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Are all classical superintegrable systems in two-dimensional space linearizable?

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Several examples of classical superintegrable systems in a two-dimensional space are shown to possess hidden symmetries leading to their linearization. They include those determined fifty years ago in the work of Friš *et al.* [Phys. Lett. **13**, 354–356 (1965)], their generalizations, and the more recent Tremblay-Turbiner-Winternitz system [F. Tremblay *et al.*, J. Phys. A: Math. Theor. **42**, 242001 (2009)]. We conjecture that all classical superintegrable systems in the two-dimensional space have hidden symmetries that make them linearizable. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4974264]

I. INTRODUCTION

Fifty years ago in a seminal paper,[1] the authors considered Hamiltonian systems with Hamiltonian given either in Cartesian coordinates, i.e.,

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + V(x_1, x_2), \tag{1}$$

or in polar coordinates, i.e.,

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r, \varphi). \tag{2}$$

Their purpose was to determine all the potentials such that the corresponding Hamiltonian system admits two first integrals that are quadratic in the momenta, in addition to the Hamiltonian. No assumption about the separation of variables in the Hamilton-Jacobi equation was made *a priori*. Four independent potentials were found and it was proven that the corresponding Hamilton-Jacobi equation was separable in at least two different coordinate systems. Two of the four potentials were given in Cartesian coordinates,

$$V_{\mathrm{I}}(x_1, x_2) = \frac{\omega^2}{2}\left(x_1^2 + x_2^2\right) + \frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2}, \tag{3a}$$

$$V_{\mathrm{II}}(x_1, x_2) = \frac{\omega^2}{2}\left(4x_1^2 + x_2^2\right) + \beta_1 x_1 + \frac{\beta_2}{x_2^2}, \tag{3b}$$

while the other two were given in polar coordinates,

$$V_{\mathrm{III}}(r, \varphi) = \frac{\alpha}{r} + \frac{1}{r^2}\left(\frac{\beta_1}{\cos^2\left(\frac{\varphi}{2}\right)} + \frac{\beta_2}{\sin^2\left(\frac{\varphi}{2}\right)}\right), \tag{3c}$$

$$V_{\mathrm{IV}}(r, \varphi) = \frac{\alpha}{r} + \frac{1}{\sqrt{r}}\left(\beta_1 \cos\left(\frac{\varphi}{2}\right) + \beta_2 \sin\left(\frac{\varphi}{2}\right)\right). \tag{3d}$$

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These four cases belong to the class of two-dimensional superintegrable systems, namely, those Hamiltonian systems that admit three first integrals. Actually these four cases are also maximally superintegrable. In fact a Hamiltonian system with n degrees of freedom is called superintegrable if allows $n + 1$ integrals, and maximally superintegrable if the integrals are $2n - 1$. For $n = 2$, the two definitions coincide.

More details and insights into classical and quantum superintegrability can be found in a recent review.[2]

In any undergraduate text of Mechanics, e.g., Ref. 3, it is shown that the Kepler problem in polar coordinates is linearizable, namely, that one can exactly transform its nonlinear equations of motion into the equation of an harmonic linear oscillator. In Ref. 4, it was shown that such a linearization can be achieved by means of the reduction method that was proposed in Ref. 5 in order to find hidden symmetries of the Kepler problem. Moreover, the reduction method was successfully applied to generalizations of the Kepler problem with and without drag in order to find their hidden linearity, although not all of them admit a Lagrangian description.[6]

In 2009, a new two-dimensional superintegrable system was determined,[7] and it has been known since as the Tremblay-Turbiner-Winternitz (TTW) system.

In 2011, a two-dimensional superintegrable system such that the corresponding Hamilton-Jacobi equation does not admit the separation of variables in any coordinates was studied in Ref. 8. In Ref. 9, it was found that its Lagrangian equations can be transformed into a linear third-order equation by applying the reduction method.[5]

In the present paper, we show that the Lagrangian equations corresponding to the potentials $V_{\rm I}, V_{\rm II}, V_{\rm III}$ and their generalizations (not necessary superintegrable) are all linearizable by means of their hidden symmetries. The hidden symmetries that we determined are more general than those considered in Ref. 10, which *really are symmetries of the Hamiltonian, in the sense that they are canonical transformations where both positions and momenta change, and that leave the Hamiltonian function unchanged.*

We also prove that the TTW system is linearizable, and determine the hidden linearity of the Hamiltonian equations with Hamiltonian

$$H_G = \frac{1}{2}(p_1^2 + p_2^2) + \beta_1\sqrt{x_1} + \beta_2\sqrt{x_2}, \tag{4}$$

namely, Case iii (p. 1012) among the superintegrable systems that are separable in Cartesian coordinates and admit a third-order integral of motion as derived by Gravel in Ref. 11. This case was suggested to us by Ian Marquette.

We do not consider the Hamiltonian system with potential $V_{\rm IV}$ because it is a subcase of the linearizable systems determined in Ref. 4, where the following Newtonian equations were considered:

$$\ddot{r} - r\dot{\varphi}^2 + g = 0, \tag{5}$$
$$r\ddot{\varphi} + 2\dot{r}\dot{\varphi} + h = 0, \tag{6}$$

with

$$g = \frac{U''(\varphi) + U(\varphi)}{r^2} + 2\frac{W'(\varphi)}{r^{3/2}}, \qquad h = \frac{W(\varphi)}{r^{3/2}}. \tag{7}$$

The Hamiltonian system with potential $V_{\rm IV}$ corresponds to the substitution Ref. 6,

$$U = \alpha, \qquad W = \frac{1}{2}\left(\beta_1 \sin\left(\frac{\varphi}{2}\right) - \beta_2 \cos\left(\frac{\varphi}{2}\right)\right). \tag{8}$$

See also Ref. 12. Therefore, all the details of the linearization of the Hamiltonian system with potential $V_{\rm IV}$ can be found in Ref. 4.

We would like to remark that the hidden linearity that we find in superintegrable systems is regardless of the separability of the corresponding Hamilton-Jacobi equation: an example has been shown in Ref. 9 as stated above.

Let us then show that the classical two-dimensional Kepler problem in Cartesian coordinates is linearizable by means of hidden symmetries. The linearization of the three-dimensional Kepler problem in Cartesian coordinates was dealt in Ref. 13. The Lagrangian of the two-dimensional

Kepler problem in Cartesian coordinates is

$$L_K = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{\mu}{\sqrt{x_1^2 + x_2^2}}. \tag{9}$$

It admits two Noether symmetries, translation in time t and rotation of the plane x_1, x_2, that yield the conservation of energy and of angular momentum, respectively. The Hamiltonian (namely, the energy) of the two-dimensional Kepler problem in Cartesian coordinates is

$$H_K = \frac{1}{2}(p_1^2 + p_2^2) - \frac{\mu}{\sqrt{x_1^2 + x_2^2}}, \tag{10}$$

and the corresponding Hamiltonian equations are:

$$\begin{aligned} \dot{x}_1 &= p_1, \\ \dot{x}_2 &= p_2, \\ \dot{p}_1 &= -\frac{\mu x_1}{(x_1^2 + x_2^2)^{3/2}}, \\ \dot{p}_2 &= -\frac{\mu x_2}{(x_1^2 + x_2^2)^{3/2}}. \end{aligned} \tag{11}$$

We apply the reduction method[5] by choosing $x_2 = y$ as a new independent variable. Then system (11) becomes a system of three first-order equations, i.e.,

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}y} &= \frac{p_1}{p_2}, \\ \frac{\mathrm{d}p_1}{\mathrm{d}y} &= -\frac{\mu x_1}{p_2(x_1^2 + y^2)^{3/2}}, \\ \frac{\mathrm{d}p_2}{\mathrm{d}y} &= -\frac{\mu y}{p_2(x_1^2 + y^2)^{3/2}}. \end{aligned} \tag{12}$$

Noether's theorem gives the conservation of angular momentum, i.e.,

$$x_2 p_1 - x_1 p_2 = A_0 = \text{const.} \tag{13}$$

We can derive p_2, i.e.,

$$p_2 = \frac{x_2 p_1 - A_0}{x_1}, \tag{14}$$

that replaced into system (12) yields a system of two first-order equations, i.e.,

$$\frac{\mathrm{d}x_1}{\mathrm{d}y} = \frac{x_1 p_1}{y p_1 + A_0}, \tag{15}$$

$$\frac{\mathrm{d}p_1}{\mathrm{d}y} = -\frac{\mu x_1^2}{(x_1^2 + y^2)^{3/2}(y p_1 + A_0)}. \tag{16}$$

If we further get p_1 from (15) and replace it into Equation (16), then we obtain the following second-order equation in the unknown $x_1 = x_1(y)$:

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}y^2} = -\frac{\mu\left(x_1 - y\dfrac{\mathrm{d}x_1}{\mathrm{d}y}\right)^3}{A_0^2(x_1^2 + y^2)^{3/2}}, \tag{17}$$

that admits an eight-dimensional Lie point symmetry algebra, and therefore it is linearizable.[14] The linearizing transformation is obtained by means of Lie's canonical representation of a two-dimensional abelian intransitive subalgebra.[14] One such subalgebra is that generated by the following two operators:

$$x_1\left(y\partial_y + x_1\partial_{x_1}\right), \qquad y\left(y\partial_y + x_1\partial_{x_1}\right), \tag{18}$$

that we have to put in the canonical form $\partial_{\tilde{x}_1}, \tilde{y}\partial_{\tilde{x}_1}$. Consequently, the transformation

$$\tilde{y} = \frac{y}{x_1} = \frac{x_2}{x_1}, \qquad \tilde{x}_1 = -\frac{1}{x_1} + \frac{\mu\sqrt{x_1^2 + y^2}}{A_0^2 x_1} = -\frac{1}{x_1} + \frac{\mu\sqrt{x_1^2 + x_2^2}}{A_0^2 x_1} \tag{19}$$

takes Equation (17) into the free particle equation,

$$\frac{\mathrm{d}^2\tilde{x}_1}{\mathrm{d}\tilde{y}^2} = 0. \tag{20}$$

Thus, we have shown that there are hidden symmetries in the two-dimensional Kepler problem in Cartesian coordinates that takes the Kepler system (11) into the linear equation (20).

All the superintegrable systems that we consider in the present paper are in the real Euclidean space. In a forthcoming paper,[15] we will show that many known superintegrable systems in the space of non-constant curvature are also linearizable, e.g., the three superintegrable systems for the Darboux space of Type I determined in Ref. 16.

We conclude with a conjecture, namely, that all two-dimensional superintegrable systems are linearizable by means of their hidden symmetries.

II. LINEARITY OF THE LAGRANGIAN EQUATIONS WITH POTENTIALS V_{I}, V_{II}, AND V_{III}

The Lagrangian corresponding to the Hamiltonian (1) in Cartesian coordinates is

$$L = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) - V(x_1, x_2), \tag{21}$$

while the Lagrangian corresponding to the Hamiltonian (2) in polar coordinates is

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r, \varphi). \tag{22}$$

Remark 1. We have applied Douglas' method[17] to the Lagrangian equations corresponding to the four potentials. The two potentials V_{I} and V_{II} lead to many different Lagrangians, while in the case of potentials V_{III} and V_{IV}, there exists only one Lagrangian in analogy with Kepler's problem. ■

A. The potential V_I

The Lagrangian equations corresponding to the Lagrangian (21) with $V = V_{\mathrm{I}}$ are

$$\begin{aligned} \ddot{x}_1 &= -\omega^2 x_1 + \frac{2\beta_1}{x_1^3}, \\ \ddot{x}_2 &= -\omega^2 x_2 + \frac{2\beta_2}{x_2^3}. \end{aligned} \tag{23}$$

This Lagrangian admits three Noether symmetries generated by the operators

$$\begin{aligned} \Sigma_1 = \partial_t, \quad \Sigma_2 &= \cos(2\omega t)\partial_t - \omega x_1 \sin(2\omega t)\partial_{x_1} - \omega x_2 \sin(2\omega t)\partial_{x_2}, \\ \Sigma_3 &= \sin(2\omega t)\partial_t + \omega x_1 \cos(2\omega t)\partial_{x_1} + \omega x_2 \cos(2\omega t)\partial_{x_2}, \end{aligned} \tag{24}$$

which correspond to the algebra $sl(2,\mathbb{R})$. The application of Noether's theorem yields three first integrals. From Σ_1 comes the Hamiltonian, i.e.,

$$H_I = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) + \frac{\omega^2}{2}(x_1^2 + x_2^2) + \frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2}, \tag{25}$$

and from Σ_2 and Σ_3 the following two time-dependent integrals,

$$K_2 = \left[\frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2} + \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{\omega^2}{2}(x_1^2 + x_2^2)\right]\cos(2\omega t) + \omega(x_1\dot{x}_1 + x_2\dot{x}_2)\sin(2\omega t) \tag{26}$$

and

$$K_3 = \left[\frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2} + \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{\omega^2}{2}(x_1^2 + x_2^2)\right] \sin(2\omega t) - \omega(x_1\dot{x}_1 + x_2\dot{x}_2)\cos(2\omega t), \tag{27}$$

respectively.

Remark 2. Another time-independent first integral can be obtained by the following ubiquitous combination:

$$H_I^2 - K_2^2 - K_3^2 = \omega^2\left(2\beta_1 + 2\beta_2 + 2\beta_1\frac{x_2^2}{x_1^2} + 2\beta_2\frac{x_1^2}{x_2^2} + (x_2\dot{x}_1 - x_1\dot{x}_2)^2\right). \tag{28}$$

Such a combination can be found in other instances where a couple of time-dependent first integrals is derived from Noether's theorem. ∎

The presence of the algebra $sl(2,\mathbb{R})$ suggests to eliminate the two parameters β_1 and β_2 by raising the order, as it was done in Ref. 18 in the case of the isotonic oscillator. We solve system (23) with respect to β_1 and β_2, i.e.,

$$\begin{aligned}\beta_1 &= \frac{1}{2}(x_1^3\ddot{x}_1 + \omega^2 x_1^4),\\ \beta_2 &= \frac{1}{2}(x_2^3\ddot{x}_2 + \omega^2 x_2^4),\end{aligned} \tag{29}$$

and then we differentiate them with respect to t in order to get the following two third-order equations:

$$\begin{aligned}\dddot{x}_1 &= -\frac{\dot{x}_1}{x_1}(4\omega^2 x_1 + 3\ddot{x}_1),\\ \dddot{x}_2 &= -\frac{\dot{x}_2}{x_2}(4\omega^2 x_2 + 3\ddot{x}_2).\end{aligned} \tag{30}$$

This system admits a thirteen-dimensional Lie point symmetry algebra generated by the following operators:

$$\begin{gathered}\Gamma_1 = \cos(2\omega t)\partial_t - \omega\sin(2\omega t)\left(x_1\partial_{x_1} + x_2\partial_{x_2}\right),\ \Gamma_2 = \sin(2\omega t)\partial_t + \omega\cos(2\omega t)\left(x_1\partial_{x_1} + x_2\partial_{x_2}\right),\\ \Gamma_3 = \partial_t,\ \Gamma_4 = \frac{\cos(2\omega t)}{x_1}\,\partial_{x_1},\ \Gamma_5 = \frac{\sin(2\omega t)}{x_1}\,\partial_{x_1},\ \Gamma_6 = \frac{\cos(2\omega t)}{x_2}\,\partial_{x_2},\ \Gamma_7 = \frac{\sin(2\omega t)}{x_2}\,\partial_{x_2},\\ \Gamma_8 = \frac{x_2^2}{x_1}\,\partial_{x_1},\ \Gamma_9 = x_1\,\partial_{x_1},\ \Gamma_{10} = \frac{1}{x_1}\,\partial_{x_1},\ \Gamma_{11} = \frac{x_1^2}{x_2}\,\partial_{x_2},\ \Gamma_{12} = x_2\partial_{x_2},\ \Gamma_{13} = \frac{1}{x_2}\,\partial_{x_2}.\end{gathered} \tag{31}$$

Therefore system (30) is linearizable. In order to find the linearizing transformation we could use the method in Refs. 19 and 20 based on the classification of the four-dimensional abelian subalgebras.[21] Instead we recall that the following linear system, namely, the derivative with respect to t of the equations of a two-dimensional isotropic oscillator with frequency 2ω:

$$\begin{aligned}\dddot{u}_1 &= -4\omega^2\dot{u}_1,\\ \dddot{u}_2 &= -4\omega^2\dot{u}_2,\end{aligned} \tag{32}$$

admits a thirteen-dimensional Lie point symmetry algebra generated by the following operators:

$$\begin{gathered}\overline{\Gamma}_1 = \cos(2\omega t)\partial_t - 2\omega\sin(2\omega t)\left(u_1\partial_{u_1} + u_2\partial_{u_2}\right),\ \overline{\Gamma}_2 = \sin(2\omega t)\partial_t + 2\omega\cos(2\omega t)\left(u_1\partial_{u_1} + u_2\partial_{u_2}\right),\\ \overline{\Gamma}_3 = \partial_t,\ \overline{\Gamma}_4 = \cos(2\omega t)\,\partial_{u_1},\ \overline{\Gamma}_5 = \sin(2\omega t)\,\partial_{u_1},\ \overline{\Gamma}_6 = \cos(2\omega t)\,\partial_{u_2},\ \overline{\Gamma}_7 = \sin(2\omega t)\,\partial_{u_2},\\ \overline{\Gamma}_8 = u_2\,\partial_{u_1},\ \overline{\Gamma}_9 = u_1\,\partial_{u_1},\ \overline{\Gamma}_{10} = \partial_{u_1},\ \overline{\Gamma}_{11} = u_1\,\partial_{u_2},\ \overline{\Gamma}_{12} = u_2\,\partial_{u_2},\ \overline{\Gamma}_{13} = \partial_{u_2}.\end{gathered} \tag{33}$$

Consequently, if we make the following transformation:

$$u_1 = \frac{x_1^2}{2}, \qquad u_2 = \frac{x_2^2}{2}, \tag{34}$$

system (23) becomes the linear system (32).

More recently the following generalization of the potential V_{I} has been proposed and proved superintegrable:[22–25]

$$V_{\mathrm{I}}^{\mathrm{gen}}(x_1, x_2) = \frac{\omega_1^2}{2} x_1^2 + \frac{\omega_2^2}{2} x_2^2 + \frac{\beta_1}{x_1^2} + \frac{\beta_2}{x_2^2}. \tag{35}$$

Applying the same procedure as described above to the corresponding Lagrangian equations, i.e.,

$$\begin{aligned} \ddot{x}_1 &= -\omega_1^2 x_1 + \frac{2\beta_1}{x_1^3}, \\ \ddot{x}_2 &= -\omega_2^2 x_2 + \frac{2\beta_2}{x_2^3}, \end{aligned} \tag{36}$$

yields the following system of two third-order equations:

$$\begin{aligned} \dddot{x}_1 &= -\frac{\dot{x}_1}{x_1}(4\omega_1^2 x_1 + 3\ddot{x}_1), \\ \dddot{x}_2 &= -\frac{\dot{x}_2}{x_2}(4\omega_2^2 x_2 + 3\ddot{x}_2), \end{aligned} \tag{37}$$

which admits a nine-dimensional Lie point symmetry algebra generated by the operators Γ_3, Γ_4, Γ_5, Γ_6, Γ_7, Γ_8, Γ_{10}, Γ_{11}, and Γ_{13} in (31). Indeed by applying again the transformation (34), we obtain that the system (37) is transformed into the following linear system:

$$\begin{aligned} \dddot{u}_1 &= -4\omega_1^2 \dot{u}_1, \\ \dddot{u}_2 &= -4\omega_2^2 \dot{u}_2, \end{aligned} \tag{38}$$

namely, the derivative with respect to t of the equations of a two-dimensional anisotropic oscillator.

B. The potential V_{II}

The Lagrangian equations corresponding to the Lagrangian (21) with $V = V_{\mathrm{II}}$ are

$$\ddot{x}_1 = -4\omega^2 x_1 - \beta_1, \tag{39a}$$

$$\ddot{x}_2 = -\omega^2 x_2 + \frac{2\beta_2}{x_2^3}. \tag{39b}$$

This Lagrangian admits three Noether symmetries generated by the following operators:

$$\Upsilon_1 = \partial_t, \quad \Upsilon_2 = \sin(2\omega t)\partial_{x_1}, \quad \Upsilon_3 = \cos(2\omega t)\partial_{x_1}, \tag{40}$$

that is the algebra $A_{3,6} \simeq \langle \Upsilon_1/(2\omega), \Upsilon_3, \Upsilon_2 \rangle$ in the classification given in Ref. 21. The application of Noether's theorem yields three first integrals. From Υ_1 comes the Hamiltonian, i.e.,

$$H_{\mathrm{II}} = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) + \frac{\omega^2}{2}(4x_1^2 + x_2^2) + \beta_1 x_1 + \frac{\beta_2}{x_2^2}, \tag{41}$$

and from Υ_2 and Υ_3 the following two time-dependent integrals:

$$Y_2 = \cos(2\omega t)\beta_1 + 4\cos(2\omega t)\omega^2 x_1 - 2\sin(2\omega t)\omega \dot{x}_1 \tag{42}$$

and

$$Y_3 = -2\cos(2\omega t)\omega \dot{x}_1 - \sin(2\omega t)\beta_1 - 4\sin(2\omega t)\omega^2 x_1, \tag{43}$$

respectively.

Remark 3. The following combination of (42) and (43) yields the Hamiltonian for the Equation (39a) only, i.e.,

$$H_1 = \frac{Y_2^2 + Y_3^2}{8\omega^2} = \frac{1}{2}\dot{x}_1^2 + 2\omega^2 x_1^2 + \beta_1 x_1 + \frac{\beta_1^2}{8\omega^2}. \tag{44}$$

The following combination of (44) and the Hamiltonian (41) yields the Hamiltonian for the Equation (39b) only, i.e.,

$$H_2 = H_{\rm II} - \frac{Y_2^2 + Y_3^2}{8\omega^2} = \frac{1}{2}\dot{x}_2^2 + \frac{1}{2}\omega^2 x_2^2 + \frac{\beta_2}{x_2^2} - \frac{\beta_1^2}{8\omega^2}. \tag{45}$$

Of course, the addition/subtraction of the constant $\frac{\beta_1^2}{8\omega^2}$ does not influence either the Hamiltonian H_1 or H_2. ■

We solve system (39a) and (39b) with respect to β_1 and β_2, i.e.,

$$\begin{aligned} \beta_1 &= -\ddot{x}_1 - 4\omega^2 x_1, \\ \beta_2 &= \frac{1}{2}(x_2^3\ddot{x}_2 + \omega^2 x_2^4), \end{aligned} \tag{46}$$

and then we take the derivative with respect to t in order to get the following system of two third-order equations:

$$\begin{aligned} \dddot{x}_1 &= -4\omega^2\dot{x}_1, \\ \dddot{x}_2 &= -\frac{\dot{x}_2}{x_2}(4\omega^2 x_2 + 3\ddot{x}_2). \end{aligned} \tag{47}$$

It should not be a surprise that this system admits a thirteen-dimensional Lie symmetry algebra. Consequently, the transformation

$$u_1 = x_1, \qquad u_2 = \frac{x_2^2}{2} \tag{48}$$

takes system (47) into the linear system (32), namely, that obtained by taking the derivative with respect to t of the equations of a two-dimensional isotropic oscillator with frequency 2ω.

C. The potential $V_{\rm III}$

The Lagrangian equations corresponding to the Lagrangian (22) with $V = V_{\rm III}$ are

$$\begin{aligned} \ddot{r} &= \frac{\alpha}{r^2} + r\dot{\varphi}^2 + \frac{2}{r^3}\left(\frac{\beta_1}{\cos^2\left(\frac{\varphi}{2}\right)} + \frac{\beta_2}{\sin^2\left(\frac{\varphi}{2}\right)}\right), \\ \ddot{\varphi} &= -\frac{2}{r}\dot{r}\dot{\varphi} - \frac{1}{r^4}\left(\frac{\beta_1\sin\left(\frac{\varphi}{2}\right)}{\cos^3\left(\frac{\varphi}{2}\right)} - \frac{\beta_2\cos\left(\frac{\varphi}{2}\right)}{\sin^3\left(\frac{\varphi}{2}\right)}\right). \end{aligned} \tag{49}$$

This Lagrangian admits one Noether symmetry, i.e., translation in t, and the Noether theorem yields the Hamiltonian. We now write the two second-order Lagrangian equations (49) as the following four first-order equations:

$$\begin{aligned} \dot{w}_1 &= w_3, \\ \dot{w}_2 &= w_4, \\ \dot{w}_3 &= \frac{\alpha}{w_1^2} + w_1 w_4^2 + \frac{2}{w_1^3}\left(\frac{\beta_1}{\cos^2\left(\frac{w_2}{2}\right)} + \frac{\beta_2}{\sin^2\left(\frac{w_2}{2}\right)}\right), \\ \dot{w}_4 &= -\frac{2}{w_1}w_3 w_4 - \frac{1}{w_1^4}\left(\frac{\beta_1\sin\left(\frac{w_2}{2}\right)}{\cos^3\left(\frac{w_2}{2}\right)} - \frac{\beta_2\cos\left(\frac{w_2}{2}\right)}{\sin^3\left(\frac{w_2}{2}\right)}\right), \end{aligned} \tag{50}$$

with the identification

$$(w_1, w_2, w_3, w_4) \equiv (r, \varphi, \dot{r}, \dot{\varphi}). \tag{51}$$

We apply the reduction method[5] by choosing $w_2 = y$ as the new independent variable, and consequently the following system of three first-order equations is obtained:

$$\frac{dw_1}{dy} = \frac{w_3}{w_4}, \tag{52}$$

$$\frac{dw_3}{dy} = \frac{\alpha}{w_1^2 w_4} + w_1 w_4 + \frac{2}{w_1^3 w_4}\left(\frac{\beta_1}{\cos^2\left(\frac{y}{2}\right)} + \frac{\beta_2}{\sin^2\left(\frac{y}{2}\right)}\right), \tag{53}$$

$$\frac{dw_4}{dy} = -\frac{2}{w_1}w_3 - \frac{1}{w_1^4 w_4}\left(\frac{\beta_1 \sin\left(\frac{y}{2}\right)}{\cos^3\left(\frac{y}{2}\right)} - \frac{\beta_2 \cos\left(\frac{y}{2}\right)}{\sin^3\left(\frac{y}{2}\right)}\right). \tag{54}$$

We derive w_3 from Equation (52), i.e.,

$$w_3 = w_4 \frac{dw_1}{dy}, \tag{55}$$

and consequently Equation (54) becomes

$$\frac{dw_4}{dy} + \frac{2w_4}{w_1}\frac{dw_1}{dy} + \frac{1}{w_1^4 w_4}\left(\frac{\beta_1 \sin\left(\frac{y}{2}\right)}{\cos^3\left(\frac{y}{2}\right)} - \frac{\beta_2 \cos\left(\frac{y}{2}\right)}{\sin^3\left(\frac{y}{2}\right)}\right) = 0, \tag{56}$$

which can be simplified by means of the following transformation, i.e.,

$$w_4 = \frac{r_4}{w_1^2}, \tag{57}$$

with r_4 a new function of y that then has to satisfy the following equation:

$$\frac{dr_4}{dy} = -\frac{1}{r_4}\left(\frac{\beta_1 \sin\left(\frac{y}{2}\right)}{\cos^3\left(\frac{y}{2}\right)} - \frac{\beta_2 \cos\left(\frac{y}{2}\right)}{\sin^3\left(\frac{y}{2}\right)}\right). \tag{58}$$

Its general solution is easily obtained to be

$$r_4 = \pm\sqrt{a_1 - 2\left(\frac{\beta_1 \sin\left(\frac{y}{2}\right)}{\cos\left(\frac{y}{2}\right)} + \frac{\beta_2 \cos\left(\frac{y}{2}\right)}{\sin\left(\frac{y}{2}\right)}\right)}, \tag{59}$$

with a_1 an arbitrary constant. Finally, Equation (53) becomes the following second-order differential equation:

$$\begin{aligned}\frac{d^2 w_1}{dy^2} &= \frac{2}{w_1}\left(\frac{dw_1}{dy}\right)^2 + \frac{w_1(\alpha w_1 + a_1)\sin^2\left(\frac{y}{2}\right)\cos^2\left(\frac{y}{2}\right)}{a_1 \sin^2\left(\frac{y}{2}\right)\cos^2\left(\frac{y}{2}\right) - 2\beta_1 \sin^2\left(\frac{y}{2}\right) - 2\beta_2\cos^2\left(\frac{y}{2}\right)} \\ &+ \frac{\beta_1 \sin^4\left(\frac{y}{2}\right) - \beta_2\cos^4\left(\frac{y}{2}\right)}{\sin\left(\frac{y}{2}\right)\cos\left(\frac{y}{2}\right)\left(a_1 \sin^2\left(\frac{y}{2}\right)\cos^2\left(\frac{y}{2}\right) - 2\beta_1\sin^2\left(\frac{y}{2}\right) - 2\beta_2\cos^2\left(\frac{y}{2}\right)\right)}\frac{dw_1}{dy}.\end{aligned} \tag{60}$$

This equation admits an eight-dimensional Lie point symmetry algebra, which means that it is linearizable. The linearizing transformation is obtained by means of Lie's canonical representation of a two-dimensional abelian intransitive subalgebra.[14] One such subalgebra is that generated by the following two operators:

$$\begin{aligned}\Xi_1 &= (2\beta_1 - 2\beta_2 - \cos(y)a_1)w_1^2\partial_{w_1}, \\ \Xi_2 &= \sqrt{4\beta_2\cos(y) - 4\beta_1\cos(y) + a_1\cos^2(y) + 4\beta_1 + 4\beta_2 - a_1}\,w_1^2\partial_{w_1},\end{aligned} \tag{61}$$

that we have to put in the canonical form $\partial_{\tilde{w}_1}, \tilde{y}\partial_{\tilde{w}_1}$. Therefore the transformation

$$\tilde{y} = -\frac{\sqrt{4\beta_2\cos(y) - 4\beta_1\cos(y) + a_1\cos^2(y) + 4\beta_1 + 4\beta_2 - a_1}}{-2\beta_1 + 2\beta_2 + \cos(y)a_1}, \tag{62}$$

$$\tilde{w}_1 = \frac{1}{(-2\beta_1 + 2\beta_2 + \cos(y)a_1)w_1} \tag{63}$$

takes Equation (60) into a linear equation of the type

$$\frac{d^2\tilde{w}_1}{d\tilde{y}^2} = \mathfrak{F}(\tilde{y}). \tag{64}$$

Actually (63) suggests the simpler transformation

$$u = \frac{1}{w_1}, \tag{65}$$

that applied to Equation (60) yields the following linear equation:

$$\ddot{u} = \frac{\dot{u}\left(\beta_1 \sin^4\left(\frac{y}{2}\right) - \beta_2 \cos^4\left(\frac{y}{2}\right)\right) - (\alpha - a_1 u)\sin^3\left(\frac{y}{2}\right)\cos^3\left(\frac{y}{2}\right)}{\left(a_1 \sin^2\left(\frac{y}{2}\right)\cos^2\left(\frac{y}{2}\right) - 2\beta_1 \sin^2\left(\frac{y}{2}\right) - 2\beta_2 \cos^2\left(\frac{y}{2}\right)\right)\sin\left(\frac{y}{2}\right)\cos\left(\frac{y}{2}\right)}. \tag{66}$$

There exists a generalization of the potential V_{III}, i.e.,

$$V_{\mathrm{III}}^{\mathrm{gen}}(r,\varphi) = \frac{A}{r} + \frac{f(\varphi)}{r^2}, \tag{67}$$

where f is an arbitrary function of φ. Then the corresponding equations are

$$\ddot{r} = r\dot{\varphi}^2 + \frac{A}{r^2} + \frac{2f(\varphi)}{r^3}, \tag{68}$$

$$\ddot{\varphi} = -2\frac{\dot{r}}{r}\dot{\varphi} - \frac{f'(\varphi)}{r^2}, \tag{69}$$

where prime indicates the derivative of f with respect to φ. Introducing the new variables w_1, w_2, w_3, w_4 as in (51) yields the following Hamilton equations:

$$\dot{w}_1 = w_3, \tag{70}$$

$$\dot{w}_2 = \frac{w_4}{w_1^2}, \tag{71}$$

$$\dot{w}_3 = \frac{Aw_1 + 2f(w_2) + w_4^2}{w_1^3}, \tag{72}$$

$$\dot{w}_4 = -\frac{f'(w_2)}{w_1^2}. \tag{73}$$

We apply the reduction method[5] by choosing $w_2 = y$ as the new independent variable, and consequently the following system of three first-order equations is obtained:

$$\frac{dw_1}{dy} = \frac{w_3 w_1^2}{w_4}, \tag{74}$$

$$\frac{dw_3}{dy} = \frac{Aw_1 + 2f(y) + w_4^2}{w_1 w_4}, \tag{75}$$

$$\frac{dw_4}{dy} = -\frac{f'(y)}{w_4}. \tag{76}$$

Equation (76) can be integrated to give

$$w_4 = \pm\sqrt{J - 2f(y)}, \tag{77}$$

with J an arbitrary constant. Finally, eliminating w_3 by means of (74), i.e.,

$$w_3 = \frac{w_4}{w_1^2}\frac{dw_1}{dy}, \tag{78}$$

yields the following second-order equation for $w_1 = w_1(y)$:

$$\frac{d^2 w_1}{dy^2} = \frac{2(J - 2f(y))\left(\frac{dw_1}{dy}\right)^2 + f'(y)w_1\frac{dw_1}{dy} + Aw_1^3 + w_1^2 J}{w_1(J - 2f(y))}. \tag{79}$$

This equation is linearizable since it admits an eight-dimensional Lie symmetry algebra. As in the case of Equation (60), the transformation (65) yields the linear equation

$$\ddot{u} = \frac{f'(y)\dot{u} - Ju - A}{J - 2f(y)}. \tag{80}$$

A particular case of the potential (67) is a deformed Kepler-Coulomb potential dependent on an indexing parameter k studied in Ref. 26, which corresponds to

$$f(\varphi) = \frac{\alpha k^2}{4\cos^2\left(\frac{k}{2}\varphi\right)} + \frac{\beta k^2}{4\sin^2\left(\frac{k}{2}\varphi\right)}.$$

III. THE TREMBLAY-TURBINER-WINTERNITZ SYSTEM

We now consider the superintegrable Tremblay-Turbiner-Winternitz (TTW) system,[7] namely, an Hamiltonian system with a potential that generalizes V_{I} in (3a), i.e.,

$$V_{\mathrm{TTW}}(r,\varphi) = \omega^2 r^2 + \frac{k^2}{r^2}\left(\frac{\beta_1}{\cos^2(k\varphi)} + \frac{\beta_2}{\sin^2(k\varphi)}\right). \tag{81}$$

The corresponding Lagrangian, i.e.,

$$L_{TTW} = \frac{1}{4}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \omega^2 r^2 - \frac{k^2}{r^2}\left(\frac{\beta_1}{\cos^2(k\varphi)} + \frac{\beta_2}{\sin^2(k\varphi)}\right), \tag{82}$$

yields the following Lagrangian equations:

$$\ddot{r} = -4\omega^2 r + r\dot{\varphi}^2 + \frac{4k^2}{r^3}\left(\frac{\beta_1}{\cos^2(k\varphi)} + \frac{\beta_2}{\sin^2(k\varphi)}\right), \tag{83a}$$

$$\ddot{\varphi} = -\frac{2}{r}\dot{r}\dot{\varphi} - \frac{4k^3}{r^4}\left(\frac{\beta_1\sin(k\varphi)}{\cos^3(k\varphi)} - \frac{\beta_2\cos(k\varphi)}{\sin^3(k\varphi)}\right), \tag{83b}$$

that admit a three-dimensional Lie point symmetry algebra sl$(2,\mathbb{R})$ spanned by

$$\begin{gathered}\Sigma_1 = \partial_t, \quad \Sigma_2 = \cos(4\omega t)\partial_t - 2\omega\sin(4\omega t) r\partial_r, \\ \Sigma_3 = \sin(4\omega t)\partial_t + 2\omega\cos(4\omega t) r\partial_r,\end{gathered} \tag{84}$$

which are also Noether symmetries of the Lagrangian (82). The application of Noether's theorem yields three first integrals, one being the Hamiltonian, i.e.,

$$H_{TTW} = \frac{1}{4}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \omega^2 r^2 + \frac{k^2}{r^2}\left(\frac{\beta_1}{\cos^2(k\varphi)} + \frac{\beta_2}{\sin^2(k\varphi)}\right). \tag{85}$$

The other two first integrals depend on t, i.e.,

$$\begin{aligned}K_{2TTW} = {} & \left[\frac{1}{4}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \omega^2 r^2 + \frac{k^2}{r^2}\left(\frac{\beta_1}{\cos^2(k\varphi)} + \frac{\beta_2}{\sin^2(k\varphi)}\right)\right]\cos(4\omega t) \\ & + \omega r\dot{r}\sin(4\omega t),\end{aligned} \tag{86}$$

$$\begin{aligned}K_{3TTW} = {} & \left[\frac{1}{4}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \omega^2 r^2 + \frac{k^2}{r^2}\left(\frac{\beta_1}{\cos^2(k\varphi)} + \frac{\beta_2}{\sin^2(k\varphi)}\right)\right]\sin(4\omega t) \\ & - \omega r\dot{r}\cos(4\omega t).\end{aligned} \tag{87}$$

Remark 4. Another time-independent first integral can be obtained by the following combination:

$$H_{TTW}^2 - K_{2TTW}^2 - K_{3TTW}^2 = r^4\dot{\varphi}^2 + 4k^2\left(\frac{\beta_1}{\cos^2(k\varphi)} + \frac{\beta_2}{\sin^2(k\varphi)}\right). \tag{88}$$

■

The presence of the algebra $sl(2,\mathbb{R})$ suggests to eliminate the two parameters β_1 and β_2 by raising the order. We solve system (83) with respect to β_1 and β_2, and then we take the derivative with respect to t which yields the following two third-order equations:

$$r\dddot{r} + \dot{r}(16\omega^2 r + 3\ddot{r}) = 0, \tag{89}$$

$$\begin{aligned}
&\cos(k\varphi)\sin(k\varphi)r^2\dddot{\varphi} + 3\cos^2(k\varphi)kr^2\dot{\varphi}\ddot{\varphi} + 6\cos^2(k\varphi)kr\dot{r}\dot{\varphi}^2 \\
&+ 16\cos(k\varphi)\sin(k\varphi)k^2\omega^2r^2\dot{\varphi} - 4\cos(k\varphi)\sin(k\varphi)k^2r^2\dot{\varphi}^3 + 4\cos(k\varphi)\sin(k\varphi)k^2r\ddot{r}\dot{\varphi} \\
&+ 6\cos(k\varphi)\sin(k\varphi)r\dot{r}\ddot{\varphi} + 2\cos(k\varphi)\sin(k\varphi)r\ddot{r}\dot{\varphi} \\
&+ 6\cos(k\varphi)\sin(k\varphi)\dot{r}^2\dot{\varphi} - 3\sin^2(k\varphi)kr^2\dot{\varphi}\ddot{\varphi} - 6\sin^2(k\varphi)kr\dot{r}\dot{\varphi}^2 = 0.
\end{aligned} \tag{90}$$

Equation (89) admits a seven-dimensional Lie symmetry algebra generated by the following operators:

$$\begin{aligned}
&X_1 = \partial_t,\ X_2 = \cos(4\omega t)\partial_t - 2\omega\sin(4\omega t)r\partial_r,\ X_3 = \sin(4\omega t)\partial_t + 2\omega\cos(4\omega t)r\partial_r, \\
&X_4 = \frac{\cos(4\omega t)}{r}\partial_r,\ X_5 = \frac{\sin(4\omega t)}{r}\partial_r,\ X_6 = r\partial_r,\ X_7 = \frac{1}{r}\partial_r,
\end{aligned} \tag{91}$$

and consequently it is linearizable. We find that a two-dimensional non-abelian intransitive subalgebra is that generated by X_6 and X_7, and following Lie's classification,[14] if we transform them into their canonical form, i.e., $\partial_u, u\partial_u$, then we obtain that the new dependent variable is given by

$$u = \frac{r^2}{2},$$

and consequently Equation (89) becomes

$$\dddot{u} = -16\omega^2\dot{u}, \tag{92}$$

namely, the derivative with respect to t of the equation of a linear harmonic oscillator with frequency 4ω. Thus, the general solution of (89) is

$$r = \sqrt{a_1 + a_2\cos(4\omega t) + a_3\sin(4\omega t)}. \tag{93}$$

Equation (90) is also linearizable since it admits a seven-dimensional Lie symmetry algebra generated by

$$\Omega = s_1(t)\partial_t + \frac{-\cos^2(k\varphi)s_2(t) + 2ks_3(t)}{2\cos(k\varphi)\sin(k\varphi)k}\partial_\varphi, \tag{94}$$

with s_1, s_2, s_3 that satisfy the following seventh-order linear system:

$$r^2\dddot{s}_1 + 4\dot{s}_1\ddot{r}k^2r - 4\dot{s}_1\ddot{r}r + 16\dot{s}_1k^2\omega^2r^2 - 8\dot{r}\ddot{r}k^2s_1 + 8\dot{r}\ddot{r}s_1 - 32\dot{r}k^2\omega^2s_1r + 32\dot{r}\omega^2s_1r = 0, \tag{95a}$$

$$r^2\dot{s}_2 - \ddot{s}_1r^2 + 2\dot{s}_1r\dot{r} + 2r\ddot{r}s_1 - 2\dot{r}^2s_1 = 0, \tag{95b}$$

$$r^2\dddot{s}_3 + 6\ddot{s}_3\dot{r}r + 4\dot{s}_3\ddot{r}k^2r + \dot{s}_3\ddot{r}r + 6\dot{s}_3\dot{r}^2 + 16\dot{s}_3k^2\omega^2r^2 = 0, \tag{95c}$$

with r given in (93). Similarly to Equation (89), we find that a two-dimensional non-abelian intransitive subalgebra is generated by the operators

$$-\frac{1}{2k}\cot(k\varphi)\partial_\varphi, \quad \frac{2}{\sin(2k\varphi)}\partial_\varphi, \tag{96}$$

that put into the canonical form yield the new dependent variable

$$v = -\frac{1}{2k}\cos^2(k\varphi),$$

and consequently Equation (90) becomes linear, i.e.,

$$\dddot{v} = -\frac{6\dot{r}}{r}\ddot{v} - \frac{2}{r^2}\left(3\dot{r}^2 + 8k^2\omega^2r^2 + \left(2k^2+1\right)r\ddot{r}\right)\dot{v}. \tag{97}$$

Remark 5. The TTW system admits closed orbits if k is rational, as it has been shown by various methods in Refs. 27–29. We observe that Equation (97) yields solutions of (83) in terms of hypergeometric and trigonometric functions if k is rational, although the linearization that we have achieved remains valid even for k irrational. ■

IV. A GRAVEL'S SUPERINTEGRABLE SYSTEM

The Hamiltonian equations corresponding to Hamiltonian (4) are

$$\begin{aligned}\dot{x}_1 &= p_1,\\ \dot{x}_2 &= p_2,\\ \dot{p}_1 &= -\frac{\beta_1}{2\sqrt{x_1}},\\ \dot{p}_2 &= -\frac{\beta_2}{2\sqrt{x_2}}.\end{aligned} \tag{98}$$

We make the substitution

$$x_1 = r_1^2, \qquad x_2 = r_2^2, \tag{99}$$

with r_1, r_2 two new variables in order to render system (98) more amenable to a computer algebraic software such as REDUCE, i.e.,

$$\begin{aligned}\dot{r}_1 &= \frac{p_1}{2r_1},\\ \dot{r}_2 &= \frac{p_2}{2r_2},\\ \dot{p}_1 &= -\frac{\beta_1}{2r_1},\\ \dot{p}_2 &= -\frac{\beta_2}{2r_2}.\end{aligned} \tag{100}$$

Then we derive r_1 and r_2 from the third and fourth equation of system (100), respectively, i.e.,

$$r_1 = -\frac{\beta_1}{2\dot{p}_1}, \qquad r_2 = -\frac{\beta_2}{2\dot{p}_2}, \tag{101}$$

and consequently we obtained the following two second-order equations:

$$\ddot{p}_1 = -\frac{2}{\beta_1^2} p_1 \dot{p}_1^3, \tag{102}$$

$$\ddot{p}_2 = -\frac{2}{\beta_2^2} p_2 \dot{p}_2^3. \tag{103}$$

Each equation admits an eight-dimensional Lie symmetry algebra and therefore is linearizable.[14] The linearizing transformation is obtained by means of Lie's canonical representation of a two-dimensional abelian intransitive subalgebra.[14] One such subalgebra is that generated by the following two operators:

$$p_j \partial_t, \qquad \partial_t, \qquad (j = 1, 2) \tag{104}$$

that we have to put in the canonical form $\partial_{\tilde{p}_j}, \tilde{t}_j \partial_{\tilde{p}_j}$. Consequently, the transformations

$$\tilde{t}_1 = \frac{1}{p_1}, \qquad \tilde{p}_1 = \frac{t}{p_1} - \frac{p_1^2}{3\beta_1^2} \tag{105}$$

and

$$\tilde{t}_2 = \frac{1}{p_2}, \qquad \tilde{p}_2 = \frac{t}{p_2} - \frac{p_2^2}{3\beta_2^2} \tag{106}$$

take Equations (102) and (103) into the free particle equation

$$\frac{\mathrm{d}^2 \tilde{p}_1}{\mathrm{d}\tilde{t}_1^2} = 0, \tag{107}$$

and

$$\frac{\mathrm{d}^2\tilde{p}_2}{\mathrm{d}\tilde{t}_2^2} = 0, \tag{108}$$

respectively. Therefore, the general solution of system (102) and (103) is

$$\frac{t}{p_1} - \frac{p_1^2}{3\beta_1^2} - \frac{A_1}{p_1} - A_2 = 0, \qquad \frac{t}{p_2} - \frac{p_2^2}{3\beta_2^2} - \frac{A_3}{p_2} - A_4 = 0 \tag{109}$$

with $A_i (i = 1, \dots, 4)$ arbitrary constants.

V. CONCLUSIONS

In this paper, we have considered classical superintegrable systems on two-dimensional real Euclidean space E_2 and shown that they possess hidden symmetries leading to linearization. Actually, we have also demonstrated that some of their generalizations are also linearizable, e.g., Lagrangian equations (68) and (69). Moreover, we have determined hidden linearity regardless of the separation of variables and the degree of the known first integrals.

In Ref. 30, each classical superintegrable system in Ref. 1 was shown to admit an exactly solvable quantum mechanical counterpart, namely, quantum system *characterized by the fact that in its solution space one can indicate explicitly an infinite flag of functional linear spaces, which is preserved by the Hamiltonian.* The authors conjectured *that the property of exact solvability will remain valid for higher dimensional superintegrable systems with integrals given by second order differential operators*. At present, it is not known if classical superintegrable systems with third-order integrals admit an exactly solvable quantum counterpart.

Several classical superintegrable systems have been determined in two-dimensional non-Euclidean spaces, i.e., in two-dimensional space with non-constant curvature. Examples of such systems are the Perlick system,[31] the Taub-NUT system,[32] superintegrable systems for the Darboux space of Type I,[16] and others,[33,34]

In a forthcoming paper,[15] we will show that also superintegrable systems in a two-dimensional non-Euclidean space can be reduced to linear equations by means of their hidden symmetries.

Consequently, we conclude with a conjecture, namely, that all two-dimensional superintegrable systems are linearizable by means of their hidden symmetries.

It remains an open-problem to see if linear equations are hidden in (maximally?) superintegrable systems in $N > 2$ dimensions, regardless of the separability of the corresponding Hamilton-Jacobi equation, and the degree of the known first integrals.

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