Multiparameter quantum metrology with discrete-time quantum walks

Mostafa Annabestani [*]
Faculty of Physics, Shahrood University of Technology, P. O. Box 3619995161, Shahrood, Iran

Majid Hassani [†]
LIP6, CNRS, Sorbonne Université, 4 Place Jussieu, 75005 Paris, France

Dario Tamascelli [‡]
Quantum Technology Lab & Applied Quantum Mechanics Group, Dipartimento di Fisica Aldo Pontremoli, Università degli Studi di Milano, I-20133 Milano, Italy

Matteo G. A. Paris [§]
Quantum Technology Lab & Applied Quantum Mechanics Group, Dipartimento di Fisica Aldo Pontremoli, Università degli Studi di Milano, I-20133 Milano, Italy and INFN, Sezione di Milano, I-20133 Milano, Italy

(Received 13 October 2021; accepted 19 May 2022; published 6 June 2022)

We address multiparameter quantum estimation for one-dimensional discrete-time quantum walks and its applications to quantum metrology. We use the quantum walker as a probe for the unknown parameters encoded in its coin degrees of freedom. We find an analytic expression of the quantum Fisher information matrix for the most general coin operator, and show that only two out of the three coin parameters can be accessed. We also prove that the resulting two-parameter coin model is asymptotically classical, i.e., the Uhlmann curvature vanishes. Finally, we apply our findings to relevant case studies, including the simultaneous estimation of charge and mass in the discretized Dirac model.

DOI: 10.1103/PhysRevA.105.062411

I. INTRODUCTION

Quantum enhanced metrology [1–5] is among the most promising quantum technologies. Squeezing-enhanced optical interferometry [6,7] has been recently exploited in gravitational wave detectors [8,9], whereas quantum probes have carved their place into experimental investigation of delicate systems [10]. Several other applications of quantum enhanced sensors have been also suggested [11–20]. Quantum metrology has its foundations in quantum estimation theory [1–4,7,21–26], which assesses the ultimate precision in the estimation of unknown parameters characterizing quantum systems and operations.

Since the early stages of quantum simulations, quantum walks have provided a formidable tool for both the determination of the computational power of quantum computers, and the study of discrete quantum systems [27–31]. In fact, being the quantum analog of classical random walks (on configuration spaces), quantum walks in either the discrete- or continuous-time version [32] provide a simple but powerful instrument to define and characterize quantum algorithms [33] and communication protocols in the same way as classical random walks are used to analyze randomized algorithms. On the other side, discrete time and space version of fundamental equations, such as the Dirac equation, can be interpreted as coined discrete-time quantum walks on lattices [34–41]. In view of their use in the realization of quantum protocols, the characterization of quantum walks at the quantum level is a necessary step. Indeed, quantum metrological schemes have been proposed, and precision benchmarks have been obtained, for either the discrete- [42] or continuous-time quantum walks [43].

In this work, we assess the ultimate precision attainable in the determination of the unknown coin operator in a discrete-time quantum walk. Since three parameters are necessary, in the general setting, to define a unitary operator acting on a two-level system, multiparameter quantum estimation theory will be exploited to determine bounds on the efficiency of unbiased estimators of the coin operator. As we will see by exploiting our analytic expression of the quantum Fisher information matrix, only two out of the three parameters defining the coin operator may be actually accessed. On the other hand, we prove that for the resulting two-parameter coin model there is no incompatibility, i.e., the Uhlmann curvature vanishes and the model is asymptotically classical. This fact implies that the quantum walker can be used as an optimal probe in the multiparameter quantum metrology, which yields to the compatible model of estimation [44].

The paper is organized as follows. We will first briefly introduce multiparameter quantum estimation and define the relevant quantities in Sec. II. Section III is devoted to coined discrete-time quantum walks. In Sec. IV we present our main

[*]Annabestani@shahroodut.ac.ir
[†]Corresponding author: majid.hassani@lip6.fr
[‡]dario.tamascelli@unimi.it
[§]matteo.paris@fisica.unimi.it

results about the analytic expression of the quantum Fisher information matrix and Uhlmann curvature (or incompatibility) matrix. We proceed in Sec. V with presenting some relevant examples, which allow us to establish the compatibility of our findings with what is already present in the literature, and to illustrate the scope and range of our results. Section VI closes the paper with some concluding remarks.

II. MULTIPARAMETER QUANTUM ESTIMATION

Quantum estimation theory deals with the assessment of the ultimate estimation precision attainable in the presence of quantum resources, such as coherence and entanglement. The estimation process can be ideally seen as follows: a quantum system, or probe, is prepared in a particular initial state, evolved under the action of an Hamiltonian, or Liouvillian, having unknown parameters, and then measured. The goal of this procedure is to gain as much information as possible on the unknown parameters, which determine the dynamics of the probe. For fixed initial state and evolution, the amount of information that is accessed upon measurement depends on the measurement itself, whose outcome is then suitably processed by means of an estimator. Analogously to the classical Cramèr-Rao bound, which limits the precision of an estimator in terms of a quantity independent of the estimator itself, i.e., the Fisher information, the quantum Cramèr-Rao bound poses a bound to the ultimate precision of parameter estimation in the presence of quantum resources. In the case of a single unknown parameter θ, once indicated by $\{\rho_\theta\}$ the family of quantum states depending on θ the ultimate precision of any unbiased estimator $\widehat{\theta}$ for θ is given by quantum Cramèr-Rao inequality:

$$\sigma^2[\widehat{\theta}] \geqslant \frac{1}{\mathcal{F}(\theta)}, \tag{1}$$

where σ^2 is the variance of the estimator and $F(\theta)$ is the quantum Fisher information (QFI) defined as

$$\mathcal{F}(\theta) = \mathrm{Tr}\big[\rho_\theta L_\theta^2\big]. \tag{2}$$

L_θ is the symmetric logarithmic derivative (SLD) implicitly defined by

$$\frac{\partial \rho_\theta}{\partial \theta} = \frac{1}{2}\{L_\theta, \rho_\theta\}, \tag{3}$$

and $\{\cdot\}$ denotes the anticommutator. The QFI is therefore, as its classical counterpart, independent of the measurement. We moreover remark that the optimal measurement, i.e. the one saturating the quantum Cramér-Rao inequality, is the projector operator, which can be constructed by the eigenvectors of the SLD [2].

These results and definitions can be extended to the multiparameter estimation scenario [45–49], namely when the unknown parameters to be jointly estimated are more than one. In this case we indicate by $\{\varrho_\Theta\}$ the statistical model, with $\Theta = (\theta_1, \theta_2, \ldots, \theta_n)$ and $\theta_i \in \mathbb{R}$. The Cramér-Rao bound (1) for multiparameter estimation is expressed as [50]

$$\mathrm{Cov}(\Theta) \geqslant \mathcal{F}^{-1}, \tag{4}$$

where $\mathrm{Cov}(\Theta)$ is the $n \times n$ covariance matrix; $\mathcal{F}$ denotes instead the quantum information Fisher matrix (QFIm) with elements defined as

$$\mathcal{F}_{\mu\nu} = \tfrac{1}{2}\mathrm{Tr}[\varrho_\Theta\{L_\mu, L_\nu\}], \tag{5}$$

where L_μ denotes the SLD with respect to the parameter μ. From now on we omit the Θ subscript wherever clear from the context for ease of notation.

Differently from the single-parameter case, in the multiparameter scenario the quantum limit given by the matrix inequality (4) is not achievable in general [44,51–56]. This fact has its root in the noncommuting nature of the operator algebra, preventing the simultaneous measurement of arbitrary observables with arbitrary accuracy and leading to tradeoffs for the precision of the individual estimators.

A most useful scalar bound can be obtained by introducing a real and positive weight matrix W. This yields

$$\mathrm{Tr}[\mathrm{Cov}(\Theta)W] \geqslant \mathrm{C}^S(\Theta, W), \tag{6}$$

where

$$\mathrm{C}^S(\Theta, W) = \mathrm{Tr}[\mathcal{F}^{-1}W], \tag{7}$$

also known as the symmetric bound. A tighter scalar bound was derived by Holevo [45,48], which can be numerically calculated by means of linear semidefinite programming as Ref. [53]. Recently, it has been proved that the Holevo bound $\mathrm{C}^H(\Theta, W)$ can be upper bounded by the symmetric bound as follows [52]:

$$\mathrm{C}^S(\Theta, W) \leqslant \mathrm{C}^H(\Theta, W) \leqslant (1 + \mathcal{R})\mathrm{C}^S(\Theta, W). \tag{8}$$

The quantity $\mathcal{R}$ is defined as

$$\mathcal{R} = \|i\mathcal{F}^{-1}\mathcal{D}\|_\infty, \tag{9}$$

where $\|A\|_\infty$ is the largest eigenvalue of the matrix A, and $0 \leqslant \mathcal{R} \leqslant 1$. The coefficient $\mathcal{R}$ measures the amount of incompatibility of the unknown parameters, and is in fact defined in terms of the Uhlmann curvature matrix

$$\mathcal{D}_{\mu\nu} = -\frac{i}{2}\mathrm{Tr}[\varrho_\Theta[L_\mu, L_\nu]]. \tag{10}$$

Strictly speaking, multiparameter quantum metrology corresponds to simultaneous estimation of multiple parameters using a single quantum system to probe a quantum dynamics with unknown parameters. Of course, separate experiments may be also exploited, and in this case each parameter is independently estimated. This means that in every estimation run all parameters except one are considered perfectly known. The symmetric bound is not generally achievable in the simultaneous estimation, corresponding to the fact that in simultaneous estimation one uses only the resources of one of the separate schemes.

The compatibility conditions are as follows: (i) The Uhlmann curvature matrix vanishes, $\mathcal{D}_{\mu\nu} = 0$; this requirement ensures the existence of compatible measurements and the saturability of the symmetric bound; (ii) The QFIm is a diagonal matrix, i.e., $\mathcal{F}_{\mu\nu} = 0$ for all $\mu \neq \nu$, which implies that the different parameters can be estimated independently; and (iii) There exists a single probe state that maximizes the QFIs for all parameters. When these compatibility conditions are fulfilled, the performance of the simultaneous and separate schemes will be equal to each other. Such models are referred to as compatible models [44]. In compatible models each of parameters is estimated independently with ultimate precision, and typically require fewer resources with respect to separate schemes [44,57]. In this work, we prove that discrete-time quantum walks provide a compatible model for multiparameter quantum metrology.

III. DISCRETE-TIME QUANTUM WALK

In a discrete-time quantum walk (DTQW) on a one-dimensional lattice, at each time step a walker moves between nearest-neighbor sites of the lattice with an amplitude that depends on the state of a two-level system playing the role of a coin [29,30]. Between different moves, moreover, the state of the coin can be modified by some operator. The Hilbert space $\mathcal{H}_p$ of the walker is spanned by the elements of the position basis $\{|x\rangle \mid x \in \mathbb{Z}\}$, where $|x\rangle$ indicates that the walker is on the xth site of the lattice. A basis for the space $\mathcal{H}_c$ of the coin is instead provided by the eigenstates $\{|0\rangle, |1\rangle\}$ of the Pauli matrix σ_z, and the complete lattice-coin space is $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_c$.

In this basis, the evolution of the quantum walk is therefore determined by the repeated application to an initial state of the form

$$|\Psi(0)\rangle = \sum_{x,j} c_{x,j}(0)|x\rangle_p \otimes |j\rangle_c \tag{11}$$

of the operator

$$U_\Theta = S(\mathbb{1} \otimes C_\Theta), \tag{12}$$

where

$$S = \sum_x |x+1\rangle\langle x| \otimes |0\rangle\langle 0| + |x-1\rangle\langle x| \otimes |1\rangle\langle 1| \tag{13}$$

is the conditional shift operator and C_Θ is the coin operator. Neglecting an overall phase factor the most general form of the coin operator, i.e., an element of $U(2)$, is given by

$$C_\Theta = C_{\theta,\alpha,\beta} = \begin{pmatrix} e^{i\alpha}\cos\theta & e^{i\beta}\sin\theta \\ -e^{-i\beta}\sin\theta & e^{-i\alpha}\cos\theta \end{pmatrix}. \tag{14}$$

The parameters θ, α and β are the unknown parameters addressed by our multiparameter estimation problem. The state of the quantum walker after t steps is equal to

$$|\Psi_\Theta(t)\rangle = U_\Theta^t|\Psi(0)\rangle. \tag{15}$$

For our purposes it is expedient to work with a diagonal representation of the shift operator. To this end we define a new basis for the position space by applying the Fourier transformation [58]

$$|k\rangle = \sum_x e^{ikx}|x\rangle. \tag{16}$$

The set $\{|k\rangle\}$, $-\pi \leqslant k \leqslant \pi$ satisfies the completeness condition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dk\ |k\rangle\langle k| = \mathbb{1}, \tag{17}$$

and, moreover,

$$\delta(k-k') = \frac{1}{2\pi}\sum_x e^{-i(k-k')x}. \tag{18}$$

The unitary operator in k space is given by

$$U_\Theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk\ |k\rangle\langle k| \otimes u_k(\Theta), \tag{19}$$

with

$$u_k(\Theta) = (e^{-ik}|0\rangle\langle 0| + e^{ik}|1\rangle\langle 1|)C_\Theta. \tag{20}$$

Replacing Eq. (19) in Eq. (15) yields

$$|\Psi_\Theta(t)\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk\ |k\rangle \otimes \big|\varphi_k^t(\Theta)\big\rangle, \tag{21}$$

where $|\varphi_k^t(\Theta)\rangle = u_k^t|\varphi_k(0)\rangle$ and

$$|\varphi_k(0)\rangle = \langle k|\Psi(0)\rangle, \tag{22}$$

is the amplitude of initial state in k space. We observe that we have replaced $u_k^t(\Theta)$ by u_k^t to simplify the notation. It is clear from Eq. (21) that in the k space the state of the walker after t steps will be block diagonal and that the parameter dependence appears only in the coin part.

IV. MAIN RESULT

Equation (21) is the pure state of DTQW after t steps, which is defined on the whole Hilbert space, namely the coin and the position space. By exploiting the fact that for pure states the relations ($\varrho^2 = \varrho = |\Psi\rangle\langle\Psi|$) and $L_\mu = 2\partial_\mu\varrho = 2(|\partial_\mu\Psi\rangle\langle\Psi| + |\Psi\rangle\langle\partial_\mu\Psi|)$ hold, it is possible to determine the QFIm and the Uhlmann curvature matrix; their elements are given by

$$\mathcal{F}_{\mu\nu}[|\Psi\rangle\langle\Psi|] = 4\,\Re(\langle\partial_\mu\Psi|\partial_\nu\Psi\rangle - \langle\partial_\mu\Psi|\Psi\rangle\langle\Psi|\partial_\nu\Psi\rangle), \tag{23}$$

$$\mathcal{D}_{\mu\nu}[|\Psi\rangle\langle\Psi|] = 4\,\Im(\langle\partial_\mu\Psi|\partial_\nu\Psi\rangle - \langle\partial_\mu\Psi|\Psi\rangle\langle\Psi|\partial_\nu\Psi\rangle), \tag{24}$$

where $\Re$ and $\Im$ denote, respectively, the real and imaginary part, and $\partial_\mu = \frac{\partial}{\partial\theta_\mu}$. The first derivative of Eq. (21) is

$$|\partial_\mu\Psi_\Theta(t)\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk\ |k\rangle \otimes \big|\partial_\mu\varphi_k^t(\Theta)\big\rangle, \tag{25}$$

where

$$|\partial_\mu\varphi_k^t(\Theta)\rangle = \sum_{m=0}^{t-1} u_k^{m+1}\,O_\mu\,u_k^{m+1\,\dagger}\big|\varphi_k^t(\Theta)\big\rangle, \tag{26}$$

and $O_\mu = u_k^\dagger\,\partial_\mu u_k$ (see Appendix A for details on the derivation). One can define the superoperator, $\mathcal{A}_k$ on the coin space of the walker as

$$\sum_{m=0}^{t-1} u_k^{m+1}\,O_\mu\,u_k^{m+1\,\dagger} \equiv \sum_{m=0}^{t-1}\mathcal{A}_k^{m+1}(O_\mu) = \mathcal{A}'_k(O_\mu). \tag{27}$$

This yields

$$\begin{aligned}&\langle\partial_\mu\Psi|\partial_\nu\Psi\rangle - \langle\partial_\mu\Psi|\Psi\rangle\langle\Psi|\partial_\nu\Psi\rangle \\ &\quad= \int_{-\pi}^{\pi}\frac{dk}{2\pi}\,\langle\mathcal{A}'_k(O_\mu^\dagger)\mathcal{A}'_k(O_\nu)\rangle_t \\ &\quad\quad - \left(\int_{-\pi}^{\pi}\frac{dk}{2\pi}\,\langle\mathcal{A}'_k(O_\mu^\dagger)\rangle_t\right)\left(\int_{-\pi}^{\pi}\frac{dk}{2\pi}\,\langle\mathcal{A}'_k(O_\nu)\rangle_t\right),\end{aligned} \tag{28}$$

where

$$\langle \bullet \rangle_t = \langle \varphi_k^t(\Theta) | \bullet | \varphi_k^t(\Theta) \rangle = \mathrm{Tr}[\bullet\, \mathcal{A}_k^t(\varrho_0)], \tag{29}$$

with $\varrho_0 = |\varphi_k(0)\rangle\langle\varphi_k(0)|$. In order to extract simple analytic relations for $\mathcal{F}_{\mu\nu}$ and $\mathcal{D}_{\mu\nu}$, we adopt the superoperator formalism [59,60].

Any two-dimensional Hermitian (anti-Hermitian) operator such as O can be represented in terms of Pauli matrices $\{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$ as follows:

$$O = \tfrac{1}{2}(o_0 \mathbb{1} + o_x\sigma_x + o_y\sigma_y + o_z\sigma_z), \tag{30}$$

where the coefficients o_i are determined by the Hilbert-Schmidt product $\mathrm{Tr}[Oe_i]$ of O with the ith element of a basis for the space of 2×2 matrices. We set $o_i = \mathrm{Tr}[O\sigma_i]$, $i \in \{0, x, y, z\}$, and $\sigma_0 = \mathbb{1}$. The coefficients of the above expansion can be regarded as the elements of four-dimensional column vector

$$|O) = \begin{pmatrix} o_0 \\ o_x \\ o_y \\ o_z \end{pmatrix} \equiv (o_0, \vec{o})^T, \tag{31}$$

in which $\vec{o}$ is nothing else than the Bloch vector. In Appendix B we report some simple, but useful, properties of this representation.

Theorem 1. The elements of the QFIm $\mathcal{F}_{\mu\nu}$ and of the Uhlmann curvature matrix $\mathcal{D}_{\mu\nu}$ are

$$\mathcal{F}_{\mu\nu} = t^2\left\{\int_{-\pi}^{\pi}\frac{dk}{2\pi}\,(O_\mu|\tilde{\mathcal{A}}_k^{\mathbb{1}}|O_\nu) - \left(\int_{-\pi}^{\pi}\frac{dk}{2\pi}\,(O_\mu|\tilde{\mathcal{A}}_k^{\mathbb{1}}|\varrho_0)\right)\left(\int_{-\pi}^{\pi}\frac{dk}{2\pi}\,(\varrho_0|\tilde{\mathcal{A}}_k^{\mathbb{1}}|O_\nu)\right)\right\} + O(t), \tag{32}$$

$$\mathcal{D}_{\mu\nu} = \int_{-\pi}^{\pi}\frac{dk}{2\pi}\,(\vec{o}'_\mu \times \vec{o}'_\nu)\cdot\vec{r}, \tag{33}$$

where $O_i = u_k^\dagger\,\partial_i u_k$, $i \in \{\alpha, \beta, \theta\}$, u_k represents the evolution operator, and $\varrho_0 = |\varphi_k(0)\rangle\langle\varphi_k(0)|$ denotes the initial state of the DTQW in k space; $\vec{r}$ and $\vec{o}'_{\mu/\nu}$ are the Bloch vectors of $|R) = \tilde{\mathcal{A}}_k^t|\varrho_0)$ and the imaginary part of $|O'_{\mu/\nu}) = \tilde{\mathcal{A}}'_k|O_{\mu/\nu})$, respectively [see (31)]. The matrix $\tilde{\mathcal{A}}_k^{\mathbb{1}}$ is instead defined as

$$\tilde{\mathcal{A}}_k^{\mathbb{1}} = N\begin{pmatrix} \frac{1}{N} & 0 & 0 & 0 \\ 0 & \sin^2(k-\beta) & -\cos(k-\beta)\sin(k-\beta) & \cot\theta\sin(k-\beta)\sin(k-\alpha) \\ 0 & -\cos(k-\beta)\sin(k-\beta) & \cos^2(k-\beta) & -\cot\theta\cos(k-\beta)\sin(k-\alpha) \\ 0 & \cot\theta\sin(k-\beta)\sin(k-\alpha) & -\cot\theta\cos(k-\beta)\sin(k-\alpha) & \cot^2\theta\sin^2(k-\alpha) \end{pmatrix},$$
$$N = \frac{\sin^2\theta}{1-\cos^2\theta\cos^2(k-\alpha)}. \tag{34}$$

The proof of the theorem is reported in Appendix C.

Here in the following we describe some relevant consequences of our main result.

Corollary 1. Only the parameters θ and α of the coin operator can be estimated.

Proof. By exploiting the definition $O_i = u_k^\dagger\,\partial_i u_k$, one can easily show that

$$|O_\theta) = 2i\begin{pmatrix} 0 \\ -\sin(\alpha-\beta) \\ \cos(\alpha-\beta) \\ 0 \end{pmatrix}, \tag{35}$$

$$|O_\alpha) = i\begin{pmatrix} 0 \\ \cos(\alpha-\beta)\,\sin 2\theta \\ \sin(\alpha-\beta)\,\sin 2\theta \\ 2\cos^2\theta \end{pmatrix}, \tag{36}$$

$$|O_\beta) = i\begin{pmatrix} 0 \\ \cos(\alpha-\beta)\,\sin 2\theta \\ \sin(\alpha-\beta)\,\sin 2\theta \\ -2\sin^2\theta \end{pmatrix}. \tag{37}$$

It is thus straightforward to show that

$$\tilde{\mathcal{A}}_k^{\mathbb{1}}|O_\beta) = 0. \tag{38}$$

Corollary 2. The maximum value of the diagonal elements of the QFIm may be obtained when the second term of the $\mathcal{F}_{\mu\nu}$ in Eq. (32) vanishes. In turn, at variance with the first one, this term depends on the initial state and may be tuned by choosing the proper preparation of the probe. The maximum value of the diagonal elements of the QFIm is given by

$$\mathcal{F}_{\mu\mu} = t^2\int_{-\pi}^{\pi}\frac{dk}{2\pi}\,(O_\mu|\tilde{\mathcal{A}}_k^{\mathbb{1}}|O_\mu) = \begin{cases} 4t^2\sin\theta(1+\sin\theta)^{-1} & \mu=\theta, \\ 4t^2(1-\sin\theta) & \mu=\alpha, \end{cases} \tag{39}$$

which explicitly depends the single parameter θ. In addition, it exists a probe preparation making the QFIm diagonal.

To see this more explicitly, we notice that the off-diagonal elements ($\mu \neq \nu$) of $\tilde{\mathcal{A}}_k^{\mathbb{1}}$ vanishes (see the explicit form of $\tilde{\mathcal{A}}_k^{\mathbb{1}}$ in Appendix D)

$$\int_{-\pi}^{\pi}\frac{dk}{2\pi}\,(O_\mu|\tilde{\mathcal{A}}_k^{\mathbb{1}}|O_\nu) = 0. \tag{40}$$

and thus the off-diagonal elements of the QFIm contain only terms dependent on the initial state, which may be set to zero by a suitable choice of the initial preparation of the probe.

Corollary 3. As in any local estimation problem, the actual ultimate precision, as quantified by the QFIm, depends on the true value of the parameters. One may wonder whether there are values for which both the diagonal elements $\mathcal{F}_{\theta\theta}$ and $\mathcal{F}_{\alpha\alpha}$ of the QFIm (see Corollary 2), are maximized. In order to check whether such values exist, one should solve $\mathcal{F}_{\theta\theta} = \mathcal{F}_{\alpha\alpha}$, which indeed admits solutions $\sin\theta = -g$ and $\sin\theta = -g^{-1}$, where $g = \frac{1+\sqrt{5}}{2}$ and $g^{-1} = \frac{1-\sqrt{5}}{2}$ are the so-called golden and the golden conjugate (silver) ratio, respectively. Since $g > 1$, we have $\theta = \arcsin(-g^{-1})$.

Corollary 4. If the walker is initially localized on a site of the lattice, the elements of the QFIm are explicitly given by

$$\mathcal{F}_{\theta\theta} = \frac{4t^2}{1+\sin\theta}\left(\sin\theta - \frac{(\hat{n}\times\vec{r})^2_{\hat{z}}}{1+\sin\theta}\right), \tag{41}$$

$$\mathcal{F}_{\phi\phi} = 4t^2(1-\sin\theta)\left(1 - \frac{(\hat{n}\cdot\vec{r})^2}{1+\sin\theta}\right), \tag{42}$$

$$\mathcal{F}_{\theta\phi} = \frac{-4t^2(1-\sin\theta)}{\cos\theta(1+\sin\theta)}(\hat{n}\cdot\vec{r})(\hat{n}\times\vec{r})_{\hat{z}}, \tag{43}$$

where $\phi = \alpha - \beta$, $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, $\vec{r}$ is the Bloch vector of the coin initial state, and $(\cdot)_{\hat{z}}$ indicates the third component of the vector. In the expressions (41)–(43) we have moreover neglected an $O(t)$ additional term that becomes irrelevant for large enough times t. On the other side, the choice $\vec{r} = \partial_\theta\hat{n}$ makes the t^2 term in Eq. (43) vanish thus leaving a QFIm having $O(t)$ off-diagonal elements, and $O(t^2)$ diagonal elements.

Proof. All of $|O_i\rangle$s depend on θ and $\phi = \alpha - \beta$ see Eqs. (35)–(37). In addition, for any local initial state in the position space

$$|\Psi(0)\rangle = |x_0\rangle_p \otimes |\chi\rangle_c, \tag{44}$$

and $\varrho_0 = |\varphi_k(0)\rangle\langle\varphi_k(0)| = |\chi\rangle\langle\chi|$ does not depend on k. The integration in Eq. (32) is thus taken over $\tilde{\mathcal{A}}_k^{\mathbb{1}}$ and the solution of the integral depends on θ and ϕ (see Appendix D). It follows that the elements of the QFIm have only two independent parameters (θ and ϕ). Equations (41), (42), and (43) can be easily derived by calculating Eq. (32) with $\varrho_0 \equiv |\varrho_0\rangle = (1, \vec{r})^T$.

These results make it clear that for a localized initial state of the walker, the optimal choice of the coin state (see Corollary 2) leads to a QFIm where the off-diagonal elements become negligible, for large values of t, with respect to the diagonal elements. However, since the off-diagonal terms do not vanish, such optimal choice does not lead to a compatible model. On the other side, here below we show that we can exploit the state dependence of the second term of the QFIm (32) to obtain such a compatible quantum multiparameter model.

Theorem 2. There exists an optimal choice for the initial state of the quantum walker for which we may maximize both the diagonal elements of the QFIm and set to zero the off-diagonal elements, thus obtaining on optimal compatible model. This optimal choice corresponds to a coin-position entangled initial state

$$|\Psi'(0)\rangle = \frac{1}{\sqrt{2}}(|x_{1,p}0_c\rangle + |x_{2,p}1_c\rangle), \tag{45}$$

were x_1 and x_2 indicate two points in the position space and $d = |x_1 - x_2|$ is an odd number.

We refer the reader to Appendixes C and D for a proof of the theorem. Here we limit ourselves to observe that, for this choice of the initial state, the initial-state-dependent term (46) of the QFIm, namely

$$\int_{-\pi}^{\pi}\frac{dk}{2\pi}\,(O_\mu|\tilde{\mathcal{A}}_k^{\mathbb{1}}|\varrho_0'), \quad \mu\in\{\theta,\alpha\}, \tag{46}$$

vanishes. Moreover, $|\varrho_0')$ does depend on k but does not depend on the unknown parameters and, remarkably enough, the elements of the Uhlmann curvature matrix (33) vanish.

A (discrete-time) quantum walker can thus be used as a probe in a multiparameter quantum metrology scenario in which the unknown parameters are encoded on the coin space of the walker. The evolution is governed by the unitary operation of the DTQW. Our results show that by suitably choosing the initial state, a DTQW probe yields to a compatible model, where all parameters are estimated independently with the ultimate precision and consuming fewer resources. Moreover, optimal entangled initial states does not depend on the unknown parameters, at variance with optimal local initial states, see Corollary 4. In the following, we illustrate some relevant applications of our results by means of a few examples.

V. CASE STUDIES

In order to gain insight about the applications of the above general results, let us now consider a few examples. In particular, we aim to show the applicability of our results in DTWQ-related quantum metrology problems.

1. *Single-parameter quantum metrology*

We start by applying our results to a single-parameter quantum statistical model. Without loss of generality in what follows we set $\alpha = \beta = 0$, and focus on the remaining parameter θ. In this case, Eq. (41) simplifies to

$$\mathcal{F}_\theta = f_{\vec{r}}(\theta)t^2 = \frac{4\sin\theta\left[1+\sin\theta\left(1-r_y^2\right)\right]}{(1+\sin\theta)^2}t^2. \tag{47}$$

The ultimate bound to precision, as determined by the QFI, is thus monotonically increasing with time. For a fixed value of the unknown parameter θ, the QFI is a function of the initial state, and it is maximized by the set of states lying in the x-z plane (for which we have $r_y = 0$), whereas the ±1 eigenstates of σ_y minimize the QFI. As shown in Fig. 1, at any fixed time the ratio between the QFIs corresponding to the best and worst choices of the initial state is larger than one, and can lead to a $\sqrt{2}$ gain factor in the standard deviation of the efficient estimator of θ. The minimum value of the $\mathcal{F}_\theta$ agrees with the results in Ref. [42], where numerical evaluation for each initial state of the coin was required.

2. *Two-parameter quantum metrology with special initial states of the coin*

Let us now consider the following initial state of the walker $|\Psi_1(0)\rangle = \frac{1}{\sqrt{2}}(|0_p0_c\rangle + |1_p1_c\rangle)$. This corresponds to an initially delocalized walker, over the site $x = 0$ and $x = 1$ in the position space, and entangled with the coin. Substituting $|\Psi_1(0)\rangle$ in Eq. (22) yields

$$|\varphi_k'(0)\rangle = \frac{1}{\sqrt{2}}(|0_c\rangle + e^{-ik}|1_c\rangle), \tag{48}$$

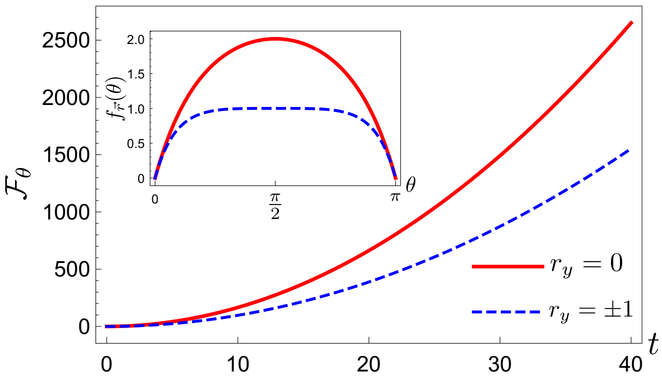

FIG. 1. The QFI (47) for $\theta = \pi/4$ as a function of time when a maximizing ($r_y = 0$, solid red line) or a minimizing (eigenstates of σ_y, dashed blue line) initial state is selected. In the inset the prefactor $f_{\vec{r}}(\theta)$ for the same maximizing (solid red line) and minimizing (dashed blue line) initial states as a function of θ.

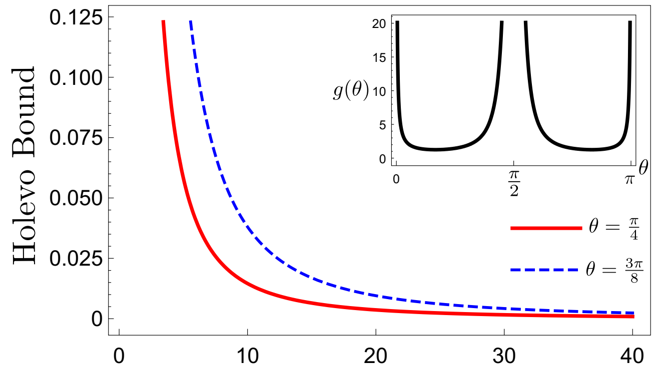

FIG. 2. The Holevo bound in Eq. (54) as a function of time for $\theta = \pi/4$ and $\theta = 3\pi/8$. The quantity $g(\theta)$ from the same equation is shown as a function of θ in the inset.

and

$$|\varrho'_0) = |\varphi'_k(0)\rangle\langle\varphi'_k(0)| = (1, \cos k, -\sin k, 0)^T. \quad (49)$$

By exploiting Eq. (32) one can evaluate the QFIm as follows:

$$\mathcal{F}_1 = 4t^2 \begin{pmatrix} \frac{\sin\theta}{1+\sin\theta} & 0 \\ 0 & 1-\sin\theta \end{pmatrix}, \quad (50)$$

where the initial-state-dependent terms of Eq. (32) vanish

$$\int_{-\pi}^{\pi} \frac{dk}{2\pi} (O_\theta|\tilde{\mathcal{A}}_k^{\mathbb{1}}|\varrho'_0) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{2i\cos(k-\alpha)\sin(2k-\beta)\sin^2\theta}{1-\cos^2(k-\alpha)\cos^2\theta} = 0, \quad (51)$$

and

$$\int_{-\pi}^{\pi} \frac{dk}{2\pi} (O_\alpha|\tilde{\mathcal{A}}_k^{\mathbb{1}}|\varrho'_0) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{-i\sin(k-\alpha)\sin(2k-\beta)\sin 2\theta}{1-\cos^2(k-\alpha)\cos^2\theta} = 0. \quad (52)$$

In other words, having an initially entangled state makes the off-diagonal elements of the $\mathcal{F}$ and the $\mathcal{D}$ to vanish, and maximizes its diagonal elements (see Theorems 1 and 2). The model is thus compatible and the inequalities (8) are saturated, so that

$$C^H(\Theta, W) = C^S(\Theta, W) = \mathrm{Tr}[\mathcal{F}_1^{-1}W]. \quad (53)$$

In particular, by assuming $W = \mathbb{1}$, $\mathrm{Tr}[\mathcal{F}_1^{-1}W]$ gives the sum of the mean-square errors for each of the unknown parameters. One can thus calculate the Holevo bound

$$C_1^H(\Theta, W) = \frac{1}{t^2}g(\theta) = \frac{1}{t^2}\frac{\sin\theta + \cos^2\theta}{4\sin\theta(1-\sin\theta)}. \quad (54)$$

Figure 2 shows the Holevo bound for different values of θ. One asymptotically gains a factor $\frac{1}{t^2}$, i.e., a quadratic enhancement in precision.

As another example, we consider the following initial state:

$$|\Psi_2(0)\rangle\langle\Psi_2(0)| = |0\rangle\langle 0| \otimes |\gamma\rangle\langle\gamma|$$
$$|\gamma\rangle\langle\gamma| = \tfrac{1}{2}(\mathbb{1} + \cos\gamma\sigma_x + \sin\gamma\sigma_y). \quad (55)$$

The localized initial condition of the walker sets us once again under the conditions for Corollary 4 to apply. This time, however, the initial state of the coin is parametrized by γ. The elements of the QFIm of the full space can be derived by means of (41)–(43) and are

$$\mathcal{F}_{2,\theta\theta} = \frac{4t^2}{1+\sin\theta}\left[\sin\theta - \frac{\sin^2\theta}{1+\sin\theta}\sin^2(\gamma-\phi)\right], \quad (56)$$

$$\mathcal{F}_{2,\alpha\alpha} = 4t^2(1-\sin\theta)\left[1 - \frac{\sin^2\theta}{1+\sin\theta}\cos^2(\gamma-\phi)\right], \quad (57)$$

$$\mathcal{F}_{2,\theta\alpha} = \frac{-4t^2(1-\sin\theta)}{\cos\theta(1+\sin\theta)}[\sin^2\theta\sin(\gamma-\phi)\cos(\gamma-\phi)]. \quad (58)$$

Note that by tuning γ in the initial state $|\Psi_2(0)\rangle$ [Eq. (55)], one finds the optimal value $\gamma = \phi$, which maximizes $\mathcal{F}_{2,\theta\theta}$ and makes the off-diagonal terms to vanish. On the other hand $\mathcal{F}_{2,\alpha\alpha}$ is not maximum. Hence for this local initial state the compatibility conditions are not fulfilled.

3. *Joint estimation of two components of the magnetic field*

Let us now apply our formalism to estimate the components of a magnetic field [15,51,61]. In order to do that, we consider a quantum walk in which the coin operation is defrermined by a magnetic field having two unknown components $\vec{B} = (0, b_2, b_3)$, where $B = \sqrt{b_2^2 + b_3^2}$. The magnetic field acts on a two-level system according to the following unitary evolution:

$$e^{-iB\hat{B}.\sigma} = \cos B\mathbb{1} - i\frac{\sin B}{B}b_2\sigma_y - i\frac{\sin B}{B}b_3\sigma_z. \quad (59)$$

To estimate the unknown components of the magnetic field within our formalism, we encode them on the coin operator of the DTQW [see Eq. (14)[as follows:

$$\sin\theta = -\frac{\sin B}{B}b_2, \quad \tan\alpha = -\frac{\tan B}{B}b_3, \quad \beta = 0. \quad (60)$$

This implies that estimating the unknown parameters of the coin provides the components of the magnetic field.

4. Joint estimation of mass and charge in the Dirac equation

Our formalism may be used to jointly estimate physical quantities such as mass and charge. In order to show this possibility, we consider the Dirac Hamiltonian in $(1+1)$ dimensions in the presence of an electromagnetic field (in Planck units $\hbar = c = 1$)

$$[i\gamma^{\mu}(\partial_{\mu} - iqA_{\mu}) - m]\psi = 0, \tag{61}$$

where q and m denote the charge and the mass of a spinless particle, A_{μ} is the vector potential, and the γ^{μ} denote Dirac gamma matrices with $\mu = 0, 1$, satisfying the anticommutation relation $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}\mathbb{1}$, in which $g^{\mu\nu} = \mathrm{diag}(1, -1)$ and $\mathbb{1}$ is the 2×2 identity matrix. We choose $\gamma^0 = \sigma_x$, $\gamma^1 = -i\sigma_y$, and $A^{\mu} = (0, A_x)^T$, i.e., we assume a zero scalar potential. With the above assumptions, the Dirac Hamiltonian, Eq. (61), rewrites as

$$i\partial_t\psi = H_D\psi = (-i\sigma_z\partial_x + qA_x\sigma_z + m\sigma_x)\psi. \tag{62}$$

The unitary evolution of the Dirac Hamiltonian for small ϵ is given by [35]

$$\begin{aligned}|\psi(t+\epsilon)\rangle &= e^{-iH_D\epsilon}|\psi(t)\rangle \\ &= e^{-\epsilon\sigma_z\partial_x}e^{-i\epsilon(qA_x\sigma_z + m\sigma_x)}|\psi(t)\rangle + O(\epsilon^2),\end{aligned} \tag{63}$$

where in the last line we have employed the Lie-Trotter product formula. Equation (63) shows that the evolution induced by the Dirac Hamiltonian corresponds to a DTQW where the first exponential term is the translational operator and the second one denotes the coin operator. Moreover, it indicates that the mass and charge of the particle corresponds to the coin parameters as follows:

$$\begin{aligned}\sin\theta &= \frac{-m}{\sqrt{q^2A_x^2 + m^2}}\sin\left(\epsilon\sqrt{q^2A_x^2 + m^2}\right) \\ &\simeq -m\epsilon, \\ \tan\alpha &= \frac{-qA_x}{\sqrt{q^2A_x^2 + m^2}}\tan\left(\epsilon\sqrt{q^2A_x^2 + m^2}\right) \\ &\simeq -qA_x\epsilon, \\ \beta &= \frac{\pi}{2}.\end{aligned} \tag{64}$$

DTQW thus represents a convenient tool to simulate the evolution of a Dirac particle, and to simultaneously estimate the mass and the charge of the particle via coin parameter estimation. Moreover, the vanishing of the Uhlmann curvature ensures that the mass and the charge may be jointly estimated using a single walker. One can easily see that the Lie-Trotter approximation used to derive Eq. (63) introduces errors of the order $O(\epsilon^2 t)$ after t steps of the quantum walk. As long as the simulation time $T = t\epsilon$ satisfies $\epsilon \ll T \ll (1/\epsilon)$, therefore, all the constraints for the validity of Theorem 1 and of the Lie-Trotter approximations hold.

VI. CONCLUSIONS

In this paper, we have addressed multiparameter quantum estimation for one-dimensional discrete-time quantum walks. In particular, we have explored the possibility of exploiting quantum walk to address the full statistical model, namely the simultaneous estimation of different unknown parameters. We have found the analytic expression of the quantum Fisher information matrix for the most general coin operator, and then exploited our findings to demonstrate that a (discrete-time) quantum walker can be used as an optimal probe if the unknown parameters of the statistical model are encoded in the coin space.

We have shown that for the full model, only two out of the three parameters defining the coin operator can be actually accessed, and proved that the resulting two-parameter coin model is asymptotically classical, i.e., the Uhlmann curvature vanishes. Finally, we have applied our findings to relevant case studies, including the simultaneous estimation of two components of a magnetic field and of the charge and the mass of a particle in the discretized Dirac model. Our results clarify the role of coin parameters in discrete quantum walks, and pave the way for further investigation in systems with more than a walker.

ACKNOWLEDGMENT

M.G.A.P. acknowledges INdAM-GNFM.

APPENDIX A: PROOF OF EQ. (26)

Here we present the proof of Eq. (26). The first derivative of the coin part ($|\varphi_k^t(\Theta)\rangle$) is given by

$$\left|\partial_{\mu}\varphi_k^t(\Theta)\right\rangle = \partial_{\mu}u_k^t|\chi\rangle_c = \sum_{m=0}^{t-1} u_k^m\,\mathbb{1}\,(\partial_{\mu}u_k)\,u_k^{t-m-1}|\chi\rangle_c = \sum_{m=0}^{t-1} u_k^m\,u_k u_k^{\dagger}\,(\partial_{\mu}u_k)\,u_k^{\dagger m+1}u_k^t|\chi\rangle_c = \sum_{m=0}^{t-1} u_k^{m+1}\,(u_k^{\dagger}\,\partial_{\mu}u_k)\,u_k^{\dagger m+1}\left|\varphi_k^t(\Theta)\right\rangle. \tag{A1}$$

APPENDIX B: USEFUL RELATIONS IN THE CALCULATION OF THE BLOCH REPRESENTATION

In the main text, we apply the Bloch representation for any two-dimensional Hermitian (anti-Hermitian) operator. Here we recall some properties of the trace in this representation which help us to derive our results. Let us consider three 2×2 matrices A, B, and C as follows:

$$A = \tfrac{1}{2}(a_0\mathbb{1} + \vec{a}\cdot\vec{\sigma}), \quad B = \tfrac{1}{2}(b_0\mathbb{1} + \vec{b}\cdot\vec{\sigma}), \quad C = \tfrac{1}{2}(c_0\mathbb{1} + \vec{c}\cdot\vec{\sigma}).$$

By exploiting Eq. (30), it is straightforward to show that

$$\mathrm{Tr}[AB] = \frac{1}{2}(A^\dagger|B) = \frac{1}{2}(a_0b_0 + \vec{a}\cdot\vec{b}), \tag{B1}$$

$$\begin{aligned}\mathrm{Tr}[ABC] &= \frac{1}{4}(i\vec{a}\cdot(\vec{b}\times\vec{c}) + a_0(B^\dagger|C) + b_0(C^\dagger|A) + c_0(A^\dagger|B) - 2a_0b_0c_0)\\ &= \frac{1}{4}(i\vec{a}\cdot(\vec{b}\times\vec{c}) + a_0(\vec{b}\cdot\vec{c}) + b_0(\vec{a}\cdot\vec{c}) + c_0(\vec{a}\cdot\vec{b}) + a_0b_0c_0).\end{aligned} \tag{B2}$$

Using Eq. (B2), one can obtain following identities:

$$\mathrm{Tr}[A\{B,C\}] = \mathrm{Tr}[ABC + ACB] \overset{(B2)}{=} \frac{1}{2}[a_0(\vec{b}\cdot\vec{c}) + b_0(\vec{a}\cdot\vec{c}) + c_0(\vec{a}\cdot\vec{b}) + a_0b_0c_0], \tag{B3}$$

$$\mathrm{Tr}[A[B,C]] = \mathrm{Tr}[ABC - ACB] \overset{(B2)}{=} \frac{1}{2}i\vec{a}\cdot(\vec{b}\times\vec{c}). \tag{B4}$$

APPENDIX C: PROOFS FOR THEOREM 1 AND THEOREM 2

In this Appendix we prove Theorem 1 and Theorem 2. In order to do that, let consider the effect of the unitary evolution u_k [Eq. (20)] on an arbitrary operator, namely O, as follows:

$$O' = u_k O u_k^\dagger \equiv \mathcal{A}_k(O), \tag{C1}$$

where $\mathcal{A}_k$ denotes a superoperator. From Eqs. (20), (30), and (31), one obtains

$$O' = u_k O u_k^\dagger = \frac{1}{2}\begin{pmatrix} e^{-i(k-\alpha)}\cos\theta & e^{-i(k-\beta)}\sin\theta \\ -e^{i(k-\beta)}\sin\theta & e^{i(k-\alpha)}\cos\theta\end{pmatrix}\begin{pmatrix} o_0+o_z & o_x - io_y \\ o_x + io_y & o_0 - o_z\end{pmatrix}\begin{pmatrix} e^{i(k-\alpha)}\cos\theta & -e^{-i(k-\beta)}\sin\theta \\ e^{i(k-\beta)}\sin\theta & e^{-i(k-\alpha)}\cos\theta\end{pmatrix}, \tag{C2}$$

with four-dimensional column vector representation as

$$|O') = \begin{pmatrix} o_0 \\ \cos^2\theta\,[o_x\cos(2k-2\alpha) - o_y\sin(2k-2\alpha)] - \sin^2\theta\,[o_x\cos(2k-2\beta) + o_y\sin(2k-2\beta)] - \sin 2\theta\,\cos(2k-\alpha-\beta)\,o_z \\ \cos^2\theta\,[o_x\cos(2k-2\alpha) + o_y\sin(2k-2\alpha)] + \sin^2\theta\,[o_y\cos(2k-2\beta) - o_x\sin(2k-2\beta)] - \sin 2\theta\,\sin(2k-\alpha-\beta)\,o_z \\ \sin 2\theta\,[o_x\cos(\alpha-\beta) + o_y\sin(\alpha-\beta)] + \cos 2\theta\,o_z\end{pmatrix}. \tag{C3}$$

We the define $|O') = \tilde{\mathcal{A}}_k|O)$ where

$$\tilde{\mathcal{A}}_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2k-2\alpha)\cos^2\theta - \cos(2k-2\beta)\sin^2\theta & -\sin(2k-2\alpha)\cos^2\theta - \sin(2k-2\beta)\sin^2\theta & -\cos(2k-\alpha-\beta)\sin 2\theta \\ 0 & \sin(2k-2\alpha)\cos^2\theta - \sin(2k-2\beta)\sin^2\theta & \cos(2k-2\alpha)\cos^2\theta + \cos(2k-2\beta)\sin^2\theta & -\sin(2k-\alpha-\beta)\sin 2\theta \\ 0 & \cos(\alpha-\beta)\sin 2\theta & \sin(\alpha-\beta)\sin 2\theta & \cos 2\theta\end{pmatrix}. \tag{C4}$$

In order to arrive at Eq. (28), we need to calculate $\mathcal{A}'_k = \sum_{\mu=0}^{t-1}\mathcal{A}_k^{\mu+1}$ and $\mathcal{A}_k^t$. The spectral decomposition of $\tilde{\mathcal{A}}_k$ yields

$$\tilde{\mathcal{A}}_k = |\lambda_1)(\lambda_1| + |\lambda_2)(\lambda_2| + e^{2i\omega}|\lambda_3)(\lambda_3| + e^{-2i\omega}|\lambda_4)(\lambda_4|, \tag{C5}$$

i.e., the eigenvalues of $\tilde{\mathcal{A}}_k$ are

$$\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = e^{2i\omega}, \quad \lambda_4 = e^{-2i\omega}, \tag{C6}$$

in which $\cos\omega \equiv \cos(k-\alpha)\cos\theta$. From whence

$$\begin{aligned}\tilde{\mathcal{A}}'_k = \sum_{\mu=0}^{t-1}\tilde{\mathcal{A}}_k^{\mu+1} &= t(|\lambda_1)(\lambda_1| + |\lambda_2)(\lambda_2|) + \sum_{\mu=0}^{t-1}e^{2i\omega(\mu+1)}|\lambda_3)(\lambda_3| + \sum_{\mu=0}^{t-1}e^{-2i\omega(\mu+1)}|\lambda_4)(\lambda_4|\\ &= t(|\lambda_1)(\lambda_1| + |\lambda_2)(\lambda_2|) + \frac{e^{2i\omega}(1-e^{2i\omega t})}{1-e^{2i\omega}}|\lambda_3)(\lambda_3| + \frac{e^{-2i\omega}(1-e^{-2i\omega t})}{1-e^{-2i\omega}}|\lambda_4)(\lambda_4|,\end{aligned} \tag{C7}$$

and

$$\tilde{\mathcal{A}}_k^t = |\lambda_1)(\lambda_1| + |\lambda_2)(\lambda_2| + e^{2i\omega t}|\lambda_3)(\lambda_3| + e^{-2i\omega t}|\lambda_4)(\lambda_4|. \tag{C8}$$

in which $\tilde{\mathcal{A}}_k^{\mathbb{1}} = |\lambda_1)(\lambda_1| + |\lambda_2)(\lambda_2|$ is the projector of $\tilde{\mathcal{A}}_k$ on the subspace corresponding to the eigenvalue 1 and

$$|\lambda_1) = \frac{1}{\sqrt{2(1-\cos\omega)}}\begin{pmatrix}\cos(k-\alpha)\cos\theta - 1 \\ \sin(k-\beta)\sin\theta \\ -\cos(k-\beta)\sin\theta \\ \sin(k-\alpha)\cos\theta\end{pmatrix}, \tag{C9}$$

$$|\lambda_2) = \frac{1}{\sqrt{2(1+\cos\omega)}}\begin{pmatrix}\cos(k-\alpha)\cos\theta+1\\ \sin(k-\beta)\sin\theta\\ -\cos(k-\beta)\sin\theta\\ \sin(k-\alpha)\cos\theta\end{pmatrix}. \tag{C10}$$

By defining $\mathcal{A}'_k(O^\dagger_\mu)\equiv A,\ \mathcal{A}'_k(O_\nu)\equiv B, \mathcal{A}^t_k(\varrho_0)\equiv C$, and using (B2), the first term of Eq. (28) is

$$\int_{-\pi}^{\pi}\frac{dk}{2\pi}\,\mathrm{Tr}\big[\mathcal{A}'_k(O^\dagger_\mu)\mathcal{A}'_k(O_\nu)\mathcal{A}^t_k(\varrho_0)\big] = \frac{1}{4}\int_{-\pi}^{\pi}\frac{dk}{2\pi}\,(i\vec{a}\cdot(\vec{b}\times\vec{c}) + (O_\mu|\tilde{\mathcal{A}}'^{\dagger}_k\tilde{\mathcal{A}}'_k|O_\nu)) \tag{C11}$$

and the integrals appearing in the last term of Eq. (28) are

$$\int_{-\pi}^{\pi}\frac{dk}{2\pi}\,\mathrm{Tr}\big[\mathcal{A}^t_k(\varrho_0)\mathcal{A}'_k(O_x)\big] = \frac{1}{2}\int_{-\pi}^{\pi}\frac{dk}{2\pi}\,\big(\varrho_0|\tilde{\mathcal{A}}^{t\,\dagger}_k\tilde{\mathcal{A}}'_k|O_x\big). \tag{C12}$$

Note that $\mathcal{A}'_k$ and $\mathcal{A}_k$ are trace preserving super operator, so we have $a_0 = b_0 = 0$ and $c_0 = 1$.

Since $\mathcal{A}_k$ is real and $|O_{\mu/\nu})$ are pure imaginary, all elements of (C11) and (C12) are real, except $i\vec{a}\cdot(\vec{b}\times\vec{c})$, which is pure imaginary. So according to the definitions of (23) and (24)

$$\mathcal{F}_{\mu\nu} = \int_{-\pi}^{\pi}\frac{dk}{2\pi}\,(O_\mu|\tilde{\mathcal{A}}'^{\dagger}_k\tilde{\mathcal{A}}'_k|O_\nu) - \int_{-\pi}^{\pi}\frac{dk}{2\pi}\,(O_\mu|\tilde{\mathcal{A}}'^{\dagger}_k\tilde{\mathcal{A}}^t_k|\varrho_0)\int_{-\pi}^{\pi}\frac{dk}{2\pi}\,\big(\varrho_0|\tilde{\mathcal{A}}^{t\,\dagger}_k\tilde{\mathcal{A}}'_k|O_\nu\big) \tag{C13}$$

$$\mathcal{D}_{\mu\nu} = \int_{-\pi}^{\pi}\frac{dk}{2\pi}\,(\vec{o'_\mu}\times\vec{o'_\nu})\cdot\vec{r}, \tag{C14}$$

in which $\vec{r}$ and $\vec{o'_{\mu/\nu}}$ are Bloch vectors of $|R) = \tilde{\mathcal{A}}^t_k|\varrho_0)$ and the imaginary part of $|O'_{\mu/\nu}) = \tilde{\mathcal{A}}'_k|O_{\mu/\nu})$, respectively [see (31)].

Now we show that, for the large enough t, only the subspace spanned by $|\lambda_1)$ and $|\lambda_2)$ needs to be considered, whereas the subspaces spanned by $|\lambda_3)$ and $|\lambda_4)$ can be neglected in Eqs. (C13) and (C14). To see this, let us rewrite (C7) and (C8) as

$$\tilde{\mathcal{A}}'_k = t\mathcal{L}_1 + f\mathcal{L}_3 + f^*\mathcal{L}_4, \tag{C15}$$

$$\tilde{\mathcal{A}}^t_k = \mathcal{L}_1 + e^{2i\omega t}\mathcal{L}_3 + e^{-2i\omega t}\mathcal{L}_4, \tag{C16}$$

in terms of projectors

$$\mathcal{L}_1 = \mathcal{A}^{\mathbb{1}}_k = |\lambda_1)(\lambda_1| + |\lambda_2)(\lambda_2|,\quad \mathcal{L}_3 = |\lambda_3)(\lambda_3|,\quad \mathcal{L}_4 = |\lambda_4)(\lambda_4|, \tag{C17}$$

in which

$$f = \frac{e^{2i\omega}(1-e^{2i\omega t})}{1-e^{2i\omega}}. \tag{C18}$$

Note that the projectors $\mathcal{L}_i$ are orthogonal, i.e.,

$$\mathcal{L}_i\mathcal{L}_j = \mathcal{L}_i\delta_{i,j}, \tag{C19}$$

in which $\delta_{i,j}$ is Kronecker delta. By using (C15) and (C19) we have

$$(O_\mu|\tilde{\mathcal{A}}'^{\dagger}_k\tilde{\mathcal{A}}'_k|O_\nu) = t^2(O_\mu|\tilde{\mathcal{A}}^{\mathbb{1}}_k|O_\nu) + |f|^2(O_\mu|\mathcal{L}_3+\mathcal{L}_4|O_\nu). \tag{C20}$$

for the first term of (C13) and

$$\big(\varrho_0|\tilde{\mathcal{A}}^{t\,\dagger}_k\tilde{\mathcal{A}}'_k|O_x\big) = t(\varrho_0|\tilde{\mathcal{A}}^{\mathbb{1}}_k|O_x) + e^{-2i\omega t}f(\varrho_0|\mathcal{L}_3|O_x) + e^{2i\omega t}f^*(\varrho_0|\mathcal{L}_4|O_x), \tag{C21}$$

for the integrals appearing in the second term.

According to the stationary phase theorem, for the large values of t any terms proportional to $e^{\pm 2i\omega t}$ scale with $\frac{1}{\sqrt{t}}$. So the largest error of neglecting oscillation terms (any terms proportional to $|f|^2$, $e^{-2i\omega t}f$ or $e^{2i\omega t}f^*$) scale with 1. Therefore putting (C20) and (C21) into (C13) and using stationary phase theorem, we have

$$\mathcal{F}_{\mu\nu} = t^2\left\{\int_{-\pi}^{\pi}\frac{dk}{2\pi}\,(O_\mu|\tilde{\mathcal{A}}^{\mathbb{1}}_k|O_\nu) - \int_{-\pi}^{\pi}\frac{dk}{2\pi}\,(O_\mu|\tilde{\mathcal{A}}^{\mathbb{1}}_k|\varrho_0)\int_{-\pi}^{\pi}\frac{dk}{2\pi}\,(\varrho_0|\tilde{\mathcal{A}}^{\mathbb{1}}_k|O_\nu)\right\} + O(t) \tag{C22}$$

which has relative error of order t^{-1}. We should note that for the diagonal $\mathcal{F}_{xx}$ (optimal initial state in Theorem 2) the relative error of approximation will be t^{-2}.

Let us turn our attention to the Uhlmann curvature matrix $\mathcal{D}_{\mu\nu}$. Since

$$|\rho_t) = \mathcal{A}^t_k|\rho_0) \tag{C23}$$

and all the elements of $|\rho_0)$, $|\rho_t)$, $|\lambda_1)$, and $|\lambda_2)$ are real, the elements of

$$e^{2i\omega t}|\lambda_3) + e^{-2i\omega t}|\lambda_4) \tag{C24}$$

are real, that is $|\lambda_3) = |\lambda_4)^* \equiv |R_2) + i|R_3)$. Furthermore we can write $\mathcal{L}_1 \equiv |R_1\rangle\langle R_1|$, where $|R_x)$ are real four-vectors and the orthogonality of $|\lambda_x)$ implies $|R_x)$ to be orthogonal.

Since both $\tilde{\mathcal{A}}_k$ and $\tilde{\mathcal{A}}_k^{\mathbb{1}}$ are trace preserve (all off-diagonal elements in the first row and the first column are 0), the remaining subspace of $\tilde{\mathcal{A}}_k$, which spanned by $|R_2)$ and $|R_3)$ can not change the trace, so the first element of $|R_2)$ and $|R_3)$ must be 0 and consequently the Bloch vectors of $\vec{r}_x$ [(31)] are orthogonal too, i.e.,

$$\vec{r}_i \cdot (\vec{r}_j \times \vec{r}_k) = \varepsilon_{ijk} r_i r_j r_k, \tag{C25}$$

in which $r_i = |\vec{r}_i|$ and ε_{ijk} is Levi-Civita symbol.

By expressing $\tilde{\mathcal{A}}_k$, $\tilde{\mathcal{A}}_t$ and $|O_x)$ in $|R_x\rangle$ basis and using (C25), we have

$$(\vec{\sigma}_\mu \times \vec{\sigma}_\nu)\cdot\vec{r} = 4te^{-2i\omega t} f[\beta_2(\alpha_{3\mu}\alpha_{1\nu} - \alpha_{1\mu}\alpha_{3\nu}) + \beta_3(\alpha_{1\mu}\alpha_{2\nu} - \alpha_{2\mu}\alpha_{1\nu})] - 4|f|^2\beta_1(\alpha_{3\mu}\alpha_{2\nu} - \alpha_{2\mu}\alpha_{3\nu}) \tag{C26}$$

with $\alpha_{ix} = \vec{r}_i \cdot \vec{o}_x$ and $\beta_i = \vec{r}_i \cdot \vec{\varrho}_0$, which, thanks to the stationary phase theorem, scale as $O(t)$ in (C14). Equation (C26) makes it clear that it is impossible to set the proper local initial state to have a compatible model. In fact, for the diagonal $\mathcal{F}$ with maximum elements we need a local initial state satisfies $(O_x|\tilde{\mathcal{A}}_k^{\mathbb{1}}|\varrho_0) = 0$, which is equivalent to $\vec{r}_1 \cdot \vec{\varrho}_0 = 0$. So at least one of $\beta_2 = \vec{r}_2 \cdot \vec{\varrho}_0$ or $\beta_3 = \vec{r}_3 \cdot \vec{\varrho}_0$ is not 0, which keeps time-dependent term in $\mathcal{D}_{\mu\nu}$ [see (C26)]. Inversely by setting $\vec{\varrho}_0$ perpendicular to both $\vec{r}_2$ and $\vec{r}_3$, the coefficients $\beta_2 = \beta_3 = 0$ which eliminate time-dependent term of $\mathcal{D}_{\mu\nu}$, but this kind of $\vec{\varrho}_0$ can not be perpendicular to $\vec{r}_1$ too and $\mathcal{F}$ is not diagonal. As we observed, only by starting from an entangled initial state we can diagonalize the $\mathcal{F}$ and eliminate the time dependency of $\mathcal{D}$ at the same time.

APPENDIX D: VANISHING EQ. (45) FOR COIN-POSITION ENTANGLED INITIAL STATES WITH ODD SEPARATION IN THE POSITION SPACE

We prove that by choosing coin-position entangled initial states with odd separation in the position space, Eq. (45) vanishes. In order to show that, let consider the explicit form of $\int_{-\pi}^{\pi}\frac{dk}{2\pi}\tilde{\mathcal{A}}_k^{\mathbb{1}}$ as

$$\int_{-\pi}^{\pi}\frac{dk}{2\pi}\tilde{\mathcal{A}}_k^{\mathbb{1}} = \frac{\sin\theta}{\sin\theta+1}\begin{pmatrix} \frac{\sin\theta+1}{\sin\theta} & 0 & 0 & 0\\ 0 & \sin\theta\cos^2\phi+\sin^2\phi & \sin\phi\cos\phi(\sin\theta-1) & \cos\phi\cos\theta\\ 0 & \sin\phi\cos\phi(\sin\theta-1) & \sin\theta\sin^2\phi+\cos^2\phi & \sin\phi\cos\theta\\ 0 & \cos\phi\cos\theta & \sin\phi\cos\theta & \cos\theta\cot\theta \end{pmatrix}. \tag{D1}$$

and the analytic solution of integrals are

$$\int_{-\pi}^{\pi}\frac{dk}{2\pi}(O_\theta|\tilde{\mathcal{A}}_k^{\mathbb{1}}|\varrho_0') = -i(1+(-1)^d)\tan\theta\left(\frac{1-\sin\theta}{\cos\theta}\right)^{d-1}\sin[\beta+\alpha\,(d-1)] \tag{D2}$$

and

$$\int_{-\pi}^{\pi}\frac{dk}{2\pi}(O_\alpha|\tilde{\mathcal{A}}_k^{\mathbb{1}}|\varrho_0') = i(1+(-1)^d)\sin\theta\left(\frac{1-\sin\theta}{\cos\theta}\right)^{d-1}\cos[\beta+\alpha\,(d-1)]. \tag{D3}$$

Notice that for odd d the integrals are zero, whereas for even d this is no longer true except for some specific values of β. On the other hand, for $d \to \infty$ the integrals (D2) and (D3) vanish also for even d, and we have a compatible model by choosing an entangled initial states with enough large separation $d = |x_1 - x_2|$ between the spatial components [see Eq. (45)].

[1] V. Giovannetti, S. Lloyd, and L. Maccone, Nature Photonics **5**, 222 (2011).
[2] M. G. A. Paris, Int. J. Quantum Inf. **07**, 125 (2009).
[3] C. L. Degen, F. Reinhard, and P. Cappellaro, Rev. Mod. Phys. **89**, 035002 (2017).
[4] B. M. Escher, R. L. de Matos Filho, and L. Davidovich, Nature Phys. **7**, 406 (2011).
[5] D. Braun, G. Adesso, F. Benatti, R. Floreanini, U. Marzolino, M. W. Mitchell, and S. Pirandola, Rev. Mod. Phys. **90**, 035006 (2018).
[6] C. M. Caves, Phys. Rev. D **23**, 1693 (1981).
[7] R. Demkowicz-Dobrzański, M. Jarzyna, and J. Kołodyński, Prog. Opt. **60**, 345 (2015).
[8] F. Acernese *et al.*, Phys. Rev. Lett. **123**, 231108 (2019).
[9] M. Tse *et al.*, Phys. Rev. Lett. **123**, 231107 (2019).
[10] M. A. Taylor and W. P. Bowen, Phys. Rep. **615**, 1 (2016).
[11] D. Budker and M. Romalis, Nature Phys. **3**, 227 (2007).
[12] M. Koschorreck, M. Napolitano, B. Dubost, and M. W. Mitchell, Phys. Rev. Lett. **104**, 093602 (2010).
[13] W. Wasilewski, K. Jensen, H. Krauter, J. J. Renema, M. V. Balabas, and E. S. Polzik, Phys. Rev. Lett. **104**, 133601 (2010).
[14] R. J. Sewell, M. Koschorreck, M. Napolitano, B. Dubost, N. Behbood, and M. W. Mitchell, Phys. Rev. Lett. **109**, 253605 (2012).
[15] F. Troiani and M. G. A. Paris, Phys. Rev. Lett. **120**, 260503 (2018).
[16] A. D. Ludlow, M. M. Boyd, J. Ye, E. Peik, and P. O. Schmidt, Rev. Mod. Phys. **87**, 637 (2015).
[17] A. Louchet-Chauvet, J. Appel, J. J. Renema, D. Oblak, N. Kjaergaard, and E. S. Polzik, New J. Phys. **12**, 065032 (2010).

[18] E. M. Kessler, P. Kómár, M. Bishof, L. Jiang, A. S. Sørensen, J. Ye, and M. D. Lukin, Phys. Rev. Lett. **112**, 190403 (2014).
[19] D. Tamascelli, C. Benedetti, H.-P. Breuer, and M. G. A. Paris, New J. Phys. **22**, 083027 (2020).
[20] D. Tamascelli, C. Benedetti, S. Olivares, and M. G. A. Paris, Phys. Rev. A **94**, 042129 (2016).
[21] V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004).
[22] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett. **96**, 010401 (2006).
[23] E. Polino, M. Valeri, N. Spagnolo, and F. Sciarrino, AVS Quantum Sci. **2**, 024703 (2020).
[24] M. Hassani, C. Macchiavello, and L. Maccone, Phys. Rev. Lett. **119**, 200502 (2017).
[25] A. T. Rezakhani, M. Hassani, and S. Alipour, Phys. Rev. A **100**, 032317 (2019).
[26] L. Seveso, F. Albarelli, M. G. Genoni, and M. G. A. Paris, J. Phys. A: Math. Theor. **53**, 02LT01 (2019).
[27] R. P. Feynman, Found. Phys. **16**, 507 (1986).
[28] K. R. Parthasarathy, J. Appl. Probab. **25**, 151 (1988).
[29] Y. Aharonov, L. Davidovich, and N. Zagury, Phys. Rev. A **48**, 1687 (1993).
[30] J. Kempe, Contemp. Phys. **44**, 307 (2003).
[31] A. M. Childs and J. Goldstone, Phys. Rev. A **70**, 022314 (2004).
[32] E. Farhi and S. Gutmann, Phys. Rev. A **58**, 915 (1998).
[33] A. Ambainis, SIAM J. Comput. **37**, 210 (2007).
[34] C. M. Chandrashekar, S. Banerjee, and R. Srikanth, Phys. Rev. A **81**, 062340 (2010).
[35] P. Arrighi, V. Nesme, and M. Forets, J. Phys. A: Math. Theor. **47**, 465302 (2014).
[36] A. Mallick and C. M. Chandrashekar, Sci. Rep. **6**, 25779 (2016).
[37] P. Arnault and F. Debbasch, Physica A **443**, 179 (2016).
[38] A. Mallick, S. Mandal, and C. M. Chandrashekar, Eur. Phys. J. C **77**, 85 (2017).
[39] P. Arrighi, G. Di Molfetta, I. Márquez-Martín, and A. Pérez, Phys. Rev. A **97**, 062111 (2018).
[40] P. Arnault, A. Pérez, P. Arrighi, and T. Farrelly, Phys. Rev. A **99**, 032110 (2019).
[41] S. De Vincenzo, Results Phys. **15**, 102598 (2019).
[42] S. Singh, C. M. Chandrashekar, and M. G. A. Paris, Phys. Rev. A **99**, 052117 (2019).
[43] F. Zatelli, C. Benedetti, and M. G. A. Paris, Entropy **22**, 1321 (2020).
[44] S. Ragy, M. Jarzyna, and R. Demkowicz-Dobrzański, Phys. Rev. A **94**, 052108 (2016).
[45] A. Holevo, J. Multivariate Anal. **3**, 337 (1973).
[46] A. Holevo, Rep. Math. Phys. **12**, 251 (1977).
[47] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976), Vol. 3.
[48] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (Springer Science & Business Media, Berlin, 2011), Vol. 1.
[49] C. Helstrom, Phys. Lett. A **25**, 101 (1967).
[50] M. Hayashi and K. Matsumoto, J. Math. Phys. **49**, 102101 (2008).
[51] T. Baumgratz and A. Datta, Phys. Rev. Lett. **116**, 030801 (2016).
[52] A. Carollo, B. Spagnolo, A. A. Dubkov, and D. Valenti, J. Stat. Mech.: Theory Exp. (2019) 094010.
[53] F. Albarelli, J. F. Friel, and A. Datta, Phys. Rev. Lett. **123**, 200503 (2019).
[54] F. Albarelli, M. Barbieri, M. Genoni, and I. Gianani, Phys. Lett. A **384**, 126311 (2020).
[55] R. Demkowicz-Dobrzański, W. Górecki, and M. Guţă, J. Phys. A: Math. Theor. **53**, 363001 (2020).
[56] S. Razavian, M. G. Paris, and M. G. Genoni, Entropy **22**, 1197 (2020).
[57] L. J. Fiderer, T. Tufarelli, S. Piano, and G. Adesso, PRX Quantum **2**, 020308 (2021).
[58] A. Nayak and A. Vishwanath, arXiv:quant-ph/0010117.
[59] T. A. Brun, H. A. Carteret, and A. Ambainis, Phys. Rev. A **67**, 032304 (2003).
[60] M. Annabestani, S. J. Akhtarshenas, and M. R. Abolhassani, Phys. Rev. A **81**, 032321 (2010).
[61] I. Apellaniz, I. Urizar-Lanz, Z. Zimborás, P. Hyllus, and G. Tóth, Phys. Rev. A **97**, 053603 (2018).