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A SYSTEM OF SUPERLINEAR ELLIPTIC EQUATIONS IN A CYLINDER

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Contents

Abstract

This dissertation is primarily concerned with the study of existence of positive solutions of nonlinear elliptic boundary value problems. An important way to deal with the problem is the study of a priori estimates of positive solutions. We will adapt a classical idea which was introduced by Brezis and Turner and, together with a fixed point theorem, we will derive the existence result of a superlinear elliptic system which defined on a cylinder.

First we present some of the history of this problem, along with the necessary mathematical background. We present the main technical tools: Hardy's inequality, regularity theory and maximum principle as well as the work of Brezis and Turner. They treated a general superlinear elliptic problem and obtained the existence of positive solutions for nonlinear term having an asymptotic growth s^γ with $1 < \gamma < \frac{n+1}{n-1}$.

In the novel part we apply Brezis and Turner's technique to a specific elliptic system. We study the L^p regularity theory, Hardy's inequality on a cylinder and with growth conditions imposed on the nonlinear term. In particular, we will find that the nonlinear term embeds into different L^p spaces as the dimension n varies. We point out that there is a regularizing effect in the system which leads to a larger exponent than the Brezis-Turner exponent.

Introduction

The question of positive solutions of nonlinear elliptic boundary value problems is the subject of a large literature [1, 4, 10, 12, 13, 16, 17, 22, 26, 28, 35, 39, 43, 47, 54]. Such problems arise in the theory of nonlinear diffusion generated by nonlinear sources [38, 41, 46], in the theory of thermal ignition of a chemically active mixture of gases [33], in quantum field theory and mechanical statistics [9, 19, 58], in nonlinear heat generation [46], in nuclear or chemical reactor theory [1, 15, 20, 42, 46] and so forth. Therefore, positive solutions are often of main interest. This kind of problem can be studied via various methods: roughly speaking, the proofs in [1, 15, 22] were based on iteration procedures which require certain growth restrictions on the nonlinear term and the boundary condition. In [4, 12, 54], the authors deal with the problem with variational methods since the operator has a variational structure. Moreover, other references obtain the existence results with topological arguments especially when the equation has no variational structure. The main difficulty when using a topological approach lies in the need of obtaining a priori bounds of the solutions.

We first review some classical results about the existence of positive solutions for the scalar equation

$$\begin{cases} -\Delta u &= f(u), & \text{in } \Omega \subset \mathbb{R}^n, \\ u &= 0, & \text{on } \partial\Omega, \\ u &> 0, & \text{in } \Omega, \\ u &\in C^2(\bar{\Omega}), \end{cases} \tag{0.1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$. In 1965, Pohozaev [53] proved that if the non-linearity term $f(u)$ grows as or faster than u^σ, where $\sigma = (N+2)/(N-2)$, and under the geometric assumption on the set Ω of being star-shaped, then the problem (0.1) may not have any positive solution. This problem was also investigated by Ambrosetti and Rabinowitz [4]; they obtained an existence result for (0.1) with $f(u)$ growing less fast than u^σ. They provided a very general variational approach (see also [54]) in which the real valued continuously differentiable functional I corresponds to the energy of the equation, and the solution of (0.1) to its critical points. On the other hand, as we mentioned above, many researchers applied topological arguments and have developed various methods to obtain a priori bounds. (see [13, 28, 35]). A priori bounds of the solutions can give information on the structure of the full set of solutions, however, [39] showed that even if the equation does have positive solutions, a priori bounds may fail. In light of the foregoing facts, we recall some pioneering works and some recent results based on this approach. In the case $n = 2$, Turner [59, 60] treated

$$-\Delta u = f(x,u), \quad x \in U$$

where U is a bounded, simply-connected domain in $\mathbb{R}^2$. The L^∞ a priori bound for positive solutions of the equation was obtained if

$$Au^\beta \leq f(x,u) \leq u^\beta + B$$

for constants $A > 0, B > 0$ and β satisfying $1 < \beta < 3$. For general operators and in higher dimensions $n \geq 2$, Nussbaum [51] considered the following nonlinear elliptic boundary value problem

$$\begin{cases} L(u) + f(x,u) &= 0, \quad \text{on } \Omega \\ u + \gamma \cdot (\frac{\partial u}{\partial \nu}) &= 0, \quad \text{on } \partial\Omega \end{cases}$$

where Ω is a smooth bounded region in $\mathbb{R}^n$, $u : \bar{\Omega} \to \mathbb{R}$ is a C^2 real valued function on $\bar{\Omega}$, $\gamma : \partial\Omega \to \mathbb{R}$ is a $C^{1,\lambda}$ function (that is, γ is Hölder continuous with Hölder constant λ) which is either positive on $\partial\Omega$ or identically 0 and ν denotes the outward normal vector to Ω at $x \in \partial\Omega$. The drawback of the main result is that the existence of positive solutions is established under the requirement that $f(x,u) \leq B + |u|^\sigma$, where σ must be less than $n/(n-1)$ if the function γ above is identically zero on $\partial\Omega$, and n is the dimension of the space. Soon after, a powerful way to obtain uniform bounds of positive solutions for the following problem (0.2) (permitting the nonlinear term to depend also on the gradient of u) was developed by Brezis and Turner [13]; their results also included the previously obtained existence results for the same problem and in their work the proofs of the required bounds were considerably simplified. More precisely, they treated the general problem

$$\begin{cases} Lu &= g(x,u,Du), \quad x \in \Omega \\ u &= 0, \quad x \in \partial\Omega \end{cases} \tag{0.2}$$

where Ω is a smooth, bounded domain in $\mathbb{R}^n$ and g is a non-negative function. They combined the Hardy-Sobolev inequality with interpolation and obtained the existence of positive solutions for g having growth u^γ with $1 < \gamma < \frac{n+1}{n-1}$, a wider range of the growth condition of the nonlinear term compared to the one in [51]. This exponent is also the so-called Brezis-Turner exponent. In 1980, de Figueiredo, Lions and Nussbaum [28] examined the existence of positive solutions of problem (0.1). They obtained the a priori bounds under the assumption

$$\lim_{u \to +\infty} f(u)u^{-\sigma} = 0, \quad \text{where } \sigma = \frac{N+2}{N-2}.$$

and requiring also assumptions on the primitive of f. Note that in view of the non-existence result by Pohozaev, this assumption seems optimal. They first studied the case when Ω is convex, then the general case, and finally the case where Ω has some geometrical properties. One drawback of their methods is that the arguments depend strongly on symmetry properties of the Laplacian and it is not clear whether these arguments can be extended to general, second-order, elliptic semi-linear equations as in [13]. Afterwards, Gidas and Spruck [35] used a scaling ("blow-up") argument deriving a priori bounds for positive solutions of the non-linear elliptic boundary value problem

$$\begin{cases} \frac{\partial}{\partial x_j}(a^{ij}(x)\frac{\partial u}{\partial x_j}) + b_j(x)u_{x_j}(x) + f(x,u) = 0 & \text{in } \Omega \\ u(x) = \varphi(x) & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a C^1 boundary. The nonlinearity $f(x,u)$ is continuous in $x \in \bar{\Omega}$, and for some $1 < \alpha < \frac{n+2}{n-2}$

$$\lim_{t \to +\infty} \frac{f(x,t)}{t^\alpha} = h(x)$$

uniformly in $x \in \bar{\Omega}$, where $h(x)$ is continuous and strictly positive in $\bar{\Omega}$. The proof was done by contradiction via "blow up", defining a scaled function $v_k(y) = \lambda_k^{\frac{2}{\alpha-1}} u^k(x)$ ($\lambda_k \to 0$ as $k \to +\infty$) and to reduce the problem of a priori bounds to global existence/nonexistence results of Liouville type for $n \geq 2$.

In view of the advances in this area, it is natural to ask whether it is possible to obtain the corresponding results for systems of equations. In fact, many existence results proved by the a priori estimate of the scalar equation (0.1) have been extended to the corresponding systems, such as elliptic nonlinear coupled systems [16–18, 29, 30, 32]. In 1984, Cosner [16] considered the problem of existence of positive solutions for semi-linear systems which are not necessarily variational. His main results, which are based on the cooperativity (quasimonotonicity) of the nonlinearities, are extensions to system of the fundamental paper of [13]. In [17], Clément, de Figueiredo, and Mitidieri used a method developed in [28] for the case of one equation and L^∞ a priori bounds were obtained. For another coupled system studied by de Figueiredo and Yang [32], the difficulties of obtaining the a priori bounds were due to the presence of gradients in the nonlinear terms. The authors have to use some norm with weights depending on the distance to the boundary of the domain. They obtained the a priori bounds via the so called blow-up method which was introduced by Gidas-Spruck [35] for the scalar case. In dimension two [30], de Figueiredo, do Ó, and Ruf derived a priori estimates for positive solutions for nonlinearities which are allowed to have a faster growth than the pure exponential. The article adapted methods to the case of systems introduced by Brezis-Merle [11] to treat the scalar case. In [18], the authors found L^∞ a priori bounds with different exponent assumption imposed on the nonlinear term, the technique used in their work is based on the work of Brezis and Turner. In their paper the Brezis-Turner exponent assumption is replaced by conditions that involve two curves in the (p, q) plane.

Working along this line, we will focus on a special coupled system of equations on a cylindrical domain $\Omega = \Omega' \times (0, a) \subset \mathbb{R}^n (n \geq 3)$, with $x = (x', x_n) \in \Omega$ and we will adapt Brezis-Turner's method to find a priori bounds for positive solutions of the system. The particularity of this system is that it couples two unknowns which are defined on *different domains*: the unknown u is defined in the whole cylinder Ω, while v is defined at the bottom Ω' of the cylinder. One may think of Ω as a jar containing a gas $u(x)$ interacting with a fluid $v(x')$ at the bottom. Other models may come from biology: e.g. insects $u(x)$ in a cylindrical habitat interacting with plants $v(x')$ at the bottom Ω'. We have not seen such type of coupled systems in the literature. Of course, one can consider many different versions of such couplings.

This article is organized as follows. In the first chapter, we recall some necessary information about Sobolev space. We will present the L^p regularity ([36], chapter 9) theory, and will establish the L^p regularity on the cylinder as we will see in the third chapter. We introduce the important Hardy's inequality which will be extended in the third chapter as well. We also present some properties of the first eigenfunction of the Laplacian which will be a test function in obtaining the a priori bounds of positive solutions. At the end of this chapter, we briefly inform on some simple concepts related to fixed point theorems which are frequently used to prove the existence of positive solutions to nonlinear elliptic equations.

In the second chapter, we mainly introduce the work of Brezis-Turner [13]. The proofs depend upon a priori estimates for solutions of elliptic problems and existence theorems for "positive" operators. Using a weighted Sobolev embedding inequality and a bound on the growth of the non-linearity f with respect to u, they first gave a uniform bound of the H^1 norm of u. With the aid of this result and a bootstrap argument, they obtained the L^∞ a priori bounds and thus the existence result.

In the last chapter, we consider the mentioned coupled system of equations on a cylindrical domain. The proof follows the idea of Brezis-Turner [13] in the second chapter. We consider two separated cases which depend on the growth of the nonlinearity in the second equation, and derive the existence of the positive solutions to the system. It is interesting to note that the maximal exponent in the article of Brezis-Turner was $\frac{n+1}{n-1}$. Our maximal exponent is larger, which is due to the regularizing effect of the inverted operator $(-\Delta)^{-1}$.

1 Preliminary Knowledge of Second Order Elliptic Partial Differential Equations

Some of the background material needed for Chapters 2 and 3 will be presented here ([25] chapter 6.1.1).

1.1 Elliptic equation, $W^{k,p}$ Space and Embedding Theorem

1.1.1 Elliptic equation

In this section we will mainly present the boundary-value problem

$$\begin{cases} Lu &= f \quad \text{in } U \\ u &= 0 \quad \text{on } \partial U, \end{cases} \tag{1.1}$$

where U is an open, bounded subset of $\mathbb{R}^n$ and $u : \bar{U} \to \mathbb{R}$ is the unknown, $u = u(x)$. Here $f : U \to \mathbb{R}$ is given, and L denotes a second-order partial differential operator having either the form

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u \tag{1.2}$$

or else

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u, \tag{1.3}$$

for given coefficient functions a^{ij}, b^i, c $(i, j = 1, \cdots, n)$.

We say that the PDE $Lu = f$ is in *divergence form* if L is given by (1.2), and is in *non-divergence form* provided L is given by (1.3). The requirement that $u = 0$ on ∂U in (1.1) is sometimes called *Dirichlet boundary condition*.

Remark 1.1. *If the highest order coefficients $a^{ij}(i, j = 1, \cdots, n)$ are C^1 functions, then an operator given in divergence form can be rewritten into non-divergence structure, and vice versa. Indeed the divergence form equation* (1.2) *becomes*

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} \tilde{b}^i(x)u_{x_i} + c(x)u \tag{1.4}$$

for $\tilde{b}^i := b^i - \sum_{j=1}^{n} a^{ij}_{x_j}(i = 1, \cdots, n)$, and (1.4) *is obviously in non-divergence form. We will see, however, there are definite advantages to considering the two different representations of L separately. The divergence form is most natural for energy methods, based upon integration by parts, and the non-divergence form is most appropriate for maximum principle techniques.*

We henceforth assume as well the symmetry condition

$$a^{ij} = a^{ji}(i, j = 1, \cdots, n).$$

Definition 1.1 ([25], chapter 6.1.1)**.** *We say the partial differential operator L is (uniformly) elliptic if there exists a constant $\theta > 0$ such that*

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \tag{1.5}$$

for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$.

Ellipticity thus means that for each point $x \in U$, the symmetric $n\times n$ matrix $\mathbf{A}(x) = [a^{ij}(x)]$ is positive definite, with smallest eigenvalue greater than or equal to θ.

1.1.2 Definition and elementary properties of the space $W^{k,p}$

Let $U \subset \mathbb{R}^n$ be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 1.2 ([25], chapter 5)**.** *The Sobolev space of order $k \in \mathbb{N}$*

$$W^{k,p}(U)$$

consists of all the locally summable functions $u : U \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

Remark 1.2. *(i) If $p = 2$, we usually write*

$$H^k(U) = W^{k,2}(U) \quad (k = 0, 1, \cdots).$$

The letter H is used, since $H^k(U)$ is a Hilbert space. Note that $H^0(U) = L^2(U)$.

(ii) We henceforth identify functions in $W^{k,p}(U)$ which coincide a.e.

Definition 1.3 ([25], chapter 5)**.** *If $u \in W^{k,p}(U)$, we define its norm to be*

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\Sigma_{|\alpha|\leq k} \int_U |D^\alpha u|^p dx\right)^{1/p} & (1 \leq p < \infty) \\ \Sigma_{|\alpha|\leq k} \text{ess sup}_U |D^\alpha u| & (p = \infty). \end{cases}$$

Definition 1.4 ([25], chapter 5)**.** *(i) Let $\{u_m\}_{m=1}^{\infty}$, $u \in W^{k,p}(U)$. We say u_m converges to u in $W^{k,p}(U)$, written*

$$u_m \to u \quad \textit{in } W^{k,p}(U),$$

provided

$$\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(U)} = 0.$$

(ii) We write

$$u_m \to u \quad \text{in } W^{k,p}_{loc}(U)$$

to mean

$$u_m \to u \quad \text{in } W^{k,p}(V)$$

for each $V \subset\subset U$.

Definition 1.5 ([25], chapter 5)**.** *We denote by*

$$W_0^{k,p}(U)$$

the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$.

Thus $u \in W_0^{k,p}(U)$ if and only if there exist functions $u_m \in C_c^\infty(U)$ such that $u_m \to u$ in $W^{k,p}(U)$. We interpret $W_0^{k,p}(U)$ as comprising those functions $u \in W^{k,p}(U)$ such that

$$\text{“}D^\alpha u = 0 \text{ on } \partial U\text{” for all } |\alpha| \leq k-1.$$

NOTATION. It is customary to write

$$H_0^k(U) = W_0^{k,2}(U).$$

In the following we display several basic inequalities for functions in Sobolev space ([25], appendix B.2).

Cauchy's inequality.

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2} \quad (a, b \in \mathbb{R}).$$

Proof.

$$0 \leq (a-b)^2 = a^2 - 2ab + b^2.$$

□

Cauchy's inequality with ϵ.

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon} \quad (a, b > 0, \epsilon > 0).$$

Proof. Write

$$ab = ((2\epsilon)^{1/2} a)\Big(\frac{b}{(2\epsilon)^{1/2}}\Big)$$

and apply Cauchy's inequality. □

Young's inequality. Let $1 < p, q < \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad (a, b > 0).$$

Proof. The mapping $x \to e^x$ is convex, and consequently

$$ab = e^{\log a + \log b} = e^{\frac{1}{p}\log a^p + \frac{1}{q}\log b^q} \leq \frac{1}{p}e^{\log a^p} + \frac{1}{q}e^{\log b^q} = \frac{a^p}{p} + \frac{b^q}{q}.$$

$\square$

Young's inequality with ϵ.

$$ab \leq \epsilon a^p + C(\epsilon)b^q \quad (a, b > 0, \epsilon > 0)$$

for $C(\epsilon) = (\epsilon p)^{-q/p}q^{-1}$.

Proof. Write

$$ab = ((\epsilon p)^{1/p}a)\Big(\frac{b}{(\epsilon p)^{1/p}}\Big)$$

and apply Young's inequality. $\square$

Hölder's inequality. Assume $1 \leq p, q \leq \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then if $u \in L^p(U)$, $v \in L^q(U)$, we have

$$\int_U |uv| dx \leq \|u\|_{L^p(U)}\|v\|_{L^p(U)}.$$

Proof. The conclusion is obvious if $p = 1$ or $p = \infty$; therefore we assume $1 < p < \infty$. By homogeneity, we may assume $\|u\|_{L^p} = \|v\|_{L^q} = 1$. Then Young's inequality implies for $1 < p, q < \infty$ that

$$\int_U |uv| dx \leq \frac{1}{p}\int_U |u|^p dx + \frac{1}{q}\int_U |v|^q dx = 1 = \|u\|_{L^p}\|v\|_{L^q}.$$

$\square$

Poincaré inequality. ([7], Corollary 9.19) Suppose that $1 \leq p < \infty$ and U is a bounded open set. Then there exists a constant C (depending on U and p) such that

$$\|u\|_{L^p(U)} \leq C\|\nabla u\|_{L^p(U)} \quad \forall u \in W_0^{1,p}(U).$$

Theorem 1.1 (General Embedding theorem for $W^{k,p}(U)$, [36] Theorem 7.26)**.** *Let U be a bounded $C^{0,1}$ domain in $\mathbb{R}^n$ (see Definition 1.6 below). Then,*

(i) if $kp < n$, the space $W^{k,p}(U)$ is continuously embedded in L^{p^*}, $p^* = np/(n - kp)$, and compactly embedded in $L^q(U)$ for any $q < p^*$;

(ii) if $0 \leq m < k - \frac{n}{p} < m + 1$, the space $W^{k,p}(\Omega)$ is continuously embedded in $C^{m,\alpha}(\bar{\Omega})$, $\alpha = k - n/p - m$, and compactly embedded in $C^{m,\beta}(\bar{\Omega})$ for any $\beta < \alpha$, where $C^{m,\beta}(\bar{\Omega})$ is the Hölder space of functions whose m-th derivative is β-Hölder continuous.

Remark 1.3. *For more general Sobolev inequalities, see [2] chapter 4.*

1.2 Hardy's Inequality

1.2.1 The one-dimensional Hardy's inequality

Lemma 1.1 ([7], exercise 8.8)**.** *Let* $I = (0,1)$ *and let* $v \in W^{1,p}(I)$ *with* $1 < p < \infty$. *If* $v(0) = 0$, *then* $\dfrac{v(x)}{x} \in L^p(0,1)$ *and*

$$\Big\|\frac{v(x)}{x}\Big\|_{L^p(0,1)} \le \frac{p}{p-1}\|v'\|_{L^p(0,1)}. \tag{1.6}$$

Proof. Given $u \in C_c^\infty(I)$, define Tu by

$$Tu(x) = \frac{1}{x}\int_0^x u(t)dt \quad \text{for } x \in (0,1].$$

We first prove $T \in \mathcal{L}(L^p, L^p)$. Set $\varphi(x) = \int_0^x u(t)dt$, it is obvious that $\varphi(x) \in C^1(I)$, $\varphi(0) = 0$ and $\varphi'(x) = u(x)$. We compute

$$\begin{aligned}
\int_0^1 |Tu(x)|^p dx &= \int_0^1 \frac{|\varphi(x)|^p}{x^p} dx \\
&= -\frac{1}{p-1}\int_0^1 |\varphi(x)|^p d(\frac{1}{x^{p-1}}) \\
&= -\frac{1}{p-1}|\varphi(1)|^p + \frac{p}{p-1}\int_0^1 \frac{|\varphi(x)|^{p-1}}{x^{p-1}}\varphi'(x)(\text{sign}\varphi(x))dx.
\end{aligned}$$

Using Hölder's inequality, we obtain

$$\begin{aligned}
\int_0^1 |Tu(x)|^p dx &\le \frac{p}{p-1}\int_0^1 \Big|\frac{\varphi(x)}{x}\Big|^{p-1}|\varphi'(x)|dx \\
&\le \frac{p}{p-1}\Big[\int_0^1 \Big|\frac{\varphi(x)}{x}\Big|^{(p-1)\cdot\frac{p}{p-1}} dx\Big]^{\frac{p-1}{p}}\Big[\int_0^1 |u(x)|^p dx\Big]^{\frac{1}{p}} \\
&\le \frac{p}{p-1}\Big[\int_0^1 |Tu(x)|^p dx\Big]^{\frac{p-1}{p}}\Big[\int_0^1 |u(x)|^p dx\Big]^{\frac{1}{p}}
\end{aligned}$$

that is,

$$\|Tu(x)\|_{L^p(I)} \le \frac{p}{p-1}\|u(x)\|_{L^p(I)} \quad \forall u \in C_c^\infty(I). \tag{1.7}$$

(1.7) implies T is a linear bounded operator in $C_c^\infty(I)$. Now we assume $u_m \in C_c^\infty(I)$ and $u_m \to u$ in $L^p(I)$ as $m \to \infty$. Since $\{u_m\}$ is a Cauchy sequence, then we have

$$\|Tu_m - Tu_n\|_{L^p} \le \frac{p}{p-1}\|u_m - u_n\|_{L^p} \to 0, \tag{1.8}$$

which implies Tu_m is a Cauchy sequence in $L^p(I)$ as well. Denote the limit point of Tu_m in L^p as g, then

$$\|g\|_{L^p} = \lim_{m\to\infty} \|Tu_m\|_{L^p} = \| \lim_{m\to\infty} Tu_m\|_{L^p} = \|T \lim_{m\to\infty} u_m\|_{L^p} = \|Tu\|_{L^p}$$

this means that T is a bounded operator from $L^p(I)$ to $L^p(I)$. Therefore,

$$\|Tu_m - Tu\|_{L^p} \leq \|T\|\|u_m - u\|_{L^p}, \tag{1.9}$$

let $m \to \infty$ in (1.9), we get $g = Tu$, thus $\|Tu\|_{L^p} \leq \|u\|_{L^p}$ i.e.,

$$\Big\|\frac{\varphi(x)}{x}\Big\|_{L^p(I)} \leq \frac{p}{p-1}\|\varphi'(x)\|_{L^p(I)} \tag{1.10}$$

for any $u \in L^p(I)$. Since $v \in W^{1,p}(I)$, then $v' \in L^p(I)$, so we can substitute $u(x)$ with $v'(x)$ above. Besides,

$$\varphi(x) = \int_0^x u(t)dt = \int_0^x v'(t)dt = v(x) - v(0) = v(x),$$

then going back to (1.10), we then obtain (1.6). □

Remark 1.4. *For a more general result, see Theorem 5.2 [45].*

1.2.2 The N-dimensional Hardy's inequality

Theorem 1.2 (Improved Hardy's Inequality; Theorem 4.1 [14])**.** *For any bounded domain U in $\mathbb{R}^n$ ($0 \in U$), any dimension $n \geq 2$ and for every $u \in H_0^1(U)$ we have*

$$\int_U |\nabla u|^2 dx \geq H \int_U \frac{u^2}{|x|^2} dx + H_2 (\frac{\omega_n}{|U|})^{\frac{2}{n}} \int_U u^2 dx,$$

where $H = H(n) = \frac{(n-2)^2}{4} (n \geq 3)$. The result for $n = 2$ is just the Poincaré inequality with the constant H_2, the first eigenvalue of the Laplacian in the unit ball in $n = 2$. Both constants are optimal when U is a ball. ω_n denotes the measure of the unit ball.

Proof. (i) The first step is to make a symmetrization that replaces U by a ball B_R with the same volume,

$$\omega_n R^n = |U|,$$

and the function u by its symmetric rearrangement u^* [5, 44]. To construct the rearrangement of u, we need some notations. For $t \in \mathbb{R}$, the level set $\{u > t\}$ is defined as

$$\{u > t\} = \{x \in U \mid u(x) > t\}.$$

Then the distribution function of u is given by

$$\mu_u(t) = |\{u > t\}|.$$

here μ denote the Lebesgue measure. With $u(x) \geq 0$, we first associate a function $u^{\#}(x)$ depending on $|x|$ by the requirement

$$\mu\{x \in B_R \mid u^* > t\} = \mu\{x \in U \mid u > t\} \text{ for every } t \geq 0$$

and defined on $[0, |U|]$ by the following formula:

$$u^{\#}(x) = \begin{cases} \inf\{t \geq 0 \mid \mu_u(t) \leq x\}, & 0 < x < |U|, \\ \operatorname{ess\,sup} u, & x = 0, \\ \operatorname{ess\,inf} u, & x = |U|. \end{cases}$$

Then the spherically symmetric and decreasing rearrangement $u^* : B_R \to \mathbb{R}$ is defined by

$$u^*(x) = u^{\#}(\omega_n |x|^n), \quad x \in B_R.$$

It is well-known that the rearrangement does not change the L^2-norm, decreases the $H_0^1(U)$ norm and increases the integral $\int (u^2/|x|^2)dx$ [6]. Hence, it is enough to prove the result in the symmetric case. Moreover, a simple scaling allows to consider the case $R = 1$.

(ii) Let us tackle the main part of the proof, proving the inequality for radial functions in the ball $B = B_1(0)$ in $\mathbb{R}^n$, $n \geq 3$. We define the new variable

$$v(r) = u(r) r^{(n-2)/2}, \quad r = |x|.$$

It is easy to compute

$$v'(r) = u'(r) \cdot r^{\frac{n-2}{2}} + \frac{n-2}{2} u(r) \cdot r^{\frac{n-4}{2}}$$

$$|v'(r)|^2 = r^{n-2} u'(r)^2 + \frac{(n-2)^2}{4} r^{n-4} u^2(r) + (n-2) r^{\frac{2n-6}{2}} u(r) u'(r)$$

$$v(r)v'(r) = r^{n-2} u(r) u'(r) + \frac{n-2}{2} r^{n-3} u^2(r)$$

Now, we have a “magical” computation:

$$\begin{aligned}
\int_B |\nabla u|^2 dx - H \int_B \frac{u}{r^2} dx &= \int_0^1 \int_{\partial B_r} |\nabla u(r)|^2 ds dr - H \int_0^1 \int_{\partial B_r} \frac{u(r)}{r^2} ds dr \\
&= n\omega_n \Big(\int_0^1 r^{n-1} |\nabla u|^2 dr - H \int_0^1 \frac{u(r)}{r^2} \cdot r^{n-1} dr \Big) \\
&= n\omega_n \Big(\int_0^1 r^{n-1} |\nabla u|^2 dr - \frac{(n-2)^2}{4} \int_0^1 u(r) r^{n-3} dr \Big) \\
&= n\omega_n \Big[\int_0^1 (v')^2 r dr - (n-2) \int_0^1 v(r) v'(r) dr \Big]
\end{aligned}$$

Taking for instance $u \in C_0^1(B)$ the last integral is zero and we get

$$\int_B |\nabla u|^2 dx - H \int_B \frac{u}{r^2} dx = n\omega_n \int_0^1 (v'(r))^2 r dr. \tag{1.11}$$

This is where Poincaré's inequality in two dimension enters:

$$\int_0^1 (v'(r))^2 r dr \geq H_2 \int_0^1 v(r)^2 r dr. \tag{1.12}$$

We finally observe that

$$\int_B u^2(x) dx = n\omega_n \int_0^1 v(r)^2 r dr, \tag{1.13}$$

Combining (1.11), (1.12) with (1.13), we have

$$\int_B |\nabla u|^2 dx \geq H \int_B \frac{u^2}{|x|^2} dx + H_2 \int_B u^2 dx.$$

Since we only consider the unit ball above and the eigenvalue of the laplacian in two dimension is related to the radius of the ball, which implies for $R > 1$

$$\int_{B_R} |\nabla u|^2 dx \geq H \int_{B_R} \frac{u^2}{|x|^2} dx + H_2(\frac{1}{R^2}) \int_{B_R} u^2 dx.$$

The last remark consists in removing the restriction $u \in C_0^1(B)$ and this is done by density. $\square$

1.2.3 Hardy's inequality involving the distance to the boundary

In this section, we introduce the N-dimensional Hardy's inequality involving the distance to the boundary, the main tool that we are going to use is the method of local coordinates (Chapter 8 [45]). We will first introduce the required knowledge and list the main theorem we use afterwards. This inequality is often used as a preliminary step in obtaining a priori estimate of solutions to some partial differential equations as we will see in chapter 1.2.4 and chapter 2.2.

Definition 1.6 (see Chapter 4.2 [45]). *A bounded domain U is said to be of class $C^{0,\kappa}$ (notation $U \in C^{0,\kappa}, 0 < \kappa \leq 1$), if the following conditions are fulfilled:*
(i) There exists a finite number m of coordinate systems

$$(y_i', y_{iN}), \quad y_i' = (y_{i1}, y_{i2}, \cdots, y_{iN-1})$$

and the same number of continuous functions $a_i = a_i(y_i')$ defined on the closure of $(N-1)$-dimensional cubes

$$\Delta_i = \{y_i'; |y_{ij}| < \mu \text{ for } \mu > 0, j = 1, 2, \cdots, N-1\} \ (i = 1, 2, \cdots, m)$$

so that for each point $x \in \partial U$ there is at least one $i \in \{1, 2, \cdots, m\}$ such that

$$x = (y_i', y_{iN}) \text{ and } y_{iN} = a_i(y_i').$$

(ii) The function a_i are continuous and satisfy Hölder condition on closed cubes $\bar{\Delta}_i$ with the exponent κ (and with a constant $A > 0$), that is, if

$$|a_i(y_i') - a_i(z_i')| \leq A|y_i' - z_i'|^\kappa.$$

holds for $y_i', z_i' \in \bar{\Delta}_i$ $(i = 1, 2, \cdots, m)$.

(iii) There exists a positive number $\beta < 1$ such that the sets B_i, defined by the relation

$$B_i = \{(y_i', y_{iN}); y_i' \in \Delta_i, a_i(y_i') - \beta < y_{iN} < a_i(y_i') + \beta\}, \tag{1.14}$$

satisfy

$$U_i = B_i \cap U = \{(y_i', y_{iN}); y_i' \in \Delta_i, a_i(y_i') - \beta < y_{iN} < a_i(y_i')\} \tag{1.15}$$

and

$$\Gamma_i = B_i \cap \partial U = \{(y_i', y_{iN}); y_i' \in \Delta_i, y_{iN} = a_i(y_i')\} \tag{1.16}$$

$i = (1, 2, \cdots, m)$.

Partition of Unity. The system

$$\{B_1, B_2, \cdots, B_m\}, \tag{1.17}$$

where B_i are the sets given by the formula (1.14), form a covering of the boundary ∂U. Let us denote by B_0 such an open set in $\mathbb{R}^N$ that

$$\bar{B}_0 \subset U \text{ and } U = B_0 \cup \bigcup_{i=1}^m U_i,$$

where U_i are the sets defined by (1.15). Then the system

$$\{B_0, B_1, \cdots, B_m\}$$

forms a covering of the closure $\bar{U}$ of the domain U.

Let us denote by

$$\{\phi_0, \phi_1, \cdots, \phi_m\}$$

a *partition of unity* corresponding to the covering (1.17), that is, let

$$\begin{aligned} &\phi_i \in C^\infty(\mathbb{R}^N), \ \text{supp } \phi_i \in B_i, \ 0 \leq \phi_i(x) \leq 1, \\ &\sum_{i=0}^m \phi_i(x) = 1 \text{ for } x \in \bar{U}. \end{aligned} \tag{1.18}$$

As

$$\partial U = \bigcup_{i=1}^{m} \Gamma_i,$$

where $\Gamma_i = B_i \cap \partial U$(see (1.16)), and $\phi_0(x) = 0$ for $x \in \partial U$, we have

$$\sum_{i=1}^{m} \phi_i(x) = 1 \text{ for } x \in \partial U.$$

Remark 1.5 (Chapter 4.5 [45])**.** *(i) Definition 1.6 together with the partition of unity* $\{\phi_0, \phi_1, \cdots, \phi_m\}$ *make it possible to apply the method of local coordinates: instead of investigating a function* $u \in W^{k,p}(U;\sigma)$ *in the domain* U*, we investigate the function*

$$v_i = u\phi_i$$

in the "cylinder" U_i *- see Figure 1.*

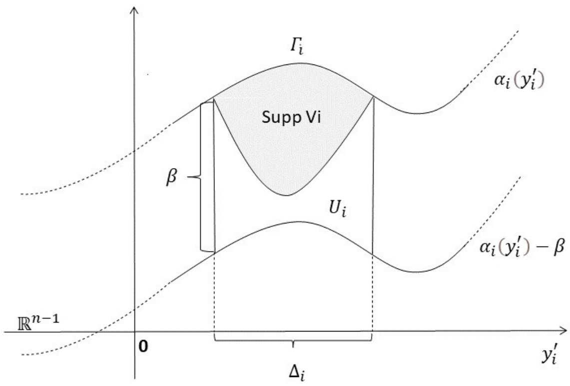

Figure 1

This procedure is successively applied to each $i \in \{1, 2, \cdots, m\}$*, while for* $i = 0$*, we base our argument on the fact that* $\bar{B}_0 \subset U$*, so that our special weight functions* $\sigma = \delta(x)$ *satisfy the inequality*

$$0 < c_1 \leq \sigma(x) \leq c_2, x \in B_0,$$

and, consequently, for $v_0 = u\phi_0$ *we can use the results which hold for the classical Sobolev spaces* $W^{k,p}(B_0)$*. Finally, we exploit the fact that the last property of the functions* ϕ_i *in* (1.18) *yields*

$$u(x) = \sum_{i=0}^{m} v_i(x) \text{ for } x \in U.$$

(ii) The function a_i *is defined on* $\bar{\Delta}_i$*, but we can extend it (continuously) to the whole space* $\mathbb{R}^{N-1}$ *and consider a half-space*

$$G_i = \{y = (y_i', y_{iN}); y_i' \in \mathbb{R}^{N-1}, y_{iN} < a_i(y_i')\} \tag{1.19}$$

instead of the cylinder U_i*.*

Now let us present an elementary but important result for the domains of the type (1.19).

Lemma 1.2 (Lemma 4.6 [45])**.** *Let $a = a(x')$ be a function defined on $\mathbb{R}^{N-1}$ and satisfying the Hölder condition with the exponent κ, $0 < \kappa \le 1$ and with a constant A, $A > 0$:*

$$|a(x') - a(y')| \le A|x' - y'|^{\kappa} \tag{1.20}$$

for all x', $y' \in \mathbb{R}^{N-1}$.

Further, let

$$\begin{aligned} G &= \{x = (x', x_N) \in \mathbb{R}^N; x' \in \mathbb{R}^{N-1}, x_N < a(x')\}, \\ G_1 &= \{x = (x', x_N) \in \mathbb{R}^N; x' \in \mathbb{R}^{N-1}, a(x') - 1 < x_N < a(x')\}, \\ \Gamma &= \{x = (x', x_N) \in \mathbb{R}^N; x' \in \mathbb{R}^{N-1}, x_N = a(x')\} \end{aligned}$$

and let us denote

$$\begin{aligned} r(x) &= \operatorname{dist}(x, \Gamma), \\ \rho(x) &= \rho(x', x_N) = |a(x') - x_N|. \end{aligned}$$

Then

$$\left[\frac{\rho(x)}{1+A}\right]^{1/\kappa} \le r(x) \le \rho(x) \quad \text{for } x \in G_1. \tag{1.21}$$

Proof. We have $G_1 \subset G$ and $\Gamma = \partial G$. As $r(x)$ is the distance of the point x from the boundary Γ of the "half-space" G and $\rho(x)$ is the distance of point x from Γ "in the direction of the x_N-axis", we have evidently

$$r(x) \le \rho(x)$$

which proves the second inequality in (1.21). To be more intuitive, see Figure 2.

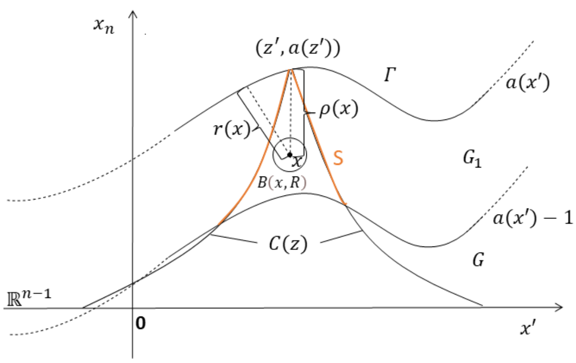

Figure 2

Now let $z = (z', a(z')) \in \Gamma$. The set

$$C(z) = \{(x', x_N) \in \mathbb{R}^N; x_N < a(z') - A|z' - x'|^{\kappa}\}$$

is a cusp with its vertex at the point $z \in \Gamma$. We shall know that

$$C(z) \subset G. \tag{1.22}$$

The Hölder continuous inequality (1.20) yields

$$a(z') - a(x') \leq A|z' - x'|^{\kappa}$$

and hence

$$a(x') \geq a(z') - A|z' - x'|^{\kappa} \text{ for all } x' \in \mathbb{R}^{N-1}.$$

Now if $x \in C(z)$, then taking into account the definition of the cusp $C(z)$ we obtain

$$a(z') - A|z' - x'|^{\kappa} > x_N,$$

which together with the preceding inequality yields

$$a(x') > x_N, \text{that is, } x \in G.$$

Let us now fix a point $x \in G_1$ and choose a point $z \in \Gamma$ in the following way: $z = (x', a(x'))$. Let us consider the cusp $C(z)$. We introduce the notation

$$S = \partial C(z) \cap \bar{G}_1,$$

that is,

$$S = \{(y', y_N) \in \mathbb{R}^N; y_N = a(x') - A|x' - y'|^{\kappa},\ a(y') - 1 \leq y_N \leq a(y')\},$$

and

$$d(x) = \operatorname{dist}(x, S).$$

It follows from (1.22) that

$$d(x) \leq r(x);$$

thus it suffices to prove the inequality

$$d(x) \geq \left[\frac{\rho(x)}{1 + A}\right]^{1/\kappa}. \tag{1.23}$$

To this aim, let us denote $R = \left[\frac{\rho(x)}{1 + A}\right]^{1/\kappa}$ (the point x is fixed in G_1 which implies $\rho(x) < 1$ and hence $R < 1$) and consider the closed ball

$$B(x, R) = \{y; |x - y| \leq R\}.$$

For $y \in B(x,R)$ the inequalities $|x'-y'| \le |x-y| \le R$ and $|x_N - y_N| \le |x-y| \le R \le R^\kappa$ hold and, consequently,

$$\begin{aligned} y_N - a(x') + A|x'-y'|^\kappa &= y_N - x_N + x_N - a(x') + A|x'-y'|^\kappa \\ &\le |y_N - x_N| - (a(x') - x_N) + A|x'-y'|^\kappa \\ &\le R^\kappa - \rho(x) + AR^\kappa \\ &= (1+A)R^\kappa - \rho(x) \\ &= 0. \end{aligned}$$

Hence

$$y_N \le a(x') - A|x'-y'|^\kappa,$$

that is,

$$y \in \overline{C(z)}.$$

In other words, $B(x,R) \subset \overline{C(z)}$, which implies $d(x) \ge R$. This proves (1.23) and, consequently, (1.21) as well. □

Theorem 1.3 (Theorem 8.4 [45]). *Let $U \in C^{0,\kappa}$, $0 < \kappa \le 1$. Let $1 < p < \infty$. Then*

$$W_0^{1,p}(U) \hookrightarrow L^p(U;\delta), \tag{1.24}$$

where $\delta(x) = \operatorname{dist}(x, \partial U)$ and

$$\|u\|_{p;\delta} = \Big(\int_U \big|\frac{u(x)}{\delta(x)}\big|^p dx\Big)^{1/p}.$$

Proof. Since the set $C_0^\infty(U)$ is dense in $W_0^{1,p}(U;\delta)$, it suffices to establish the estimate for functions

$$u \in C_0^\infty(U).$$

Since $U \in C^{0,\kappa}$, we can apply the local coordinates method as presented in Definition 1.6 and the *partition of unity*. Let u be such a smooth function and denote $v_i = u\phi_i$.

Now we estimate

$$\begin{aligned} \|v_i\|_{p;\delta}^p &= \int_U \Big|\frac{v_i(x)}{\delta(x)}\Big|^p dx = \int_{U_i} \Big|\frac{v_i(y)}{\delta(y)}\Big|^p dy \\ &= \int_{\Delta_i} dy_i' \int_{a_i(y_i')-\beta}^{a_i(y_i')} \Big|\frac{v_i(y_i', y_{iN})}{\delta(y)}\Big|^p dy_{iN}, \end{aligned} \tag{1.25}$$

then the first inequality in (1.21) yields,

$$\delta(y)^{-p} \le \big(\frac{1}{1+A}\big)^{-p}[a_i(y_i') - y_{iN}]^{-p}. \tag{1.26}$$

Consequently, using (1.26) and substituting $t = a_i(y_i') - y_{iN}$, we obtain the following formula from (1.25):

$$\begin{aligned} \|v_i\|_{p;\delta}^p &\leq C_0 \int_{\Delta_i} dy_i' \int_{a_i(y_i')-\beta}^{a_i(y_i')} \Big|\frac{v_i(y_i', y_{iN})}{[a_i(y_i') - y_{iN}]}\Big|^p dy_{iN} \\ &= C_0 \int_{\Delta_i} dy_i' \int_0^\beta \Big|\frac{v_i(y_i', a_i(y_i') - t)}{t}\Big|^p dt \\ &= C_0 \int_{\Delta_i} dy_i' \int_0^\infty \Big|\frac{v_i(y_i', a_i(y_i') - t)}{t}\Big|^p dt \end{aligned} \tag{1.27}$$

where $C_0 = (1+A)^p$. (We have also used the fact that $\mathrm{supp} v_i \subset U_i + \Gamma_i$ and hence $v_i(y_i', a_i(y_i') - t) = 0$ for $t \geq \beta$). Next we estimate the inner integral on the right hand side of the inequality (1.27) by the Hardy's inequality in one dimension (see Remark 1.4). Since with regard to the choice $u \in C_0^\infty(U)$ we also have $v_i(y_i', a_i(y_i') - t) = 0$ for small $t \geq 0$. Therefore we have the estimate

$$\int_0^\infty \Big|\frac{v_i(y_i', a_i(y_i') - t)}{t}\Big|^p dt \leq (\frac{p}{p-1})^p \int_0^\infty \Big|\frac{\partial v_i}{\partial y_{iN}}(y_i', a_i(y_i') - t)\Big|^p dt.$$

Again we can integrate only from 0 to β in the last integral; after substituting $t = a_i(y_i') - y_{iN}$ and integrating the resulting inequality with respect to y_i' over Δ_i we obtain from (1.27)

$$\|v_i\|_{p;\delta}^p \leq C_1 \int_{U_i} \Big|\frac{\partial v_i}{\partial y_{iN}}(y)\Big|^p dy \leq C_1 \|v_i\|_{1,p}^p \tag{1.28}$$

with $C_1 = C_0 p^p |p-1|^p$.

The inequality (1.28) holds for $i = 1, 2, \cdots, m$. On the other hand, it also holds for the function $v_0 = u\phi_0$, since supp $\phi_0 \subset B_0$ and $\bar{B}_0 \subset U$ hence $\delta(x)$ bounded from above and below (see Remark 1.5). As

$$u = \sum_{i=0}^m u\phi_i = \sum_{i=0}^m v_i,$$

(1.28) finally yields the estimate

$$\|u\|_{p;\delta} \leq \sum_{i=0}^m \|v_i\|_{p;\delta} \leq C\|u\|_{1,p}, \tag{1.29}$$

which holds (with a constant $C = C_0 + C_1 + \cdots + C_m$ independent of the function u) for every $u \in C_0^\infty(U)$.

Now let $u \in W^{1,p}(U;\delta)$ and $\{u_n\}_{n=1}^\infty$ a sequence of functions $u_n \in C_0^\infty(U)$ that converges to the function u. Then every u_n satisfies the estimate (1.29). Passing here to the limit with $n \to \infty$, we find that the estimate (1.29) holds for $u \in W^{1,p}(U)$ as well, and this completes the proof of the embedding (1.24).

□

1.2.4 A generalized Hardy-Sobolev inequality

Proposition 1.1 (Lemma 2.2 [13]). *For $v \in H_0^1(U)$ and $0 \le \tau \le 1$, one has*

$$\|\frac{v}{\delta^\tau}\|_{L^q} \le C\|Dv\|_{L^2}$$

where $\dfrac{1}{q} = \dfrac{1}{2} - \dfrac{1-\tau}{N}$.

Proof. From Hölder's inequality,

$$\begin{aligned} \|\frac{v}{\delta^\tau}\|_{L^q(U)} &\le \|\frac{v^\tau}{\delta^\tau}\|_{L^r(U)} \cdot \|v^{1-\tau}\|_{L^s}; \quad \frac{1}{q} = \frac{1}{r} + \frac{1}{s} \\ &= \|\frac{v}{\delta}\|^\tau_{L^{\tau r}} \|v\|^{1-\tau}_{L^{(1-\tau)s}} \end{aligned} \tag{1.30}$$

Choosing $\tau r = 2$ and $\frac{1}{(1-\tau)s} = \frac{1}{2} - \frac{1}{N}$, then applying Theorem 1.3 and Sobolev's embedding theorem to the respective terms in (1.30) we obtain

$$\|\frac{v}{\delta^\tau}\|_{L^q} \le C\|Dv\|^\tau_{L^2}\|Dv\|^{1-\tau}_{L^2}. \tag{1.31}$$

Then (1.31) becomes the desired inequality. □

Remark 1.6. *Observe that the extreme case $\tau = 0$ is the Sobolev embedding therem $H_0^1(U) \subset L^{2^*}(U)$, where $2^* = 2N/(N-2)$. The other extreme case $\tau = 1$ is a fact already observed in Lions-Magenes [48] p.76, that the behavior of a function $u \in H_0^1(U)$ near the boundary ∂U is such that u/δ in $L^2(U)$.*

1.3 Strong solutions and regularity

For the general form (1.2), a strong solution of the equation

$$Lu = f \tag{1.32}$$

is a twice weakly differentiable function on U satisfying the equation (1.32) almost everywhere in U. We now address the question as to whether a weak solution u of the (1.32) is in fact a strong solution: this is the regularity problem for the weak solutions.

1.3.1 L^2 regularity theory

We as always assume that $U \subset \mathbb{R}^n$ is a bounded, open set. Suppose also $u \in H_0^1(U)$ is a weak solution of (1.32). We require the uniform ellipticity condition (1.5) and will, as necessary, make various additional assumptions about the smoothness of the coefficients a^{ij}, b^i, c.

Theorem 1.4 (Interior H^2-regularity [25], chapter 6.3.1)**.** *Assume*

$$a^{ij} \in C^1(U), b^i, c \in L^\infty(U) \ (i, j = 1, \cdots, n)$$

and

$$f \in L^2(U).$$

Suppose furthermore that $u \in H^1(U)$ *is a weak solution of* (1.32). *Then*

$$u \in H^2_{loc}(U);$$

and for each open subset $V \subset\subset U$ *we have the estimate*

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}), \tag{1.33}$$

the constant C *depending only on* V, U, *and the coefficients of* L.

Remark 1.7. *Note carefully that we do not require* $u \in H^1_0(U)$; *that is, we are not necessarily assuming the boundary condition* $u = 0$ *on* ∂U *in the trace sense.*

For simplicity, we only present the main steps here.

Proof. 1. Fix any open set $V \subset\subset U$, and choose an open set W such that $V \subset\subset W \subset\subset U$. Then select a smooth function ζ satisfying

$$\begin{cases} \zeta \equiv 1 \text{ on } V, \zeta \equiv 0 \text{ on } \mathbb{R}^n - W \\ 0 \leq \zeta \leq 1. \end{cases}$$

we call ζ a cut-off function. Its purpose in the subsequent calculations will be to restrict all expressions to the subset W, which is a positive distance away from ∂U. This is necessary as we have no information concerning the behavior of u near ∂U. An interesting technical point is to take a suitable test function v as we will see in the following calculation.

2. Since u is a weak solution of (1.32), we have

$$\sum_{i,j=1}^{n} \int_U a^{ij} u_{x_i} v_{x_j} dx = \int_U \tilde{f} v dx, \quad \forall v \in H^1_0(U) \tag{1.34}$$

where

$$\tilde{f} := f - \sum_{i=1}^{n} b^i u_{x_i} - cu. \tag{1.35}$$

3. Now let $|h| > 0$ be small, choose $k \in \{1, \cdots, n\}$, and then substitute

$$v := -D_k^{-h}(\zeta^2 D_k^h u) \tag{1.36}$$

into (1.34), where $D_k^h u$ denotes the difference quotient

$$D_k^h u(x) = \frac{u(x+he_k) - u(x)}{h},$$

here $h \in \mathbb{R}, h \neq 0, e_i = (0, \cdots, 0, 1, \cdots, 0) = i^{th}$ standard coordinate vector. We write the resulting expression as

$$A = B, \tag{1.37}$$

for

$$A := \sum_{i,j=1}^{n} \int_U a^{ij} u_{x_i} v_{x_j} dx$$

and

$$B := \int_U \tilde{f} v \, dx \tag{1.38}$$

4. Estimate of A. By the properties of difference quotient and integration by parts, we find

$$\begin{aligned} A &= \sum_{i,j=1}^{n} \int_U a^{ij,h} D_k^h u_{x_i} D_k^h u_{x_j} \zeta^2 \, dx \\ &\quad + \sum_{i,j=1}^{n} \int_U \left[a^{ij,h} D_k^h u_{x_i} D_k^h u 2\zeta\zeta_{x_j} + (D_k^h a^{ij}) u_{x_i} D_k^h u_{x_j} \zeta^2 + (D_k^h a^{ij}) u_{x_i} D_k^h u 2\zeta\zeta_{x_j} \right] dx \\ &= A_1 + A_2. \quad (a^{ij,h} = a^{ij}(x+he_k)) \end{aligned} \tag{1.39}$$

The uniform ellipticity condition implies

$$A_1 \geq \theta \int_U \zeta^2 |D_k^h Du|^2 dx. \tag{1.40}$$

Furthermore, by the assumption of a^{ij}, b^i, c and Cauchy's inequality with $\epsilon = \frac{\theta}{2}$, we obtain the inequality

$$|A_2| \leq \frac{\theta}{2} \int_U \zeta^2 |D_k^h Du|^2 dx + C \int_U |Du|^2 dx. \tag{1.41}$$

(1.39), (1.40) and (1.41) imply finally

$$A \geq \frac{\theta}{2} \int_U \zeta^2 |D_k^h Du|^2 dx - C \int_U |Du|^2 dx. \tag{1.42}$$

The essential point in employing Cauchy's inequality here is that the higher order term occurs on the right-hand side in (1.41) with a smaller coefficient than the same term in A_1, and so the contribution on the right-hand side can be absorbed in A_1. The benefits to do this is to keep the second order term on the left which can be controlled by the lower order term and the inhomogeneous term.

5. Estimate of B. Recalling (1.35), (1.36) and (1.38), we estimate

$$|B| \leq C \int_U (|f| + |Du| + |u|)|v| dx. \tag{1.43}$$

Likewise, we apply the properties of the difference quotient and Cauchy's inequality with $\epsilon = \dfrac{\theta}{4}$ to B to obtain

$$|B| \leq \frac{\theta}{4} \int_U \zeta^2 |D_k^h Du|^2 dx + C \int_U f^2 + u^2 + |Du|^2 dx.$$

6. We finally combine (1.37), (1.42) and (1.43), to discover

$$\int_V |D_k^h Du|^2 dx \leq \int_U \zeta^2 |D_k^h Du|^2 dx \leq C \int_U f^2 + u^2 + |Du|^2 dx \tag{1.44}$$

for $k = 1, \cdots, n$ and all sufficiently small $|h| \neq 0$. (1.44) implies $Du \in H^1_{loc}(U)$, and thus $u \in H^2_{loc}(U)$, with the estimate

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}). \tag{1.45}$$

7. We now refine estimate (1.45) by noting that if $V \subset\subset W \subset\subset U$, then the same argument shows

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(W)} + \|u\|_{H^1(W)}), \tag{1.46}$$

for an appropriate constant C depending on V, W, etc. Choose a new cutoff function ζ satisfying

$$\begin{cases} \zeta \equiv 1 \text{ on } W, \text{spt } \zeta \in U, \\ 0 \leq \zeta \leq 1. \end{cases}$$

Now set $v = \zeta^2 u$ in (1.34) and perform elementary calculations, to discover

$$\int_U \zeta^2 |Du|^2 dx \leq C \int_U f^2 + u^2 dx.$$

Thus

$$\|u\|_{H^1(W)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}).$$

This inequality and (1.46) yield (1.33). □

Theorem 1.5 (Boundary H^2-regularity [25], chapter 6.3.2)**.** *Assume*

$$a^{ij} \in C^1(\bar{U}), b^i, c \in L^\infty(U) \ (i, j = 1, \cdots, n) \tag{1.47}$$

and

$$f \in L^2(U).$$

Suppose that $u \in H_0^1(U)$ is a weak solution of the elliptic boundary-value problem

$$\begin{cases} Lu &= f \quad \text{in } U \\ u &= 0 \quad \text{on } \partial U. \end{cases} \tag{1.48}$$

Assume finally

$$\partial U \text{ is } C^2.$$

Then

$$u \in H^2(U),$$

and we have the estimate

$$\|u\|_{H^2(U)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}), \tag{1.49}$$

the constant C depending only on U and the coefficients of L.

Remark 1.8. *(i) If $u \in H_0^1(U)$ is the unique weak solution of* (1.48)*, estimate* (1.49) *simplifies to read*

$$\|u\|_{H^2(U)} \leq C\|f\|_{L^2(U)}.$$

(ii) Observe also that in contrast to Theorem 1.4, we are now assuming $u = 0$ along ∂U.

The proof of this theorem is to turn the boundary estimate to the interior estimate, once we straighten out the boundary, we can perform the same technique as in the proof of the interior regularity (we will see why soon). As we will not discuss details about the proof, we will only give an outline here.

Proof. 1. We first investigate the special case that U is a half-ball:

$$U = B^0(0,1) \cap \mathbb{R}^n_+. \tag{1.50}$$

Set $V := B^0(0, \frac{1}{2}) \cap \mathbb{R}^n_+$. Then select a smooth cutoff function ζ satisfying

$$\begin{cases} \zeta \equiv 1 \text{ on } B(0, \frac{1}{2}), \ \zeta \equiv 0 \text{ on } \mathbb{R}^n - B(0,1), \\ 0 \leq \zeta \leq 1. \end{cases}$$

So $\zeta \equiv 1$ on V and ζ vanishes near the curved part of ∂U.

2. Since u is a weak solution of (1.48), we have the same formula as (1.34) and (1.35). Now we take

$$v := -D_k^{-h}(\zeta^2 D_k^h u).$$

Similar to interior H^2-regularity, for $k \in \{1, \cdots, n-1\}$, we have the same estimate

$$\int_V |D_k^h Du|^2 dx \leq C \int_U f^2 + u^2 + |Du|^2 dx$$

which implies

$$u_{x_k} \in H^1(V) \ (k = 1, \cdots, n-1),$$

with the estimate

$$\sum_{\substack{k,l=1 \\ k+l<2n}}^{n} \|u_{x_k x_l}\|_{L^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}). \tag{1.51}$$

Since v is not defined along the x_n direction, we argue the L^2-norm of $u_{x_n x_n}$ over V separately. Recalling the definition of L, we can rewrite the equation as non-divergence form as (1.4), so that

$$a^{nn} u_{x_n x_n} = -\sum_{\substack{i,j=1 \\ i+j<2n}}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} \tilde{b}^i u_{x_i} + cu - f \tag{1.52}$$

for $\tilde{b}^i := b^i - \sum_{j=1}^n a_{x_j}^{ij} (i = 1, \cdots, n)$. According to the uniform ellipticity condition, $\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$ for all $x \in U$, $\xi \in \mathbb{R}^n$. We set $\xi = e_n = (0, \cdots, 0, 1)$ to conclude

$$a^{nn}(x) \geq \theta > 0 \tag{1.53}$$

for all $x \in U$. We then combine (1.47), (1.52) and (1.53) to discover

$$|u_{x_n x_n}| \leq C(\sum_{\substack{i,j=1 \\ i+j<2n}}^{n} |u_{x_i x_j}| + |Du| + |u| + |f|)$$

in U. Utilizing this estimate in inequality (1.51), we conclude $u \in H^2(V)$ and

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$$

for some appropriate constant C.

3. We now drop the assumption that U is a half-ball and so has the special form (1.50). In the general case we choose any point $x_0 \in \partial U$ and notice that since ∂U is C^2, by "straighten out the boundary", there exists some C^2 function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$, such that

$$U \cap B(x^0, r) = \{x \in B(x^0, r) \mid x_n > \gamma(x_1, \cdots, x_{n-1})\}$$

for some $r > 0$ and write

$$y = \phi(x), \quad x = \psi(y).$$

Choose $s > 0$ so small that the half ball $U' := B^0(0,s) \cap \{y_n > 0\}$ lies in $\phi(U \cup B(x^0, r))$. Finally define

$$u'(y) := u(\psi(y)) \quad (y \in U').$$

It is straightforward to check

$$u' \in H^1(U')$$

and

$$u' = 0 \text{ on } \partial U' \cap \{y_n = 0\}.$$

Furthermore, by changing variable, we can show u' is a weak solution of

$$L'u' = f' \text{ in } U', \tag{1.54}$$

for

$$f'(y) := f(\psi(y))$$

and

$$L'u' := -\sum_{k,l=1}^{n} (a'^{kl} u'_{yk})_{yl} + \sum_{k=1}^{n} b'^{k} u'_{yk} + c'u',$$

where

$$a'^{kl}(y) := \sum_{r,s=1}^{n} a^{rs}(\psi(y))\phi^k_{x_r}(\psi(y))\phi^l_{x_s}(\psi(y)) \quad (k, l = 1, \cdots, n),$$

$$b'^{k}(y) := \sum_{r=1}^{n} b^r(\psi(y))\phi^k_{x_r} \quad (k = 1, \cdots, n),$$

and

$$c'(y) := c(\psi(y))$$

for $y \in U', k, l = 1, \cdots, n$. The operator L' is uniformly elliptic in U',

$$\sum_{k,l=1}^{n} a'^{kl}(y)\xi_k\xi_l \geq \theta'|\xi|^2 \tag{1.55}$$

for some $\theta' > 0$ and all $y \in U'$, $\xi \in \mathbb{R}^n$. Since ϕ and ψ are C^2, the coefficients a'^{kl} are C^1.

4. In view of (1.54) and (1.55), we may apply the results from the first two steps in the proof above to ascertain that $u' \in H^2(V')$, with the bound

$$\|u'\|_{H^2(V')} \leq C(\|f'\|_{L^2(U')} + \|u'\|_{L^2(U')}).$$

Consequently

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$$

for $V := \psi(V')$.

Since ∂U is compact, we can as usual cover ∂U with finitely many sets $V_1, \cdots, V_N$ as above. We sum the resulting estimates, along with the interior estimate, to find $u \in H^2(U)$, with the inequality (1.49). $\square$

For higher interior and boundary L^2 regularity, one can refer to [25] chapter 6.3. In the next section we list the interior and global L^p estimates for the second derivatives of elliptic equations of the form (1.3).

1.3.2 L^p regularity theory

In this section we will introduce the theory that weak solutions of the second order elliptic partial equation (1.3) have second order weak derivatives which belong to any L^p space. The approach is based on the a priori estimate of solutions. These estimates provide compactness results that are essential for the existence and regularity theory. We will start from a Newtonian potential, then use the method of "frozen coefficients" to generalize it to uniformly elliptic equations. (see [21], chapter 3.)

Let U be a bounded domain in $\mathbb{R}^n$ and f a function in $L^p(U)$ for some $p \geq 1$. Recall that the Newtonian potential of f is the function $w = Nf$ defined by the convolution.

$$w(x) = \int_U \Gamma(x-y)f(y)dy,$$

where Γ is the fundamental solution of Laplace's equation given by

$$\Gamma(x-y) = \Gamma(|x-y|) = \begin{cases} \dfrac{1}{n(2-n)\omega_n}|x-y|^{2-n}, & n > 2 \\ \dfrac{1}{2\pi}\log|x-y|, & n = 2. \end{cases}$$

Proposition 1.2 ([21], Theorem 3.1.1; [36], Theorem 9.9)**.** *Let $f \in L^p(U)$, $1 < p < \infty$, and let w be the Newtonian potential of f. Then $w \in W^{2,p}(U)$, $\Delta w = f$ a.e. and*

$$\|D^2 w\|_{L^p(U)} \leq C\|f\|_{L^p(U)}$$

where C depends only on n and p. Furthermore, when $p = 2$ we have

$$\int_{\mathbb{R}^n} |D^2 w| = \int_U f^2.$$

Proof. First we consider $f \in C_0^\infty(U) \subset C_0^\infty(\mathbb{R}^n)$, then we have $w \in C^\infty(\mathbb{R}^n)$ and

$$\Delta w = f(x), \forall x \in \mathbb{R}^n.$$

Write $w = Nf$, where N is a bounded mapping from L^p into itself for $1 \leq p < \infty$ ([36], Lemma 7.12). Next, for fixed i, j, define the linear operator T as

$$Tf = D_{ij}Nf = D_{ij}\int_{\mathbb{R}^n} \Gamma(x-y)f(y)dy, \qquad i,j = 1, \cdots, n.$$

To prove the lemma, it is equivalent to show that

$$T : L^p(U) \to L^p(U) \tag{1.56}$$

is a bounded linear operator. Indeed, if the sequence $\{f_m\} \subset C_0^\infty(U)$ converges to f in $L^p(U)$, and there exists a sequence $\{w_m\}$ such that $\Delta w_m = f_m$, then

$$\|w_m\|_{L^p(U)} = \|Nf_m\|_{L^p(U)} \leq C\|f_m\|_{L^p(U)} \tag{1.57}$$

and

$$\|D_{ij}w_m\|_{L^p(U)} = \|Tf_m\|_{L^p(U)} \leq C\|f_m\|_{L^p(U)}. \tag{1.58}$$

By Nirenberg-Gagliardo interpolation inequality, (1.57) and (1.58) we have for $\epsilon > 0$

$$\|Dw_m\|_{L^p(U)} \leq \epsilon\|D^2w_m\|_{L^p(U)} + C(U,\epsilon)\|w_m\|_{L^p(U)} \leq C\|f_m\|_{L^p(U)},$$

and so

$$\|w_m\|_{W^{2,p}(U)} \leq C\|f_m\|_{L^p(U)}.$$

Furthermore, it is easy to see that

$$\|w_m - w_n\|_{W^{2,p}(U)} \leq C\|f_m - f_n\|_{L^p(U)},$$

which implies that $\{w_m\}$ is a Cauchy sequence in $W^{2,p}(U)$, thus $w_m \to w'$ in $W^{2,p}(U)$. Since $\Delta w_m = f_m$, then $\Delta w' = f$ by letting $m \to \infty$, therefore, $w' = w$.

The proof of (1.56) can actually be applied to more general operators. To this end, we introduce the concept of weak type and strong type operators.

Define

$$\mu_f(t) = |\{x \in U \big| |f(x)| > t\}|.$$

For $p \geq 1$, the weak L^p space $L^p_w(U)$ is the collection of functions f that satisfy

$$\|f\|^p_{L^p_w(U)} = \sup\{\mu_f(t)t^p, \ \forall t > 0\} < \infty.$$

An operator $T : L^p(U) \to L^q(U)$ is of strong type (p,q) if

$$\|Tf\|_{L^q(U)} \leq C\|f\|_{L^p(U)}, \forall f \in L^p(U).$$

T is of weak type (p,q) if

$$\|Tf\|_{L^q_w(U)} \leq C\|f\|_{L^p(U)}, \forall f \in L^p(U).$$

Outline of the proof of (1.56).

We decompose the proof of Proposition 1.2 into the proofs of the following four lemmas:

Lemma 1.3. *$T : L^2(U) \to L^2(U)$ is a bounded linear operator. i.e. T is of strong type (2, 2).*

Secondly, we use the Calderon-Zygmund Decomposition Lemma to prove

Lemma 1.4. *T is of weak type (1,1).*

Thirdly, we employ the Marcinkiewicz Interpolation Theorem to derive

Lemma 1.5. *T is of strong type (r, r) for any $1 < r \leq 2$.*

Finally, by duality, we conclude

Lemma 1.6. *T is of strong type (p, p) for $1 < p < \infty$.*

Given the space limitation, one can find the whole proof in appendix A.1. □

The L^p estimates for solutions of Poisson's equation follow immediately from Proposition 1.2.

Corollary 1.1 ([36], Corollary 9.10)**.** *Let U be a domain in $\mathbb{R}^n$, $u \in W_0^{2,p}(U)$, $1 < p < \infty$. Then*

$$\|D^2 u\|_{L^p(U)} \leq C\|\Delta u\|_{L^p(U)} \tag{1.59}$$

where $C = C(n, p)$. If $p = 2$,

$$\|D^2 u\|_{L^2(U)} = \|\Delta u\|_{L^2(U)}.$$

Theorem 1.6 (L^p interior estimate [36], Theorem 9.11)**.** *Let U be an open set in $\mathbb{R}^n$ and $u \in W_{loc}^{2,p}(U) \cap L^p(U)$, $1 < p < \infty$, a strong solution of the equation (1.3) in U where the coefficients of L satisfy, for positive constants λ, Λ,*

$$\begin{aligned} &a^{ij} \in C^0(U), b^i, c \in L^\infty(U), f \in L^p(U); \\ &a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \forall \xi \in \mathbb{R}^n; \\ &|a^{ij}|, |b^i|, |c| \leq \Lambda, \end{aligned} \tag{1.60}$$

where $i, j = 1, \cdots, n$. Then for any domain $U' \subset\subset U$,

$$\|u\|_{W^{2,p}(U')} \leq C(\|u\|_{L^P(U)} + \|f\|_{L^P(U)}), \tag{1.61}$$

where C depends on n, p, λ, Λ, U', U and the moduli of continuity of the coefficients a^{ij} on U'.

Proof. Here comes the method of "Frozen coefficients". For a fixed point $x_0 \in U'$, we let L_0 denote the constant coefficient operator given by

$$L_0 u = a^{ij}(x_0) D_{ij} u.$$

By means of a linear transformation, we obtain from Corollary 1.1 the estimate

$$\|D^2 v\|_{L^p(U)} \leq \frac{C}{\lambda}\|L_0 v\|_{L^p(U)} \tag{1.62}$$

for any $v \in W_0^{2,p}(U)$, where $C = C(n,p)$ as in (1.59). Consequently, if v has support in a ball $B_R = B_R(x_0) \subset\subset U$, we have

$$L_0 v = (a^{ij}(x_0) - a^{ij}) D_{ij} v + a^{ij} D_{ij} v,$$

and by (1.62)

$$\|D^2 v\|_{L^p} \le \frac{C}{\lambda} (\sup_{B_R} |a - a(x_0)| \|D^2 v\|_{L^p} + \|a^{ij} D_{ij} v\|_{L^p}).$$

Since a is uniformly continuous on U', the first term on the right could be 'eaten' by the term on the left. Precisely, there exists a positive number δ such that

$$|a - a(x_0)| \le \lambda / 2C$$

if $|x - x_0| < \delta$, and hence

$$\|D^2 v\|_{L^p} \le C \|a^{ij} D_{ij} v\|_{L^p}$$

provided $R \le \delta$, where $C = C(n, p, \lambda)$.

For $\sigma \in (0,1)$, we now introduce a cutoff function $\eta \in C_0^2(B_R)$ satisfying $0 \le \eta \le 1$, $\eta = 1$ in $B_{\sigma R}$, $\eta = 0$ for $|x| \ge \sigma' R$, $\sigma' = (1+\sigma)/2$, $|D\eta| \le 4/(1-\sigma)R$, $|D^2 \eta| \le 16/(1-\sigma)^2 R^2$. Then, if $u \in W_{loc}^{2,p}(U)$ satisfies $Lu = f$ in U and $v = \eta u$, we obtain

$$\begin{aligned} \|D^2 u\|_{L^p(B_{\sigma R})} &\le C \|\eta a^{ij} D_{ij} u + 2 a^{ij} D_i \eta D_j u + u a^{ij} D_{ij} \eta\|_{L^p(B_R)} \\ &\le C(\|f\|_{L^p(B_R)} + \tfrac{1}{(1-\sigma)R} \|Du\|_{L^p(B_{\sigma' R})} + \tfrac{1}{(1-\sigma)^2 R^2} \|u\|_{L^p(B_R)}) \end{aligned} \tag{1.63}$$

provided $R \le \delta \le 1$, where $C = C(n, p, \lambda, \Lambda)$.

Introducing the weighted semi-norms

$$\phi_k = \sup_{0<\sigma<1} (1-\sigma)^k R^k \|D^k u\|_{L^p(B_{\sigma R})}, \qquad k = 0, 1, 2,$$

by (1.63), we therefore have

$$\phi_2 \le C(R^2 \|f\|_{L^p(B_R)} + \phi_1 + \phi_0). \tag{1.64}$$

Besides, ϕ_k satisfy an interpolation inequality

$$\phi_1 \le \epsilon \phi_2 + \frac{C}{\epsilon} \phi_0 \tag{1.65}$$

for any $\epsilon > 0$, where $C = C(n)$. Using (1.65) in (1.64), we then get

$$\phi_2 \le C(R^2 \|f\|_{L^p(B_R)} + \phi_0),$$

that is,

$$\|D^2 u\|_{L^p(B_{\sigma R})} \le \frac{C}{(1-\sigma)^2 R^2} (R^2 \|f\|_{L^p(B_R)} + \|u\|_{L^p(B_R)}),$$

where $C = C(n, p, \lambda, \Lambda)$ and $0 < \sigma < 1$.

The desired estimate (1.61) follows by taking $\sigma = 1/2$ and covering U' with a finite number of balls of radius $R/2$ for $R \le \min\{\delta, \mathrm{dist}(U', \partial U)\}$. □

To extend the preceding interior estimates to the entire domain it is necessary to have estimates that are meaningful near the boundary. These can be obtained provided the boundary values of the solution and the boundary itself are of a certain smoothness. We first consider the case of a flat boundary portion. Letting

$$U^+ = U \cap \mathbb{R}^n_+ = \{x \in U \mid x_n > 0\},$$
$$(\partial U)^+ = (\partial U) \cap \mathbb{R}^n_+ = \{x \in \partial U \mid x_n > 0\}.$$

Lemma 1.7 ([36], Lemma 9.12)**.** *Let $u \in W^{2,p}(U^+) \cap W^{1,p}_0(U^+)$, $f \in L^p(U^+)$, $1 < p < \infty$, satisfy $\Delta u = f$ weakly in U^+ with $u = 0$ near $(\partial U)^+$. Then*

$$\|D^2 u\|_{L^P(U^+)} \le C\|f\|_{L^p(U^+)}, \tag{1.66}$$

where $C = C(n, p)$.

Proof. We extend u and f to all of $\mathbb{R}^n_+$ by setting $u = f = 0$ in $\mathbb{R}^n_+ - U$, and then to all of $\mathbb{R}^n$ by odd reflection, that is, by setting

$$u(x', x_n) = -u(x', -x_n), \quad f(x', x_n) = -f(x', -x_n)$$

for $x_n < 0$, where $x' = (x_1, \cdots, x_{n-1})$. Then the extended functions, say $\tilde{u}$ and $\tilde{f}$, satisfy $\Delta\tilde{u} = \tilde{f}$ weakly in $\mathbb{R}^n$ because $D_n\tilde{u}$ converge to zero as x_n close to 0. Since $\tilde{u}$ also has compact support in $\mathbb{R}^n$, the regularization $u_h \in C^\infty_0(\mathbb{R}^n) \to u$ in $W^{2,p}(\mathbb{R}^n)$ as $h \to 0$, and satisfies $\Delta u_h = f_h$ in $\mathbb{R}^n$. Hence applying Corollary 1.1, and the estimate (1.66) follows with constant C twice that in (1.59). □

Theorem 1.7 (L^p Boundary Estimates [36], Theorem 9.13)**.** *Let U be a domain in $\mathbb{R}^n$ with a $C^{1,1}$ boundary portion $T \subset \partial U$. Let $u \in W^{2,p}(U)$, $1 < p < \infty$, be a strong solution of $Lu = f$ in U with $u = 0$ on T, in the sense of $W^{1,p}(U)$, where L satisfies* (1.60) *with $a^{ij} \in C^0(U \cup T)$. Then, for any domain $U' \subset\subset U \cup T$,*

$$\|u\|_{W^{2,p}(U')} \le C(\|u\|_{L^p(U)} + \|f\|_{L^p(U)}) \tag{1.67}$$

where C depends on n, p, λ, Λ, U', U and the moduli of continuity of the coefficients a^{ij} on U'.

Proof. Since $T \in C^{1,1}$, for each point $x_0 \in T$ there is a neighborhood $\mathscr{N} = \mathscr{N}_{x_0}$ and a diffeomorphism $\psi = \psi(x_0)$ from $\mathscr{N}$ onto the unit ball $B = B_1(0)$ in $\mathbb{R}^n$ such that $\psi(\mathscr{N} \cap U) \subset \mathbb{R}^n_+$, $\psi(\mathscr{N} \cap \partial U) \subset \partial\mathbb{R}^n_+$, $\psi \in C^{1,1}(\mathscr{N})$, $\psi^{-1} \in C^{1,1}(B)$. Writing $y = \psi(x) = (\psi_1(x), \cdots, \psi_n(x))$, $\tilde{u}(y) = u(x), x \in \mathscr{N}, y \in B$, we have

$$\tilde{L}\tilde{u} = -\tilde{a}^{ij}D_{ij}\tilde{u} + \tilde{b}^i D_i\tilde{u} + \tilde{c}\tilde{u} = \tilde{f}$$

in B^+, where

$$\tilde{a}^{ij}(y) = \frac{\partial\psi_i}{\partial x_r}\frac{\partial\psi_j}{\partial x_s}a^{rs}(x), \quad \tilde{b}^i(y) = \frac{\partial^2\psi_i}{\partial x_r \partial x_s}a^{rs}(x) + \frac{\partial\psi_i}{\partial x_r}b^r(x),$$

$$\tilde{c}(y) = c(x), \quad \tilde{f}(y) = f(x)$$

so that $\tilde{L}$ satisfies conditions similar to (1.60) with constants $\tilde{\lambda}$, $\tilde{\Lambda}$ depending on λ, Λ and ψ. Furthermore, $\tilde{u} \in W^{2,p}(B^+)$, and $\tilde{u} = 0$ on $B \cap \partial\mathbb{R}^n_+$ in the sense of $W^{1,p}(B^+)$. By straightening out the boundary, we now apply Theorem 1.6 with the ball $B_R(x_0)$ replaced by the half ball $B^+_R(0)$ and with Lemma 1.7 used in place of Corollary 1.1. We obtain thus,

$$\|D^2\tilde{u}\|_{L^p(B^+_{\sigma R})} \leq \frac{C}{(1-\sigma)^2 R^2}\{R^2\|\tilde{f}\|_{L^p(B^+_R)} + \|\tilde{u}\|_{L^p(B^+_R)}\}$$

provided $R \leq \delta \leq 1$, where C depending on n, p, λ, Λ and ψ; and δ depends on the moduli of continuity of a^{ij} at x_0 and also on ψ. Taking $\sigma = \frac{1}{2}$ and $\tilde{\mathscr{N}} = \tilde{\mathscr{N}}_{x_0} = \psi^{-1}(B_{\delta/2})$ we therefore have on returning to our original coordinates.

$$\|D^2 u\|_{L^p(\tilde{\mathscr{N}})} \leq C(\|u\|_{L^p(\tilde{\mathscr{N}})} + \|f\|_{L^p(\tilde{\mathscr{N}})})$$

where $C = C(n, p, \lambda, \Lambda, \delta, \psi)$. Finally, by covering $U' \cap T$ with a finite number of such neighborhoods $\tilde{\mathscr{N}}$, and using also the interior estimate (1.61), we obtain the desired estimate. $\square$

Remark 1.9. *When $T = \partial U$ in Theorem 1.7 we may take $U' = U$ to obtain a global $W^{2,p}(U)$ estimate.*

Theorem 1.8 (Interior and Boundary Regularity [36], Theorem 9.15)**.** *Let U be a $C^{1,1}$ domain in $\mathbb{R}^n$, and let the operator L be strictly elliptic in U with coefficients $a^{ij} \in C^0(\bar{U})$, b^i, $c \in L^\infty$, with $i, j = 1, \cdots, n$ and $c \geq 0$. Then, if $f \in L^p(U)$ and $\phi \in W^{2,p}(U)$, with $1 < p < \infty$, the Dirichlet problem $Lu = f$ in U, $u - \phi \in W^{1,p}_0(U)$ has a unique solution $u \in W^{2,p}(U)$.*

Proof. The treatment here to deduce the regularity result for the Dirichlet problem for strong solutions is based on the L^2 regularity. We shall need the following regularity result which is a refinement of Theorem 1.6 and Theorem 1.7.

Lemma 1.8. *In addition to the hypotheses of Theorem 1.7, suppose that $f \in L^q(U)$ for some $q \in (p, \infty)$. Then, $u \in W^{2,q}_{loc}(U \cup T)$, $u = 0$ on T in the sense of $W^{1,q}(U)$, and consequently, u satisfies the estimate (1.67) with p replaced by q.*

Proof. We first treat the interior case when T is empty. Returning to the proof of Theorem 1.6, we fix a ball $B_R = B_R(x_0)$ and a cutoff function η, and set $v = \eta u$, $g = a^{ij} D_{ij} v$, so that

$$L_0 v = (a^{ij}(x_0) - a^{ij}(x)) D_{ij} v + g,$$

and

$$\begin{aligned} a^{ij} D_{ij} v &= a^{ij} D_{ij}(\eta u) \\ &= a^{ij} D_i(D_j \eta u + \eta D_j u) \\ &= a^{ij} D_{ij}\eta u + a^{ij} D_j \eta D_i u + a^{ij} D_i \eta D_j u + a^{ij} \eta D_{ij} u \\ &= a^{ij} D_{ij}\eta u + a^{ij} D_j \eta D_i u + a^{ij} D_i \eta D_j u + \eta(b^i D_i u + cu - f) \end{aligned} \tag{1.68}$$

Since $u \in W^{2,p}(U)$, $\eta \in C_0^2(B_R)$, then all the terms but f in the last equation of (1.68) are in $W^{1,p}(B_R)$, moreover, $f \in L^q(U)$, it follows from the Sobolev embedding theorem that $g \in L^r(U)$ where $\frac{1}{r} = \max\{(1/q, (1/p) - (1/n))\}$. By means of the linear transformation, the operator L_0 becomes Laplacian, and hence

$$\Delta \tilde{v} = (\delta^{ij} - \tilde{a}^{ij}(x)) D_{ij}\tilde{v} + \tilde{g}.$$

where $\tilde{v}$, $\tilde{a}^{ij}$, $\tilde{g}$ correspond to v, a^{ij} g, respectively. By taking the Newtonian potential, we then obtain the equation

$$\tilde{v} = N[(\delta^{ij} - \tilde{a}^{ij}(x)) D_{ij}\tilde{v}] + N\tilde{g}.$$

Consequently, the function v satisfies an equation of the form

$$v = Tv + h. \tag{1.69}$$

where $h \in W^{2,r}(B_R)$. By virtue of the Calderón-Zygmund estimate (appendix A) T is a bounded linear mapping from $W^{2,p}(B_R)$ into itself for any $p \in (1, \infty)$. As in the proof of Theorem 1.6, $R \leq \delta$ we must have $\|T\| \leq \frac{1}{2}$. Therefore, (1.69) has a unique solution $v \in W^{2,p}(B_R)$ for any $p \in [1, r]$. In fact, if not, we assume there exists v_1, v_2 satifying (1.69) respectively, that is,

$$v_1 = Tv_1 + h, \quad v_2 = Tv_2 + h$$

then,

$$v_1 - v_2 = T(v_1 - v_2)$$

and the contraction mapping principle yields that $v_1 = v_2$. Because of $h \in W^{2,r}(B_R)$, the solution is in $W^{2,p}(B_R)$ for any $p \in [1, r]$. Hence, $\eta u \in W^{2,r}(U)$, and, since $x_0 \in U$ is arbitrary, we obtain $u \in W^{2,r}_{loc}(U)$. If now $r = q$, we are done. Otherwise, the desired interior regularity follows by using the Sobolev embedding theorem and repeating the above argument. The case of local boundary regularity is handled similarly with $x_0 \in T$ and the ball replaced by the half-ball $B_R^+(0)$ as in the proof of Theorem 1.7. $\square$

The uniqueness assertion of Theorem 1.8 follows from the following lemma and Lemma 1.8.

Lemma 1.9. *Let $Lu = f$ in a bounded domain U and $u \in C^0(\bar{U}) \cap W^{2,n}_{loc}(U)$. Then*

$$\sup_U u \leq \sup_{\partial U} u^+ + C\|f/\mathscr{D}^*\|_{L^n(U)}$$

where $\mathscr{D}$ denotes the determinant of $[a^{ij}]$ and $\mathscr{D}^* = \mathscr{D}^{1/n}$, C is a constant depending only on n, diam U and $\|b/\mathscr{D}^*\|_{L^n(U)}$.

Because of space limitation the details of the proof of Lemma 1.9 will not be dealt with here, one can refer to [36], Theorem 9.1.

Proof of the uniqueness. If the operator L satisfies the hypotheses of Theorem 1.8 and the functions $u, v \in W^{2,p}(U)$ satisfy $Lu = Lv$ in U, $u - v \in W_0^{1,p}(U)$, we have by lemma 1.8, $u - v \in W^{2,q}(U) \cap W_0^{1,q}(U)$ for all $1 < q < \infty$. Now using Lemma 1.9, we conclude $u = v$. $\square$

From the uniqueness, we can derive an apriori bound which is independent of u.

Lemma 1.10. *Let the operator L satisfy the hypotheses of Theorem 1.8. Then there exists a constant C such that*

$$\|u\|_{W^{2,p}(U)} \leq C\|Lu\|_{L^p(U)} \tag{1.70}$$

for all $u \in W^{2,p}(U) \cap W_0^{1,p}(U), 1 < p < \infty$.

Proof. We argue by contradiction. If (1.70) is not true, there must exist a sequence $\{v_m\} \subset W^{2,p}(U) \cap W_0^{1,p}(U)$ satisfying

$$\|v_m\|_{L^p(U)} = 1; \quad \|Lv_m\|_{L^p(U)} \to 0.$$

By virtue of the a priori estimate (Theorem 1.7)

$$\|v\|_{W^{2,p}(U)} \leq C(\|v\|_{L^p(U)} + \|Lv\|_{L^p(U)}) \leq C$$

and the weak compactness of bounded sets in $W^{2,p}(U)$, there exists a sub-sequence, which we relabel as $\{v_m\}$, converging weakly to a function $v \in W^{2,p}(U) \cap W_0^{1,p}(U)$ satisfying $\|v\|_{L^p(U)} = 1$. Since

$$\int_U gD^\alpha v_m \to \int_U gD^\alpha v$$

for all $|\alpha| \leq 2$ and $g \in L^{p/(p-1)}(U)$, we must have

$$\int_U gLv = 0$$

for all $g \in L^{p/(p-1)}(U)$; hence $Lv = 0$ and $v = 0$ by the uniqueness assertion, which contradicts the condition $\|v\|_{L^p(U)} = 1$. $\square$

We are now in a position to prove Theorem 1.8. If $p \geq 2$, say $f \in L^p, p \in [2, \infty)$, due to L^2 regularity theory, we know $u \in W^{2,2}(U)$, then by lemma 1.8, we have $u \in W^{2,p}(U)$. In the case $1 < p < 2$, taking $\{f_m\} \subset L^2(U)$ such that $f_m \to f$ in $L^p(U)$ and $-\Delta u_m = f_m$, $u_m = 0$ on the boundary of U. By L^2 regularity theory again, we have $u_m \in W^{2,2}(U)$. Thus $u_m \in W^{2,p}(U) \cap W_0^{1,p}(U)$ since $1 < p < 2$. We then infer from Lemma 1.10 that

$$\|u_m\|_{W^{2,p}(U)} \leq C\|f_m\|_{L^p(U)} \leq C,$$

because $f_m \to f$ in $L^p(U)$. Consequently, there exists a sub-sequence of $\{u_m\}$ converging weakly to a function u in $W^{2,p}(U) \cap W_0^{1,p}(U)$ that satisfies $Lu = f$ in U. $\square$

1.4 Maximum Principle

1.4.1 The weak maximum principle

Theorem 1.9 ([25], chapter 6.4.1). *Assume $u \in C^2(U) \cap C(\bar{U})$ and*

$$c \equiv 0 \quad \text{in } U.$$

(i) If

$$Lu \leq 0 \quad \text{in } U, \tag{1.71}$$

then

$$\max_{\bar{U}} = \max_{\partial U} u.$$

(i) If

$$Lu \geq 0 \quad \text{in } U,$$

then

$$\min_{\bar{U}} = \min_{\partial U} u.$$

Remark 1.10. *It is convenient to introduce the following terminology suggested by the maximum principle: a function satisfying $Lu = 0$ $(\geq 0, \leq 0)$ in U is a solution (subsolution, supersolution) of $Lu = 0$ in U. When L is the Laplacian, these terms correspond respectively to harmonic, sub-harmonic and super-harmonic functions.*

Proof. 1. First suppose $Lu < 0$ in U and there exists a point $x_0 \in U$ with $u(x_0) = \max_{\bar{U}} u$. At this maximum point x_0, we have $Du(x_0) = 0$, $D^2u(x_0) \leq 0$. Since L is elliptic, $[a^{ij}(x_0)]$ is symmetric and positive definite, and can be diagonalized through some orthogonal matrix such that at x_0, $-\sum_{i,j=1}^n a^{ij} u_{x_i} u_{x_j} \geq 0$. Thus at x_0, $Lu(x_0) = -a^{ij}(x_0) D_{ij} u(x_0) \geq 0$, contradicts the condition $Lu < 0$.

2. In the general case that (1.71) holds, write

$$u^\epsilon(x) := u(x) + \epsilon e^{\gamma x_1} \quad (x \in U),$$

where $\gamma > 0$ will be selected below and $\epsilon > 0$. Recall that the uniform ellipticity condition implies $a^{ii}(x) \geq \theta > 0$ $(i = 1, \cdots, n, x \in U)$. Therefore

$$\begin{aligned} Lu^\epsilon &= Lu + \epsilon L(e^{\gamma x_1}) \\ &\leq \epsilon e^{\gamma x_1}(-\gamma^2 a^{11} + \gamma b^1) \\ &\leq \epsilon e^{\gamma x_1}(-\gamma^2 \theta + \|b\|_{L^\infty} \gamma) \\ &< 0 \text{ in } U, \end{aligned}$$

provided we choose $\gamma > 0$ sufficiently large. Hence for any $\epsilon > 0$, $L(u + \epsilon e^{\gamma x_1}) < 0$ in U so that according to step 1

$$\max_{\bar{U}}(u + \epsilon e^{\gamma x_1}) = \max_{\partial U}(u + \epsilon e^{\gamma x_1}).$$

Letting $\epsilon \to 0$, we see that $\max_{\bar{U}} u = \max_{\partial U} u$ as asserted in the theorem.

3. Since $-u$ is a sub-solution whenever u is a super-solution, assertion (ii) follows. □

Next, we present the following theorem by modifying the maximum principle to allow for a non-negative zeroth-order coefficient c. Remember that $u^+ = \max(u, 0)$, $u^- = -\min(u, 0)$.

Theorem 1.10 (Weak maximum principle for $c \geq 0$, [25], chapter 6.4.1)**.** *Assume $u \in C^2(U) \cap C(\bar{U})$, and $c \geq 0$ in U.*

(i) If

$$Lu \leq 0 \quad \textit{in } U,$$

then

$$\max_{\bar{U}} u \leq \max_{\partial U} u^+. \tag{1.72}$$

(ii) Likewise, if

$$Lu \geq 0 \quad \textit{in } U,$$

then

$$\min_{\bar{U}} u \geq -\max_{\partial U} u^-.$$

Remark 1.11. *So in particular, if $Lu = 0$ in U, then*

$$\max_{\bar{U}} |u| = \max_{\partial U} |u|.$$

Proof. Let u be a sub-solution and set $V := \{x \in U \mid u(x) > 0\}$. Then

$$Ku := Lu - cu \leq -cu \leq 0 \quad \text{in } V.$$

The operator K has no zeroth-order term and consequently Theorem 1.9 implies that the maximum of u on $\bar{V}$ must be achieved on ∂V and also on ∂U, hence

$$\max_{\bar{V}} u = \max_{\partial V} u = \max_{\partial U} u^+.$$

This gives (1.72) in the case that $V \neq \emptyset$. Otherwise $u \leq 0$ everywhere in U, and (1.72) follows likewise.

Assertion (ii) follows from (i) applied to $-u$. □

1.4.2 The strong maximum principle

The next lemma substantially strengthens the foregoing assertions, by demonstrating that a sub-solution u cannot attain its maximum at an interior point of a connected region at all, unless u is constant. This statement is the *strong maximum principle*, which depends on the following subtle analysis of the outer normal derivative $\frac{\partial u}{\partial \nu}$ at a boundary maximum point.

Lemma 1.11 (Hopf's Lemma, [25], chapter 6.4.2)**.** *Assume $u \in C^2(U) \cap C^1(\bar{U})$ and*

$$c \equiv 0 \quad in\ U.$$

Suppose further

$$Lu \leq 0 \quad in\ U$$

and there exists a point $x^0 \in \partial U$ such that

$$u(x^0) > u(x) \quad for\ all\ x \in U. \tag{1.73}$$

Assume finally that U satisfies the interior ball condition at x^0; that is, there exists an open ball $B \subset U$ with $x^0 \in \partial B$.

(i) Then

$$\frac{\partial u}{\partial \nu}(x^0) > 0,$$

where ν is the outer unit normal to B at x^0.

(ii) if

$$c \geq 0 \quad in\ U,$$

the same conclusion holds provided

$$u(x_0) \geq 0.$$

Remark 1.12. *The importance of (i) is the strict inequality: that $\frac{\partial u}{\partial \nu}(x^0) \geq 0$ is obvious. Note that the interior ball condition automatically holds if ∂U is C^2.*

Proof. 1. Assume $c \geq 0$. We may further assume $B = B^0(0, r)$ for some radius $r > 0$. Define

$$v(x) := e^{-\gamma |x|^2} - e^{-\gamma r^2} \qquad (x \in B(0, r))$$

for $\gamma > 0$ as selected below. Using the uniform ellipticity condition, we compute

$$\begin{aligned} Lv &= -\sum_{i,j=1}^{n} a^{ij} v_{x_i x_j} + \sum_{i=1}^{n} b^i v_{x_i} + cv \\ &= e^{-\gamma|x|^2} \sum_{i,j=1}^{n} a^{ij}(-4\gamma^2 x_i x_j + 2\gamma \delta_{ij}) - e^{-\gamma|x|^2} \sum_{i=1}^{n} b^i 2\gamma x_i + c(e^{-\gamma|x|^2} - e^{-\gamma r^2}) \\ &\leq e^{-\gamma|x|^2}(-4\theta\gamma^2|x|^2 + 2\gamma tr\mathbf{A} + 2\gamma|\mathbf{b}||x| + c), \end{aligned}$$

for $\mathbf{A} = [a^{ij}]$, $\mathbf{b} = [b^1, \cdots, b^n]$. Consider next the open annular region $R := B^0(0,r) - B(0,r/2)$. We have

$$Lv \leq e^{-\gamma|x|^2}(-\theta\gamma^2 r^2 + 2\gamma tr\mathbf{A} + 2\gamma|\mathbf{b}|r + c) \leq 0 \tag{1.74}$$

in R, provided $\gamma > 0$ is fixed large enough.

2. In view of (1.73), there exists a constant $\epsilon > 0$ so small that

$$u(x^0) \geq u(x) + \epsilon v(x), \quad x \in \partial B(0, r/2). \tag{1.75}$$

In addition,

$$u(x^0) \geq u(x) + \epsilon v(x), \quad x \in \partial B(0, r), \tag{1.76}$$

since $v \equiv 0$ on $\partial B(0,r)$.

From (1.74) we see

$$L(u + \epsilon v - u(x^0)) \leq -cu(x^0) \leq 0 \ \text{ in } R,$$

and from (1.75), (1.76) we observe

$$u + \epsilon v - u(x^0) \leq 0 \ \text{ on } \partial R.$$

In view of the weak maximum principle Theorem 1.10, $u + \epsilon v - u(x^0) \leq 0$ in R. But $u(x^0) + \epsilon v(x^0) - u(x^0) = 0$, and so

$$\frac{\partial u}{\partial \nu}(x^0) + \epsilon \frac{\partial v}{\partial \nu}(x^0) \geq 0.$$

Consequently,

$$\frac{\partial u}{\partial \nu}(x^0) \geq -\epsilon \frac{\partial v}{\partial \nu}(x^0) = -\frac{\epsilon}{r} Dv(x^0) \cdot x^0 = 2\gamma\epsilon r e^{-\gamma r^2} > 0$$

as required. □

We are now in a position to derive the following *strong maximum principle.*

Theorem 1.11 ([25], chapter 6.4.2). *Assume $u \in C^2(U) \cap C(\bar{U})$ and $c \equiv 0$ in U. Suppose U is connected, open and bounded.*

(i) If

$$Lu \leq 0 \quad \text{in } U$$

and u attains its maximum over $\bar{U}$ at an interior point, then u is constant within U.

(ii) Similarly, if

$$Lu \geq 0 \quad \text{in } U$$

and u attains its minimum over $\bar{U}$ at an interior point, then u is constant within U.

Proof. Write $M := \max_{\bar{U}} u$ and $C := \{x \in U \mid u(x) = M\}$. Then if $u \neq M$, set

$$V := \{x \in U \mid u(x) < M\}.$$

Choose a point $y \in V$ satisfying $\text{dist}(y, C) < \text{dist}(y, \partial U)$, and let B denote the largest ball with center y whose interior lies in V. Then there exists some point $x^0 \in C$, with $x^0 \in \partial B$. Clearly V satisfies the interior ball condition at x^0, whence Hopf's Lemma, (i), implies $\frac{\partial u}{\partial \nu}(x^0) > 0$. But this is a contradiction: since u attains its maximum at $x^0 \in U$, we have $Du(x^0) = 0$. □

If the zeroth-order term c is non-negative, we have this version of the strong maximum principle:

Theorem 1.12 ([25], chapter 6.4.2)**.** *Assume $u \in C^2(U) \cap C(\bar{U})$ and $c \geq 0$ in U. Suppose also that U is connected.*

(i) If

$$Lu \leq 0 \quad \textit{in } U$$

and u attains a non-negative maximum over $\bar{U}$ at an interior point, then

$$u \textit{ is constant within } U.$$

(ii) Similarly, if

$$Lu \geq 0 \quad \textit{in } U$$

and u attains a non-positive minimum over $\bar{U}$ at an interior point, then

$$u \textit{ is constant within } U.$$

.

The proof of Theorem 1.12 is like the one above, except that we use statement (ii) in Hopf's Lemma.

1.5 Eigenvalues and Eigenfunctions of the Laplacian

For this section, one can refer to [50]. The classical eigenvalue problem for the Laplace operator $\Delta := \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ is a problem as follows:

$$\begin{cases} \Delta u + \lambda u = 0, & \text{in } U \\ u = 0, & \text{on } \partial U \end{cases}$$

where u is a sufficiently smooth real valued function, $u : U \to R$ and $x_1, x_2, \cdots, x_n$ are the coordinates for a bounded domain $U \subset \mathbb{R}^n$. We look for pairs (λ, u) consisting of a real number λ called an eigenvalue of the Laplacian and a function $u \in C^2(U)$ called an eigenfunction. Such eigenvalue, eigenfunction pairs have some very nice properties, one fact of particular interest is that they form an orthonormal basis for $L^2(U)$. In particular, we deal with solutions u in the Sobolev space $H_0^1(U)$ that obey the following equation for all test functions $v \in H_0^1(U)$:

$$\int_U \nabla u \nabla v dx = \lambda \int_U uv dx,$$

and we refer this equation as "*weak eigenvalue equation*".

Lemma 1.12 ([50], Lemma 2.1)**.** *If u_1 and u_2 are eigenfunctions with eigenvalues λ_1 and λ_2 respectively and if $\lambda_1 \neq \lambda_2$ then $\langle u_1, u_2 \rangle_2 = 0$ and moreover $\langle \nabla u_1, \nabla u_2 \rangle_2 = 0$.*

Proof. Since u_1 and u_2 are both eigenfunctions, they satisfy the eigenvalue equation by definition. Plugging in $v = u_2$ into the eigenvalue equation for u_1 and $v = u_1$ into the eigenvalue equation for u_2 gives

$$\int_U \nabla u_1 \cdot \nabla u_2 dx = \lambda_1 \int_U u_1 u_2 dx$$

$$\int_U \nabla u_2 \cdot \nabla u_1 dx = \lambda_2 \int_U u_2 u_1 dx.$$

Subtracting the second equation from the first gives

$$(\lambda_1 - \lambda_2) \int_U u_2 u_1 dx = 0,$$

so the condition $\lambda_1 \neq \lambda_2$ allows us to conclude $\int_U u_2 u_1 = \langle u_1, u_2 \rangle_2 = 0$ as desired. Finally, notice that $\langle \nabla u_1, \nabla u_2 \rangle = \int_U \nabla u_1 \nabla u_2 = \lambda_1 \int_U u_1 u_2 dx = 0$, too. $\square$

Consider now the functionals from $H_0^1(U) \to \mathbb{R}$,

$$F(u) = \int_U |\nabla u|^2 dx = \|\nabla u\|^2_{L^2(U)},$$

$$G(u) = \int_U u^2 dx - 1 = \|u\|^2_{L^2(U)} - 1.$$

These functionals have an intimate relationship with the eigenvalue problem. The following results makes this precise.

Lemma 1.13 ([50], Lemma 2.2)**.** *If $u \in H_0^1(U)$ is a local extremum of the functional F subject to the condition $G(u) = 0$, then u is an eigenfunction with eigenvalue $\lambda = F(u)$.*

Proof. The proof of this relies on the Lagrange multiplier theorem in the calculus of variations setting. The Lagrange multiplier theorem states that if F and G are C^1-functionals on a Banach space X, and if $x \in X$ is a local extremum for the functional F subject to the condition that $G(x) = 0$ then either $G'(x)y = 0$ for all $y \in X$ or there exists some $\lambda \in \mathbb{R}$ so that $F'(x)y = \lambda G'(x)y$ for all $y \in X$.

We use this theorem with the space $H_0^1(U)$ serving the role of our Banach space, and F, G as defined above playing the role of the functionals under consideration. We compute

$$\begin{aligned}
F'(u)v &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}(F(u+\epsilon v) - F(u)) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\bigg(\int_U |\nabla u + \epsilon \nabla v|^2 dx - \int_U |\nabla u|^2 dx \bigg) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\bigg(\int_U \left(|\nabla u|^2 + 2\epsilon \nabla u \cdot \nabla v + \epsilon^2 |\nabla v|^2 - |\nabla u|^2\right) dx \bigg) \\
&= \lim_{\epsilon \to 0} \int_U \left(2\nabla u \cdot \nabla v dx + \epsilon |\nabla v|^2\right) dx \\
&= 2\int_U \nabla u \cdot \nabla v \, dx.
\end{aligned}$$

A similar calculation yields

$$G'(u)v = 2\int_U uv \, dx.$$

Notice that $G'(u)u = 2\|u\|_{L^2(U)} = 2$ by the constraint $G(u) = 0$. This means that $G'(u)v$ is not identically zero for all $v \in H_0^1(U)$. Hence, since u is given to be a local extremum of F subject to $G(u) = 0$, the Lagrange multiplier theorem tells us that there exists a λ so that for all $v \in H_0^1(U)$ we have

$$F'(u)v = 2\int_U \nabla u \cdot \nabla v dx = \lambda G'(u)v = 2\lambda \int_U uv dx.$$

Hence u is an eigenfunction of eigenvalue λ as desired. Moreover,

$$F(u) = \int_U |\nabla u|^2 dx = \lambda \int_U u^2 dx = \lambda,$$

since $G(u) = 0$ is given. □

Lemma 1.14 ([50], Theorem 2.3)**.** *There exists some $u \in H_0^1(U)$ so that u is a global minimum for F subject to the constraint $G(u) = 0$.*

Proof. Let us denote by $\mathscr{C}$ the constraint set we are working on, namely $\mathscr{C} = \{u \in H_0^1(U) : G(u) = 0\}$. Notice $\mathscr{C}$ is the set of unit L^2-norm functions. Let $I = \inf\{F(u) : u \in \mathscr{C}\}$ be the infimum of F taken over this constraint set. We will prove that this infimum is actually achieved at some point $u \in \mathscr{C}$. By the definition of an infimum, we can find a minimizing sequence $\{u_k\}_{k=1}^{\infty} \subset \mathscr{C}$ so that $\lim_{k\to\infty} F(u_k) = I$ for $k \in \mathbb{N}$, which implies, $F(u_k) = \int_U |\nabla u_k|^2 \leq I + 1 \leq C$.

Since $F(u_k)$ could be considered as the norm of $H_0^1(U)$, this shows that $\{u_k\}$ is a bounded sequence in $H_0^1(U)$. According to the Banach-Alaoglu theorem, there exists a sub-sequence which we still mark as $\{u_k\}$ for notational ease, converging to $\bar{u}$ weakly in $H_0^1(U)$. On the other hand, by the Rellich-Kondrachov embedding theorem, we know $u_k \to \bar{u}$ in $L^2(U)$. Since $H_0^1(U)$ is a Hilbert space, and the norm of the Hilbert space is weakly lower semi-continuous, we have $F(\bar{u}) \leq \lim_{k\to\infty} \inf F(u_k) = \lim_{k\to\infty} F(u_k) = I$. Moreover, using the fact that $\|\bar{u}\|_{L^2(U)} = \lim_{k\to\infty} \|u_k\|_{L^2(U)} = 1$, we can see that $\bar{u} \in \mathscr{C}$. Since $I = \inf\{F(u) : u \in \mathscr{C}\}$, obviously $F(\bar{u}) \geq I$. Hence, $F(\bar{u}) = I$ achieves the minimum for F restricted to $\mathscr{C}$ as desired. □

Remark 1.13 ([50], Remark 2.4). *The lemma above shows that $\bar{u}$ is a global minimum of F subject to $G(u) = 0$. In particular then, it is a local extremum for F subject to $G(u) = 0$, so applying the result of Lemma 1.13 informs us that u is an eigenfunction with eigenvalue $\lambda = F(u)$. Since this is the smallest possible value of F subject to $G(u) = 0$, this is the smallest possible eigenvalue one could obtain. For this reason we shall call this eigenvalue λ_1 and the associated eigenfunction u_1.*

Remark 1.14 ([50], Remark 2.5). *By the definition of F, we notice that for any $u \in H_0^1(U)$ and any scalar $c \in \mathbb{R}$, we have $F(cu) = c^2F(u)$. This homogeneity for scalars means that we can remove the condition $G(u) = 0$ from our consideration by normalizing F by $\|u\|_{L^2(U)}$. Notice that*

$$\begin{aligned} F\Big(\frac{u}{\|u\|_{L^2(U)}}\Big) &= \int_U |\nabla\Big(\frac{u}{\|u\|_{L^2(U)}}\Big)|^2 dx \\ &= \frac{\int_U |\nabla u|^2}{\int_U |u|^2} dx. \end{aligned}$$

Since $\|\frac{u}{\|u\|_{L^2(U)}}\|_{L^2(U)} = 1$, minimizing $F(u)$ subject to $\|u\|_{L^2(U)} = 1$ is the same as minimizing the quotient $\frac{\int_U |\nabla u|^2 dx}{\int_U |u|^2 dx}$ with u running in all of $H_0^{1,2}(U)$. This quotient is known as the Rayleigh quotient. This gives us a more notationally concise way to write down our smallest eigenvalue

$$\lambda_1 = \inf_{u \in H_0^{1,2}(U)} \frac{\int_U |\nabla u|^2 dx}{\int_U |u|^2 dx}.$$

To find the next eigenvalue, we can do something very similar. We first notice that according to Lemma 1.12, the second smallest eigenvalue will have an eigenfunction that is orthogonal to u_1, so we can restrict the search for this eigenfunction to the subspace $X_1 = \text{span}\{u_1\}^{\perp} = \{u \in H_0^1(U) : \langle u, u_1 \rangle_2 = 0\}$. Since this is the null space of the continuous operator $\langle \cdot, u_1 \rangle_2$, this is a closed subspace of $H_0^1(U)$ and hence can be thought of as a Hilbert space in its own norm. By modifying the proof of Lemma 1.14 slightly by using X_1 as our Banach space rather than all $H_0^1(U)$, we see that any $u \in X_1$ that is a local extreme point for F subject to $G(u) = 0$ will be an eigenfunction of eigenvalue $\lambda = F(u)$. By modifying the argument of Lemma 1.14 slightly by changing the restriction set $\mathscr{C}$ to be $\mathscr{C} = \{u \in X_1 : G(u) = 0\}$, the identical argument shows that there is some $u \in \mathscr{C}$ that achieves the minimum for F on this restricted set. This will be an extremum for F on X_1 subject to the restriction $G(u) = 0$, so by modified Lemma 1.13 this will be an eigenfunction, call it u_2. By arguments similar to the above, we find the associated eigenvalue λ_2 is

$$\begin{aligned} \lambda_2 &= \min\{F(u) : u \in \mathscr{C} \subset X_1\} \\ &= \inf_{u \in X_1} \frac{\int_U |\nabla u|^2 dx}{\int_U u^2 dx}. \end{aligned}$$

Since $X_1 \subset H_0^1(U)$, the Rayleigh quotient definition above tells us immediately that $\lambda_1 \leq \lambda_2$. Repeating this same idea inductively, we can define $X_n = \text{span}\{u_1, u_2, \cdots, u_n\}^\perp = \{u \in H_0^{1,2}(U) : \langle u, u_i \rangle = 0, \forall i \in 1, \cdots, n\}$ and by appropriately modifying Lemma 1.13 and Lemma 1.14 we will be able to justify the fact that the $(n+1)$th eigenvalue can be found by

$$\lambda_{n+1} = \inf_{u \in X_n} \frac{\int_U |\nabla u|^2 dx}{\int_U u^2 dx}.$$

Moreover we can always find a normalized eigenfunction u_{n+1} that achieves this lower bound. Since $H_0^1(U) \supset X_1 \supset X_2 \cdots$, we can see that this generates a sequence of eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots$ and eigenfunction u_1, u_2, u_3, $\cdots$ which are generated in such a way that they are all mutually orthogonal with respect to the $L^2(U)$ inner product. Moreover, these eigenfunctions have been normalized so that $\|u_n\|_{L^2(U)} = 1$ and also we have that $\|\nabla u_n\|_{L^2(U)} = \lambda_n \|u\|_{L^2(U)} = \lambda_n$. The following theorem shows that these eigenvalues tend to infinity.

Lemma 1.15 ([50], Theorem 2.6)**.** $\lim_{n\to\infty} \lambda_n = \infty$.

Proof. Since the sequence λ_i is non-decreasing, the only way that they could not tend to infinity is if they are bounded above. Suppose by contradiction that there is some constant M such that $\lambda_n < M$ for all $n \in \mathbb{N}$. Notice then that

$$\begin{aligned} \|\nabla u_n\|^2_{L^2(U)} &= \int_U \nabla u_n \cdot \nabla u_n dx \\ &= \lambda_n \int_U u^2 dx \\ &= \lambda_n \\ &\leq M, \end{aligned}$$

where we have used the eigenvalue equation with $v = u_n$ and the fact that $\|u_n\|_{L^2(U)} = 1$. Notice now that the sequence of eigenfunctions is bounded in $H_0^1(U)$. By the Rellich compactness theorem, we can find a convergent sub-sequence u_{n_k} converging to some element of $L^2(U)$. This sub-sequence, being convergent, is an L^2-Cauchy sequence, meaning in particular that

$\|u_{n_k} - u_{n_{k+1}}\|^2_{L^2(U)} \to 0$ as $n \to \infty$. But orthonormality of u_n prohibits this as we have

$$\begin{aligned}
\|u_{n_k} - u_{n_{k+1}}\|^2_{L^2(U)} &= \|u_{n_k}\|^2_{L^2(U)} - 2\langle u_{n_k}, u_{n_{k+1}}\rangle + \|u_{n_{k+1}}\|^2_{L^2(U)} \\
&= 1 - 0 + 1 \\
&= 2.
\end{aligned}$$

This contradiction shows that our original assumption that the eigenvalues are bounded above by some M is impossible. Since the eigenvalues are non-decreasing, this is enough to show $\lim_{n\to\infty} \lambda_n = \infty$, as desired. □

1.6 Some Degree Theory about Compact Mappings and Cones

1.6.1 Brouwer degree and Leray-Schauder degree

The topological degree (in short, degree) of a map is a classical tool which is very useful for solving functional equations. It was introduced by L. Brouwer for finite dimension and extended by J. Leray and J. Schauder to infinite dimension [3, 8, 49]. We will simply carry out the construction of the degree. We start from Brouwer degree, the finite-dimensional version of Leray–Schauder degree. Its context is euclidean spaces. We will list certain properties of it which we will need in order to extend to the Leray–Schauder degree theory in the following. However, Brouwer degree theory wasn't about fixed points at all. The existence property does not produce fixed points but instead zeros, that is, solutions to the equation $f(x) = \mathbf{0} \in \mathbb{R}^n$. There's no mystery about the connection between zeros and fixed points. Leray–Schauder theory seeks conditions that imply that a map f has a fixed point, that is, $f(x) = x$. It takes place in the setting of a map from a subset of a normed linear space X instead of a euclidean space $\mathbb{R}^n$. Moreover, an important hypothesis will be added to the map f itself, namely, that it is a compact map.

Brouwer degree and its properties:

We will just give an outline of the procedure usually followed to define the degree. For details about the standard construction, see the references given at [3, 8].

Let us assume that:

(a) U is an open bounded set in $\mathbb{R}^n$, with boundary ∂U;

(b) f is a continuous map from $\bar{U}$ to $\mathbb{R}^n$;

(c) p is a point in $\mathbb{R}^n$ such that $p \notin \partial U$.

To each triple (f, U, p) satisfying $(a) - (c)$, one can associate an integer $\deg(f, U, p)$, called degree of f. First one considers a C^1 map f and a regular value p. Let us recall that, by definition, p is said to be a regular value for f , if the Jacobian $J_f(x)$ is different from zero for every $x \in f^{-1}(p)$. The Jacobian is the determinant of the matrix $f'(x)$. If p is a regular value then the set $f^{-1}(p)$ is finite and one can define the degree by setting

$$\deg(f, U, p) = \sum_{x \in f^{-1}(p)} \operatorname{sgn} J_f(x). \tag{1.77}$$

In order to extend the preceding definition to any continuous function f and any point p, one uses an approximation procedure. First, in order to approximate p with regular values p_k one applies the Sard theorem.

Theorem 1.13 (Sard theorem [27])**.** *Let $f \in C^1(U, \mathbb{R}^n)$ and set $\mathcal{S}_f = \{x \in U : J_f(x) = 0\}$. Then $f(\mathcal{S}_f)$ is a set of zero measure.*

For a proof, see Lemma 1.1.4 [27]. The set $\mathcal{S}_f$ is called the set of singular points of f. Any u such that $f(u) = p$ is called a non-singular solution of the equation $f = p$, provided $u \notin \mathcal{S}_f$. According to the Sard theorem, there exists a sequence $p_k \notin \mathcal{S}_f$, such that $p_k \to p$. Hence it makes sense to consider the $\deg(f, U, p_k)$, given by (1.77). Moreover, one can show that, for k large, $\deg(f, U, p_k)$ is a constant which is independent of the approximating sequence p_k. Hence one can define the degree of $f \in C^1(U, \mathbb{R}^n) \cap C(\bar{U}, \mathbb{R}^n)$ at any p by setting $\deg(f, U, p) = \lim\limits_{k\to\infty} \deg(f, U, p_k)$. Similarly, for general continuous $f \in C(\bar{U}, \mathbb{R}^n)$, let $f_k \in C^1(U, \mathbb{R}^n) \cap C(\bar{U}, \mathbb{R}^n)$ be such that $f_k \to f$ uniformly on $\bar{U}$. If k goes large, then any (f_k, U, p) satisfies (a)–(c) and one can consider the degree $deg(f_k, U, p)$. Once more, one can show that $\lim \deg(f_k, U, p)$ does not depend upon the choice of the sequence f_k and thus one can define the degree of f by setting $\deg(f, U, p) = \lim\limits_{k\to\infty} \deg(f_k, U, p)$.

Given f defined as above, we define the following properties:

(1) (Normality) $\deg(I, U, p) = 1$ if and only if $p \in U$, where I denotes the identity mapping;

(2) (Solvability) if $\deg(f, U, p) \neq 0$, then $f(x) = p$ has a solution in U;

(3) (Homotopy) If $f_t(x) : [0,1] \times \bar{U} \to \mathbb{R}^n$ is continuous and $p \notin \cap_{t\in[0,1]} f_t(\partial U)$, then $\deg(f_t, U, p)$ does not depend on $t \in [0,1]$;

(4) (Additivity) Suppose that U_1, U_2 are two disjoint subsets of U and $p \notin f(\bar{U} \setminus (U_1 \cup U_2))$. Then $\deg(f, U, p) = \deg(f, U_1, p) + \deg(f, U_2, p)$;

(5) (Excision property) Let U_0 be an open subset of U and $p \notin f(\bar{U} \setminus (U_0))$, then $\deg(f, U, p) = \deg(f, U_0, p)$.

Next, we will define the Leray-Schauder degree, namely the degree for maps $f \in C(X, X)$, where X is a Banach space and $f : \bar{U} \to X$ is a compact perturbation of the identity $I = I_X$. This means that there is a compact map K in X that contains $f(\bar{U})$.

Leray-Schauder degree and its properties:

An important consequence of the compactness of $f : \bar{U} \to X$ is described by the following result. For a subset C of $\bar{U}$, let

$$r(C) = \inf\{\|x - f(x)\| : x \in C\}$$

where inf denotes the infimum (greatest lower bound) of the set.

The key to moving from the finite-dimensional world of the Brouwer degree to the more general Leray-Schauder degree is the following Schauder projection lemma. It says that although the concept of a normed linear space is quite general, the compact subsets within a normed linear space are surprisingly special: they are"almost" finite-dimensional.

Lemma 1.16 (Schauder Projection Lemma, Theorem 4.2 [8])**.** *Let K be a compact subset of normed linear space X, with metric d induced by the norm. Given $\epsilon > 0$, there exists a finite subset F of X and a map $P : K \to con(F)$, called the Schauder projection, such that $d(P(x), x) < \epsilon$ for all $x \in K$.*

An evident consequence of this lemma is the following lemma:

Lemma 1.17 (see Lemma 10.2 [8])**.** *Let X be a normed linear space and let K be a compact subset of X. Given $\epsilon > 0$, there exists a finite-dimensional subspace X_ϵ of X, the span of a finite ϵ-net for K, and a map $P_\epsilon : K \to X_\epsilon$ such that $\|P_\epsilon(x) - x\| < \epsilon$ for all $x \in K$.*

Now if we start with a compact map $f : \bar{U} \to K$, a compact set K containing $f(\bar{U})$ and $\epsilon > 0$, we have the composition

$$P_\epsilon f : \bar{U} \xrightarrow{f} f(\bar{U}) \subset K \xrightarrow{P_\epsilon} X_\epsilon.$$

We suppose that f has no fixed points on ∂U. Let $I : X \to X$ and $I_\epsilon : X_\epsilon \to X_\epsilon$ be the identity maps. Let $U_\epsilon = U \cap X_\epsilon$ and define $f_\epsilon : U_\epsilon \to X_\epsilon$ to be the restriction of $P_\epsilon f$ to $\bar{U}_\epsilon$. Furthermore, $I_\epsilon - f_\epsilon$ has no zeros on ∂U_ϵ. The Leray-Schauder degree $\deg(I - f, U)$ of $I - f$ on $\bar{U}$ is defined by

$$\deg(I - f, U) = \deg(I_\epsilon - f_\epsilon, U_\epsilon)$$

where the symbol on the right hand side is the Brouwer degree and we require $\epsilon < \frac{r}{2}$ $(r = r(\partial U))$ (see p73, [8]). But unlike the definition of the Brouwer degree, we made a number of choices in defining the Leray–Schauder degree that, potentially, could change the value of the degree. The following theorem will show that the Leray–Schauder degree is independent of those choices.

Theorem 1.14. *The definition of Leray-Schauder degree $\deg(I - f, U)$ is independent of the choices made: the compact set K containing $f(\bar{U})$, the positive number ϵ, the subspace X_ϵ, and the map $P_\epsilon : K \to X_\epsilon$, provided only that $\epsilon < \frac{r}{2}$.*

Proof. For a detailed proof, see Theorem 10.5 [8]. □

We will not list and demonstrate properties of the Leray–Schauder degree, properties which, basically, are consequences of the corresponding properties of the Brouwer degree. So it has the same properties (1)–(5) as the finite dimensional degree.

1.6.2 Fixed point theory

For the section below, one can refer to [26] chapter 3.

Let C be a closed convex subset of a Banach space X, and $W \subset C$ a relatively open subset of C, that is, $W = \mathcal{O} \cap C$ for some open subset $\mathcal{O}$ of X. Let $\phi : \bar{W} \to C$ be a compact mapping such that $\phi(x) \neq x$ for $\bar{W} \setminus W$. Associated with each such mapping we define an integer $i_C(\phi, W)$, called the *fixed point index* of ϕ, as follows. By a theorem of Dugundji [24] the mapping ϕ has a compact extension $\tilde{\phi} : \bar{\mathcal{O}} \to C$. Then define

$$i_C(\phi, W) = \deg(I - \tilde{\phi}, \mathcal{O}, 0), \tag{1.78}$$

where deg on the right side denotes Leray-Schauder degree. To see that this is in fact a good definition we have to settle the three following points:

(i) $\tilde{\phi} \neq x$ for all $x \in \partial\mathcal{O}$,

(ii) the degree in the right side of (1.78) is independent of the particular extension $\tilde{\phi}$,

(iii) it does not depend either on the particular open set $\mathcal{O}$.

These facts are easily proved using the homotopy invariance of the degree and the excision property. In fact,

(i) if $\tilde{\phi}(x) = x$ for $x \in \partial(\mathcal{O} \setminus C) \cap \partial\mathcal{O}$, this contradicted with $\tilde{\phi} : \bar{\mathcal{O}} \to C$ since $x \notin C$. Besides, $\phi(x) \neq x$ for $x \in \partial W$, $\partial\mathcal{O} = (\partial W \setminus \partial C) \cup (\partial(\mathcal{O} \setminus C) \cap \partial\mathcal{O})$ which implies $\tilde{\phi}(x) \neq x$ for $x \in \partial\mathcal{O}$.

(ii) Suppose there is another compact extension of ϕ, $\tilde{\phi}' : \bar{\mathcal{O}} \to C$. Let $H(t, x) = t\tilde{\phi} + (1-t)\tilde{\phi}'$ for $(t, x) \in [0, 1] \times \bar{W}$. Since $\tilde{\phi}$ and $\tilde{\phi}'$ are all compact mappings, then $H(t, x)$ is a compact mapping. By the definition, $\tilde{\phi}(x) = \tilde{\phi}'(x)$ for $x \in \partial W$, then $H(t, x) = \tilde{\phi}' \neq x$ for $x \in \partial W$ and $t \in [0, 1]$. Then by the homotopy invariance of the Leray-Schauder degree we know $i_C(\tilde{\phi}', W) = \deg(I - \tilde{\phi}', \mathcal{O}, 0) = i_C(H(0, x), W) = i_C(H(1, x), W) = \deg(I - \tilde{\phi}, \mathcal{O}, 0) = i_C(\tilde{\phi}, W)$.

(iii) Assume, there exists some other set M such that $W = M \cap C$ and the compact extension $\tilde{\phi}' : \bar{M} \to C$, since $\bar{M} \setminus W \notin C$, we see $\tilde{\phi}'(x) \neq x$, for $x \in \bar{M} \setminus W$. Moreover, $\tilde{\phi}'(x) \neq x$ for $\bar{W} \setminus W$, by the excision property of the degree, we have $\deg(I - \tilde{\phi}', W, 0) = \deg(I - \tilde{\phi}', M, 0)$, similarly, $\deg(I - \tilde{\phi}, \mathcal{O}, 0) = \deg(I - \tilde{\phi}, W, 0)$. By (ii), we have $\deg(I - \tilde{\phi}, W, 0) = \deg(I - \tilde{\phi}', W, 0) = \deg(I - \tilde{\phi}', M, 0) = \deg(I - \tilde{\phi}, \mathcal{O}, 0)$.

The usual properties of the Leray-Schauder degree are transferred immediately to the fixed point index. So we have the following properties.

I) Normalization. Let $\phi : \bar{W} \to W$ be a constant mapping, that is, $\phi(x) = a \in W$ for all $x \in \bar{W}$ and some fixed $a \in W$. Then $i_C(\phi, W) = 1$.

II) Additivity. Let W_1 and W_2 be two disjoint (relatively) open subsets of W, and $\phi : \bar{W} \to C$ a compact mapping such that $\phi(x) \neq x$ for all $x \in \bar{W} \setminus (W_1 \cup W_2)$. Then

$$i_C(\phi, W) = i_C(\phi, W_1) + i_C(\phi, W_2).$$

III) Homotopy invariance. Let $I \subset \mathbb{R}$ be a compact interval and $h : I \times \bar{W} \to C$ a compact mapping such that $h(t,x) \neq x$ for all $x \in \bar{W} \setminus W$ and all $t \in I$. Then $i_C(h(t,\cdot),W)$ =constant for $t \in I$.

IV) Excision. Let $V \subset W$ be relatively open, and $\phi : \bar{W} \to C$ be a compcat mapping such that $\phi(x) \neq x$ for $x \in \bar{W} \setminus V$. Then $i_C(\phi, V) = i_C(\phi, W)$.

V) Solution property. $i_C(\phi, W) \neq 0 \Longrightarrow \exists x \in W$ such that $\phi(x) = x$.

We shall apply the previous facts to the case when C is a cone. Let us recall that a cone C in a Banach space X is a closed subset of X such that

(i) if $x, y \in C$ and $\alpha, \beta \geq 0$, then $\alpha x + \beta y \in C$,

(ii) if $x \in C$ and $x \neq 0$, then $-x \notin C$.

Theorem 1.15 ([26], Theorem 3.1)**.** *Let C be a cone in a Banach space X, and $\phi : C \to C$ a compact mapping. Assume that there are real numbers r, $R > 0$ such that*

(1) $x \neq t\phi(x)$ for $0 \leq t \leq 1$ and $\|x\| = r, x \in C$,

(2) there exists a compact mapping $F : \bar{B}_R \times [0,+\infty) \to C$ such that $F(x,0) = \phi(x)$ for $\|x\| = R, F(x,t) \neq x$ for $\|x\| = R$ and $t \geq 0$, and $F(x,t) = x$ has no solution $x \in \bar{B}_R$ for $t \geq t_0$.

Then: i) (1) $\Longrightarrow i_C(\phi, B_r) = 1$ and ii) (2) $\Longrightarrow i_C(\phi, B_R) = 0$.

Proof. i) Let $H(t,x) = t\phi(x), t \in [0,1]$, since $\phi(x)$ is compact, then $H(t,x)$ is compact as well. Also since, $t\phi(x) \neq x$ for $t \in [0,1]$ and $x \in \partial B_r$, then $H(t,x) \neq x$ for $t \in [0,1]$ and $x \in \partial B_r$. Thus by the homotopy invariance of the degree, $i_C(\mathbf{0}, B_r) = \deg(I, B_r, 0) = i_C(\phi(x), B_r) = \deg(I - \phi(x), B_r, 0)$=1.

ii) Let us denote by $F_t : \bar{B}_R \times [0,+\infty) \to C$ the mapping $F_t = F(t,x)$. Since $F(t,x) \neq x$ for $x = R$ and $t \geq 0$, by the homotopy invariance of the degree, $i_C(F_t, B_R)$ =const, then the assumption $F(x,t) = x$ has no solution for $x \in \bar{B}_R$ for $t \geq t_0$ implies the constant is 0. On the other hand, suppose $H(t,x) = tF_0(x) + (1-t)\phi(x)$ $(0 \leq t \leq 1)$, apparently, $H(t,x) \neq x$ for $x \in \bar{B}_R \setminus B_R$, because $F_0(x) = \phi(x) \neq x$ for $x \in \partial B_R$ and all the $t \in [0,1]$. Therefore from the homotopy invariance property of the index, $i_C(\phi, B_R) = \deg(I - \phi, B_R, 0) = i_C(H(0,x), B_R) = i_C(H(1,x), B_R) = \deg(I - F_0, B_R, 0) = i_C(F_0, B_R) = 0$. □

A sufficient condition for (2) in Theorem 1.15:

(2') There exists $v \in C \setminus \{0\}$ such that $x \neq \phi(x) + tv$ for $\|x\| = R$ and $t \geq 0$.

Proof. Let $\mu = \sup\{\|\phi(x)\| : x \in C, \|x\| \leq R\}$ and take $t_0 > \frac{R+\mu}{\|v\|}$. Taking $F(t,x) = \phi(x) + \psi(t)v$, where

$$\psi(t) = \begin{cases} t, & 0 \leq t \leq t_0 \\ t_0, & t \geq t_0. \end{cases}$$

Now we prove $F(t,x)$ satisfies all the conditions in (2). It is easy to see that when $t=0$, $F(0,x)=\phi(x)$. Next we prove $F(x,t)\neq x$ for $\|x\|=R$ and $t\geq 0$. In fact, suppose $F(t,x)=x=\phi(x)+\psi(t)v$ for $\|x\|=R$, then $\|\phi(x)+\psi(t)v\|=\|x\|=R$. For $0\leq t\leq t_0$,

$$t\|v\|-\|\phi(x)\|\leq\|\phi(x)+\psi(t)v\|=R$$

that is,

$$t\|v\|\leq R+\|\phi\|\leq R+\sup_{\|x\|=R}\|\phi\|\leq R+\mu$$

since $0\leq t\leq t_0$, we have $t_0\leq\frac{R+\mu}{\|v\|}$ which contradicts with the assumption $t_0>\frac{R+\mu}{\|v\|}$. When $t\geq t_0$, similarly, we will get $t_0\leq\frac{R+\mu}{\|v\|}$ which is a contradiction as well. Likewise, $F(t,x)\neq x$ for $x\in\bar{B}_R$ and $t\geq t_0$. □

2 On a Class of Superlinear Elliptic Problems

2.1 Problem description

In this section, we introduce some work of Brezis and Turner [13] in which they considered a class of super-linear elliptic problems

$$\begin{cases} Lu &= g(x,u,Du), \quad x\in U \\ u &= 0, \quad\quad\quad\quad\; x\in\partial U \end{cases} \tag{2.1}$$

where U is a smooth, bounded domain in $\mathbb{R}^N$, and L is a linear elliptic operator having a maximum principle which will be specified in the proof and defined as:

$$Lu=-\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial u}{\partial x_j}\big)+\sum_{i=1}^{N}b_i(x)\frac{\partial u}{\partial x_i}+c(x)u$$

where a_{ij}, b_i and c are smooth and $\sum\limits_{i,j=1}^{N}a_{ij}\xi_i\xi_j\geq\nu\sum\limits_{i=1}^{N}\xi_i^2$ for some $\nu>0$. In addition, Du is the gradient of u, and g is a non-negative function which, with respect to the variable u, satisfies the following growth conditions: if λ_1 is the first eigenvalue of the formal adjoint L', then $u^{-1}g(x,u,p)$ is required to be less than λ_1 for u near zero and greater than λ_1 for u near ∞; moreover, with $\beta=\frac{N+1}{N-1}$, we suppose that $u^{-\beta}g(x,u,p)$ approaches to 0 as $u\to+\infty$. The main theorem is stated as below:

Theorem 2.1. *Let $g(x,u,p)$ be a continuous, non-negative function defined on $\bar{U}\times\mathbb{R}_+\times\mathbb{R}^N$ and suppose:*

1) $\lim\limits_{u\to+\infty}\frac{g(x,u,p)}{u}>\lambda_1$

2) $\lim\limits_{u\to+\infty} \frac{g(x,u,p)}{u^\beta} = 0, \quad \beta = \frac{N+1}{N-1}$

3) $\overline{\lim\limits_{u\to 0}} \frac{g(x,u,p)}{u} < \lambda_1$*, the three conditions holding uniformly for* $x \in \bar{U}, p \in \mathbb{R}^N$*. Then there exists a solution* $u > 0$ *of*

$$\left\{ \begin{array}{rcll} Lu &=& g(x,u,Du) & x \in U \\ u &=& 0 & x \in \partial U \end{array} \right. \tag{2.2}$$

such that $u \in W^{2,q}$ *for all* $q < \infty$*.*

For a g satisfying these conditions uniformly in its remaining variables, the existence of a positive solution was proved. The results obtained in [13] include those previously obtained for the problem (2.1) (see [51, 59, 60]) and the proofs of the required bounds are considerably simplified.

2.2 A priori bound for positive solutions

In what follows U denotes a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with a smooth boundary ∂U. We use $\delta = \delta(x)$ to denote the distance from x to ∂U. We will use C for a generic constant.

In what follows we let J denote the function satisfying

$$\left\{ \begin{array}{rcll} L'J &=& \lambda_1 J, & x \in U \\ J &=& 0, & x \in \partial U \end{array} \right.$$

where λ_1 is the lowest eigenvalue of L' and where J is normalized so that $\int_U J^2 dx = 1$. It is known that $J > 0$ in U and it follows from the strong maximum principle that $J(x) \geq C\delta(x)$ with $C > 0$. In fact, $\forall x_0 \in \partial U$, by Hopf's lemma,

$$\frac{\partial J(x_0)}{\partial \nu} < 0$$

where ν is the outer unit normal at x_0. That is, for $x = x_0 - \delta(x)\nu$

$$\frac{\partial J(x_0)}{\partial \nu} = \lim_{x\to x_0} \frac{J(x_0) - J(x)}{\delta(x)} = \lim_{\delta(x)\to 0} \frac{-J(x)}{\delta(x)} < 0.$$

So $\forall \epsilon_0 > 0$, $\exists\ \delta'(x_0)$ and C_1, such that when $\delta(x) < \delta'(x_0)$,

$$\frac{J(x)}{\delta(x)} \geq C_1 > 0. \tag{2.3}$$

Now let $f(x) = \dfrac{J(x)}{\delta(x)}$, $x = x_0 + s\nu$, $0 < s < \delta'(x_0)$, since $f(x)$ is continuous for $x \in U$, then for the $\epsilon_0 > 0$ above, $\exists\ \sigma(x_0)$, and for $|x - y| = |x_0 - y_0| < \sigma(x_0)$, we have

$$|f(x) - f(y)| = \left| \frac{J(x)}{\delta(x)} - \frac{J(y)}{\delta(y)} \right| < \epsilon_0 \tag{2.4}$$

where $y = y_0 + s\nu$, $y_0 \in \partial U$. Taking $\epsilon_0 = \frac{C_1}{2}$ and $\sigma'(x_0) = \min\{\delta'(x_0), \sigma(x_0)\}$, therefore, for $\delta(x) < \sigma'(x_0)$ and $|x - y| = |x_0 - y_0| < \sigma'(x_0)$, from (2.3) and (2.4) it follows that

$$\frac{J(y)}{\delta(y)} > \frac{J(x)}{\delta(x)} - \frac{C_1}{2} \geq C_1 - \frac{C_1}{2} = \frac{C_1}{2} > 0.$$

Since $\{\mathscr{N}_{\sigma'(x_0)}(x_0) \cap \partial U\}_{x_0 \in \partial U}$ ($\mathscr{N}_{\sigma'(x_0)}(x_0)$ is the neighborhood of x_0 with radius $\sigma'(x_0)$) can cover ∂U, and ∂U is compact, by finite covering theorem there exist finitely many points $\{x_n\}_{n=1}^N \in \partial U$ and $\sigma'(x_n)$ such that $\{\mathscr{N}_{\sigma'(x_n)} \cap \partial U\}_{n=1}^N$ is able to cover ∂U. Hence, we choose $\delta_{\min} = \min\{\sigma'(x_n)\}$, for $\delta(x) =$dist$\{x, \partial U\} < \delta_{\min}$, and get

$$J(x) > \frac{C_1}{2}\delta(x).$$

On the other hand, for $\delta(x) \geq \delta_{\min}$, we observe that $J(x)$ is continuous and $\{x \mid x \in \delta(x) \geq \delta_{\min}\}$ is compact, thus, there exists a constant C_2 such that

$$J(x) \geq C_2 \geq C_2 \cdot \frac{\delta(x)}{\operatorname{diam}\{U\}} = \frac{C_2}{\operatorname{diam}\{U\}} \cdot \delta(x).$$

In conclusion, for $x \in U$, there exists a constant $C = \min\{C_1, \frac{C_2}{\operatorname{diam}\{U\}}\} > 0$ such that $J(x) \geq C\delta(x)$.

The basic a priori bound that Brezis-Turner use is the following.

Theorem 2.2. *Let $f(x,u)$ be a continuous, non-negative function defined on $\bar{U} \times [0, \infty)$ and suppose:*

(1) $\varliminf_{u \to \infty} \frac{f(x,u)}{u} > \lambda_1$ *uniformly for $x \in \bar{U}$.*

(2) $\lim_{u \to \infty} \frac{f(x,u)}{u^\beta} = 0$ *uniformly for $x \in \bar{U}$, where $\beta = \frac{N+1}{N-1}$.*

Then there is a constant K such that if $u \in H_0^1(U)$ is non-negative and satisfies

$$\begin{cases} Lu &= f(x,u) + tJ \quad x \in U \\ u &= 0 \qquad\qquad\quad\; x \in \partial U \end{cases} \tag{2.5}$$

we have $u \in L^\infty$ and

$$\|u\|_{L^\infty} \leq K,$$

where K is independent of $t \geq 0$.

In the proof of the above theorem one uses the Hardy-Sobolev inequality (see Chapter 1.2, Proposition 1.1).

Proof. *Step 1:* There is a K_1 such that if u satisfies (2.5) for a $t \geq 0$, we have $t \leq K_1$ and

$$\int_U \delta(x) f(x,u) dx \leq K_1.$$

It follows from the first assumption on f in the theorem that there is a $K_0 > \lambda_1$ and a u_0 such that $f(x,u) \geq K_0 u$ for $u \geq u_0$. Since f is continuous for $0 \leq u \leq u_0$, then

$$f(x,u) \geq K_0 u - C, \quad \forall u \geq 0. \tag{2.6}$$

Multiplying J on both sides of (2.5) and integrating by parts we obtain

$$\lambda_1 \int_U uJdx = \int_U f(x,u)Jdx + \int_U tJ^2 dx. \tag{2.7}$$

Using (2.6) we have

$$\lambda_1 \int_U uJdx \geq K_0 \int_U uJdx - C + \int_U tJ^2 dx,$$

therefore

$$\int_U tJ^2 \leq (\lambda_1 - K_0) \int_U uJdx + C \leq C,$$

this implies that t is bounded. Then

$$(K_0 - \lambda_1) \int_U uJdx \leq - \int_U tJ^2 dx + C \leq C,$$

since $K_0 > \lambda_1$, it follows that $\int_U uJdx$ is bounded, then back to (2.7), and due to $J(x) \geq C\delta(x)$ we see

$$\int_U f\delta dx \leq \int_U fJdx \leq C = K_1.$$

Step 2: We show that under the assumptions of Theorem 2.2, there is a constant K_2 such that

$$\|u\|_{H^1} \leq K_2$$

for every non-negative solution of (2.5).

Multiplying (2.5) by u and using Gårding's inequality ([55], Theorem 9.17) we obtain

$$\nu' \|Du\|_{L^2}^2 \leq C\|u\|_{L^2}^2 + \int_U f(x,u)udx + K_1 \int_U Judx.$$

By (2.6), we know that

$$K_0 u^2 \leq f(x,u)u + Cu,$$

thus using Cauchy's inequality we get for $\epsilon > 0$

$$\begin{aligned}\nu' \|Du\|^2_{L^2} &\leq \int_U f(x,u) u dx + C\|u\|_{L^1} \\ &\leq \int_U f(x,u) u dx + \epsilon \int_U u^2 dx + C(\epsilon) \\ &\leq \int_U f(x,u) u dx + \epsilon \int_U |Du|^2 dx + C(\epsilon),\end{aligned}$$

where we have used the Poincaré inequality. Thus, choosing $\epsilon = \frac{\nu'}{2}$ we get

$$\frac{\nu'}{2}\|Du\|^2_{L^2} \leq \int_U f(x,u) u dx + C. \tag{2.8}$$

Now to estimate the first term on the right we write

$$\int_U f(x,u) u dx = \int_U (\delta^\alpha f^\alpha)(f^{1-\alpha} \cdot \frac{u}{\delta^\alpha}) dx$$

where $0 < \alpha < 1$ will be determined later. Applying Hölder's inequality to the right side, we observe

$$\int_U f(x,u) u dx \leq \|\delta f\|^\alpha_{L^1} (\int_U f \ \frac{u^{\frac{1}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}} dx)^{1-\alpha}. \tag{2.9}$$

From the first step we know that the first term of (2.9) is bounded. By hypothesis (2) of the theorem, for each $\epsilon > 0$, there is a C_ϵ such that $f(x,u) \leq \epsilon u^\beta + C_\epsilon$. Then

$$\int_U f(x,u) u dx \leq \epsilon \Big[\int_U \frac{u^{\beta + \frac{1}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}} \Big]^{1-\alpha} + C_\epsilon \Big[\int_U \frac{u^{\frac{1}{1-\alpha}}}{\delta^{\frac{\alpha}{1-\alpha}}} \Big]^{1-\alpha}.$$

Now choosing α such that $\frac{1}{1-\alpha} = \beta = \frac{N+1}{N-1}$, that is $\alpha = \frac{2}{N+1}$, we get from (2.8) and (2.9) we get

$$\frac{\nu'}{2}\|Du\|^2_{L^2} \leq \epsilon \left\| \frac{u}{\delta^{\alpha/2}} \right\|^2_{L^{\frac{2}{1-\alpha}}} + C_\epsilon \left\| \frac{u}{\delta^\alpha} \right\|_{L^{\frac{1}{1-\alpha}}} + C. \tag{2.10}$$

Applying Proposition 1.1 with $\tau = \alpha/2$ and $\tau = \alpha$ we have

$$\left\| \frac{u}{\delta^{\frac{\alpha}{2}}} \right\|_{L^q} \leq C\|Du\|_{L^2}$$

and

$$\left\| \frac{u}{\delta^\alpha} \right\|_{L^r} \leq C\|Du\|_{L^2}$$

where $\frac{1}{q} = \frac{1}{2} - \frac{1-\alpha/2}{N}$ and $\frac{1}{r} = \frac{1}{2} - \frac{1-\alpha}{N}$, that is $q = \frac{2}{1-\alpha} = \frac{2(N+1)}{N-1}$ and $r = \frac{2N(N+1)}{N^2-N+2} > \frac{1}{1-\alpha}$. We can then conclude from (2.10) that

$$\|Du\|_{L^2} \leq C.$$

Step 3: First we claim $u \in L^\infty(U)$. This can be derived by a bootstrap argument. Since $u \in H_0^1(U)$, it follows from the Sobolev embedding theorem that $u \in L^{2^*}(U)$, where $2^* = \frac{2N}{N-2}$. It is readily seen that $u^\beta \in L^{\frac{2N(N-1)}{(N-2)(N+1)}}$. Then from $f(x,u) \leq \epsilon u^\beta + C_\epsilon$ we see that

$$\|f\|_{L^{\frac{2N(N-1)}{(N-2)(N+1)}}} \leq \|\epsilon u^\beta + C_\epsilon\|_{L^{\frac{2N(N-1)}{(N-2)(N+1)}}} \leq \epsilon \|u^\beta\|_{L^{\frac{2N(N-1)}{(N-2)(N+1)}}} + C \leq C,$$

So $f \in L^{\frac{2N(N-1)}{(N-2)(N+1)}}$. Note that u is a solution of (2.5), then according to L^p theory we have $u \in W^{2,\frac{2N(N-1)}{(N-2)(N+1)}}$ and

$$\|u\|_{W^{2,\frac{2N(N-1)}{(N-2)(N+1)}}} \leq \|f\|_{L^{\frac{2N(N-1)}{(N-2)(N+1)}}} + C.$$

Using the Sobolev embedding theorem again, then $u \in L^{(\frac{2N(N-1)}{(N-2)(N+1)})^*}$. Observe that $\frac{2N(N-1)}{(N-2)(N+1)} > 2$, thus $(\frac{2N(N-1)}{(N-2)(N+1)})^* > 2^*$, which implies that u is now in a better L^p space, in return, u^β will be in a better L^p space, so is f, then getting back to the regularity theory, u will be in a space better than $L^{(\frac{2N(N-1)}{(N-2)(N+1)})^*}$. By iteration procedure, finally, $u \in L^\infty(U)$.

Since $u \in L^\infty$, we have

$$\|u\|_{L^\infty} \leq C\|u\|_{W^{2,p}} \leq C\|f(x,u) + K_1 J\|_{L^p}$$

for any $p > N/2$. In particular, since $N(\beta - 1) < 2^*$, we choose $N < p < 2^*/(\beta - 1)$, then

$$\begin{aligned}
\|u\|_{L^\infty} &\leq \epsilon \|u^\beta\|_{L^p} + C_\epsilon \\
&= \epsilon \Big(\int_U u^p \cdot u^{p\beta - p} dx\Big)^{\frac{1}{p}} + C_\epsilon \\
&\leq \epsilon \|u\|_{L^\infty} \Big(\int_U u^{p\beta - p}\Big)^{\frac{1}{p}} + C_\epsilon \\
&= \epsilon \|u\|_{L^\infty} \|u\|^{\beta-1}_{L^{p(\beta-1)}} + C_\epsilon \\
&\leq \epsilon \|u\|_{L^\infty} \|Du\|^{\beta-1}_{L^2} + C_\epsilon.
\end{aligned}$$

It follows from the second step that $\|Du\|_{L^2} \leq C$, and then choosing $\epsilon > 0$ such that $\epsilon \cdot C \leq \frac{1}{2}$, we conclude that

$$\|u\|_{L^\infty} \leq K.$$

□

2.3 The existence of solution

Proof of Theorem 2.1. We can always assume that L satisfies the strong maximum principle as well as the version given by Stampacchia ([56], P.1, P.18, Theorem 3.1) for $H_0^1(U)$ solutions. We shall find the solution in C^1. For $u \geq 0$ in C^1 let $w = F(u)$ be the solution of $Lw = g(x, u, Du), w = 0$ on ∂U. If there exists a positive fixed point of F such that $F(u) = u$, then this point satisfies (2.2). We complete the proof of Theorem 2.1 by applying the following fixed point theorem which is an variant of Theorem 1.15. □

Theorem 2.3. *(see [26], Theorem 3.1 or [37]) A compact mapping F acting in the cone of nonnegative functions will have a fixed point u with $0 < r \leq \|u\|_{C^1} \leq R < \infty$ provided*

1) $F(u) \neq su, s \geq 1$ for $\|u\|_{C^1} = r$ and

2) $F(u) \neq u - t\tilde{J}, t \geq 0$, for $\|u\|_{C^1} = R$,

where $\tilde{J} = L^{-1}J > 0$.

To see that the mapping F given about is a compact mapping, one first verifies that w depends continuously on u. Further, for u in a bounded set in C^1, by the second condition of Theorem 2.1, $g(x, u, Du)$ lies in a bounded set in L^∞, then using the L^p theory again, w lies in a bounded set in $W^{2,q}$, and hence by compact embedding theorem in a compact set in C^1 for $q > N$. Moreover, by the maximum principle for H_0^1 ([57], B.6), $w \geq 0$. In fact, for $u \neq 0$, we can find a smooth function $\phi \neq 0$ such that $g(x, u, Du) \geq \phi \geq 0$. By Lax-Milgram theorem, there exists a weak solution w' satisfying

$$\begin{cases} Lw' &= \phi, \quad \text{in } U \\ w' &= 0, \quad \text{on } U. \end{cases}$$

Since $\phi \geq 0$, due to the strong minimum principle for H_0^1, $w' > 0$ in U. On the other hand,

$$L(w - w') = g - \phi \geq 0,$$

applying the maximum principle for H_0^1, we obtain $w \geq w' > 0$ in U. Therefore, $F : C^1 \to C^1$ is a compact mapping in the nonnegative cone.

Now we verify the two conditions in Theorem 2.3.

1) holds for a small r. We argue by contradiction. Suppose that $Fu = su$ with $s \geq 1$ and $\|u\|_{C^1} = r$. Then

$$Lw = L(F(u)) = L(su) = sLu = g(x, u, Du).$$

Therefore,

$$\int_U Lu \cdot J dx = \int_U uL'J dx = \lambda_1 \int_U uJ dx = s^{-1} \int_U g(x, u, Du) J dx.$$

Since $\overline{\lim\limits_{u\to 0}} \frac{g(x,u,p)}{u} < \lambda_1$, now choose γ and small r such that

$$\frac{g(x, u, p)}{u} \leq \gamma < \lambda_1 \quad \text{for } 0 < u \leq r. \tag{2.11}$$

Consequently,

$$\lambda_1 \int_U uJdx = s^{-1} \int_U g(x,u,Du)Jdx \le \gamma \int_U uJdx.$$

From (2.11), $\gamma < \lambda_1$, this implies

$$\int_U uJdx = 0, \text{ and so } u \equiv 0,$$

a contradiction.

2) we argue by contradiction again, assuming $F(u) = u - t\tilde{J}$, then

$$L(F(u)) = Lw = g(x,u,Du) = Lu - tJ,$$

that is,

$$Lu = g(x,u,Du) + tJ.$$

By Theorem 2.2,

$$t \le K_1 \text{ and } \|u\|_{L^\infty} \le K,$$

then getting back to equation (2.5), from step 3 of Theorem 2.2 and applying L^p regularity we conclude that

$$\|u\|_{C^1} \le \|u\|_{W^{2,p}} \le \|f + tJ\|_{L^p} \le R^1, \forall p < \infty.$$

So taking any $R > R^1$, when $\|u\|_{C^1} = R$, $F(u) \ne u - t\tilde{J}$, the proof is completed.

3 A Coupled System of Elliptic Equations in A Cylinder

3.1 Motivations and overview of the problem

In this section we consider a system of equations on a cylindrical domain $\Omega = \Omega' \times (0,a) \subset \mathbb{R}^n (n \ge 3)$, with $x = (x', x_n) \in \Omega$ and Ω' is smooth. The particularity of this system is that it couples two variables $u(x)$ and $v(x')$ which are defined on *different domains*. We can think of Ω as a jar or a cylindrical habitat containing two interacting substances or species: the substance $u(x)$ (say a gas, insects, birds...) is distributed in the interior of the jar or habitat Ω, while the substance $v(x')$ (say a fluid, plants, worms ...) is located at the bottom $\Omega' \times \{0\}$ of the jar or on the ground of the habitat. A simple model of such an interacting system is

$$\begin{cases} -\Delta_{(n)} u(x) & = h(x) v(x')^\gamma \ , \ x \in \Omega \\ -\Delta_{(n-1)} v(x') & = \int_0^a u^\eta (x', x_n)\, dx_n \ , \ x' \in \Omega' \\ u(x) = 0 \ , & x \in \partial\Omega' \times [0,a]; \ \ \partial_\nu u(x) = 0, \ x \in \Omega' \times \{0, a\} \\ v(x') = 0 \ , & x' \in \partial\Omega' \end{cases} \tag{3.1}$$

where $\Delta_{(n)} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $x' = (x_1, \cdots, x_{n-1})$, ν denotes the exterior normal to the boundary $\partial\Omega$, and γ, η are exponents with $\eta \geq 1$, $\gamma > 1$ at the same time.

Here, we assume that the vertically cumulated effect of the substance $u(x)$, $x \in \Omega$, interacts with the substance $v(x')$ on the bottom Ω', hence the term $\int_0^a u^\eta(x', x_n)\, dx_n$ in the second equation; on the other hand, the substance $v(x')$ at the bottom Ω' interacts with the substance $u(x)$ via a continuous coefficient function $h : \bar{\Omega} \to \mathbb{R}^+$, which we may consider decreasing with increasing height x_n.

Roughly speaking, for fixed $u \in L^2(\Omega)$, the operator $\Delta_{(n-1)}$ with Dirichlet boundary condition in the second equation of (3.1) is invertible (see section 3.2), and we can insert the expression

$$v(x') = (-\Delta_{(n-1)})^{-1}\big(\int_0^a u^\eta(x', x_n)\, dx_n\big)$$

into the first equation of the system, to obtain the non-local equation

$$\begin{cases} -\Delta_{(n)} u(x) = h(x)\big[(-\Delta_{(n-1)})^{-1}(\int_0^a u^\eta(x', x_n)\, dx_n)\big]^\gamma \\ u(x) = 0 \ \text{ for } \ x \in \partial\Omega' \times [0, a] \ \ ; \ \ \partial_\nu u(x) = 0 \ \text{ for } \ x \in \Omega' \times \{0, a\}. \end{cases} \tag{3.2}$$

Our aim is to prove the following result:

Theorem 3.1. *[34] Suppose that $\Omega := \Omega' \times (0, a) \subset \mathbb{R}^n$ is a bounded open domain. Furthermore*

1) if $1 \leq \eta < \frac{4n}{(n-1)(n-2)}$, then assume that $1 < \gamma\eta \leq \frac{2n+2}{n}$;
if $\eta \geq \frac{4n}{(n-1)(n-2)}$, then assume that $1 < \gamma\eta \leq \frac{n+1}{n-1} + \frac{2n\gamma}{(n-1)^2}$.

2) $h \in C(\overline{\Omega}, \mathbb{R}^+)$, with $h_m := \min\{h(x), x \in \overline{\Omega}\} > 0$.

Then equation (3.2), and hence system (3.1), has a positive solution $u \in W^{2,q}(\Omega), 1 \leq q < \infty$.

Remark 3.1. *Notice that for $n = 3, 4$, we are always in case 1), since then*

$$\frac{2n+2}{n} < \frac{4n}{(n-1)(n-2)}.$$

3.2 Some properties of Laplacian

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Consider

$$-\Delta : H_0^1(\Omega) \cap H^2(\Omega) \to L^2(\Omega)$$

where Δ is the Laplace operator $\sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

1. existence and uniqueness

$\forall f \in L^2(\Omega)$, the problem

$$\begin{cases} -\Delta u &= f, \quad x \in \Omega \\ u &= 0, \quad x \in \partial\Omega \end{cases} \tag{3.3}$$

has a unique weak solution. In fact, $\forall \varphi \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u \nabla v dx \leq \|u\|_{H_0^1(\Omega)} \|\varphi\|_{H_0^1(\Omega)}$$

and by the Poincaré inequality

$$C\|u\|_{H_0^1(\Omega)}^2 \leq \int_\Omega |\nabla u|^2 dx$$

with $C > 0$. By Lax-Milgram theorem, there exists a unique solution of (3.3) in $H_0^1(\Omega)$. Therefore, $-\Delta$ is invertible. Moreover, using regularity theory, we know the solution belongs to $H^2(\Omega)$ as well (see chapter 6.3 [25]).

2. $-\Delta$ is self-adjoint

Definition 3.1. *([7], chapter 6.4) A bounded operator $T \in \mathcal{L}(H)$ is said to be self-adjoint if $T^\star = T$, i.e.,*

$$(Tu, v) = (u, Tv) \quad \forall u, v \in H,$$

where H is a Hilbert space.

First, $-\Delta$ is a bounded operator from $H^2(\Omega) \cap H_0^1(\Omega)$ to $L^2(\Omega)$. Since $\forall f \in L^2(\Omega)$ and for a bounded set $\|u\|_{H^2(\Omega)\cap H_0^1(\Omega)} \leq C$,

$$\|f\|_{L^2(\Omega)} = \| - \Delta u\|_{L^2(\Omega)} \leq \|u\|_{H^2(\Omega)\cap H_0^1(\Omega)} \leq C.$$

Second, for every $u, v \in H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega)$,

$$< -\Delta u, v >= -\int_\Omega \Delta u \cdot v dx = \int_\Omega \nabla u \cdot \nabla v dx = -\int_\Omega u \cdot \Delta v dx =< u, -\Delta v >$$

thus $-\Delta$ is self-adjoint on $L^2(\Omega)$.

3. $(-\Delta)^{-1}$ is linear and self-adjoint

$(-\Delta)^{-1}$ is linear: Suppose $-\Delta(\alpha u_1) = \alpha f_1$, $-\Delta(\beta u_2) = \beta f_2$, since $-\Delta$ is linear and invertible, then $\forall \alpha, \beta \in \mathbb{R}$

$$-\Delta(\alpha u_1 + \beta u_2) = \alpha f_1 + \beta f_2, \quad u_1 = (-\Delta)^{-1}(f_1), \quad u_2 = (-\Delta)^{-1}(f_2)$$

and hence

$$(-\Delta)^{-1}(\alpha f_1 + \beta f_2) = \alpha u_1 + \beta u_2 = \alpha(-\Delta)^{-1} f_1 + \beta(-\Delta)^{-1} f_2.$$

This shows that $(-\Delta)^{-1}$ is linear.

$(-\Delta)^{-1}$ is self-adjoint (one can also refer to [7], Theorem 8.22):

From 1, we see $-\Delta : H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ is invertible, so we can define: $(-\Delta)^{-1} : L^2(\Omega) \to H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega)$. $\forall f, g \in L^2(\Omega)$, we assume

$$u = -(\Delta)^{-1} f, \quad v = (-\Delta)^{-1} g.$$

Since

$$\int_\Omega (-\Delta u) v = \int_\Omega f v, \quad \int_\Omega (-\Delta v) u = \int_\Omega g u,$$

then

$$\int_\Omega f (-\Delta)^{-1} g = \int_\Omega f v = \int_\Omega g u = \int_\Omega g (-\Delta)^{-1} f.$$

3.3 L^P regularity on the cylinder

The proof of Theorem 3.1 depends on a priori estimates of the solutions and a related existence theorem. The L^p theory presented here is to pave the way to get the a priori bound. In this part we will concentrate on showing that a weak solution of the equation

$$\begin{cases} -\Delta_{(n)} u &= f(x) \quad x \in \Omega \\ u(x', x_n) &= 0, \qquad x' \in \partial\Omega' \times [0, a] \\ \partial_{x_n} u(x', x_n) &= 0\ , \qquad x_n \in \{0, a\}. \end{cases} \tag{3.4}$$

with $f \in L^p(\Omega)(1 < p < \infty)$, will also be a strong solution which is twice weakly differentiable. The proof of the regularity is based on the a priori estimates below. In view of the mixed boundary conditions and the special shape of the domain, we will do an even reflection on the bottom of the cylinder to reduce the problem to a familiar case for which we can refer to the ninth chapter in [36].

3.3.1 L^p a priori estimate

We define the space $H^1_{cyl}(\Omega)$ as the closure in $H^1(\Omega)$ of the set $C^1_{cyl}(\Omega) = \{u \in C^1(\Omega) \mid u(x) = 0,\ x \in \partial\Omega' \times [0, a]\}$. Correspondingly, $W^{1,p}_{cyl} = \{u \in W^{1,p}(\Omega) \mid u(x) = 0, x \in \partial\Omega' \times [0, a]\}$ and $W^{2,p}_{cyl} = \{u \in W^{2,p}(\Omega) \mid u(x) = 0, x \in \partial\Omega' \times [0, a]\}$.

Interior estimate:

Lemma 3.1. *Assume $u \in W^{2,p}_{loc}(\Omega) \cap L^p(\Omega)$, $1 < p < \infty$, a strong solution of the equation* (3.4)*, then for $f \in L^p(\Omega)$ and for any open domain $\Omega_i \subset\subset \Omega$,*

$$\|u\|_{W^{2,p}(\Omega_i)} \le C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}),$$

where $C = C(n, p, \Omega_i, \Omega)$.

The proof of this lemma follows from the same proof of Theorem 1.6.

Estimate on the bottom and the top:

Lemma 3.2. *Assume $u \in W^{2,p}_{cyl}(\Omega)$, $1 < p < \infty$, a strong solution of (3.4), then for $f \in L^p(\Omega)$ and for any open domain $\Omega_b \subset\subset \Omega \cup \Big\{\Omega' \times \{0\}\Big\}$ or $\Omega_t \subset\subset \Omega \cup \Big\{\Omega' \times \{a\}\Big\}$*

$$\|u\|_{W^{2,p}(\Omega_b)} \le C_b(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)})$$

or

$$\|u\|_{W^{2,p}(\Omega_t)} \le C_t(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)})$$

where $C_b = C(n, p, \Omega_b, \Omega)$, $C_t = C(n, p, \Omega_t, \Omega)$.

Proof. We extend u and f to $\Omega' \times (-a, a)$ by even reflection, that is, by setting

$$u(x', x_n) = u(x', -x_n), \qquad f(x', x_n) = f(x', -x_n)$$

for $x_n < 0$. It follows that the extended functions, say $\tilde{u}$ and $\tilde{f}$, satisfy the same equation of (3.4) weakly in $\Omega' \times (-a, a)$. To prove this we take an arbitrary test function $\varphi \in C^1_{cyl}(\Omega' \times (-a, a))$, then since u is a weak solution of (3.4) on Ω, we have

$$\int_{\Omega' \times (0,a)} \nabla u \nabla \phi \, dx = \int_{\Omega' \times (0,a)} f \phi \, dx, \quad \forall\ \phi \in C^1_{cyl}(\Omega' \times (0, a)). \tag{3.5}$$

As $\varphi \in C^1$ in $\Omega' \times (0, a)$ and $\varphi = 0$ on $\partial\Omega'$, we can take $\phi = \varphi$ in $\Omega' \times (0, a)$, then

$$\int_{\Omega' \times (0,a)} \nabla u \nabla \varphi \, dx = \int_{\Omega' \times (0,a)} f \varphi \, dx. \tag{3.6}$$

On the other hand, due to the even reflection, from (3.5), we get

$$\int_{\Omega' \times (-a,0)} \nabla u \nabla \phi' \, dx = \int_{\Omega' \times (-a,0)} f \phi' \, dx, \quad \forall\ \phi' \in C^1_{cyl}(\Omega' \times (-a, 0)),$$

then taking $\phi' = \varphi$ in $\Omega' \times (-a, 0)$, so

$$\int_{\Omega' \times (-a,0)} \nabla u \nabla \varphi \, dx = \int_{\Omega' \times (-a,0)} f \varphi \, dx. \tag{3.7}$$

(3.6)+(3.7), we obtain

$$\begin{aligned}
\int_{\Omega' \times (0,a)} \nabla u \nabla \varphi \, dx + \int_{\Omega' \times (-a,0)} \nabla u \nabla \varphi \, dx &= \int_{\Omega' \times (-a,a)} \nabla \tilde{u} \nabla \varphi \, dx \\
&= \int_{\Omega' \times (0,a)} f \varphi \, dx + \int_{\Omega' \times (-a,0)} f \varphi \, dx \\
&= \int_{\Omega' \times (-a,a)} \tilde{f} \varphi \, dx.
\end{aligned}$$

Consequently, we have

$$\int_{\Omega'\times(-a,a)} \nabla\tilde{u}\nabla\varphi\,dx = \int_{\Omega'\times(-a,a)} \tilde{f}\varphi\,dx \quad \forall\varphi\in C^1_{cyl}(\Omega'\times(-a,a)).$$

Besides, $\tilde{u}=0$, $x\in\partial\Omega'\times[-a,a]$ and $\frac{\partial\tilde{u}}{\partial x_n}\big|_{x_n=-a} = -\frac{\partial\tilde{u}}{\partial x_n}\big|_{x_n=a} = -\frac{\partial u}{\partial x_n}\big|_{x_n=a} = 0$, so that $\tilde{u}$ is a weak solution of (3.4) in $\Omega'\times(-a,a)$. By the evenness of $\tilde{u}$, we also have $\frac{\partial\tilde{u}}{\partial x_n}\big|_{x_n=0} = 0$. Since Ω_b is a compact subset of $\Omega'\times(-a,a)$, so we are able to apply the interior estimate to Ω_b and thus get the desired estimate. The result for the estimate on the top of Ω can then be obtained by substituting Ω_b with Ω_t. □

Estimate on the side:

Lemma 3.3. *Assume $u\in W^{2,p}_{cyl}(\Omega)$, $1<p<\infty$, a strong solution of (3.4), then for any domain $\Omega_s \subset\subset \{\overline{\Omega'}\times(0,a)\}$,*

$$\|u\|_{W^{2,p}(\Omega_s)} \le C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}).$$

where $C = C(n,p,\Omega_s,\Omega)$.

Proof. Since $u(x) = 0, x'\in\partial\Omega'$, the proof follows from the boundary L^p estimate of Theorem 1.7. □

Estimate on the edge $\partial\Omega'\times\{0,a\}$:

Lemma 3.4. *Assume $u\in W^{2,p}_{cyl}(\Omega)$, $1<p<\infty$, a strong solution of (3.4), then for $f\in L^p(\Omega)$ and for any open domain $\Omega_e \subset\subset \Omega\cup\{\overline{\Omega'}\times\{0\}\}$,*

$$\|u\|_{W^{2,p}(\Omega_e)} \le C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}), \tag{3.8}$$

where $C = C(n,p,\Omega_e,\Omega)$.

Proof. In the proof of Lemma 3.2, we extended u and f to $\Omega'\times(-a,a)$ by even reflection, and we proved the extended function $\tilde{u}$ is a weak solution of (3.4) in $\Omega'\times(-a,a)$ with f replaced by $\tilde{f}$. In this case, each point $x_0\in\partial\Omega'\times\{0\}$ is the boundary point of $\Omega'\times(-a,a)$ on the side, we then can proceed as in the proof of Lemma 3.3 with Ω_s replaced by $\Omega_S \subset\subset \{\overline{\Omega'}\times(-a,a)\}$, since $\Omega_e\subset\Omega_S$, we have

$$\begin{aligned} \|u\|_{W^{2,p}(\Omega_e)} \le \|\tilde{u}\|_{W^{2,p}(\Omega_S)} &\le C\big(\|\tilde{u}\|_{L^p(\Omega'\times(-a,a))} + \|\tilde{f}\|_{L^p(\Omega'\times(-a,a))}\big) \\ &\le C\big(2\|u\|_{L^p(\Omega'\times(0,a))} + 2\|f\|_{L^p(\Omega'\times(0,a))}\big). \end{aligned}$$

We therefore derive

$$\|u\|_{W^{2,p}(\Omega_e)} \le C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}).$$

□

Combined with all these eistmates, we get the following result.

Global L^p estimate and regularity:

Lemma 3.5. *Assume $u \in W^{2,p}_{cyl}(\Omega)$, $1 < p < \infty$, satisfying* (3.4), *if $f \in L^p(\Omega)$, then*

$$\|u\|_{W^{2,p}(\Omega)} \leq C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}),$$

where $C = C(n, p, \Omega)$.

Proof. (see a similar proof of Theorem 2.2.3 [61]) From the boundary estimate we conclude that for $x_0 \in \partial\Omega$, there exists a neighborhood $U(x_0)$ such that

$$\begin{aligned} \|u\|_{W^{2,p}(U(x_0)\cap\Omega)} &\leq \|u\|_{W^{2,p}(\Omega_s)} + \|u\|_{W^{2,p}(\Omega_b)} + \|u\|_{W^{2,p}(\Omega_t)} + \|u\|_{W^{2,p}(\Omega_e)} \\ &\leq C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}). \end{aligned} \tag{3.9}$$

According to the Heine-Borel theorem, there exists a finite open covering $U_1, \cdots, U_N$ to cover $\partial\Omega$. Denote $K = \Omega \setminus \bigcup_{i=1}^{N} U_i$, then K is a closed subset of Ω and there exists a subdomain $U_0 \subset\subset \Omega$ such that $U_0 \supset K$. Lemma 3.1 shows that

$$\|u\|_{W^{2,p}(U_0)} \leq C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}). \tag{3.10}$$

Using the theorem on the partition of unity, we can choose functions $\eta_0, \eta_1, \cdots, \eta_N$ such that

$$0 \leq \eta_i \leq 1, \quad \forall x \in U_i \ (i = 0, 1, \cdots, N), \tag{3.11}$$

$$\sum_{i=0}^{N} \eta(x) = 1, \quad x \in \bar{\Omega}. \tag{3.12}$$

Thus

$$\begin{aligned} \|u\|_{W^{2,p}(\Omega)} = \Big\| \sum_{i=0}^{N} \eta_i u \Big\|_{W^{2,p}(\Omega)} &\leq \sum_{i=0}^{N} \|\eta_i u\|_{W^{2,p}(\Omega)} \\ &\leq C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}). \end{aligned} \tag{3.13}$$

□

In the next lemma we eliminate the dependence of u on the right.

Lemma 3.6 (A better a priori L^p estimate, cf. [21], Lemma 3.2.1)**.** *Assume $u \in W^{2,p}_{cyl}(\Omega)$, $1 < p < \infty$, satisfying* (3.4), *if $f \in L^p(\Omega)$, then*

$$\|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \tag{3.14}$$

where $C = C(n, p, \Omega)$.

Proof. We argue by contradiction. If (3.14) is not true, then $\forall N$, $\exists u_N \in W^{2,p}_{cyl}(\Omega)$, $f_N \in L^p(\Omega)$, such that

$$\left\{\begin{array}{rcll} -\Delta_{(n)} u_N &=& f_N, & x \in \Omega \\ u_N(x', x_n) &=& 0, & x' \in \partial\Omega' \\ \partial_{x_n} u_N(x', x_n) &=& 0\ , & x_n \in \{0, a\} \end{array}\right. \tag{3.15}$$

but

$$\|u_N\|_{W^{2,p}(\Omega)} \geq N \|f_N\|_{L^p(\Omega)}. \tag{3.16}$$

Let

$$v_N = \frac{u_N}{\|u_N\|_{L^p(\Omega)}}, \quad g_N = \frac{f_N}{\|u_N\|_{L^p(\Omega)}}, \tag{3.17}$$

then

$$\left\{\begin{array}{rcll} -\Delta_{(n)} v_N &=& g_N, & x \in \Omega \\ v_N(x', x_n) &=& 0, & x' \in \partial\Omega' \\ \partial_{x_n} v_N(x', x_n) &=& 0\ , & x_n \in \{0, a\} \end{array}\right. \tag{3.18}$$

and

$$\|v_N\|_{L^p(\Omega)} = 1, \quad \|v_N\|_{W^{2,p}(\Omega)} = \frac{\|u_N\|_{W^{2,p}(\Omega)}}{\|u_N\|_{L^p(\Omega)}}. \tag{3.19}$$

From the global estimate Lemma 3.5 and (3.16) we have

$$\begin{aligned} \|v_N\|_{W^{2,p}(\Omega)} &\leq C(\|g_N\|_{L^p(\Omega)} + \|v_N\|_{L^p(\Omega)}) \\ &\leq C\big(\frac{\|f_N\|_{L^p(\Omega)}}{\|u_N\|_{L^p(\Omega)}} + 1\big) \\ &\leq \frac{C}{N}\frac{\|u_N\|_{W^{2,p}(\Omega)}}{\|u_N\|_{L^p(\Omega)}} + C \\ &= \frac{C}{N}\|v_N\|_{W^{2,p}(\Omega)} + C \end{aligned}$$

taking $N > C$, then

$$\|v_N\|_{W^{2,p}(\Omega)} \leq C. \tag{3.20}$$

Following from Rellich-Kondrachov theorem (cf. [2], Theorem 6.3), $W^{2,p}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ compactly. That is there exists a subsequence such that

$$\|v_N - v\|_{L^p(\Omega)} \to 0, \quad \|\nabla v_N - \nabla v\|_{L^p(\Omega)} \to 0. \tag{3.21}$$

Since v_N satisfies (3.18) weakly, then

$$\int_\Omega \nabla v_N \nabla \varphi \, dx = \int_\Omega g_N \varphi \, dx, \quad \forall \varphi \in C^\infty_{cyl}(\Omega). \tag{3.22}$$

From (3.21), we have $v_N \rightharpoonup v$ in $W^{1,p}_{cyl}(\Omega)$, and hence

$$\int_\Omega \nabla v_N \nabla \varphi \, dx \to \int_\Omega \nabla v \nabla \varphi \, dx, \qquad N \to \infty.$$

On the other hand, since

$$\|g_N\|_{L^p(\Omega)} = \frac{\|f_N\|_{L^p(\Omega)}}{\|u_N\|_{L^p(\Omega)}} \leq \frac{1}{N} \frac{\|u_N\|_{W^{2,p}(\Omega)}}{\|u_N\|_{L^p(\Omega)}} = \frac{1}{N}\|v_N\|_{W^{2,p}(\Omega)}$$

and (3.20), we see $\|g_N\|_{L^p(\Omega)} \to 0$ as $N \to \infty$ and then $g_N \rightharpoonup 0$ in $L^p(\Omega)$, which implies $\forall \varphi \in C^\infty_{cyl}(\Omega)$

$$\int_\Omega g_N \varphi \, dx \to 0, \qquad N \to \infty.$$

So,

$$\int_\Omega \nabla v \nabla \varphi \, dx = 0, \quad \forall \varphi \in C^\infty_{cyl}(\Omega), \quad v \in W^{1,p}_{cyl}(\Omega).$$

as $N \to \infty$ in (3.22). Hence v weakly satisfies

$$\left\{ \begin{array}{rcll} -\Delta_{(n)} v &=& 0, & x \in \Omega \\ v &=& 0, & x' \in \partial\Omega' \\ \partial_{x_n} v &=& 0\,, & x_n \in \{0, a\} \end{array} \right. \tag{3.23}$$

In the following we prove $v = 0$. We first consider the case when $p \geq 2$, because one can rather easily show the uniqueness of the weak solutions by multiplying both sides of the equation by the solution itself and integrating by parts.

case $p \geq 2$:

Multyplying with v on both sides of the first equation in (3.23), we get $\int_\Omega |\nabla v|^2 \, dx = 0$, so $\nabla v = 0$, combing with the boundary condition then $v = 0$, which contradicts with $\|v_N\|_{L^p(\Omega)} \to \|v\|_{L^p(\Omega)} = 1$.

case $1 < p < 2$:

Let $\{v_k\}$ be a sequence of $C^\infty_{cyl}(\Omega)$ functions such that

$$v_k \to v$$

in $W^{1,p}_{cyl}(\Omega)$, as $k \to \infty$. For each k, consider the equation

$$\left\{ \begin{array}{rcll} -\Delta y_k &=& F_k(x), & x \in \Omega \\ y_k &=& 0, & x' \in \partial\Omega' \\ \partial_{x_n} y_k &=& 0, & x_n \in \{0, a\} \end{array} \right. \tag{3.24}$$

where $F_k(x) := \nabla \cdot \frac{|\nabla v_k|^{p/q} \nabla v_k}{\sqrt{1+|\nabla v_k|^2}}$, $\frac{1}{p} + \frac{1}{q} = 1$. For $1 < p < 2$, q is greater than 2, and since $F_k(x)$ is smooth enough, it is in $L^q(\Omega)$. And hence Lemma 3.7 guarantees the existence of a strong solution y_k to equation (3.24). Multiplying v on both sides of (3.24) and applying integration by parts to (3.24), we have

$$0 = -\int_\Omega y_k \cdot \Delta v = -\int_\Omega \Delta y_k \cdot v = \int_\Omega F_k(x) v = -\int_\Omega \frac{|\nabla v_k|^{p/q} \nabla v_k}{\sqrt{1+|\nabla v_k|^2}} \cdot \nabla v. \tag{3.25}$$

Next, we show

$$\frac{|\nabla v_k|^{p/q} \nabla v_k}{\sqrt{1+|\nabla v_k|^2}} \to \frac{|\nabla v|^{p/q} \nabla v}{\sqrt{1+|\nabla v|^2}} \tag{3.26}$$

in L^q. In fact, since $v_k \to v$ in $W^{1,p}_{cyl}$, then

$$\|\nabla v_k - \nabla v\|_{L^p} \to 0,$$

since L^p convergence implies pointwise convergence, so that $|\nabla v_k| \to |\nabla v|$ a.e.. Besides,

$$\begin{aligned}
\int_\Omega \left| \sqrt{1+|\nabla v_k|^2} - \sqrt{1+|\nabla v|^2} \right| &= \int_\Omega \frac{\left| |\nabla v_k|^2 - |\nabla v|^2 \right|}{\sqrt{1+|\nabla v_k|^2} + \sqrt{1+|\nabla v|^2}} \\
&\leq \int_\Omega \frac{\left| |\nabla v_k|^2 - |\nabla v|^2 \right|}{\sqrt{|\nabla v_k|^2} + \sqrt{|\nabla v|^2}} \\
&= \int_\Omega \left| |\nabla v_k| - |\nabla v| \right| \\
&\leq \int_\Omega |\nabla v_k - \nabla v| \\
&= \|\nabla v_k - \nabla v\|_{L^1} \\
&\leq C \|\nabla v_k - \nabla v\|_{L^p} \\
&\to 0,
\end{aligned}$$

thus, for the same reason mentioned before, there exists a sub-sequence such that $\sqrt{1+|\nabla v_k|^2} \to \sqrt{1+|\nabla v|^2}$ a.e., which implies,

$$\frac{|\nabla v_k|^{p/q} \nabla v_k}{\sqrt{1+|\nabla v_k|^2}} \to \frac{|\nabla v|^{p/q} \nabla v}{\sqrt{1+|\nabla v|^2}}$$

a.e.. On the other hand, since

$$\left| \frac{|\nabla v_k|^{p/q} \nabla v_k}{\sqrt{1+|\nabla v_k|^2}} \right| \leq |\nabla v_k|^{p/q} \in L^q$$

according to the dominated convergence theorem, we complete the proof of (3.26). Therefore, $\frac{|\nabla v_k|^{p/q}\nabla v_k}{\sqrt{1+|\nabla v_k|^2}} \rightharpoonup \frac{|\nabla v|^{p/q}\nabla v}{\sqrt{1+|\nabla v|^2}}$ in L^q. Letting $k \to \infty$ in (3.25), we obtain $\nabla v = 0$ almost everywhere. Getting back to equation (3.23), we deduce that $v = 0$, which is a contradiction to $\|v\|_{L^p} = 1$. □

3.3.2 regularity

With the better a priori estimate above we can get the following result:

Lemma 3.7. *If $f \in L^p(\Omega)$ with $1 < p < \infty$, then the problem* (3.4) *has a unique solution $u \in W^{2,p}(\Omega)$.*

Proof. The existence of the strong solution follows from the Theorem 9.15 [36] in the same way. Here we present the main points of the proof. We start from the L^2 regularity.

L^2 interior regularity: If $f \in L^2(\Omega)$, $u \in H^1_{cyl}(\Omega)$ is a weak solution of (3.4), then $u \in H^2_{cyl,loc}(\Omega)$, and for each open subset $V \subset\subset \Omega$ we have the estimate

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}),$$

the constant C depending only on V, Ω. The proof of L^2 interior regularity is the same as Theorem 1.4, section 1.3.1. One can also refer to Theorem 1 ([25] section 6.3.1).

In order to get the boundary regularity, we extend u and f to $\Omega' \times (-a, a)$ as we did in Lemma 3.2. The extended function $\tilde{u}$ and $\tilde{f}$ satisfy the same equation of (3.4) weakly in $\Omega' \times (-a, a)$. Since the bottom of $\Omega' \times (0, a)$ is inside of $\Omega' \times (-a, a)$ after the extension, then the proof of regularity near the bottom $\overline{\Omega'} \times \{0\}$ is the same as L^2 interior regularity. Considering $u = 0$ on $\partial\Omega' \times [0, a]$, then the regularity near the side and also the edge $(\partial\Omega' \times \{0, a\})$ of the cylinder is the same as Theorem 4 ([25] section 6.3.2). Thus we have:

L^2 boundary regularity: If $f \in L^2(\Omega)$, $u \in H^1_{cyl}(\Omega)$ is a weak solution of (3.4), then $u \in H^2(\Omega)$, and we have the estimate

$$\|u\|_{H^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}),$$

the constant C depending only on Ω.

We are now in a position to prove Lemma 3.7 with $2 < p < \infty$. In fact, given that we have the same L^p a priori estimate as in chapter 9 [36], the interior regularity result follows directly from Lemma 9.16 [36]. After doing the even reflection, the case of local boundary regularity is handled similarly as the Lemma 9.16 [36] as well.

The way we prove the case $1 < p < 2$ is the same as Theorem 1.8, taking $\{f_m\} \subset L^2(U)$ such that $f_m \to f$ in $L^p(U)$ and $-\Delta u_m = f_m$, $u_m = 0$ on the boundary of U. By L^2 regularity theory again, we have $u_m \in W^{2,2}(U)$. Thus $u_m \in W^{2,p}(U) \cap W^{1,p}_0(U)$ since $1 < p < 2$. We then infer from Lemma 3.6 that

$$\|u_m\|_{W^{2,p}(U)} \leq C\|f_m\|_{L^p(U)} \leq C,$$

because $f_m \to f$ in $L^p(U)$. Consequently, there exists a sub-sequence of $\{u_m\}$ converging weakly to a function u in $W^{2,p}(U) \cap W_0^{1,p}(U)$ that satisfies $Lu = f$ in U.

For the uniqueness, assume $u_1, u_2 \in W^{2,p}(\Omega)$ both the strong solution of (3.4). Let $u = u_1 - u_2$, then $u \in W^{2,p}(\Omega)$ and satisfies (3.23) weakly with v replaced by u. From Lemma 3.6,

$$\|u\|_{W^{2,p}(\Omega)} \leq 0,$$

therefore, $u = 0$ a.e. in Ω, that is, $u_1 = u_2$. □

3.4 Generalized Hardy's inequality

Lemma 3.8. *For $u \in H^1_{cyl}(\Omega)$, we have*

$$\int_\Omega |\nabla u|^2 \, dx \geq C \int_\Omega \left| \frac{u}{\delta_{n-1}} \right|^2 dx$$

where $\delta_{n-1} = \delta(x')$ denotes the distance of x to $\partial\Omega'$, $\Omega' \subset \mathbb{R}^{n-1}$.

Proof. Notice that $\displaystyle\int_\Omega \frac{u^2(x)}{\delta^2(x')} \, dx = \int_0^a \int_{\Omega'} \frac{u^2(x)}{\delta^2(x')} \, dx' dx_n$; we start the proof from the inner integral. Consider $u_n \in C^\infty_{cyl}(\Omega)$; for fixed x_n, $u_n(x', x_n)$ is a function of x', then by Hardy's inequality [52] we have

$$\int_{\Omega'} \frac{u_n^2(x', x_n)}{\delta^2(x')} \, dx' \leq C \int_{\Omega'} \left| \frac{\partial u_n(x', x_n)}{\partial x'} \right|^2 dx',$$

and then integrating along x_n direction,

$$\begin{aligned} \int_0^a \int_{\Omega'} \frac{u_n^2(x)}{\delta^2(x')} \, dx' dx_n &\leq C \int_0^a \int_{\Omega'} \left| \frac{\partial u_n(x)}{\partial x'} \right|^2 dx' dx_n + \int_0^a \int_{\Omega'} \left| \frac{\partial u_n(x)}{\partial x_n} \right|^2 dx' dx_n \\ &\leq C \int_\Omega \left| \frac{\partial u_n(x)}{\partial x} \right|^2 dx. \end{aligned} \tag{3.27}$$

Since $C^\infty_{cyl}(\Omega)$ is dense in $H^1_{cyl}(\Omega)$, for $u \in H^1_{cyl}(\Omega)$, there exists functions $u_n(x) \in C^\infty_{cyl}(\Omega)$ such that

$$\int_\Omega \Big| \nabla (u_n(x) - u(x)) \Big|^2 dx \to 0, \quad \int_\Omega |u_n(x) - u(x)|^2 \, dx \to 0$$

as $n \to \infty$. This implies that $\{u_n\}$ is a Cauchy sequence in $H^1_{cyl}(\Omega)$, then there exist n_ϵ such that for $n, m \geq n_\epsilon$,

$$\int_\Omega \Big| \nabla (u_n(x) - u_m(x)) \Big|^2 dx \leq \epsilon.$$

Notice $u_n - u_m \in C^\infty_{cyl}(\Omega)$, we substitute u_n with $u_n - u_m$ in (3.27), then

$$\int_0^a \int_{\Omega'} \frac{|u_n(x) - u_m(x)|^2}{\delta^2(x')} \, dx' dx_n \le C \int_\Omega \Big|\nabla(u_n(x) - u_m(x))\Big|^2 dx \le \epsilon,$$

which implies that $\{\frac{u_n(x)}{\delta(x')}\}$ is a Cauchy sequence in $L^2(\Omega)$ and hence

$$\frac{u_n(x)}{\delta(x')} \to y,$$

for some $y \in L^2(\Omega)$. It remains to show $y = \frac{u(x)}{\delta(x')}$. Since $\delta(x')$ is bounded, we have that

$$u_n(x) \to y\delta(x'), \text{ in } L^2(\Omega).$$

In fact,

$$\begin{aligned}
\int_\Omega |u_n(x) - \delta(x')y|^2 \, dx' dx_n &= \int_\Omega \Big|\frac{u_n(x)}{\delta(x')} \cdot \delta(x') - \delta(x')y\Big|^2 dx \\
&= \int_\Omega |\delta(x')|^2 \Big|\frac{u_n(x)}{\delta(x')} - y\Big|^2 dx \\
&\le C \int_\Omega \Big|\frac{u_n(x)}{\delta(x')} - y\Big|^2 dx \\
&\to 0,
\end{aligned}$$

and since $u_n(x) \to u(x)$ in $L^2(\Omega)$, we conclude that indeed $y = \frac{u(x)}{\delta(x')}$. Then we complete the proof by letting $n \to \infty$ in (3.27). $\square$

The next lemma is a variant of the Hardy's inequality.

Lemma 3.9. *There exists $C > 0$ such that for $n \ge 3$, and $0 \le \tau \le 1$, we have*

$$\Big\|\frac{u}{\delta^\tau_{n-1}}\Big\|_{L^q(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H^1_{cyl}(\Omega)$$

where $\dfrac{1}{q} = \dfrac{1}{2} - \dfrac{1-\tau}{n}$.

Proof. By the Hölder inequality,

$$\begin{aligned}
\Big\|\frac{u}{\delta^\tau_{n-1}}\Big\|_{L^q(\Omega)} &= \Big(\int_\Omega (\frac{u^\tau}{\delta^\tau_{n-1}} \cdot u^{1-\tau})^q \, dx\Big)^{\frac{1}{q}} \\
&\le \Big(\Big(\int_\Omega (|\frac{u}{\delta_{n-1}}|^{\tau q})^{\frac{r}{q}} \, dx\Big)^{\frac{q}{r}}\Big)^{\frac{1}{q}} \cdot \Big(\Big(\int_\Omega (|u|^{(1-\tau)q})^{\frac{s}{q}} \, dx\Big)^{\frac{q}{s}}\Big)^{\frac{1}{q}} \\
&= \Big\|\frac{u^\tau}{\delta^\tau_{n-1}}\Big\|_{L^r(\Omega)} \cdot \big\|u^{1-\tau}\big\|_{L^s(\Omega)} \\
&= \Big\|\frac{u}{\delta_{n-1}}\Big\|^\tau_{L^{\tau r}(\Omega)} \|u\|^{1-\tau}_{L^{(1-\tau)s}(\Omega)}
\end{aligned} \tag{3.28}$$

where $\dfrac{1}{q} = \dfrac{1}{r} + \dfrac{1}{s}$. We choose $\tau r = 2$, and $\dfrac{1}{(1-\tau)s} = \dfrac{1}{2} - \dfrac{1}{n}$, thus,

$$\frac{1}{q} = \frac{1}{s} + \frac{\tau}{2} = \frac{1}{2} - \frac{1-\tau}{n}.$$

Applying Lemma 3.8 and Sobolev's embedding theorem to the respective term in (3.28) we obtain

$$\Big\| \frac{u}{\delta_{n-1}^{\tau}} \Big\|_{L^q(\Omega)} \leq C \|Du\|_{L^2(\Omega)}^{\tau} \, \|Du\|_{L^2(\Omega)}^{1-\tau} \tag{3.29}$$

Then (3.29) becomes the desired inequality. □

3.5 The a priori bound for mixed boundary problem

In what follows we let J_1' denote the first positive eigenfunction satisfying

$$\begin{cases} -\Delta_{(n-1)} J_1' &= \lambda_1' \, J_1', \quad x' \in \Omega' \\ \quad J_1'(x') &= 0, \qquad\quad x' \in \partial\Omega' \end{cases}$$

where λ_1' is the first eigenvalue of $-\Delta_{(n-1)}$ and J_1' is normalized so that $\int_{\Omega'} |J_1'|^2 \, dx' = 1$. Furthermore, $J_1(x)$ is the eigenfunction to the corresponding Laplacian equation in Ω, with $J_1(x', x_n) := J_1'(x'), x_n \in (0, a)$, that is $J_1(x', x_n)$ is constant with respect to the variable x_n and satisfies

$$\begin{cases} -\Delta_{(n)} J_1 = \lambda_1' \, J_1 \ , & x \in \Omega \\ J_1(x) = 0 \ , & x \in \partial\Omega' \times [0, a] \\ \partial_{x_n} J_1(x) = 0 \ , & x_n \in \{0, a\} \end{cases}$$

Remark 3.2. *It is known that $J_1'(x') > 0$ in Ω', and as proved in section 2.2, $J_1'(x') \geq C\delta_{n-1}(x')$ with $C > 0$. Note that $\int_\Omega |J_1(x)|^2 \, dx = a$.*

The basic *a priori* bound we prove is the following.

Theorem 3.2. *Suppose that $h(x) \geq h_m > 0$. Furthermore,*

if $1 < \eta < \dfrac{4n}{(n-1)(n-2)}$, then suppose that $1 < \gamma\eta \leq \frac{2n+2}{n}$;

if $\eta \geq \dfrac{4n}{(n-1)(n-2)}$, then suppose that $1 < \gamma\eta \leq \dfrac{n+1}{n-1} + \dfrac{2n\gamma}{(n-1)^2}$.

Then there is a constant K such that for any $u \in H^1_{cyl}(\Omega)$ non-negative and satisfying weakly

$$\begin{cases} -\Delta_{(n)} u = h(x) \big[(-\Delta_{(n-1)})^{-1} \big(\int_0^a u^\eta(x', x_n) \, dx_n \big) \big]^\gamma + tJ_1 & x \in \Omega \\ u(x', x_n) = 0, & x' \in \partial\Omega' \\ \partial_{x_n} u(x', x_n) = 0 \ , & x_n \in \{0, a\} \end{cases} \tag{3.30}$$

we have $u \in L^\infty(\Omega)$ *and*

$$\|u\|_{L^\infty(\Omega)} \le K,$$

where K *is independent of* $t \ge 0$.

We first prove some lemmas.

Lemma 3.10. *Under the assumption of Theorem 3.2, there is a constant* $K_1 > 0$ *such that for* $u \in H^1_{cyl}(\Omega)$ *satisfying (3.30), for a* $t \ge 0$, *we have* $t \le K_1$ *and*

$$\int_\Omega f(x,u)\,\delta_{n-1}(x)\,dx \le K_1\,,$$

where $f(x,u) := h(x)\big[(-\Delta_{(n-1)})^{-1}\big(\int_0^a u^\eta(x',x_n)\,dx_n\big)\big]^\gamma$.

Proof. Since $u \in H^1_{cyl}(\Omega)$ is a weak solution of (3.30), we have

$$\int_\Omega \nabla u\,\nabla\varphi(x)\,dx = \int_\Omega f(x,u)\varphi(x)\,dx + t\int_\Omega J_1\,\varphi(x)\,dx\ ,\ \forall\ \varphi \in H^1_{cyl}(\Omega) \tag{3.31}$$

Taking $\varphi = J_1$ we get

$$\int_\Omega \nabla u\,\nabla J_1\,dx = \int_\Omega f(x,u)J_1\,dx + t\int_\Omega |J_1|^2\,dx.$$

Note that $\partial\Omega = \partial\Omega' \times [0,a] \cup (\Omega' \times \{0,a\})$. The left side of the equation yields, using that $u|_{\partial\Omega'\times[0,a]} = 0$ and $\partial_\nu J_1|_{\Omega'\times\{0,a\}} = 0$

$$\begin{aligned}\int_\Omega \nabla u\cdot\nabla J_1\,dx &= \int_{\partial\Omega} u\,\partial_\nu J_1\,dx - \int_\Omega u\Delta_{(n)}J_1\,dx\\ &= -\int_\Omega u\Delta_{(n)}J_1\,dx\\ &= \lambda_1'\int_\Omega uJ_1\,dx.\end{aligned}$$

Since by assumption $h(x)$ has the positive lower bound h_m, then

$$\begin{aligned}
\lambda_1' \int_\Omega u J_1\, dx &= \int_\Omega f(x,u) J_1\, dx + t \int_\Omega |J_1|^2\, dx \\
&= \int_\Omega h(x) \big[(-\Delta_{(n-1)})^{-1} \big(\textstyle\int_0^a u^\eta(x)\, dx_n\big)\big]^\gamma J_1\, dx + t \int_\Omega |J_1|^2\, dx \\
&\geq h_m \int_\Omega \big[(-\Delta_{(n-1)})^{-1} \big(\textstyle\int_0^a u^\eta(x)\, dx_n\big)\big]^\gamma J_1\, dx + t \int_\Omega |J_1|^2\, dx \\
&= h_m \int_{\Omega\cap\{[(-\Delta_{(n-1)})^{-1}\int_0^a u^\eta(x)\,dx_n]<k\}} \big[(-\Delta_{(n-1)})^{-1} \big(\textstyle\int_0^a u^\eta(x)\, dx_n\big)\big]^\gamma J_1\, dx \\
&\quad + h_m \int_{\Omega\cap\{[(-\Delta_{(n-1)})^{-1}\int_0^a u^\eta(x)\,dx_n]\geq k\}} \big[(-\Delta_{(n-1)})^{-1} \big(\textstyle\int_0^a u^\eta(x)\, dx_n\big)\big]^\gamma J_1\, dx \\
&\quad + t \int_\Omega |J_1|^2\, dx,
\end{aligned}$$

where $k > 0$ will be chosen below. Since we consider non-negative solutions, then $\int_0^a u^\eta(x', x_n)\, dx_n$ is non-negative, and by the maximum principle, $(-\Delta_{(n-1)})^{-1}\big(\int_0^a u^\eta\, dx_n\big)$ is non-negative. Therefore

$$\begin{aligned}
\lambda_1' \int_\Omega u J_1\, dx &\geq h_m \int_0^a dx_n \int_{\Omega'\cap\{[(-\Delta_{(n-1)})^{-1}\int_0^a u^\eta(x)\,dx_n]\geq k\}} \big[(-\Delta_{(n-1)})^{-1} \big(\textstyle\int_0^a u^\eta(x)\, dx_n\big)\big]^\gamma J_1'\, dx' \\
&\quad + t \int_\Omega |J_1|^2\, dx \\
&\geq h_m \cdot a \cdot k^{\gamma-1} \cdot \int_{\Omega'\cap\{[(-\Delta_{(n-1)})^{-1}\int_0^a u^\eta(x)\,dx_n]\geq k\}} \big[(-\Delta_{(n-1)})^{-1} \big(\textstyle\int_0^a u^\eta(x)\, dx_n\big)\big] J_1'\, dx' \\
&\quad + t \int_\Omega |J_1|^2\, dx \\
&= h_m \cdot a \cdot k^{\gamma-1} \cdot \Big\{ \int_{\Omega'} \big[(-\Delta_{(n-1)})^{-1} \big(\textstyle\int_0^a u^\eta(x)\, dx_n\big)\big] J_1'\, dx' \\
&\quad - \int_{\Omega'\cap\{[(-\Delta_{(n-1)})^{-1}\int_0^a u^\eta(x)\,dx_n]<k\}} \big[(-\Delta_{(n-1)})^{-1} \big(\textstyle\int_0^a u^\eta(x)\, dx_n\big)\big] J_1'\, dx' \Big\} \\
&\quad + t \int_\Omega |J_1|^2\, dx \\
&\geq h_m \cdot a \cdot k^{\gamma-1} \cdot \Big\{ \int_{\Omega'} \big[(-\Delta_{(n-1)})^{-1} \big(\textstyle\int_0^a u^\eta(x)\, dx_n\big)\big] J_1'\, dx' - C(k) \Big\} \\
&\quad + t \int_\Omega |J_1|^2\, dx.
\end{aligned}$$

For given $\epsilon > 0$, choose $k : h_m \cdot a \cdot k^{\gamma-1} \geq (\lambda_1')^2 + \epsilon$, thus

$$
\begin{aligned}
\lambda_1' \int_\Omega u J_1\, dx \ &\geq\ [(\lambda_1')^2+\epsilon]\cdot \int_{\Omega'} \big[(-\Delta_{(n-1)})^{-1}\big(\textstyle\int_0^a u^\eta(x)\, dx_n\big)\big] J_1'\, dx' - C \\
&\quad + t\int_\Omega |J_1|^2\, dx \\
&=\ [(\lambda_1')^2+\epsilon]\cdot \int_{\Omega'} \big[\big(\textstyle\int_0^a u^\eta(x)\, dx_n\big)\big]\cdot \big[(-\Delta_{(n-1)})^{-1} J_1'\big]\, dx' - C \\
&\quad + t\int_\Omega |J_1|^2\, dx \\
&=\ [(\lambda_1')^2+\epsilon]\cdot \int_{\Omega'} \big(\textstyle\int_0^a u^\eta(x)\, dx_n\big)\big[\frac{1}{\lambda_1'} J_1'\big]\, dx' + t\int_\Omega |J_1|^2\, dx - C \\
&=\ \frac{(\lambda_1')^2+\epsilon}{\lambda_1'}\int_\Omega u^\eta(x) J_1\, dx + t\int_\Omega |J_1|^2\, dx - C \\
&=\ (\lambda_1' + \frac{\epsilon}{\lambda_1'})\int_\Omega u^\eta(x) J_1\, dx + t\int_\Omega |J_1|^2\, dx - C \\
&=\ (\lambda_1' + \frac{\epsilon}{\lambda_1'})\big\{\int_{\Omega\cap\{u\leq 1\}} u^\eta(x) J_1\, dx + \int_{\Omega\cap\{u>1\}} u^\eta(x) J_1\, dx\big\} + t\int_\Omega |J_1|^2\, dx - C \\
&\geq\ (\lambda_1' + \frac{\epsilon}{\lambda_1'})\int_{\Omega\cap\{u>1\}} u^\eta(x) J_1\, dx + t\int_\Omega |J_1|^2\, dx - C \\
&\geq\ (\lambda_1' + \frac{\epsilon}{\lambda_1'})\int_\Omega u(x) J_1\, dx + t\int_\Omega |J_1|^2\, dx - C.
\end{aligned}
$$

Hence,

$$
C \geq t\int_\Omega |J_1|^2\, dx + \frac{\epsilon}{\lambda_1'}\int_\Omega u J_1\, dx
$$

which implies t is bounded, and also

$$
\int_\Omega u(x) J_1\, dx < C.
$$

Since $\lambda_1' \int_\Omega u J_1\, dx = \int_\Omega f(x,u) J_1\, dx + t\int_\Omega |J_1|^2\, dx$, we see that also $\int_\Omega f(x,u) J_1\, dx$ is bounded, and using Remark 3.2 we obtain,

$$
\int_\Omega f(x,u)\delta_{n-1}(x')\, dx \leq C\int_\Omega f(x,u) J_1\, dx < K_1\ ,
$$

this completes the proof of Lemma 3.10. □

Next, we show a Poincaré type inequality in $W^{1,p}_{cyl}(\Omega)$.

Lemma 3.11. *There exists a constant $c > 0$ such that*

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}_{cyl}(\Omega). \tag{3.32}$$

Proof. We may assume $u \in C^\infty_{cyl}(\Omega)$ and $(0, x_2, \ldots, x_n) \in \partial\Omega'$, then

$$\begin{aligned} |u(x_1, x_2, \cdots, x_n)| &= |u(x_1, x_2, \cdots, x_n) - u(0, x_2, \cdots, x_n)| \\ &= \Big|\int_0^{x_1} \frac{d}{dt} u(t, x_2, \cdots, x_n)\, dt\Big|, \end{aligned}$$

therefore Hölder's inequality yields

$$\begin{aligned} |u|^p &= \Big|\int_0^{x_1} \frac{d}{dt} u(t, x_2, \cdots, x_n)\, dt\Big|^p \\ &\leq \Big|\int_0^{x_1} 1^q dt\Big|^{\frac{p}{q}} \Big|\int_0^{x_1} \big|\frac{\partial u}{\partial t}(t, x_2, \cdots, x_n)\big|^p\, dt\Big|, \qquad \frac{1}{p} + \frac{1}{q} = 1 \\ &\leq C\Big|\int_0^{x_1} \big|\frac{\partial u}{\partial t}(t, x_2, \cdots, x_n)\big|^p\, dt\Big|. \end{aligned}$$

Taking the integration over Ω on both sides, we get

$$\int_\Omega |u|^p\, dx \leq C \int_\Omega \int_0^{x_1} \big|\frac{\partial u}{\partial t}(t, x_2, \cdots, x_n)\big|^p\, dt dx,$$

and applying Fubini's theorem to the right hand side of the inequality,

$$\begin{aligned} \int_\Omega |u|^p\, dx &\leq C \int_0^{x_1} \int_\Omega \big|\frac{\partial u}{\partial x_1}(x_1, x_2, \cdots, x_n)\big|^p\, dx dt \\ &\leq C \int_0^{x_1} \int_\Omega |\nabla u|^p\, dx dt \\ &\leq C\|\nabla u\|^p_{L^p(\Omega)}\ , \end{aligned}$$

since Ω' is bounded. Now assuming $u_n \in C^\infty_{cyl}(\Omega)$ converging to u in $W^{1,p}_{cyl}(\Omega)$, from the result above we have

$$\int_\Omega |u_n|^p\, dx \leq C\|\nabla u_n\|^p_{L^p(\Omega)}\ ,\ \forall\ n \in \mathbb{N}\ .$$

Letting n go to infinity, we conclude that

$$\int_\Omega |u|^p\, dx \leq C\|\nabla u\|^p_{L^p(\Omega)}.$$

$\square$

In the next Lemma we prove an H^1-*a priori bound* for any weak non-negative solution of equation (3.30).

Lemma 3.12. *Under the assumptions of Theorem 3.2 there is a constant K_2 such that*

$$\|u\|_{H^1(\Omega)} \le K_2$$

for every non-negative weak solution of (3.30).

Proof. Taking $\varphi = u \in H^1_{cyl}(\Omega)$ in (3.31) we obtain

$$\|\nabla u\|^2_{L^2(\Omega)} \le \int_\Omega f(x,u)u\,dx + K_1 \int_\Omega J_1 u\,dx.$$

Applying the Hölder inequality and the Poincaré inequality (3.32) to the second term on the right hand side we get

$$\begin{aligned} \|\nabla u\|^2_{L^2(\Omega)} &\le \int_\Omega f(x,u)u\,dx + K_1\|J_1\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \\ &\le \int_\Omega f(x,u)u\,dx + C\|\nabla u\|_{L^2(\Omega)}. \end{aligned} \tag{3.33}$$

Next, for $0<\alpha<1$, by Hölder's inequality we get

$$\begin{aligned} \int_\Omega f(x,u)u\,dx &= \int_\Omega \left(\delta^\alpha_{n-1} f^\alpha(x,u)\right)\left(f^{1-\alpha}(x,u)\cdot\frac{u}{\delta^\alpha_{n-1}}\right)dx \\ &\le \left\|\delta^\alpha_{n-1} f^\alpha(x,u)\right\|_{L^{\frac{1}{\alpha}}(\Omega)} \left\| f^{1-\alpha}(x,u)\cdot\frac{u}{\delta^\alpha_{n-1}}\right\|_{L^{\frac{1}{1-\alpha}}(\Omega)} \\ &= \left\|\delta_{n-1} f(x,u)\right\|^\alpha_{L^1(\Omega)} \Big(\int_\Omega f(x,u)\,\frac{u^{\frac{1}{1-\alpha}}}{\delta_{n-1}^{\frac{\alpha}{1-\alpha}}}\,dx\Big)^{1-\alpha}. \end{aligned} \tag{3.34}$$

We now distinguish the two cases:

Case 1: $1 \le \eta < \frac{4n}{(n-1)(n-2)}$

We first show that for each $\epsilon>0$ there is a C_ϵ such that

$$\|f(x,u)\|_{L^\infty(\Omega)} \le \epsilon\|u\|^{\beta_n\eta}_{L^{s\eta}(\Omega)} + C_\epsilon\ , \tag{3.35}$$

where

$$\frac{n-1}{2} < s \le \frac{2^*}{\eta},\quad \beta_n := \frac{1}{\eta}\cdot\frac{2n+2}{n}. \tag{3.36}$$

In fact, since $u \in H^1_{cyl}(\Omega)$, according to Sobolev inequality, we know $u \in L^q(\Omega)$, $(q \le 2^* = \frac{2n}{n-2})$, and because

$$\begin{aligned}
\Big\| \int_0^a u^\eta \, dx_n \Big\|^s_{L^s(\Omega')} &= \int_{\Omega'} \Big(\int_0^a u^\eta \, dx_n \Big)^s dx' \\
&= \int_{\Omega'} \Big(\int_0^a u^\eta \cdot 1 \, dx_n \Big)^s dx' \\
&\le \int_{\Omega'} \Big(\big(\int_0^a u^{\eta s} \, dx_n \big) \cdot \big(\int_0^a 1^\theta \, dx_n \big)^{\frac{s}{\theta}} \Big) dx' \\
&\le C \int_{\Omega'} \big(\int_0^a u^{\eta s} \, dx_n \big) dx' \\
&= C \|u\|^{s\eta}_{L^{s\eta}(\Omega)}
\end{aligned} \tag{3.37}$$

where $\frac{1}{s} + \frac{1}{\theta} = 1, (s, \theta > 1)$ and $s\eta \le 2^*$, we see that $\int_0^a u^\eta \, dx_n \in L^s(\Omega')$. Next, using that $(-\Delta_{(n-1)})^{-1}$ is a continuous operator from $L^s(\Omega') \to W^{2,s}(\Omega')$, $s \le \frac{2n}{n-2} \cdot \frac{1}{\eta}$, we are able to use the Morrey embedding inequality in $\Omega' \subset \mathbb{R}^{n-1}$ and we have, for $s > (n-1)/2$,

$$\begin{aligned}
\|f(\cdot, u)\|_{L^\infty(\Omega)} &\le \max_{x \in \bar{\Omega}} \{h(x)\} \, C \, \big\| \big[(-\Delta_{(n-1)})^{-1} \big(\textstyle\int_0^a u^\eta(x) \, dx_n \big) \big] \big\|^\gamma_{L^\infty(\Omega')} \\
&\le C \, \big\| \big[(-\Delta_{(n-1)})^{-1} \big(\textstyle\int_0^a u^\eta(x) \, dx_n \big) \big] \big\|^\gamma_{W^{2,s}(\Omega')} \, , \ s > (n-1)/2 \\
&\le C \big\| \textstyle\int_0^a u^\eta(x) \, dx_n \big\|^\gamma_{L^s(\Omega')} \\
&= C \Big(\int_{\Omega'} \big(\textstyle\int_0^a u^\eta(x) \, dx_n \big)^s dx' \Big)^{\gamma/s} \\
&\le C \Big(\int_{\Omega'} \big(\int_0^a 1^\theta \, dx_n \big)^{\frac{s}{\theta}} \cdot \big(\int_0^a |u(x)|^{s\eta} \, dx_n \big) dx' \Big)^{\gamma/s}, \quad \frac{1}{s} + \frac{1}{\theta} = 1 \\
&\le C \, a^{\gamma/\theta} \big(\int_\Omega |u(x)|^{s\eta} \, dx \big)^{\gamma/s} \\
&\le C \|u\|^{\gamma\eta}_{L^{s\eta}(\Omega)}.
\end{aligned} \tag{3.38}$$

Therefore $f(x, u) \in L^\infty(\Omega)$ for fixed $u \in H^1_{cyl}(\Omega)$. Due to the condition of Theorem 3.2, it follows that $1 < \gamma < \beta_n$ and we conclude that

$$\lim_{\|u\|_{L^{s\eta}(\Omega)} \to \infty} \frac{\|f(x, u)\|_{L^\infty(\Omega)}}{\|u\|^{\beta_n \eta}_{L^{s\eta}(\Omega)}} = 0,$$

which means that for $\epsilon > 0$ small, there exists $M_\epsilon > 0$ such that $\|f(x, u)\|_{L^\infty(\Omega)} \le \epsilon \|u\|^{\beta_n \eta}_{L^{s\eta}(\Omega)}$, for $\|u\|_{L^{s\eta}(\Omega)} \ge M_\epsilon$. This shows (3.35).

Next, with the aid of Lemma 3.10 and due to $(a+b)^l \le a^l + b^l$ $(a,b \ge 0,\ 0<l<1)$, we deduce from (3.34)

$$
\begin{aligned}
\int_\Omega f(x,u)u\,dx &\le K_1^\alpha \Big(\int_\Omega f(x,u)\, \frac{u^{\frac{1}{1-\alpha}}}{\delta_{n-1}^{\frac{\alpha}{1-\alpha}}}\,dx \Big)^{1-\alpha} \\
&\le C\, \|f(\cdot,u)\|_{L^\infty(\Omega)}^{1-\alpha} \left[\int_\Omega \frac{u^{\frac{1}{1-\alpha}}}{\delta_{n-1}^{\frac{\alpha}{1-\alpha}}}\,dx \right]^{1-\alpha} \\
&\le \epsilon\, C\, \|u\|_{L^{s\eta}(\Omega)}^{\beta_n\eta(1-\alpha)} \left[\int_\Omega \frac{u^{\frac{1}{1-\alpha}}}{\delta_{n-1}^{\frac{\alpha}{1-\alpha}}}\,dx \right]^{1-\alpha} + C_\epsilon \left[\int_\Omega \frac{u^{\frac{1}{1-\alpha}}}{\delta_{n-1}^{\frac{\alpha}{1-\alpha}}}\,dx \right]^{1-\alpha}.
\end{aligned} \tag{3.39}
$$

Now we choose $0 < \alpha = \dfrac{n+2}{2n+2} < 1$, so that $\beta_n\eta + \dfrac{1}{1-\alpha} = \dfrac{2}{1-\alpha}$. From (3.33), (3.34) and (3.39), we get by the Sobolev inequality for $\Omega \subset \mathbb{R}^n$

$$
\begin{aligned}
\|\nabla u\|_{L^2(\Omega)}^2 &\le \epsilon\, C\, \|u\|_{L^{s\eta}(\Omega)}^{\beta_n\eta(1-\alpha)} \Big\| \frac{u}{\delta_{n-1}^\alpha} \Big\|_{L^{\frac{1}{1-\alpha}}(\Omega)} + C_\epsilon \Big\| \frac{u}{\delta_{n-1}^\alpha} \Big\|_{L^{\frac{1}{1-\alpha}}(\Omega)} + C\big\|\nabla u\big\|_{L^2(\Omega)} \\
&\le \epsilon\, C\, \big\|\nabla u\big\|_{L^2(\Omega)} \Big\| \frac{u}{\delta_{n-1}^\alpha} \Big\|_{L^{\frac{1}{1-\alpha}}(\Omega)} + C_\epsilon \Big\| \frac{u}{\delta_{n-1}^\alpha} \Big\|_{L^{\frac{1}{1-\alpha}}(\Omega)} + C\big\|\nabla u\big\|_{L^2(\Omega)}.
\end{aligned} \tag{3.40}
$$

Applying Lemma 3.9 with $\tau = \alpha$ we have

$$
\Big\| \frac{u}{\delta_{n-1}^\alpha} \Big\|_{L^q(\Omega)} \le C \|\nabla u\|_{L^2(\Omega)},
$$

where $\dfrac{1}{q} = \dfrac{1}{2} - \dfrac{1-\alpha}{n}$, i.e. $q = \dfrac{1}{1-\alpha}$ by the choice of α above. We can then conclude from (3.40) that

$$
\|\nabla u\|_{L^2(\Omega)} \le C,
$$

and the proof of Lemma 3.12 is complete in this case since also $\|u\|_{L^2(\Omega)} \le C$ by Lemma 3.11. Note that the choice of s in (3.36) is possible for $1 \le \eta < \frac{4n}{(n-1)(n-2)}$.

Case 2: $\eta \ge \frac{4n}{(n-1)(n-2)}$

We show that for $1 \le \gamma < \beta_n := \frac{n^2-1}{(n-1)^2\eta - 2n}$

$$
\big\|f(\cdot,u)\big\|_{L^r(\Omega)} \le \epsilon\, \big\|u\big\|_{L^{\rho\eta}(\Omega)}^{\beta_n\eta} + C_\epsilon, \tag{3.41}
$$

where

$$
1 \le \rho \le \frac{2^*}{\eta}, r > 1 \text{ and } \gamma\, r \le \frac{1}{\frac{1}{\rho} - \frac{2}{n-1}} = \rho^{\star} \quad (\frac{1}{\rho} > \frac{2}{n-1}). \tag{3.42}
$$

Here $\rho^{\star} = \frac{\rho(n-1)}{n-1-2\rho}$ denotes the critical Sobolev exponent for the embedding $W^{2,\rho}(\Omega') \subset L^{\rho^{\star}}(\Omega')$, $\Omega' \subset \mathbb{R}^{n-1}$.

In fact, first $u \in H^1_{cyl}(\Omega)$ which implies $u \in L^{\rho}(\Omega)$, $1 \le \rho \le 2^* \cdot \frac{1}{\eta}$, then as in (3.37), $\int_0^a u^\eta \, dx \in L^\rho(\Omega')$. By L^p regularity in Ω', we have $v \in W^{2,\rho}(\Omega')$. Then if $\gamma r \le \rho^{\star}$, we again have $\|v\|_{L^{\gamma r}(\Omega')} \le C\|v\|_{W^{2,\rho}(\Omega')}$ by the Sobolev embedding theorem. After this we have

$$
\begin{aligned}
\|f(\cdot,u)\|^r_{L^r(\Omega)} &= \int_\Omega (h(x))^r \cdot \Big[(-\Delta_{(n-1)})^{-1}\Big(\int_0^a u^\eta(x',x_n)\,dx_n\Big)\Big]^{\gamma r} dx \\
&\le C\int_{\Omega'} \Big[(-\Delta_{(n-1)})^{-1}\Big(\int_0^a u^\eta(x',x_n)\,dx_n\Big)\Big]^{\gamma r} dx' \\
&= C\|v\|^{\gamma r}_{L^{\gamma r}(\Omega')} \\
&\le C\|v\|^{\gamma r}_{W^{2,\rho}(\Omega')} \\
&\le C\Big\|\int_0^a u^\eta(x',x_n)\,dx_n\Big\|^{\gamma r}_{L^\rho(\Omega')} \\
&= C\Big(\int_{\Omega'}\big(\int_0^a u^\eta(x',x_n)dx_n\big)^\rho dx'\Big)^{\frac{\gamma r}{\rho}} \\
&\le C\Big(\int_{\Omega'}\int_0^a u^{\rho\eta}(x',x_n)dx_n dx'\Big)^{\frac{\gamma r}{\rho}} \\
&= C\|u\|^{\gamma r\eta}_{L^{\rho\eta}(\Omega)}.
\end{aligned}
\tag{3.43}
$$

Since $\rho \le \frac{2^*}{\eta}$, we have $f(x,u) \in L^r(\Omega)$ for fixed $u \in H^1_{cyl}(\Omega)$ and

$$
\|f(\cdot,u)\|_{L^r(\Omega)} \le C\|u\|^{\gamma\eta}_{L^{\rho\eta}(\Omega)} \,, \tag{3.44}
$$

and hence, for $1 < \gamma\eta < \beta_n\eta$ and every $\epsilon > 0$ there exists C_ϵ such that

$$
\|f(\cdot,u)\|_{L^r(\Omega)} \le \epsilon\,\|u\|^{\beta_n\eta}_{L^{\rho\eta}(\Omega)} + C_\epsilon.
$$

This shows (3.41).

From (3.34), (3.41), Lemma 3.10 and Hölder inequality we now deduce

$$
\begin{aligned}
\int_\Omega f(x,u)\,u\,dx &\le C\Big(\int_\Omega f(x,u)\,\frac{u^{\frac{1}{1-\alpha}}}{\delta_{n-1}^{\frac{\alpha}{1-\alpha}}}\,dx\Big)^{1-\alpha} \\
&\le C\Bigg(\|f(\cdot,u)\|_{L^r(\Omega)}\,\Bigg\|\frac{u^{\frac{1}{1-\alpha}}}{\delta_{n-1}^{\frac{\alpha}{1-\alpha}}}\Bigg\|_{L^h(\Omega)}\Bigg)^{1-\alpha} \\
&\le \big(\epsilon\|u\|_{L^{\rho\eta}(\Omega)}^{\beta_n\eta(1-\alpha)}+C_\epsilon\big)\,\Bigg\|\frac{u^{\frac{1}{1-\alpha}}}{\delta_{n-1}^{\frac{\alpha}{1-\alpha}}}\Bigg\|_{L^h(\Omega)}^{1-\alpha} \\
&= \epsilon\|u\|_{L^{\rho\eta}(\Omega)}^{\beta_n\eta(1-\alpha)}\Bigg\|\frac{u}{\delta_{n-1}^{\alpha}}\Bigg\|_{L^{\frac{h}{1-\alpha}}(\Omega)}+C_\epsilon\Bigg\|\frac{u}{\delta_{n-1}^{\alpha}}\Bigg\|_{L^{\frac{h}{1-\alpha}}(\Omega)},
\end{aligned}
\tag{3.45}
$$

where $\dfrac{1}{r}+\dfrac{1}{h}=1, r>1, h>1$. Again applying Lemma 3.9 with $\tau=\alpha$ and $q=\frac{h}{1-\alpha}$, we get

$$
\Bigg\|\frac{u}{\delta_{n-1}^{\alpha}}\Bigg\|_{L^{\frac{h}{1-\alpha}}(\Omega)}\le C\|\nabla u\|_{L^2(\Omega)},
\tag{3.46}
$$

where $\dfrac{1-\alpha}{h}=\dfrac{1}{2}-\dfrac{1-\alpha}{n}$, and thus $1-\alpha=\dfrac{nh}{2(n+h)}$. Since $0<1-\alpha<1$, so $0<\dfrac{nh}{2(n+h)}<1$, which implies

$$
1<h<\frac{2n}{n-2}=2^*,\quad r>\frac{2n}{n+2}.
$$

Then as before, we take

$$
\beta_n\eta=\frac{1}{1-\alpha}=2\big(\frac{1}{h}+\frac{1}{n}\big)=2\big(1-\frac{1}{r}+\frac{1}{n}\big)\,.
\tag{3.47}
$$

Now that $\rho\le\dfrac{2^*}{\eta}$, from (3.33), (3.45), (3.46) and (3.47) we get

$$
\begin{aligned}
\|\nabla u\|_{L^2(\Omega)}^2 &\le \epsilon\|\nabla u\|_{L^2(\Omega)}\Bigg\|\frac{u}{\delta_{n-1}^{\alpha}}\Bigg\|_{L^{\frac{h}{1-\alpha}}(\Omega)}+C_\epsilon\Bigg\|\frac{u}{\delta_{n-1}^{\alpha}}\Bigg\|_{L^{\frac{h}{1-\alpha}}(\Omega)}+C\|\nabla u\|_{L^2(\Omega)} \\
&\le \epsilon\|\nabla u\|_{L^2(\Omega)}^2+C_\epsilon\|\nabla u\|_{L^2(\Omega)}+C\|\nabla u\|_{L^2(\Omega)}.
\end{aligned}
\tag{3.48}
$$

We can then conclude from (3.48) that

$$
\|\nabla u\|_{L^2(\Omega)}\le C.
$$

Now combining (3.47) with $\gamma\,r\le\rho^{\star}$ and $\gamma<\beta_n$, we are going to find a best r to have the largest γ. So first we take $\rho=\frac{2^*}{\eta}$. Thus

$$\begin{aligned}\gamma \le (\frac{2^*}{\eta})^{\star} \cdot \frac{1}{r} &= \frac{1}{\frac{\eta}{2^*} - \frac{2}{n-1}} \cdot \frac{1}{r}\\ &= \frac{\frac{2n}{n-2}(n-1)}{(n-1)\eta - 2\frac{2n}{n-2}} \cdot \frac{1}{r}\\ &= \frac{2n(n-1)}{(n-1)(n-2)\eta - 4n} \cdot \frac{1}{r}.\end{aligned}$$

Since β_n is increasing with respect to r and the largest γ is decreasing with respect to r, we can let

$$\frac{2n(n-1)}{(n-1)(n-2)\eta - 4n} \cdot \frac{1}{r} = \frac{1}{\eta} \cdot 2(1 - \frac{1}{r} + \frac{1}{n})$$

and derive

$$r = \frac{n(2n^2 - 4n + 2)\eta - 4n^2}{(n^2-1)(n-2)\eta - 4n(n+1)}\ ,\ (\eta \ge \frac{4n}{(n-1)(n-2)}), \tag{3.49}$$

and thus, from (3.47)

$$\beta_n = \frac{n^2-1}{(n-1)^2\eta - 2n}\ ,\ (\eta \ge \frac{4n}{(n-1)(n-2)}).$$

Like the first case, the choice of ρ in (3.42) is possible for the second case of Theorem 3.2. Based on the above two cases ($1 \le \eta < \frac{4n}{(n-1)(n-2)}$, $\eta \ge \frac{4n}{(n-1)(n-2)}$), the proof of Lemma 3.12 is complete. □

Proof of Theorem 3.2. Likewise, we consider two cases:

Case 1: $1 \le \eta < \frac{4n}{(n-1)(n-2)}$, $1 < \gamma\eta \le \frac{2n+2}{n}$.

By (3.38), we know $f(x,u) \in L^\infty(\Omega)$ for any $u \in H^1_{cyl}(\Omega)$ weak solution of (3.30). According to Lemma 3.7 , for any fixed u, we have $u \in W^{2,p}(\Omega)$, for any $p > 1$ and since J_1 is a known smooth function, we have by Lemma 3.6 the estimate

$$\|u\|_{W^{2,p}(\Omega)} \le C\,\|f(\cdot,u)\|_{L^p(\Omega)} + \|K_1 J_1\|_{L^p(\Omega)} \le C\,\|f(\cdot,u)\|_{L^\infty(\Omega)} + C.$$

Choosing $p > \frac{n}{2}$, we get by Morrey's inequality

$$\|u\|_{L^\infty(\Omega)} \le C\|u\|_{W^{2,p}(\Omega)} \le C\|f(\cdot,u)\|_{L^\infty(\Omega)} + C.$$

In particular, due to (3.35) and Lemma 3.12, for $\frac{n-1}{2} < s \le \frac{2^*}{\eta}$

$$\begin{aligned}\|u\|_{L^\infty(\Omega)} &\le C\,\|u\|^{\gamma\eta}_{L^{s\eta}(\Omega)} + C\\ &\le C\,\|Du\|^{\gamma\eta}_{L^2(\Omega)} + C\\ &\le C.\end{aligned}$$

So that

$$\|u\|_{L^\infty(\Omega)} \le K.$$

Case 2: $\eta \ge \frac{4n}{(n-1)(n-2)}$, $1 < \gamma\eta \le \frac{n+1}{n-1} + \frac{2n\gamma}{(n-1)^2}$.

Similarly, for any fixed $u \in H^1_{cyl}(\Omega)$ weak solution of (3.30), according to (3.43), $f(x,u) \in L^r(\Omega)$, so $u \in W^{2,r}(\Omega)$ by Lemma 3.7, and by (3.14) with $p = r$ we have

$$\|u\|_{W^{2,r}(\Omega)} \le C\|f(x,u) + K_1 J_1\|_{L^r(\Omega)}. \tag{3.50}$$

Next, we have by the Sobolev inequality that $u \in L^\mu(\Omega)$, for $\mu \le r^* = \dfrac{1}{\frac{1}{r} - \frac{2}{n}} = \dfrac{nr}{n-2r}$. By (3.43) and (3.50) and Sobolev embedding theorem

$$\begin{aligned}
\|u\|_{L^\mu(\Omega)} \le C\,\|u\|_{W^{2,r}(\Omega)} &\le C\,\|f(\cdot,u)\|_{L^r(\Omega)} + C,\\
&\le C\,\|u\|^{\gamma r\eta}_{L^{\rho\eta}(\Omega)} + C\\
&\le C\,\|Du\|^{\gamma r\eta}_{L^2(\Omega)} + C\\
&\le C.
\end{aligned}$$

So finally we get

$$\|u\|_{L^\mu(\Omega)} \le C, \tag{3.51}$$

where $2^* \le \mu \le r^*$.

Notice for $\frac{n}{r} = 2$, we get $\eta = \frac{4n}{n^2-5n+2} > \frac{4n}{(n-1)(n-2)}$, where r is given by (3.49). We denote this η as η'. Hence when $1 \le \eta < \eta'$, thus $2 > \frac{n}{r}$, then Morrey's embedding theorem implies $r^* = \infty$, and then we are done.

Next, suppose that $\eta' \le \eta \le 2^*$, it then follows that $2 \le \frac{n}{r}$. Then we will get an improved uniform L^p bound of $f(x,u)$ by showing an improved uniform L^p bound of u. To see this we first consider

$$(\frac{r^*}{\eta})^\star = \frac{1}{\frac{\eta}{r^*} - \frac{2}{n-1}} = \frac{1}{\frac{\eta}{r} - \frac{2\eta}{n} - \frac{2}{n-1}} = \frac{n(n-1)r}{(n-1)(n\eta - 2\eta r) - 2nr}.$$

Similarly, in the case $2 > \dfrac{(n-1)\eta}{r^*}$, we can replace $(\frac{r^*}{\eta})^\star$ by $+\infty$. In the case $2 \le \dfrac{(n-1)\eta}{r^*}$, we compute

$$\begin{aligned}
\big\|f(\cdot,u)\big\|_{L^{\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}}(\Omega)} &= \Big\|h(x)\Big[(-\Delta_{(n-1)})^{-1}\Big(\int_0^a u^\eta(x',x_n)\,dx_n\Big)\Big]^\gamma\Big\|_{L^{\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}}(\Omega)} \\
&= \bigg(\int_\Omega h(x)^{\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}}\Big[(-\Delta_{(n-1)})^{-1}\Big(\int_0^a u^\eta(x',x_n)\,dx_n\Big)\Big]^{(\frac{r^*}{\eta})^{\star}}dx\bigg)^{\frac{\gamma}{(\frac{r^*}{\eta})^{\star}}} \\
&\le C\bigg(\int_{\Omega'}\Big[(-\Delta_{(n-1)})^{-1}\Big(\int_0^a u^\eta(x',x_n)\,dx_n\Big)\Big]^{(\frac{r^*}{\eta})^{\star}}dx'\bigg)^{\frac{\gamma}{(\frac{r^*}{\eta})^{\star}}} \\
&= C\Big\|\Big[(-\Delta_{(n-1)})^{-1}\Big(\int_0^a u^\eta(x',x_n)\,dx_n\Big)\Big]\Big\|^\gamma_{L^{(\frac{r^*}{\eta})^{\star}}(\Omega')} \\
&= C\Big\|v(x')\Big\|^\gamma_{L^{(\frac{r^*}{\eta})^{\star}}(\Omega')} \\
&\le C\Big\|v(x')\Big\|^\gamma_{w^{2,\frac{r^*}{\eta}}(\Omega')} \\
&\le C\Big\|\int_0^a u^\eta(x',x_n)\,dx_n\Big\|^\gamma_{L^{\frac{r^*}{\eta}}(\Omega')} \\
&= C\Big(\int_{\Omega'}(\int_0^a u^\eta(x',x_n)dx_n)^{\frac{r^*}{\eta}}\,dx'\Big)^{\frac{\gamma\eta}{r^*}} \\
&\le C\Big\|u(x',x_n)\Big\|^{\gamma\eta}_{L^{r^*}(\Omega)},
\end{aligned} \tag{3.52}$$

From (3.51), we deduce

$$\big\|f(\cdot,u)\big\|_{L^{\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}}(\Omega)} \le C.$$

Noting that

$$\begin{aligned}
\frac{(\frac{r^*}{\eta})^{\star}}{\gamma} &= \frac{1}{\frac{\eta}{r^*}-\frac{2}{n-1}}\cdot\frac{1}{\gamma} \\
&= \frac{1}{\frac{\eta}{r}-\frac{2\eta}{n}-\frac{2}{n-1}}\cdot\frac{1}{\gamma} \\
&> \frac{n(n-1)r}{(n-1)(n\eta-2\eta r)-2nr}\cdot\frac{\eta(n^2-2n+1)-2n}{n^2-1} \qquad (\gamma<\beta_n),
\end{aligned}$$

hence

$$\frac{(\frac{r^*}{\eta})^{\star}}{\gamma} > \frac{n(n-1)r}{(n-1)(n\eta-2\eta r)-2nr}\cdot\frac{\eta(n^2-2n+1)-2n}{n^2-1} > r \tag{3.53}$$

where the last inequality follows by elementary calculations, using (3.49), we see that $f(\cdot,u)$ is bounded in an improved L^p space, if $2 \leq \frac{(n-1)\eta}{r^*}$. Then taking $p = \frac{(\frac{r^*}{\eta})^{\star}}{\gamma}$, by (3.53) and the Sobolev inequality, we have,

$$\|u\|_{L^{\left(\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}\right)^*}(\Omega)} \leq \|u\|_{W^{2,\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}}(\Omega)} \leq \left\|f(\cdot,u)\right\|_{L^{\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}}(\Omega)} + C \leq C, \tag{3.54}$$

where $\left(\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}\right)^* = \frac{1}{\frac{\gamma}{(\frac{r^*}{\eta})^{\star}} - \frac{2}{n}}$. From (3.53), we see $\left(\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}\right)^* > r^*$, which means we get a better uniform L^p bound of u. Afterwards, we repeat the computation of (3.52) and get

$$\left\|f(x,u)\right\|_{L^{\frac{\left(\frac{\left(\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}\right)^*}{\eta}\right)^{\star}}{\gamma}}(\Omega)} \leq \|u\|_{L^{\left(\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}\right)^*}(\Omega)} \leq C.$$

Iterating (3.52)-(3.54), finally, we will derive

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

Thus, we have completed the proof of Theorem 3.2. $\square$

3.6 Fixed point theorem and existence of the positive solution

In this section we complete the proof of Theorem 3.1. We first show a maximum principle for the Poisson equation with mixed boundary conditions:

Lemma 3.13 ([23]). *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be the cylinder in (3.2) and let Γ_1, Γ_2 be a partition of $\partial\Omega$, with $\Gamma_1 = \partial\Omega' \times [0,a]$, $\Gamma_2 = \Omega' \times \{0,a\}$. Let $g \in C_0^\infty(\Omega)$, $g \geq 0$, $g \not\equiv 0$, and let u denote the solution of*

$$\begin{cases} -\Delta u &= g \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma_1 \\ \dfrac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma_2 \end{cases} \tag{3.55}$$

where ν is the outer unit normal vector to $\partial\Omega$. Then the solution of (3.55) satisfies:

$$u \geq 0 \text{ in } \overline{\Omega}.$$

Proof. If the claim were not true, then there would exists a $x_0 \in \overline{\Omega}$ such that $u(x_0) < 0$. Without loss of generality, we suppose $u(x_0) = \min_{x\in\overline{\Omega}} u(x) < 0$. By the assumption, we know $x_0 \notin \Gamma_1$. Next we show $x_0 \notin \Gamma_2$; otherwise, we may assume that $x_0 \in \Omega' \times \{0\}$ or $\Omega' \times \{a\}$, by interior regularity, since $g \in C_0^\infty(\Omega)$, we obtain $u \in C^\infty(\Omega)$ and u in $W^{2,p}(\Omega)(1 \le p < \infty)$. In addition, $W^{2,p}(\Omega) \hookrightarrow C^1(\bar{\Omega})$([2] Theorem 4.12, PART II), so we have $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Since $\Omega' \times \{0\}$ or $\Omega' \times \{a\}$ is flat, Ω satisfies the interior ball condition at x_0, and from Hopf's lemma we have $\dfrac{\partial u(x_0)}{\partial \nu} < 0$, which contradicts the assumption on Γ_2. So x_0 is an interior point of Ω. But due to the maximum principle, u cannot have a negative minimum in Ω. □

We are now in the position to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. For every fixed $u \ge 0$ in $C^1(\Omega)$, by Lax-Milgram theorem we know there exists a unique solution for equation (3.2), which we denote by w_u. That is, $-\Delta_{(n)} w_u = f(x, u)$, with $f(x, u) = h(x)[(-\Delta_{(n-1)})^{-1} \int_0^a u^\eta(x)\, dx_n]^\gamma$. To solve problem (3.2), we define the mapping $u \to w_u =: F(u)$. If there is a fixed point of F in $C^1(\Omega)$ such that $F(u) = u$, we are done. Now we check that F satisfies the following fixed point theorem ([37] Theorem 3.1, see also [26] Theorem 3.1).

F: $C^1(\Omega) \to C^1(\Omega)$ a compact mapping, acting in the cone of non-negative functions, will have a fixed point u with $0 < r \le \|u\|_{C^1(\Omega)} \le R < \infty$ provided

1) $Fu \ne s'u, s' \ge 1$ for $\|u\|_{C^1(\Omega)} = r$ and

2) $Fu \ne u - t\tilde{J}_1, t \ge 0$, for $\|u\|_{C^1(\Omega)} = R$,

where $\tilde{J}_1 = (-\Delta_{(n)})^{-1} J_1$.

Step 1: $F : C^1(\Omega) \to C^1(\Omega)$ is compact. Let $\mathcal{A} \subset C^1(\Omega)$ be a bounded set, for $u \in \mathcal{A}$ we have

$$
\begin{aligned}
\|f(x,u)\|_{L^\infty(\Omega)} &= \left\| h(x)\left[(-\Delta_{(n-1)})^{-1}\left(\int_0^a u^\eta(x',x_n)\,dx_n\right)\right]^\gamma \right\|_{L^\infty(\Omega)} \\
&\le C\max_{x\in\overline{\Omega}}\{h(x)\}\left\|(-\Delta_{(n-1)})^{-1}\left(\int_0^a u^\eta(x)\,dx_n\right)\right\|^\gamma_{L^\infty(\Omega')} \\
&\le C\left\|(-\Delta_{(n-1)})^{-1}\left(\int_0^a u^\eta(x)\,dx_n\right)\right\|^\gamma_{W^{2,s}(\Omega')}\ ,\ s>(n-1)/2 \\
&\le C\left\|\int_0^a u^\eta(x)\,dx_n\right\|^\gamma_{L^s(\Omega')} \\
&\le C\|u\|^{\gamma\eta}_{L^{s\eta}(\Omega)} \\
&\le C\|u\|^{\gamma\eta}_{L^\infty(\Omega)} \\
&\le C\|u\|^{\gamma\eta}_{C^1(\Omega)} \\
&\le C,
\end{aligned}
\tag{3.56}
$$

thus $f(x,u)\in L^\infty(\Omega)$ and $\{f(x,u), u\in\mathcal{A}\}$ is uniformly bounded. Since $-\Delta_{(n)}w_u = f(x,u)$, by Lemma 3.6 and Lemma 3.7, $w_u\in W^{2,q}(\Omega)$, q large enough, and lies in a bounded set in $W^{2,q}(\Omega)$. Then by Morrey's inequality, we get for $q>n$, $w_u\in C^{1,\gamma'}(\Omega)$, that is

$$\|w_u\|_{C^{1,\gamma'}(\bar{\Omega})} \le C\|w_u\|_{W^{2,q}(\Omega)} \le C\|f(\cdot,u)\|_{L^q(\Omega)} + C \le C\|f(\cdot,u)\|_{L^\infty(\Omega)} + C \le C,$$

where $\gamma' = 1-\frac{n}{q}$. Therefore we have for every x, y in Ω, and $\forall u\in\mathcal{A}$

$$|Dw_u(x) - Dw_u(y)| \le C|x-y|^{\gamma'}.$$

Hence $\forall\epsilon>0$, we take $\delta = (\frac{\epsilon}{C})^{\gamma'/1}$ then, if $|x-y|<\delta$, $\{w_u\}$ satisfies

$$|Dw_u(x) - Dw_u(y)| \le C|x-y|^{\gamma'} < \epsilon$$

which means $\{w_u, u\in\mathcal{A}\}$ is uniformly bounded and equicontinuous in $C^1(\Omega)$. According to Arzelà-Ascoli theorem, it is in a compact set in $C^1(\Omega)$.

Moreover, it is easy to see that F is continuous, since it is a composition of continuous maps. Precisely, we would like to prove $\forall\epsilon>0$, there exists a δ, such that when $\|u-u_0\|_{C^1}<\delta$, we have $\|w-w_0\|_{C^1}<\epsilon$, where

$$
\begin{cases}
-\Delta_{(n)}w = f(x,u) \\
-\Delta_{(n)}w_0 = f(x,u_0).
\end{cases}
\tag{3.57}
$$

We notice that $\|w-w_0\|_{C^1(\Omega)} \leq \|w-w_0\|_{C^{1,\gamma}(\bar{\Omega})} \leq C\|w-w_0\|_{W^{2,q}(\Omega)}(q>N)$. From (3.57) we also have

$$-\Delta_{(n)}(w-w_0) = f(x,u) - f(x,u_0),$$

by regularity theory, Lagrange mean value theorem, and (3.56), $\forall \epsilon > 0$, we take $\delta = \min\{1,\epsilon\}$, for $\|u-u_0\|_{C^1} < \delta$,

$$\begin{aligned}
\|w-w_0\|_{w^{2,q}} &\leq C\|f(x,u)-f(x,u_0)\|_{L^\infty} \\
&= C\Big\|h(x)[(-\Delta_{(n-1)})^{-1}\int_0^a u^\eta(x)\,dx_n]^\gamma - h(x)[(-\Delta_{(n-1)})^{-1}\int_0^a u_0^\eta(x)\,dx_n]^\gamma\Big\|_{L^\infty} \\
&\leq C\Big\|[(-\Delta_{(n-1)})^{-1}\int_0^a u^\eta(x)\,dx_n]^\gamma - [(-\Delta_{(n-1)})^{-1}\int_0^a u_0^\eta(x)\,dx_n]^\gamma\Big\|_{L^\infty} \quad (h(x) \text{ is bounded}) \\
&\leq C\Big\|\gamma\cdot\xi^{\gamma-1}\cdot[(-\Delta_{(n-1)})^{-1}\int_0^a (u^\eta-u_0^\eta)(x)\,dx_n]\Big\|_{L^\infty} \\
&\quad \text{(apply Lagrange mean value theorem)} \\
&\leq C\Big(\|(-\Delta_{(n-1)})^{-1}\int_0^a u^\eta(x)\,dx_n\|_{L^\infty} + \|(-\Delta_{(n-1)})^{-1}\int_0^a u_0^\eta(x)\,dx_n\|_{L^\infty}\Big)^{\gamma-1} \\
&\quad \cdot\Big\|(-\Delta_{(n-1)})^{-1}\int_0^a (u^\eta-u_0^\eta)(x)\,dx_n\Big\|_{L^\infty} \\
&\quad \text{(enlarge } \xi \text{ as } \|(-\Delta_{(n-1)})^{-1}\textstyle\int_0^a u^\eta(x)\,dx_n\|_{L^\infty} + \|(-\Delta_{(n-1)})^{-1}\int_0^a u_0^\eta(x)\,dx_n\|_{L^\infty}.\text{)} \\
&\leq C(\|u\|_{C^1}^\eta + \|u_0\|_{C^1}^\eta)^{\gamma-1}\cdot\Big(\eta(\|u\|_{L^\infty}+\|u_0\|_{L^\infty})^{\eta-1}\|u-u_0\|_{C^1}\Big) \text{ apply (3.56)} \\
&\leq C\Big((1+\|u_0\|_{C^1})^\eta + \|u_0\|_{C^1}^\eta\Big)^{\gamma-1}\cdot(1+2\|u_0\|_{C^1})^{\eta-1}\cdot\|u-u_0\|_{C^1} \\
&\leq C\epsilon
\end{aligned}$$

where ξ is between $(-\Delta_{(n-1)})^{-1}\int_0^a u^\eta(x)dx_n$ and $(-\Delta_{(n-1)})^{-1}\int_0^a u_0^\eta(x)dx_n$. Thus, F is continuous. Hence, F is a compact mapping from $C^1(\Omega)$ to $C^1(\Omega)$.

Step 2: F maps the non-negative cone in $C^1(\Omega)$ into itself. For this we are going to prove that when u is fixed non-negative, then w_u is non-negative. Indeed, w_u satisfies

$$\begin{cases} -\Delta_{(n)}w_u(x) = f(x,u), & x\in\Omega \\ w_u(x) = 0, & x\in\partial\Omega'\times[0,a] \\ \partial_{x_n}w_u(x) = 0, & x\in\Omega'\times\{0,a\}, \end{cases} \tag{3.58}$$

where $f(x,u) = f(x) = h(x)[(-\Delta_{(n-1)})^{-1}\int_0^a u^\eta(x',x_n)dx_n]^\gamma$. By (3.56), $f\in L^\infty(\Omega)$ so that $f\in L^p(\Omega)$ for any $p>1$ when u is fixed in $C^1(\Omega)$.

We assume

$$\begin{cases} -\Delta_{(n)} w_{u_n}(x) = f_n, & x \in \Omega \\ w_{u_n}(x) = 0, & x \in \partial\Omega' \times [0,a] \\ \partial_{x_n} w_{u_n}(x) = 0, & x \in \Omega' \times \{0,a\}, \end{cases} \tag{3.59}$$

where $f_n \in C_0^\infty(\Omega)$, $f_n \geq 0$, $\|f_n - f\|_{L^p(\Omega)} \to 0$ $(1 \leq p < \infty)$. Applying Lemma 3.13, we get

$$w_{u_n} \geq 0, \quad \forall n \in \mathbb{N}.$$

On the other hand, subtracting (3.58) from (3.59), we get

$$\begin{cases} -\Delta_{(n)}(w_{u_n}(x) - w_u(x)) = f_n - f, & x \in \Omega \\ w_{u_n}(x) - w_u(x) = 0, & x \in \partial\Omega' \times [0,a] \\ \partial_{x_n}(w_{u_n}(x) - w_u(x)) = 0, & x \in \Omega' \times \{0,a\}. \end{cases}$$

Since $f_n - f \in L^\infty(\Omega)$, by Lemma 3.7, we have $w_{u_n} - w_u \in W^{2,p}(\Omega)$, p large enough. Then by Lemma 3.6 and Morrey's inequality we have $w_{u_n} - w_u \in C^{1,\gamma'}(\Omega)$ and, $\|w_{u_n} - w_u\|_{C^{1,\gamma'}(\Omega)} \leq C\|w_{u_n} - w_u\|_{W^{2,p}(\Omega)} \leq C\|f_n - f\|_{L^p(\Omega)}$ for $p > n$. So, $\|w_{u_n} - w_u\|_{C^{1,\gamma'}(\Omega)} \leq C\|f_n - f\|_{L^p(\Omega)}$. Furthermore

$$\lim_{n\to\infty} \|w_{u_n} - w_u\|_{C^{1,\gamma'}(\Omega)} \leq C \lim_{n\to\infty} \|f_n - f\|_{L^p(\Omega)} = 0,$$

which implies,

$$\lim_{n\to\infty} \{\sup_{x\in\bar{\Omega}} |(w_{u_n} - w_u)(x)| + \sup_{x\in\bar{\Omega}} |(Dw_{u_n} - Dw_u)(x)|\} = 0$$

so,

$$w_{u_n} \to w_u \qquad \forall x \in \Omega.$$

Since $w_{u_n} \geq 0$, then $w_u \geq 0$.

Next we verify the two conditions 1) and 2).

1) holds for $r < (\frac{1}{C})^{\frac{1}{\gamma\eta-1}+1}$, where C will be determined later. If not, we suppose there exists $s' \geq 1$ and u with $\|u\|_{C^1(\Omega)} = r$ such that $Fu = s'u$. Since $-\Delta_{(n)}F(u) = f(x,u)$, we obtain

$$-\Delta_{(n)}(Fu) = -\Delta_{(n)}(s'u) = f(x,u)$$

then

$$-\Delta_{(n)}u = \frac{1}{s'}f(x,u).$$

Multiplying by u and taking the integral over Ω on both sides, we have,

$$\int_\Omega -\Delta_{(n)}u \cdot u = \frac{1}{s'}\int_\Omega f(x,u)\cdot u \leq \int_\Omega f(x,u)\cdot u. \tag{3.60}$$

Case 1: $1 \leq \eta < \frac{4n}{(n-1)(n-2)}$, $1 < \gamma\eta \leq \frac{2n+2}{n}$; by (3.38), Hölder inequality and (3.60) we get

$$\int_\Omega |\nabla u|^2\,dx \leq \int_\Omega f(x,u)\cdot u\,dx \leq C\|f(x,u)\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)} \leq C\|u\|^{\gamma\eta}_{L^{s\eta}(\Omega)}\|u\|_{L^2(\Omega)}.$$

From (3.36), the Sobolev embedding inequality and Lemma 3.11 we derive,

$$\|Du\|^2_{L^2(\Omega)} \leq C\|Du\|^{\gamma\eta+1}_{L^2(\Omega)}. \tag{3.61}$$

and hence

$$\left(\frac{1}{C}\right)^{\frac{1}{\gamma\eta-1}} \leq \|Du\|_{L^2(\Omega)} \leq C\|Du\|_{L^\infty(\Omega)}.$$

However, by assumption

$$\left(\frac{1}{C}\right)^{\frac{1}{\gamma\eta-1}+1} > r = \|u\|_{C^1(\Omega)} \geq \|Du\|_{L^\infty(\Omega)}$$

which is a contradition.

Case 2: $\eta \geq \frac{4n}{(n-1)(n-2)}$, $1 < \gamma\eta \leq \frac{n+1}{n-1} + \frac{2n\gamma}{(n-1)^2}$; from (3.44) and (3.60), we have

$$\int_\Omega |\nabla u|^2\,dx \leq \int_\Omega f(x,u)\cdot u\,dx \leq C\|f(x,u)\|_{L^r(\Omega)}\|u\|_{L^h(\Omega)} \leq C\|u\|^{\gamma\eta}_{L^{2^*}(\Omega)}\|u\|_{L^h(\Omega)}, \tag{3.62}$$

where $\dfrac{1}{r} + \dfrac{1}{h} = 1$. Moreover, since $r > 2$, so $h < 2 < 2^*$. Then by the Sobolev embedding inequality, we have the same result as (3.61) . Thus 1) will follow by the same proof.

For 2), we show that there exists $R_1 > 0$ such that there is no solution of $F(u) = u - t\tilde{J}_1$ with $\|u\|_{C^1(\Omega)} \geq R_1, \forall t \geq 0$. Indeed, suppose $u \in H^1_{cyl}(\Omega)$ a solution of $F(u) = u - t\tilde{J}_1$, then $-\Delta_{(n)}F(u) = f(x,u)$, that is,

$$-\Delta_{(n)}u = f(x,u) + tJ_1. \tag{3.63}$$

then by Theorem 3.2, $\|u\|_{L^\infty(\Omega)} \leq K$, K independent of $t \geq 0$. We conclude that for any $1 < q < \infty$,

$$\|u\|_{C^1(\Omega)} < \|u\|_{C^{1,\gamma'}(\Omega)} \leq C\|u\|_{W^{2,q}(\Omega)} \leq \|f(x,u)\|_{L^\infty(\Omega)} \leq C\|u\|^{\gamma\eta}_{L^\infty(\Omega)} \leq C\cdot K^{\gamma\eta} = R_1.$$

So for any $R > R_1$, $F(u) \neq u - t\tilde{J}_1$. □

A Appendix

Lemma A.1. *$T : L^2(U) \to L^2(U)$ is a bounded linear operator. i.e. T is of strong type (2, 2).*

Proof. (see [36], theorem 9.9) First we consider $f \in C_0^\infty(U) \subset C_0^\infty(\mathbb{R}^n)$, then we have $w \in C^\infty(\mathbb{R}^n)$ and satisfies

$$\Delta w = f(x), \forall x \in \mathbb{R}^n.$$

Consequently, for any ball B_R containing the support of f,

$$\int_{B_R} (\Delta w)^2 = \int_{B_R} f^2.$$

Applying Green's first identity twice, we obtain

$$\begin{aligned}
\int_{B_R} |D^2 w|^2 dx = \sum_{i,j=1}^{n} \int_{B_R} D_{ij}w \cdot D_{ij}w dx &= \sum_{i,j=1}^{n} \Big(\int_{\partial B_R} D_i w \cdot D_{ij} w dx - \int_{B_R} D_j(D_{ij}w) \cdot D_i w dx \Big) \\
&= \sum_{i,j=1}^{n} \int_{\partial B_R} D_i w \cdot D_{ij} w dx - \sum_{i,j=1}^{n} \int_{B_R} D_{ijj} w \cdot D_i w dx \\
&= \sum_{i,j=1}^{n} \int_{\partial B_R} D_i w \cdot D_{ij} w dx \\
&\quad - \sum_{i,j=1}^{n} \Big(\int_{\partial B_R} D_{jj} w \cdot D_i w dx - \int_{B_R} D_{jj} w \cdot D_{ii} w dx \Big) \\
&= \sum_{i,j=1}^{n} \int_{\partial B_R} D_i w \cdot D_{ij} w dx \\
&\quad - \Big(\sum_{i=1}^{n} \int_{\partial B_R} \Delta w \cdot D_i w dx - \int_{B_R} (\Delta w)^2 dx \Big) \\
&= \int_{\partial B_R} Dw \cdot \frac{\partial}{\partial \nu} Dw dx + \int_{B_R} (\Delta w)^2 dx.
\end{aligned}$$

Since

$$D_i w = \int_U D_i \Gamma(x-y) f(y) dy, \quad D_{ij} w = \int_U D_{ij} \Gamma(x-y) f(y) dy,$$

and

$$|D_i \Gamma(x-y)| \le C|x-y|^{1-n}, \quad |D_{ij}\Gamma(x-y)| \le C|x-y|^{-n},$$

we have

$$Dw = O(R^{1-n}), \quad D^2 w = O(R^{-n})$$

uniformly on ∂B_R as $R \to \infty$, whence follows the identity

$$\int_{R^n} |D^2 w|^2 = \int_U f^2,$$

which means

$$\|Tf\|_{L^2(U)} \le \|f\|_{L^2(U)}, \quad \forall f \in C_0^\infty(U). \tag{A.1}$$

For arbitrary $f \in L^2(U)$, we pick a sequence $\{f_k\} \subset C_0^\infty(\Omega)$ that converges to f in $L^2(\Omega)$, so that $\{f_k\}$ is a Cauchy sequence in $L^2(U)$ and for $k, l \to \infty$, $\epsilon > 0$, $\|f_k - f_l\|_{L^2(U)} < \epsilon$. From (A.1),

$$\|Tf_k - Tf_l\|_{L^2(U)} \le \|f_k - f_l\|_{L^2(U)} < \epsilon \tag{A.2}$$

it follows that $\{Tf_k\}$ is a Cauchy sequence in $L^2(U)$ and it will converge to a point in $L^2(U)$. We denote the unique limit point as Tf. We complete the proof by taking the limit in (A.2). $\square$

Lemma A.2. *T is of weak type (1,1).*

Before the proof, we need the following well-known Calderón-Zygmund's Decomposition Lemma.

Lemma A.3 (Calderón-Zygmund's Decomposition Lemma)**.** *For $f \in L^1(\mathbb{R}^n)$, fixed $\alpha > 0$, $\exists\ E, G$ such that*
(i) $\mathbb{R}^n = E \cup G$, $E \cap G = \emptyset$;
(ii) $|f(x)| \leq \alpha$, a.e. $x \in E$;
(iii) $G = \bigcup\limits_{k=1}^{\infty} Q_k$, $\{Q_k\}$: disjoint cubes such that

$$\alpha < \frac{1}{|Q_k|} \int_{Q_k} |f(x)| dx \leq 2^n \alpha.$$

Proof. Since $\int_{\mathbb{R}^n} f(x)dx$ is finite, for a given $\alpha > 0$, one can pick a cube Q_0 sufficiently large, such that

$$\int_{Q_0} f(x)dx \leq \alpha |Q_0|.$$

Divide Q_0 into 2^n equal sub-cubes with disjoint interior. Those sub-cubes Q satisfying

$$\int_{Q} f(x)dx \leq \alpha |Q|$$

are similarly sub-divided, and this process is repeated infinitely. Let $\mathcal{Q}$ denote the set of sub-cubes of Q thus obtained that satisfy

$$\int_{Q} f(x)dx > \alpha |Q|.$$

For each $Q \in \mathcal{Q}$, let $\tilde{Q}$ be its predecessor, i.e., Q is one of the 2^n sub-cubes of $\tilde{Q}$. Then obviously, we have $|\tilde{Q}|/|Q| = 2^n$, and consequently,

$$\alpha < \frac{1}{|Q|} \int_{Q} f(x)dx \leq \frac{1}{|Q|} \int_{\tilde{Q}} f(x)dx \leq \frac{1}{|Q|} \alpha |\tilde{Q}| = 2^n \alpha.$$

Let

$$G = \bigcup_{Q \in \mathcal{Q}} Q, \text{and } E = Q_0 \setminus G.$$

Then (iii) follows immediately. To see (ii), noticing that each point of E lies in a nested sequence of cubes Q with diameters tending to zero and satisfying

$$\int_{Q} f(x)dx \leq \alpha |Q|,$$

now by Lebesgue's Differentiation Theorem, we have

$$f(x) \leq \alpha, \ \textit{a.e.} \text{ in } E.$$

This completes the proof of the lemma. $\square$

Proof of Lemma A.2. For any $f \in L^1(U)$, to apply the Calderón-Zygmund's Decomposition Lemma, we first extend f to vanish outside U. For any given $\alpha > 0$, fix a large cube Q_0 in $\mathbb{R}^n$, such that

$$\int_{Q_0} |f(x)|dx \leq \alpha |Q_0|.$$

We show that

$$\mu_{Tf}(\alpha) := \left|\{x \in \mathbb{R}^n \big| |Tf(x)| \geq \alpha\}\right| \leq C \frac{\|f\|_{L^1(U)}}{\alpha}. \tag{A.3}$$

Split the function f into the "good" part g and "bad" part b: $f = g + b$, where

$$g(x) = \begin{cases} f(x) & for \ x \in E \\ \dfrac{1}{|Q_k|}\displaystyle\int_{Q_k} f(x)dx & for \ x \in Q_k, k = 1, 2, \cdots. \end{cases}$$

Since the operator T is linear, $Tf = Tg + Tb$; and therefore

$$\mu_{Tf}(\alpha) \leq \mu_{Tg}(\frac{\alpha}{2}) + \mu_{Tb}(\frac{\alpha}{2}).$$

We will estimate $\mu_{Tg}(\frac{\alpha}{2})$ and $\mu_{Tb}(\frac{\alpha}{2})$ separately. The estimate of the first one is easy, because $g \in L^2$. To estimate $\mu_{Tb}(\frac{\alpha}{2})$, we divide $\mathbb{R}^n$ into two parts: G^* and $E^* := \mathbb{R}^n \setminus G^*$ (See below for the precise definition of G^*). We will show that

(a) $|G^*| \leq \dfrac{C}{\alpha}\|f\|_{L^1(U)}$ and

(b) $|\{x \in E^* \mid |Tb(x)| \geq \dfrac{\alpha}{2}\}| \leq \dfrac{C}{\alpha}\displaystyle\int_{E^*} |Tb(x)|dx \leq \dfrac{C}{\alpha}\|f\|_{L^1(U)}$.

These will imply the desired estimate for $\mu_{Tb}(\frac{\alpha}{2})$.

Obviously, from the definition of g, we have

$$|g(x)| \leq 2^n\alpha, \ \text{almost everywhere} \tag{A.4}$$

and

$$b(x) = 0 \text{ for } x \in E, \ \text{and} \int_{Q_k} b(x)dx = 0 \text{ for } k = 1, 2, \cdots.$$

We first estimate μ_{Tg}. By Lemma A.1 and (A.4), we derive

$$\mu_{Tg}(\frac{\alpha}{2}) \leq \frac{4}{\alpha^2}\int_{\mathbb{R}^n} g^2(x)dx \leq \frac{2^{n+2}}{\alpha}\int_{\mathbb{R}^n} |g(x)|dx \leq \frac{2^{n+2}}{\alpha}\int_{\mathbb{R}^n} |f(x)|dx. \tag{A.5}$$

We then estimate μ_{Tb}. Let

$$b_k(x) = \begin{cases} b(x) & \text{for } x \in Q_k \\ 0 & \text{elsewhere.} \end{cases}$$

Then

$$Tb = \sum_{k=1}^{\infty} Tb_k.$$

For each fixed k, let $\{b_{km}\} \subset C_0^\infty(Q_k)$ be a sequence converging to b_k in $L^2(U)$ satisfying

$$\int_{Q_k} b_{km}(x)dx = \int_{Q_k} b_k(x)dx = 0. \tag{A.6}$$

From the expression

$$Tb_{km}(x) = \int_{Q_k} D_{ij}\Gamma(x-y)b_{km}(y)dy,$$

one can see that due to the singularity of $D_{ij}\Gamma(x-y)$ in Q_k and the fact that b_{km} may not be bounded in Q_k, one can only estimate $Tb_{km}(x)$ when x is of a positive distance away from Q_k. For this reason, we cover Q_k by a bigger ball B_k which has the same center as Q_k, and the radius of the ball δ_k is the same as the diameter of Q_k. We now estimate the integral in the complement of B_k:

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus B_k} |Tb_{km}|(x)dx &= \int_{Q_0\setminus B_k} \Big| \int_{Q_k} D_{ij}\Gamma(x-y)b_{km}(y)dy \Big| dx \\
&= \int_{Q_0\setminus B_k} \Big| \int_{Q_k} [D_{ij}\Gamma(x-y) - D_{ij}\Gamma(x-\bar{y})]b_{km}(y)dy \Big| dx \\
&\leq C\delta_k \int_{Q_0\setminus B_k} \frac{1}{|x|^{n+1}}dx \cdot \Big| \int_{Q_k} b_{km}(y)dy \Big| \qquad (A.7) \\
&\leq C_1\delta_k \int_{\delta_k}^{\infty} \frac{1}{r^2}dr \cdot \int_{Q_k} |b_{km}(y)|dy \\
&\leq C_2 \int_{Q_k} |b_{km}(y)|dy
\end{aligned}$$

where $\bar{y}$ is the center of the cube Q_k. One small trick here is to add a term (which is 0 by (A.6)):

$$\int_{Q_k} D_{ij}\Gamma(x-\bar{y})b_{km}(y)dy$$

to produce a helpful factor δ_k by applying the mean value theorem to the difference:

$$D_{ij}\Gamma(x-y) - D_{ij}\Gamma(x-\bar{y}) = (y-\bar{y}) \cdot D(D_{ij}\Gamma)(x-\xi) \leq \delta_k |D(D_{ij}\Gamma)(x-\xi)|.$$

Now letting $m \to \infty$ in (A.7), we obtain

$$\int_{\mathbb{R}^n \setminus B_k} |Tb_k(x)| dx \leq C \int_{Q_k} |b_k(y)| dy.$$

Let

$$G^* = \bigcup_{k=1}^{\infty} B_k \text{ and } E^* = \mathbb{R}^n \setminus G^*.$$

It follows that

$$\begin{aligned} \int_{E^*} |Tb(x)| dx &\leq C \sum_{k=1}^{\infty} \int_{\mathbb{R}^n \setminus G^*} |Tb_k| dx \leq C \sum_{k=1}^{\infty} \int_{\mathbb{R}^n \setminus B_k} |Tb_k| dx \\ &\leq C \sum_{k=1}^{\infty} \int_{Q_k} |b_k(y)| dy \leq C \int_{\mathbb{R}^n} |f(x)| dx. \end{aligned} \tag{A.8}$$

Obviously

$$\mu_{Tb}(\frac{\alpha}{2}) \leq |G^*| + |\{x \in E^* \mid Tb(x) \geq \frac{\alpha}{2}\}|. \tag{A.9}$$

By (iii) in the Calderón-Zygmund's Decomposition Lemma, we have

$$|G^*| = \sum_{k=1}^{\infty} |B_k| = C \sum_{k=1}^{\infty} |Q_k| \leq \frac{C}{\alpha} \sum_{k=1}^{\infty} \int_{Q_k} |f(x)| dx = \frac{C}{\alpha} \int_{\mathbb{R}^n} |f(x)| dx. \tag{A.10}$$

Write

$$E^*_\alpha = \{x \in E^* \mid |Tb(x)| \geq \frac{\alpha}{2}\}.$$

Then by (A.8), we derive

$$|E^*_\alpha| \frac{\alpha}{2} \leq \int_{E^*_\alpha} |Tb(x)| dx \leq \int_{E^*} |Tb(x)| dx \leq C \int_{\mathbb{R}^n} |f(x)| dx. \tag{A.11}$$

Now the desired inequality (A.3) is a direct consequence of (A.5), (A.9), (A.10), and (A.11). This completes the proof of the Lemma. □

Lemma A.4. *T is of strong type (r, r) for any* $1 < r \leq 2$.

Proof. In the previous lemmas, we have shown that the operator T is of weak type (1, 1) and strong type (2, 2) (of course also weak type (2, 2)). Now Lemma A.4 is a direct consequence of the Marcinkiewicz interpolation theorem in the following restricted form: interpolation □

Lemma A.5. *Let T be a linear operator from $L^p(U) \cap L^q(U)$ into itself with $1 \le p < q < \infty$. If T is of weak type (p, p) and weak type (q, q), then for any $p < r < q$, T is of strong type (r, r). More precisely, if there exist constants B_p and B_q, such that, for any $t > 0$,*

$$\mu_{Tf}(t) \le \Big(\frac{B_p \|f\|_p}{t}\Big)^p \text{and } \mu_{Tf}(t) \le \Big(\frac{B_q \|f\|_q}{t}\Big)^q, \forall f \in L^p(U) \cap L^q(U),$$

then

$$\|Tf\|_r \le C B_p^{\theta} B_q^{1-\theta} \|f\|_r, \forall f \in L^p(U) \cap L^q(U),$$

where

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$$

and C depends only on $p, q,$ and r.

Proof of Lemma A.5. For any number $s > 0$, let

$$g(x) = \begin{cases} f(x) & \text{if } |f(x)| \le s \\ 0 & \text{if } |f(x)| > s. \end{cases}$$

We split f into the good part g and the bad part $b : f(x) = g(x) + b(x)$. Then

$$|Tf(x)| \le |Tg(x)| + |Tb(x)|,$$

and hence

$$\begin{aligned} \mu(t) \equiv \mu_{Tf}(t) &\le \mu_{Tg}(\frac{t}{2}) + \mu_{Tb}(\frac{t}{2}) \\ &\le \left(\frac{2B_q}{t}\right)^q \int_U |g(x)|^q dx + \left(\frac{2B_p}{t}\right)^p \int_U |b(x)|^p dx. \end{aligned}$$

It follows that

$$\begin{aligned} \int_U |Tf|^r dx = \int_0^{\infty} \mu(t) d(t^r) &= r \int_0^{\infty} t^{r-1} \mu(t) dt \\ &\le r(2B_q)^q \int_0^{\infty} t^{r-1-q} \Big(\int_{|f| \le s} |f(x)|^q dx \Big) dt \\ &\quad + r(2B_p)^p \int_0^{\infty} t^{r-1-p} \Big(\int_{|f| > s} |f(x)|^p dx \Big) dt \\ &\equiv r(2B_q)^q I_q + r(2B_p)^p I_p. \end{aligned} \tag{A.12}$$

Let $s = t/A$ for some positive number A to be fixed later. Then

$$\begin{aligned} I_q &= A^{r-q} \int_0^\infty s^{r-1-q} \Big(\int_{|f| \le s} |f(x)|^q dx \Big) ds \\ &= A^{r-q} \int_U |f(x)|^q \Big(\int_{|f|}^\infty s^{r-1-q} ds \Big) dx \qquad \text{(A.13)} \\ &= \frac{A^{r-q}}{q-r} \int_U |f(x)|^r dx. \end{aligned}$$

Similarly,

$$\begin{aligned} I_p &= A^{r-p} \int_0^\infty s^{r-1-p} \Big(\int_{|f| > s} |f(x)|^p dx \Big) ds \\ &= \int_U |f(x)|^p \int_0^{|f|} s^{r-1-p} ds \qquad \text{(A.14)} \\ &= \frac{A^{r-p}}{r-p} \int_U |f(x)|^r dx. \end{aligned}$$

Combining (A.12), (A.13) and (A.14), we derive

$$\int_U |Tf(x)|^r dx \le rF(A) \int_U |f(x)|^r dx, \qquad \text{(A.15)}$$

where

$$F(A) = \frac{(2B_q)^q A^{r-q}}{q-r} + \frac{(2B_p)^p A^{r-p}}{r-p}.$$

By elementary calculus, one can easily verify that the minimum of $F(A)$ is

$$A = 2B_q^{q/(q-p)} B_p^{p/(p-q)}.$$

For this value of A, (A.15) becomes

$$\int_U |Tf(x)|^r dx \le r2^r (\frac{1}{q-r} + \frac{1}{r-p}) B_q^{q(r-p)/(q-p)} B_p^{p(q-r)/(q-p)} \int_U |f(x)|^r dx.$$

Letting

$$C = 2\Big(\frac{r}{q-r} + \frac{r}{r-p} \Big)^{\frac{1}{r}}, \quad \text{and } \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q},$$

we arrive immediately at

$$\|Tf\|_{L^r(U)} \le C B_p^\theta B_q^{1-\theta} \|f\|_{L^r(U)}.$$

This completes the proof of the lemma.

□

Lemma A.6. *T is of strong type (p, p) for $1 < p < \infty$.*

Proof. From the previous lemma, we know that, for any $1 < r \leq 2$, we have

$$\|Tg\|_{L^r(U)} \leq C_r \|g\|_{L^r(U)}. \tag{A.16}$$

Let

$$< f, g >= \int_U f(x)g(x)dx$$

be the duality between f and g. Then it is easy to verify that

$$< g, Tf >=< Tg, f > . \tag{A.17}$$

Given any $2 < p < \infty$, let $r = \frac{p}{p-1}$, i.e. $\frac{1}{r} + \frac{1}{p} = 1$. Obviously, $1 < r < 2$. It follows from (A.16) and (A.17) that

$$\begin{aligned}
\|Tf\|_{L^p} &= \sup_{\|g\|_{L^r}=1} < g, Tf >= \sup_{\|g\|_{L^r}=1} < Tg, f > \\
&\leq \sup_{\|g\|_{L^r}=1} \|f\|_{L^p} \|Tg\|_{L^r} \leq C_r \|f\|_{L^p}.
\end{aligned}$$

This completes the proof of the Lemma.

□

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