CHARACTERIZING SOME POLARIZED FANO FIBRATIONS VIA HILBERT CURVES

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Abstract. The Hilbert curve of a complex polarized manifold (X, L) is the complex affine plane curve of degree $\dim(X)$ defined by the Hilbert-like polynomial $\chi(xK_X + yL)$, where K_X is the canonical bundle of X and x and y are regarded as complex variables. A natural expectation is that this curve encodes several properties of the pair (X, L). In particular, the existence of a fibration of X over a variety of smaller dimension induced by a suitable adjoint bundle to L translates into the fact that the Hilbert curve has a quite special shape. Along this line, Hilbert curves of special varieties like Fano manifolds with low coindex, as well as fibrations over low dimensional varieties having such a manifold as general fiber, endowed with appropriate polarizations, are investigated. In particular, several polarized manifolds relevant for adjunction theory are completely characterized in terms of their Hilbert curves.

Introduction

A polarized manifold (n-fold) is a pair (X, L) consisting of a smooth projective variety X of dimension $n \geq 2$ and an ample line bundle L on X.

The Hilbert curve $\Gamma = \Gamma_{(X,L)}$ of a polarized manifold (X, L) was introduced in [1] and further studied in [8], [9]. It is the affine plane curve of degree $n = \dim X$ defined by $p(x, y) = 0$, where p is the complexified of the polynomial expression provided by the Riemann–Roch theorem for the Euler–Poincaré characteristic $\chi(xK_X + yL)$, regarding x and y as complex variables. Clearly $p \in \mathbb{Q}[x, y]$ is a numerical polynomial. As shown in [1], Γ encodes interesting properties of the pair (X, L); in particular, it is sensitive to the possibility of fibering X over a variety of smaller dimension via an adjoint bundle to L. This makes polarized manifolds arising in adjunction theory [2] very interesting from the point of view of their Hilbert curves.

In this paper, inspired by the study of Hilbert curves of projective bundles over a smooth curve made in [8], we provide a unifying perspective of the Hilbert curves of these special varieties. Since Fano manifolds are the building blocks of these varieties, we first address in Section 2 the study of pairs (X, L), where X is a Fano n-fold and $L = \frac{r}{\iota_X}(-K_X)$, ι_X being the index of X and r any positive integer. For r and m two coprime integers such that $m > \frac{n+1}{2}$, we provide a characterization of pairs (X, L) as above with $\iota_X = m$ in terms of their Hilbert curves and the condition $\mathrm{rk}\langle K_X, L\rangle = 1$ (Theorem 2.3). Next, in Sections 3 and 4 we characterize Fano fibrations of low coindex endowed with a polarization inducing on the general fiber the same situation as in Section 2. In Section 3, some ideas used in [8] to deal with the case of projective bundles are further developed and lead to a complete characterization of $\mathbb{P}$-bundles (Theorem 3.3), $\mathbb{Q}$-fibrations (Theorem 3.8) and del Pezzo fibrations (Theorem 3.10) with a polarization as above in terms of their Hilbert curves,

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the condition $\mathrm{rk}\langle K_X, L\rangle = 2$ and the nefness of a suitable adjoint bundle. In particular, this generalizes [8, Theorem 4.1] and [9, Theorem 6]. As a consequence of Theorem 3.3 it turns out that conjecture $\mathrm{C}(n,r)$ stated in [8], and proved there for $r = 1$, is true also for $r \geq 2$, provided that $rK_X + nL$ is nef (this requirement is not needed if L is r-very ample, see Remark 3.9(i)). Finally, in Section 4 we consider Fano fibrations of low coindex over a normal variety of dimension ≥ 2. Here we relate the equation of the Hilbert curve of the fibration with that of a suitable divisor, which is in turn a Fano fibration of the same coindex but of smaller dimension and by induction we reduce to the case of fibrations over a smooth curve, which allows us to apply the results in Section 3. Several computations have been done with the help of Maple 15.

1. Background material

Varieties considered in this paper are defined over the field $\mathbb{C}$ of complex numbers. We use the standard notation and terminology from algebraic geometry. Tensor products of line bundles are denoted additively. The pullback of a vector bundle $\mathscr{F}$ on a manifold X by an embedding $Y \hookrightarrow X$ is simply denoted by $\mathscr{F}_Y$. We denote by K_X the canonical bundle of a manifold X. The symbol $\equiv$ will stand for numerical equivalence.

A *Fano n-fold* is a manifold X of dimension n such that $-K_X$ is an ample line bundle. The *index* ι_X of X is defined as the greatest positive integer which divides $-K_X$ in $\mathrm{Pic}(X)$, the Picard group of X, while by the *coindex* of X we simply mean the nonnegative integer $c_X := \dim X + 1 - \iota_X$. Moreover, we say that a polarized n-fold (X, L) is a *del Pezzo manifold* (respectively a *Mukai n-fold*) if $K_X + (n-1)L = \mathscr{O}_X$ (respectively $K_X + (n-2)L = \mathscr{O}_X$).

We say that a polarized n-fold (X, L) is a *Fano fibration of coindex* $n - m + 1 - t$ if there exists a surjective morphism with connected fibers $\varphi : X \to Y$ onto a normal variety Y of dimension $m < n$ such that $K_X + tL = \varphi^* H$ for some ample line bundle H on Y and positive integer t. In particular, a *scroll* (X, L) is a Fano fibration of coindex 0, a *quadric fibration* is a Fano fibration of coindex 1, and so on. We say that a polarized n-fold (X, L) is a $\mathbb{P}$*–bundle* over a normal variety Y if $X = \mathbb{P}(\mathscr{F})$ for some vector bundle $\mathscr{F}$ on Y and L is any ample line bundle on X; we say that (X, L) is a $\mathbb{Q}$*–fibration* over Y if X is endowed with a surjective morphism $X \to Y$ whose general fiber is a smooth quadric hypersurface and L is any ample line bundle on X.

For the notion and the general properties of the *Hilbert curve* (HC for short) associated to a polarized manifold we refer to [1]. Here we just recall some basic facts. Let (X, L) be a polarized n-fold. For any line bundle D on X consider the expression of the Euler–Poincaré characteristic $\chi(D)$ provided by the Riemann–Roch theorem

$$\chi(D) = \frac{1}{n!}D^n - \frac{1}{2(n-1)!}K_X \cdot D^{n-1} + \text{terms of lower degree} \tag{1}$$

(a polynomial of degree n in the Chern class of D, whose coefficients are polynomials in the Chern classes of X [7, Theorem 20.3.2]). Let p (or $p_{(X,L)}$ to avoid possible ambiguity) be the complexified polynomial of $\chi(D)$, when we set $D = xK_X + yL$, with x, y complex numbers, namely $p(x, y) := \chi(xK_X + yL)$. The Hilbert curve of (X, L) is the complex affine plane curve $\Gamma = \Gamma_{(X,L)}$ of degree n defined by $p(x, y) = 0$ [1, Section 2]. We refer to $p(x, y) = 0$ as the *canonical equation* of Γ. Clearly, $p_{(X,L)}(x, y) = p_{(X,L')}(x, y)$ if $L \equiv L'$, hence two numerically equivalent polarizations on X give rise to the same HC. If $\mathrm{rk}\langle K_X, L\rangle = 2$ in $\mathrm{Num}(X)$, and we consider $\mathrm{N}(X) := \mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ as a complex affine space, then Γ is the section of the

Hilbert variety of X ([1, §2]) with the plane $\mathbb{A}^2 = \mathbb{C}\langle K_X, L\rangle$, generated by the classes of K_X and L. On the other hand, if $\mathrm{rk}\langle K_X, L\rangle = 1$ in $\mathrm{Num}(X)$, Γ loses this meaning, the plane of coordinates (x, y) being only formal. We will refer to this situation as the *degenerate case*. Recall that Γ is invariant under the Serre involution $D \mapsto K_X - D$ acting on $\mathrm{N}(X)$. Furthermore, to deal with points at infinity, sometimes it is convenient to consider also the projective Hilbert curve $\overline{\Gamma} \subset \mathbb{P}^2$, namely the projective closure of Γ. In this case we use x, y, z as homogeneous coordinates on $\mathbb{P}^2$, $z = 0$ representing the line at infinity. Given a point $(x, y) \in \mathbb{A}^2$, we write $(x : y : 1)$ to denote the same point when regarded as a point of $\mathbb{P}^2$. Moreover, we denote by $p_0(x, y, z)$ the homogeneous polynomial associated with $p(x, y)$ (i.e. $p(x, y) = p_0(x, y, 1)$), which defines the plane projective curve $\overline{\Gamma}$. Note that

$$p_0(x,y,0) = \frac{1}{n!}(xK_X + yL)^n \tag{2}$$

in view of (1). This will be used over and over. Another fact of frequent use will be the following.

Remark 1.1. Let (X, L) be a polarized n-fold with $n \geq 3$ and suppose that $\sigma K_X + \tau L$ is nef and not big for some positive integers σ, τ. Then there exists a morphism $\varphi : X \to Y$ onto a normal variety Y with $\dim Y < n$ such that $\sigma K_X + \tau L = \varphi^* D$ for a nef line bundle D on Y. Actually, we can write $\sigma K_X + \tau L = K_X + M$ where M is an ample line bundle. This is obvious for $\sigma = 1$, while for $\sigma \geq 2$ we have

$$M = (\sigma - 1)K_X + \tau L = \frac{\sigma - 1}{\sigma}\big(\sigma K_X + \tau L\big) + \frac{\tau}{\sigma}\ L.$$

Thus M is ample being the sum of a nef and an ample $\mathbb{Q}$-line bundles. Then by the Kawamata–Shokurov base-point free theorem the linear system $|m\,(\sigma K_X + \tau L)\,|$ is effective and base-point free for $m \gg 0$. Hence it defines a morphism $\Phi : X \to \mathbb{P}^N$, where the image has dimension $< n$, since $\sigma K_X + \tau L$ is not big. The morphism $\varphi : X \to Y$ is defined by the Stein-factorization of Φ.

2. The case $\mathrm{rk}\langle K_X, L\rangle = 1$: High index Fano manifolds

In this section we aim at a characterization of Fano manifolds of high index polarized by a multiple of the fundamental divisor in terms of their Hilbert curves. Let X be a Fano n-fold of index ι_X with $n \geq 2$. Then there exists an ample line bundle H (a fundamental divisor) on X such that $-K_X = \iota_X H$. Recall that $1 \leq \iota_X \leq n+1$. From now on our setting for Fano polarized manifolds (X, L) will be the following:

$$\begin{gathered} X \text{ is a Fano } n\text{-fold with } n \geq 2 \text{ and } L := rH \text{ for some positive integer } r, \\ \text{where } H := \frac{1}{\iota_X}(-K_X) \ \text{ is the fundamental divisor.} \end{gathered} \tag{3}$$

Let (X, L) be as in (3). Then $rK_X + \iota_X L = \mathcal{O}_X$, which implies $\mathrm{rk}\langle K_X, L\rangle = 1$. Let $p(x, y) = 0$ be the canonical equation of the Hilbert curve $\Gamma_{(X,L)}$. Recalling that $p(x, y) = \chi(xK_X + yL)$, we get

$$p(x, y) = \chi((ry - \iota_X x)H) = \chi(tH) =: q(t),$$

where $t := ry - \iota_X x$. Moreover, note that $q(t) = \chi(K_X + (t + \iota_X)H) = h^0(tH)$ for $t \geq 1 - \iota_X$, by the Kodaira vanishing theorem. Thus $q(t)$ vanishes at all negative integer values

$-(\iota_X - 1), \dots, -2, -1$ (if any, i.e. if $\iota_X \geq 2$). As a consequence, $q(t)$ is divisible by $(t+i)$ for every $i = 1, \dots, \iota_X - 1$ and therefore we can write

$$q(t) = \varphi(t) \cdot \psi(t) \ ,$$

where $\varphi(t) := \sum_{j=0}^{c_X} a_j t^j$ is a polynomial in t of degree c_X, the coindex of X, and

$$\psi(t) := \begin{cases} 1 & \text{if } \iota_X = 1 \ , \\ \prod_{i=1}^{\iota_X - 1}(t+i) & \text{if } 2 \leq \iota_X \leq n+1 \ . \end{cases}$$

Observe that we need $c_X + 1$ linearly independent linear conditions on $q(t)$ to determine the polynomial $\varphi(t)$. So, for $s = 0, 1, ..., c_X$, we see that $\varphi(s) \cdot \prod_{i=1}^{\iota_X - 1}(s+i) = q(s) = h^0(sH)$, that is,

$$a_{c_X} s^{c_X} + a_{c_X - 1} s^{c_X - 1} + \cdots + a_1 s + a_0 = \varphi(s) = \frac{h^0(sH)}{\prod_{i=1}^{\iota_X - 1}(s+i)} \ .$$

This gives the following system of $c_X + 1$ linear equations in the $c_X + 1$ unknowns $a_0, a_1, ..., a_{c_X}$

$$U \cdot \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{c_X} \end{pmatrix} = \begin{pmatrix} \frac{h^0(\mathscr{O}_X)}{\delta(0)} \\ \frac{h^0(H)}{\delta(1)} \\ \vdots \\ \frac{h^0(c_X H)}{\delta(c_X)} \end{pmatrix} \ , \tag{4}$$

where U is the $(c_X + 1) \times (c_X + 1)$ Vandermonde matrix

$$U := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & \cdots & 2^{c_X - 1} & 2^{c_X} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & c_X & \cdots & (c_X)^{c_X - 1} & (c_X)^{c_X} \end{pmatrix} \ , \tag{5}$$

and

$\delta(u) := \prod_{i=1}^{\iota_X - 1}(u+i)$ for any $u \in \mathbb{Z}_{\geq 0}$ if $\iota_X \geq 2$, $\delta(u) := 1$ for any $u \in \mathbb{Z}_{\geq 0}$ if $\iota_X = 1$.

The above discussion can be summarized by Algorithm 1 below.

Algorithm 1 The Hilbert curve Γ of a Fano n-fold X of index ι_X for $L := \frac{r}{\iota_X}(-K_X)$ with $r \in \mathbb{Z}_{\geq 1}$

Require: r, ι_X, n
Ensure: Γ
if $n > 1, 0 < \iota_X \leq n+1$ **then**
  $c_X \leftarrow n - \iota_X + 1$
  procedure RHilbPolynF(ι_X)
    if $\iota_X = 1$ **then** $\delta(l) \leftarrow 1$
      else $\delta(l) \leftarrow \prod_{h=1}^{\iota_X - 1}(l+h)$
    end if
    for $j = 0, ..., c_X$ **do** $b_j \leftarrow \frac{1}{\delta(j)} h^0\left(\frac{j}{\iota_X}(-K_X)\right)$
    end for
    $U \leftarrow$ `Vandermonde Matrix of` $\{0, 1, ..., c_X\}$
    $(a_0, a_1, ..., a_{c_X}) \leftarrow (b_0, b_1, ..., b_{c_X}) \cdot U^{-1}$
    $R_{(X,L)}(x,y) \leftarrow (\sum_{k=0}^{\mu} a_k (ry - mx)^k)$
    return $R_{(X,L)}(x,y)$
  end procedure
end if

Lemma 2.1. *Let (X, L) be as in (3). Then*

$$p(x,y) = \left(\sum_{i=0}^{c_X} a_i (ry - \iota_X x)^i \right) \cdot \prod_{i=1}^{\iota_X - 1} (ry - \iota_X x + i) \ , \tag{6}$$

where $(a_0, a_1, \dots, a_{c_X})$ is the solution of (4).

The next result will be useful to characterize Fano n-folds of large indexes.

Lemma 2.2. *Let (X, L) be a polarized n-fold with $n \geq 2$ and let r, m be two positive integers with $\gcd(r, m) = 1$. If $\mathrm{rk}\langle K_X, L\rangle = 1$ and $p_0(r, m, 0) = 0$, then X is Fano of index $\iota_X = km$ and $L = krH$ for some positive integer k, where H is the fundamental divisor.*

Proof. Since $\mathrm{rk}\langle K_X, L\rangle = 1$, we have $K_X + aL = \mathscr{O}_X$ for some $a \in \mathbb{Q}$. Moreover, (2) gives

$$0 = n!\ p_0(r, m, 0) = (rK_X + mL)^n = (m - ra)^n L^n,$$

hence $a = \frac{m}{r} > 0$. Therefore X is a Fano n-fold such that $rK_X + mL = \mathscr{O}_X$. Let ι_X be the index of X so that $-K_X = \iota_X H$ for some ample $H \in \mathrm{Pic}(X)$. Note that $\mathrm{Pic}(X)$ is torsion free. Moreover, we have $mL = r(-K_X) = r\iota_X H = A$ for some ample line bundle A on X. Write $m = \sigma m'$ and $r\iota_X = \sigma s'$ for some positive integers m', s', where $\sigma := \gcd(m, r\iota_X)$. Then A is divisible by m and $r\iota_X$ in $\mathrm{Pic}(X)$ and this implies that $A = m's'\sigma M$ for some ample line bundle M on X, because $\gcd(m', r) = \gcd(m, s') = 1$. Thus we get $L = s'M, H = m'M$ and then $-K_X = \iota_X m' M$. Since ι_X is the index of X, we conclude that $m' = 1$. Hence $M = H$, $L = s'H$, $m = \sigma = \gcd(m, r\iota_X)$. As $\gcd(m, r) = 1$, we deduce that m divides ι_X, that is, $\iota_X = km$ for some positive integer k. So we get

$$\mathscr{O}_X = rK_X + mL = r(-\iota_X H) + ms'H = r(-kmH) + ms'H = m(s' - kr)H,$$

i.e. $s' = kr$, hence $L = krH$. □

For Fano n-folds of sufficiently large index, we thus get the following characterization.

Theorem 2.3. *Let (X, L) be a polarized n-fold with $n \geq 2$ and let r, m be two positive integers such that $\gcd(r, m) = 1$ and $m > \frac{n+1}{2}$. Then X is a Fano n-fold of index $\iota_X = m$, $L := rH$ with $H := \frac{-K_X}{m}$, and $(a_0, \dots, a_{n+1-m})$ is a solution of (4) if and only if the following two conditions are satisfied:*

a) $\mathrm{rk}\langle K_X, L\rangle = 1$, *and*

b) $p_{(X,L)}(x, y) = (a_{n+1-m}(ry - mx)^{n+1-m} + \cdots + a_1(ry - mx) + a_0) \cdot \prod_{i=1}^{m-1}(ry - mx + i)$.

Proof. The "only if" part follows easily from Lemma 2.1. To prove the converse, note that $m \geq 2$, hence $p_{(X,L)}(x, y)$ is divisible by $(ry - mx + 1)$, and therefore $p_0(r, m, 0) = 0$. Due to a), by applying Lemma 2.2 we know that X is Fano of index $\iota_X = km$ and $L = krH$ for some positive integer k. Combining the assumption with the upper bound for ι_X we get $k\frac{n+1}{2} < km \leq n + 1$. Thus $k = 1$, hence $L = rH$ and the assertion follows from Lemma 2.1 in view of (3). □

Remark 2.4. Given a Fano n-fold X of index $\iota_X \geq n - 2$, it is known that there exists a smooth element $Y \in |H|$. This is obvious for $\iota_X = n+1$ and n; it follows from Fujita's theory of del Pezzo manifolds [6, §8] for $\iota_X = n - 1$, and from a result of Mella [11] for $\iota_X = n - 2$. Note that $-K_Y = (\iota_X - 1)H_Y$ by adjunction. In particular, if $n \geq 3$ and (X, H) is a del Pezzo n-fold, then (Y, H_Y) is a del Pezzo $(n - 1)$-fold, and similarly, if $n \geq 4$ and (X, H) is a Mukai n-fold, then (Y, H_Y) is a Mukai $(n-1)$-fold. A consequence of this fact is that for $\iota_X \geq n-2$ we can always apply an inductive argument up to the surface case to compute $h^0(tH)$ for $t = 1, \ldots, c_X \leq 3$. This gives the column vector on the right hand side of (4).

Remark 2.5. We would like to stress that Theorem 2.3 contains an explicit characterization of $(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(r)), (\mathbb{Q}^n, \mathscr{O}_{\mathbb{Q}^n}(r))$ for $n \geq 2$, and of polarized n-folds (X, L) with X a Fano manifold of index $\iota_X = n-1$ and $L := \frac{r}{n-1}(-K_X)$, r being a positive integer such that $\gcd(r, n-1) = 1$, for $n \geq 4$. As to $n = 3$, let r be an odd positive integer and assume that $X \neq \mathbb{P}^3$. Then X is a Fano 3-fold of index $\iota_X = 2$ and $L = \frac{r}{2}(-K_X)$ if and only if $\mathrm{rk}\langle K_X, L\rangle = 1$ and

$$p_{(X,L)}(x, y) = \left(\frac{d}{6}(ry-2x)^2 + \frac{d}{3}(ry-2x) + 1\right)(ry-2x+1),$$

where $d = \left(\frac{1}{2}(-K_X)\right)^3 \neq 8$. The "only if" part follows from Lemma 2.1 and Remark 2.4. As to the "if part", recalling Lemma 2.2, we see that $4 = n+1 \geq \iota_X = k(n-1)$. Hence $k \leq 2$. If $k = 1$ we are done; on the other hand if $k = 2$, then $\iota_X = 4$, hence $X = \mathbb{P}^3$ by the Kobayashi–Ochiai theorem, but this gives a contradiction. Finally, a similar characterization result holds also for $n = 2$, provided that $(-K_X)^2 \neq 9$.

A natural conjecture is the following.

Conjecture. Let (X, L) be a polarized n-fold, suppose that $\mathrm{rk}\langle K_X, L\rangle = 1$ and let $\Gamma \subset \mathbb{A}^2$ be its Hilbert curve. Let r be a positive integer with $\gcd(n+1, r) = 1$, and suppose that the Hilbert curve Γ of (X, L) consists of n distinct lines (symmetric with respect to the origin), parallel each other with slope $\frac{n+1}{r}$ and evenly spaced. Then $(X, L) = \left(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(r)\right)$.

The above Conjecture is true in view of Theorem 2.3. Moreover, let us observe that a consequence of the next result (Theorem 2.6) is that this Conjecture is still true provided that $rK_X + (n+1)L$ is nef regardless of the assumption $\mathrm{rk}\langle K_X, L\rangle = 1$. This change of perspective will be the starting point for the discussion in the next section of a similar, but less elementary, conjecture (see [8, Conjecture $C(n,r)$]) for the case $\mathrm{rk}\langle K_X, L\rangle = 2$.

Theorem 2.6. *Let (X, L) be a polarized n-fold with $n \geq 2$ and let r be a positive integer such that $\gcd(r, n+1) = 1$. Then $(X, L) = (\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(r))$ if and only if $rK_X + (n+1)L$ is nef and $p_{(X,L)}(x, y) = \frac{1}{n!}\prod_{i=1}^{n}(ry - (n+1)x + i)$.*

Proof. In view of Theorem 2.3 we only need to prove the "if part" showing that

$$\mathrm{rk}\langle K_X, L\rangle = 2 \tag{7}$$

cannot happen. First let $n = 2$. By comparing the expression of $p_{(X,L)}(x, y)$ in the statement with that holding for any polarized surface (X, L), we see that $K_X^2 = 9$, $K_X \cdot L = -3r$ and $\chi(\mathscr{O}_X) = 1$. The last two conditions imply that X is a rational surface and then the first condition says that $X = \mathbb{P}^2$, which contradicts (7). Let $n \geq 3$. The expression of $p_{(X,L)}(x, y)$ shows that the point $(r : n+1 : 0)$ belongs to the projective closure of the HC, $\overline{\Gamma_{(X,L)}} \subset \mathbb{P}^2$, hence

$$0 = n!\, p_0(r, n+1, 0) = (rK_X + (n+1)L)^n$$

by (2). Therefore $rK_X + (n+1)L$ is nef but not big. By Remark 1.1 we know that there exists a morphism $\varphi : X \to Y$ with $\dim Y < \dim X$ such that $rK_X + (n+1)L = \varphi^* D$ for some nef line bundle D on Y. Then $rK_F + (n+1)L_F = \mathscr{O}_F$ by adjunction, where F is a general fiber of φ. Thus $-K_F = \frac{n+1}{r}L_F$ and since L_F is ample we conclude that F is a

Fano manifold. Moreover, we have $-K_F \cdot \gamma = (n+1)\frac{L_F \cdot \gamma}{r} \geq n+1$ for every rational curve $\gamma \subset F$, because the assumption $\gcd(r, n+1) = 1$, together with the fact that $-K_F \cdot \gamma$ is an integer, implies that r must divide the positive integer $L_F \cdot \gamma$. Then the index i_F of F satisfies $\dim X + 1 \geq \dim F + 1 \geq i_F \geq n+1$, i.e. $\dim F = \dim X$. So Y is a point and $X = F = \mathbb{P}^n$, which contradicts (7) again. ☐

3. Case $\mathrm{rk}\langle K_X, L\rangle = 2$: Fano fibrations over curves

Here and in the next section we characterize Fano fibrations of low coindex endowed with an ample line bundle making the general fiber a polarized Fano manifold as in Section 2.

First of all, we describe a method for obtaining the canonical equation of the HC of a Fano fibration over a smooth curve. Let (X, L) be a Fano fibration over a smooth irreducible curve C via a morphism $\varphi : X \to C$, let F be a general fiber of φ, let r be any positive integer and suppose that $rK_X + \iota_F L = \varphi^*\mathscr{A}$ for some non-trivial line bundle $\mathscr{A}$ on C. Thus $rK_X + \iota_F L \equiv tF$, where $t = \deg \mathscr{A} \neq 0$. Consider the following exact sequences defined by the restriction to F:

$$0 \to xK_X + yL - F \to xK_X + yL \to xK_F + yL_F \to 0 \ ,$$
$$0 \to xK_X + yL - 2F \to xK_X + yL - F \to xK_F + yL_F \to 0 \ ,$$
$$\vdots$$
$$0 \to xK_X + yL - |t|F \to xK_X + yL - (|t|-1)F \to xK_F + yL_F \to 0 \ .$$

Let ε be the sign of t, so that $|t| = \varepsilon t$. Then, due to the additivity of the Euler–Poincaré characteristic χ for exact sequences, by recursion we get

$$p_{(X,L)}(x, y) = p_{(X,L)}(x - \varepsilon r, y - \varepsilon\iota_F) + \varepsilon t \cdot p_{(F,L_F)}(x, y) \ .$$

Since F is Fano with $rK_F + \iota_F L_F = \mathscr{O}_F$, we know that

$$p_{(F,L_F)}(x, y) = R_F(x, y) \cdot \prod_{j=1}^{\iota_F - 1} \left(ry - \iota_F x + j\right) \ ,$$

where $R_F(x, y)$ is the first factor of (6) in Lemma 2.1. Thus, we finally obtain that

$$R(x, y) = R(x - \varepsilon r, y - \varepsilon\iota_F) + \varepsilon t \cdot R_F(x, y), \tag{8}$$

where $p_{(X,L)}(x, y) = R(x, y) \cdot \prod_{j=1}^{\iota_F - 1}\left(ry - \iota_F x + j\right)$. So, (8) with further suitable conditions allows us to determine $R(x, y)$, once we know $R_F(x, y)$.

The above discussion can be summarized by Algorithm 2 below.

A different approach consists in reducing the computation of $p_{(X,L)}(x, y)$ to the case $r = 1$. It relies on the following technical result.

Lemma 3.1. *Let $\psi : X \to Y$ be a morphism between irreducible projective varieties with $\dim X \geq \dim Y$. Assume that there are positive, coprime integers σ, τ, and an ample line bundle L on X such that $\sigma K_X + \tau L = \psi^* D$ for a line bundle D on Y. Let p, q be two positive integers such that $\sigma p - \tau q = 1$ and let A be a line bundle on Y such that $\sigma A + qD$ and $\tau A + pD$ are both nef. Then the following properties hold:*

(a) *$\mathscr{L} := qK_X + pL + \psi^* A$ is an ample line bundle on X;*
(b) *$K_X + \tau\mathscr{L} = \psi^*(\tau A + pD)$ and τ is the nefvalue of $(X, \mathscr{L})$;*

Algorithm 2 The Hilbert curve Γ of a Fano fibration $\pi : X \to C$ with $\dim X = n$ and fiber F

Require: F, r, ι_F, n, t
Ensure: Γ
if $n > 1, 0 < \iota_F \leq n$ **then**
  Find $R(x, y)$ with suitable conditions such that
  if $t > 0$ **then**
    Consider $R(x, y) = R(x - r, y - \iota_F) + t\cdot$RHilbPolynF$(\iota_F)$
  end if
  if $t < 0$ **then**
    Consider $R(x, y) = R(x + r, y + \iota_F) - t\cdot$RHilbPolynF$(\iota_F)$
  end if
  $p_{(X,L)}(x, y) \leftarrow R(x, y) \cdot \prod_{k=1}^{\iota_F - 1}(ry - \iota_F x - k)$
  return $\Gamma : p_{(X,L)}(x, y) = 0$
end if

(*c*) *if* $\dim X > \dim Y$*, then any general fiber* F *of* ψ *is a Fano variety and* $L_F = \sigma\mathscr{L}_F$*.*

In particular, assume that $\tau A + pD$ *is ample. If* $\psi : X \to Y$ *is a surjective morphism with* $\dim X > \dim Y$ *and connected fibers,* X *is a manifold and* Y *is a normal variety, then* $(X, \mathscr{L})$ *is a Fano fibration of coindex* $\dim X - \dim Y + 1 - \tau$*. Moreover, if* D *is nef then we can take* $A = \mathscr{O}_X$ *and in this case* $p_{(X,L)}$ *and* $p_{(X,\mathscr{L})}$ *are related as follows:*

$$\text{(j)} \qquad p_{(X,L)}(x, y) = p_{(X,\mathscr{L})}\left(x + \left(\frac{1 - p\sigma}{\tau}\right)\frac{y}{p}\ ,\ \frac{y}{p}\right)\ ,$$

$$\text{(jj)} \qquad p_{(X,\mathscr{L})}(x, y) = p_{(X,L)}\left(x - \left(\frac{1 - p\sigma}{\tau}\right)y\ ,\ py\right)\ .$$

Proof. Going over the proof of [2, Lemma 1.5.6], note that

$$\sigma\mathscr{L} = \sigma q K_X + (1 + \tau q)L + \psi^*(\sigma A) = L + \psi^*(\sigma A + qD)\ ,$$

$$K_X + \tau\mathscr{L} = K_X + \tau q K_X + \tau p L + \psi^*(\tau A) = \sigma p K_X + \tau p L + \psi^*(\tau A) = \psi^*(\tau A + pD)\ .$$

This gives (a) and (b), keeping in mind that $\tau A + pD$ is nef on Y. To obtain (c), let F be a general fiber of ψ. Then $-K_F = \tau\mathscr{L}_F$ and $\sigma K_F + \tau L_F = \mathscr{O}_F$. Thus F is a Fano variety and

$$\tau L_F = \sigma(-K_F) = \sigma(\tau\mathscr{L}_F) = \tau\sigma\mathscr{L}_F\ ,$$

hence $L_F = \sigma\mathscr{L}_F$. The final part of the statement follows easily from (a) and (b). □

A consequence of Lemma 3.1 is the following result which extends facts well-known for $r = 1$ (see [5, (2.12)], [2, Proposition 3.2.1], [6], or [5, (11.8)]) and $(n, r) = (3, 2)$ (see [5, Theorem 3′]).

Proposition 3.2. *Let* (X, L) *be a polarized* n*-fold with* $n \geq 3$ *and let* r *be a positive integer.*

1) *Assume that* $\gcd(r, n) = 1$*. Then* X *admits a surjective morphism* $\pi : X \to C$ *over a smooth curve* C *with connected fibers such that* $(F, L_F) = (\mathbb{P}^{n-1}, \mathscr{O}_{\mathbb{P}^{n-1}}(r))$ *for any general fiber* F *of* π *if and only if* $X = \mathbb{P}(\mathscr{E})$ *for an ample vector bundle* $\mathscr{E}$ *of rank* n *on* C *and* $L \equiv r\xi + bF$*, where* ξ *is the tautological line bundle of* $\mathscr{E}$ *and* b *is a suitable integer.*
2) *Assume that* $\gcd(r, n - 1) = 1$*. Then* (X, L) *is a* $\mathbb{Q}$*-fibration over a smooth curve* C *with* $L_F = \mathscr{O}_{\mathbb{Q}^{n-1}}(r)$ *for any general fiber* $F \cong \mathbb{Q}^{n-1}$ *if and only if there exist a vector bundle* $\mathscr{V}$ *of rank* $n + 1$ *and line bundles* $\mathscr{A}$*,* $\mathscr{B}$ *on* C *such that* $P := \mathbb{P}(\mathscr{V})$ *contains* X *as a smooth divisor in the linear system* $|2\xi + \widetilde{\pi}^*\mathscr{A}|$*, where* ξ *is the tautological line bundle of* $\mathscr{V}$*,* $\widetilde{\pi} : P \to C$ *is the bundle projection and* $L = (r\xi + \widetilde{\pi}^*\mathscr{B})_X$*.*

Proof. 1) The "if" part is obvious. To prove the converse, note that if (F, L_F) is the pair $(\mathbb{P}^{n-1}, \mathscr{O}_{\mathbb{P}^{n-1}}(r))$ for any general fiber F of $\pi : X \to C$, then $(rK_X+nL)_F = rK_F+nL_F = \mathscr{O}_F$ and this gives $rK_X + nL = \pi^* D$ for some line bundle D on C. Since $\gcd(r, n) = 1$, we know from Lemma 3.1 that there exists an ample line bundle $\mathscr{L}$ on X such that $(X, \mathscr{L})$ is a Fano fibration of coindex $\dim X - \dim Y + 1 - n = 0$. Thus $(X, \mathscr{L})$ is a scroll over C via π with $(F, \mathscr{L}_F) = (\mathbb{P}^{n-1}, \mathscr{O}_{\mathbb{P}^{n-1}}(1))$ for any fiber F of π. Therefore, from [5, (2.12)] (or [2, Proposition 3.2.1]) we deduce that $X = \mathbb{P}(\mathscr{E})$, where $\mathscr{E} = \pi_* \mathscr{L}$ is an ample vector bundle of rank n on C; moreover, $\mathscr{L} = \xi$ and $L \equiv r\xi + bF$, as in the statement.

2) The "if part" is obvious, the fibration morphism being $\widetilde{\pi}|_X$. To see the converse, let $\pi : X \to C$ be the fibration morphism. Since $\gcd(r, n - 1) = 1$, by arguing as in case 1), we get an ample line bundle $\mathscr{L}$ on X such that $(X, \mathscr{L})$ is a quadric fibration via π. Then, as $n \geq 3$, the assertion follows from [6, (11.8), case b1-Q)] by taking $\mathscr{V} := \pi_* \mathscr{L}$ and noting that $\xi_X = \mathscr{L}$. □

In line with Theorems 2.3 and 2.6, we obtain the following result which improves [8, Proposition 4.1] and generalizes [8, Corollary 4.1].

Theorem 3.3. *Let (X, L) be a polarized n-fold with $n \geq 2$, let $d := L^n$, $q := h^1(\mathscr{O}_X)$ and consider a positive integer r such that $\gcd(r, n) = 1$. Then $X = \mathbb{P}(\mathscr{E})$ for a vector bundle $\mathscr{E}$ of rank n over a smooth curve C of genus q, and up to numerical equivalence, $L = r\xi + bF$ with $r(2q - 2 + e) + bn > 0$, where $e := \deg \mathscr{E}$, ξ is the tautological line bundle of $\mathscr{E}$ and $F \cong \mathbb{P}^{n-1}$ is a fiber of $\mathbb{P}(\mathscr{E}) \to C$ if and only if the following three conditions are satisifed:*
i) $\mathrm{rk}\langle K_X, L\rangle = 2$;
ii) $p_{(X,L)}(x, y) := \frac{1}{n!} \left(n(2q - 2) \left(x - \frac{1}{2}\right) + \frac{d}{r^{n-1}} y\right) \cdot \prod_{i=1}^{n-1} (ry - nx + i)$;
iii) $rK_X + nL$ *is nef.*

Proof. To prove the "only if" part, first note that ii) follows from [8, Proposition 2.1]. Moreover, an easy computation shows that $rK_X + nL \equiv tF$ with $t := r(2q - 2 + e) + bn > 0$. Hence we get iii). To show that also i) holds, by contradiction suppose that $\mathrm{rk}\langle K_X, L\rangle = 1$. Then there exist two integers h, k with $h \neq 0$ and such that $hK_X + kL \equiv \mathscr{O}_X$. This gives $(nh - rk)L \equiv htF$, but this contradicts the ampleness of L. To prove the converse, assume that $p_{(X,L)}(x, y)$ is as in ii) and that i) and iii) hold. Since $p_0(r, n, 0) = 0$, we see from (2) that $(rK_X + nL)^n = 0$. Thus, $rK_X + nL$ is nef but not big. Let $\varphi : X \to Y$ be the morphism as in Remark 1.1 with $\dim Y < \dim X$ and $rK_X + nL = \varphi^* D$ for some nef line bundle D on Y. Since $\gcd(r, n) = 1$, by applying Lemma 3.1 with $\psi = \varphi$ and $(\sigma, \tau) = (r, n)$, we know that there exists an ample line bundle $\mathscr{L}$ on X such that $K_X + n\mathscr{L}$ is nef but not big. Thus, by [2, Proposition 7.2.2] $(X, \mathscr{L})$ is: either (A1) $(\mathbb{Q}^n, \mathscr{O}_{\mathbb{Q}^n}(1))$, or (A2) a scroll over a smooth curve C of genus q. Case (A1) cannot occur due to i), while in case (A2) we have $X = \mathbb{P}(\mathscr{E})$ for a vector bundle $\mathscr{E}$ of rank n on C and $\mathscr{L} \equiv \xi + b'F$ for some integer b', where ξ is the tautological line bundle of $\mathscr{E}$ and $F \cong \mathbb{P}^{n-1}$ is any fiber of $\mathbb{P}(\mathscr{E}) \to C$. Write $L \equiv a\xi + bF$ for some integers a, b, and recall from Lemma 3.1 that $L_F = r\mathscr{L}_F$. Thus $a = r$, that is, $L \equiv r\xi + bF$. Hence $rK_X + nL \equiv (r(2q - 2 + e) + bn) F$, where $e = \deg \mathscr{E}$ and $r(2q - 2 + e) + bn \geq 0$ due to iii). Finally, note that equality cannot occur, otherwise

(B) $$-rK_X \equiv nL\ ,$$

hence $\mathrm{rk}\langle K_X, L\rangle = 1$, contradicting i). □

In line with the Conjecture stated at the end of the previous section, let us recall here the following conjecture from [8, §3].

Conjecture $C(n, r)$**.** Let (X, L) be a polarized manifold of dimension n, suppose that $\mathrm{rk}\langle K_X, L\rangle = 2$ and let $\Gamma \subset \mathbb{A}^2$ be its Hilbert curve. Let r be a positive integer with $\gcd(n, r) = 1$, and suppose that the following three conditions are satisfied:

- $(\frac{1}{2}, 0) \in \Gamma$;
- Γ consists of n distinct lines: $n-1$ parallel lines $l_1, \dots, l_{n-1}$ having $P'_\infty = (r : n : 0)$ as point at infinity, and another line l_0 having $P''_\infty = (d : 2nr^{n-1}(1-q) : 0)$ as point at infinity, where $d := L^n$ and $q := h^1(\mathscr{O}_X)$;
- up to a reordering, the $n-1$ parallel lines $l_1, \dots, l_{n-1}$ fulfill the following condition: l_i has the same distance from l_{i-1} and from l_{i+1} for every $i = 2, \dots, n-2$.

Then X is a projective bundle over a smooth curve and either $L_F = \mathscr{O}_{\mathbb{P}^{n-1}}(r)$ for every fiber F of X, or $n = 2, q = 0, X \not\cong \mathbb{P}^1 \times \mathbb{P}^1$, and $L_F = \mathscr{O}_{\mathbb{P}^1}(\frac{d}{2r})$.

In particular, Theorem 3.3 has the following immediate consequences.

Corollary 3.4. *Conjecture $C(n, r)$ is true under the assumption that $rK_X + nL$ is nef.*

Corollary 3.5. *Let (X, L) be a polarized n-fold with $n \geq 2$, let $d := L^n$ and consider a positive integer r such that $\gcd(r, n) = 1$. Suppose that $q := h^1(\mathscr{O}_X) > 0$. Then (X, L) is as in Theorem 3.3 if and only if the following two conditions are satisifed:*
ii′) $p_{(X,L)}(x, y) := \frac{1}{n!}\left(n(2q-2)\left(x - \frac{1}{2}\right) + \frac{d}{r^{n-1}}y\right) \cdot \prod_{i=1}^{n-1}(ry - nx + i)$;
iii′) *$rK_X + nL$ is nef.*

In particular, for $r = 1$, we obtain that among the polarized n-folds (X, L) with positive irregularity, a scroll (X, L) over a smooth curve is uniquely characterized by its polynomial $p_{(X,L)}(x, y)$.

Note that the last assertion is not true with the HC in place of $p_{(X,L)}(x, y)$ ([8, Remark 3.1]).

Proof. The first part of the statement follows by arguing as in the proof of Theorem 3.3, except for the fact that cases (A1) and (B) can be ruled out by the assumption $q > 0$ instead of $\mathrm{rk}\langle K_X, L\rangle = 2$. Finally, when $r = 1$, observe that the nefness of $K_X + nL$ follows immediately from adjunction theory [2, Theorem 7.2.1] and the hypothesis $q > 0$. $\square$

Remark 3.6. (i) The hypothesis $\mathrm{rk}\langle K_X, L\rangle = 2$ in Theorem 3.3 can be replaced by $rK_X + nL \not\equiv \mathscr{O}_X$. Therefore, the assertion of Conjecture C(n, r) remains true by replacing the assumption $\mathrm{rk}\langle K_X, L\rangle = 2$ with the requirement that $rK_X + nL$ is nef but not trivial.

(ii) Referring to Conjecture C(n, r) again, let (X, L) be a polarized n-fold with $n \geq 2$ for which $p_{(X,L)}(x, y)$ is as in case ii) of Theorem 3.3. If $\mathrm{rk}\langle K_X, L\rangle = 1$ then $(X, L) = (\mathbb{Q}^n, \mathscr{O}_{\mathbb{Q}^n}(r))$. Actually, $K_X + aL \equiv \mathscr{O}_X$ for some $a \in \mathbb{Q}$. The expression of $p_{(X,L)}(x, y)$ shows that $p_0(r, n, 0) = 0$, hence $(rK_X + nL)^n = 0$ by (2). Thus $(n - ar)^n L^n = 0$, that is, $a = \frac{n}{r}$ and then $rK_X + nL \equiv \mathscr{O}_X$. Hence X is a Fano n-fold. Let H be the fundamental divisor on X. Hence $-K_X = \iota_X H$ and by Lemma 2.2, we have $\iota_X = kn$ for some positive integer k. Moreover, we have $k = 1$ because $1 \leq k = \frac{\iota_X}{n} \leq \frac{n+1}{n}$. By [2, Theorem 3.1.6] we conclude that $(X, H) = (\mathbb{Q}^n, \mathscr{O}_{\mathbb{Q}^n}(1))$, i.e. $(X, L) = (\mathbb{Q}^n, \mathscr{O}_{\mathbb{Q}^n}(r))$. In particular, this result shows that in Conjecture C(n, r) the hypothesis $\mathrm{rk}\langle K_X, L\rangle = 2$ cannot be dropped, and that Theorem 3.3 holds again with the condition $\mathrm{Pic}(X) \neq \mathbb{Z}$ in place of i).

Notation 3.7. According to Proposition 3.2, a $\mathbb{Q}$-fibration (X, L) over a smooth curve is described by the following data: $C, \pi, \mathscr{E}, \mathscr{A}, \mathscr{B}$ and r. We set

$$g := g(C), \quad e := \deg \mathscr{E}, \quad a := \deg \mathscr{A}, \quad b := \deg \mathscr{B}\ .$$

By the canonical bundle formula for $\mathbb{P}$-bundles and adjunction, we get $K_X + (n-1)\xi_X = \pi^*(K_C + \det \mathscr{E} + \mathscr{A})$, hence

$$rK_X + (n-1)L = \pi^*\left(r\left(K_C + \det \mathscr{E} + \mathscr{A}\right) + (n-1)\mathscr{B}\right) \ .$$

Therefore, $rK_X + (n-1)L \equiv tF$, where

$$t := r(2g-2+e+a) + (n-1)b \ . \tag{9}$$

Clearly, if $rK_X + (n-1)L$ is nef and $q > 0$, or $\mathrm{rk}\langle K_X, L\rangle = 2$, then $t > 0$.

The following result provides a characterization of $\mathbb{Q}$-fibrations in terms of their HC, generalizing Proposition 3 and Theorem 6 of [9].

Theorem 3.8. *Let (X, L) be a polarized n-fold with $n \geq 3$ and consider a positive integer r such that $\gcd(r, n-1) = 1$. Then (X, L) is a $\mathbb{Q}$-fibration as in Notation 3.7 with $t > 0$ if and only if the following three conditions are satisfied:*

j) $\mathrm{rk}\langle K_X, L\rangle = 2$*;*

jj) $p_{(X,L)}(x, y)$ *is*

$$\frac{1}{n!}\Big[(1-n)\big(2nc+2e+(n+1)a\big)x^2 + 2\big((nc-(n-2)e+a)r - n(n-1)b\big)xy +$$
$$+\big((2e+a)r^2 + 2nbr\big)y^2 + (n-1)\big(2nc+2e+(n+1)a\big)x +$$
$$-\big((nc-(n-2)e+a)r - n(n-1)b\big)y - \frac{n(n-1)}{2}c\Big] \cdot \prod_{j=1}^{n-2}\Big(ry-(n-1)x+j\Big) \ ,$$

where $c := 2g-2$;

jjj) $rK_X + (n-1)L$ *is nef.*

Proof. Let (X, L) be a $\mathbb{Q}$-fibration as in 3.7. Then (X, ξ_X) is a quadric fibration over C. The same equation as in [9, Proposition 3], rewritten in terms of coordinates $(x', y') = (\frac{1}{2}+u, v)$ is the following (taking into account that b in [9] is our $-a$):

$$p_{(X,\xi_X)}(x', y') = \frac{1}{n!}\Bigg[(1-n)\Big(2nc+2e+(n+1)a\Big)x'^2 + 2\Big(nc-(n-2)e+a\Big)x'y'$$
$$+(2e+a)y'^2 + (n-1)\Big(2nc+2e+(n+1)a\Big)x' - \Big(nc-(n-2)e+a\Big)y'$$
$$-\binom{n}{2}c\Bigg] \cdot \prod_{i=1}^{n-2}\Big(y'-(n-1)x'+i\Big) = 0. \tag{10}$$

Recalling that $L \equiv r\xi_X + bF$ and 3.7, we have $(t-(n-1)b)F \equiv r(K_X + (n-1)\xi_X)$. Thus

$$F \equiv \frac{r}{t-(n-1)b}(K_X + (n-1)\xi_X) \ ,$$

and substituting this expression of F into $xK_X + y(r\xi_X + bF)$, we get

$$p_{(X,L)}(x, y) = \chi(xK_X + yL) = \chi(\overline{x}K_X + \overline{y}\xi_X) = p_{(X,\xi_X)}(\overline{x}, \overline{y}) \ ,$$

where

$$(\overline{x}, \overline{y}) = \left(x + \frac{br}{t-(n-1)b}\, y \ , \ \frac{tr}{t-(n-1)b}\, y\right)$$

Then the expression of $p_{(X,L)}(x,y)$ follows from (10) by replacing (x',y') with $(\overline{x},\overline{y})$ as above and taking into account (9). To prove the converse, let $p_{(X,L)}(x,y)$ be as in jj) and write it as $R(x,y)\cdot\prod_{j=1}^{n-2}\big(ry-(n-1)x+j\big)$. Then $R(x,y)=Ax^2+Bxy+Cy^2+Ex+Gy+H$, where $A:=\frac{1}{n!}\left[(1-n)(2nc+2e+(n+1)a)\right]$, $B:=\frac{1}{n!}\left[-2enr+2cnr+2bn-2n^2b+4re+2ar\right]$ and $C:=\frac{1}{n!}\left[2er^2+r^2a+2nbr\right]$. Recalling (2), from the equality

$$\frac{1}{n!}(K_X+yL)^n=p_0(1,y,0)=R_0(1,y,0)\cdot\prod_{j=1}^{n-2}\Big(ry-(n-1)\Big)=(A+By+Cy^2)\cdot[ry+(1-n)]^{n-2}\ ,$$

where $R_0(x,y,z)$ is the homogeneous polynomial associated with $R(x,y)$, we deduce that

$$L^n=Cn!r^{n-2},\quad K_X\cdot L^{n-1}=(n-1)!\big(Br+C(n-2)(1-n)\big)r^{n-3},$$

$$K_X^2\cdot L^{n-2}=2(n-2)!\left(Ar^2+B(n-2)r(1-n)+C\frac{(n-2)(n-3)}{2}(1-n)^2\right)r^{n-4}\ .$$

A computation with Maple shows that $(rK_X+(n-1)L)^2\cdot L^{n-2}=r^2K_X^2\cdot L^{n-2}+2r(n-1)K_X\cdot L^{n-1}+(n-1)^2L^n=0$. On the other hand, $\frac{1}{n!}(rK_X+(n-1)L)^n=p_0(r,n-1,0)=0$ by (2). Therefore

$$(rK_X+(n-1)L)^n=0\qquad\text{and}\qquad(rK_X+(n-1)L)^2\cdot L^{n-2}=0\ .$$

Since $rK_X+(n-1)L$ is nef (by jjj)) but not big, let $\varphi:X\to Y$ be the morphism as in Remark 1.1 with $\dim Y<\dim X$ and $rK_X+nL=\varphi^*D$ for some nef line bundle D on Y. Since $\gcd(r,n-1)=1$, by applying Lemma 3.1 with $\psi=\varphi$ and $(\sigma,\tau)=(r,n-1)$, we know that there exists an ample line bundle $\mathscr{L}$ on X such that $K_X+(n-1)\mathscr{L}=p(rK_X+(n-1)L)=\varphi^*D$ for some ample line bundle D on Y. Since $(\varphi^*D)^2\cdot L^{n-2}=0$, we see that $\dim Y\leq 1$. Moreover, since $\mathrm{rk}\langle K_X,\mathscr{L}\rangle=\mathrm{rk}\langle K_X,L\rangle=2$ by j), we get $\dim Y=1$ and from [2, Theorem 7.3.2] we conclude that $(X,\mathscr{L})$ is a quadric fibration over the smooth curve Y. Then Lemma 3.1 (c) allows us to conclude. $\square$

Remark 3.9. (i) Assume that L is r-very ample on X (see [2, p. 225]). Then $L\cdot\gamma\geq r$ for any curve $\gamma\subset X$ (see [3, Corollary (1.3)]). So, if $rK_X+\delta L$ is not nef for $\delta=n$ or $n-1$, then $rK_X+(\delta+\epsilon)L$ is nef but not ample for some $\epsilon>0$. Hence by Mori theory and [2, Lemma 6.4.2], there exists an extremal rational curve C on X such that $(rK_X+(\delta+\epsilon)L)\cdot C=0$ and $-K_X\cdot C$ is the length $\ell(R)$ of the extremal ray $R:=\mathbb{R}_+[C]$. This gives

$$n+1\geq -K_X\cdot C=(\delta+\epsilon)\ \frac{L\cdot C}{r}\geq\delta+\epsilon>\delta\ .$$

Hence either (a) $\delta=n$ and $\ell(R)=n+1$, or (b) $\delta=n-1$ and $\ell(R)=n$ or $n+1$. In case (a), by [12, (2.4.1)] we have $\mathrm{Pic}(X)=\mathbb{Z}$, which implies $\mathrm{rk}\langle K_X,L\rangle=1$. This shows that if L is r-very ample and $\mathrm{rk}\langle K_X,L\rangle=2$, then rK_X+nL is nef; in particular Conjecture $\mathrm{C}(n,r)$ is true if L is r-very ample. In case (b), by [12, (2.4)] we have either $\mathrm{Pic}(X)=\mathbb{Z}$, which implies $\mathrm{rk}\langle K_X,L\rangle=1$, or $\mathrm{Pic}(X)=\mathbb{Z}\oplus\mathbb{Z}$ and the contraction of R defines a morphism $\rho_R:X\to B$ onto a smooth curve B whose general fiber F is a Fano $(n-1)$-fold with $\mathrm{Pic}(F)=\mathbb{Z}$. Let Γ be any rational curve on F. Then the nefvalue morphism associated to (X,L) with nefvalue $\frac{n-1+\epsilon}{r}$ contracts Γ, hence $(rK_X+(n-1+\epsilon)L)\cdot\Gamma=0$. This shows that $-K_F\cdot\Gamma=-K_X\cdot\Gamma=(n-1+\epsilon)\ \frac{L\cdot\Gamma}{r}>n-1$, i.e. $-K_F\cdot\Gamma\geq n=\dim F+1$. Thus by [2, Theorem 6.3.14] we get $F\cong\mathbb{P}^{n-1}$. Since distinct general fibers of ρ_R are numerically equivalent, from Proposition 3.2 we deduce that $X=\mathbb{P}(\mathscr{V})$ for a vector bundle $\mathscr{V}$ of rank

n on B. The above argument shows that if L is r-very ample on X with $\gcd(r, n-1) = 1$, $\mathrm{rk}\langle K_X, L\rangle = 2$ and X is not a $\mathbb{P}^{n-1}$-bundle over a smooth curve, then $rK_X + (n-1)L$ is nef.

(ii) When $r = 1$ (see [9, Theorem 6]), we do not need to assume that $K_X + (n-1)L$ is nef, provided that (X, L) is not a scroll over a smooth curve and $\mathrm{rk}\langle K_X, L\rangle = 2$. Actually, under these assumptions, this property comes from [2, Proposition 7.2.2 and Theorem 7.2.4].

(iii) Taking $r = 1$ and $b = 0$ in Theorem 3.8, the polynomial $p_{(X,L)}(x, y)$ coincides with that given in [9, Proposition 3] once expressed in terms of the coordinates (x, y).

(iv) For polarized n-folds (X, L) with $n \geq 3$ and $q := h^1(\mathscr{O}_X) > 0$, we have the same result as in Theorem 3.8 by dropping out condition j).

Finally, let us give here also a characterization of Fano fibrations of coindex 2 over smooth curves in the case $r = 1$.

Theorem 3.10. *Let (X, L) be a polarized n-fold with $n \geq 3$. If (X, L) is a Fano fibration of coindex 2 over a smooth curve of genus g such that $K_X + (n-2)L \equiv tF$ for some positive integer t, where F is a general fiber of X, then*

$$p_{(X,L)}(x, y) = \left(\sum_{0 \leq i+j \leq 3} c_{ij} x^i y^j\right) \cdot \prod_{i=1}^{n-3} \Big(y - (n-2)x + i\Big) ,$$

with

$$\begin{aligned}
c_{30} &= (n-2)^2 \left(\frac{t\delta}{(n-1)!} - (n-2)\frac{d}{n!}\right) , \; c_{21} = -(n-2)\left(\frac{2t\delta}{(n-1)!} - 3(n-2)\frac{d}{n!}\right) , \\
c_{12} &= \frac{t\delta}{(n-1)!} - 3(n-2)\frac{d}{n!} , \; c_{03} = \frac{d}{n!} , \; c_{20} = -\frac{3}{2}(n-2)^2\left(\frac{t\delta}{(n-1)!} - (n-2)\frac{d}{n!}\right),
\end{aligned}$$

$$\begin{aligned}
c_{11} &= (n-2)\left(\frac{2t\delta}{(n-1)!} - 3(n-2)\frac{d}{n!}\right) , \; c_{02} = -\frac{1}{2}\left(\frac{t\delta}{(n-1)!} - 3(n-2)\frac{d}{n!}\right) , \\
c_{10} &= \frac{1}{2}(n-2)^2\left(\frac{t\delta}{(n-1)!} - (n-2)\frac{d}{n!}\right) + 2\frac{g-1}{(n-3)!} , \\
c_{01} &= \frac{\chi}{(n-2)!} + \frac{1}{2}\frac{t\delta}{(n-1)!} - \frac{d}{2(n!)}(3n-4) + \frac{g-1}{(n-3)!} , \; c_{00} = -\frac{g-1}{(n-3)!} ,
\end{aligned}$$

where $d := L^n$, $\delta := F \cdot L^{n-1}$ and $\chi := \chi(L) = \frac{1}{2}(d - t\delta + 2t) - n(g-1)$. Conversely, assume that either $q > 0$, or $\mathrm{rk}\langle K_X, L\rangle = 2$. Suppose that $K_X + (n-2)L$ is nef and that $p_{(X,L)}(x, y)$ is as above for integers d, δ, χ, t, g such that $\chi = \frac{1}{2}(d - t\delta + 2t) - n(g-1)$. Then (X, L) is a Fano fibration of coindex 2 over a smooth curve of genus g such that $K_X + (n-2)L \equiv tF$, $\chi(L) = \chi$, $L^n = d$ and $F \cdot L^{n-1} = \delta$ for any general fiber F of X.

Proof. Let (X, L) be a Fano fibration of coindex 2 over a smooth curve C via a morphism $\varphi : X \to C$ and let F be a general fiber. By [1, Theorem 6.1], we know that

$$p_{(X,L)}(x, y) = R(x, y) \cdot \prod_{i=1}^{\iota_F - 1} \big(y - \iota_F x + i\big) = R(x, y) \cdot \prod_{i=1}^{n-3} \Big(y - (n-2)x + i\Big) ,$$

where $R(x,y)=\sum_{0\leq i+j\leq 3} c_{ij}x^iy^j$ for some $c_{ij}\in\mathbb{Q}$. Since $R(x,y)=-R(1-x,-y)$ we have the relations

$$c_{30}=4c_{00}+2c_{10}\ ,c_{21}=-c_{11}\ ,c_{12}=-2c_{02}\ ,c_{20}=-6c_{00}-3c_{10}\ . \tag{11}$$

By Theorem 2.5 applied to the pair (F,L_F), we deduce that

$$p_{(F,L_F)}(x,y)=\left(\frac{\delta}{(n-1)!}\,(y-(n-2)x)^2+\frac{(n-2)\delta}{(n-1)!}\,(y-(n-2)x)+\frac{1}{(n-3)!}\right)\cdot\prod_{i=1}^{n-3}(y-(n-2)x+i)\ .$$

Now apply (8) and Algorithm 2, noting that the polynomial $R_F(x,y)$ appearing in (8) is just the first factor of $p_{(F,L_F)}(x,y)$. Next the use of the following relations

$$1-g=\chi(\mathscr{O}_C)=\chi(\mathscr{O}_X)=p_{(X,L)}(0,0),\ \ \chi(L)=p_{(X,L)}(0,1),\ \ \frac{L^n}{n!}=p_0(0,1,0),$$

coming from $\varphi_*\mathscr{O}_X=\mathscr{O}_C$, the projection formula and (2), allow us to express the c_{ij}'s in terms of d,δ,t,χ,g. Finally, from the relation

$$t+1-g=\chi(\varphi_*(tF))=\chi(tF)=p_{(X,L)}(1,n-2)=R(1,n-2)(n-3)!\ ,$$

by using Maple we get $\chi:=\chi(L)=\frac{1}{2}(d-t\delta+2t)-n(g-1)$. This gives the first part of the statement. Now, suppose that (X,L) is a polarized n-fold for $n\geq 3$, satisfying $q>0$, or $\mathrm{rk}\langle K_X,L\rangle=2$, with $K_X+(n-2)L$ nef and such that $p_{(X,L)}(x,y)$ is as in the statement for some integers d,δ,χ,t,g, where $\chi=\frac{1}{2}(d-t\delta+2t)-n(g-1)$. The expression of $p_{(X,L)}(x,y)$ combined with (2) shows that $(K_X+(n-2)L)^n=p_0(1,n-2,0)=0$. Thus, $K_X+(n-2)L$ is nef but not big. By Remark 1.1, there exists a morphism $\varphi:X\to Y$ onto a normal variety Y with $\dim Y<\dim X$ such that $K_X+(n-2)L=\varphi^*D$ for some nef line bundle D on Y. Recalling (2), we have

$$p_0(x,1,0)=\frac{1}{n!}\big(xK_X+L\big)^n=\frac{1}{n!}\left(L^n+\binom{n}{1}K_X\cdot L^{n-1}x+\binom{n}{2}K_X^2\cdot L^{n-2}x^2+\dots\right).$$

On the other hand, we have $p_0(x,1,0)=(c_{30}x^3+c_{21}x^2+c_{12}x+c_{03})\cdot\big(1-(n-2)x\big)^{n-3}$, i.e.

$$p_0(x,1,0)=c_{03}+\left[-c_{03}\binom{n-3}{1}(n-2)+c_{12}\right]x\ +$$
$$+\ \left[c_{21}-c_{12}\binom{n-3}{1}(n-2)\ +\ c_{03}\binom{n-3}{2}(n-2)^2\right]x^2+\dots\ .$$

Comparing the coefficients of the three terms of lowest degree in x in the two expressions, we get

$$L^n\ =\ n!\ c_{03}\ ,\quad K_X\cdot L^{n-1}\ =\ (-1)^{n-3}(n-1)!\left[c_{03}\binom{n-3}{1}(n-2)(-1)^n+c_{12}(-1)^{n-1}\right],$$
$$K_X^2\cdot L^{n-2}\ =\ 2(n-2)!\left[c_{21}-c_{12}\binom{n-3}{1}(n-2)+c_{03}\binom{n-3}{2}(n-2)^2\right]$$

and then a computation with Maple shows that

$$(K_X+(n-2)L)^2\cdot L^{n-2}=K_X^2\cdot L^{n-2}+2(n-2)K_X\cdot L^{n-1}+(n-2)^2L^n=0\ .$$

Therefore, by arguing as in the proof of Theorem 3.8, we see that there exists an ample line bundle $\mathscr{L}$ on X such that $K_X+(n-2)\mathscr{L}=\varphi^*D'$, for some ample line bundle D' on Y, and that $(\varphi^*D')^2\cdot L^{n-2}=0$. Hence $\dim Y\leq 1$. Since by hypothesis we have either $q>0$, or $\mathrm{rk}\langle K_X, \mathscr{L}\rangle = \mathrm{rk}\langle K_X, L\rangle = 2$, it follows that $\dim Y = 1$. Therefore, we can conclude that $K_X + (n-2)L = \varphi^* D$ for some ample line bundle D on Y, because $\mathscr{L} = L + \varphi^* A$ for some line bundle A on Y, $K_X + (n-2)L$ is nef and by assumption we have either $q > 0$, or $\mathrm{rk}\langle K_X, L\rangle = 2$. Thus (X, L) is a Fano fibration of coindex 2 over the smooth curve Y whose genus $g(Y)$ is g, because

$$1 - g(Y) = \chi(\mathscr{O}_Y) = \chi(\mathscr{O}_X) = p_{(X,L)}(0,0) = (n-3)!\; c_{00} = 1 - g\ .$$

Moreover, writing $K_X + (n-2)L \equiv t'F$ for some positive integer t', where F is a general fiber of φ, by the relation $\chi = \frac{1}{2}(d - t\delta + 2t) - n(g-1)$ we see that

$$t' + 1 - g = \chi(t'F) = \chi(K_X + (n-2)L) = p_{(X,L)}(1, n-2) = t + 1 - g\ ,$$

i.e. $t' = t$. Finally, in view of the above expressions of L^n and $K_X \cdot L^{n-1}$, we have $F \cdot L^{n-1} = \frac{1}{t}(K_X \cdot L^{n-1} + (n-2)L^n) = \delta$. $\square$

Remark 3.11. A result similar to Theorem 3.10 can be obtained for $rK_X + (n-2)L \equiv tF$ with r a positive integer such that $\gcd(r, n-2) = 1$ by using Lemma 3.1. Notice that in this case the corresponding expression of $p_{(X,L)}(x,y)$ is very intricate.

4. Case $\mathrm{rk}\langle K_X, L\rangle = 2$: Fano fibrations over varieties

First of all, let us describe a procedure to obtain the canonical equation of the Hilbert curve for Fano fibrations of low coindex over a normal variety of dimension $m \geq 2$.

More precisely, let X be a manifold of dimension n and let $\pi : X \to Y$ be a morphism onto a normal variety Y of dimension $m < n$ whose general fiber F is a Fano manifold. Let L be an ample line bundle on X such that $rK_X + \iota_F L = \pi^* A$ for some ample line bundle A on Y, with r and ι_F coprime. Then there exists an integer $s \gg 0$ such that sA is very ample on Y. Thus $s(rK_X + \iota_F L) = \pi^* sA$ is spanned on X and then we can take a smooth irreducible element $V \in |\pi^* sA|$. Consider the following exact sequence

$$0 \to xK_X + yL + (x-1)V \to xK_X + yL + xV \to xK_V + yL_V \to 0\ .$$

Since $V \in |srK_X + s\iota_F L|$, we have

$$p_{(X,L)}\big(x(sr+1), y + sx\iota_F\big) = p_{(X,L)}\big((x-1)sr + x, y + s\iota_F(x-1)\big) + p_{(V,L_V)}(x,y)\ , \tag{12}$$

with $rK_V + \iota_F L_V = \pi_{|V}{}^*(rs+1)A$. By using general coordinates (x', y'), we can write

$$p_{(X,L)}\big(x', y'\big) = R_X(x', y') \cdot \prod_{i=1}^{\iota_F - 1} \big(ry' - \iota_F x' + i\big)\ ,$$

for some polynomial R_X of degree $n - \iota_F + 1$. Letting $(x', y') = (x(sr+1), y + sx\iota_F)$ and $((x-1)sr + x, y + s\iota_F(x-1))$ respectively, we obtain

$$p_{(X,L)}\big(x(sr+1), y + sx\iota_F\big) = R_X(x(sr+1), y + sx\iota_F) \cdot \prod_{i=1}^{\iota_F - 1} \big(ry - \iota_F x + i\big)\ ,$$

$$p_{(X,L)}\big((x-1)sr + x, y + s\iota_F(x-1)\big) = R_X((x-1)sr + x, y + s\iota_F(x-1)) \cdot \prod_{i=1}^{\iota_F - 1} \big(ry - \iota_F x + i\big)\ .$$

Similarly, for the pair (V, L_V) we have

$$p_{(V,L_V)}\big(x,y\big) = R_V(x,y) \cdot \prod_{i=1}^{\iota_F - 1} \big(ry - \iota_F x + i\big) \ .$$

Thus (12) gives

$$R_X\big(x + srx, y + s\iota_F x\big) = R_X\big((x+srx) - sr, (y + s\iota_F x) - s\iota_F\big) + R_V\big(x,y\big) \ . \tag{13}$$

Set $M := \begin{pmatrix} sr+1 & s\iota_F \\ 0 & 1 \end{pmatrix}$, $\vec{v} := (-sr, -s\iota_F)$ and $\vec{x} := (x,y)$. Then (13) becomes

$$R_X\big(\vec{x}M\big) = R_X\big(\vec{x}M + \vec{v}\big) + R_V\big(\vec{x}\big) \ . \tag{14}$$

For any $j \in \{0, ..., m-2\}$, denote by X_j the pull-back via π of the transverse intersection of j general elements of $|sA|$. Then $X_0 := X$, $X_1 := V$ and equation (14) can be rewritten as

$$R_{X_0}\big(\vec{x}M\big) = R_{X_0}\big(\vec{x}M + \vec{v}\big) + R_{X_1}\big(\vec{x}\big) \ .$$

By an inductive argument, we obtain $rK_{X_j} + \iota_F L_{X_j} = {\pi_{|X_j}}^*(jrs+1)A$ and

$$R_{X_j}\big(\vec{x}M\big) = R_{X_j}\big(\vec{x}M + \vec{v}\big) + R_{X_{j+1}}\big(\vec{x}\big) \tag{15}$$

for $j \in \{1, ..., m-2\}$. Equation (15) says that if we know the term $R_{X_{j+1}}\big(\vec{x}\big)$, it is possible to go back to the term $R_{X_j}\big(\vec{x}\big)$, and so on.

Finally, consider the case $j = m-1$ and for simplicity set $W := X_{m-1}$ and $R\big(\vec{x}\big) := R_W\big(\vec{x}\big)$. Then $p_{(W,L_W)}(\vec{x}) = R(\vec{x}) \cdot \prod_{k=1}^{\iota_F - 1} \big(ry - \iota_F x + k\big)$. Note that W is a smooth variety of dimension $n - m + 1$ endowed with a morphism $\varphi := \pi_{|X_{m-1}} : W \to C$ onto a smooth irreducible curve C which is the transverse intersection of $m-1$ general elements of $|sA|$. Then $rK_W + \iota_F L_W = \varphi^* \mathscr{A}$ for some ample line bundle $\mathscr{A}$ on C. Hence $rK_W + \iota_F L_W \equiv tF$ for some positive integer t. Thus, for a general fiber F of φ, by considering the following exact sequences

$$\begin{gathered}
0 \to xK_W + yL_W - F \to xK_W + yL_W \to xK_F + yL_F \to 0 \ , \\
0 \to xK_W + yL_W - 2F \to xK_W + yL_W - F \to xK_F + yL_F \to 0 \ , \\
\vdots \\
0 \to xK_W + yL_W - tF = (x-r)K_W + (y - \iota_F)L_W \to xK_W + yL_W - (t-1)F \to xK_F + yL_F \to 0 \ ,
\end{gathered}$$

we get

$$\begin{gathered}
\chi(xK_W + yL_W) = \chi(xK_W + yL_W - F) + \chi(xK_F + yL_F) \ , \\
\chi(xK_W + yL_W - F) = \chi(xK_W + yL_W - 2F) + \chi(xK_F + yL_F) \ , \\
\vdots \\
\chi(xK_W + yL_W - (t-1)F) = \chi((x-r)K_W + (y-\iota_F)L_W) + \chi(xK_F + yL_F) \ .
\end{gathered}$$

So, having in mind that $p_{(W,L_W)}(a,b) = \chi(aK_W + bL_W)$ and $p_{(F,L_F)}(x,y) = \chi(xK_F + yL_F)$, we deduce that

$$p_{(W,L_W)}(x,y) = p_{(W,L_W)}(x - r, y - \iota_F) + t \ p_{(F,L_F)}(x,y) \ . \tag{16}$$

Since F is a Fano manifold with $rK_F + \iota_F L_F = \mathscr{O}_F$, by Proposition 2.1 we know that

$$p_{(F,L_F)}(x,y) = R_F(x,y) \cdot \prod_{i=1}^{\iota_F - 1} \left(ry - \iota_F x + i\right) \ ,$$

for a suitable polynomial R_F of degree $n - m - \iota_F + 1$. Then equation (16) with the above relations yields

$$R(x,y) = R(x-r, y-\iota_F) + t\ R_F(x,y). \tag{17}$$

Therefore, once we know $R_F(x,y)$, from (17) we can find $R(\vec{x}) := R(x,y)$. In conclusion, by (15), going back by induction, we can obtain all the $R_{X_j}\left(\vec{x}\right)$'s.

This completes the procedure and gives rise to Algorithm 3 below.

Algorithm 3 The Hilbert curve Γ of a Fano fibration $\pi : X \to Y$ of fiber F with $m = \dim Y \geq 2$

Require: A, F, r, ι_F, n, m
Ensure: Γ
if $n > m, 0 < \iota_F \leq n - m + 1$ **then**
  $c_F \leftarrow n - m - \iota_F + 1$
  $s \leftarrow 1$
  repeat
    $s \leftarrow s + 1$
  until sA is very ample
  $t \leftarrow s^{m-1}(rs+1)^{\frac{m(m-1)}{2}} A^m$
  `Find` $R_{X_{m-1}}(x,y)$ `such that` $R_{X_{m-1}}(x,y) = R_{X_{m-1}}(x-r, y-\iota_F) + t\cdot$`RHilbPolynF`$(\iota_F)$
  $M \leftarrow \begin{pmatrix} sr+1 & s\iota_F \\ 0 & 1 \end{pmatrix}$
  $\vec{v} \leftarrow (-sr, -s\iota_F)$
  $\vec{x} \leftarrow (x,y)$
  $j \leftarrow m$
  repeat
    $j \leftarrow j - 1$
    `Find` $R_{X_{j-1}}(x,y)$ `such that` $R_{X_{j-1}}(\vec{x}M) = R_{X_{j-1}}(\vec{x}M + \vec{v}) + R_{X_j}(\vec{x})$
  until $j = 1$
  $p_{(X,L)}(x,y) \leftarrow R_{X_0}(x,y) \cdot \prod_{k=1}^{\iota_F - 1}(ry - \iota_F x + k)$
  return $\Gamma : p_{(X,L)}(x,y) = 0$
end if

Remark 4.1. A Fano fibration of dimension n and coindex 0 over a normal surface is in fact a projective bundle $\mathbb{P}(\mathscr{V})$ for some ample vector bundle $\mathscr{V}$ of rank $n-1$ over a smooth surface in view of [4, (3.2.1)] and [5, Lemma (2.12)].

As an example, we apply the above method to scrolls over a smooth surface (see also [10, Theorem 3.1] with $u = x - \frac{1}{2}, v = y$).

Example 4.2. Let $X := \mathbb{P}(\mathscr{V})$ for some ample vector bundle $\mathscr{V}$ of rank $n-1$ over a smooth surface S. Let L be the tautological line bundle and let $F \cong \mathbb{P}^{n-2}$ be a fiber of the bundle projection $\pi : X \to S$. In this case $\iota_F = n - 1$. So, letting $r = 1$, we have

$$p_{(X,L)}(x,y) = R(x,y) \cdot \prod_{i=1}^{\iota_F - 1} \left(ry - \iota_F x + i\right) = R(x,y) \cdot \prod_{i=1}^{n-2} \left[y - (n-1)x + i\right] \ ,$$

where, due to the invariance of $p_{(X,L)}$ under the Serre involution, the polynomial $R(x,y)$ has the following expression:

$$R(x,y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 - a_{11}x - a_{12}y + a_{00}\ , \tag{18}$$

with $a_{00} = \frac{\chi(\mathscr{O}_X)}{(n-2)!}$, for some $a_{11}, a_{12}, a_{22} \in \mathbb{Q}$. Since we are assuming that (X,L) is a scroll over S, we have $K_X + (n-1)L = \pi^*A$ for some ample line bundle A on S. Moreover,

$$A := K_S + \det \mathscr{V} \tag{19}$$

by the canonical line bundle formula. Let s be a positive integer such that sA is very ample. Let $C \in |sA|$ be any smooth curve and let $V = \pi^{-1}(C)$. In the present case (13) becomes

$$R\big((1+s)x, y+s(n-1)x\big) = R\big((1+s)x - s, (y+s(n-1)x) - s(n-1)\big) + R_{(V,L_V)}\big(x,y\big)\ .$$

Note that (V, L_V) is a scroll over C via $\pi_{|V} : V \to C$. Then by Theorem 3.3 we know that

$$R_{(V,L_V)}\big(x,y\big) = \frac{2q-2}{(n-2)!}\ x + \frac{d}{(n-1)!}\ y - \frac{q-1}{(n-2)!}\ .$$

Here $q = g(C) = 1 + \frac{s}{2}(K_S \cdot A + sA^2)$ by the genus formula and

$$d = L_V^{n-1} = (\det \mathscr{V})C = sA \cdot (A - K_S) \tag{20}$$

by the Chern-Wu relation and (19). So, by using Maple and comparing $R\big((1+s)x, y+s(n-1)x\big) - R\big((1+s)x - s, (y+s(n-1)x) - s(n-1)\big)$ with $R_{(V,L_V)}\big(x,y\big)$, we obtain

$$\begin{aligned}
a_{11} &= \frac{n-1}{2(s^2+s)n!}\big\{\left[-2\chi(\mathscr{O}_S)(n-1)^2 + 2\chi(L)(n-1)\right](s^2+s) - d(n+1)s + n(2q-2) - d\big\}\\
&= \frac{n-1}{2n!}\left[-2\chi(\mathscr{O}_S)(n-1)^2 + 2\chi(L)(n-1) + (n+1)K_S \cdot A - A^2\right]\ ,\\
a_{12} &= \frac{1}{2s(n!)}\big\{\left[2\chi(\mathscr{O}_S)(n-1)^2 - 2\chi(L)(n-1)\right]s + d\big\}\\
&= \frac{1}{2(n!)}\left[2\chi(\mathscr{O}_S)(n-1)^2 - 2\chi(L)(n-1) - K_S \cdot A + A^2\right]\ ,\\
a_{22} &= \frac{1}{2s(n!)}\big\{\left[-2\chi(\mathscr{O}_S)(n-1) + 2\chi(L)\right]s + d\big\} = \frac{1}{2(n!)}\left[-2\chi(\mathscr{O}_S)(n-1) + 2\chi(L) - K_S \cdot A + A^2\right]\ .
\end{aligned}$$

Note that to get the final expressions of the a_{ij}'s we used (20) and the fact that $-d(n+1)s + n(2q-2) - d = (s^2+s)[(n+1)K_S \cdot A - A^2]$.

Actually, Fano fibrations of coindex 0 over a smooth surface can be characterized by means of their Hilbert curves.

Theorem 4.3. *Let (X,L) be a polarized n-fold with $n \geq 3$. If (X,L) is a Fano fibration of coindex 0 over a smooth surface S such that $K_X + (n-1)L = \pi^*A$ for some ample line bundle A on S, then*

$$\begin{aligned}
p_{(X,L)}(x,y) = \frac{1}{n!}\Bigg\{\ &\frac{n-1}{2}\left[-2\chi_0(n-1)^2 + 2\chi(n-1) + (n+1)k - h\right]x^2\ +\\
&+ \left[2\chi_0(n-1)^2 - 2\chi(n-1) - k + h\right]xy\ +\ \frac{1}{2}\left[-2\chi_0(n-1) + 2\chi - k + h\right]y^2\ +\\
&+ \frac{1}{2}\left[-2\chi_0(n-1) + 2\chi - k + h\right]y^2\ +\ \frac{n-1}{2}\left[2\chi_0(n-1)^2 - 2\chi(n-1) - (n+1)k + h\right]x\ +
\end{aligned}$$

$$+ \frac{1}{2}\left[-2\chi_0(n-1)^2 + 2\chi(n-1) + k - h\right] y \ + \ n(n-1)\chi_0 \ \Bigg\} \cdot \prod_{i=1}^{n-2}\Big(y-(n-1)x+i\Big) \ ,$$

where $\chi_0 := \chi(\mathscr{O}_S), \chi := \chi(L), k := K_S \cdot A$ *and* $h := A^2 > 0$. *Conversely, assume that* $\mathrm{rk}\langle K_X, L\rangle = 2$ *and* $K_X + (n-1)L$ *is nef. If* $p_{(X,L)}(x,y)$ *is as above for some integers* χ_0, χ, k, h *with* $h > 0$*, then* (X,L) *is a Fano fibration of coindex* 0 *over a smooth surface.*

Proof. Keeping in mind Remark 4.1, the "only if" part of the statement follows from Example 4.2 once we consider that $X = \mathbb{P}(\mathscr{E})$ for some ample vector bundle $\mathscr{E}$ of rank $n-1$ on S, L being the tautological line bundle, and $A = K_S + \det\mathscr{E}$. Thus assume that (X,L) is a polarized n-fold with $n \geq 3$ and $\mathrm{rk}\langle K_X, L\rangle = 2$ for which $K_X + (n-1)L$ nef and $p_{(X,L)}(x,y)$ is as in the statement. Note that $(K_X + (n-1)L)^n = n! p_0(1, n-1, 0) = 0$ by (2). Hence $K_X + (n-1)L$ is nef but not big. Thus by Remark 1.1 there exists a morphism $\varphi : X \to Y$ onto a normal variety Y with $\dim Y < n$ such that

$$K_X + (n-1)L = \varphi^* D \tag{21}$$

for some nef line bundle D on Y. Write $p_{(X,L)}(x,y) = R(x,y) \cdot \prod_{i=1}^{n-2}\big(y-(n-1)x+i\big)$, where $R(x,y)$ is as in (18). From (2), it follows that

$$p_0(x,1,0) = \frac{1}{n!}\left[L^n + \binom{n}{1}K_X \cdot L^{n-1}x + \binom{n}{2}K_X^2 \cdot L^{n-2}x^2 + \binom{n}{3}K_X^3 \cdot L^{n-3}x^3 + \dots\right] .$$

On the other hand, we have $p_0(x,1,0) = (a_{11}x^2 + 2a_{12}x + a_{22}) \cdot [1-(n-1)x]^{n-2}$, i.e.

$$p_0(x,1,0) = a_{22} + \left[-a_{22}\binom{n-2}{1}(n-1) + 2a_{12}\right] x +$$

$$+ \ \left[a_{22}\binom{n-2}{2}(n-1)^2 - 2a_{12}\binom{n-2}{1}(n-1) + a_{11}\right] x^2 +$$

$$+ \left[-a_{22}\binom{n-2}{3}(n-1)^3 + 2a_{12}\binom{n-2}{2}(n-1)^2 - a_{11}\binom{n-2}{1}(n-1)\right] x^3 \ + \dots \ .$$

Comparing the coefficients of the four terms of lowest degree in x in the two expressions of $p_0(x,1,0)$, a computation with Maple shows that

$$(K_X+(n-1)L)^3 \cdot L^{n-3} = K_X^3 \cdot L^{n-3} + 3(n-1)K_X^2 \cdot L^{n-2} + 3(n-1)^2 K_X \cdot L^{n-1} + (n-1)^3 L^n = 0,$$

hence $\dim Y \leq 2$. Note that $\dim Y = 1$ or 2, because $\mathrm{rk}\langle K_X, L\rangle = 2$. Since the nefvalue of (X,L) is $n-1$ in view of (21), we deduce by [2, Theorem 7.3.2] and [6, (11.8)] that (X,L) is either (i) a scroll over a smooth surface or (ii) a quadric fibration over a smooth curve of genus q. Assume we are in case (ii). Then comparing the coefficients of x^2, xy, y^2 in $R(x,y)$ of $p_{(X,L)}$ given in the statement with those provided by Theorem 3.8 for $r = 1$ (see formula (10)), we get the following equalities:

$$\frac{n-1}{2}\left[-2\chi_0(n-1)^2+2\chi(n-1)+(n+1)k-h\right]=(1-n)\left[2n(2q-2)+2e+(n+1)a\right] \ ,$$
$$\frac{1}{2}\left[2\chi_0(n-1)^2-2\chi(n-1)-k+h\right]=n(2q-2)-(n-2)e+a \ ,$$
$$\frac{1}{2}\left[-2\chi_0(n-1)+2\chi-k+h\right]=2e+a \ .$$

A check with Maple shows that the above three equations imply $h=0$, but this is impossible because h is assumed to be a positive integer. Thus (X,L) is as in case (i), i.e. (X,L) is a Fano fibration of coindex 0 over a smooth surface. □

A result similar to Theorem 4.3 holds also for $rK_X+(n-1)L$ with $\gcd(r,n-1)=1$. Finally, summing-up the above results and the proof of [8, Proposition 5.1], we can deduce also the following result comparable with Corollary 3.4 for $(n,r)=(3,2)$.

Corollary 4.4. *Conjecture $C(3,2)$ is true provided that (X,L) does not contain (-1)-planes.*

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