Unification in lax logic

Silvio Ghilardi [*], Giacomo Lenzi [†]
In honor of Antonio Di Nola's 75^{th} birthday

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Abstract

In this paper, we focus on the intuitionistic propositional logic extended with a *local* operator [23] (also called *nucleus* [22]); such logic is commonly named *lax logic* after [9]. We prove that unification is finitary in this logic and supply algorithms for computing a basis of unifiers and for recognizing admissibility of inference rules, following analogous known results for intuitionistic logic.

Keywords: nuclei, lax logic, unification theory, admissible inference rules.

1 Introduction

It is well-known that the algebraic counterpart of intuitionistic logic is given by Heyting algebras, which are among the most important examples of residuated lattices, whereas the algebraic counterpart of the logic we study in this paper (namely, lax logic) is given by Heyting algebras with nuclei. So we believe that this paper sits both in the scope of the journal and in the area of interest of Antonio Di Nola.

More precisely, in this paper we consider the extension of intuitionism obtained by adding it a unary modal operator j (a nucleus) subject to the axiom schemata (2.1) below. This logic comes from Lawvere's suggestion that a Grothendieck topology appears in a natural way as a modal operator; as such, this geometrically inspired modality (and the related algebraic and

[*]Università degli Studi di Milano, Department of Mathematics, via Cesare Saldini 50, 20133 Milano, email silvio.ghilardi@unimi.it

[†]University of Salerno, Department of Mathematics, via Giovanni Paolo II, 84084 Fisciano (SA), email gilenzi@unisa.it

categorical counterparts) found numerous applications in topos theory [23, 28, 29] and point-free topology [22].

The investigation of the basic properties (completeness, finite model property, decidability) of such logic at the propositional level was first undertaken in [5, 6] and later on in [16]. In more recent times, the logic was rediscovered (and renamed as *lax logic*) inside the area of applications to hardware verification [24, 9, 25]. The problem there is the timing analysis of logical circuits, implication in lax logic is interpreted as follows: $A \to B$ holds if A gives rise to B in a bounded amount of time.

In the last two decades, there have been increasing interest in lax logic, from various sources. We give here a list (far from exhaustive) of examples.

In [27] the authors give a new presentation of lax logic and finds that the lax modality is already expressible using possibility and necessity.

In [17] the author gives two proof-search calculi for lax logic. The first calculus is useful for enumerating without redundancy all proofs in the logic; especially where proof-search is for natural deductions. The other calculus builds on the propositional fragment of the first calculus to give a decision procedure for propositional lax logic, so useful for hardware verification.

In [10] we find an interpretation of first order lax logic in the theorem prover HOL.

[8] shows that lax logic can be faithfully embedded into the underlying intuitionistic logic and discusses (computational) properties of the embedding. Using the proposed polynomial-time computable embedding, PSPACE-completeness of the provability problem of propositional lax logic is shown.

In the area of algebraic logic, there have been further intersting recent contributions. In [3], the authors associate to each superintuitionistic logic its downward and upward subframizations, and characterize them by embedding si-logics into the extensions of the propositional lax logic PLL.

In [7] a study is performed of what we call the $\{\to, j\}$-fragment of lax logic. The paper studies the algebraic correspondent of this fragment, called Lax Hilbert algebras, and their dual objects, lax Hilbert spaces.

In [4] a full duality theorem is given for Heyting algebras with nuclei and in [2] duality techniques are exploited in order to show the somewhat surprising result that the disjunction-free fragment of lax logic is locally finite.

Our main result and plan of the paper

Despite the progresses described above, it seems that the unification problem for lax logic has not been settled yet, whereas unification for intuitionism has been investigated in [12] (for a survey on results in unification theory in modal and description logics, the reader is referred to [1]).

The main result of [12] is the following:

Theorem 1.1. *([12], Theorem 5, p. 871) Intuitionism has finitary unification type.*

Likewise in this paper we prove:

Theorem 1.2. *Lax logic has finitary unification type.*

The proof is given in the following sections. Actually, our proof of Theorem 1.2 is similar to the one of Theorem 1.1, so we will stress mostly the points where the two proofs differ.

The paper is entirely self-contained: to this aim, we supply in Section 2 below an essential background on propositional lax logic, tailored to the semantic results we need in the main sections.

2 Basic syntax and semantics

The formulae of intuitionistic propositional logic are obtained from a countable set of atomic propositions (atoms) $x_1, \ldots, y_1, \ldots$ by applying them the connectives $\top, \bot, \wedge, \vee, \rightarrow$. For a complete Hilbert style axiomatization of intuitionistic propositional calculus (IPC), the reader is referred to textbooks like [26].

The language of propositional lax logic (PLL) includes also an additional unary operator j and an axiomatization for such logic is obtained by adding to the axiomatization of (IPC) the following axiom schemata

$$A \rightarrow jA, \quad jjA \leftrightarrow jA, \quad j(A \wedge B) \leftrightarrow jA \wedge jB \; . \tag{2.1}$$

Given a formula A, we write $\vdash_{PLL} A$ if A is provable in (PLL). More generally, we write $A_1, \ldots, A_n \vdash_{PLL} B$ to denote $\vdash_{PLL} A_1 \wedge \ldots \wedge A_n \rightarrow B$.

From an algebraic point of view, we consider *Heyting algebras with nuclei*, i.e. Heyting algebras $\mathcal{H} = (H, \sqcap, \sqcup, \Rightarrow, 1, 0)$ endowed with a unary operator j satisfying the following conditions

$$a \leq j(a), \quad j(j(a)) = j(a), \quad j(a \sqcap b) = j(a) \sqcap j(b) \tag{2.2}$$

for every a, b in the support H of $\mathcal{H}$ (the partial order relation $a_1 \leq a_2$ is defined as usual as $a_1 \sqcap a_2 = a_1$). We recall that a Heyting algebra is a bounded distributive lattice endowed with an extra operation $\Rightarrow$ satisfying the following condition

$$a \sqcap b \leq c \ \text{ iff } \ a \leq b \Rightarrow c \tag{2.3}$$

for all a, b, c in its support (it is well-known that condition (2.3) can be formulated equationally, so that Heyting algebras are an equational class).

A *valuation* V for (PLL)-formulae is a map associating with every atom x an element $V(x)$ of the support of an Heyting algebra with nucleus. The valuation V is then extended to the set of all formulae by induction in the obvious way. A Lindenbaum construction easily gives the following

Theorem 2.1. *For every formula A, we have that $\vdash_{PLL} A$ iff $V(A) = 1$ for every valuation V.*

The above result can be improved, by restricting to *finite* valuations, i.e. to valuations whose codomain is a finite algebra. We give below a quite short proof of this result, first we need a preliminary Lemma:

Lemma 2.1. *Every finitely generated bounded distributive lattice endowed with a nucleus j (i.e. with a unary operation satisfying conditions* (2.2)*) is finite.*

Proof. The lemma holds because all elements of such a structure can be written as meets of joins of elements which are either generators or obtained applying j to a disjunction of generators. It is so because the equality

$$j(a \sqcup j(b)) = j(a \sqcup b)$$

follows from (2.2). □

Theorem 2.2 (Finite Model Property)**.** *[5, 6, 16, 9] For every formula A, we have that $\vdash_{PLL} A$ iff $V(A) = 1$ for every finite valuation V.*

Proof. Let A be such that $\not\vdash_{PLL} A$ and consider a valuation V into a Heyting algebra with nucleus $(\mathcal{H}, j)$ such that $V(A) \neq 1$. Consider the (finite, by Lemma 2.1) bounded distributive latttice with nucleus generated by the elements of the kind $V(B)$ where B is a subformula of A. Every finite distributive lattice is a Heyting algebra, hence we have a sub-Heyting algebra with nucleus $(\mathcal{H}', j)$ of $(\mathcal{H}, j)$, including the valuation of all the subformulae of A. The inclusion preserves joins, meets, $0, 1, j$, but not the implication:

in fact $\mathcal{H}'$ has its own implication, that we denote by $\Rightarrow'$. However, if for a, b in the support of $\mathcal{H}'$ it happens that $a \Rightarrow b$ belongs to the support of $\mathcal{H}'$, then $a \Rightarrow b$ coincides with $a \Rightarrow' b$: to see this, notice that $a \Rightarrow b$ is the unique element satisfying (2.3) for all c in the support of $\mathcal{H}$, hence it is also the unique element satisfying (2.3) for all c in the support of $\mathcal{H}'$. Thus, we can restrict V in the codomain to the support of $\mathcal{H}'$ and get a finite valuation V' such that $V'(A) \neq 1$. $\square$

By Post's theorem (a set is recursive iff both it and its complement are recursively enumerable), we get decidability:

Corollary 2.1. *Given lax formulae $A_1, \ldots, A_n, B$, it is decidable whether $A_1, \ldots, A_n \vdash_{PLL} B$ holds or not.*

We now want to recover from Theorem 2.2 a Kripke-style semantics for (PLL). In fact, whereas various semantics for (PLL) have been proposed in the literature (in particular, the semantics in terms of Grothendieck coverings remains the most natural and appropriate one),[1] for the kind of propositional logic applications considered in this paper, we focus on the simplest and the most manageable semantics.

A *lax Kripke frame* is a triple

$$P_S \ = \ (P, \leq, S)$$

where P is a poset and S is a subset of P. A lax Kripke frame as above induces a Heyting algebra withy nucleus $\mathcal{H}(P_S)$ by taking the Heyting algebra $\downarrow P$ of downward closed subsets (downsets)[2] of P ordered by set-theoretic inclusion and by endowing this algebra with the nucleus defined in the following way for every downset $D \in \downarrow P$:

$$j_S(D) \ = \ \{p \in P \mid \forall s \in S \ (s \leq p \Rightarrow s \in D)\} \ . \tag{2.4}$$

Not all the nuclei on the downward closed sets of a poset P are of this kind, however we shall see that this is the case when P is finite. Since all finite Heyting algebras (up to isomorphism) are of the kind $\downarrow P$ for some finite poset P, we obtain in this way a complete characterization of finite Heyting

[1]See [11] for a comparison of the different Grothendieck coverings that can be used to prove the completeness theorem via canonical models. The lax Kripke frames explained in this section correspond to special kinds of Grothendieck coverings (called 'covering by points' in [11]).

[2]We say that $D \subseteq P$ is downward closed iff for all $p, q \in P$, if $p \in D$ and $q \leq p$ then $q \in D$.

algebras with nuclei (this is indeed part of a full duality theorem for Heyting algebras with nuclei, see [4]).

To prove such characterization (Theorem 2.3 below) we specialize some techniques from [2]. We need some notation concerning an Heyting algebra of the kind $\downarrow P$. For $p \in P$, we let $\downarrow p = \{q \in P \mid q \leq p\}$ and $p^c = \{q \in P \mid q \not\geq p\}$. For an abitrary downset D, we have

$$p \in D \quad \text{iff} \quad \downarrow p \subseteq D \tag{2.5}$$

$$p \notin D \quad \text{iff} \quad D \subseteq p^c \ . \tag{2.6}$$

Thus, $D = \bigcup_{p \in D} \downarrow p$ and $D = \bigcap_{p \notin D} p^c$. If P is finite, the elements of the kind $\downarrow p$ are precisely the join-irreducible elements of $\downarrow P$, whereas the elements of the kind p^c are precisely the meet-irreducible elements of $\downarrow P$.[3]

The minimal downsets of the kind p^c for $p \notin D$ are called the *meet-irreducible components* of D.

Lemma 2.2 ([2], Lemma 3.14)**.** *Let P be a finite poset and let j be a nucleus on $\downarrow P$; if p_c is a meet-irreducible component of some downset D such that $D = j(D)$, then $j(p^c) = p^c$.*

Proof. Let $q_1^c, \dots, q_n^c$ be the meet-irreducible components of D; for every $i = 1, \dots, n$ we have

$$j(q_1^c) \cap \cdots \cap j(q_n^c) = j(q_1^c \cap \cdots \cap q_n^c) = j(D) = D \subseteq q_i^c \ .$$

Since q_i^c is meet-irreducible, there is $k = 1, \dots, n$ such that $q_k^c \subseteq j(q_k^c) \subseteq q_i^c$; by minimality, $q_k^c = q_i^c$ and so $q_i^c = j(q_i^c)$. □

Theorem 2.3. *Let $(\mathcal{H}, j)$ be a finite Heyting algebra with nucleus. Then there exists a finite lax Kripke frame $P_S = (P, \leq, S)$ such that $(\mathcal{H}, j)$ is isomorphic to $\mathcal{H}(P_S)$.*

Proof. It is a basic well-known fact (see e.g. [15],Theorem 2.1) that $\mathcal{H}$ is isomorphic to the Heyting algebra $\downarrow P$ of the downsets of a finite poset P. Hence, we only need to identify some $S \subseteq P$ such that $\mathcal{H}(P_S)$ is isomorphic to $(\downarrow P, j)$. We take[4]

$$S \ = \ \{q \in P \mid j(q^c) = q^c\}$$

[3]Recall that an element a of a lattice is said to be join-irreducible iff the relation $a \leq b_1 \sqcup \cdots \sqcup b_n$ (for some $n \geq 0$ and $b_1, \dots, b_n$) implies $a \leq b_i$ for some i. The definition of a meet-irreducible element is dual.

[4]In view of (2.6), this is the same as taking S to be $\{q \in P \mid \forall D \in \downarrow P \ (q \in j(D) \to q \in D)\}$. Such alternative definition of S matches the definition of the 'covering by points' Grothendieck topology in the canonical model [11].

Now for $p \in P$ and $D \in \downarrow P$, we have that $p \notin j(D)$ iff $j(D) \subseteq p^c$ iff there is some meet-irreducible component q^c of $j(D)$ such that $j(D) \subseteq q^c \subseteq p^c$. Taking into consideration Lemma 2.2 and the fact that $j(j(D)) = j(D)$, this happens iff there is $q \in S$ such that $j(D) \subseteq j(q^c) = q^c \subseteq p^c$, i.e. iff there is $q \in S$ such that $D \subseteq q^c \subseteq p^c$ (in fact, since $j(q^c) = q^c$, the inclusion $j(D) \subseteq q^c$ is equivalent to the inclusion $D \subseteq q^c$). However $q^c \subseteq p^c$ holds iff $q \leq p$; thus, in conclusion we have that $p \notin j(D)$ iff there is $q \in S$ such that $q \notin D$ and $q \leq p$ iff $p \notin j_S(D)$. Since p and D are arbitrary, this shows that $j = j_S$, as wanted. □

Let us call a *finite lax Kripke model* (or simply a *finite lax model* or a *finite Kripke model*) any valuation V whose codomain is an algebra of the kind $\mathcal{H}(P_S)$, for a finite P_S. Putting together our results, we get the following completeness/finite model property theorem for (PLL) with respect to finite lax Kripke frames:

Theorem 2.4. *For every formula A, we have that $\vdash_{PLL} A$ iff $V(A) = 1$ for every finite lax Kripke model V.*

It is customary to reformulate Kripke semantics in terms of forcing; this is a purely mechanical translation, taking in mind that "the point $p \in P$ of the model induced by the valuation V forces a formula A" must be defined in such a way that it is equivalent to "$p \in V(A)$". Also, notice that the forcing relation at $p \in P$ defined in this way only depends on the restriction of the model to the points $q \leq p$. This is why sometimes it is preferred to formulate Kripke semantics using *rooted* posets (a poset $(P, \leq)$ is rooted when there is $\rho \in P$ such that $\downarrow \rho = P$). We directly give below such reformulation in terms of rooted posets, leaving the reader to convince himself that Theorem 2.5 below is just a rewording of Theorem 2.4 above. We adopted such rewording to make our notation fully compliant with [12]; for the same reason, we prefer to introduce Kripke models defined only over finite subsets of the set of atomic propositions. We nevertheless maintain the name '(lax) Kripke models' for the models introduced below, despite these light and immaterial differences.

For a finite set of atoms X, we let $F(X)$ the set of all (PLL)-formulae built up from atoms in X (we often write $A(X)$ to mean that $A \in F(X)$).

A *lax Kripke model* (or, simply, a *Kripke model* or, again, a *lax model*) is a tuple

$$u = (X, P, \leq, S_u, V) \tag{2.7}$$

where X is a finite set of atoms, $(P, \leq)$ is a finite rooted poset, $S_u \subseteq P$[5] and $V : X \longrightarrow \downarrow P$ is a function.

Often the elements of P are called 'worlds' of the model and $\leq$ is called the 'accessibility relation'. When u is a model as in (2.7), we say that u is a model over X or that it is an X-model.

Given such a model, it is possible to define the forcing relation $u \models_p A$ (where $p \in P$ and $A \in F(X)$) as follows, by induction on A:

- $u \models_p \top$ always;
- $u \not\models_p \bot$ always;
- $u \models_p x$ iff $p \in V(x)$ for all $x \in X$;
- $u \models_p A \wedge B$ if and only if $u \models_p A$ and $u \models_p B$;
- $u \models_p A \vee B$ if and only if $u \models_p A$ or $u \models_p B$;
- $u \models_p A \to B$ if and only if for every point $q \leq p$ of P, if $u \models_q A$, then $u \models_q B$;
- $u \models_p jA$ iff for every point $s \leq p$ belonging to S_u we have that $u \models_s A$.

When we write $u \models A$ (without specifying p), we mean $u \models_\rho A$, where ρ is the root of P. Actually, it is always possible to avoid the specification of p by considering submodels: for $p \in P$, we denoted by u_p the model obtained from u by restricting P and S_u to the subset $\{q \in P \mid q \leq p\}$ (p will be the root of this restricted model). Then it is easily seen that we have

$$u \models_p A \quad \text{iff} \quad u_p \models A$$

for every formula A (we shall systematically adopt this model restriction notation in the paper).

The *semantics of a formula* $A \in F(X)$ (denoted by A^*) is the class of models over X validating A; formally, we define

$$A^* = \{u = (X, P, \leq, S_u, V) \mid u \models A\} \quad . \tag{2.8}$$

Then we can reformulate the completeness theorem 2.4 in the style of [12] (Theorem 1 p. 862) as follows:

Theorem 2.5. *For every formulae $A, B \in F(X)$, we have that $A \vdash_{PLL} B$ iff $A^* \subseteq B^*$.*

[5]Note that S_u is not necessarily a downset of P (otherwise we would collapse our logic inside intuitionism with a distinguished atom). However, it is still true that lax formulas are interpreted as downsets.

3 Unification and Projectivity

A *substitution* is a map $\sigma : X \to F(Y)$, where X, Y are finite sets of atoms. A substitution extends naturally to a map from $F(X)$ to $F(Y)$. The composition of the substitutions $\sigma : F(X) \to F(Y)$ and $\tau : F(Y) \to F(Z)$ is the substitution $\tau\sigma : F(X) \to F(Z)$ defined by $(\tau\sigma)(x) := \tau(\sigma(x))$ for all $x \in X$. A substitution $\sigma_1 : F(X) \to F(Y)$ is less general than $\sigma_2 : F(X) \to F(Z)$, written $\sigma_1 \leq \sigma_2$, if there is $\tau : F(Z) \to F(Y)$ such that $\vdash_{PLL} \tau(\sigma_2(x)) \leftrightarrow \sigma_1(x)$.

We say that the substitution σ is a *unifier* of A if $\vdash_{PLL} \sigma(A)$.

A set S of unifiers for A is said to be a *complete* set of unifiers for A if every unifier for A is less general than a member of S. A complete set of unifiers for A is said to be a *basis* of unifiers for A if and only if its members are pairwise incomparable with respect to the preorder $\leq$. A unifier a for A is said to be a *most general unifier* (mgu) for A if and only if $\{\sigma\}$ is a complete set of unifiers for A.

A logic has *finitary unification type* if every every unifiable formula has a finite complete set of unifiers. Among logics with finitary unification types, we can mention intuitionistic logic [12] and the modal systems $K4, S4, Grz, GL$ [13]; however the modal system K lacks this property [20].

Given $\sigma : F(X) \to F(Y)$ and u a model over a finite set Y of atoms, $\sigma^*(u)$ is the model on the same frame but over the atoms X, such that $\sigma^*(u)_p \models x$ if and only if $u_p \models \sigma(x)$, for all p in the domain of u and for all $x \in X$ (by an easy induction, this implies

$$\sigma^*(u)_p \models A \text{ iff } u_p \models \sigma(A) \tag{3.1}$$

for all $A \in F(X)$). Notice that for composition of substitutions, we have the following equality $(\tau\sigma)(u) = \sigma^*(\tau^*(u))$; for restrictions to submodels, we have $\sigma^*(u)_p = \sigma^*(u_p)$ for all p in the domain of u.

Projective formulas

A lax formula $A(X)$ is said to be *projective* if there is a unifier $\sigma : F(X) \to F(X)$ of A such that

$$A \vdash_{PLL} x \leftrightarrow \sigma(x) \tag{3.2}$$

for every atom $x \in X$. By the replacement theorem (which holds in lax logic), the previous formula implies

$$A \vdash_{PLL} B \leftrightarrow \sigma(B) \tag{3.3}$$

for every lax formula $B \in F(X)$. A unifier σ of a formula A satisfying (3.2) is sais to be a *projective unifier* of A; it is immediate to see that projective unifiers of A are most general unifiers of A.

Like in [12], it is easily seen that substitutions satisfying (3.2) (independently on the fact whether they unify A or not) are closed under composition.

Moreover, like in [12], given a formula $A \in F(X)$ and given a subset a of X, we define the substitution

$$\theta_A^a(x) = \begin{cases} A \to x, & \text{if } x \in a \\ A \wedge x, & \text{if } x \notin a. \end{cases}$$

θ_A^a satisfies (3.2) and the same holds for every composition of such substitutions.

The aim of this Section is to give a semantic characterization of projective formulae. Such characterization is in term of the extension property and is the same as the characterization given in [12] for intuitionistic projective formulae (of course, the extension property now refers to lax Kripke models, but this is the only formal difference). Proofs follows the same schema and the same arguments as in [12], so they are reported here only in a synthetic way.

Lemma 3.1. *Let $A(X)$ be a lax formula, $a \subseteq X$, $x \in X$. We have:*

1. $\vdash_{PLL} \theta_A^a(x) \leftrightarrow \theta_A^a(\theta_A^a(x))$;
2. *for every other $b \subseteq X$, $A \leftrightarrow \theta_A^b(A) \vdash_{PLL} \theta_A^b(\theta_A^a(x)) \leftrightarrow \theta_A^a(x)$.*

The semantic analog of the previous lemma is the following:

Lemma 3.2. *Let A be a lax formula over X, $a \subseteq X$, u be a lax model. We have:*

1. *if $u \models A$ then $(\theta_A^a)^*(u) = u$*
2. *if $u \not\models A$ then $(\theta_A^a)^*(u)(\rho) \subseteq a$*
3. $(\theta_A^a)^*((\theta_A^a)^*(u)) = (\theta_A^a)^*(u)$
4. *given another $b \subseteq X$, if for all $p \in u$*

   $$(\theta_A^b)^*(u_p) \models A \quad \textit{iff} \quad u_p \models A$$

   then $(\theta_A^a)^*((\theta_A^b)^*(u)) = (\theta_A^a)^*(u)$.

A lax model is a *variant* of a model on the same lax frame if they are equal or they differ only on the semantics of the atoms at the root.

Lemma 3.3. *Let A be a lax formula over X and let u be a lax model such that $u \not\models A$ but $u_p \models A$ for every nonroot p. If there is a variant u' of u which is a model of A, then for some $a \subseteq X$ we have both $(\theta_A^a)^*(u) \models A$ and $(\theta_A^a)^*(u)(\rho) = a$.*

Like [12] we define a substitution θ_A as follows. Let a_i be an ordering of all subsets of X such that $a_i \subseteq a_j$ implies $i \leq j$. Let

$$\theta_A = (\theta_A^{a_s}) \dots (\theta_A^{a_1}).$$

Note that

$$\theta_A^*(u) = (\theta_A^{a_1}) \dots (\theta_A^{a_s})^*(u)$$

for every lax model u over X. In this way we obtain:

Lemma 3.4. *Let A be a lax formula and u a lax model such that $\theta_A^*(u) \models A$. Let $i = 1, \dots, s$ such that $a_i \subseteq \theta_A^*(u)(\rho)$. Then*

$$(\theta_A^{a_i})^* \dots (\theta_A^{a_s})^*(u) \models A.$$

A class K of lax models has the *extension property* if for every model u such that $u_p \in K$ for every nonroot p in the domain of u, there is a variant of u which belongs to K.

The projectivity criterion of [12] (Theorem 5, page 866) for intuitionism extends to lax logic, because the sets of special worlds S appearing in the models play no role in the proofs. In particular we have:

Theorem 3.1. *(see [12], Theorem 5, page 866) If A is a lax formula, then the following properties are equivalent:*

1. *θ_A unifies A.*
2. *A is projective.*
3. *A^* has the extension property.*

Proof. $1 \to 2$ follows because θ_A satisfies (3.2).

$2 \to 3$ Suppose σ is a unifier for A satisfying (3.2) and take a lax model u such that $u_p \in A^*$ for every nonroot p. Then $\sigma^*(u) \in A^*$. It is sufficient to show that $\sigma^*(u)$ is a variant of u: this follows from the fact that $\sigma^*(u)_p =$

$\sigma^*(u_p)$ for all $p \in u$ and from the fact that if $w \in A^*$ then $\sigma^*(w) = w$ for every lax model w, because σ satisfies (3.2).

$3 \to 1$ let u be a lax model. Our aim is to show $\theta_A^*(u) \models A$. Suppose not. We can assume $\theta_A^*(u_p) \models A$ for all nonroot p, since u is finite.

Since A^* has the extension property, by Lemma 3.3 there is $i = 1, \ldots, s$ such that

$$(\theta_A^{a_i})^*(\theta_A^*(u)) \models A$$

and

$$a_i = (\theta_A^{a_i})^*(\theta_A^*(u))(\rho),$$

hence $a_i \subseteq \theta_A^*(u)(p)$ for every nonroot p because lax models are monotonic and because of Lemma 3.2. By Lemma 3.4 we infer

$$(\theta_A^{a_i})^* \ldots (\theta_A^{a_s})^*(u_p) \models A$$

for all nonroot p. Now we can repeatedly apply the hypothesis of Lemma 3.2(4) to all Kripke models

$$(\theta_A^{a_j})^* \ldots (\theta_A^{a_i})^* \ldots (\theta_A^{a_s})^*(u) \ (1 < j \leq i)$$

relatively to $\theta_A^a = \theta_A^{a_i}$ and $\theta_A^b = \theta_A^{a_{j-1}}$ (in fact, A is true at points different from the root and false at the root both for $(\theta_A^{a_j})^* \ldots (\theta_A^{a_i})^* \ldots (\theta_A^{a_s})^*(u)$ and for $(\theta_A^{a_{j-1}})^*(\theta_A^{a_j})^* \ldots (\theta_A^{a_i})^* \ldots (\theta_A^{a_s})^*(u)$). With the same calculations of [12] we obtain the equation

$$(\theta_A^{a_i})^*(\theta_A)^*(u) = (\theta_A^{a_i})^* \ldots (\theta_A^{a_s})^*(u)$$

because of Lemma 3.2(3). So from $(\theta_A^{a_i})^*(\theta_A)^*(u) \models A$ and the above equation, we get $(\theta_A^{a_i})^* \ldots (\theta_A^{a_s})^*(u) \models A$, hence also $\theta_A^*(u) \models A$ by Lemma 3.2(1), contradiction. □

Corollary 3.1. *The projectivity of a lax formula A is decidable.*

Proof. Let A be a formula over a finite set X of atoms. A is projective if and only if θ_A unifies A, i.e. iff $\vdash_{PLL} \theta_A(A)$, which is decidable because (PLL) is decidable. □

We conclude the section by supplying some information which is specific for (PLL). There are projective formulae in lax logic which are not intuitionistic formulae; an example is

$$jx \to x$$

whose most general unifier is the substitution mapping x to $(jx \to x) \to x$. However, the following proposition says in particular that, up to logical equivalence, the main connective of a projective formula cannot be j:

Proposition 3.1. *If P is projective, then for every formula A we have*

$$\vdash_{PLL} P \to jA \quad \Longleftrightarrow \quad \vdash_{PLL} P \to A \ .$$

Proof. One side is trivial, as $\vdash_{PLL} A \to jA$. For the other side, suppose that we have $\vdash_{PLL} P \to jA$ but that there is a model v such that $v \models P$ and $v \not\models A$. Let ρ be the root of v and let v' be an extension of v with a new root ρ' such that $v' \models P$ and ρ' is in the set of the selected worlds $S_{v'}$ relatively to the lax Kripke frame underlying v': such v' exists because P^* has the extension property. Then we obtain $v' \models P$, $v' \models jA$ and also $v' \models A$ because the root of v' is in $S_{v'}$. By monotonicity of the forcing relation, it follows that $v'_\rho = v \models A$, a contradiction. □

4 Lax simulation

To carry out the proof of Theorem 1.2 we need a notion of simulation in lax logic. The idea is to modify the simulation for intuitionism of [12] so to take account of the j operator. We define simulation in an alternating sequence of steps, where the odd steps are as in [12] and the even steps take into account the j operator.

Definition 4.1. *Let u, v be X-models; for $n \geq 0$ we define the relations $u \sim_n v$ and $u \leq_n v$ as follows, by induction on n.*

- *We say $v \sim_n u$ if $v \leq_n u$ and $u \leq_n v$.*
- *We say $v \leq_0 u$ if every atom of X true in the root of u is true in the root of v.*
- *We say $v \leq_{2n+1} u$ if $v \leq_{2n} u$ and for every $p \in S_v$ there is $q \in S_u$ such that $v_p \leq_{2n} u_q$.*
- *We say $v \leq_{2n+2} u$ if $v \leq_{2n+1} u$ and for every $p \in dom(v)$ there is $q \in dom(u)$ such that $v_p \sim_{2n+1} u_q$.*

Lemma 4.1. *For every n, X there are only finitely many non $\sim_n$-equivalent lax models over X.*

Proof. By induction. □

Lemma 4.2. *For every finite set of atoms X, for every X-model u and for every $n \geq 0$, there exists a lax formula X^n_u (called the n-*characteristic formula of *u) such that for every X-model v we have that*

$$v \models X^n_u \quad \textit{if and only if} \quad v \leq_n u \ .$$

Proof. For $n = 0$, X_u^0 is the conjunction of the atoms of X true in the root of u.

For the even case $2n + 2$ we take (as in [12]):

$$X_u^{2n+2} := X_u^{2n+1} \wedge \bigwedge_{\{u' \mid \forall p \in P.\ u' \not\sim_{2n+1} u_p\}} (X_{u'}^{2n+1} \to \bigvee_{\{w \mid u' \not\leq_{2n+1} w\}} X_w^{2n+1}).$$

For the odd case $2n + 1$, we take:

$$X_u^{2n+1} := X_u^{2n} \wedge j(\bigvee_{r \in S_u} X_{u_r}^{2n})$$

Now we prove by induction on n that for every X-model v we have

$$v \leq_n u \iff v \models X_u^n.$$

The even case is similar to [12]. Let us check the odd case.

Suppose $v \leq_{2n+1} u$; since $v \leq_{2n} u$, by induction we have $v \models X_u^{2n}$. To show that $v \models j(\bigvee_{r \in S_u} X_{u_r}^{2n})$, take $p \in S_v$; from $v \leq_{2n+1} u$, we get that $v_p \leq_{2n} u_q$ for some $q \in S_u$, thus we have $v_p \models X_{u_q}^{2n}$, yielding what required.

Vice versa, suppose that $v \models X_u^{2n+1}$; this implies that $v \models X_u^{2n}$ and also that $v \leq_{2n} u$. Pick now $p \in S_v$; since $v \models j(\bigvee_{r \in S_u} X_{u_r}^{2n})$, we have $v_p \models \bigvee_{r \in S_u} X_{u_r}^{2n}$, which means that there is $q \in S_u$ such that $v_p \models X_{u_q}^{2n}$, that is there is $q \in S_u$ such that $v_p \leq_{2n} u_q$. ☐

5 Invariance

A set K of finite models over X is said to be $\leq_n$*-invariant* iff $u \in K$ and $v \leq_n u$ imply $v \in K$. Likewise we speak of $\sim_n$ invariance.

Lemma 5.1. *There are only finitely many $\leq_n$-invariant sets of X-models for every X and for every $n \geq 0$.*

Proof. This holds because every $\leq_n$-invariant set is defined by a union of sets of of the kind $(X_u^n)^*$ and the latter are finitely many. ☐

We say that a formula A is $\leq_n$-invariant ($\sim_n$-invariant) iff the set of models A^* is $\leq_n$-invariant (resp. $\sim_n$-invariant).

Lemma 5.2. *For every $n \geq 0$, we have that $\leq_n$-invariance implies $\sim_n$-invariance and $\leq_m$ invariance for every $m > n$.*

Proof. Obvious, because $\leq_m \subseteq \leq_n$ and $\sim_n \subseteq \leq_n$. $\square$

Lemma 5.3. *Invariance has the following further properties:*

1. *The atoms $x \in X$ and $\top$, $\bot$ are $\leq_0$-invariant;*
2. *if A, B are $\leq_n$-invariant, then $A \wedge B$, $A \vee B$ are $\leq_n$-invariant;*
3. *if A, B are $\sim_{2n+1}$-invariant, then $A \to B$ is $\leq_{2n+2}$-invariant (in particular, if A, B are $\leq_{2n+1}$-invariant, then $A \to B$ is $\leq_{2n+2}$-invariant);*
4. *if A is $\leq_{2n}$-invariant, then jA is $\leq_{2n+1}$-invariant.*

Proof. The first two statements are easy.

For the third, we reason by contradiction. Suppose A, B are $\sim_{2n+1}$ invariant, $u \models A \to B$ and $v \leq_{2n+2} u$. Suppose for a contradiction that $v \not\models A \to B$. Then there is $z \leq v$ such that $v_z \models A$ and $v_z \not\models B$. By the definition of $\leq_{2n+2}$ there is $z' \leq u$ with $u_{z'} \sim_{2n+1} v_z$. So $u_{z'} \models A$ and $u_{z'} \not\models B$, contrary to the fact that $u \models A \to B$.

For the fourth, we again reason by contradiction. Suppose A is $\leq_{2n}$ invariant. Suppose $u \models jA$ and $v \leq_{2n+1} u$. Suppose by contradiction $v \not\models jA$. Then there is $p \in S_v$ such that $v_p \not\models A$. By the definition of $\leq_{2n+1}$, there is $q \in S_u$ with $v_p \leq_{2n} u_q$. But $u_q \models A$, so $v_p \models A$, a contradiction. $\square$

From the lemma we infer:

Theorem 5.1. *For every formula A there is n such that A is $\leq_n$-invariant. It is enough to take $n = 2 * N(A)$, where $N(A)$ is the maximum number of nested occurrences of the symbols $\to, j$ in A.*

Proof. By induction. $\square$

We remark that the bound for $n = 2 * N(A)$ given in Theorem 5.1 is not optimal. For instance, the $4 = 2 * 2$ bound for jjx is due to the following chain of arguments: one begins by observing that x is $\leq_0$-invariant; then one infers by Lemma 5.3.4 that jx is $\leq_1$-invariant, hence also $\leq_2$-invariant; finally, again by Lemma 5.3.4, one could conclude that jjx is $\leq_3$-invariant and also $\leq_4$-invariant. However, it is easily seen that jjx is $\leq_1$-invariant (being logically equivalent to jx).

For the purpose of this paper, we do not need to compute optimal bounds for $\leq_n$-invariance, the information that some bound exists being sufficient. However, better bounds could improve algorithms for computing projective approximations and complete sets of unifiers. In addition, it is interesting to notice that, applying only $\wedge, \vee, j$, an odd invariance bound does not increase. For this reason, we conclude this section by relating invariance to a syntactically defined 'complexity' measure (we make this further investigation just for the sake of completeness, the result we obtain will not be used in the sequel).

We define the complexity $c(A)$ of a formula A, by induction as follows:

- if A is an atom or $\bot, \top$, $c(A) = 0$;
- if A is $B \vee C$, then $c(A) = max(c(B), c(C))$;
- if A is $B \wedge C$, then $c(A) = max(c(B), c(C))$;
- if A is $B \to C$, then $c(A)$ is the smallest even number strictly greater then $c(B), c(C)$;
- if A is $j(B)$, then $c(A)$ is the smallest odd number greater or equal to $c(B)$.

Lemma 5.4. *If $c(jA) = c(A)$, then there is a formula B whose complexity is strictly less than $c(A)$ and such that jA is logically equivalent to jB.*

Proof. We use the fact that

$$j(C \vee jD) \leftrightarrow j(C \vee D) \tag{5.1}$$

is a valid formula (the argument is similar to the argument used in the proof of Lemma 2.1). We first rewrite jA as $\bigwedge_i j(\bigvee_k A_{ik})$, where the main connective of the formulae A_{ik} is either j or $\to$. Since $c(jA) = c(A)$, the A_{ik} whose main connective is $\to$ have complexities less than $c(A)$: we let A'_{ik} be A_{ik} for such A_{ik}. The A_{ik} that are of the kind jA'_{ik} have complexity less or equal to $c(A)$. According to (5.1), we have that jA is logically equivalent to $\bigwedge_i j(\bigvee_k A'_{ik})$. We can repeat the above procedure until possible, i.e. until all the A'_{ik} have complexity less than $c(A)$. The procedure terminates because we can associate with any formula of the kind $\bigwedge_i j(\bigvee_k A_{ik})$ the multiset of the numbers of the j operators occurring in the subformulae A_{ik}. □

Using the above lemma together with Lemmas 5.2 and 5.3, one can easily prove by induction the following improvement of Theorem 5.1:

Proposition 5.1. *If $c(A) \leq n$, then A is $\leq_n$-invariant. Vice versa, if A is $\leq_n$-invariant, then there exists A' which is logically equivalent to A and is such that $c(A') \leq n$.*

Proof. For the vice versa claim, take A' to be the disjunction of the (finitely many) characteristic formulae X^n_u, varying u among the models of A. □

6 On the extension property

A class K of lax models over the same set of atoms X is *stable* if $u \in K$ implies $u_p \in K$ for every point p of u. Let $\langle K \rangle_n$ be the closure of K under $\leq_n$, i.e. is the smallest $\leq_n$-invariant class extending K.

Lemma 6.1. *(see also [12], Lemma 3, page 870) Let K be a stable class of lax models over a finite set X of atoms. If K has the extension property, so does $\langle K \rangle_{2n+2}$ for every $n \geq 0$.*

Proof. Let u be such that $u_p \in \langle K \rangle_{2n+2}$ for all nonroot p. We look for a variant of u belonging to $\langle K \rangle_{2n+2}$. By definition, every nonroot p is such that there is $v^p \in K$ such that $u_p \leq_{2n+2} v^p$; as K is stable, for every nonroot p there is $w^p \in K$ with

$$u_p \sim_{2n+1} w^p \tag{6.1}$$

(w^p will be a submodel of v^p).

Take the model w_0 given by the disjoint union of the models w^p plus a root where every atom is false, and the root belongs to S_{w_0} if and only if the root of u belongs to S_u. A variant w' of w_0 belongs to K since K has the extension property.

Define now a variant u' of u by putting $u'(\rho) = w'(\rho)$. Note that the root of u' is in $S_{u'}$ if and only if the root of w' is in $S_{w'}$; also $u'(\rho) \subseteq u'(p) = u(p)$ for all nonroot $p \in dom(u') = dom(u)$ because $w'(\rho) \subseteq w^p(\rho) = u(p)$, as $u_p \sim_0 w^p$.

It is enough to show that

$$u' \leq_{2n+2} w'$$

and by construction, it is sufficient to show

$$u' \sim_{2n+1} w'$$

(in particular, this implies $u' \leq_{2n+1} w'$, as required by Definition 4.1, even case). So we prove that

$$u' \sim_k w'$$

for all $k = 0, \ldots, 2n+1$ by induction on k.

For $k = 0$, $u' \sim_0 w'$ holds by definition of u'. For $k > 0$ we must show (this is different from [12]):

(i) if k is even, for all $p \in dom(u')$ there is $q \in dom(w')$ with $u'_p \sim_{k-1} w'_q$;
(ii) if k is even, for all $q \in dom(w')$ there is $p \in dom(u')$ with $u'_p \sim_{k-1} w'_q$;

(iii) if k is odd, for all $p \in S_{u'}$ there is $q \in S_{w'}$ with $u'_p \leq_{k-1} w'_q$;
(iv) if k is odd, for all $q \in S_{w'}$ there is $p \in S_{u'}$ with $u'_p \leq_{k-1} w'_q$.

The case (i) follows by (6.1) and induction (induction is needed when p is the root). The case (ii) follows again by induction when q is the root; if q is not the root, then q belongs to the domain of some $w^{\tilde{p}}$ for some $\tilde{p} \in dom(u')$ different from the root. By (6.1), we have $u'_{\tilde{p}} = u_{\tilde{p}} \sim_{2n+1} w_{\tilde{q}} = w'_{\tilde{q}}$, where $\tilde{q}$ is the root of $w^{\tilde{p}}$. Then $u'_{\tilde{p}} \sim_{2n} w'_{\tilde{q}}$, so there is p in the domain of $u_{\tilde{p}}$ with $w'_q \sim_{2n-1} u'_p$. Since k is even, from $k \leq 2n+1$, we get $k \leq 2n$ and $k-1 \leq 2n-1$, so we are done.

Consider the case (iii). So let $k > 0$ be even. Let us pick $p \in S_{u'}$ and distinguish two subcases.

(iii.1) Suppose that p is the root of u'. If p is the root, since the root of w' is in $S_{w'}$, we have $u_p = u' \sim_{k-1} w'$ by induction and (iii) is satisfied.

(iii.2) Suppose instead that p is different from the root of u'; then, since $u'_p = u_p \sim_{2n+1} w^p$ by (6.1), there is $q \in S_{w'}$ such that $u'_p \leq_{2n} w'_q$ and we are done since $k - 1 \leq 2n + 1 - 1 = 2n$.

In the case (iv), we argue as in (iii.1) if q is the root of $S_{w'}$; if not, then q belongs to the domain of some $w^{\tilde{p}}$ for some $\tilde{p} \in dom(u')$ different from the root. By (6.1), we have $u'_{\tilde{p}} = u_{\tilde{p}} \sim_{2n+1} w_{\tilde{q}} = w'_{\tilde{q}}$, where $\tilde{q}$ is the root of $w^{\tilde{p}}$. This implies that there is $p \in S_{u'}$ such that $u'_p \leq_{2n} w'_q$ and we are done since $k - 1 \leq 2n + 1 - 1 = 2n$.

Summing up, $u' \leq_{2n+2} w'$ and $u' \in \langle K \rangle_{2n+2}$. Hence $\langle K \rangle_{2n+2}$ has the extension property. $\square$

7 End of the proof

As in [12], given two models u, v on the same set of atoms X, we write $u \sim_\infty v$ if $u \sim_n v$ holds for every $n \geq 0$.

Lemma 7.1. *For rooted finite models u, v on X, we have that $u \sim_\infty v$ if and only if for every p there is q such that $u_p \sim_\infty v_q$ and viceversa.*

It follows that if $u(\rho), v(\rho)$ both force the same atoms from X and both are in S_u (resp. S_v) or both not in S_u (resp. S_v), then in order to conclude that $u \sim_\infty v$ holds, it is sufficient to show that for every nonroot p there is q such that $u_p \sim_\infty v_q$ and viceversa.

Proof. The claim is due to the fact that the domains of our models are finite. Thus $u \sim_{2n+2} v$ for every $n \geq 0$ implies that for every $p \in dom(u)$ there is $q \in dom(v)$ such that $u_p \sim_{2n+1} v_q$ (and vice versa); for infinitely many n, this q must be the same, thus there is $q \in dom(v)$ (independent on n) such that $u_p \sim_{2n+1} v_q$ (and vice versa). From $u_p \sim_{2n+1} v_q$, since $\sim_{2n} \supseteq \sim_{2n+1}$, we also obtain $u_p \sim_{2n} v_q$ for all n. In conclusion, for every p there is q such that $u_p \sim_\infty v_q$ and vice versa.

This proves the left-to-right part of the first claim; the right-to-left part of the first claim is proved by showing that (under the assumptiom that for every p there is q such that $u_p \sim_\infty v_q$ and vice versa) we have that $v \leq_n u$ and $u \leq_n v$ hold for every n, by induction on n. Let us show for instance $v \leq_n u$ in the odd $n > 0$ case: pick $p \in S_v$ and consider $\tilde{q} \in S_u$ such that $v_p \sim_\infty u_{\tilde{q}}$. Then we have $v_p \leq_n u_{\tilde{q}}$ and so there is $q \in S_u$ such that $v_p \leq_{n-1} u_q$, as required.

To show the second claim, it is sufficient again to check by induction on $n \geq 0$ that (under the hypotheses of the claim) we have $v \leq_n u$ and $u \leq_n v$. □

Now the proof of Theorem 1.2 is concluded as follows.

Theorem 7.1. *Each unifiable lax formula $A(x)$ admits a finite complete set of unifiers.*

Proof. Let A be $\leq_n$-invariant with unifier $\sigma : F(X) \to F(Y)$. Note that we can suppose n even and positive, see Theorem 5.1.

As in [12], the theorem is proved once we find a projective formula B which is $\leq_n$-invariant such that $B \vdash_{PLL} A$ and σ is a unifier for B, because then σ would be less general than the mgu of B and the latter would also be a unifier for A.

Define the set

$$K = \{v \textit{ model on } X \mid \textit{ there is } u \textit{ on } Y \textit{ s.t. } v \sim_\infty \sigma^*(u)\}$$

K is stable by Lemma 7.1 (recall the equalities $\sigma^*(u)_p = \sigma^*(u_p)$ coming from the definition of σ^*). We show that K has the extension property.

Suppose v is a model on X such that $v_q \in K$ for every nonroot q. Let $u^q \in K$ such that $\sigma^*(u^q) \sim_\infty v_q$. Call u the disjoint sum of the u^q plus a root where all the atoms are false, and the root is in S_u if and only if the root of v is in S_v.

Define a variant v' of v by $v'(\rho) = \sigma^*(u)(\rho)$. Note that the root of v' is in $S_{v'}$ if and only if the root of $\sigma^*(u)$ is in $S_u = S_{\sigma^*(u)}$. By Lemma 7.1, we have $v' \sim_\infty \sigma^*(u)$. So $v' \in K$ and K has the extension property.

Since n is even and positive, we can apply Lemma 6.1 to $\langle K\rangle_n$. So $\langle K\rangle_n$ has the extension property and has the form B^* where B is $\leq_n$-invariant and B is projective by Theorem 3.1. Since B is $\leq_n$-invariant for a fixed n, the search for B ranges over a finite set (recall that our n comes from the $\leq_n$-invariance of A).

Moreover σ unifies B because for every lax model u on Y, we have $\sigma^*(u) \in K \subseteq \langle K \rangle_n = B^*$, which implies $u \in (\sigma(B))^*)$ by (2.8) and (3.1): in other words, $\sigma(B)$ is provable in PLL as $(\sigma(B))^*$ contains all models.

It remains to show that $B \vdash_{PLL} A$. As σ unifies A we have $\sigma^*(u) \models A$ for every lax model u on Y, so $K \subseteq A^*$. But A is $\leq_n$ invariant, so $\langle K \rangle_n \subseteq A^*$ and $\langle K \rangle_n = B^* \subseteq A^*$, that is $B \vdash_{PLL} A$. $\square$

We notice that the finite complete set mentioned in Theorem 7.1 *can be effectively computed*. In fact, given a formula A, we can easily compute an even number n such that A is $\leq_n$-invariant (see Theorem 5.1). Then one can go through the formulae which are $\leq_n$-invariant (their representatives can be syntactically enumerated via Proposition 5.1), select those which are projective and imply A and finally take their most general unifiers (given by Theorem 3.1.1). This algorithm is quite heavy (actually non elementary), however we believe that it can be improved along the lines of [14].

Notice also that for projective formulae P_1, P_2, we have that the mgu θ_{P_1} of P_1 is less general, as a substitution, than the mgu of P_2 iff $\vdash_{PLL} P_1 \to P_2$ (this is easily established, see in any case the argument in [12]). Thus we can introduce the notion of a projective approximation [12, 13].

Given a formula A, a *projective approximation* of A is a finite set of projective formulae Π_A such that:

1. every $P \in \Pi_A$ implies A, i.e. $\vdash_{PLL} P \to A$;
2. if $P_1, P_2 \in \Pi_A$ and $\vdash_{PLL} P_1 \to P_2$, then $P_1 = P_2$;
3. for every projective formula Q such that $\vdash_{PLL} Q \to A$, there is $P \in \Pi_A$ such that $\vdash_{PLL} Q \to P$.

The content of the proof of Theorem 7.1 is that projective approximations exist, are computable for every formula A and can be taken to be formed by formulae having the same even invariance as A;[6] moreover, every unifier of A is less general then the mgu of some $P \in \Pi_A$. In particular, *A is unifiable iff it has a non empty projective approximation.*

Projective approximations can be used to decide admissibility of rules. An inference rule

$$\text{from } A \text{ infer } B \tag{7.1}$$

[6]It is immediate to see that two projective approximations are equal up to provable equivalence in (PLL).

is *admissible* iff every unifier of A is also a unifier of B (the rule is *derivable* iff $\vdash_{PLL} A \to B$).

Theorem 7.2. *The inference rule* (7.1) *is admissible iff we have* $\vdash_{PLL} P \to B$ *for every* $P \in \Pi_A$.

Proof. Immediate from the above results and definitions (notice also that for a projective formula P, we have that $\vdash_{PLL} P \to B$ iff $\vdash_{PLL} \theta_P(B)$, where θ_P is a projective unifier of P). ☐

As an example, notice that the inference rule

$$\text{from } jx \text{ infer } x$$

is admissible (and not derivable), because $\{x\}$ is a projective approximation of jx, according to Proposition 3.1.

8 Conclusions

We have studied unification and admissibility for propositional lax logic (in the parameter-free case) and we showed that this logic essentially behaves like intuitionistic logic. Altough our approach closely follows [12] and consequently gives a non elementary algorithm for admissibility, we are confident that an improved calculus similar to that introduced in [14] should produce a more practical algorithm and that an optimal coNExpPTime complexity bounds should be available by applying the techniques from [19] (see also the recent paper [21] for a deep picture concerning complexity results of unification and admissibility in various modal logics, including the extensions where parameters are included).

Another interesting research direction would consist in designing analytic calculi for admissibility (in the style of [18]) specific for lax logic.

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