) \rangle > 0\},$$

so that Λ_+ is the interior of the polar of $\Phi(E)$ (cf. [17, Proposition 2.5]), contains Ω' , and coincides with Ω' if and only if $\mathbf{b} < \mathbf{0}$ (cf. [18, Corollary 2.58]). Then, for every $\lambda \in \Lambda_+$ the quotient $\mathcal{N}/\ker \lambda$ is isomorphic to a Heisenberg group (to $\mathbb{R}$, if $n = 0$), so that the Stone–von Neumann theorem (cf., e.g., [23, Theorem 1.50]) implies that there is (up to unitary equivalence) a unique irreducible (continuous) unitary representation π_λ of $\mathcal{N}$ in a Hilbert space H_λ such that $\pi_\lambda(0, x) = e^{-i\langle \lambda, x \rangle}$ for every $x \in F$. One may choose $H_\lambda = \text{Hol}(E) \cap L^2(e^{-2\langle \lambda, \Phi \rangle} \cdot \mathcal{H}^{2n})$ and

$$\pi_\lambda(\zeta, x)\psi(\omega) := e^{(\lambda_c, -ix + 2\Phi(\omega, \zeta) - \Phi(\zeta))} \psi(\omega - \zeta)$$

for every $(\zeta, x) \in \mathcal{N}$, for every $\psi \in H_\lambda$, and for every $\omega \in E$. If we define $P_{\lambda,0}$ as the orthogonal projector of H_λ onto the space of constant functions, then (cf. [18, Proposition 1.15])

$$\text{Tr}(P_{\lambda,0}\pi_\lambda(\zeta, x)) = e^{-\langle \lambda_c, ix + \Phi(\zeta) \rangle}$$

for every $(\zeta, x) \in \mathcal{N}$ and the mapping

$$L^1_{\text{CR}}(\mathcal{N}) \ni f \mapsto (\pi_\lambda(f)) \in \prod_{\lambda \in \Lambda_+} \mathcal{L}(H_\lambda)$$

induces an isometric isomorphism

$$L^2_{\text{CR}}(\mathcal{N}) \rightarrow c \int_{\Lambda_+}^{\oplus} \mathcal{L}^2(H_\lambda) P_{\lambda,0} \Delta_{\Omega'}^{-\mathbf{b}}(\lambda) d\lambda,$$

where $L^1_{\text{CR}}(\mathcal{N})$ and $L^2_{\text{CR}}(\mathcal{N})$ denote the spaces of CR elements of $L^1(\mathcal{N})$ and $L^2(\mathcal{N})$, respectively, $\mathcal{L}(H_\lambda)$ and $\mathcal{L}^2(H_\lambda)$ denote the spaces of endomorphisms and Hilbert–Schmidt endomorphisms of H_λ , respectively, and $c > 0$ is a suitable constant (cf. [18, Corollary 1.17 and Propositions 1.19 and 2.30] or [17, Propositions 2.4 and 2.6]).

2.3 Lattices

We endow D with a complete Riemannian metric which is invariant under the group of affine automorphisms of D (e.g., the Bergman metric, cf. [18, §2.5]), and denote by d the corresponding distance. Observe that we may identify $\rho : D \rightarrow \Omega$ with the projection of D onto its quotient by the action of bD by isometries, so that Ω may be endowed with the quotient (complete) Riemannian metric, which is then T_+ -invariant (but not necessarily invariant under all linear automorphisms of Ω). We endow Ω' with the Riemannian metric induced by means of the diffeomorphism $\Omega \ni (t \cdot e_\Omega) \rightarrow (e_{\Omega'} \cdot t) \in \Omega'$, and denote by d_Ω

and $d_{\Omega'}$ the induced distances on Ω and Ω' , respectively. We denote by B , B_{Ω} , and $B_{\Omega'}$ the balls on D , Ω , and Ω' , respectively.

By an (δ, R) -lattice on Ω , for $\delta > 0$ and $R > 1$, we mean a family $(h_k)_{k \in K}$ of elements of Ω such that the balls $B_{\Omega}(h_k, \delta)$ are pairwise disjoint while the balls $B_{\Omega}(h_k, R\delta)$ cover Ω . For example, the $(\delta, 2)$ -lattices are the maximal families satisfying $d_{\Omega}(h_k, h_{k'}) \geq 2\delta$ for every $k \neq k'$. We define (δ, R) -lattices on Ω' analogously.

By a (δ, R) -lattice on D we mean a family $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$ such that the balls $B((\zeta_{j,k}, z_{j,k}), \delta)$ are pairwise disjoint, the balls $B((\zeta_{j,k}, z_{j,k}), R\delta)$ cover D , and there is a (δ, R) -lattice $(h_k)_{k \in K}$ on Ω such that $h_k = \rho(\zeta_{j,k}, z_{j,k})$ for every $j \in J$ and for every $k \in K$. Arguing as in [18, Lemma 2.55] (where the Bergman metric on D is considered), one may prove that $(\delta, 4)$ -lattices exist for every $\delta > 0$.

2.4 Bergman spaces

Definition 2.2 Take $\mathbf{s} \in \mathbb{R}^r$ and $p, q \in (0, \infty]$. We define (cf. (1))

$$L_{\mathbf{s}}^{p,q}(D) := \{f : f \text{ is measurable, } \|h \mapsto \Delta_{\Omega}^{\mathbf{s}}(h)\|_{L^p(\mathcal{N})} \|L^q(v_{\Omega})\| < \infty\},$$

modulo the space of negligible functions, where $f_h : \mathcal{N} \ni (\zeta, x) \mapsto f(\zeta, x + i\Phi(\zeta) + ih)$ for every $h \in \Omega$ and for every $f : D \rightarrow \mathbb{C}$. We define $L_{\mathbf{s},0}^{p,q}(D)$ as the closure of $C_c(D)$ in $L_{\mathbf{s}}^{p,q}(D)$.

We define

$$A_{\mathbf{s}}^{p,q}(D) := \text{Hol}(D) \cap L_{\mathbf{s}}^{p,q}(D) \quad \text{and} \quad A_{\mathbf{s},0}^{p,q}(D) := \text{Hol}(D) \cap L_{\mathbf{s},0}^{p,q}(D),$$

endowed with the corresponding topology.

We observe explicitly that $A_{\mathbf{s},0}^{p,q}(D) \neq \{0\}$ (resp. $A_{\mathbf{s}}^{p,q}(D) \neq \{0\}$) if and only if $\mathbf{s} > \frac{1}{2q}\mathbf{m}$ (resp. $\mathbf{s} \geq 0$ if $q = \infty$), and that $A_{\mathbf{0}}^{p,\infty}(D)$ is the Hardy space $H^p(D)$ (cf. [18, Proposition 3.5]).

By [18, Proposition 3.11], $A_{\mathbf{s}}^{2,2}(D)$ is a reproducing kernel Hilbert space for every $\mathbf{s} > \frac{1}{4}\mathbf{m}$, with reproducing kernel

$$((\zeta, z), (\zeta', z')) \mapsto c'_{\mathbf{s}} B_{(\zeta', z')}^{\mathbf{b}+\mathbf{d}-2\mathbf{s}}(\zeta, z),$$

where $c'_{\mathbf{s}}$ is a suitable (non-zero) constant and

$$B_{(\zeta', z')}^{\mathbf{s}'}(\zeta, z) := \Delta_{\Omega}^{\mathbf{s}'} \left(\frac{z - \bar{z}'}{2i} - \Phi(\zeta, \zeta') \right)$$

for every $(\zeta, z), (\zeta', z') \in D$ and for every $\mathbf{s}' \in \mathbb{R}^r$. We then denote by $P_{\mathbf{b}+\mathbf{d}-2\mathbf{s}}$ the orthogonal projector of $L_{\mathbf{s}}^{2,2}(D)$ onto $A_{\mathbf{s}}^{2,2}(D)$, so that

$$P_{\mathbf{s}'} f(\zeta, z) = c_{\mathbf{s}'} \int_D f \overline{B_{(\zeta, z)}^{\mathbf{s}'}} (\Delta_{\Omega}^{-\mathbf{s}'} \circ \rho) d\nu_D$$

for every $f \in C_c(D)$, for every $(\zeta, z) \in D$, and for every $\mathbf{s}' < \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$, where $c_{\mathbf{s}'} := c'_{(\mathbf{b}+\mathbf{d}-\mathbf{s}')/2}$.

Notice that, even though this choice may seem unnatural, it has the advantage of simplifying the notation.

2.5 Besov spaces

Denote by $\mathcal{S}(\mathcal{N})$ the Schwartz space on $\mathcal{N}$ and define, for every compact subset K of Ω' ,

$$\mathcal{S}_{\Omega}(\mathcal{N}, K) := \{ \varphi \in \mathcal{S}(\mathcal{N}) : \varphi \text{ is CR, } \forall \lambda \in \Lambda_+ \pi_{\lambda}(\varphi) = \chi_K(\lambda) P_{\lambda,0} \pi_{\lambda}(\varphi) P_{\lambda,0} \}$$

and

$$\mathcal{S}_{\Omega,L}(\mathcal{N}, K) := \mathcal{S}(\mathcal{N}) * \mathcal{S}_{\Omega}(\mathcal{N}, K),$$

endowed with the topology induced by $\mathcal{S}(\mathcal{N})$, and

$$\mathcal{S}_{\Omega}(\mathcal{N}) = \lim_{\substack{\longrightarrow \\ K}} \mathcal{S}_{\Omega}(\mathcal{N}, K) \quad \text{and} \quad \mathcal{S}_{\Omega,L}(\mathcal{N}) = \lim_{\substack{\longrightarrow \\ K}} \mathcal{S}_{\Omega,L}(\mathcal{N}, K),$$

endowed with their locally convex topologies.

We denote by $\mathcal{S}'_{\Omega,L}(\mathcal{N})$ the dual of the conjugate of $\mathcal{S}_{\Omega,L}(\mathcal{N})$. Then, the mapping

$$\mathcal{F}_{\mathcal{N}} : \mathcal{S}_{\Omega}(\mathcal{N}) \ni \varphi \mapsto [\lambda \mapsto \text{Tr}(\pi_{\lambda}(\varphi))] \in C_c^{\infty}(\Omega')$$

is an isomorphism, and there is $c > 0$ such that

$$(\mathcal{F}_{\mathcal{N}}^{-1} \psi)(\zeta, x) = c \int_{\Omega'} \psi(\lambda) e^{\langle \lambda_{\mathbb{C}}, i x - \Phi(\zeta) \rangle} \Delta_{\Omega'}^{-\mathbf{b}}(\lambda) d\lambda \quad (2)$$

for every $\psi \in C_c^{\infty}(\Omega')$ and for every $(\zeta, x) \in \mathcal{N}$ (cf. [18, Proposition 4.2], but also [17, § 5]).

Definition 2.3 Take $p, q \in (0, \infty]$ and $\mathbf{s} \in \mathbb{R}^r$. Given a (δ, R) -lattice $(\lambda_k)_{k \in K}$ on Ω' , and a bounded family $(\varphi_k)_{k \in K}$ of positive elements of $C_c^{\infty}(\Omega')$ such that

$$\sum_{k \in K} \varphi_k(\cdot t_k^{-1}) \geq 1$$

on Ω' , where $t_k \in T_+$ and $\lambda_k = e_{\Omega'} \cdot t_k$ for every $k \in K$, we define $B_{p,q}^{\mathbf{s}}(\mathcal{N}, \Omega)$ (resp. $\dot{B}_{p,q}^{\mathbf{s}}(\mathcal{N}, \Omega)$) as the space of $u \in \mathcal{S}'_{\Omega,L}(\mathcal{N}, \Omega)$ such that

$$(\Delta_{\Omega'}^{\mathbf{s}}(\lambda_k) u * \psi_k) \in \ell^q(K; L^p(\mathcal{N})) \quad (\text{resp. } (\Delta_{\Omega'}^{\mathbf{s}}(\lambda_k) u * \psi_k) \in \ell_0^q(K; L_0^p(\mathcal{N}))),$$

where $\psi_k := \mathcal{F}_{\mathcal{N}}^{-1}(\varphi_k(\cdot t_k^{-1}))$ for every $k \in K$.⁶

Notice that $\dot{B}_{\mathbf{s}}^{p,q}(\mathcal{N}, \Omega)$ is the closure of (the canonical image of) $\mathcal{S}_{\Omega,L}(\mathcal{N})$ in $B_{\mathbf{s}}^{p,q}(\mathcal{N}, \Omega)$, and that the definition of both spaces does not depend on the choice of (λ_k) and (φ_k) by [18, Lemma 4.14 and Theorem 4.23]. In addition, the closure of $\mathcal{S}_{\Omega,L}(\mathcal{N})$ in $\mathcal{S}(\mathcal{N})$ is (cf. [17, Theorem 5.13])

⁶ As before, $L_0^p(\mathcal{N})$ is the closure of $C_c(\mathcal{N})$ in $L^p(\mathcal{N})$, while $\ell_0^q(K; L_0^p(\mathcal{N}))$ is the closure of $L_0^p(\mathcal{N})^{(K)}$ in $\ell^q(K; L_0^p(\mathcal{N}))$. In addition, $u * \psi_k \in \mathcal{S}'(\mathcal{N})$ is defined so that $\langle u * \psi_k | \tau \rangle = \langle u | \tau * \psi_k^* \rangle$ for every $\tau \in \mathcal{S}(\mathcal{N})$. The definition is well posed since $\psi_k^* \in \mathcal{S}_{\Omega}(\mathcal{N})$, so that $\tau * \psi_k^* \in \mathcal{S}_{\Omega,L}(\mathcal{N})$. It is then readily seen that $u * \psi_k$ is actually a function of class C^{∞} .

$$\widetilde{\mathcal{S}}_{\Omega}(\mathcal{N}) := \{\varphi \in \mathcal{S}(\mathcal{N}) : \varphi \text{ is CR}, \forall \lambda \in \Lambda_+ \setminus \Omega' \pi_{\lambda}(\varphi) = 0\}$$

and $B_{p,q}^s(\mathcal{N}, \Omega)$ embeds canonically into the dual of the conjugate of $\widetilde{\mathcal{S}}_{\Omega}(\mathcal{N})$ (which is a quotient of $\mathcal{S}'(\mathcal{N})$) for every $s \in \mathbb{R}'$ and for every $p, q \in (0, \infty]$ (cf. [16, Proposition 7.12]).

One may then define a canonical sesquilinear form

$$B_{p,q}^s(\mathcal{N}, \Omega) \times B_{p',q'}^{-s-(1/p-1)_+(\mathbf{b}+\mathbf{d})}(\mathcal{N}, \Omega) \rightarrow \mathbb{C}$$

so that

$$\langle u|u' \rangle := \sum_k \langle u * \psi_k | u' * \psi_k \rangle,$$

where (ψ_k) is as in Definition 2.3 and $\sum_k (\mathcal{F}_{\mathcal{N}} \psi_k)^2 = 1$ on Ω' . This definition does *not* depend on the choice of (ψ_k) (cf. [18, Proposition 4.20]), and identifies $B_{p',q'}^{-s-(1/p-1)_+(\mathbf{b}+\mathbf{d})}(\mathcal{N}, \Omega)$ with the dual of $\dot{B}_{p,q}^s(\mathcal{N}, \Omega)$ (cf. [18, Theorem 4.23]). By analogy, we define the weak topology $\sigma_{p,q}^s$ as the weak topology $\sigma(B_{p,q}^s(\mathcal{N}, \Omega), \dot{B}_{p',q'}^{-s-(1/p-1)_+(\mathbf{b}+\mathbf{d})}(\mathcal{N}, \Omega))$ with respect to this sesquilinear form.

2.6 An extension operator

Observe that $H^2(D) = A_0^{2,\infty}(D)$ is a reproducing kernel Hilbert space, with reproducing (Cauchy–Szegő) kernel

$$((\zeta, z), (\zeta', z')) \mapsto c_0 B_{(\zeta', z')}^{\mathbf{b}+\mathbf{d}}(\zeta, z).$$

We then define a continuous linear mapping (cf. [18, Lemma 2.51, Theorem 5.2])

$$\mathcal{E} : B_{p,q}^{-s}(\mathcal{N}, \Omega) \ni u \mapsto [(\zeta, z) \mapsto c_0 \langle u | (B_{(\zeta,z)}^{\mathbf{b}+\mathbf{d}})_0 \rangle] \in A_{s-(\mathbf{b}+\mathbf{d})/p}^{\infty,\infty}(D)$$

for $s > \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$, so that $(B_{(\zeta,z)}^{\mathbf{b}+\mathbf{d}})_0 \in \dot{B}_{p',q'}^{-s-(1/p-1)_+(\mathbf{b}+\mathbf{d})}(\mathcal{N}, \Omega)$ for every $(\zeta, z) \in D$. In addition, $(\mathcal{E}u)_h$ converges to u in $B_{p,q}^{-s}(\mathcal{N}, \Omega)$ if $u \in \dot{B}_{p,q}^{-s}(\mathcal{N}, \Omega)$, in the weak topology $\sigma_{p,q}^s$ if $u \in B_{p,q}^{-s}(\mathcal{N}, \Omega)$. In particular, $\mathcal{E}$ is one-to-one.

We define

$$\widetilde{A}_s^{p,q}(D) := \mathcal{E}(B_{p,q}^s(\mathcal{N}, \Omega)) \quad \text{and} \quad \widetilde{A}_{s,0}^{p,q}(D) := \mathcal{E}(\dot{B}_{p,q}^s(\mathcal{N}, \Omega)),$$

endowed with the corresponding topology. We denote by $\widetilde{\sigma}_s^{p,q}$ the topology on $\widetilde{A}_s^{p,q}(D)$ and $\widetilde{A}_{s,0}^{p,q}(D)$ induced by the weak topology $\sigma_{p,q}^{-s}$ on $B_{p,q}^{-s}(\mathcal{N}, \Omega)$.

The following result summarizes [18, Proposition 5.4 and Corollary 5.11].

Proposition 2.4 *Take $p, q \in (0, \infty]$ and $s > \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$ such that $s > \frac{1}{2q}\mathbf{m}$ (resp. $s \geq 0$ if $q = \infty$). Then, there are continuous inclusions*

$$\mathcal{E}(\mathcal{S}_{\Omega,L}(\mathcal{N})) \subseteq \widetilde{A}_{s,0}^{p,q}(D) \subseteq \widetilde{A}_s^{p,q}(D) \quad (\text{resp. } \mathcal{E}(\mathcal{S}_{\Omega,L}(\mathcal{N})) \subseteq A_s^{p,q}(D) \subseteq \widetilde{A}_s^{p,q}(D)).$$

If, in addition,

$$s > \frac{1}{2q} \mathbf{m} + \left(\frac{1}{2 \min(p, p')} - \frac{1}{2q} \right)_+ \mathbf{m}',$$

then $A_{s,0}^{p,q}(D) = \tilde{A}_{s,0}^{p,q}(D)$ and $A_s^{p,q}(D) = \tilde{A}_s^{p,q}(D)$.

We are able to improve the first half of [18, Proposition 5.18] following [3, Proposition 4.34] and [8]. For clarity of presentation, we postpone the proof until Sect. 7.

Proposition 2.5 Take $p, q \in (0, \infty]$ and $s > \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q'} \mathbf{m}'$. If $\tilde{A}_{s,0}^{p,q}(D) = A_{s,0}^{p,q}(D)$, then $s > \left(\frac{1}{2p} - \frac{1}{2q} \right)_+ \mathbf{m}'$. If, in addition, $n = 0$, then $s > \left(\frac{1}{2 \min(2,p)} - \frac{1}{2q} \right)_+ \mathbf{m}'$.

Remark 2.6 The second part of Proposition 2.5 is actually [18, (2) of Proposition 5.18], which, in turn, extends [3, Proposition 4.34]. The argument which leads to the proof of these results is based on the interplay between modulation ‘in space’ and traslation ‘in frequency’ when using the (Euclidean) Fourier transform. Since this interplay is no longer valid when using the non-commutative Fourier transform on $\mathcal{N}$, we do not know if this result holds for $n > 0$.

When $n = 0$ and $r = 2$, so that Ω is either (isomorphic to) $(\mathbb{R}_+^*)^2$ or a Lorentz cone, then combining [8, Theorems 6.6 and 6.8] (the latter being a consequence of [15, Theorem 1.2]) with [18, Theorem 5.10], we see that $A_s^{p,q}(D) = \tilde{A}_s^{p,q}(D)$ and $A_{s,0}^{p,q}(D) = \tilde{A}_{s,0}^{p,q}(D)$, provided that

$$s > \frac{1}{2q} \mathbf{m} + \frac{1}{2} \left(\frac{1}{\min(2,p)} - \frac{1}{q} \right)_+ \mathbf{m}', \frac{\mathbf{d}}{p} + \frac{1}{2q'} \mathbf{m}'.$$

In addition, since in this case $\alpha \mathbf{m} + \beta \mathbf{m}' = \sup(\alpha \mathbf{m}, \beta \mathbf{m}')$ for every $\alpha, \beta \geq 0$ (cf. [18, Definition 2.8]), the preceding sufficient conditions are also necessary, thanks to Proposition 2.5.

We now recall another result concerning the dual of $\tilde{A}_{s,0}^{p,q}(D)$, namely [18, Proposition 5.12].⁷

Proposition 2.7 Take $p, q \in (0, \infty]$, $s > \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{q'} \mathbf{m}'$ and $s' > \frac{1}{p'}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q''} \mathbf{m}'$. Define $\mathbf{s}'' := \mathbf{s} + \mathbf{s}' - \left(\frac{1}{p} - 1 \right)_+ (\mathbf{b} + \mathbf{d})$, and assume that $\mathbf{s}'' > \frac{1}{2} \mathbf{m}$. Then, the sesquilinear form

$$\mathcal{E}(\mathcal{S}_{\Omega,L}(\mathcal{N})) \times \mathcal{E}(\mathcal{S}_{\Omega,L}(\mathcal{N})) \ni (f, g) \mapsto \int_D f \bar{g} (\Delta_{\Omega}^{\mathbf{s}'' - \mathbf{b} - \mathbf{d}} \circ \rho) \, d\nu_D$$

is well defined and extends to a unique continuous sesquilinear form on $\tilde{A}_{s,0}^{p,q}(D) \times \tilde{A}_{s'}^{p',q'}(D)$ which is continuous on the second factor with respect to $\tilde{\sigma}_{s'}^{p',q'}$ through $\mathcal{E}$. The extended

⁷ We observe explicitly that there is a mistake in the statement of the cited result, which we correct here. The proof is unchanged, besides the corresponding correction.

sesquilinear form induces an antilinear isomorphism of $\widetilde{A}_{s'}^{p',q'}(D)$ onto $\widetilde{A}_{s,0}^{p,q}(D)'$, and there is $c \neq 0$ such that

$$\langle \mathcal{E}u | \mathcal{E}(u' * I_{\Omega}^{-s''}) \rangle = c \langle u | u' \rangle$$

for every $u, u' \in \mathcal{S}_{\Omega,L}(\mathcal{N})$.

3 Continuity of $P_{s',+}$

In this section we extend to homogeneous Siegel domains of type II an interesting characterization of the boundedness of the operator

$$P_{s',+} : f \mapsto \int_D f(\zeta, z) |B_{(\zeta,z)}^{s'}| \Delta_{\Omega}^{-s'}(\rho(\zeta, z)) \, d\nu_D(\zeta, z)$$

provided in [2, Theorem 4.10] for irreducible symmetric tube domains ($\mathbf{s} \in \mathbb{R}\mathbf{d}$ and $\mathbf{s}' = \mathbf{d} - q\mathbf{s}$).

Theorem 3.1 *Take $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$ and $p, q \in [1, \infty]$. Then, $P_{s',+}$ induces an endomorphism of $L_{s,0}^{p,q}(D)$ (resp. $L_s^{p,q}(D)$) if and only if*

$$T : C_c(\Omega) \ni f \mapsto \Delta_{\Omega}^{\mathbf{s}} \int_{\Omega} f(h) \Delta_{\Omega}^{\mathbf{s}' - (\mathbf{b} + \mathbf{d})}(\cdot + h) \Delta_{\Omega}^{\mathbf{b} + \mathbf{d} - \mathbf{s} - \mathbf{s}'}(h) \, d\nu_{\Omega}(h)$$

induces an endomorphism of $L_{\Omega}^q(\nu_{\Omega})$ (resp. $L^q(\nu_{\Omega})$).

For the continuity of operators such as T , see [18, Lemma 3.35], but also [25] for a much more general treatment.

Before we pass to the proof, we need a lemma, which extends [2, Lemma 4.11]. Cf. also [29].

Lemma 3.2 *Take $\mathbf{s} \in \mathbb{C}^r$, and let U be a compact neighbourhood of $(0, 0)$ in $\mathcal{N}$. Then, there are two constants $C, c > 0$ such that*

$$\int_{U^2} \left| \left(B_{(\zeta,z)}^{s'} \right)_{h'}(\zeta', x') \right| d(\zeta', x') \geq C \Delta_{\Omega}^{\mathbf{s}' - (\mathbf{b} + \mathbf{d})}(h + h')$$

for every $(\zeta, z) \in D$ and for every $h' \in \Omega$ such that $(\zeta, \operatorname{Re} z) \in U$ and such that $|\rho(\zeta, z)|, |h'| \leq c$.

Proof Take an open neighbourhood U' of 0 in E and an open neighbourhood V' of 0 in F such that $U' \times V' \subseteq U$. Since Φ is proper, we may assume that $U' = \Phi^{-1}(\Phi(U'))$. Then, [18, Proposition 2.30] implies that there is a constant $C_1 > 0$ such that

$$\begin{aligned}
& \int_{U^2} \left| \left(B_{(\zeta, x + i\Phi(\zeta) + ih)}^{s'} \right)_{h'}(\zeta', x') \right| d(\zeta', x') \geq \int_{(\zeta, x) \in U} \left| \left(B_{(0, ih)}^{s'} \right)_{h'}((\zeta, x)^{-1}(\zeta', x')) \right| d(\zeta', x') \\
&= \int_U \left| \left(B_{(0, ih)}^{s'} \right)_{h'}(\zeta', x') \right| d(\zeta', x') \\
&= 2^{-s'} \int_U |\Delta_{\Omega}^{s'}(h + h' + \Phi(\zeta') - ix')| d(\zeta', x') \\
&\geq C_1 \int_{\Phi(U') \times V'} |\Delta_{\Omega}^{s'}(h + h' + h'' - ix')| d(I_{\Omega}^{-b} \otimes \mathcal{H}^m)(h'', x')
\end{aligned}$$

for every $(\zeta, x) \in U$ and for every $h, h' \in \Omega$. In addition, take $R > 0$ and observe that by [18, Corollary 2.51] there is a constant $C_2 \geq 1$ such that

$$\frac{1}{C_2} \leq \frac{|h|}{|h'|} \leq C_2$$

for every $h, h' \in \Omega$ with $d_{\Omega}(h, h') \leq R$. If we take $c > 0$ so that $\bar{B}_F(0, 3cC_2) \subseteq V'$, then

$$B_{\Omega}(h + h' + h'', R) \subseteq V'$$

for every $h, h', h'' \in \Omega$ with $|h|, |h'|, |h''| \leq c$. Since we may assume that $\Phi(U') \subseteq B_F(0, c)$, this implies that

$$\begin{aligned}
& \int_{U^2} \left| \left(B_{(\zeta, x + i\Phi(\zeta) + ih)}^{s'} \right)_{h'}(\zeta', x') \right| d(\zeta', x') \\
&\geq C_1 \int_{\Phi(U')} \int_{B_{\Omega}(h + h' + h'', R)} \left| \Delta_{\Omega}^{s'}(h + h' + h'' - ix') \right| dx' dI_{\Omega}^{-b}(h'')
\end{aligned}$$

for every $(\zeta, x) \in U$ and for every $h, h' \in \Omega \cap \bar{B}_F(0, c)$. Now, by homogeneity,

$$\int_{B_{\Omega}(h''', R)} \left| \Delta_{\Omega}^{s'}(h''' - ix') \right| dx' = C_3 \Delta_{\Omega}^{\text{Re } s' - d}(h''')$$

for every $h''' \in \Omega$, where $C_3 := \int_{B_{\Omega}(e_{\Omega}, R)} |\Delta_{\Omega}^{s'}(e_{\Omega} - ix')| dx'$. In addition,

$$\int_{\Phi(U')} \Delta_{\Omega}^{\text{Re } s' - d}(h + h' + h'') dI_{\Omega}^{-b}(h'') = J(h + h') \Delta_{\Omega}^{\text{Re } s' - b - d}(h + h'),$$

where

$$J(t \cdot e_{\Omega}) := \int_{t^{-1} \cdot \Phi(U')} \Delta_{\Omega}^{\text{Re } s' - d}(e_{\Omega} + h'') dI_{\Omega}^{-b}(h'')$$

for every $t \in T_+$. Now, define $Q := \{t \in T_+ : t \cdot e_{\Omega} \in e_{\Omega} - \bar{\Omega}\}$, and observe that

$$t \cdot [\Omega \cap (e_{\Omega} - \bar{\Omega})] \subseteq \Omega \cap (e_{\Omega} - \bar{\Omega} - \bar{\Omega}) = \Omega \cap (e_{\Omega} - \bar{\Omega})$$

for every $t \in Q$, so that Q is bounded in $\mathcal{L}(F)$ and $QQ \subseteq Q$. Now, by [18, §2.6], there is $e' \in \bar{\Omega}$ such that I_{Ω}^{-b} is concentrated on $T_+ \cdot e'$ and has support $\overline{T_+ \cdot e'}$ so that, by homogeneity, $\Phi(E) \supseteq T_+ \cdot e'$. Observe that $\Phi(U')$ is a neighbourhood of 0 in $\Phi(E)$ since Φ is

proper, thanks to [12, Proposition 10 of Chapter I, §5, No. 4]. Since $Q \cdot e'$ is relatively compact, this implies that there is $R' > 0$ such that $R'(Q \cdot e') \subseteq \Phi(U')$, so that

$$t^{-1} \cdot \Phi(U') \supseteq R'[t^{-1} \cdot (Q \cdot e')] \supseteq R'[t^{-1} \cdot (tQ \cdot e')] = R'(Q \cdot e')$$

for every $t \in Q$, so that

$$J(h) \geq \int_{R'Q \cdot e'} \Delta_{\Omega}^{\text{Re } s' - d}(e_{\Omega} + h'') dI_{\Omega}^{-b}(h'') > 0$$

for every $h \in Q \cdot e_{\Omega} = \Omega \cap (e_{\Omega} - \overline{\Omega})$. The conclusion follows, since we may take c so small that $\overline{B}_F(0, 2c) \cap \Omega \subseteq \Omega \cap (e_{\Omega} - \overline{\Omega})$. $\square$

Proof of Theorem 3.1 Assume first that T induces an endomorphism of $L_0^q(\nu_{\Omega})$ (resp. $L^q(\nu_{\Omega})$). Observe that, since T is an integral operator with a *positive* kernel, the endomorphism of $L_0^q(\nu_{\Omega})$ (resp. $L^q(\nu_{\Omega})$) induced by T is still an integral operator with the same kernel. Observe that [25, Theorem 1.4] implies that $s' < b + d - \frac{1}{2}m'$. Then, for every $f \in L_{s,0}^{p,q}(D)$ (resp. $f \in L_s^{p,q}(D)$)

$$\begin{aligned} |P_{s',+}f(\zeta, z)| &\leq \int_{\Omega} \int_{\mathcal{N}} |f_{h'}(\zeta', x')| \left| \left(B_{(0,ih')}^{s'} \right)_h \left((\zeta', x')^{-1}(\zeta, x) \right) \right| d(\zeta', x') \Delta_{\Omega}^{b+d-s'}(h') d\nu_{\Omega}(h') \\ &= \int_{\Omega} \left(|f_{h'}| * \left| \left(B_{(0,ih')}^{s'} \right)_h \right| \right) (\zeta, x) \Delta_{\Omega}^{b+d-s'}(h') d\nu_{\Omega}(h') \end{aligned}$$

for every $(\zeta, z) \in D$, where $h := \rho(\zeta, z)$ and $x := \text{Re } z$, so that, by Young's inequality

$$\begin{aligned} \Delta_{\Omega}^s(h) \|(P_{s',+}f)_h\|_{L^p(\mathcal{N})} &\leq C_1 \Delta_{\Omega}^s(h) \int_{\Omega} \|f_{h'}\|_{L^p(\mathcal{N})} \Delta_{\Omega}^{s'-(b+d)}(h+h') \Delta_{\Omega}^{b+d-s'}(h') d\nu_{\Omega}(h') \\ &= C_1 T(\Delta_{\Omega}^s \|f\|_{L^p(\mathcal{N})})(h) \end{aligned}$$

for every $h \in \Omega$, where $C_1 > 0$ is a suitable constant (cf. [18, Lemma 2.39]). Therefore, $P_{s',+}$ maps $L_{s,0}^{p,q}(D)$ (resp. $L_s^{p,q}(D)$) in $L_s^{p,q}(D)$. To conclude the proof of this implication, take $f \in C_c(D)$, and observe that $(P_{s',+}f)_h \in L_0^p(\mathcal{N})$ for every $h \in \Omega$, thanks to the preceding remarks. In addition, the mapping $h \mapsto \Delta_{\Omega}^s(h) \|f_h\|_{L^p(\mathcal{N})} \in \mathbb{R}$ clearly belongs to $C_c(\Omega)$, so that $P_{s',+}f \in L_{s,0}^{p,q}(D)$.

Conversely, assume that $P_{s',+}$ induces an endomorphism of $L_{s,0}^{p,q}(D)$ (resp. $L_s^{p,q}(D)$). As for T , the endomorphism of $L_{s,0}^{p,q}(D)$ (resp. $L_s^{p,q}(D)$) induced by $P_{s'}$ is an integral operator with the same kernel. Fix a compact neighborhood U of $(0, 0)$ in $\mathcal{N}$ and take $C, c > 0$ as in Lemma 3.2. Fix two positive function $\tau_1, \tau'_1 \in C_c(F)$ and a positive function $\tau_2 \in C_c(\mathcal{N})$ so that $\tau_1 = 1$ on $B_F(0, c)$, τ'_1 is non-zero and supported in $B_F(0, c)$, and $\tau_2 = 1$ on U^2 .

Take a positive $f \in L_0^q(\nu_{\Omega})$ (resp. $f \in L^q(\nu_{\Omega})$), and define $g \in L_{s,0}^{p,q}(D)$ (resp. $g \in L_s^{p,q}(D)$) so that

$$g_h = \tau_1(h) \Delta_{\Omega}^{-s}(h) f(h) \tau_2$$

for every $h \in \Omega$. Then,

$$\begin{aligned}
& (P_{s',+}g)_h(\zeta, x) \\
&= \int_{\Omega} \int_{\mathcal{N}} g_{h'}(\zeta', x') \left| \left(B_{(\zeta, x+i\Phi(\zeta)+ih)}^{s'} \right)_h(\zeta', x') \right| d(\zeta', x') \Delta_{\Omega}^{\mathbf{b}+\mathbf{d}-s'}(h') d\nu_{\Omega}(h') \\
&\geq \int_{\Omega \cap B_F(0,c)} \int_{U^2} \left| \left(B_{(\zeta, x+i\Phi(\zeta)+ih)}^{s'} \right)_h(\zeta', x') \right| d(\zeta', x') f(h') \Delta_{\Omega}^{\mathbf{b}+\mathbf{d}-s-s'}(h') d\nu_{\Omega}(h') \\
&\geq C \int_{\Omega \cap B_F(0,c)} f(h') \Delta_{\Omega}^{s'-(\mathbf{b}+\mathbf{d})}(h+h') \Delta_{\Omega}^{\mathbf{b}+\mathbf{d}-s-s'}(h') d\nu_{\Omega}(h') \\
&= C \Delta_{\Omega}^{-s}(h) T(\chi_{\Omega \cap B_F(0,c)} f)
\end{aligned}$$

for every $(\zeta, x) \in U$ and for every $h \in \Omega \cap B_F(0, c)$. Therefore,

$$\Delta_{\Omega}^s(h) \|(P_{s',+}g)_h\|_{L^p(\mathcal{N})} \geq C \mathcal{H}^{2n+m}(U) T(\chi_{B_F(0,c)} f)(h)$$

for every $h \in \Omega \cap B_F(0, c)$. Therefore, there is a constant $C_2 > 0$ such that

$$\|T(\tau_1' f)\|_{L^q(\nu_{\Omega})} \leq C_2 \|\tau_1 f\|_{L^q(\nu_{\Omega})}$$

for every (positive) $f \in L^q(\nu_{\Omega})$. By homogeneity, this implies that

$$\|T(\tau_1'(R \cdot) f)\|_{L^q(\nu_{\Omega})} \leq C_2 \|\tau_1(R \cdot) f\|_{L^q(\nu_{\Omega})}$$

for every $f \in L^q(\nu_{\Omega})$ and for every $R > 0$. In addition, the preceding remarks also imply that $T(\tau_1'(R \cdot) f) \in L_0^q(\nu_{\Omega})$ if $f \in L_0^q(\nu_{\Omega})$. We have thus proved that T induces an endomorphism of $L_0^q(\nu_{\Omega})$ (resp. a continuous linear mapping $L_0^q(\nu_{\Omega}) \rightarrow L^q(\nu_{\Omega})$). To conclude, it suffices to observe that, since T is an integral operator with a positive kernel, if T induces a continuous linear mapping $L_0^q(\nu_{\Omega}) \rightarrow L^q(\nu_{\Omega})$, then it also induces an endomorphism of $L^q(\nu_{\Omega})$ with the same expression and the same norm. $\square$

Corollary 3.3 *Take $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$ and $p, q \in [1, \infty]$ such that $P_{s',+}$ induces an endomorphism of $L_{s,0}^{p,q}(D)$ (resp. $L_s^{p,q}(D)$). Then, the following hold:*

- $\mathbf{s} > \frac{1}{2q} \mathbf{m}, \frac{1}{2q'} \mathbf{m}'$;
- $\mathbf{b} + \mathbf{d} - (\mathbf{s} + \mathbf{s}') > \frac{1}{2q'} \mathbf{m}, \frac{1}{2q} \mathbf{m}'$.

Observe that, when $r \leq 2$, the above necessary conditions are also sufficient, since in this case $\alpha \mathbf{m} + \beta \mathbf{m}' = \sup(\alpha \mathbf{m}, \beta \mathbf{m}')$ for every $\alpha, \beta \geq 0$ (cf. [18, Definition 2.8 and Corollary 5.23], or [25, Theorem 1.4]). The same holds also if D is irreducible and symmetric, and $\mathbf{s}$ and $\mathbf{s}'$ are parallel to $\mathbf{d}$. In this case, also $\mathbf{b}$ and $\mathbf{m} + \mathbf{m}'$ are parallel to $\mathbf{d}$, so that

$$\mathbf{s} > \frac{1}{2q} \mathbf{m} + \frac{1}{2q'} \mathbf{m}' \iff \mathbf{s} > \frac{1}{2q} \mathbf{m}, \frac{1}{2q'} \mathbf{m}'$$

and

$$\mathbf{b} + \mathbf{d} - (\mathbf{s} + \mathbf{s}') > \frac{1}{2q'} \mathbf{m} + \frac{1}{2q} \mathbf{m}' \iff \mathbf{b} + \mathbf{d} - (\mathbf{s} + \mathbf{s}') > \frac{1}{2q'} \mathbf{m}, \frac{1}{2q} \mathbf{m}'$$

since there are $j \neq k$ such that $m_j = m'_k = 0$ (cf. [18, Definition 2.8 and Corollary 5.23], or [25, Theorem 1.4]).

Proof This is a consequence of Theorem 3.1 and either [18, Proposition 5.20] or [25, Theorem 1.4]. $\square$

4 Equivalences

In this section we prove the equivalence of various notions of atomic decompositions, the continuity of Bergman projectors, and the determination of boundary values.

Proposition 4.1 Take $p, q \in (0, \infty]$ and $\mathbf{s} > \frac{\mathbf{b}+\mathbf{d}}{p} + \frac{1}{2q'}\mathbf{m}'$. Then, $\tilde{A}_{\mathbf{s},0}^{p,q}(D) = A_{\mathbf{s}}^{p,q}(D)$ if and only if $\tilde{A}_{\mathbf{s},0}^{p,q}(D) \subseteq A_{\mathbf{s}}^{p,q}(D)$. In particular, the following conditions are equivalent:

- (1) $\tilde{A}_{\mathbf{s},0}^{p,q}(D) = A_{\mathbf{s},0}^{p,q}(D)$;
- (2) $\mathbf{s} > \mathbf{0}$ and $\tilde{A}_{\mathbf{s}}^{p,q}(D) = A_{\mathbf{s}}^{p,q}(D)$.

We first need a technical lemma.

Lemma 4.2 There is a sequence (Ψ_j) of linear mappings $\mathcal{S}'_{\Omega,L}(\mathcal{N}) \rightarrow \mathcal{S}_{\Omega,L}(\mathcal{N})$ such that, for every $p, q \in (0, \infty]$ and for every $\mathbf{s} \in \mathbb{R}^r$, (Ψ_j) induces an equicontinuous sequence of endomorphisms of $B_{p,q}^{\mathbf{s}}(\mathcal{N}, \Omega)$, and $\Psi_j u$ converges to u in $\sigma_{p,q}^{\mathbf{s}}$ for every $u \in B_{p,q}^{\mathbf{s}}(\mathcal{N}, \Omega)$.

Proof Fix a (δ, R) -lattice $(\lambda_k)_{k \in K}$ on Ω' for some $\delta > 0$ and some $R > 1$, and a bounded family (φ_k) of positive elements of $C_c^\infty(B_{\Omega'}(e_{\Omega'}, R\delta))$ such that

$$\sum_{k \in K} \varphi_k(\cdot t_k^{-1}) = 1$$

on Ω' , where $t_k \in T_+$ and $\lambda_k = e_{\Omega'} \cdot t_k$ for every $k \in K$. Define $\psi_k := \mathcal{F}_{\mathcal{N}}^{-1}(\varphi_k(\cdot t_k^{-1}))$ for every $k \in K$. In addition, fix $\tau \in \mathcal{S}_{\Omega}(\mathcal{N})$ such that $\tau(0, 0) = 1$, and define $\tau_j := \tau(2^{-j} \cdot)$ for every $j \in \mathbb{N}$ (so that $\tau_j(\zeta, x) = \tau(2^{-j/2}\zeta, 2^{-j}x)$ for every $(\zeta, x) \in \mathcal{N}$ and $\mathcal{F}_{\mathcal{N}}\tau_j = (\mathcal{F}_{\mathcal{N}}\tau)(2^j \cdot)$). Observe that, for every $k \in K$, there is $j \in \mathbb{N}$ such that $B_{\Omega'}(\lambda_k, R\delta) + 2^{-j'} \text{Supp}(\mathcal{F}_{\mathcal{N}}\tau) \subseteq B_{\Omega'}(\lambda_k, 2R\delta)$ for every $j' \geq j$. Therefore, there is an increasing sequence (K_j) of finite subsets of K , whose union is K , such that

$$B_{\Omega'}(\lambda_k, R\delta) + \text{Supp}(\mathcal{F}_{\mathcal{N}}\tau_j) \subseteq B_{\Omega'}(\lambda_k, 2R\delta)$$

for every $k \in K_j$ and for every $j \in \mathbb{N}$. Define

$$\Psi_j u := \sum_{k \in K_j} (u * \psi_k) \tau_j$$

for every $u \in \mathcal{S}'_{\Omega,L}(\mathcal{N})$, and observe that $\Psi_j u \in \mathcal{S}_{\Omega,L}(\mathcal{N})$ by [18, Proposition 4.19 and Corollary 4.6]. For every $k \in K$, define

$$K'_k := \{k' \in K : d_{\Omega'}(\lambda_k, \lambda_{k'}) \leq 3R\delta\},$$

and observe that there is $N \in \mathbb{N}$ such that $\text{Card}(K'_k) \leq N$ for every $k \in K$, thanks to [18, Proposition 2.56]. Therefore,

$$(\Psi_j u) * \psi_k = \sum_{k' \in K_j \cap K_k} [(u * \psi_{k'}) \tau_j] * \psi_k$$

for every $j \in \mathbb{N}$, for every $k \in K$, and for every $u \in B_{p,q}^s(\mathcal{N}, \Omega)$. Fix $\varphi' \in C_c^\infty(\Omega')$ such that $\varphi' = 1$ on $B_{\Omega'}(e_{\Omega'}, 5R\delta)$ and observe that, if we define $\psi'_k := \mathcal{F}_{\mathcal{N}}^{-1}(\varphi'(\cdot t_k^{-1}))$, then

$$[(u * \psi_{k'}) \tau_j] * \psi'_k = (u * \psi_{k'}) \tau_j$$

for every $k' \in K_j \cap K_k$, for every $j \in \mathbb{N}$, for every $k \in K$, and for every $u \in \mathcal{S}'_{\Omega,L}(\mathcal{N})$, thanks to [18, Corollary 4.6]. Now, [18, Corollary 4.10] implies that there is a constant $C_1 > 0$ such that

$$\|u' * \psi_k\|_{L^p(\mathcal{N})} \leq C_1 \|u'\|_{L^p(\mathcal{N})}$$

for every $u' \in \mathcal{S}'_{\Omega,L}(\mathcal{N})$ such that $u' = u' * \psi'_k$ for every $k \in K$. Set $C_2 := \|\tau\|_{L^\infty(\mathcal{N})}$. Then,

$$\begin{aligned} \|(\Psi_j u) * \psi_k\|_{L^p(\mathcal{N})} &\leq N^{(1/p-1)_+} C_1 \sum_{k' \in K_j \cap K_k} \|(u * \psi_{k'}) \tau_j\|_{L^p(\mathcal{N})} \\ &\leq N^{(1/p-1)_+} C_1 C_2 \sum_{k' \in K_j \cap K_k} \|u * \psi_{k'}\|_{L^p(\mathcal{N})} \end{aligned}$$

for every $u \in B_{p,q}^s(\mathcal{N}, \Omega)$, for every $k \in K$, and for every $j \in \mathbb{N}$. Now, by [18, Corollary 2.49], there is a constant $C_3 > 0$ such that

$$\frac{1}{C_3} \Delta_{\Omega'}^s(\lambda_k) \leq \Delta_{\Omega'}^s(\lambda_{k'}) \leq C_3 \Delta_{\Omega'}^s(\lambda_k)$$

for every $k \in K$ and for every $k' \in K_k$, so that

$$\|\Delta_{\Omega'}^s(\lambda_k) \|(\Psi_j u) * \psi_k\|_{L^p(\mathcal{N})} \|_{\ell^q(K)} \leq C_4 \|\Delta_{\Omega'}^s(\lambda_k)\| \|u * \psi_k\|_{L^p(\mathcal{N})} \|_{\ell^q(K_j)}$$

for every $j \in \mathbb{N}$ and for every $u \in B_{p,q}^s(\mathcal{N}, \Omega)$, where $C_4 := N^{(1/p-1)_+ + \max(1/q, 1)} C_1 C_2 C_3$. Thus, the Ψ_j are equicontinuous on $B_{p,q}^s(\mathcal{N}, \Omega)$. Then, take $u \in B_{p,q}^s(\mathcal{N}, \Omega)$ and let us prove that $(\Psi_j u)$ converges to u in $\sigma_{p,q}^s$ for $j \rightarrow \infty$. By [18, Corollary 4.25] it will suffice to prove convergence in $\mathcal{S}'_{\Omega,L}(\mathcal{N})$. Then, we are reduced to proving that $\sum_{k \in K_j} (\eta \tau_j) * \psi_k$ converges to η for $j \rightarrow \infty$, for every $\eta \in \mathcal{S}_{\Omega,L}(\mathcal{N})$. Nonetheless, it is clear that

$$\sum_{k \in K_j} (\eta \tau_j) * \psi_k = \eta \tau_j$$

if j is sufficiently large (cf. [18, Corollary 4.6]), so that the proof is complete. $\square$

Proof of Proposition 4.1 (1) $\implies$ (2). Observe that $A_{s,0}^{p,q}(D) = \tilde{A}_{s,0}^{p,q}(D) \neq \{0\}$, so that $s > 0$ by [18, Proposition 3.5]. Then, it will suffice to prove that, if $\tilde{A}_{s,0}^{p,q}(D) \subseteq A_s^{p,q}(D)$, then $\tilde{A}_s^{p,q}(D) = A_s^{p,q}(D)$, which will prove also the first assertion. Take (Ψ_j) as in Lemma 4.2. Then,

$$\lim_{j \rightarrow \infty} \mathcal{E}\Psi_j u = \mathcal{E}u$$

pointwise on D , for every $u \in B_{p,q}^{-s}(\mathcal{N}, \Omega)$, thanks to [18, Lemma 5.1] and Lemma 4.2.

In addition, the sequence $(\mathcal{E}\Psi_j u)$ is bounded in $\tilde{A}_{s,0}^{p,q}(D) = A_{s,0}^{p,q}(D)$, so that, by lower semi-continuity,

$$\|\mathcal{E}u\|_{A_s^{p,q}(D)} \leq \liminf_{j \rightarrow \infty} \|\mathcal{E}\Psi_j u\|_{A_s^{p,q}(D)} < \infty.$$

Hence, $\tilde{A}_s^{p,q}(D) \subseteq A_s^{p,q}(D)$, whence $\tilde{A}_s^{p,q}(D) = A_s^{p,q}(D)$ by Proposition 2.4.

(2) $\implies$ (1). Observe first that, since $A_s^{p,q}(D) = \tilde{A}_s^{p,q}(D) \neq \{0\}$ and $s > 0$, [18, Proposition 3.5] implies that $s > \frac{1}{2q} \mathbf{m}$ so that, in particular, $A_{s,0}^{p,q}(D) \neq \{0\}$. Then, take $f \in \tilde{A}_{s,0}^{p,q}(D)$, and observe that there is a sequence (f_j) of elements of $\mathcal{E}(\mathcal{S}_{\Omega,L}(\mathcal{N}))$ which converges to f in $\tilde{A}_{s,0}^{p,q}(D)$, hence in $\tilde{A}_s^{p,q}(D) = A_s^{p,q}(D)$, thanks to [18, Theorem 4.23]. Since $f_j \in A_{s,0}^{p,q}(D)$ for every $j \in \mathbb{N}$, thanks to Proposition 2.4, this implies that $f \in A_{s,0}^{p,q}(D)$. The assertion follows by Proposition 2.4 as before. $\square$

We now recall (and extend) various notions of atomic decomposition introduced in [18].

Definition 4.3 Take $p, q \in (0, \infty]$ and $s, s' \in \mathbb{R}^r$. We say that (weak) property $(L)_{s,s'}^{p,q}$ holds if for every $\delta_0 > 0$ there are a $(\delta, 4)$ -lattice $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$ on D , with $\delta \in (0, \delta_0]$, such that the mapping

$$\Psi : \mathcal{L}^{p,q}(J, K) \ni \lambda \mapsto \sum_{j,k} \lambda_{j,k} B_{(\zeta_{j,k}, z_{j,k})}^{s'} \Delta_{\Omega}^{(\mathbf{b}+\mathbf{d})/p-s-s'}(h_k) \in \text{Hol}(D),$$

with $h_k := \rho(\zeta_{j,k}, z_{j,k})$ for every $(j, k) \in J \times K$, is well defined (with locally uniform convergence of the sum) and maps $\mathcal{L}^{p,q}(J, K)$ continuously into $A_s^{p,q}(D)$.

We say that strong property $(L)_{s,s'}^{p,q}$ holds if Ψ has the preceding properties for every (δ, R) -lattice on D , with $\delta > 0$ and $R > 1$.

We say that strong property $(L')_{s,s'}^{p,q}$ holds if strong property $(L)_{s,s'}^{p,q}$ holds and for every $R_0 > 1$ there is $\delta_0 > 0$ such that $\Psi(\mathcal{L}^{p,q}(J, K)) = A_s^{p,q}(D)$ whenever $(\zeta_{j,k}, z_{j,k})$ is a (δ, R) -lattice with $\delta \in (0, \delta_0]$ and $R \in (1, R_0]$.

We define weak and strong properties $(L)_{s,s',0}^{p,q}$ and $(L')_{s,s',0}^{p,q}$ analogously, replacing $A_s^{p,q}(D)$ with $A_{s,0}^{p,q}(D)$.

Finally, if $s > \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q}\mathbf{m}'$, then we define (weak or strong) properties $(\tilde{L})_{s,s',0}^{p,q}$, $(\tilde{L}')_{s,s',0}^{p,q}$, $(\tilde{L})_{s,s'}^{p,q}$, and $(\tilde{L}')_{s,s'}^{p,q}$, replacing the spaces $A_{s,0}^{p,q}(D)$ and $A_s^{p,q}(D)$ with $\tilde{A}_{s,0}^{p,q}(D)$ and $\tilde{A}_s^{p,q}(D)$, respectively.

Proposition 4.4 Take $p, q \in (0, \infty]$ and $s, s' \in \mathbb{R}^r$. If $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$ is a (δ, R) -lattice on D for some $\delta > 0$ and some $R > 1$ and the mapping

$$\Psi : \mathbb{C}^{(J \times K)} \ni \lambda \mapsto \sum_{j,k} \lambda_{j,k} B_{(\zeta_{j,k}, z_{j,k})}^{s'} \Delta_{\Omega}^{(\mathbf{b}+\mathbf{d})/p-s-s'}(h_k) \in \text{Hol}(D)$$

induces a continuous linear mapping $\ell_0^{p,q}(J, K) \rightarrow A_s^{p,q}(D)$, then it also induces a continuous linear mapping $\ell^{p,q}(J, K) \rightarrow A_s^{p,q}(D)$, defined in the same way with locally uniform convergence of the sum.

In particular, the following conditions are equivalent:

- (1) weak (resp. strong) property $(L)_{s,s',0}^{p,q}$ holds;
- (2) $s > 0$, and weak (resp. strong) property $(L)_{s,s'}^{p,q}$ holds;

Proof The equivalence of (1) and (2) follow easily from the first assertion and [18, Proposition 2.41 and Lemma 3.29].

Then, let us prove the first assertion. By [18, Lemma 3.29], $\tilde{A}_{s-(1/p-1)_+(\mathbf{b}+\mathbf{d})}^{p'',q''}(D)$ and $\tilde{A}_{(\mathbf{b}+\mathbf{d})/p-s-s',0}^{p',q'}(D)$ are well defined, while $A_{s-(1/p-1)_+(\mathbf{b}+\mathbf{d})}^{p'',q''}(D), A_{(\mathbf{b}+\mathbf{d})/p-s-s'}^{p',q'}(D) \neq \{0\}$ (notice that $p'' = \max(1, p)$ and $q'' = \max(1, q)$). Define $V := A_{(\mathbf{b}+\mathbf{d})/p-s-s'}^{p',q'}(D) \cap \tilde{A}_{(\mathbf{b}+\mathbf{d})/p-s-s',0}^{p',q'}(D)$, endowed with the topology induced by $A_{(\mathbf{b}+\mathbf{d})/p-s-s'}^{p',q'}(D)$, and observe that V is a closed subspace of $A_{(\mathbf{b}+\mathbf{d})/p-s-s'}^{p',q'}(D)$ by the continuity of the inclusion $A_{(\mathbf{b}+\mathbf{d})/p-s-s'}^{p',q'}(D) \subseteq \tilde{A}_{(\mathbf{b}+\mathbf{d})/p-s-s',0}^{p',q'}(D)$. Denote by $i_1 : A_{s-(1/p-1)_+(\mathbf{b}+\mathbf{d})}^{p'',q''}(D) \rightarrow V$ and $i_2 : \tilde{A}_{s-(1/p-1)_+(\mathbf{b}+\mathbf{d})}^{p'',q''}(D) \rightarrow \tilde{A}_{(\mathbf{b}+\mathbf{d})/p-s-s',0}^{p',q'}(D)$ the continuous antilinear mapping and the antilinear isomorphism induced by the sesquilinear form (cf. Proposition 2.7)

$$(f, g) \mapsto \int_D f \bar{g} (\Delta_{\Omega}^{-s'} \circ \rho) \, dv_D.$$

Denote by $i_3 : V \rightarrow \tilde{A}_{(\mathbf{b}+\mathbf{d})/p-s-s',0}^{p',q'}(D)$ and $i_4 : A_{s-(1/p-1)_+(\mathbf{b}+\mathbf{d})}^{p'',q''}(D) \rightarrow \tilde{A}_{s-(1/p-1)_+(\mathbf{b}+\mathbf{d})}^{p'',q''}(D)$ the canonical (continuous linear) mappings, so that i_3 is one-to-one and has a dense image (since V contains $\mathcal{E}(\mathcal{S}_{\Omega,L}(\mathcal{N}))$, cf. [18, Theorem 4.23] and Proposition 2.4). Then, i_3 is one-to-one and

$$i_1 = i_3 \circ i_2 \circ i_4.$$

In addition, using [18, Theorem 3.22] and the canonical mappings $\ell^{p,q}(J, K) \rightarrow \ell^{p',q''}(J, K) \rightarrow \ell^{p',q'}(J, K)'$, we see that $i_3 \circ i_2 \circ \mathcal{P}$ extends to a continuous linear mapping

$$\ell^{p,q}(J, K) \ni \lambda \mapsto \sum_{j,k} \lambda_{j,k} (i_3 \circ i_2) (B_{(\zeta_{j,k}, \zeta_{j,k})}^{s'}) \Delta_{\Omega}^{(\mathbf{b}+\mathbf{d})/p-s-s'}(h_k) \in V',$$

with the sum converging in the weak topology $\sigma(V', V)$ (cf. the proof of [18, Proposition 3.39]). Observe that, by compactness,

$$\lim_{H, \mathfrak{U}} \sum_{(j,k) \in H} \lambda_{j,k} B_{(\zeta_{j,k}, \zeta_{j,k})}^{s'} \Delta_{\Omega}^{(\mathbf{b}+\mathbf{d})/p-s-s'}(h_k)$$

converges in $\tilde{\mathcal{O}}_{s-(1/p-1)_+(\mathbf{b}+\mathbf{d})}^{p'',q''}$ for every ultrafilter $\mathfrak{U}$ on the set of finite subsets of $J \times K$ which is finer than the section filter associated with $\subseteq$. Since $i_3 \circ i_2$ is one-to-one, this proves that the sum

$$\sum_{j,k} \lambda_{j,k} B_{(\zeta_{j,k}, z_{j,k})}^{s'} \Delta_{\Omega}^{(\mathbf{b}+\mathbf{d})/p-s-s'}(h_k),$$

converges in $\tilde{\sigma}_{s-(1/p-1)_+(\mathbf{b}+\mathbf{d})}^{p'',q''}$ for every $\lambda \in \ell^{p,q}(J, K)$ (cf. [12, Proposition 2 of Chapter I, §7, No. 1]). Observe that, since the $B_{(\zeta,z)}^{s'}$ stay in a compact subset of $\tilde{A}_{s-(1/p-1)_+(\mathbf{b}+\mathbf{d}),0}^{p'',q''}(D)$ as (ζ, z) stays in a compact subset of D (cf. [18, Lemmas 3.29 and 5.15]), by means of [18, Proposition 3.13 and Lemma 3.29] we see that convergence in $\tilde{\sigma}_{s-(1/p-1)_+(\mathbf{b}+\mathbf{d})}^{p'',q''}$ implies convergence in $\text{Hol}(D)$. Thus, the sum

$$\sum_{j,k} \lambda_{j,k} B_{(\zeta_{j,k}, z_{j,k})}^{s'} \Delta_{\Omega}^{(\mathbf{b}+\mathbf{d})/p-s-s'}(h_k)$$

converges in $\text{Hol}(D)$ for every $\lambda \in \ell^{p,q}(J, K)$, and defines a continuous linear function (of λ) from $\ell^{p,q}(J, K)$ into $A_s^{p,q}(D)$. $\square$

We are now ready to prove the main result of this section.

Theorem 4.5 *Take $p, q \in [1, \infty]$ and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$ such that the following conditions hold:*

- $\mathbf{s} > \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$;
- $\mathbf{s} + \mathbf{s}' < \frac{1}{p}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q}\mathbf{m}'$.

Then, the following conditions are equivalent:

- (1) $A_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D) = \tilde{A}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$;
- (2) *the sesquilinear mapping*

$$\tilde{A}_{\mathbf{s},0}^{p,q}(D) \times A_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D) \ni (f, g) \mapsto \langle \mathcal{E}^{-1}f | (\mathcal{E}^{-1}g) * I_{\Omega}^{\mathbf{b}+\mathbf{d}-\mathbf{s}'} \rangle,$$

induces an antilinear isomorphism of $A_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$ onto $\tilde{A}_{\mathbf{s},0}^{p,q}(D)'$;

- (3) *strong property $(\tilde{L})_{\mathbf{s},\mathbf{s}',0}^{p,q}$ holds;*
- (3') *weak property $(\tilde{L})_{\mathbf{s},\mathbf{s}',0}^{p,q}$ holds;*
- (4) *strong property $(\tilde{L})_{\mathbf{s},\mathbf{s}'}^{p,q}$ holds;*
- (4') *weak property $(\tilde{L})_{\mathbf{s},\mathbf{s}'}^{p,q}$ holds.*

If, in addition, $\mathbf{s}' < \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$, then the preceding conditions are equivalent to the following ones:

- (2') *the sesquilinear mapping*

$$\mathcal{E}(S_{\Omega,L}(\mathcal{N})) \times \mathcal{E}(S_{\Omega,L}(\mathcal{N})) \ni (f, g) \mapsto \int_D f \bar{g} (\Delta_{\Omega}^{-s'} \circ \rho) \, d\nu_D,$$

extended to $\tilde{A}_{\mathbf{s},0}^{p,q}(D) \times \tilde{A}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$ as in Proposition 2.7, induces an antilinear isomorphism of $A_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$ onto $\tilde{A}_{\mathbf{s},0}^{p,q}(D)'$;

(5) $P_{s'}$ induces a continuous linear mapping of $L_{s,0}^{p,q}(D)$ onto $\tilde{A}_{s,0}^{p,q}(D)$;

(6) $P_{s'}$ induces a continuous linear mapping of $L_s^{p,q}(D)$ onto $\tilde{A}_s^{p,q}(D)$ such that

$$P_{s'}f(\zeta, z) = c_{s'} \int_D \overline{f B_{(\zeta, z)}^{s'}} (\Delta_{\Omega}^{-s'} \circ \rho) d\nu_D$$

for every $f \in L_s^{p,q}(D)$ and for every $(\zeta, z) \in D$;

(6') $P_{s'}$ induces a continuous linear mapping of $L_{s,0}^{p,q}(D)$ into $\tilde{A}_s^{p,q}(D)$.

Let us remark explicitly that the conditions $\mathbf{s} > \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$ and $\mathbf{s} + \mathbf{s}' < \frac{1}{p}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q}\mathbf{m}'$ mean that the spaces $\tilde{A}_s^{p,q}(D)$ and $\tilde{A}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$, respectively, are defined, whereas the condition $\mathbf{s}' < \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$ means that the weighted Bergman projector $P_{s'}$ is defined.

In the proof of Theorem 4.5 we shall make use of several duality arguments which are analogous to those employed in the proof of [18, Proposition 3.39].⁸ We shall present them once in the following extension of the implication (5') $\implies$ (1).

Proposition 4.6 Take $p, q \in (0, \infty]$, $\mathbf{s} > \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$, and $\mathbf{s}' \in \mathbb{R}^r$. Then, weak (resp. strong) property $(\tilde{L})_{s,s',0}^{p,q}$ holds if and only if weak (resp. strong) property $(\tilde{L})_{s,s'}^{p,q}$ holds, in which case $(\mathbf{b} + \mathbf{d})/\min(1, p) - \mathbf{s} - \mathbf{s}' > \frac{1}{p'}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q''}\mathbf{m}'$ and $A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D) = \tilde{A}_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$.

Proof Since $B_{(\zeta,z)}^{s'} \in \tilde{A}_{s,0}^{p,q}(D)$ if and only if $B_{(\zeta,z)}^{s'} \in \tilde{A}_{s,0}^{p,q}(D)$ for every $(\zeta, z) \in D$ by [18, Lemma 5.15], weak (resp. strong) property $(\tilde{L})_{s,s'}^{p,q}$ implies weak (resp. strong) property $(\tilde{L})_{s,s',0}^{p,q}$. In addition, if either one of the preceding conditions hold, then $\mathbf{s} + \mathbf{s}' < \frac{1}{p}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q}\mathbf{m}'$, so that $(\mathbf{b} + \mathbf{d})/\min(1, p) - \mathbf{s} - \mathbf{s}' > \frac{1}{p'}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q''}\mathbf{m}'$. Then, assume that weak property $(\tilde{L})_{s,s',0}^{p,q}$ holds, and let us prove that $A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D) = \tilde{A}_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$. By [18, Theorem 3.22], there is $\delta_0 > 0$ such that, if $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$ is a $(\delta, 4)$ -lattice on D with $\delta \in (0, \delta_0]$, and we define $h_k := \rho(\zeta_{j,k}, z_{j,k})$ for every $j \in J$ and for every $k \in K$,

$$S : \text{Hol}(D) \ni f \mapsto (\Delta_{\Omega}^{(\mathbf{b}+\mathbf{d})/p-\mathbf{s}-\mathbf{s}'}(h_k)f(\zeta_{j,k}, z_{j,k})) \in \mathbb{C}^{J \times K},$$

then S induces an isomorphism of $A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$ onto a closed subspace of $\mathcal{C}^{p',q'}(J, K)$, and $S^{-1}(\mathcal{C}^{p',q'}(J, K)) \cap A_{(\mathbf{b}+\mathbf{d})/p-\mathbf{s}-\mathbf{s}'}^{\infty,\infty}(D) = A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$. Since weak property $(\tilde{L})_{s,s',0}^{p,q}$ holds, we may assume that the mapping

⁸ We remark explicitly that the assumption of this latter result are mistakenly stated, and should be: $\mathbf{s} > \frac{1}{p}\mathbf{m}, \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$, $\mathbf{s}' < \frac{1}{p'}(\mathbf{b} + \mathbf{d}) - \frac{1}{2p'}\mathbf{m}'$, and $\mathbf{s} + \mathbf{s}' < \frac{1}{\min(1,p)}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q'}\mathbf{m}'$ or $\mathbf{s} + \mathbf{s}' \leq \frac{1}{\min(1,p)}(\mathbf{b} + \mathbf{d})$ if $q' = \infty$. In addition, for conclusion (3) in the cited result to hold, one has to assume further that $p, q \geq 1$. Nonetheless, [18, Corollary 3.40] still holds, thanks to Corollaries 4.7 and 4.14.

$$\Psi : \mathcal{L}_0^{p,q}(J, K) \ni \lambda \mapsto \sum_{j,k} \lambda_{j,k} B_{(\zeta_{j,k}, z_{j,k})}^{s'} \Delta_{\Omega}^{(\mathbf{b}+\mathbf{d})/p-s-s'}(h_k) \in \widetilde{A}_{s,0}^{p,q}(D)$$

is well defined and continuous. Now, denote by $\langle \cdot | \cdot \rangle$ the sesquilinear form on $\widetilde{A}_{s,0}^{p,q}(D) \times A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-s-s'}^{p',q'}(D)$ defined in (2) of Proposition 2.7. Then,

$$\langle \Psi(\lambda) | f \rangle = \sum_{j,k} \lambda_{j,k} \Delta_{\Omega}^{(\mathbf{b}+\mathbf{d})/p-s-s'}(h_k) \langle (\mathcal{E}^{-1} B_{(\zeta_{j,k}, z_{j,k})}^{s'}) * I_{\Omega}^{s''} | \mathcal{E}^{-1} f \rangle = c_{s'} \langle \lambda | Sf \rangle$$

for every $\lambda \in \mathbb{C}^{(J \times K)}$ and for every $f \in A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-s-s'}^{p',q'}(D)$, where $s'' = \mathbf{b} + \mathbf{d} - s'$, since $(\mathcal{E}^{-1} B_{(\zeta_{j,k}, z_{j,k})}^{s'}) * I_{\Omega}^{s''} = c_{s'} (B_{(\zeta_{j,k}, z_{j,k})}^{\mathbf{b}+\mathbf{d}})_0$ for a suitable $c_{s'} \neq 0$ by the proof of [18, Lemma 5.15].

Hence, $Sf \in \mathcal{L}^{p',q'}(J, K)$ for every $f \in \widetilde{A}_{(\mathbf{b}+\mathbf{d})/\min(1,p)-s-s'}^{p',q'}(D)$. Since

$\widetilde{A}_{(\mathbf{b}+\mathbf{d})/\min(1,p)-s-s'}^{p',q'}(D) \subseteq A_{(\mathbf{b}+\mathbf{d})/p-s-s'}^{\infty,\infty}(D)$ by Proposition 2.4, this implies that $\widetilde{A}_{(\mathbf{b}+\mathbf{d})/\min(1,p)-s-s'}^{p',q'}(D) \subseteq A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-s-s'}^{p',q'}(D)$, whence our assertion.

To conclude, we only need to show that weak (resp. strong) property $(\widetilde{L})_{s,s',0}^{p,q}$ implies weak (resp. strong) property $(\widetilde{L})_{s,s'}^{p,q}$. The proof is now similar to that of Proposition 4.4, and proceeds by means of the duality between the closed subspace $\widetilde{A}_{(\mathbf{b}+\mathbf{d})/\min(1,p)-s-s',0}^{p',q'}(D)$ of $A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-s-s'}^{p',q'}(D)$ and the space $\widetilde{A}_{s-(1/p-1),+(\mathbf{b}+\mathbf{d})}^{p'',q''}(D)$. $\square$

Proof of Theorem 4.5 (1) $\iff$ (2). This is a consequence of the continuous inclusion $A_{\mathbf{b}+\mathbf{d}-s-s'}^{p',q'}(D) \subseteq \widetilde{A}_{\mathbf{b}+\mathbf{d}-s-s'}^{p',q'}(D)$.

(2) $\implies$ (3). This follows by transposition, using [18, Theorem 3.22] and [13, Corollary 3 to Theorem 1 of Chapter IV, §4, No. 2, and Proposition 5 of Chapter IV, §1, No. 3] (cf. Proposition 4.6 and [18, Proposition 3.39]).

(1) $\implies$ (4). This follows by transposition since $\widetilde{A}_s^{p,q}(D)$ canonically identifies with the dual of the closed subspace $\widetilde{A}_{\mathbf{b}+\mathbf{d}-s-s',0}^{p',q'}(D)$ of $A_{\mathbf{b}+\mathbf{d}-s-s'}^{p',q'}(D)$, using [18, Theorem 3.22] and [13, Corollary 1 to Theorem 1 of Chapter IV, §4, No. 2] (cf. Proposition 4.6 and [18, Proposition 3.39]).

(3) $\implies$ (3'); (4) $\implies$ (4'). Obvious.

(4') $\iff$ (3'); (3') $\implies$ (1). This is a consequence of Proposition 4.6.

Now, assume that $s' < \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$.

(2) $\iff$ (2'). This is clear by Proposition 2.7.

(1) $\implies$ (6'). Take $f \in L_s^{p,q}(D) \cap L_{(\mathbf{b}+\mathbf{d}-s')/2}^{2,2}(D)$ and $\varphi \in \mathcal{E}(\mathcal{S}_{\Omega,L}(\mathcal{N}))$. Observe that $P_{s'} f \in A_{(\mathbf{b}+\mathbf{d}-s')/2}^{2,2}(D) = \widetilde{A}_{(\mathbf{b}+\mathbf{d}-s')/2}^{2,2}(D)$, so that there is $u \in B_{2,2}^{(s'-\mathbf{b}-\mathbf{d})/2}(\mathcal{N}, \Omega)$ such that $P_{s'} f = \mathcal{E}u$. Then, by Proposition 2.7, there is a constant $c \neq 0$ such that

$$\begin{aligned} |\langle u | \varphi \rangle| &= \left| c \int_D \mathcal{E}u \overline{\mathcal{E}(\varphi * I_{\Omega}^{s'-\mathbf{b}-\mathbf{d}})} (\Delta_{\Omega}^{-s'} \circ \rho) \, \mathrm{d}v_D \right| \\ &= \left| c \int_D f \overline{\mathcal{E}(\varphi * I_{\Omega}^{s'-\mathbf{b}-\mathbf{d}})} (\Delta_{\Omega}^{-s'} \circ \rho) \, \mathrm{d}v_D \right| \\ &\leq |c| \|f\|_{L_s^{p,q}(D)} \|\mathcal{E}(\varphi * I_{\Omega}^{s'-\mathbf{b}-\mathbf{d}})\|_{A_{\mathbf{b}+\mathbf{d}-s-s'}^{p',q'}(D)}. \end{aligned}$$

Now, by (1) and [18, Theorem 4.26], there are constants $C_1, C'_1 > 0$ such that

$$\|\mathcal{E}(\varphi * I_{\Omega}^{s'-b-d})\|_{A_{b+d-s-s'}^{p',q'}(D)} \leq C_1 \|\varphi * I_{\Omega}^{s'-b-d}\|_{B_{p',q'}^{s+s'-b-d}(\mathcal{N},\Omega)} \leq C'_1 \|\varphi\|_{B_{p',q'}^s(\mathcal{N},\Omega)}$$

for every $u \in \mathcal{S}_{\Omega,L}(\mathcal{N})$, for some fixed norms on $B_{p',q'}^{s+s'-b-d}(\mathcal{N},\Omega)$ and $B_{p',q'}^s(\mathcal{N},\Omega)$. Therefore, by means of [18, Theorem 4.23] we see that $u \in B_{p,q}^{-s}(\mathcal{N},\Omega)$, that is, $P_{s'}f \in \tilde{A}_s^{p,q}(D)$, and that there is a constant $C_2 > 0$ such that

$$\|P_{s'}f\|_{\tilde{A}_s^{p,q}(D)} \leq C_2 \|f\|_{L_s^{p,q}(D)}$$

for every $f \in L_s^{p,q}(D) \cap L_{(b+d-s')/2}^{2,2}(D)$. Then, (3') follows.

(6') $\implies$ (5). Observe that $B_{(\zeta,z)}^{s'} \in \tilde{A}_{s,0}^{p,q}(D)$ for every $(\zeta, z) \in D$, thanks to [18, Lemma 5.15]. In addition, using [18, Theorem 2.47, Corollary 5.11, and Proposition 5.13], we see that the mapping $D \ni (\zeta, z) \mapsto B_{(\zeta,z)} \in \tilde{A}_{s,0}^{p,q}(D)$ is actually continuous. Therefore, [14, Proposition 8 of Chapter VI, §1, No. 2] implies that $P_{s'}f \in \tilde{A}_{s,0}^{p,q}(D)$ for every $f \in C_c(D)$, so that $P_{s'}$ induces a continuous linear mapping of $L_{s,0}^{p,q}(D)$ into $\tilde{A}_{s,0}^{p,q}(D)$. Now, take $s'' \in \mathbb{N}_{\Omega'}$ so that $s + s'' > \frac{1}{2q}\mathbf{m} + \left(\frac{1}{2\min(p,p')} - \frac{1}{2q}\right)_+ \mathbf{m}'$, so that $A_{s+s'',0}^{p,q}(D) = \tilde{A}_{s+s'',0}^{p,q}(D)$ by [18, Corollary 5.11]. In addition, observe that, by [18, Proposition 2.29], there is a constant $c \neq 0$ such that

$$P_{s'}(f(\Delta_{\Omega}^{s''} \circ \rho)) * I_{\Omega}^{-s''} = cP_{s'-s''}f \quad (3)$$

for every $f \in C_c(D)$. Therefore, [18, Proposition 5.13] implies that $P_{s'-s''}$ induces a continuous linear mapping of $L_{s+s'',0}^{p,q}(D)$ into $\tilde{A}_{s+s'',0}^{p,q}(D) = A_{s+s'',0}^{p,q}(D)$. Then, $P_{s'-s''}(L_{s+s'',0}^{p,q}(D)) = \tilde{A}_{s+s'',0}^{p,q}(D)$, so that, by [18, Proposition 5.13] again, $P_{s'}(L_{s,0}^{p,q}(D)) = \tilde{A}_{s,0}^{p,q}(D)$.

(5) $\implies$ (1). Take $f \in C_c(D)$ and $\varphi \in \mathcal{S}_{\Omega,L}(\mathcal{N})$. Then, by Proposition 2.7,

$$\begin{aligned} \left| \int_D f \overline{\mathcal{E}\varphi}(\Delta_{\Omega}^{-s'} \circ \rho) \, d\nu_D \right| &= \left| \int_D P_{s'}f \overline{\mathcal{E}\varphi}(\Delta_{\Omega}^{-s'} \circ \rho) \, d\nu_D \right| \\ &\leq \|P_{s'}\|_{\mathcal{L}(L_{s,0}^{p,q}(D); \tilde{A}_{s,0}^{p,q}(D))} \|f\|_{L_s^{p,q}(D)} \|\mathcal{E}\varphi\|_{\tilde{A}_{b+d-s-s'}^{p',q'}(D)} \end{aligned}$$

with suitable choices of norms on $\tilde{A}_{s,0}^{p,q}(D)$ and $\tilde{A}_{b+d-s-s'}^{p',q'}(D)$. By the arbitrariness of f , this implies that

$$\|\mathcal{E}\varphi\|_{A_{b+d-s-s'}^{p',q'}(D)} \leq \|P_{s'}\|_{\mathcal{L}(L_{s,0}^{p,q}(D); \tilde{A}_{s,0}^{p,q}(D))} \|\mathcal{E}\varphi\|_{\tilde{A}_{b+d-s-s'}^{p',q'}(D)}$$

for every $\varphi \in \mathcal{S}_{\Omega,L}(\mathcal{N})$. Hence, $\tilde{A}_{b+d-s-s',0}^{p',q'}(D) \subseteq A_{b+d-s-s'}^{p',q'}(D)$ continuously, whence (1) by Proposition 4.1.

(5) $\implies$ (6). This follows from formula (3) (arguing as in the proof of the implication (6') $\implies$ (5)), which is readily extended to every $f \in L_s^{p,q}(D)$ taking into account the fact that, by [18, Lemma 5.15], $B_{(\zeta,z)}^{s'} \in \tilde{A}_{b+d-s-s',0}^{p',q'}(D) = A_{b+d-s-s'}^{p',q'}(D)$ for every $(\zeta, z) \in D$ (since we proved that conditions (1), (5), and (6') are equivalent), which gives convergence of $P_{s'}(f_j)$ in $\text{Hol}(D)$ when (f_j) converges almost everywhere to $f \in L_s^{p,q}(D)$ and $|f_j| \leq |f|$ for every $j \in \mathbb{N}$.

(6) $\implies$ (6'). Obvious. $\square$

Corollary 4.7 Take $p, q \in [1, \infty]$ and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$ such that the following conditions hold:

- $\mathbf{s} > \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$;
- $\mathbf{s}' < \frac{p}{2}\mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$;
- $\mathbf{s} + \mathbf{s}' < \frac{1}{p}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q}\mathbf{m}'$.

Then, the following conditions are equivalent:

- (1) $A_{\mathbf{s},0}^{p,q}(D) = \widetilde{A}_{\mathbf{s},0}^{p,q}(D)$ (resp. $A_{\mathbf{s}}^{p,q}(D) = \widetilde{A}_{\mathbf{s}}^{p,q}(D)$) and $A_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D) = \widetilde{A}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$;
- (2) $P_{\mathbf{s}'}$ induces a continuous linear mapping of $L_{\mathbf{s},0}^{p,q}(D)$ into $L_{\mathbf{s}}^{p,q}(D)$ and $\mathbf{s} > \mathbf{0}$ (resp. $\mathbf{s} \geq \mathbf{0}$);⁹
- (3) $P_{\mathbf{s}'}$ induces a continuous linear projector of $L_{\mathbf{s},0}^{p,q}(D)$ onto $A_{\mathbf{s},0}^{p,q}(D)$ (resp. of $L_{\mathbf{s}}^{p,q}(D)$ onto $A_{\mathbf{s}}^{p,q}(D)$) and of $L_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$ onto $A_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$;
- (4) $\mathbf{s} > \frac{1}{2q}\mathbf{m}$ (resp. $\mathbf{s} \geq \mathbf{0}$ if $q = \infty$) and the sesquilinear mapping

$$(f, g) \mapsto \int_D f \overline{g} (\Delta_{\Omega}^{-\mathbf{s}'} \circ \rho) \, \mathrm{d}v_D,$$

induces an antilinear isomorphism of $A_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$ onto $A_{\mathbf{s},0}^{p,q}(D)'$ (resp. onto the dual of the closed vector subspace of $A_{\mathbf{s}}^{p,q}(D)$ generated by the $B_{(\zeta,z)}^{\mathbf{s}'}$, $(\zeta, z) \in D$);

- (4') $\mathbf{s} > \frac{1}{2q}\mathbf{m}$ (resp. $\mathbf{s} \geq \mathbf{0}$ if $q = \infty$) and the sesquilinear mapping

$$(f, g) \mapsto \int_D f \overline{g} (\Delta_{\Omega}^{-\mathbf{s}'} \circ \rho) \, \mathrm{d}v_D,$$

induces an antilinear isomorphism of $A_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$ onto the dual of the closed subspace $\widetilde{A}_{\mathbf{s},0}^{p,q}(D) \cap A_{\mathbf{s}}^{p,q}(D)$ of $A_{\mathbf{s}}^{p,q}(D)$;

- (5) strong properties $(L')_{\mathbf{s},\mathbf{s}',0}^{p,q}$ (resp. $(L')_{\mathbf{s},\mathbf{s}'}^{p,q}$) and $(L')_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}',\mathbf{s}'}^{p',q'}$ hold;
- (6) weak property $(L)_{\mathbf{s},\mathbf{s}',0}^{p,q}$ (resp. $(L)_{\mathbf{s},\mathbf{s}'}^{p,q}$) holds.

This extends [2, Theorem 1.6], where the equivalence of (2) and (3) is proved when $p = q \in (1, \infty)$, $\mathbf{s} \in \mathbb{R}\mathbf{d}$, and D is an irreducible symmetric tube domain.

Proof Theorem 4.5 shows that (1) implies (2), (4'), and (5). In addition, (2) is equivalent to (3) thanks to [18, Proposition 5.21], while (5) clearly implies (6). By means of Theorem 4.5, we also see that (3) implies (1). Let us prove that (6) implies (1). Observe that (6) implies weak property $(\widetilde{L})_{\mathbf{s},\mathbf{s}',0}^{p,q}$ (resp. $(\widetilde{L})_{\mathbf{s},\mathbf{s}'}^{p,q}$), so that Theorem 4.5 implies that strong properties $(\widetilde{L}')_{\mathbf{s},\mathbf{s}',0}^{p,q}$ and $(\widetilde{L}')_{\mathbf{s},\mathbf{s}'}^{p,q}$ hold, and that $A_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D) = \widetilde{A}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$. This, together with the assumption (6), implies (1).

Next, observe that Theorem 4.5 again shows that (1) implies (4), since the closed vector subspace of $A_{\mathbf{s}}^{p,q}(D) = \widetilde{A}_{\mathbf{s}}^{p,q}(D)$ generated by the $B_{(\zeta,z)}^{\mathbf{s}'}$, $(\zeta, z) \in D$, is $\widetilde{A}_{\mathbf{s},0}^{p,q}(D)$ as strong property $(\widetilde{L}')_{\mathbf{s},\mathbf{s}',0}^{p,q}$ holds (cf. Proposition 4.1 and [18, Lemma 5.15]). In order to conclude, it then

⁹ Notice that the condition $\mathbf{s} \geq \mathbf{0}$ is empty, since it is a consequence of the continuity of $P_{\mathbf{s}'}$ on $L_{\mathbf{s},0}^{p,q}(D)$.

suffices to prove that both (4) and (4') imply (6). Arguing as in the proof of Proposition 4.6 and using [18, Theorem 3.22], this is easily established. $\square$

Corollary 4.8 Take $p, q \in [1, \infty]$, $\mathbf{s} \in \mathbb{R}^r$, and $\mathbf{s}' < \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$. Then, the following conditions are equivalent:

- (1) $P_{\mathbf{s}'}$ induces a continuous linear projector of $L_{\mathbf{s},0}^{p,q}(D)$ onto $A_{\mathbf{s},0}^{p,q}(D)$;
- (2) $\mathbf{s} > \mathbf{0}$ and $P_{\mathbf{s}'}$ induces a continuous linear projector of $L_{\mathbf{s}}^{p,q}(\bar{D})$ onto $A_{\mathbf{s}}^{p,q}(D)$.

Proof This follows from Corollary 4.7 and [18, Proposition 5.20]. $\square$

Corollary 4.9 Take $p, q \in [1, \infty]$ and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$. Then, the following conditions are equivalent:

- (1) strong properties $(L')_{\mathbf{s},\mathbf{s}',0}^{p,q}$ and $(L')_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}$ hold;
- (1') weak property $(L)_{\mathbf{s},\mathbf{s}',0}^{p,q}$ holds;
- (2) $\mathbf{s} > \mathbf{0}$ and strong properties $(L')_{\mathbf{s},\mathbf{s}'}^{p,q}$ and $(L')_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}$ hold;
- (2') $\mathbf{s} > \mathbf{0}$ and weak property $(L)_{\mathbf{s},\mathbf{s}'}^{p,q}$ holds.

Proof This follows from Corollary 4.7 and [18, Lemma 3.29]. $\square$

Corollary 4.10 Take $p, q \in [1, \infty]$, and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$ such that the following hold:

- $\mathbf{s} > \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$;
- $\mathbf{s}' < \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$;
- $\mathbf{s} + \mathbf{s}' < \mathbf{b} + \mathbf{d} - \frac{1}{2q'}\mathbf{m}, \frac{1}{p}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q}\mathbf{m}'$.

Then, the following conditions are equivalent:

- (1) $\tilde{A}_{\mathbf{s}}^{p,q}(D) = A_{\mathbf{s}}^{p,q}(D)$ and $\tilde{A}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D) = A_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$;
- (2) the sesquilinear form

$$A_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D) \times A_{\mathbf{s}}^{p,q}(D) \ni (f, g) \mapsto \int_D f \bar{g} (\Delta_{\Omega}^{-\mathbf{s}'} \circ \rho) \, d\nu_D \in \mathbb{C}$$

induces an antilinear isomorphism of $A_{\mathbf{s}}^{p,q}(D)$ onto $A_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)'$;

- (3) weak property $(L)_{\mathbf{s},\mathbf{s}'}^{p,q}$ holds;
- (3') weak property $(L)_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}$ holds;
- (4) strong properties $(L')_{\mathbf{s},\mathbf{s}'}^{p,q}$, $(L')_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}$, and $(L')_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}$ hold;
- (5) $P_{\mathbf{s}'}$ induces a continuous linear projector of $L_{\mathbf{s}}^{p,q}(D)$ onto $A_{\mathbf{s}}^{p,q}(D)$.

Notice that the condition $\mathbf{s} + \mathbf{s}' < \mathbf{b} + \mathbf{d} - \frac{1}{2q'}\mathbf{m}$ must be imposed to ensure that $A_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}',0}^{p',q'}(D) \neq \{0\}$, for otherwise condition (2) could be trivial. One may also have imposed that $A_s^{p,q}(D) \neq \{0\}$, but this condition, expressed in terms of $\mathbf{s}$, would have required to treat separately the case $q = \infty$.

The condition $\mathbf{s}' < \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$ is necessary for the operator $P_{\mathbf{s}'}$ to be defined, and is implied by the other assumptions and each one of the conditions (1), (2), (3), (3'), and (4).

Proof The equivalence of (1), (2), (3'), (4), and (5) follows from Corollary 4.7. In addition, (5) implies (3) by [18, Proposition 5.24]. It will then suffice to prove that (3) implies (3'). Then, assume that weak property $(L)_{\mathbf{s},\mathbf{s}'}^{p,q}$ holds, so that also weak property $(\tilde{L})_{\mathbf{s},\mathbf{s}'}^{p,q}$ holds and the sesquilinear form of (2) induces an antilinear isomorphism of $A_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$ onto $\tilde{A}_{\mathbf{s},0}^{p,q}(D)'$ by Proposition 4.6. In addition, Theorem 4.5 implies that strong property $(\tilde{L}')_{\mathbf{s},\mathbf{s}'}^{p,q}$ holds. This latter fact, together with weak property $(L)_{\mathbf{s},\mathbf{s}'}^{p,q}$ and the inclusion $A_s^{p,q}(D) \subseteq \tilde{A}_s^{p,q}(D)$ (cf. Proposition 2.4), implies that $A_s^{p,q}(D) = \tilde{A}_s^{p,q}(D)$, so that Corollary 4.7 implies that strong property $(L)_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}$ holds, whence (3'). $\square$

Corollary 4.11 Take $p, q \in [1, \infty]$ with $q' \leq \min(p, p')$, and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$ such that the following hold:

- $\mathbf{s} > \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$;
- $\mathbf{s}' < \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$;
- $\mathbf{s} + \mathbf{s}' < \mathbf{b} + \mathbf{d} - \frac{1}{2q'}\mathbf{m}, \frac{1}{p}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q}\mathbf{m}'$.

Then, the following conditions are equivalent:

- (1) $\tilde{A}_s^{p,q}(D) = A_s^{p,q}(D)$;
- (2) the sesquilinear form

$$A_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}',0}^{p',q'}(D) \times A_s^{p,q}(D) \ni (f, g) \mapsto \int_D f \bar{g} (\Delta_\Omega^{-\mathbf{s}'} \circ \rho) \, d\nu_D \in \mathbb{C}$$

induces an antilinear isomorphism of $A_{\mathbf{s}}^{p,q}(D)$ onto $A_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}',0}^{p',q'}(D)'$;

- (3) weak property $(L)_{\mathbf{s},\mathbf{s}'}^{p,q}$ holds;
- (3') weak property $(L)_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}',\mathbf{s}'}^{p',q'}$ holds;
- (4) strong properties $(L')_{\mathbf{s},\mathbf{s}'}^{p,q}$, $(L')_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}',\mathbf{s}',0}^{p',q'}$ and $(L')_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}$ hold;
- (5) $P_{\mathbf{s}'}$ induces a continuous linear projector of $L_s^{p,q}(D)$ onto $A_s^{p,q}(D)$.

In particular, this applies to the case $p = q \geq 2$.

Proof Apply Proposition 2.7 to show that $\tilde{A}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}',0}^{p',q'}(D) = A_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}',0}^{p',q'}(D)$, and then apply Corollary 4.10. $\square$

Corollary 4.12 Take $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$, and $\mathbf{s}'' \in \mathbb{N}_{\Omega}$, and take $p, q \in [1, \infty]$. Assume that the following conditions are satisfied:

- $\mathbf{s} + \mathbf{s}'' > \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$;
- $\mathbf{s}' < \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$;
- $\mathbf{s} + \mathbf{s}' < \frac{1}{p}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q}\mathbf{m}'$.

Then, the following conditions are equivalent:

- (1) $A_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D) = \tilde{A}_{\mathbf{b}+\mathbf{d}-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$;
- (2) $P_{\mathbf{s}'}$ induces a continuous linear mapping of $L_{\mathbf{s},0}^{p,q}(D)$ onto $\hat{A}_{\mathbf{s},\mathbf{s}',0}^{p,q}(D)$;¹⁰
- (3) $P_{\mathbf{s}'}$ induces a continuous linear mapping of $L_{\mathbf{s}}^{p,q}(D)$ onto $\hat{A}_{\mathbf{s},\mathbf{s}'}^{p,q}(D)$.

This extends [4, Theorem 1.8 (1)]. Observe that the assumption $\mathbf{s}' < \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$ cannot be replaced by the seemingly more natural assumption $\mathbf{s}' - \mathbf{s}'' < \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$, since otherwise $P_{\mathbf{s}'}$ would not be well defined.

Proof It suffices to observe that there is a constant $c_{\mathbf{s}',\mathbf{s}''} \neq 0$ such that $(P_{\mathbf{s}'}f) * I_{\Omega}^{-\mathbf{s}''} = c_{\mathbf{s}',\mathbf{s}''} P_{\mathbf{s}'-\mathbf{s}''}(f(\Delta_{\Omega}^{-\mathbf{s}''} \circ \rho))$ for every $f \in C_c(D)$, thanks to [18, Proposition 2.29], and to apply Theorem 4.5. $\square$

When $\min(p, q) < 1$ we do not know if the preceding equivalences still hold. Nonetheless, we have the following partial results.

Corollary 4.13 Take $p, q \in (0, \infty]$ and $\mathbf{s} > \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$. If $A_{\mathbf{s},0}^{p,q}(D) = \tilde{A}_{\mathbf{s},0}^{p,q}(D)$ (resp. $A_{\mathbf{s}}^{p,q}(D) = \tilde{A}_{\mathbf{s}}^{p,q}(D)$), then $A_{\mathbf{s}-(1/p-1)_+(\mathbf{b}+\mathbf{d}),0}^{p'',q''}(D) = \tilde{A}_{\mathbf{s}-(1/p-1)_+(\mathbf{b}+\mathbf{d}),0}^{p'',q''}(D)$ (resp. $A_{\mathbf{s}-(1/p-1)_+(\mathbf{b}+\mathbf{d})}^{p'',q''}(D) = \tilde{A}_{\mathbf{s}-(1/p-1)_+(\mathbf{b}+\mathbf{d})}^{p'',q''}(D)$).

Proof Observe first that [18, Corollary 5.16] implies that there is $\mathbf{s}' < \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$, $\frac{1}{p}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q}\mathbf{m}' - \mathbf{s}$ such that strong property $(L')_{\mathbf{s},\mathbf{s}',0}^{p,q}$ (resp. $(L')_{\mathbf{s},\mathbf{s}'}^{p,q}$) holds, so that Proposition 4.6 implies that $A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D) = \tilde{A}_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$. Now, consider the sesquilinear form

$$(f, g) \mapsto \int_D f \bar{g}(\Delta_{\Omega}^{-\mathbf{s}'} \circ \rho) \, dv_D,$$

and observe that it induces an antilinear isomorphism $\iota_1 : A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D) \rightarrow \tilde{A}_{\mathbf{s},0}^{p,q}(D)'$ and a continuous linear mapping $\iota_2 : A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D) \rightarrow V'$, where V is the closure of $\tilde{A}_{\mathbf{s},0}^{p,q}(D)$ in $A_{\mathbf{s}-(1/p-1)_+(\mathbf{b}+\mathbf{d})}^{p'',q''}(D)$, thanks to by [18, Proposition 3.37]. In addition, the inclusion $\iota_3 : \tilde{A}_{\mathbf{s},0}^{p,q}(D) \rightarrow V$ is continuous by [18, Propositions 3.2 and 3.7], so that

¹⁰ We define $\hat{A}_{\mathbf{s},\mathbf{s}',0}^{p,q}(D)$ as the space of $f \in \text{Hol}(D)$ such that $f * I_{\Omega}^{-\mathbf{s}''} \in \tilde{A}_{\mathbf{s}+\mathbf{s}'',0}^{p,q}(D)$. We define $\hat{A}_{\mathbf{s},\mathbf{s}'}^{p,q}(D)$ analogously.

$$\iota_1 = {}^t\iota_3 \circ \iota_2.$$

Therefore, ${}^t\iota_3$ is onto, hence an isomorphism, so that also ι_2 is an isomorphism. To conclude, observe that $B_{(\zeta,z)}^{s'} \in \tilde{A}_{s,0}^{p,q}(D) \subseteq V$ for every $(\zeta, z) \in D$, and argue by duality to show that weak property $(L)_{s-(1/p-1)_+(\mathbf{b}+\mathbf{d}),s'}^{p'',q'}$ holds (cf. Proposition 4.6). The conclusion follows from Corollary 4.7. $\square$

Corollary 4.14 *Take $p, q \in (0, \infty]$ and $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^r$ such that weak property $(L)_{\mathbf{s},\mathbf{s}',0}^{p,q}$ holds. Then,*

$$\begin{aligned} A_{s-(1/p-1)_+(\mathbf{b}+\mathbf{d}),0}^{p'',q''}(D) &= \tilde{A}_{s-(1/p-1)_+(\mathbf{b}+\mathbf{d}),0}^{p'',q''}(D), \\ A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D) &= \tilde{A}_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D), \end{aligned}$$

and the sesquilinear form

$$(f, g) \mapsto \int_D f \overline{g} (\Delta_{\Omega}^{-s'} \circ \rho) \, d\nu_D$$

induces an antilinear isomorphism of $A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$ onto V' , where V denotes the closed vector subspace of $A_{s,0}^{p,q}(D)$ generated by the $B_{(\zeta,z)}^{s'}$, $(\zeta, z) \in D$.

This result improves also [18, Proposition 3.37]. Notice that weak property $(L)_{\mathbf{s},\mathbf{s}',0}^{p,q}$ implies that $\mathbf{s} > \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$, that $\mathbf{s} + \mathbf{s}' < \frac{1}{p}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q}\mathbf{m}'$, and that $\mathbf{s}' < \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$, thanks to [18, Lemma 3.29].

Proof Take $\delta_0 > 0$ so that for every $(\delta, 4)$ -lattice $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$ on D the mapping

$$S : f \mapsto (\Delta_{\Omega}^{(\mathbf{b}+\mathbf{d})/p-\mathbf{s}-\mathbf{s}'}(h_k) f)(\zeta_{j,k}, z_{j,k})$$

induces an isomorphism of $A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$ onto a closed subspace of $\ell^{p',q'}(J, K)$, and $S^{-1}(\ell^{p',q'}(J, K)) \cap A_{(\mathbf{b}+\mathbf{d})/p-\mathbf{s}-\mathbf{s}'}^{\infty,\infty}(D) = A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$, where $h_k = \rho(\zeta_{j,k}, z_{j,k})$ for every $j \in J$ and for every $k \in K$. Observe that we may assume that $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$ is chosen so that the mapping

$$\Psi : \ell_0^{p,q}(J, K) \ni \lambda \mapsto \sum_{j,k} \lambda_{j,k} B_{(\zeta_{j,k}, z_{j,k})}^{s'} \Delta_{\Omega}^{(\mathbf{b}+\mathbf{d})/p-\mathbf{s}-\mathbf{s}'}(h_k) \in V$$

is well defined and continuous. If we denote by $\langle \cdot | \cdot \rangle$ the sesquilinear form on $\tilde{A}_{s,0}^{p,q}(D) \times \tilde{A}_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$ which extends to the sesquilinear form in the statement as in Proposition 2.7, then

$$c_{s'} \langle \Psi(\lambda) | f \rangle = \langle \lambda | Sf \rangle$$

for every $\lambda \in \mathbb{C}^{(J \times K)}$ and for every $f \in \tilde{A}_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$ by the proof of Proposition 4.6, where $c_{s'} \neq 0$ is a suitable constant. In particular, this shows that S maps $\tilde{A}_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D)$ into $\ell^{p',q'}(J, K)$ continuously. Since $\tilde{A}_{(\mathbf{b}+\mathbf{d})/\min(1,p)-\mathbf{s}-\mathbf{s}'}^{p',q'}(D) \subseteq A_{(\mathbf{b}+\mathbf{d})/p-\mathbf{s}-\mathbf{s}'}^{\infty,\infty}(D)$ by

Proposition 2.4, this implies that $\tilde{A}_{(\mathbf{b}+\mathbf{d})/\min(1,p)-s-s'}^{p',q'}(D) \subseteq A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-s-s'}^{p',q'}(D)$, so that $\tilde{A}_{(\mathbf{b}+\mathbf{d})/\min(1,p)-s-s'}^{p',q'}(D) = A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-s-s'}^{p',q'}(D)$ by Proposition 2.4. By Theorem 4.5, this implies that property $(\tilde{L})_{s-(1/p-1)_+(\mathbf{b}+\mathbf{d}),0}^{p'',q''}$ holds, so that, in particular, V is contained and dense in $\tilde{A}_{s-(1/p-1)_+(\mathbf{b}+\mathbf{d}),0}^{p'',q''}(D)$. Since $\langle \cdot | \cdot \rangle$ induces an antilinear isomorphism of $\tilde{A}_{(\mathbf{b}+\mathbf{d})/\min(1,p)-s-s'}^{p',q'}(D)$ onto $\tilde{A}_{s-(1/p-1)_+(\mathbf{b}+\mathbf{d}),0}^{p'',q''}(D)'$ by Proposition 2.7, this implies that the continuous antilinear map $\iota : A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-s-s'}^{p',q'}(D) \rightarrow V'$, induced by the sesquilinear form in the statement, is one-to-one. To prove that ι is onto, take $\lambda \in V'$, and define

$$f : D \ni (\zeta, z) \mapsto c_{s'} \overline{\langle \lambda, B_{(\zeta,z)}^{s'} \rangle} \in \mathbb{C}.$$

By [18, Lemma 3.38], $f \in A_{(\mathbf{b}+\mathbf{d})/\min(1,p)-s-s'}^{p',q'}(D)$. Since

$$\langle \lambda, B_{(\zeta,z)}^{s'} \rangle = \langle B_{(\zeta,z)}^{s'} | f \rangle$$

for every $(\zeta, z) \in D$ by [18, Lemma 3.29 and Proposition 3.13], we see that $\lambda = \iota(f)$. Finally, using sampling in V (inherited by sampling in $A_{s,0}^{p,q}(D)$, cf. [18, Theorem 3.22]), we see that strong property $(L)_{(\mathbf{b}+\mathbf{d})/\min(1,p)-s-s'}^{p',q'}$ holds (cf. Proposition 4.6), so that $A_{s-(1/p-1)_+(\mathbf{b}+\mathbf{d}),0}^{p'',q''}(D) = \tilde{A}_{s-(1/p-1)_+(\mathbf{b}+\mathbf{d}),0}^{p'',q''}(D)$ by Corollary 4.7 and Proposition 4.1. $\square$

5 Interpolation

We now prove a useful result concerning *complex* interpolation of Bergman spaces and their boundary value spaces. Cf. [7] for more information on *real* interpolation of Bergman spaces.

Theorem 5.1 Take $p_0, p_1, q_0, q_1 \in (0, \infty]$ and $s_j > \frac{1}{p_j}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q_j}\mathbf{m}'$ for $j = 0, 1$. If $\tilde{A}_{s_j,0}^{p_j,q_j}(D) = A_{s_j,0}^{p_j,q_j}(D)$ (resp. $\tilde{A}_{s_j,0}^{p_j,q_j}(D) = A_{s_j,0}^{p_j,q_j}(D)$) for $j = 0, 1$, then

$$\tilde{A}_{s_\theta,0}^{p_\theta,q_\theta}(D) = A_{s_\theta,0}^{p_\theta,q_\theta}(D) \quad (\text{resp. } \tilde{A}_{s_\theta,0}^{p_\theta,q_\theta}(D) = A_{s_\theta,0}^{p_\theta,q_\theta}(D))$$

for every $\theta \in (0, 1)$, where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \text{and} \quad s_\theta = (1-\theta)s_0 + \theta s_1.$$

Proof Define $S := \{w \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$. Take $\theta \in (0, 1)$ and $u \in \mathring{B}_{p_\theta,q_\theta}^{-s_\theta}(\mathcal{N}, \Omega)$ (resp. $u \in B_{p_\theta,q_\theta}^{-s_\theta}(\mathcal{N}, \Omega)$). By [16, Theorem 6.6], there is a bounded continuous function

$$f : \bar{S} \rightarrow \mathcal{S}'_{\Omega,L}(\mathcal{N})$$

which is holomorphic in S , equals u at θ , and maps $j + iR$ boundedly into $\mathring{B}_{p_j,q_j}^{-s_j}(\mathcal{N}, \Omega)$ (resp. $B_{p_j,q_j}^{-s_j}(\mathcal{N}, \Omega)$) for $j = 0, 1$; here, $\bar{S}$ denotes the closure of S in $\mathbb{C}$.

Take (Ψ_j) as in Lemma 4.2, and observe that [30, 2.4.6/2] implies that there are two probability measures μ_0, μ_1 on $\mathbb{R}$ such that

$$|g(\theta)|^\ell \leq \left(\int_{\mathbb{R}} |g(it)|^\ell d\mu_0(t) \right)^{1-\theta} \left(\int_{\mathbb{R}} |g(1+it)|^\ell d\mu_1(t) \right)^\theta$$

for every bounded uniformly continuous function $g: \bar{S} \rightarrow \mathbb{C}$ which is holomorphic on S , where $\ell := \min(p_0, p_1, q_0, q_1)$. Therefore,

$$\begin{aligned} e^{\varepsilon(\theta^2-\theta)} |\langle u, \Psi_j(S_{(\zeta,z)})_0 \rangle| &\leq \left(\int_{\mathbb{R}} |e^{(it)^2-(it)} \langle f(it), \Psi_j(S_{(\zeta,z)})_0 \rangle|^\ell d\mu_0(t) \right)^{1-\theta} \\ &\quad \times \left(\int_{\mathbb{R}} |e^{(1+it)^2-(1+it)} \langle f(1+it), \Psi_j(S_{(\zeta,z)})_0 \rangle|^\ell d\mu_1(t) \right)^\theta \end{aligned}$$

for every $(\zeta, z) \in D$, for every $j \in \mathbb{N}$, and for every $\varepsilon > 0$. Passing to the limit for $\varepsilon \rightarrow 0^+$, and $j \rightarrow \infty$, this implies that

$$|(\mathcal{E}u)(\zeta, z)| \leq \left(\int_{\mathbb{R}} |(\mathcal{E}f(it))(\zeta, z)|^\ell d\mu_0(t) \right)^{1-\theta} \left(\int_{\mathbb{R}} |(\mathcal{E}f(1+it))(\zeta, z)|^\ell d\mu_1(t) \right)^\theta$$

for every $(\zeta, z) \in D$. By repeated applications of Hölder's and Minkowski's integral inequalities (cf., e.g., the proof of [16, Theorem 6.6]), this implies that

$$\|\mathcal{E}u\|_{A_{s_0}^{p_0, q_0}(D)} \leq \left(\int_{\mathbb{R}} \|(\mathcal{E}f)(it)\|_{A_{s_0}^{p_0, q_0}(D)}^\ell d\mu_0(t) \right)^{1-\theta} \left(\int_{\mathbb{R}} \|(\mathcal{E}f)(1+it)\|_{A_{s_1}^{p_1, q_1}(D)}^\ell d\mu_1(t) \right)^\theta.$$

Since, by assumption, $\mathcal{E}$ induces an isomorphism of $\mathring{B}_{p_j, q_j}^{-s_j}(\mathcal{N}, \Omega)$ onto $A_{s_j, 0}^{p_j, q_j}(D)$ (resp. of $\mathring{B}_{p_j, q_j}^{-s_j}(\mathcal{N}, \Omega)$ onto $A_{s_j}^{p_j, q_j}(D)$) for $j = 0, 1$, this implies that there is a constant $C > 0$ such that

$$\|\mathcal{E}u\|_{A_{s_0}^{p_0, q_0}(D)} \leq C \sup_{j=0,1} \sup_{t \in \mathbb{R}} \|f(j+it)\|_{B_{p_j, q_j}^{-s_j}(\mathcal{N}, \Omega)}$$

once a quasi-norm on $B_{p_j, q_j}^{-s_j}(\mathcal{N}, \Omega)$ is fixed for $j = 0, 1$. By [16, Theorem 6.6] and the arbitrariness of f , this implies that $\mathcal{E}$ maps $\mathring{B}_{p_0, q_0}^{-s_0}(\mathcal{N}, \Omega)$ (resp. $B_{p_0, q_0}^{-s_0}(\mathcal{N}, \Omega)$) continuously into $A_{s_0}^{p_0, q_0}(D)$. Since $\mathcal{S}_{\Omega, L}(\mathcal{N})$ is dense in $\mathring{B}_{p_0, q_0}^{-s_0}(\mathcal{N}, \Omega)$ and $\mathcal{E}(\mathcal{S}_{\Omega, L}(\mathcal{N})) \subseteq A_{s_0, 0}^{p_0, q_0}(D)$ (resp. $\mathcal{E}(\mathcal{S}_{\Omega, L}(\mathcal{N})) \subseteq A_{s_0}^{p_0, q_0}(D)$) by Proposition 2.4, the assertion follows by means of Proposition 4.1. $\square$

Corollary 5.2 Take $p, q \in (0, \infty]$ and $s > \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$. If $\tilde{A}_{s, 0}^{p, q}(D) = A_{s, 0}^{p, q}(D)$ (resp. $\tilde{A}_s^{p, q}(D) = A_s^{p, q}(D)$), then $\tilde{A}_{s', 0}^{p, q}(D) = A_{s', 0}^{p, q}(D)$ (resp. $\tilde{A}_{s'}^{p, q}(D) = A_{s'}^{p, q}(D)$) for every $\mathbf{s}' \geq \mathbf{s}$.

Proof This follows from Theorem 5.1, since $\tilde{A}_{s'', 0}^{p, q}(D) = A_{s'', 0}^{p, q}(D)$ (resp. $\tilde{A}_{s''}^{p, q}(D) = A_{s''}^{p, q}(D)$) for $\mathbf{s}'' = \mathbf{s} + \frac{1}{\theta}(\mathbf{s}' - \mathbf{s})$ and $\theta \in (0, 1)$ sufficiently close to 0, thanks to [18, Corollary 5.11]. $\square$

With similar techniques, one may prove that also strong properties $(L)_{s, s', 0}^{p, q}$ and $(L)_{s, s'}^{p, q}$ interpolate (for fixed $\mathbf{s}'$). For $p, q \geq 1$ and for Bergman projectors, this is a consequence of Corollary 4.7.

Corollary 5.3 Take $p_0, p_1, q_0, q_1 \in [1, \infty]$, $s_0, s_1 \in \mathbb{R}^r$, and $s'_0, s'_1 < \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}'$. If $P_{s'_j}$ induces an endomorphism of $L_{s_j,0}^{p_j,q_j}(D)$ (resp. $L_{s_j}^{p_j,q_j}(D)$) for $j = 0, 1$, then $P_{s'_\theta}$ induces an endomorphism of $L_{s_\theta,0}^{p_\theta,q_\theta}(D)$ (resp. $L_{s_\theta}^{p_\theta,q_\theta}(D)$) for every $\theta \in (0, 1)$, where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad s_\theta = (1-\theta)s_0 + \theta s_1, \quad \text{and} \quad s'_\theta = (1-\theta)s'_0 + \theta s'_1.$$

Proof This follows from Corollary 4.7, Theorem 5.1, and [18, Proposition 5.20]. $\square$

Corollary 5.4 Take $p, q \in (0, \infty]$ and $\mathbf{s} \in \mathbb{R}^r$ and $\mathbf{s}' < \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$. $P_{\mathbf{s}'}$ induces an endomorphism of $L_{\mathbf{s},0}^{p,q}(D)$ (resp. $L_{\mathbf{s}}^{p,q}(D)$), then $P_{\mathbf{s}''}$ induces an endomorphism of $L_{\mathbf{s}'',0}^{p,q}(D)$ (resp. $L_{\mathbf{s}''}^{p,q}(D)$) for every $\mathbf{s}'' \geq \mathbf{s}$ and for every $\mathbf{s}''' \leq \mathbf{s} + \mathbf{s}' - \mathbf{s}''$.

Proof This follows from Corollaries 4.7 and 5.2, and [18, Proposition 5.20]. $\square$

6 Transference

In this section we prove a transference result, Corollary 6.2, showing that if the Bergman projector $P_{\mathbf{s}'}$ is bounded on the pure norm spaces $L_{\mathbf{s}}^{p,p}(F + i\Omega)$ on the tube domain $F + i\Omega$, where Ω is a homogeneous cone, then the Bergman projector $P_{\mathbf{s}'+\mathbf{b}}$ is bounded on the pure norm spaces $L_{\mathbf{s}}^{p,p}(D)$ on the corresponding homogeneous Siegel domain of Type II. We also prove a converse result, Corollary 6.4, showing that if $P_{\mathbf{s}'}$ is bounded on the mixed norm spaces $L_{\mathbf{s}}^{p,q}(D)$, then $P_{\mathbf{s}'}$ is bounded on the mixed norm spaces $L_{\mathbf{s}-\mathbf{b}/p}^{p,q}(F + i\Omega)$ on the tube domain $F + i\Omega$.

Notice that the analogous transference result [8, Theorem 2.1] holds even for $p \neq q$ since the definition of $A_{\mathbf{s}}^{p,q}(D)$ considered therein is essentially different from ours when $p \neq q$ and $n > 0$.

Theorem 6.1 Take $p \in (0, \infty]$ and $\mathbf{s} > \frac{1}{p}\mathbf{d} + \frac{1}{2p'}\mathbf{m}' + (\mathbb{R}_+^*)^r$. If $A_{\mathbf{s},0}^{p,p}(F + i\Omega) = \tilde{A}_{\mathbf{s},0}^{p,p}(F + i\Omega)$ (resp. $A_{\mathbf{s}}^{p,p}(F + i\Omega) = \tilde{A}_{\mathbf{s}}^{p,p}(F + i\Omega)$), then $A_{\mathbf{s},0}^{p,p}(D) = \tilde{A}_{\mathbf{s},0}^{p,p}(D)$ ($A_{\mathbf{s}}^{p,p}(D) = \tilde{A}_{\mathbf{s}}^{p,p}(D)$).

Proof Take $u \in \mathring{B}_{p,q}^{-\mathbf{s}}(\mathcal{N}, \Omega)$ (resp. $u \in B_{p,q}^{-\mathbf{s}}(\mathcal{N}, \Omega)$). Take a (δ, R) -lattice $(\lambda_k)_{k \in K}$ on Ω' for some $\delta > 0$ and some $R > 1$, and fix a positive $\varphi \in C_c^\infty(\Omega')$ such that

$$\sum_{k \in K} \varphi(\cdot t_k^{-1}) \geq 1$$

on Ω' , where $t_k \in T_+$ is chosen such that $\lambda_k = e_{\Omega'} \cdot t_k$. Define $\psi_k := \mathcal{F}_{\mathcal{N}}^{-1}(\varphi(\cdot t_k^{-1}))$ and $\psi'_k := \mathcal{F}_F^{-1}(\varphi(\cdot t_k^{-1}))$. Let us prove that

$$[(\mathcal{E}u)_h * \psi_k](\zeta, \cdot) = [(\mathcal{E}u)_h(\zeta, \cdot)] * \psi'_k$$

for every $\zeta \in E$ and for every $k \in K$. Assume first that $u \in S_{\Omega}(\mathcal{N})$, and observe that by (2) there is a constant $c > 0$ such that

$$\begin{aligned}
\mathcal{F}_F([(Eu)_h * \psi_k](\zeta, \cdot)) &= c\Delta_{\Omega'}^{-\mathbf{b}} e^{-\langle \cdot, \Phi(\zeta) \rangle} \mathcal{F}_{\mathcal{N}}(Eu)_h \varphi(\cdot t_k^{-1}) \\
&= \mathcal{F}_F((Eu)_h(\zeta, \cdot)) \varphi(\cdot t_k^{-1}) \\
&= \mathcal{F}_F([(Eu)_h(\zeta, \cdot)] * \psi'_k)
\end{aligned}$$

for every $\zeta \in E$ and for every $k \in K$. The assertion then follows when u is a finite sum of left translates of elements of $\mathcal{S}_{\Omega}(\mathcal{N})$. Since this set is dense in $\mathcal{S}_{\Omega,L}(\mathcal{N})$ (cf. [18, Proposition 4.5]), hence in $B_{p,p}^{-s}(\mathcal{N}, \Omega)$ for the weak topology $\sigma_{p,p}^{-s}$, the assertion follows by continuity.

In addition, by assumption there is a constant $C > 0$ such that

$$\|\Delta_{\Omega'}^s(\lambda_k)\| \|[(Eu)_h(\zeta, \cdot)] * \psi'_k\|_{L^p(F)} \|_{\ell^p(K)} \geq C \| (Eu)(\zeta, \cdot + i\Phi(\zeta) + ih) \|_{A_s^{p,p}(F+i\Omega)}$$

for every u as above, for every $h \in \Omega$, and for every $\zeta \in E$. Taking the $L^p(E)$ -norm on both sides, this gives

$$\|\Delta_{\Omega'}^s(\lambda_k)\| \|[(Eu)_h] * \psi_k\|_{L^p(\mathcal{N})} \|_{\ell^p(K)} \geq C \| (Eu)(\cdot + (0, ih)) \|_{L_s^{p,p}(D)}$$

for every u as above, and for every $h \in \Omega$. Since

$$\|u * \psi_k\|_{L^p(\mathcal{N})} \geq \|[\mathcal{E}(u * \psi_k)]_h\|_{L^p(\mathcal{N})} = \|(\mathcal{E}u)_h * \psi_k\|_{L^p(\mathcal{N})}$$

for every $h \in \Omega$, for every $k \in K$, and for every u as above by [21, Theorem 1.7], passing to the limit for $h \rightarrow 0$, $h \in \Omega$, by lower semi-continuity we infer that

$$\|\Delta_{\Omega'}^s(\lambda_k)\| \|u * \psi_k\|_{L^p(\mathcal{N})} \|_{\ell^p(K)} \geq C \| \mathcal{E}u \|_{L_s^{p,p}(D)},$$

so that $\tilde{A}_{s,0}^{p,p}(D)$ (resp. $\tilde{A}_s^{p,p}(D)$) embeds continuously into $A_s^{p,p}(D)$ by the arbitrariness of u . Since $\mathcal{E}(\mathcal{S}_{\Omega,L}(\mathcal{N}))$ is dense in $\tilde{A}_{s,0}^{p,p}(D)$ (resp. $\tilde{A}_s^{p,p}(D)$) and contained in $A_{s,0}^{p,p}(D)$ by Proposition 2.4, the assertion follows by means of Proposition 4.1. $\square$

Corollary 6.2 Take $p \in [1, \infty]$, $\mathbf{s} \in \mathbb{R}^r$, and $\mathbf{s}' < \mathbf{d} - \frac{1}{2}\mathbf{m}$. If $P_{\mathbf{s}'}$ induces a continuous linear projector of $L_{s,0}^{p,p}(F + i\Omega)$ onto $A_{s,0}^{p,p}(F + i\Omega)$ (resp. of $L_s^{p,p}(F + i\Omega)$ onto $A_s^{p,p}(F + i\Omega)$), then $P_{\mathbf{s}'+\mathbf{b}}$ induces a continuous linear projector of $L_{s,0}^{p,p}(D)$ onto $A_{s,0}^{p,p}(D)$ (resp. of $L_s^{p,p}(D)$ onto $A_s^{p,p}(D)$).

Analogous results hold for atomic decomposition, provided that $p \in [1, \infty]$. When $\mathbf{s}' = \mathbf{b} + \mathbf{d} - p\mathbf{s}$ and $p < \infty$, this is a consequence of [8, Theorem 2.1].

Proof The assertion follows from Corollary 4.7 and Theorem 6.1. $\square$

Theorem 6.3 Take $p, q \in (0, \infty]$ and $\mathbf{s} > \frac{1}{2}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$. If $A_{s,0}^{p,q}(D) = \tilde{A}_{s,0}^{p,q}(D)$ (resp. $A_s^{p,q}(D) = \tilde{A}_s^{p,q}(D)$), then $A_{s-\mathbf{b}/p,0}^{p,q}(F + i\Omega) = \tilde{A}_{s-\mathbf{b}/p,0}^{p,q}(F + i\Omega)$ (resp. $A_{s-\mathbf{b}/p}^{p,q}(F + i\Omega) = \tilde{A}_{s-\mathbf{b}/p}^{p,q}(F + i\Omega)$).

We shall postpone the proof of Theorem 6.3 until the end of this section. Here is the corollary on the associated transference result for Bergman projectors.

Corollary 6.4 Take $p, q \in [1, \infty]$, $\mathbf{s} \in \mathbb{R}^r$, and $\mathbf{s}' < \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}$. If $P_{\mathbf{s}'}$ induces a continuous linear projector of $L_{s,0}^{p,q}(D)$ onto $A_{s,0}^{p,q}(D)$ (resp. of $L_s^{p,q}(D)$ onto $A_s^{p,q}(D)$), then $P_{\mathbf{s}'}$ induces a

continuous linear projector of $L_{s-b/p,0}^{p,q}(F+i\Omega)$ onto $A_{s-b/p,0}^{p,q}(F+i\Omega)$ (resp. of $L_{s-b/p}^{p,q}(F+i\Omega)$ onto $A_{s-b/p}^{p,q}(F+i\Omega)$).

Proof The assertion follows from Theorem 6.3, Corollary 4.7, and [18, Proposition 5.20]. $\square$

Before we pass to the proof of Theorem 6.3, we need some auxiliary results.

Proposition 6.5 Take $p, q \in (0, \infty]$ and $s \in \mathbb{R}^r$. Then, there is a constant $C > 0$ such that

$$\|f(\zeta, \cdot + i\Phi(\zeta))\|_{A_{s-b/p}^{p,q}(F+i\Omega)} \leq C \|f\|_{A_s^{p,q}(D)}$$

for every $\zeta \in E$ and for every $f \in A_s^{p,q}(D)$. If, in addition, $f \in A_{s,0}^{p,q}(D)$, then $f(\zeta, \cdot + i\Phi(\zeta)) \in A_{s,0}^{p,q}(F+i\Omega)$ for every $\zeta \in E$.

Conversely, define $\iota : \text{Hol}(F+i\Omega) \rightarrow \text{Hol}(D)$ so that $\iota(f)(\zeta, z) = f(z)$ for every $(\zeta, z) \in D$. Assume that $s > \frac{1}{2p}\mathbf{m}$ (resp. $s \geq 0$ if $p = \infty$). Then, there is a constant $C' > 0$ such that

$$\|\iota(f)\|_{A_s^{p,p}(D)} = C' \|f\|_{A_{s-b/p}^{p,p}(F+i\Omega)}$$

for every $f \in A_{s-b/p,0}^{p,p}(F+i\Omega)$ (resp. $f \in A_{s-b/p}^{p,p}(F+i\Omega)$). In addition, $\iota(f) \in A_{s,0}^{p,p}(D)$ if $f \in A_{s-b/p,0}^{p,p}(F+i\Omega)$.

Proof By translation invariance, it will suffice to prove the assertion for $\zeta = 0$. Define $\ell := \min(1, p, q)$ to simplify the notation, and set

$$\varphi : E \times F \times \Omega \ni (\zeta, x, h) \mapsto (\zeta, x + i\Phi(\zeta) + ih) \in D,$$

and observe that φ is a bijection of $E \times F \times \Omega$ onto D . Observe that there are $R_0 > 0$ and $C' > 0$ such that, for every $R \in (0, R_0]$ and for every $h \in \Omega$,

$$B_{E \times F_C}((0, ih), R/C') \subseteq \varphi(B_E(0, R) \times B_F(0, R) \times B_F(h, R)) \subseteq B_{E \times F_C}((0, ih), C'R).$$

Therefore, [18, Lemma 1.24] implies that there is a constant $C_{R_0} > 0$ such that

$$|f(0, ih)|^\ell \leq C_{R_0} \int_{B_F(h, R)} \int_{B_E(0, R)} \int_{B_F(0, R)} |f_{h'}(\zeta', x')|^\ell dx' d\zeta' dh'$$

for every $h \in \Omega$ and for every $R \in (0, R_0]$ such that $\bar{B}_{E \times F_C}((0, ih), C'R) \subseteq D$. Hence,

$$|f(0, x + ih)|^\ell \leq C_{R_0} \int_{B_F(h, R)} \int_{B_E(0, R)} \int_{B_F(x, R)} |f_{h'}(\zeta', x')|^\ell dx' d\zeta' dh'$$

for every $x + ih \in F + i\Omega$ and for every $R \in (0, R_0]$ such that $\overline{B_{E \times F_{\mathbb{C}}}((0, ih), C'R)} \subseteq D$. Thus, applying Minkowski's integral inequality, the invariance of Lebesgue measure and Jensen's inequality,

$$\begin{aligned} \|f(0, \cdot + ih)\|_{L^p(F)}^{\min(1,q)} &= \left\| |f(0, \cdot + ih)|^\ell \right\|_{L^{p/\ell}(F)}^{\min(1,q)/\ell} \\ &\leq \left(C_{R_0} \int_{B_F(h,R)} \int_{B_E(0,R)} \|f_{h'}(\zeta', \cdot)\|_{L^p(\mathcal{N})}^\ell d\zeta' dh' \right)^{\min(1,q)/\ell} \\ &\leq C_{R_0}^{\min(1,q)/\ell} \int_{B_F(h,R)} \left(\int_{B_E(0,R)} \|f_{h'}(\zeta', \cdot)\|_{L^p(\mathcal{N})}^p d\zeta' \right)^{\min(1,q)/p} dh' \\ &\leq \frac{C_{R_0}^{\min(1,q)/\ell}}{\mathcal{H}^{2n}(B_E(0,R))^{\min(1,q)/p}} \int_{B_F(h,R)} \|f_{h'}\|_{L^p(\mathcal{N})}^{\min(1,q)} dh'. \end{aligned}$$

Then, there are $R > 0$ such that $B_F(e_\Omega, R') \subseteq B_\Omega(e_\Omega, R)$ for some $R' \in (0, R_0]$, and a constant $C'_{p,q,R} > 0$ such that

$$\|f(0, \cdot + ie_\Omega)\|_{L^p(F)}^{\min(1,q)} \leq C'_{p,q,R} \int_{B_\Omega(e_\Omega, R)} \left(\Delta_\Omega^{\mathbf{s}}(h') \|f_{h'}\|_{L^p(\mathcal{N})} \right)^{\min(1,q)} d\nu_\Omega(h'),$$

so that, by homogeneity,

$$\begin{aligned} &\left(\Delta_\Omega^{\mathbf{s}-\mathbf{b}/p}(h) \|f(0, \cdot + ih)\|_{L^p(F)} \right)^{\min(1,q)} \\ &\leq C'_{p,q,R} \int_{B_\Omega(h,R)} \left(\Delta_\Omega^{\mathbf{s}}(h') \|f_{h'}\|_{L^p(\mathcal{N})} \right)^{\min(1,q)} d\nu_\Omega(h'), \end{aligned}$$

for every $h \in \Omega$. Hence, by Jensen's inequality

$$\|f(0, \cdot)\|_{A_{\mathbf{s}-\mathbf{b}/p}^{p,q}(F+i\Omega)} \leq C'^{1/\min(1,q)}_{p,q,R} \|f\|_{A_{\mathbf{s}}^{p,q}(D)}$$

for every $f \in A_{\mathbf{s}}^{p,q}(D)$. The first assertion follows. In particular, the mapping $f \mapsto f(0, \cdot)$ is continuous from $A_{\mathbf{s}}^{p,q}(D)$ into $A_{\mathbf{s}-\mathbf{b}/p}^{p,q}(F+i\Omega)$, and maps $A_{\mathbf{s}}^{p,q}(D) \cap A_{\mathbf{s}}^{\min(1,p),\min(1,q)}(D)$ into $A_{\mathbf{s}-\mathbf{b}/p}^{p,q}(F+i\Omega) \cap A_{\mathbf{s}-\mathbf{b}/\min(1,p)}^{\min(1,p),\min(1,q)}(F+i\Omega) \subseteq A_{\mathbf{s}-\mathbf{b}/p,0}^{p,q}(F+i\Omega)$ (cf. [18, Proposition 3.7]). Since $A_{\mathbf{s},0}^{p,q}(D) \cap A_{\mathbf{s}}^{\min(1,p),\min(1,q)}(D)$ is dense in $A_{\mathbf{s},0}^{p,q}(D)$ by [18, Proposition 3.9], the second assertion follows.

Now, take $f \in \text{Hol}(F+i\Omega)$ and assume that $\mathbf{s} > \frac{1}{2p}\mathbf{m}$ (resp. $\mathbf{s} \geq \mathbf{0}$ if $p = \infty$). Then, [18, Proposition 2.30] implies that there is a constant $c > 0$ such that

$$\begin{aligned} \|\iota(f)_h\|_{L^p(\mathcal{N})} &= \|\zeta \mapsto \|f_{h+\Phi(\zeta)}\|_{L^p(F)}\|_{L^p(E)} \\ &= c^{1/p} \|h' \mapsto \|f_{h+h'}\|_{L^p(F)}\|_{L^p(I_\Omega^{-\mathbf{b}})}, \end{aligned}$$

so that

$$\|\iota(f)\|_{A_s^{p,p}(D)} = (c\Gamma_\Omega(ps))^{1/p} \|h \mapsto \|f_h\|_{L^p(F)}\|_{L^p(I_\Omega^s * I_\Omega^{-b})}$$

when $p < \infty$ and

$$\|\iota(f)\|_{A_s^{\infty,\infty}(D)} = \|h \mapsto \Delta_\Omega^s(h)\|_{L^\infty(F)}\|f_h\|_{L^\infty(\Omega)}$$

otherwise. The third assertion follows from [18, Proposition 2.28], while the last one need only be verified when $p = \infty$, in which case it is clear. $\square$

Lemma 6.6 *Take $\psi \in \mathcal{S}_\Omega(F)$ and $p \in (0, \infty]$. Then, there is a constant $C > 0$ such that*

$$\frac{1}{C} \|u\|_{L^p(F)} \leq \|t'(u)\|_{L^p(\mathcal{N})} \leq C \|u\|_{L^p(F)}$$

for every $u \in \mathcal{S}'_{\Omega,L}(F)$ such that $u = u * \psi \in L^p(F)$, where $t'(u)(\zeta, x) := (\mathcal{E}u)(x + i\Phi(\zeta))$ for every $(\zeta, x) \in \mathcal{N}$.

Proof Take a compact convex subset K of Ω' such that $\text{Supp}(\mathcal{F}_\mathcal{N}\psi) \subseteq K$, and observe that, if $u \in \mathcal{S}'_{\Omega,L}(F)$ and $u = u * \psi$, then $\mathcal{E}u$ is a well-defined entire function on $F_\mathbb{C}$ and

$$\|(\mathcal{E}u)_h\|_{L^p(F)} \leq \|(\mathcal{E}u)_{h'}\|_{L^p(F)} e^{H_K(h-h')}$$

for every $h, h' \in F$, thanks to [21, Theorems 1.7 and 1.10], where

$$H_K(h) := \sup_{\lambda \in -K} \langle \lambda, h \rangle$$

for every $h \in F$. Therefore,

$$\|t'(u)\|_{L^p(\mathcal{N})} = \|\zeta \mapsto \|(\mathcal{E}u)_{\Phi(\zeta)}\|_{L^p(F)}\|_{L^p(E)} \leq \|u\|_{L^p(F)} \|\zeta \mapsto e^{H_K(\Phi(\zeta))}\|_{L^p(E)}$$

and, analogously,

$$\|t'(u)\|_{L^p(\mathcal{N})} = \|\zeta \mapsto \|(\mathcal{E}u)_{\Phi(\zeta)}\|_{L^p(F)}\|_{L^p(E)} \geq \|u\|_{L^p(F)} \|\zeta \mapsto e^{-H_K(-\Phi(\zeta))}\|_{L^p(E)}.$$

Now, observe that, since K is a compact subset of Ω' , there is a constant $C_1 > 0$ such that

$$\langle \lambda, h \rangle \geq C_1 |h|$$

for every $\lambda \in K$ and for every $h \in \overline{\Omega}$, so that

$$-H_K(-\Phi(\zeta)) \leq H_K(\Phi(\zeta)) \leq -C_1 |\Phi(\zeta)|$$

for every $\zeta \in E$. Since Φ is proper and absolutely homogeneous of degree 2, there is a constant $C_2 > 0$ such that $|\Phi(\zeta)| \geq C_2 |\zeta|^2$, so that the assertion follows. $\square$

Proposition 6.7 *Take $p, q \in (0, \infty]$ and $s > \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$. Then, the mapping*

$$\iota : \text{Hol}(F + i\Omega) \ni f \mapsto [(\zeta, z) \mapsto f(z + i\Phi(\zeta))] \in \text{Hol}(D)$$

induces isomorphisms of $\widetilde{A}_{s-\mathbf{b}/p,0}^{p,q}(F + i\Omega)$ and $\widetilde{A}_{s-\mathbf{b}/p}^{p,q}(F + i\Omega)$ onto closed subspaces of $\widetilde{A}_{s,0}^{p,q}(D)$ and $\widetilde{A}_s^{p,q}(D)$, respectively.

Proof Since $\iota(\mathcal{E}(\mathcal{S}_\Omega(F))) = \mathcal{E}(\mathcal{S}_\Omega(\mathcal{N}))$, it will suffice to prove that ι induces an isomorphism of $\tilde{A}_{s-\mathbf{b}/p}^{p,q}(F + i\Omega)$ onto a closed subspace of $\tilde{A}_s^{p,q}(D)$. Take a (δ, R) -lattice $(\lambda_k)_{k \in K}$ on Ω' for some $\delta > 0$ and some $R > 1$, and fix a positive $\varphi \in C_c^\infty(\Omega')$ such that

$$\sum_{k \in K} \varphi(\cdot t_k^{-1}) \geq 1$$

on Ω' , where $t_k \in T_+$ is chosen such that $\lambda_k = e_{\Omega'} \cdot t_k$. Define $\psi_k := \mathcal{F}_F^{-1}(\varphi(\cdot t_k^{-1}))$, and observe that Lemma 6.6 and a homogeneity argument show that there is a constant $C > 0$ such that

$$\frac{1}{C} \Delta_{\Omega'}^{\mathbf{b}/p}(\lambda_k) \|u * \psi_k\|_{L^p(F)} \leq \|i'(u * \psi_k)\|_{L^p(\mathcal{N})} \leq C \Delta_{\Omega'}^{\mathbf{b}/p}(\lambda_k) \|u * \psi_k\|_{L^p(F)}$$

for every $u \in B_{p,q}^{-s+\mathbf{b}/p}(F, \Omega)$ and for every $k \in \mathbb{N}$, where $i'(u * \psi_k)(\zeta, x) = \mathcal{E}(u * \psi_k)(x + i\Phi(\zeta))$ for every $(\zeta, x) \in \mathcal{N}$. Therefore, i' induces an isomorphism of $B_{p,q}^{-s+\mathbf{b}/p}(F, \Omega)$ onto a closed subspace of $B_{p,q}^{-s}(\mathcal{N}, \Omega)$. Since clearly

$$\mathcal{E} \circ i' = \iota \circ \mathcal{E},$$

this implies that ι induces an isomorphism of $\tilde{A}_{s-\mathbf{b}/p}^{p,q}(F + i\Omega)$ onto a closed subspace of $\tilde{A}_s^{p,q}(D)$. $\square$

Proof of Theorem 6.3 Take $f \in \tilde{A}_{s-\mathbf{b}/p,0}^{p,q}(F + i\Omega)$ (resp. $f \in \tilde{A}_{s-\mathbf{b}/p}^{p,q}(F + i\Omega)$). Then, $\iota(f) \in \tilde{A}_{s,0}^{p,q}(D)$ (resp. $\iota(f) \in \tilde{A}_s^{p,q}(D)$), with the notation of Proposition 6.7. Hence, $\iota(f) \in A_{s,0}^{p,q}(D)$ (resp. $\iota(f) \in A_s^{p,q}(D)$), so that $f = \iota(f)(0, \cdot) \in A_{s-\mathbf{b}/p,0}^{p,q}(F + i\Omega)$ (resp. $f = \iota(f)(0, \cdot) \in A_{s-\mathbf{b}/p}^{p,q}(F + i\Omega)$) by Proposition 6.5. The proof is complete. $\square$

7 Proof of Proposition 2.5

We now present the proof of Proposition 2.5. We begin with some useful lemmas.

Lemma 7.1 *Let P be a (holomorphic) polynomial on $\mathbb{C}^n$ of degree k , and let C be an open convex subset of $\mathbb{C}^n$ contained in $\{z \in \mathbb{C}^n : P(z) \neq 0\}$. Then,*

$$\text{osc}_C(\text{Im } \log P) \leq k\pi.$$

Here, $\text{osc}_C f$ denotes the oscillation of a function f on the set C , that is, the diameter of $f(C)$. Notice that $\log P$ is (ambiguously) defined on C since C is convex, and that $\text{osc}_C(\text{Im } \log P)$ is unambiguously defined.

Proof We may assume that $C \neq \emptyset$, so that $P \neq 0$.

STEP I. Assume first that $n = 1$. Then, there are $a, z_1, \dots, z_k \in \mathbb{C}$, $a \neq 0$, such that

$$P(z) = a \prod_{j=1}^k (z - z_j)$$

for every $z \in \mathbb{C}$. In addition, $z_j \notin C$ for every $j = 1, \dots, k$. Choose $a' \in \mathbb{C}$ so that $e^{a'} = a$ and, for every $j = 1, \dots, k$, choose $\alpha_j \in \mathbb{R}$ such that

$$C - z_j \subseteq \ell_j := \{z \in \mathbb{C} : \operatorname{Re}(ze^{i\alpha_j}) > 0\},$$

and define

$$\log_j(z) := \log |z| + A_j(z),$$

for every $z \in \ell_j$, where $A_j(z)$ is the unique element of $(-\pi/2, \pi/2) - \alpha_j$ such that $e^{\log_j(z)} = z$. Therefore, we may assume that

$$\log P(z) = a' + \sum_{j=1}^k \log_j(z - z_j)$$

for every $z \in C$. Then,

$$\operatorname{osc}_C(\operatorname{Im} \log P) \leq \sum_{j=1}^k \operatorname{osc}_C(\operatorname{Im} \log_j(\cdot - z_j)) \leq k\pi$$

whence the result in this case.

STEP II. It suffices to observe that

$$\operatorname{osc}_C(\operatorname{Im} \log P) = \sup_{\ell} \operatorname{osc}_{C \cap \ell}(\operatorname{Im} \log P),$$

where ℓ runs through the set of (complex) affine lines in $\mathbb{C}^n$ which meet C , and to apply STEP I. $\square$

For every $\mathbf{s} \in \mathbb{C}^r$, we define $\log \Delta_{\Omega}^{\mathbf{s}}$ as the unique holomorphic function f on $\Omega + iF$ such that $f(e_{\Omega}) = 0$ and $e^f = \Delta_{\Omega}^{\mathbf{s}}$ on $\Omega + iF$. Notice that

$$\log \Delta_{\Omega}^{as+\alpha's'} = \alpha \log \Delta_{\Omega}^{\mathbf{s}} + \alpha' \log \Delta_{\Omega}^{\mathbf{s}'} \quad (4)$$

for every $\mathbf{s}, \mathbf{s}' \in \mathbb{C}^r$ and for every $\alpha, \alpha' \in \mathbb{C}$, since both sides of the asserted equality are holomorphic functions which vanish at e_{Ω} and whose exponential is $\Delta_{\Omega}^{as+\alpha's'}$ on Ω , hence on $\Omega + iF$ by holomorphy.

We define

$$|\mathbf{s}| := \sum_{j,j'} |\alpha_j| s_{jj'}$$

for every $\mathbf{s} \in \mathbb{R}^r$, where $\mathbf{s}_j = (s_{j,1}, \dots, s_{j,r})$, $j = 1, \dots, r$, is a ‘basis’ of $\mathbb{N}_{\Omega}$ over $\mathbb{N}$ (hence of $\mathbb{C}^r$ over $\mathbb{C}$), and $\mathbf{s} = \sum_j \alpha_j \mathbf{s}_j$ (such a ‘basis’ exists by [26, Theorem 2.2]). Notice that $(\mathbf{s}_j)$ is uniquely determined up to the order, so that $|\mathbf{s}|$ is well defined. In addition,

$$\sum_j |s_j| \leq |s|,$$

with equality if and only if s or $-s$ belongs to the closed convex cone generated by $\mathbb{N}_\Omega$.

Corollary 7.2 *Take $s \in \mathbb{R}^r$. Then, $\operatorname{Im} \log \Delta_\Omega^s$ is bounded on $\Omega + iF$. More precisely, $\operatorname{osc}_{\Omega+iF}(\operatorname{Im} \log \Delta_\Omega^s) \leq |s|\pi$.*

When Ω is (irreducible and) symmetric, [24, Lemma 7.3] shows that $\operatorname{Re} \Delta_\Omega^e(z) > 0$ for every $z \in \Omega + iF$, where $e_j = (\delta_{jj'})_{j'=1,\dots,r}$. In other words, one may replace $|s|$ with $\sum_j |s_j|$ in the above result. Nonetheless, the analogous estimates do *not* extend to all homogeneous cones. Counterexamples arise already when $r = 3$ and $m = 8$.

Proof The assertion follows from Lemma 7.1 for $s \in \mathbb{N}_\Omega$. The assertion then follows from (4) and the definition of $|s|$. $\square$

Lemma 7.3 *Let P be a polynomial on $\mathbb{C}$ of degree k whose zeroes are contained in $\mathbb{R}_-$. Then,*

$$\frac{1}{2^{k/2}} \leq \frac{|P(\pm ix)|}{P(x)} \leq 1$$

for every $x > 0$.

Proof Let $x_1, \dots, x_k$ be the (not necessarily distinct) zeroes of P , so that there is a non-zero $a \in \mathbb{C}$ such that

$$P(z) = a \prod_{j=1}^k (z - x_j)$$

for every $z \in \mathbb{C}$. Observe that

$$\frac{|\pm ix - x_j|}{x - x_j} = \frac{\sqrt{x^2 + x_j^2}}{x + |x_j|} \in [2^{-1/2}, 1]$$

for every $j = 1, \dots, k$ and for every $x > 0$, since $x_j \leq 0$. The assertion follows. $\square$

Corollary 7.4 *Take $s \in \mathbb{R}^r$. Then,*

$$2^{-|s|/2} \leq \frac{|\Delta_\Omega^s(x \pm iy)|}{\Delta_\Omega^s(x + y)} \leq 2^{|s|/2}$$

for every $x, y \in \Omega$.

Proof Fix $x, y \in \Omega$ and assume first that $s \in \mathbb{N}_\Omega$, that is, that Δ_Ω^s is polynomial. Define

$$P : \mathbb{C} \ni w \mapsto \Delta_\Omega^s(x + wy) \in \mathbb{C},$$

so that P is a holomorphic polynomial on $\mathbb{C}$. Since Δ_{Ω}^s is homogeneous of degree $\sum_j s_j$, the degree of P is $\sum_j s_j$. Observe that $P(w) = \Delta_{\Omega}^s(x + wy) \neq 0$ for every $w \in \mathbb{C}$ with $\operatorname{Re} w > 0$, since clearly

$$\Delta_{\Omega}^s(x + wy)\Delta_{\Omega}^{-s}(x + wy) = 1.$$

In addition, observe that

$$P(\pm iw) = (\pm i)^{\sum_j s_j} \Delta_{\Omega}^s(wy \mp ix) \neq 0$$

for every $w \in \mathbb{C}$ with $\mp \operatorname{Im} w = \operatorname{Re}(\pm iw) > 0$, for the same reason as above. Consequently, the zeroes of P are contained in $\mathbb{R}_-$, so that Lemma 7.3 implies that the assertion holds in this case.

The general case follows from (4) and the definition of $|s|$. $\square$

Lemma 7.5 Take $s_1, s_2, s_3 \in \mathbb{R}^r$ such that $s_2 > \mathbf{0}$, and $\alpha \in \mathbb{R}$. Then,

$$\int_{\Omega} \Delta_{\Omega}^{s_1}(h + h')(1 + |\log \Delta_{\Omega}^{s_2}(h + h')|)^{\alpha} \Delta_{\Omega}^{s_3}(h') \, dv_{\Omega}(h') < \infty,$$

for some (equivalently, every) $h \in \Omega$, if and only if $s_3 > \frac{1}{2}\mathbf{m}$ and either $s_1 + s_3 \leq -\frac{1}{2}\mathbf{m}'$ and $\alpha < -1$, or $s_1 + s_3 < -\frac{1}{2}\mathbf{m}'$.

Proof Observe that, by [18, Corollary 2.49], there is $\varepsilon > 0$ such that

$$\left| \frac{\Delta_{\Omega}^{s_j}(h + h')}{\Delta_{\Omega}^{s_j}(h + h'')} - 1 \right| < \frac{1}{4}$$

for every $h \in \overline{\Omega}$ and for every $h', h'' \in \Omega$ with $d_{\Omega}(h', h'') \leq \varepsilon$, and for $j = 1, 2, 3$. Consequently, if we define

$$I(h) := \int_{\Omega} \Delta_{\Omega}^{s_1}(h + h')(1 + |\log \Delta_{\Omega}^{s_2}(h + h')|)^{\alpha} \Delta_{\Omega}^{s_3}(h') \, dv_{\Omega}(h')$$

for every $h \in \Omega$, we have

$$\begin{aligned} \frac{3 \min((1 - \log(4/3))^{\alpha}, (1 + \log(4/3))^{\alpha})}{4} I(h') &\leq I(h) \\ &\leq \frac{5 \max((1 - \log(4/3))^{\alpha}, (1 + \log(4/3))^{\alpha})}{4} I(h') \end{aligned}$$

for every $h, h' \in \Omega$ with $d_{\Omega}(h, h') \leq \varepsilon$. Consequently, $I(h)$ is finite for every $h \in \Omega$ if and only if $I(h)$ is finite for some $h \in \Omega$. Next, observe that the function

$$(1 + |\log \Delta_{\Omega}^{s_2}|) \Delta_{\Omega}^{-\varepsilon s_2}$$

is bounded above on $h + \Omega$ for every $h \in \Omega$ and for every $\varepsilon > 0$, since

$$\Delta_{\Omega}^{s_2}(h + h') \geq \Delta_{\Omega}^{s_2}(h) > 0$$

for every $h' \in \Omega$ (cf. [18, Corollary 2.36]). Therefore, by means of [18, Corollary 2.22], we see that $I(e_{\Omega})$ (say) is finite when $s_3 > \frac{1}{2}\mathbf{m}$ and $s_1 + s_3 < -\frac{1}{2}\mathbf{m}'$, and that $I(e_{\Omega})$ is infinite

when either $s_3 \not\prec \frac{1}{2}\mathbf{m}$, or $s_1 + s_3 \not\prec -\frac{1}{2}\mathbf{m}'$, or $s_1 + s_3 \prec -\frac{1}{2}\mathbf{m}'$ and $\alpha \geq 0$. Consequently, we may reduce to the case in which $s_3 \succ \frac{1}{2}\mathbf{m}$, $s_1 + s_3 \prec -\frac{1}{2}\mathbf{m}'$, $s_1 + s_3 \leq -\frac{1}{2}\mathbf{m}'$, and $\alpha < 0$, and to prove that $I(e_\Omega)$ is finite if and only if $\alpha < -1$.

In order to simplify the proof, we shall now identify T_+ with Ω by means of the mapping $t \mapsto t \cdot e_\Omega$, so that hh' and h^{-1} are defined for every $h, h' \in \Omega$. Then, using [18, Lemma 2.18], we see that

$$\begin{aligned} I(h) &= \int_{h+\Omega} \Delta_\Omega^{s_1-\mathbf{d}}(h')(1 + |\log \Delta_\Omega^{s_2}(h')|)^\alpha \Delta_\Omega^{s_3+\mathbf{d}}(h'-h) \, dv_\Omega(h') \\ &= \int_{h'^{-1}h \in h+\Omega} \Delta_\Omega^{s_1-\mathbf{d}+(\mathbf{m}'-\mathbf{m})/2}(h'^{-1}h)(1 + |\log \Delta_\Omega^{s_2}(h'^{-1}h)|)^\alpha \\ &\quad \times \Delta_\Omega^{s_3+\mathbf{d}}(h'^{-1}h-h) \, dv_\Omega(h') \\ &= \Delta_\Omega^{s_3+2s_1+\mathbf{d}+\mathbf{m}'-\mathbf{m}}(h) \int_{h'h \in \Omega \cap (h-\Omega)} \Delta_\Omega^{-s_1-s_3+(\mathbf{m}-\mathbf{m}')/2}(h'h) \\ &\quad \times (1 + |\log \Delta_\Omega^{s_2}(h'h) - \log \Delta_\Omega^{2s_2}(h)|)^\alpha \Delta_\Omega^{s_3+\mathbf{d}}(h-h'h) \, dv_\Omega(h') \\ &= \Delta_\Omega^{s_3+2s_1+\mathbf{d}+(\mathbf{m}'-\mathbf{m})/2}(h) \int_{\Omega \cap (h-\Omega)} \Delta_\Omega^{-s_1-s_3+(\mathbf{m}-\mathbf{m}')/2}(h') \\ &\quad \times (1 + |\log \Delta_\Omega^{s_2}(h') - \log \Delta_\Omega^{2s_2}(h)|)^\alpha \Delta_\Omega^{s_3+\mathbf{d}}(h-h') \, dv_\Omega(h') \end{aligned}$$

for every $h \in \Omega$. Now, fix $\varepsilon' > 0$ such that

$$Q(\varepsilon') := [(1 - \varepsilon')e_\Omega + \Omega] \cap [(1 + \varepsilon')e_\Omega - \Omega] \subseteq B_\Omega(e_\Omega, \varepsilon)$$

and such that

$$|\Delta_\Omega^{2s_2}(h)| \leq \frac{1}{2}$$

for every $h \in Q(\varepsilon')$, so that, in particular,

$$\frac{1}{2} + |\log \Delta_\Omega^{s_2}(h')| \leq 1 + |\log \Delta_\Omega^{s_2}(h') - \log \Delta_\Omega^{2s_2}(h)| \leq \frac{3}{2} + |\log \Delta_\Omega^{s_2}(h')|$$

for every $h \in Q(\varepsilon')$ and for every $h' \in \Omega$. Then, setting $C_1 := 2^{-\alpha} \max_{Q(\varepsilon')} \Delta_\Omega^{s_3+2s_1+\mathbf{d}+(\mathbf{m}'-\mathbf{m})/2}$,

$$\begin{aligned} \int_{Q(\varepsilon')} I(h) \, dh &\leq C_1 \int_{Q(\varepsilon')} \int_{\Omega \cap (h-\Omega)} \Delta_\Omega^{\mathbf{d}-s_1-s_3+(\mathbf{m}-\mathbf{m}')/2}(h')(1 + |\log \Delta_\Omega^{s_2}(h')|)^\alpha \\ &\quad \times \Delta_\Omega^{s_3+\mathbf{d}}(h-h') \, dh' \, dh \\ &\leq C_1 \int_{\Omega \cap ((1+\varepsilon')e_\Omega-\Omega)} \Delta_\Omega^{s_3}(h) \, dv_\Omega(h) \int_{\Omega \cap ((1-\varepsilon')e_\Omega+\Omega)} \Delta_\Omega^{-s_1-s_3+(\mathbf{m}-\mathbf{m}')/2}(h') \\ &\quad \times (1 + |\log \Delta_\Omega^{s_2}(h')|)^\alpha \, dv_\Omega(h'). \end{aligned}$$

Now, [18, Proposition 2.19] shows that

$$\int_{\Omega \cap ((1+\varepsilon')e_\Omega - \Omega)} \Delta_\Omega^{s_3}(h) \, dv_\Omega(h) < \infty.$$

A direct computation (cf. the proof of [18, Proposition 2.19]) then shows that

$$\int_{\Omega \cap ((1+\varepsilon')e_\Omega - \Omega)} \Delta_\Omega^{-s_1-s_3+(\mathbf{m}-\mathbf{m}')/2}(h')(1 + |\log \Delta_\Omega^{s_2}(h')|)^\alpha \, dv_\Omega(h') < \infty$$

for $\alpha < -1$. Therefore, $I(h)$ is finite for some $h \in Q(\varepsilon')$, hence for every $h \in \Omega$ by the preceding remarks, provided that $\alpha < -1$.

Conversely, if $\alpha \geq -1$, then a direct computation (cf. the proof of [18, Proposition 2.19]) shows that

$$\begin{aligned} I(e_\Omega) &\geq \int_{\Omega \cap (e_\Omega/2 - \Omega)} \Delta_\Omega^{-s_1-s_3+(\mathbf{m}-\mathbf{m}')/2}(h')(1 + |\log \Delta_\Omega^{s_2}(h')|)^\alpha \Delta_\Omega^{s_3+\mathbf{d}}(e_\Omega - h') \, dv_\Omega(h') \\ &\geq 2^{\sum_j (-d_j + s_{3,j})_+} \int_{\Omega \cap (e_\Omega/2 - \Omega)} \Delta_\Omega^{-s_1-s_3+(\mathbf{m}-\mathbf{m}')/2}(h')(1 + |\log \Delta_\Omega^{s_2}(h')|)^\alpha \, dv_\Omega(h') \\ &= \infty \end{aligned}$$

(cf. [18, Corollary 2.36]), whence the conclusion by the preceding remarks. $\square$

Proof of Proposition 2.5 By [18, Proposition 5.18] and the necessary condition $\mathbf{s} > \frac{1}{2q}\mathbf{m} \geq \mathbf{0}$ (cf. [18, Proposition 3.5]), it will suffice to prove the first assertion for $p < q$ and $\mathbf{s} \geq \left(\frac{1}{2p} - \frac{1}{2q}\right)\mathbf{m}'$. The second assertion is a consequence of [18, Proposition 5.18]. Observe first that, if $\mathbf{s}' \geq \mathbf{0}$, then [18, Corollary 2.36 and Lemma 2.37] imply that

$$|\Delta_\Omega^{s'}(h+z)| \geq \Delta_\Omega^{s'}(h + \operatorname{Re} z) \geq \Delta_\Omega^{s'}(h)$$

for every $h \in \Omega$ and for every $z \in \Omega + iF$. Consequently, $|B_{(0,ie_\Omega)}^{s'}(\zeta, z)| \geq 1$ for every $(\zeta, z) \in D$, so that there is a unique $f \in \operatorname{Hol}(D)$ such that $f(0, ie_\Omega) = 0$ and

$$\exp(\exp(f)) = eB_{(0,ie_\Omega)}^{s'}$$

on D . We denote this function by $\log(1 + \log B_{(0,ie_\Omega)}^{s'})$. Thus, we may also write $(1 + \log B_{(0,ie_\Omega)}^{s'})^{s''}$ instead of $e^{s''f}$, for every $s'' \in \mathbb{C}$. Define $g^{s_1, s_2} := B_{(0,ie_\Omega)}^{s_1}(1 + \log B_{(0,ie_\Omega)}^{-s_1})^{s_2}$ for every $\mathbf{s}_1 < \mathbf{0}$ and for every $s_2 \in \mathbb{R}$.

Now, take $\mathbf{s}_3 \in \mathbb{N}_{\Omega'}$, and define $p_{\mathbf{s}_1, \mathbf{s}_3} : \mathbb{C} \rightarrow \mathbb{C}$ by

$$w \mapsto i^{s_3} \left(ws_1 + \frac{1}{2}\mathbf{m}' \right)_{s_3} := \prod_{j=1}^r i^{s_{3,j}} \left(ws_{1,j} + \frac{1}{2}m'_j \right) \cdots \left(ws_{1,j} - s_{3,j} + \frac{1}{2}m'_j + 1 \right),$$

so that $p_{\mathbf{s}_1, \mathbf{s}_3}$ is polynomial and

$$B_{(0,ie_\Omega)}^{k\mathbf{s}_1} * I_\Omega^{-s_3} = p_{\mathbf{s}_1, \mathbf{s}_3}(k) B_{(0,ie_\Omega)}^{k\mathbf{s}_1 - \mathbf{s}_3} = p_{\mathbf{s}_1, \mathbf{s}_3}(k) B_{(0,ie_\Omega)}^{k\mathbf{s}_1} B_{(0,ie_\Omega)}^{-s_3}$$

for every $k \in \mathbb{N}$, thanks to [18, Proposition 2.29]. Define $P_{\mathbf{s}_1, \mathbf{s}_3}$ as the differential operator $p_{\mathbf{s}_1, \mathbf{s}_3}(\mathcal{R})$ on $\mathbb{C}$, where $\mathcal{R}f(w) = wf'(w)$ for every $w \in \mathbb{C}$ and for every $f \in \operatorname{Hol}(\mathbb{C})$. Then, for every holomorphic polynomial φ on $\mathbb{C}$ we see that

$$(\varphi \circ B_{(0, ie_\Omega)}^{s_1}) * I_\Omega^{-s_3} = [(P_{s_1, s_3} \varphi) \circ B_{(0, ie_\Omega)}^{s_1}] B_{(0, ie_\Omega)}^{-s_3}.$$

By approximation, the preceding formula holds also with $\varphi^{(s_2)} : w \mapsto w(1 - \log w)^{s_2}$, defined in a neighbourhood of 1, in place of φ . Observe that $P_{s_1, s_3} \varphi^{(s_2)}(w) = \sum_{k \in \mathbb{N}} a_k w(1 - \log w)^{s_2 - k}$ for some $(a_k) \in \mathbb{R}^{(\mathbb{N})}$ and for every w in a neighbourhood of 1. Therefore, by holomorphy we see that

$$g^{s_1, s_2} * I_\Omega^{-s_3} = B_{(0, ie_\Omega)}^{-s_3} \sum_{k \in \mathbb{N}} a_k g^{s_1, s_2 - k}.$$

Observe that $(1 + \log B_{(0, ie_\Omega)}^{-s_1})^{s_2}$ is bounded for every $s_2 \leq 0$ by the preceding remarks, so that $g^{s_1, s_2} * I_\Omega^{-s_3} \in A_{s_1 + s_3}^{p, q}(D)$ if $B_{(0, ie_\Omega)}^{-s_3} g^{s_1, s_2} \in A_{s_1 + s_3}^{p, q}(D)$.

Now, observe that the preceding remarks show that

$$\begin{aligned} \operatorname{Re} \log(1 + \log B_{(0, ie_\Omega)}^{-s_1})(\zeta, z) &\geq \log(1 + \log |B_{(0, ie_\Omega)}^{-s_1}(\zeta, z)|) \\ &= \log(1 + \log |\Delta_\Omega^{-s_1}(e_\Omega + z/i)|) \\ &\geq \log(1 + \log \Delta_\Omega^{-s_1}(e_\Omega + \operatorname{Im} z)), \end{aligned}$$

so that

$$|(1 + \log B_{(0, ie_\Omega)}^{-s_1})^{s_2}(\zeta, z)| \leq (1 + \log \Delta_\Omega^{-s_1}(e_\Omega + \operatorname{Im} z))^{s_2}$$

for every $(\zeta, z) \in D$, when $s_2 \leq 0$. Then,

$$\begin{aligned} \|(g^{s_1, s_2} B_{(0, ie_\Omega)}^{-s_3})_h\|_{L^p(\mathcal{N})} &\leq \|(B_{(0, ie_\Omega)}^{-s_3})_h\|_{L^p(\mathcal{N})} (1 + \log \Delta_\Omega^{-s_1}(e_\Omega + h))^{s_2} \\ &= C_{s_1 - s_3, p} \Delta_\Omega^{s_1 - s_3 - (\mathbf{b} + \mathbf{d})/p} (e_\Omega + h) (1 + \log \Delta_\Omega^{-s_1}(e_\Omega + h))^{s_2} \end{aligned}$$

for a suitable constant $C_{s_1 - s_3, p} > 0$ and for every $h \in \Omega$, provided that $s_1 - s_3 < \frac{1}{p}(\mathbf{b} + \mathbf{d}) - \frac{1}{2p}\mathbf{m}'$ (cf. [18, Corollary 2.36 and Lemma 2.39]). Then, let us prove that

$$\|\Delta_\Omega^{s_1 - s_3 - (\mathbf{b} + \mathbf{d})/p} (e_\Omega + \cdot) (1 + \log \Delta_\Omega^{-s_1}(e_\Omega + \cdot))^{s_2} \Delta_\Omega^{s + s_3}\|_{L^q(v_\Omega)}$$

is finite if $\mathbf{s} + \mathbf{s}_3 > \frac{1}{2q}\mathbf{m}$, $\mathbf{s} + \mathbf{s}_1 \leq \frac{1}{p}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q}\mathbf{m}'$, and $s_2 < -1/q$. Observe that the assertion is trivial when $q = \infty$ (in which case it suffices to assume that $\mathbf{s} + \mathbf{s}_3 \geq \mathbf{0}$, $\mathbf{s} + \mathbf{s}_1 \leq \frac{1}{p}(\mathbf{b} + \mathbf{d})$ and $s_2 \leq 0$). The case $q < \infty$ is a consequence of Lemma 7.5.

Now, assume that $\mathbf{s} + \mathbf{s}_1 \leq \frac{1}{p}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q}\mathbf{m}'$, and $s_2 < -1/q$, and let us prove that $g^{s_1, s_2} \in \tilde{A}_s^{p, q}(D)$. By the preceding remarks and Proposition 2.4, if $\mathbf{s}_3 \in \mathbb{N}_{\Omega'}$ is sufficiently large, then

$$g^{s_1, s_2} * I_\Omega^{-s_3} \in A_{\mathbf{s} + \mathbf{s}_3}^{p, q}(D) = \tilde{A}_{\mathbf{s} + \mathbf{s}_3}^{p, q}(D).$$

In addition, there is $p_1 \in (p, \infty)$ such that $\mathbf{s} > \frac{1}{p_1}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q'}\mathbf{m}'$ and $\mathbf{s} + \mathbf{s}_1 < \frac{1}{p_1}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q}\mathbf{m}'$, so that $\mathbf{s}_1 < \frac{1}{p_1}(\mathbf{b} + \mathbf{d}) - \frac{1}{2p_1}\mathbf{m}'$ by the assumptions on $\mathbf{s}$, and $g^{s_1, s_2} \in A_s^{p_1, q}(D) \subseteq \tilde{A}_s^{p_1, q}(D)$ thanks to [18, Proposition 2.41] and the preceding remarks. Now, denote by $g_0^{s_1, s_2}$ the boundary values of g^{s_1, s_2} in $B_{p_1, q}^{-\mathbf{s}}(\mathcal{N}, \Omega)$, so that $g_0^{s_1, s_2} * I_\Omega^{-s_3} = (g^{s_1, s_2} * I_\Omega^{-s_3})_0 \in B_{p, q}^{-\mathbf{s} - \mathbf{s}_3}(\mathcal{N}, \Omega)$ by [18, Proposition 5.13] and the preceding

remarks. Since the mapping $u \mapsto u * I_{\Omega}^{-s_3}$ induces an isomorphism of $\mathcal{S}'_{\Omega,L}(\mathcal{N})$ which induces isomorphisms of $B_{p,q}^{-s}(\mathcal{N}, \Omega) \rightarrow B_{p,q}^{-s-s_3}(\mathcal{N}, \Omega)$ and $B_{p_1,q}^{-s}(\mathcal{N}, \Omega) \rightarrow B_{p_1,q}^{-s-s_3}(\mathcal{N}, \Omega)$ (cf. [18, Proposition 4.11, by transposition, and Theorem 4.26]), this implies that $g_0^{s_1, s_2} \in B_{p,q}^{-s}(\mathcal{N}, \Omega)$. Since $g^{s_1, s_2} = \mathcal{E}g_0^{s_1, s_2}$ (and $\mathcal{E}$ is defined in the same way in $B_{p_1,q}^{-s}(\mathcal{N}, \Omega)$ and in $B_{p,q}^{-s}(\mathcal{N}, \Omega)$), this implies that $g^{s_1, s_2} \in \widetilde{A}_s^{p,q}(D)$.

Finally, let us prove that, if $g^{s_1, s_2} \in A_s^{p,q}(D)$ and $-1/p \leq s_2 < -1/q$, then $s_1 < \frac{1}{p}(\mathbf{b} + \mathbf{d}) - \frac{1}{2p}\mathbf{m}'$. This will lead to the conclusion. In order to prove our claim, it will suffice to prove that, if $g_{e_{\Omega}}^{s_1, s_2} \in L^p(\mathcal{N})$ and $-1/p \leq s_2 < -1/q$, then $s_1 < \frac{1}{p}(\mathbf{b} + \mathbf{d}) - \frac{1}{2p}\mathbf{m}'$. By Corollaries 7.2 and 7.4, there are constants $C_2, C_3, C_4 > 0$ such that

$$\begin{aligned} (1 + \log B_{(0, ie_{\Omega})}^{-s_1})^{s_2}(\zeta, h + ie_{\Omega} + i\Phi(\zeta)) &\geq C_2(1 + |\log |\Delta_{\Omega}^{-s_1}(e_{\Omega} + (\Phi(\zeta) - ih)/2)||)^{s_2} \\ &\geq C_3(1 + |\log \Delta_{\Omega}^{-s_1}(e_{\Omega} + \Phi(\zeta)/2 + h/2)|)^{s_2} \end{aligned}$$

and then

$$|g^{s_1, s_2}(\zeta, h + ie_{\Omega} + i\Phi(\zeta))| \geq C_4 \Delta_{\Omega}^{s_1}(e_{\Omega} + \Phi(\zeta)/2 + h/2)(1 + |\log \Delta_{\Omega}^{-s_1}(e_{\Omega} + \Phi(\zeta)/2 + h/2)|)^{s_2}$$

for every $\zeta \in E$ and for every $h \in \Omega$. Therefore, it will suffice to prove that

$$\int_{E \times \Omega} \Delta_{\Omega}^{ps_1}(e_{\Omega} + \Phi(\zeta)/2 + h/2)(1 + |\log \Delta_{\Omega}^{-s_1}(e_{\Omega} + \Phi(\zeta)/2 + h/2)|)^{ps_2} d(\zeta, h) = \infty$$

if $s_1 \nless \frac{1}{p}(\mathbf{b} + \mathbf{d}) - \frac{1}{2p}\mathbf{m}'$. Observe first that, by homogeneity,

$$\begin{aligned} &\int_E \Delta_{\Omega}^{ps_1}(e_{\Omega} + \Phi(\zeta)/2 + h/2)(1 + |\log \Delta_{\Omega}^{-s_1}(e_{\Omega} + \Phi(\zeta)/2 + h/2)|)^{ps_2} d\zeta \\ &= \Delta_{\Omega}^{ps_1 - \mathbf{b}}(e_{\Omega} + h/2) \int_E \Delta_{\Omega}^{ps_1}(e_{\Omega} + \Phi(\zeta)/2)(1 + |\log \Delta_{\Omega}^{-s_1}(e_{\Omega} + \Phi(\zeta)/2) \\ &\quad + \log \Delta_{\Omega}^{-s_1}(e_{\Omega} + h/2)|)^{ps_2} d\zeta \\ &\geq C_5 \Delta_{\Omega}^{ps_1 - \mathbf{b}}(e_{\Omega} + h/2)(1 + |\log \Delta_{\Omega}^{-s_1}(e_{\Omega} + h/2)|)^{ps_2}, \end{aligned}$$

where

$$C_5 := \frac{1}{2} \int_{B_E(0, \varepsilon)} \Delta_{\Omega}^{ps_1}(e_{\Omega} + \Phi(\zeta)/2) d\zeta$$

and $\varepsilon > 0$ is chosen so that $|\log \Delta_{\Omega}^{-s_1}(e_{\Omega} + \Phi(\zeta)/2)| \leq 1$ for every $\zeta \in B_E(0, \varepsilon)$. Then,

$$\begin{aligned} &\int_{E \times \Omega} \Delta_{\Omega}^{ps_1}(e_{\Omega} + \Phi(\zeta)/2 + h/2)(1 + |\log \Delta_{\Omega}^{-s_1}(e_{\Omega} + \Phi(\zeta)/2 + h/2)|)^{ps_2} d(\zeta, h) \\ &\geq C_5 \int_{\Omega} \Delta_{\Omega}^{ps_1 - \mathbf{b} - \mathbf{d}}(e_{\Omega} + h/2)(1 + |\log \Delta_{\Omega}^{-s_1}(e_{\Omega} + h/2)|)^{ps_2} dv_{\Omega}(h) \end{aligned}$$

whence the result by Lemma 7.5. $\square$

Acknowledgements The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). This work

was partially supported by the 2022 INdAM–GNAMPA project “Holomorphic Functions in One and Several Complex Variables” (CUP_E55F22000270001).

The authors would like to thank the referee for some useful comments which helped to improve the quality of the exposition.

Funding Open access funding provided by Università degli Studi di Milano within the CRUI-CARE Agreement.

Declarations

Competing interests The authors have no competing interests to declare that are relevant to the content of this article.

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