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Darboux Integrability of Trapezoidal H^4 and H^6 Families of Lattice Equations II: General Solutions

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Abstract. In this paper we construct the general solutions of two families of quad-equations, namely the trapezoidal H^4 equations and the H^6 equations. These solutions are obtained exploiting the properties of the first integrals in the Darboux sense, which were derived in [Gubbiotti G., Yamilov R.I., *J. Phys. A: Math. Theor.* **50** (2017), 345205, 26 pages]. These first integrals are used to reduce the problem to the solution of some linear or linearizable non-autonomous ordinary difference equations which can be formally solved.

Key words: quad-equations; Darboux integrability; exact solutions; CAC

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1 Introduction

Since its introduction the integrability criterion denoted consistency around the cube (CAC) has been a source of many results in the classification of *quad-equations*. We define a quad-equation to be a relation of the form:

$$Q(x, x_1, x_2, x_{12}) = 0, \tag{1.1}$$

where $Q \in \mathbb{C}[x, x_1, x_2, x_{12}]$ is an irreducible multi-affine polynomial. This equation is defined on the four points displayed in Fig. 1 which form a *square quad graph*.

Roughly speaking the CAC approach consist in adding third direction, defined by the label 3 to a quad-equation (1.1) and extend it to a system of six equations living on the faces of a cube, usually labeled A, $\bar{A}$, B, $\bar{B}$, C and $\bar{C}$, see Fig. 2. We say that the system of six equations given by A, $\bar{A}$, B, $\bar{B}$, C and $\bar{C}$ possess the Consistency Around the Cube if three ways of computing x_{123} from $\bar{A}$, $\bar{B}$, and $\bar{C}$ coincide up to the values of x_{12}, x_{23} and x_{13} obtained from A, B and C respectively.

The CAC criterion has proved to be important in studying the integrability properties of quad-equations since from the CAC it is possible to find Bäcklund transformations [5, 10, 15, 37, 38] and, as a consequence, Lax pairs. It is well known [45] that Lax pairs and Bäcklund

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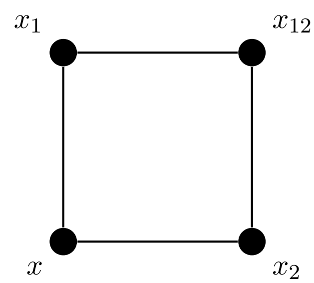

Figure 1. Quad-equation on a square.

transforms are associated with both linearizable and integrable equations. We point out that to be *bona fide* a Lax pair has to give rise to a genuine spectral problem [13], otherwise the Lax pair is called *fake Lax pair* [11, 12, 14, 28, 29]. A fake Lax pair is useless in proving (or disproving) the integrability, since it can be equally found for integrable and non-integrable equations. In the linearizable case Lax pairs must be then fake ones, even though proving this statement is usually a nontrivial task [22]. For a complete, pedagogical explanation of the CAC method we refer to [6, 32, 33].

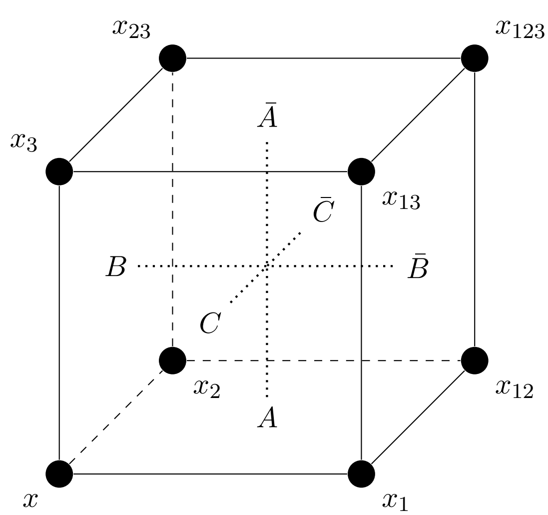

Figure 2. Equations on a cube.

Being algorithmically applicable the CAC criterion proved to be a well suited method to find and classify integrable quad-equations. The first attempt to classify, with some additional assumptions, all the quad-equations possessing CAC was carried out in [1]. The result was the existence of three classes of discrete autonomous equations with this property: the H equations, the Q equations and the A. The A equations can be transformed in particular cases of the Q equations through non-autonomous Möbius transformation, therefore they are usually removed from the general classification. Releasing one of the technical hypothesis of [1], i.e., that face of the cube (Fig. 2) carries the same equation, the same authors in [2] presented some new equations without classification purposes. A complete classification in this extended setting was then accomplished by R. Boll in a series of papers culminating in his Ph.D. Thesis [7, 8, 9]. In these papers the classification of all the consistent sextuples of quad-equations. The only technical assumption used in [7, 8, 9] is the tetrahedron property, i.e., the requirement that x_{123} is independent from x. The obtained equations may fall into three disjoint families depending on their bi-quadratics

$$h_{ij} = \frac{\partial Q}{\partial y_k}\frac{\partial Q}{\partial y_l} - Q\frac{\partial^2 Q}{\partial y_k\,\partial y_l}, \qquad Q = Q(y_1, y_2, y_3, y_4),$$

where we use a special notation for variables of Q, and the pair $\{k, l\}$ is the complement of the pair $\{i, j\}$ in $\{1, 2, 3, 4\}$. A bi-quadratic is called *degenerate* if it contains linear factors of the form $y_i - c$, where c is a constant, otherwise a bi-quadratic is called *non-degenerate*. The three families are classified depending on how many bi-quadratics are degenerate:

- Q-type equations: all the bi-quadratics are non-degenerate,
- H^4-type equations: four bi-quadratics are degenerate,
- H^6-type equations: all of the six bi-quadratics are degenerate.

Let us notice that the Q family is the same as the one introduced in [1]. The H^4 equations are divided into two subclasses: *rhombic* and *trapezoidal*, depending on their discrete symmetries. We remark that all classification results hold locally in the sense that they relate to a single quadrilateral cell or a single cube. The extension on the whole lattice $\mathbb{Z}^2$ is obtained through reflection considering an elementary cell of size 2×2. This implies that the H^4 and H^6 equations as lattice equations are non-autonomous equations with two-periodic coefficients. For more details on the construction of equations on the lattice from the single cell equations, we refer to [7, 8, 9, 44] and to the Appendix in [21].

A detailed study of all the lattice equations derived from the *rhombic* H^4 family, including the construction of their three-leg forms, Lax pairs, Bäcklund transformations and infinite hierarchies of generalized symmetries, has been presented in [44]. So there was plenty of results about the Q and the rhombic H^4 equations. On the contrary, besides the CAC property little was known about the integrability features of the *trapezoidal* H^4 equations and of the H^6 equations. Therefore these equations where thoroughly studied in a series of papers [21, 22, 23, 24, 25] with some unexpected results. First in [21] was presented their explicit non-autonomous form. Indeed it was shown that on the $\mathbb{Z}^2$ lattice with independent variables (n, m) and dependent variable $u_{n,m}$ the trapezoidal H^4 equations had the following expression

$$\begin{aligned}
{}_tH_1^{\varepsilon}\colon\ & (u_{n,m} - u_{n+1,m})(u_{n,m+1} - u_{n+1,m+1}) \\
& - \alpha_2\varepsilon^2\big(F_m^{(+)}u_{n,m+1}u_{n+1,m+1} + F_m^{(-)}u_{n,m}u_{n+1,m}\big) - \alpha_2 = 0, && \text{(1.2a)}\\
{}_tH_2^{\varepsilon}\colon\ & (u_{n,m} - u_{n+1,m})(u_{n,m+1} - u_{n+1,m+1}) + \alpha_2(u_{n,m} + u_{n+1,m} + u_{n,m+1} + u_{n+1,m+1}) \\
& + \frac{\varepsilon\alpha_2}{2}\big(2F_m^{(+)}u_{n,m+1} + 2\alpha_3 + \alpha_2\big)\big(2F_m^{(+)}u_{n+1,m+1} + 2\alpha_3 + \alpha_2\big) \\
& + \frac{\varepsilon\alpha_2}{2}\big(2F_m^{(-)}u_{n,m} + 2\alpha_3 + \alpha_2\big)\big(2F_m^{(-)}u_{n+1,m} + 2\alpha_3 + \alpha_2\big) \\
& + (\alpha_3 + \alpha_2)^2 - \alpha_3^2 - 2\varepsilon\alpha_2\alpha_3(\alpha_3 + \alpha_2) = 0, && \text{(1.2b)}\\
{}_tH_3^{\varepsilon}\colon\ & \alpha_2(u_{n,m}u_{n+1,m+1} + u_{n+1,m}u_{n,m+1}) \\
& - (u_{n,m}u_{n,m+1} + u_{n+1,m}u_{n+1,m+1}) - \alpha_3\big(\alpha_2^2 - 1\big)\delta^2 \\
& - \frac{\varepsilon^2(\alpha_2^2 - 1)}{\alpha_3\alpha_2}\big(F_m^{(+)}u_{n,m+1}u_{n+1,m+1} + F_m^{(-)}u_{n,m}u_{n+1,m}\big) = 0, && \text{(1.2c)}
\end{aligned}$$

and the H^6 equations had the following expression

$$\begin{aligned}
{}_1D_2\colon\ & \big(F_{n+m}^{(-)} - \delta_1 F_n^{(+)}F_m^{(-)} + \delta_2 F_n^{(+)}F_m^{(+)}\big)u_{n,m} \\
& + \big(F_{n+m}^{(+)} - \delta_1 F_n^{(-)}F_m^{(-)} + \delta_2 F_n^{(-)}F_m^{(+)}\big)u_{n+1,m} \\
& + \big(F_{n+m}^{(+)} - \delta_1 F_n^{(+)}F_m^{(+)} + \delta_2 F_n^{(+)}F_m^{(-)}\big)u_{n,m+1}
\end{aligned}$$

$$
\begin{aligned}
&+ \left(F_{n+m}^{(-)} - \delta_1 F_n^{(-)} F_m^{(+)} + \delta_2 F_n^{(-)} F_m^{(-)}\right) u_{n+1,m+1} \\
&+ \delta_1 \left(F_m^{(-)} u_{n,m} u_{n+1,m} + F_m^{(+)} u_{n,m+1} u_{n+1,m+1}\right) \\
&+ F_{n+m}^{(+)} u_{n,m} u_{n+1,m+1} + F_{n+m}^{(-)} u_{n+1,m} u_{n,m+1} = 0,
\end{aligned} \tag{1.3a}
$$

$$
\begin{aligned}
{}_2D_2\colon\ &\left(F_m^{(-)} - \delta_1 F_n^{(+)} F_m^{(-)} + \delta_2 F_n^{(+)} F_m^{(+)} - \delta_1 \lambda F_n^{(-)} F_m^{(+)}\right) u_{n,m} \\
&+ \left(F_m^{(-)} - \delta_1 F_n^{(-)} F_m^{(-)} + \delta_2 F_n^{(-)} F_m^{(+)} - \delta_1 \lambda F_n^{(+)} F_m^{(+)}\right) u_{n+1,m} \\
&+ \left(F_m^{(+)} - \delta_1 F_n^{(+)} F_m^{(+)} + \delta_2 F_n^{(+)} F_m^{(-)} - \delta_1 \lambda F_n^{(-)} F_m^{(-)}\right) u_{n,m+1} \\
&+ \left(F_m^{(+)} - \delta_1 F_n^{(-)} F_m^{(+)} + \delta_2 F_n^{(-)} F_m^{(-)} - \delta_1 \lambda F_n^{(+)} F_m^{(-)}\right) u_{n+1,m+1} \\
&+ \delta_1 \left(F_{n+m}^{(-)} u_{n,m} u_{n+1,m+1} + F_{n+m}^{(+)} u_{n+1,m} u_{n,m+1}\right) \\
&+ F_m^{(+)} u_{n,m} u_{n+1,m} + F_m^{(-)} u_{n,m+1} u_{n+1,m+1} - \delta_1 \delta_2 \lambda = 0,
\end{aligned} \tag{1.3b}
$$

$$
\begin{aligned}
{}_3D_2\colon\ &\left(F_m^{(-)} - \delta_1 F_n^{(-)} F_m^{(-)} + \delta_2 F_n^{(+)} F_m^{(+)} - \delta_1 \lambda F_n^{(-)} F_m^{(+)}\right) u_{n,m} \\
&+ \left(F_m^{(-)} - \delta_1 F_n^{(+)} F_m^{(-)} + \delta_2 F_n^{(-)} F_m^{(+)} - \delta_1 \lambda F_n^{(+)} F_m^{(+)}\right) u_{n+1,m} \\
&+ \left(F_m^{(+)} - \delta_1 F_n^{(-)} F_m^{(+)} + \delta_2 F_n^{(+)} F_m^{(-)} - \delta_1 \lambda F_n^{(-)} F_m^{(-)}\right) u_{n,m+1} \\
&+ \left(F_m^{(+)} - \delta_1 F_n^{(+)} F_m^{(+)} + \delta_2 F_n^{(-)} F_m^{(-)} - \delta_1 \lambda F_n^{(+)} F_m^{(-)}\right) u_{n+1,m+1} \\
&+ \delta_1 \left(F_n^{(-)} u_{n,m} u_{n,m+1} + F_n^{(+)} u_{n+1,m} u_{n+1,m+1}\right) \\
&+ F_m^{(-)} u_{n,m+1} u_{n+1,m+1} + F_m^{(+)} u_{n,m} u_{n+1,m} - \delta_1 \delta_2 \lambda = 0,
\end{aligned} \tag{1.3c}
$$

$$
\begin{aligned}
D_3\colon\ & F_n^{(+)} F_m^{(+)} u_{n,m} + F_n^{(-)} F_m^{(+)} u_{n+1,m} + F_n^{(+)} F_m^{(-)} u_{n,m+1} \\
&+ F_n^{(-)} F_m^{(-)} u_{n+1,m+1} + F_m^{(-)} u_{n,m} u_{n+1,m} + F_n^{(-)} u_{n,m} u_{n,m+1} + F_{n+m}^{(-)} u_{n,m} u_{n+1,m+1} \\
&+ F_{n+m}^{(+)} u_{n+1,m} u_{n,m+1} + F_n^{(+)} u_{n+1,m} u_{n+1,m+1} + F_m^{(+)} u_{n,m+1} u_{n+1,m+1} = 0,
\end{aligned} \tag{1.3d}
$$

$$
\begin{aligned}
{}_1D_4\colon\ & \delta_1 \left(F_n^{(-)} u_{n,m} u_{n,m+1} + F_n^{(+)} u_{n+1,m} u_{n+1,m+1}\right) \\
&+ \delta_2 \left(F_m^{(-)} u_{n,m} u_{n+1,m} + F_m^{(+)} u_{n,m+1} u_{n+1,m+1}\right) \\
&+ u_{n,m} u_{n+1,m+1} + u_{n+1,m} u_{n,m+1} + \delta_3 = 0,
\end{aligned} \tag{1.3e}
$$

$$
\begin{aligned}
{}_2D_4\colon\ & \delta_1 \left(F_n^{(-)} u_{n,m} u_{n,m+1} + F_n^{(+)} u_{n+1,m} u_{n+1,m+1}\right) \\
&+ \delta_2 \left(F_{n+m}^{(-)} u_{n,m} u_{n+1,m+1} + F_{n+m}^{(+)} u_{n+1,m} u_{n,m+1}\right) \\
&+ u_{n,m} u_{n+1,m} + u_{n,m+1} u_{n+1,m+1} + \delta_3 = 0,
\end{aligned} \tag{1.3f}
$$

where the coefficients $F_k^{(\pm)}$ are given by

$$
F_k^{(\pm)} = \frac{1 \pm (-1)^k}{2}. \tag{1.4}
$$

Then in [21] the algebraic entropy [4, 34, 42, 43] of the trapezoidal $H^4$and the H^6 equations was computed. The result of this computation showed that the *rate of growth* of all the trapezoidal H^4 (1.2) and of all H^6 equations (1.3) is *linear*. This fact according to the *algebraic entropy conjecture* [16, 34] implies linearizability. To support this result two explicit examples of linearization were given.

Remark 1.1. In [35] it was shown that sometimes it is possible to construct different consistent embedding in the $\mathbb{Z}^2$ and in $\mathbb{Z}^3$ lattices. However, in the same paper it was shown that these different embedding need not to be integrable. In this paper we will consider equations (1.2) and (1.3) which are given by the embedding procedure of [2, 7, 8, 9]. As we underlined above this procedure gives equations which, in the sense of the algebraic entropy, are only integrable or linearizable [21, 41]. Clearly, it may exist a different embedding in the $\mathbb{Z}^2$ for which the results presented in this paper do not hold. For an example where two different embedding give both rise to linearizable equations, but with different properties, see [20].

In [22] the ${}_tH_1^\varepsilon$ equation (1.2a) was studied and it was found that it possessed three-point generalized symmetries depending on arbitrary functions. This property was then linked in [23] to the fact that the ${}_tH_1^\varepsilon$ is *Darboux integrable* [3]. We say that a quad-equation on the $\mathbb{Z}^2$ lattice, possibly non-autonomous:

$$Q_{n,m}\left(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}\right) = 0, \tag{1.5}$$

is *Darboux integrable* if there exist two independent *first integrals*, one containing only shifts in the first direction and the other containing only shifts in the second direction. This means that there exist two functions

$$W_1 = W_{1,n,m}(u_{n+l_1,m}, u_{n+l_1+1,m}, \dots, u_{n+k_1,m}),$$
$$W_2 = W_{2,n,m}(u_{n,m+l_2}, u_{n,m+l_2+1}, \dots, u_{n,m+k_2}),$$

where $l_1 < k_1$ and $l_2 < k_2$ are integers, such that the relations

$$(T_n - \mathrm{Id})W_2 = 0, \tag{1.6a}$$
$$(T_m - \mathrm{Id})W_1 = 0 \tag{1.6b}$$

hold true identically on the solutions of (1.5). By T_n, T_m we denote the shift operators in the first and second directions, i.e., $T_n h_{n,m} = h_{n+1,m}$, $T_m h_{n,m} = h_{n,m+1}$, and by Id we denote the identity operator $\mathrm{Id}\, h_{n,m} = h_{n,m}$. The number $k_i - l_i$, where $i = 1, 2$, is called the *order* of the first integral W_i.

In addition to this result concerning the ${}_tH_1^\varepsilon$ equation in [23] it was proved that other quad-equations consistent around the cube, which were known to be linearizable [30, 31], were in fact Darboux integrable. These facts provide some evidence of an intimate connection between linearizable equations possessing CAC and Darboux integrability. Following these ideas in [26] it was shown that *all* the trapezoidal H^4 equations and *all* the H^6 equations are Darboux integrable. This result was proved by explicitly constructing the first integrals with a new algorithm based on those proposed in [17, 18, 27]. This new algorithm relies on the fact that in the case of non-autonomous quad-equations (1.5) with two-periodic coefficients we can, in general, represent the first integrals in the form

$$W_i = F_n^{(+)} F_m^{(+)} W_i^{(+,+)} + F_n^{(-)} F_m^{(+)} W_i^{(-,+)} + F_n^{(+)} F_m^{(-)} W_i^{(+,-)} + F_n^{(-)} F_m^{(-)} W_i^{(-,-)},$$

where $F_k^{(\pm)}$ are given by (1.4) and the $W_i^{(\pm,\pm)}$ are functions. The existence of the first integrals provides a rigorous proof of the linearizability of the trapezoidal H^4 equation (1.2) and of the H^6 equations (1.3). Indeed equation (1.6) implies that the following two transformations

$$u_{n,m} \to \tilde{u}_{n,m} = W_{1,n,m}, \tag{1.7a}$$
$$u_{n,m} \to \hat{u}_{n,m} = W_{2,n,m}, \tag{1.7b}$$

bring the quad-equation (1.5) into two *trivial linear equations*

$$\tilde{u}_{n,m+1} - \tilde{u}_{n,m} = 0, \tag{1.8a}$$
$$\hat{u}_{n+1,m} - \hat{u}_{n,m} = 0. \tag{1.8b}$$

Therefore any Darboux integrable equation is *linearizable in two different ways*, i.e., using transformation (1.7a) bringing to (1.8a) or using the transformation (1.7b) bringing to (1.8b).

In the final section of [26] it was shown, in the case of the ${}_tH_1^\varepsilon$ equation, how it is possible to find the general solution using the first integrals, applying a modification of the procedure presented in [17]. In particular we showed how it is possible to obtain a general solution using first integrals of order greater than one. We note that equations with first integrals of first order one are *trivial*, since possessing a first integral of order one means that *the equation itself is a first integral*.

In this paper we show that from the knowledge of the first integrals and from the properties of the equations it is possible to construct, maybe after some complicate algebra, the general solutions of all the remaining trapezoidal H^4 equations (1.2) and of the H^6 equations (1.3). By general solution we mean a representation of the solution of any of the equations in (1.2) and (1.3) in terms of the right number of arbitrary functions of one lattice variable n or m. Since the trapezoidal H^4 equations (1.2) and the H^6 equations (1.3) are quad-equations, i.e., the discrete analogue of second-order hyperbolic partial differential equations, the general solution must contain an arbitrary function in the n direction and another one in the m direction, i.e., a general solution is an expression of the form

$$u_{n,m} = F_{n,m}(a_n, \alpha_m), \tag{1.9}$$

where a_n and α_m are arbitrary functions of their variable. Initial conditions are then imposed through substitution in equation (1.9). Nonlinear equations usually possesses also other kinds of solutions, namely the singular solutions which satisfy only specific set of initial values. In this work we outlined when the existence of singular solutions is possible. Moreover we remark that general solutions, in the range of validity of their parameters, enclose also periodic solutions. Periodic initial values will reflect into periodic solution which will arise by fixing properly the arbitrary functions. Indeed let us consider as an example the $(N, -M)$ reduction of a quad-equation (1.5), with $N, M \in \mathbb{N}^*$ coprime [39, 40]. This implies to make the following requirement

$$u_{n+N,m-M} = u_{n,m}. \tag{1.10}$$

If we possess the general solution of the quad-equation in the form (1.9) then the periodicity requirement (1.10) is equivalent to

$$F_{n+N,m-M}(a_{n+N}, \alpha_{m-M}) = F_{n,m}(a_n, \alpha_m). \tag{1.11}$$

The existence of the associated periodic solution is subject to the ability to invert formula (1.11). When the integers N and M are not coprime a similar reasoning can be done: taking $K = \gcd(N, M)$ we have just to decompose the reduction condition into K superimposed staircases and convert the scalar condition (1.10) to a vector condition for K fields. The associated reduction will be possible if the associated system possesses a solution.

To obtain the desired solution we will need only the W_1 integrals derived in [26] and the fact that the relation (1.6b) implies $W_1 = \xi_n$ with ξ_n an arbitrary function of n. The equation $W_1 = \xi_n$ can be interpreted as an ordinary difference equation in the n direction depending parametrically on m. Then from every W_1 integral we can derive two different ordinary difference equations, one corresponding to m even and one corresponding to m odd. In both the resulting equations we can get rid of the two-periodic terms by considering the cases n even and n odd and defining

$$u_{2k,2l} = v_{k,l}, \qquad u_{2k+1,2l} = w_{k,l}, \tag{1.12a}$$

$$u_{2k,2l+1} = y_{k,l}, \qquad u_{2k+1,2l+1} = z_{k,l}. \tag{1.12b}$$

This transformation brings both equations to a *system* of *coupled difference equations*. This reduction to a system is the key ingredient in the construction of the general solutions for the trapezoidal H^4 equations (1.2) and for the H^6 equations (1.3).

We note that the transformation (1.12) can be applied to the trapezoidal H^4 equations[1] and H^6equations themselves. This casts these *non-autonomous equations with two-periodic coefficients* into *autonomous systems of four equations.* We recall that in this way some examples of direct linearization (i.e., without the knowledge of the first integrals) were produced in [21]. Finally we note that if we apply the even/odd splitting of the lattice variables given by equation (1.12) to describe a general solution we will need two arbitrary functions in both directions, i.e., we will need a total of four arbitrary functions.

In practice to construct these general solutions, we need to solve Riccati equations and non-autonomous linear equations which, in general, cannot be solved in closed form. Using the fact that these equations contain arbitrary functions we introduce new arbitrary functions so that we can solve these equations. This is usually done reducing to *total difference*, i.e., to ordinary difference equations which can be *trivially* solved. Let us assume we are given the difference equation

$$u_{n+1,m} - u_{n,m} = f_n, \tag{1.13}$$

depending parametrically on another discrete index m. Then if we can express the function f_n as a *discrete derivative*

$$f_n = g_{n+1} - g_n,$$

then the solution of equation (1.13) is simply

$$u_{n,m} = g_n + \gamma_m,$$

where γ_m is an arbitrary function of the discrete variable m. This is the simplest possible example of reduction to total difference. The general solutions will then be expressed in terms of these new arbitrary functions obtained reducing to total differences and in terms of a finite number of *discrete integrations*, i.e., the solutions of the simple ordinary difference equation

$$u_{n+1} - u_n = f_n, \tag{1.14}$$

where u_n is the unknown and f_n is an assigned function. We note that the discrete integration (1.14) is the discrete analogue of the differential equation $u'(x) = f(x)$.

To give a very simple example of the method of solution we consider how it applies to the prototypical Darboux integrable equation: *the discrete wave equation*

$$u_{n+1,m+1} + u_{n,m} = u_{n+1,m} + u_{n,m+1}. \tag{1.15}$$

It is easy to check that the discrete wave equation (1.15) is Darboux integrable with two first-order first integrals

$$W_1 = u_{n+1,m} - u_{n,m}, \tag{1.16a}$$
$$W_2 = u_{n,m+1} - u_{n,m}. \tag{1.16b}$$

From the first integrals (1.16) it is possible to construct the well known discrete d'Alembert solution as follows. From the W_1 first integral we can write $W_1 = \xi_n$ with ξ_n arbitrary function of its argument. Then we have

$$u_{n+1,m} - u_{n,m} = \xi_n. \tag{1.17}$$

[1]In fact, in the case of the trapezoidal H^4 equations (1.2), we use a simpler transformation instead of (1.12), see Section 2.3.

This means that choosing the arbitrary function as $\xi_n = a_{n+1} - a_n$, with a_n arbitrary function of its argument, we transform (1.17) into the total difference

$$u_{n+1,m} + a_{n+1} = u_{n,m} + a_n,$$

which readily implies

$$u_{n,m} = a_n + \alpha_m,$$

where α_m is an arbitrary function of its argument. This is of course the discrete analog of the d'Alembert solution of the wave equation and it is the simplest example of solution through the first integrals of a Darboux integrable equation.

Now to summarize, in this paper we prove the following result:

Theorem 1.2. *The trapezoidal H^4 equations* (1.2) *and H^6 equations* (1.3) *are exactly solvable and we can represent the solution in terms of a finite number of discrete integration* (1.14)*.*

The rest of the paper is devoted to the proof of Theorem 1.2. In Section 2 we present the general solutions of all the H^4 and H^6 equations, except the ${}_tH_1^\varepsilon$ equation (1.2a) which was treated in [26]. In particular in Section 2.1 we treat the ${}_1D_2$, ${}_2D_2$ and ${}_3D_2$ equations. In Section 2.2 we treat the D_3, ${}_1D_4$ and ${}_2D_4$ equations. In Section 2.3 we treat the ${}_tH_2^\varepsilon$ and the ${}_tH_3^\varepsilon$ equations. The partition in subsection is dictated by the procedure used to obtain the general solution, as we will explain below. Due to the technical nature of the procedures we will present only one example per type. The interested reader will find the remaining procedures of solution in Appendix A. In Section 3 we give some conclusions.

Remark 1.3. We remark that the H equations of the ABS classification [1] and their rhombic deformations [2, 7, 44] should not be Darboux integrable. This can be confirmed directly excluding the existence of integrals up to a certain order as it was done in [17] for some other equations. Moreover it was proved rigorously in [41] using the gcd-factorization method that all the equations of the ABS list [1] possess quadratic growth of the degrees. At heuristic level a similar result was presented in [21] for the rhombic H^4 equations. According to the Algebraic entropy conjecture these result means that the ABS equations and the rhombic H^4 equations are integrable, but not linearizable. Since we have recalled the fact that Darboux integrability for lattice equations implies linearizability we expect that these equations will not possess first integrals of any order. So the results obtained in [26] and in this paper about the trapezoidal H^4 and H^6 equations do not imply anything for H equations and their rhombic deformations.

2 General solutions of the H^4 and H^6 equations

In this section we present the general solutions of the H^4 equations (1.2) and of the H^6 equations (1.3). We choose to divide this section in three subsections since we have three main different kinds of procedures leading to three different representations of the solution.

First in Section 2.1 we present the general solutions of the ${}_1D_2$, ${}_2D_2$ and ${}_3D_2$ equations (1.3a)–(1.3c). In this case the construction of the general solution is carried out from the sole knowledge of the first integral and the equation acts only as a compatibility condition for the arbitrary functions obtained by solving the equations defined by the first integral. The solution therein obtained is completely explicit and no discrete integration is required.

The in Section 2.2 we present the general solution of the D_3, ${}_1D_4$ and ${}_2D_4$ equations (1.3d)–(1.3f). In this case the construction of the general solution is carried out through a series of manipulations in the equation itself and from the knowledge of the first integral. The key point will be that the equations defined by the first integrals can be reduced to a single linear equation.

The solution is no longer completely explicit since it is obtained up to two discrete integrations, one in every direction.

Finally in Section 2.3 we present the general solutions of the ${}_tH_2^\varepsilon$ equation (1.2b) and of the ${}_tH_3^\varepsilon$ equation (1.2c). In this case the construction of the general solution is carried out reducing the equation to a partial difference equation defined on six points and then using the equations defined by the first integrals. The equations defined by the first integrals are reduced to discrete Riccati equations. The solution is then given in terms of four discrete integrations.

Summing up these results and the fact that the general solution of the ${}_tH_1^\varepsilon$ equation (1.2a) was presented in [26] we prove Theorem 1.2.

2.1 The ${}_iD_2$ equations $i = 1, 2, 3$

We have that the following propositions hold true:

Proposition 2.1. *The ${}_1D_2$ equation (1.3a) is exactly solvable. If $\delta_1 \neq 0$ and $\delta \neq 0$, where δ is defined by*

$$\delta = 1 - \delta_1(1 + \delta_2) \tag{2.1}$$

the general solution is given by

$$v_{k,l} = \alpha_l - \frac{\delta_1}{\delta}\frac{b_k b_{k-1}}{b_k - b_{k-1}}(c_k - c_{k-1}), \tag{2.2a}$$

$$w_{k,l} = b_k(\beta_l + c_k) + \frac{\delta}{\delta_1}\alpha_l, \tag{2.2b}$$

$$z_{k,l} = 1 - \frac{1}{\delta_1} - b_k\frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l}, \tag{2.2c}$$

$$\begin{aligned} y_{k,l} = {} & \frac{1}{\delta_1}\frac{\beta_l\alpha_{l+1} - \beta_{l+1}\alpha_l}{\beta_{l+1} - \beta_l} + \frac{1}{\delta}\frac{b_k b_{k-1}(c_k - c_{k-1})}{b_k - b_{k-1}} \\ & + \left[\frac{(c_k - c_{k-1})b_{k-1}}{(b_k - b_{k-1})\delta_1} + \frac{c_k}{\delta_1}\right]\frac{\alpha_{l+1} - \alpha_l}{\beta_{l+1} - \beta_l}, \end{aligned} \tag{2.2d}$$

where b_k, c_k, α_l and β_l are arbitrary functions of their arguments. If $\delta = 0$ its general solution is given by

$$v_{k,l} = a_k + \beta_l, \tag{2.3a}$$

$$w_{k,l} = b_k\alpha_l, \tag{2.3b}$$

$$z_{k,l} = -b_k\frac{\alpha_{l+1} - \alpha_l}{\beta_{l+1} - \beta_l} - \delta_2, \tag{2.3c}$$

$$y_{k,l} = -(1 + \delta_2)\left(\frac{\beta_l\alpha_{l+1} - \beta_{l+1}\alpha_l}{\alpha_{l+1} - \alpha_l} + a_k\right), \tag{2.3d}$$

where a_k, b_k, α_l and β_l are arbitrary functions of their arguments. If $\delta_1 = 0$ then its general solution is given by

$$v_{k,l} = \frac{c_k - c_{k-1}}{b_k - b_{k-1}} + \alpha_l, \tag{2.4a}$$

$$w_{k,l} = c_k + \alpha_l b_k + \beta_l, \tag{2.4b}$$

$$z_{k,l} = -\delta_2 - \frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l} - b_k, \tag{2.4c}$$

$$y_{k,l} = \frac{\beta_{l+1}\alpha_l - \alpha_{l+1}\beta_l}{\alpha_{l+1} - \alpha_l} + \frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l}\frac{c_k - c_{k-1}}{b_k - b_{k-1}} + b_k\frac{c_k - c_{k-1}}{b_k - b_{k-1}} - c_k, \tag{2.4d}$$

where b_k, c_k, α_l and β_l are arbitrary functions of their arguments.

Proof. From [26] we know that the ${}_1D_2$ equation (1.3a) is Darboux integrable, and that the form of the first integral depends on the value of the parameter δ_1. We will begin with the general case when $\delta_1 \neq 0$ and $\delta \neq 0$ and then consider the particular cases.

Case $\boldsymbol{\delta \neq 0}$ and $\boldsymbol{\delta_1 \neq 0}$. In this case the W_1 first integral of the ${}_1D_2$ equation (1.3a) is given by [26]

$$
\begin{aligned}
W_1 = {} & F_n^{(+)} F_m^{(+)} \alpha \frac{[(1+\delta_2)u_{n,m} + u_{n+1,m}]\delta_1 - u_{n,m}}{[(1+\delta_2)u_{n,m} + u_{n-1,m}]\delta_1 - u_{n,m}} \\
& + F_n^{(+)} F_m^{(-)} \alpha \frac{1 + (u_{n+1,m} - 1)\delta_1}{1 + (u_{n-1,m} - 1)\delta_1} + F_n^{(-)} F_m^{(+)} \beta (u_{n+1,m} - u_{n-1,m}) \\
& - F_n^{(-)} F_m^{(-)} \beta \frac{(u_{n+1,m} - u_{n-1,m})[1 - (1 - u_{n,m})\delta_1]}{\delta_2 + u_{n,m}}.
\end{aligned} \tag{2.5}
$$

As stated in the introduction, from the relation $W_1 = \xi_n$ this first integral defines a three-point, second-order ordinary difference equation in the n direction which depends parametrically on m. From the parametric dependence we find two different three-point non-autonomous ordinary difference equations corresponding to m even and m odd. We treat them separately.

Case $\boldsymbol{m = 2l}$. If $m = 2l$ we have the following non-autonomous ordinary difference equation

$$
F_n^{(+)} \frac{[(1+\delta_2)u_{n,2l} + u_{n+1,2l}]\delta_1 - u_{n,2l}}{[(1+\delta_2)u_{n,2l} + u_{n-1,2l}]\delta_1 - u_{n,2l}} + F_n^{(-)} (u_{n+1,2l} - u_{n-1,2l}) = \xi_n,
$$

where without loss of generality we have chosen $\alpha = 1$ and $\beta = 1$. We can easily see, that once solved for $u_{n+1,2l}$ the equation is *linear*

$$
u_{n+1,2l} - \frac{F_n^{(+)}(1 - \delta_1\delta_2 - \delta_1)(1 - \xi_n)u_{n,2l}}{\delta_1} - \big(F_n^{(+)}\xi_n + F_n^{(-)}\big)u_{n-1,2l} - F_n^{(-)}\xi_n = 0. \tag{2.6}
$$

Tackling this equation directly is very difficult, but we can separate again the cases when n is even and odd and convert (2.6) into a system using the standard transformation (1.12a)

$$
w_{k,l} - \xi_{2k} w_{k-1,l} = \frac{\delta}{\delta_1} (1 - \xi_{2k})\, v_{k,l}, \tag{2.7a}
$$

$$
v_{k+1,l} - v_{k,l} = \xi_{2k+1}, \tag{2.7b}
$$

where δ is given by (2.1). Now we have two *first-order* ordinary difference equations. Equation (2.7b) is uncoupled from equation (2.7a). Furthermore, since ξ_{2k} and ξ_{2k+1} are independent functions we can write $\xi_{2k+1} = a_{k+1} - a_k$. So the second equation possesses the trivial solution[2]

$$
v_{k,l} = \alpha_l + a_k. \tag{2.8}
$$

Now introduce (2.8) into (2.7a) and solve the equation for $w_{k,l}$

$$
w_{k,l} - \xi_{2k} w_{k-1,l} = \frac{\delta}{\delta_1}(1 - \xi_{2k})(\alpha_l + a_k).
$$

We define $\xi_{2k} = b_k/b_{k-1}$ and perform the change of dependent variable: $w_{k,l} = b_k W_{k,l}$. Then $W_{k,l}$ solves the equation

$$
W_{k,l} - W_{k-1,l} = \frac{\delta}{\delta_1} \left(\frac{1}{b_k} - \frac{1}{b_{k-1}} \right) (\alpha_l + a_k).
$$

[2]From now on we use the convention of naming the arbitrary functions depending on k with Latin letters and the functions depending on l by Greek ones.

The solution of this difference equation is given by

$$W_{k,l} = \beta_l + \frac{\delta}{\delta_1}\frac{\alpha_l}{b_k} + c_k,$$

where c_k is such that

$$c_k - c_{k-1} = \frac{\delta}{\delta_1}\left(\frac{1}{b_k} - \frac{1}{b_{k-1}}\right) a_k. \tag{2.9}$$

Equation (2.9) is not a total difference, but it can be used to define a_k in terms of the arbitrary functions b_k and c_k

$$a_k = -\frac{\delta_1}{\delta}\frac{b_k b_{k-1}}{b_k - b_{k-1}}(c_k - c_{k-1}). \tag{2.10}$$

This means that we have the following solution for the system (2.7)

$$v_{k,l} = \alpha_l - \frac{\delta_1}{\delta}\frac{b_k b_{k-1}}{b_k - b_{k-1}}(c_k - c_{k-1}), \tag{2.11a}$$

$$w_{k,l} = b_k(\beta_l + c_k) + \frac{\delta}{\delta_1}\alpha_l, \tag{2.11b}$$

Case $m = 2l + 1$. If $m = 2l+1$ we have the following non-autonomous ordinary difference equation

$$F_n^{(+)}\frac{1 + (u_{n+1,2l+1} - 1)\delta_1}{1 + (u_{n-1,2l+1} - 1)\delta_1} + F_n^{(-)}\frac{(u_{n-1,2l+1} - u_{n+1,2l+1})[1 - \delta_1(1 - u_{n,2l+1})]}{\delta_2 + u_{n,2l+1}} = \xi_n, \tag{2.12}$$

We can easily see that the equation is genuinely nonlinear. However we can separate the cases when n is even and odd and convert (2.12) into a system using the standard transformation (1.12b)

$$z_{k,l} - \xi_{2k} z_{k-1,l} = \left(1 - \frac{1}{\delta_1}\right)(1 - \xi_{2k}), \tag{2.13a}$$

$$y_{k+1,l} - y_{k,l} = \frac{\xi_{2k+1}}{\delta_1}\frac{\delta - 1 + \delta_1(1 - z_{k,l})}{1 - \delta_1(1 - z_{k,l})}, \tag{2.13b}$$

where we used the definition (2.1). This is a system of two *first-order* difference equation, and equation (2.13a) is linear and uncoupled from (2.13b). As $\xi_{2k} = b_k/b_{k-1}$ we have that (2.13a) is a total difference

$$\frac{z_{k,l}}{b_k} - \frac{z_{k-1,l}}{b_{k-1}} = \left(1 - \frac{1}{\delta_1}\right)\left(\frac{1}{b_k} - \frac{1}{b_{k-1}}\right). \tag{2.14}$$

Hence the solution of (2.14) is given by

$$z_{k,l} = 1 - \frac{1}{\delta_1} + b_k \gamma_l. \tag{2.15}$$

Inserting (2.15) into (2.13b) and using the definition of ξ_{2k+1} in terms of a_k, i.e., $\xi_{2k+1} = a_{k+1} - a_k$ we obtain

$$y_{k+1,l} - y_{k,l} = -\left(\frac{1}{\delta_1} + \frac{\delta}{\delta_1^2 b_k \gamma_l}\right)(a_{k+1} - a_k). \tag{2.16}$$

We can then represent the solution of (2.16) as

$$y_{k,l} = \zeta_l + \frac{\delta d_k}{\delta_1^2 \gamma_l} - \frac{a_k}{\delta_1},$$

where d_k satisfies the first-order linear difference equation

$$d_{k+1} - d_k = \frac{a_{k+1} - a_k}{b_k}. \tag{2.17}$$

Inserting the value of a_k given by (2.10) inside (2.17) we obtain that this equation is a total difference. Then d_k is given by

$$d_k = -\frac{(c_k - c_{k-1}) b_{k-1} \delta_1}{(b_k - b_{k-1})\delta} - \frac{\delta_1 c_k}{\delta}.$$

This means that finally we have the following solutions for the fields $z_{k,l}$ and $y_{k,l}$

$$z_{k,l} = 1 - \frac{1}{\delta_1} + b_k \gamma_l, \tag{2.18a}$$

$$y_{k,l} = \zeta_l - \frac{(c_k - c_{k-1}) b_{k-1}}{(b_k - b_{k-1}) \delta_1 \gamma_l} - \frac{c_k}{\delta_1 \gamma_l} + \frac{1}{\delta} \frac{b_k b_{k-1} (c_k - c_{k-1})}{b_k - b_{k-1}}. \tag{2.18b}$$

Equations (2.11), (2.18) provide the value of the four fields, but we have too many arbitrary functions in the l direction, namely α_l, β_l, γ_l and ζ_l. Introducing (2.11), (2.18) into (1.3a) and separating the terms even and odd in n and m we obtain two independent equations

$$(\alpha_l + \delta_1 \zeta_l)\gamma_l + \beta_l = 0, \qquad (\alpha_{l+1} + \delta_1 \zeta_l)\gamma_l + \beta_{l+1} = 0, \tag{2.19}$$

which allow us to reduce by two the number of independent functions in the l direction. Solving equations (2.19) with respect to γ_l and ζ_l we obtain

$$\gamma_l = -\frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l}, \qquad \zeta_l = \frac{1}{\delta_1} \frac{\beta_l \alpha_{l+1} - \beta_{l+1} \alpha_l}{\beta_{l+1} - \beta_l}. \tag{2.20}$$

Inserting (2.20) into (2.11), (2.18) we obtain that the general solution of ${}_1D_2$ equation (1.3a) is given by (2.2), provided that $\delta_1 \neq 0$ and $\delta \neq 0$. Indeed the solution (2.2) is ill-defined if $\delta_1 = 0$ or $\delta = 0$ and we proceed to treat the relevant cases separately.

Case $\boldsymbol{\delta = 0}$. If $\delta = 0$ we can solve (2.1) with respect to δ_1

$$\delta_1 = \frac{1}{1 + \delta_2}. \tag{2.21}$$

The first integral (2.5) is not singular for δ_1 given by (2.21). The procedure of solution becomes different only when we arrive to the systems of ordinary difference equations (2.7) and (2.13). So we will present the solution of the systems in this case.

Case $\boldsymbol{m = 2l}$. If δ_1 is given by equation (2.21) the system (2.7) becomes

$$w_{k,l} - \xi_{2k} w_{k-1,l} = 0, \tag{2.22a}$$

$$v_{k+1,l} - v_{k,l} = \xi_{2k+1}. \tag{2.22b}$$

The system (2.22) is uncoupled and imposing $\xi_{2k} = b_k / b_{k-1}$ and $\xi_{2k+1} = a_{k+1} - a_k$ it is readily solved to give

$$v_{k,l} = a_k + \beta_l. \tag{2.23a}$$

$$w_{k,l} = b_k \alpha_l, \tag{2.23b}$$

Case $m = 2l + 1$. If δ_1 is given by equation (2.21) the system (2.13) becomes

$$\frac{\delta_2 + z_{k,l}}{b_k} = \frac{\delta_2 + z_{k-1,m}}{b_{k-1}}, \qquad -\frac{y_{k+1,l} - y_{k,l}}{1+\delta_2} = a_{k+1} - a_k,$$

where we used the fact that $\xi_{2k} = b_k/b_{k-1}$ and $\xi_{2k+1} = a_{k+1} - a_k$. The solution to this system is immediate and it is given by

$$z_{k,l} = b_k\gamma_l - \delta_2, \tag{2.24a}$$
$$y_{k,l} = (1+\delta_2)(\zeta_l - a_k). \tag{2.24b}$$

As in the general case we obtained the expressions of the four fields, but we have too many arbitrary functions in the l direction, namely α_l, β_l, γ_l and ζ_l. Substituting the obtained expressions (2.23), (2.24) in the equation ${}_1D_2$ (1.3a) with δ_1 given by equation (2.21) separating the even and odd terms we obtain two compatibility conditions

$$\alpha_l + \gamma_l\beta_l + \gamma_l\zeta_l = 0, \qquad \alpha_{l+1} + \gamma_l\beta_{l+1} + \gamma_l\zeta_l = 0.$$

We can solve this equation with respect to γ_l and ζ_l and we obtain

$$\gamma_l = -\frac{\alpha_{l+1} - \alpha_l}{\beta_{l+1} - \beta_l}, \qquad \zeta_l = -\frac{\beta_l\alpha_{l+1} - \beta_{l+1}\alpha_l}{\alpha_{l+1} - \alpha_l}. \tag{2.25}$$

Inserting (2.25) into (2.23), (2.24) we obtain that the general solution of ${}_1D_2$ equation (1.3a) when $\delta = 0$ is given by (2.3).

Case $\delta_1 = 0$. If $\delta_1 = 0$ the first integral (2.5) is singular. Then following [26] the ${}_1D_2$ equation (1.3a) with $\delta_1 = 0$ possesses in the direction n the following three-point, second-order integral

$$\begin{aligned} W_1^{(0,\delta_2)} &= F_n^{(+)}F_m^{(+)}\alpha\frac{u_{n+1,m} - u_{n-1,m}}{u_{n,m}} - F_n^{(+)}F_m^{(-)}\alpha(u_{n+1,m} - u_{n-1,m}) \\ &\quad + F_n^{(-)}F_m^{(+)}\beta(u_{n+1,m} - u_{n-1,m}) + F_n^{(-)}F_m^{(-)}\beta\frac{u_{n-1,m} - u_{n+1,m}}{\delta_2 + u_{n,m}}. \end{aligned} \tag{2.26}$$

In order to solve the ${}_1D_2$ equation (1.3a) in this case we use the first integral (2.26). We start separating the cases even and odd in m.

Case $m = 2l$. If $m = 2l$ we obtain from the first integral (2.26)

$$F_n^{(+)}\frac{u_{n+1,2l} - u_{n-1,2l}}{u_{n,2l}} + F_n^{(-)}(u_{n+1,2l} - u_{n-1,2l}) = \xi_n, \tag{2.27}$$

where we have chosen without loss of generality $\alpha = \beta = 1$. This equation is nonlinear. Applying the transformation (1.12a) we transform equation (2.27) into the system

$$w_{k,l} - w_{k-1,l} = \xi_{2k}v_{k,l}, \tag{2.28a}$$
$$v_{k+1,l} - v_{k,l} = \xi_{2k+1}. \tag{2.28b}$$

The system (2.28) is linear and equation (2.28b) is uncoupled from equation (2.28a). If we put $\xi_{2k+1} = a_{k+1} - a_k$ then equation (2.28b) has the solution

$$v_{k,l} = a_k + \alpha_l.$$

Substituting into (2.28a) we obtain

$$w_{k,l} - w_{k-1,l} = \xi_{2k}(a_k + \alpha_l). \tag{2.29}$$

Equation (2.29) becomes a total difference if we set

$$\xi_{2k} = b_k - b_{k-1}, \qquad a_k = \frac{c_k - c_{k-1}}{b_k - b_{k-1}} \tag{2.30}$$

and then the solution of the system (2.22) is given by

$$v_{k,l} = \frac{c_k - c_{k-1}}{b_k - b_{k-1}} + \alpha_l, \tag{2.31a}$$

$$w_{k,l} = c_k + \alpha_l b_k + \beta_l. \tag{2.31b}$$

Case $m = 2l + 1$. If $m = 2l + 1$ we obtain from the first integral (2.26)

$$F_n^{(+)}\left(u_{n-1,2l+1} - u_{n+1,2l+1}\right) - F_n^{(-)}\frac{\left(u_{n+1,2l+1} - u_{n-1,2l+1}\right)}{\delta_2 + u_{n,2l+1}} = \xi_n, \tag{2.32}$$

where we have chosen without loss of generality $\alpha = \beta = 1$. This equation is nonlinear. Applying the transformation (1.12b) we transform equation (2.32) into the system

$$z_{k-1,l} - z_{k,l} = b_k - b_{k-1}, \tag{2.33a}$$

$$y_{k,l} - y_{k+1,l} = (a_{k+1} - a_k)(\delta_2 + z_{k,l}), \tag{2.33b}$$

where we used the values of ξ_{2k} and ξ_{2k+1}. The system is now linear and equation (2.33a) is solved by

$$z_{k,l} = \gamma_l - b_k.$$

Substituting into (2.33b) we obtain

$$y_{k,l} - y_{k+1,l} = (a_{k+1} - a_k)(\delta_2 + \gamma_l - b_k).$$

Then we have that $y_{k,l}$ is given by

$$y_{k,l} = \zeta_l - (\gamma_l + \delta_2)a_k + d_k,$$

where d_k solves the ordinary difference equation

$$d_{k+1} - d_k = b_{k+1}\frac{c_{k+1} - c_k}{b_{k+1} - b_k} - c_{k+1} - b_k\frac{c_k - c_{k-1}}{b_k - b_{k-1}} + c_k. \tag{2.34}$$

In (2.34) we inserted the value of a_k according to (2.30). Equation (2.34) is a total difference and then d_k is given by

$$d_k = b_k\frac{c_k - c_{k-1}}{b_k - b_{k-1}} - c_k.$$

Then the solution of the system (2.33) is

$$z_{k,l} = \gamma_l - b_k, \tag{2.35a}$$

$$y_{k,l} = \zeta_l - (\gamma_l + \delta_2)\frac{c_k - c_{k-1}}{b_k - b_{k-1}} + b_k\frac{c_k - c_{k-1}}{b_k - b_{k-1}} - c_k. \tag{2.35b}$$

As in the general case we obtained the expressions of the four fields, but we have too many arbitrary functions in the l direction, namely α_l, β_l, γ_l and ζ_l. Substituting the obtained expressions (2.31), (2.35) in the equation ${}_1D_2$ (1.3a) with $\delta_1 = 0$ separating the even and odd terms we obtain two compatibility conditions

$$(\gamma_l + \delta_2)\alpha_l + \zeta_l + \beta_l = 0, \qquad (\gamma_l + \delta_2)\alpha_{l+1} + \beta_{l+1} + \zeta_l = 0. \tag{2.36}$$

We can solve equation (2.36) with respect to γ_l and ζ_l and to obtain

$$\gamma_l = -\delta_2 - \frac{\beta_{l+1}-\beta_l}{\alpha_{l+1}-\alpha_l}, \qquad \zeta_l = \frac{\beta_{l+1}\alpha_l - \alpha_{l+1}\beta_l}{\alpha_{l+1}-\alpha_l}. \tag{2.37}$$

Inserting (2.37) into (2.31), (2.35) we obtain that the general solution of ${}_1D_2$ equation (1.3a) when $\delta = 0$ is given by (2.4).

This discussion exhausts the possible cases. For any value of the parameters we have the general solution of the ${}_1D_2$ equation (1.3a) and this concludes the proof. ■

Proposition 2.2. *The ${}_2D_2$ equation (1.3b) is exactly solvable. If $\delta_1 \neq 0$ its general solution is given by*

$$v_{k,l} = b_k + \beta_l + \delta\frac{c_k}{\alpha_l} + \frac{1}{\delta_1} - 1 - \delta, \tag{2.38a}$$

$$w_{k,l} = \alpha_l\frac{b_{k+1}-b_k}{c_{k+1}-c_k} + \frac{1}{\delta_1} - 1 - \delta, \tag{2.38b}$$

$$\begin{aligned} z_{k,l} = {}& \delta b_k + \frac{b_{k+1}-b_k}{c_{k+1}-c_k}\left[\alpha_l(\beta_{l+1}-\beta_l) + \frac{\alpha_l^2(\beta_{l+1}-\beta_l)}{\alpha_{l+1}-\alpha_l} - \delta c_k\right] \\ & + \delta\frac{\beta_{l+1}\delta_1 - 1 - \delta_1^2\lambda + \delta\delta_1 + \delta_1}{\delta_1} + \frac{\delta\alpha_l(\beta_{l+1}-\beta_l)}{\alpha_{l+1}-\alpha_l}, \end{aligned} \tag{2.38c}$$

$$y_{k,l} = -\frac{\beta_{l+1}\delta_1 - 1 - \delta_1^2\lambda + \delta\delta_1 + \delta_1}{\delta_1^2} - \frac{\alpha_l(\beta_{l+1}-\beta_l)}{(\alpha_{l+1}-\alpha_l)\delta_1} - \frac{b_k}{\delta_1}, \tag{2.38d}$$

where b_k, c_k, α_l and β_l are arbitrary functions of their arguments. If $\delta_1 = 0$ then its general solution is given by

$$v_{k,l} = a_k\alpha_l, \tag{2.39a}$$

$$w_{k,l} = -\delta_2 - \frac{1}{\alpha_l}\frac{b_{k+1}-b_k}{a_{k+1}-a_k}, \tag{2.39b}$$

$$y_{k,l} = b_k + \beta_l, \tag{2.39c}$$

$$z_{k,l} = a_k\frac{b_{k+1}-b_k}{a_{k+1}-a_k} - b_k - \beta_l, \tag{2.39d}$$

where a_k, b_k, α_l and β_l are arbitrary functions of their arguments.

Proof. The proof of the two solution (2.38) and (2.39) proceeds as the one outlined in Proposition 2.1. The interested reader can find it in Appendix A. ■

Proposition 2.3. *The ${}_3D_2$ equation (1.3c) is exactly solvable. If $\delta_1 \neq 0$ and $\delta \neq 0$, where δ is give by (2.1), its general solution is given by*

$$v_{k,l} = -\frac{\delta_1}{\delta}b_k + c_k\alpha_l, \tag{2.40a}$$

$$w_{k,l} = \frac{\delta_1 - 1 + \delta}{\delta_1} + \frac{\delta(b_{k+1}-b_k)}{\delta\alpha_l(c_{k+1}-c_k) - (b_{k+1}-b_k)\delta_1}, \tag{2.40b}$$

$$z_{k,l} = \frac{b_k}{\delta} - \frac{1}{\delta_1}\left(\beta_l - \lambda\delta_1 - \alpha_l\frac{\beta_{l+1}-\beta_l}{\alpha_{l+1}-\alpha_l}\right) + \frac{1}{\delta}\frac{b_{k+1}-b_k}{c_{k+1}-c_k}\left(\frac{\beta_{l+1}-\beta_l}{\alpha_{l+1}-\alpha_l} + c_k\right), \tag{2.40c}$$

$$y_{k,l} = -b_k + \frac{\delta}{\delta_1}\left(\beta_l - \lambda\delta_1 - \alpha_l\frac{\beta_{l+1}-\beta_l}{\alpha_{l+1}-\alpha_l}\right), \tag{2.40d}$$

where b_k, c_k, α_l and β_l are arbitrary functions of their arguments. If $\delta = 0$ then its general solution is given by

$$v_{k,l} = c_k + b_k\alpha_l + \beta_l, \tag{2.41a}$$

$$w_{k,l} = -\delta_2 + \frac{b_{k+1} - b_k}{c_{k+1} - c_k + \alpha_l(b_{k+1} - b_k)}, \tag{2.41b}$$

$$y_{k,l} = -\frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l} - b_k, \tag{2.41c}$$

$$\begin{aligned} z_{k,l} = &-(\delta_2 + 1)\left[\left(b_{k+1} - \frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l}\right)\frac{c_{k+1} - c_k}{b_{k+1} - b_k} - c_{k+1}\right] \\ &+ (\delta_2 + 1)\frac{\beta_{l+1}\alpha_l - \alpha_{l+1}\beta_l}{\alpha_{l+1} - \alpha_l} + \lambda, \end{aligned} \tag{2.41d}$$

where b_k, c_k, α_l and β_l are arbitrary functions of their arguments. If $\delta_1 = 0$ then its general solution is given by

$$v_{k,l} = \frac{b_k + \zeta_0}{\alpha_l}, \tag{2.42a}$$

$$w_{k,l} = -\delta_2 - \alpha_l\frac{c_{k+1} - c_k}{b_{k+1} - b_k}, \tag{2.42b}$$

$$y_{k,l} = c_k + \gamma_l, \tag{2.42c}$$

$$z_{k,l} = \frac{c_{k+1} - c_k}{b_{k+1} - b_k}\zeta_0 - \gamma_l + \frac{b_k c_{k+1} - c_k b_{k+1}}{b_{k+1} - b_k}, \tag{2.42d}$$

where b_k, c_k, α_l and γ_l are arbitrary functions of their arguments and ζ_0 is a constant.

Proof. The proof of the three solution (2.40), (2.41) and (2.42) proceeds as the one outlined in Proposition 2.1. The interested reader can find it in Appendix A. ■

2.2 The D_3 and the ${}_iD_4$ equations, $i = 1, 2$

We have the following propositions:

Proposition 2.4. *The D_3 equation (1.3d) is exactly solvable. We have that the expression of the fields $y_{k,l}$ and $v_{k,l}$ is given by*

$$y_{k,l} = \alpha_l c_k + d_k + \beta_l \tag{2.43a}$$

$$\begin{aligned} v_{k,l} = &\left(\alpha_l c_k + d_k + \beta_l\right)^2 \\ &+ \left[\left(\alpha_l - \alpha_{l-1}\right)c_k + \beta_l - \beta_{l-1}\right]\left[\gamma_l + \frac{e_k - \alpha_{l-1}\left(\alpha_{l-1}c_k + 2d_k\right)}{\alpha_l - \alpha_{l-1}}\right], \end{aligned} \tag{2.43b}$$

where the function functions c_k, d_k, α_l and β_l are arbitrary functions of their arguments, whereas the function e_k and γ_l are given through the discrete integrations

$$e_{k+1} = e_k - \frac{\left(d_{k+1} - d_k\right)^2}{c_{k+1} - c_k}, \tag{2.44a}$$

$$\left(\alpha_{l+1} - \alpha_l\right)\gamma_{l+1} - \left(\alpha_l - \alpha_{l-1}\right)\gamma_l - \alpha_{l-1}\beta_{l-1} + \alpha_l\beta_l + \alpha_l\beta_{l-1} - \alpha_{l-1}\beta_l = 0. \tag{2.44b}$$

The fields $z_{k,l}$ and $w_{k,l}$ are then given in terms of $y_{k,l}$ and $v_{k,l}$ as

$$z_{k,l} = \frac{\left(y_{k+1,l}y_{k,l-1} - y_{k+1,l-1}y_{k,l}\right)y_{k,l} - \left(y_{k+1,l} + y_{k,l-1} - y_{k+1,l-1} - y_{k,l}\right)v_{k,l}}{\left(y_{k+1,l} + y_{k,l-1} - y_{k+1,l-1} - y_{k,l}\right)y_{k,l} - y_{k+1,l}y_{k,l-1} + y_{k+1,l-1}y_{k,l}} \tag{2.45a}$$

$$w_{k,l} = -\frac{y_{k+1,l}y_{k,l-1} - y_{k+1,l-1}y_{k,l}}{y_{k+1,l} + y_{k,l-1} - y_{k+1,l-1} - y_{k,l}}. \tag{2.45b}$$

Remark 2.5. We remark that we can say that equation (2.44b) defines a discrete integration for the function γ_l since it can be expressed in the form (1.14). Defining a the new function ζ_l by

$$\gamma_l = -\frac{\alpha_{l-1}\beta_{l-1} + \zeta_l}{\alpha_l - \alpha_{l-1}}$$

we have that ζ_l satisfies the following difference equation

$$\zeta_{l+1} - \zeta_l = \alpha_l\beta_{l-1} - \alpha_{l-1}\beta_l, \tag{2.46}$$

i.e., it is given by a discrete integration.

Proof. To find general of the D_3 equation (1.3d) we start from the equation itself. Applying the general transformation (1.12) to the D_3 equation (1.3d) we obtain the following system of four equations:

$$v_{k,l} + w_{k,l}y_{k,l} + w_{k,l}z_{k,l} + y_{k,l}z_{k,l} = 0, \tag{2.47a}$$
$$v_{k,l+1} + y_{k,l}w_{k,l+1} + z_{k,l}w_{k,l+1} + y_{k,l}z_{k,l} = 0, \tag{2.47b}$$
$$v_{k+1,l} + w_{k,l}y_{k+1,l} + w_{k,l}z_{k,l} + z_{k,l}y_{k+1,l} = 0, \tag{2.47c}$$
$$v_{k+1,l+1} + y_{k+1,l}w_{k,l+1} + z_{k,l}w_{k,l+1} + z_{k,l}y_{k+1,l} = 0. \tag{2.47d}$$

From the system (2.47) we have four different way for calculating $z_{k,l}$. This means that we have some compatibility conditions. Indeed from (2.47a) and (2.47c) we obtain the following equation for $v_{k+1,l}$:

$$v_{k+1,l} = \frac{(w_{k,l} + y_{k+1,l})v_{k,l}}{w_{k,l} + y_{k,l}} + \frac{(y_{k,l} - y_{k+1,l})w_{k,l}^2}{w_{k,l} + y_{k,l}}, \tag{2.48}$$

while from (2.47b) and (2.47d) we obtain the following equation for $v_{k+1,l+1}$:

$$v_{k+1,l+1} = \frac{(w_{k,l+1} + y_{k+1,l})v_{k,l+1}}{w_{k,l+1} + y_{k,l}} + \frac{(y_{k,l} - y_{k+1,l})w_{k,l+1}^2}{w_{k,l+1} + y_{k,l}}. \tag{2.49}$$

Equations (2.48) and (2.49) give rise to a compatibility condition between $v_{k+1,l}$ and its shift in the l direction $v_{k+1,l+1}$ which is given by

$$\begin{pmatrix} y_{k,l}w_{k,l+1} + y_{k+1,l+1}w_{k,l+1} + y_{k+1,l+1}y_{k,l} \\ -y_{k,l+1}w_{k,l+1} - y_{k+1,l}w_{k,l+1} - y_{k+1,l}y_{k,l+1} \end{pmatrix} \left(v_{k,l+1} - w_{k,l+1}^2\right) = 0.$$

Discarding the trivial solution $v_{k,l} = w_{k,l}^2$ we obtain the following value for the field $w_{k,l}$

$$w_{k,l} = -\frac{y_{k+1,l}y_{k,l-1} - y_{k+1,l-1}y_{k,l}}{y_{k+1,l} + y_{k,l-1} - y_{k+1,l-1} - y_{k,l}}, \tag{2.50}$$

which makes (2.48) and (2.49) compatible. Equation (2.50) gives $w_{k,l}$ in terms of $y_{k,l}$ alone and therefore it is the first part of the solution represented by (2.45b). Inserting equation (2.50) into (2.48) we are left with the following equation for $v_{k,l}$

$$v_{k+1,l} = \frac{y_{k+1,l} - y_{k+1,l-1}}{y_{k,l} - y_{k,l-1}}v_{k,l} + \frac{(y_{k+1,l-1}y_{k,l} - y_{k+1,l}y_{k,l-1})^2}{(y_{k+1,l-1} + y_{k,l} - y_{k+1,l} - y_{k,l-1})(y_{k,l} - y_{k,l-1})}. \tag{2.51}$$

Applying the transformation

$$v_{k,l} = (y_{k,l} - y_{k,l-1})V_{k,l} + y_{k,l-1}^2 \tag{2.52}$$

we can simplify (2.51) to the equation

$$V_{k+1,l} = V_{k,l} + \frac{(y_{k,l-1} - y_{k+1,l-1})^2}{y_{k+1,l-1} + y_{k,l} - y_{k+1,l} - y_{k,l-1}}. \tag{2.53}$$

To go further we need to specify the form of the field $y_{k,l}$. This can be obtained from the Darboux integrability of the D_3 equation (1.3d). From [26] we know that the D_3 equation (1.3d) possesses the following W_1 four-point, third-order integral

$$\begin{aligned} W_1 = {} & F_n^{(+)} F_m^{(+)} \alpha \frac{(u_{n+1,m} - u_{n-1,m})(u_{n+2,m} - u_{n,m})}{u_{n+1,m}^2 - u_{n,m}} \\ & + F_n^{(+)} F_m^{(-)} \alpha \frac{(u_{n+1,m} - u_{n-1,m})(u_{n+2,m} - u_{n,m})}{u_{n,m} + u_{n-1,m}} \\ & - F_n^{(-)} F_m^{(+)} \beta \frac{(u_{n+1,m} - u_{n-1,m})(u_{n+2,m} - u_{n,m})}{u_{n+1,m} - u_{n,m}^2} \\ & + F_n^{(-)} F_m^{(-)} \beta \frac{(u_{n+1,m} - u_{n-1,m})(u_{n+2,m} - u_{n,m})}{u_{n+1,m} + u_{n+2,m}}. \end{aligned} \tag{2.54}$$

Consider now the equation $W_1 = \xi_n$ with W_1 given as in (2.54). This relation defines a third-order, four-point ordinary difference equation in the n direction depending parametrically on m. In particular if we choose the case when $m = 2l + 1$ we have the equation

$$\begin{aligned} & F_n^{(+)} \frac{(u_{n+1,2l+1} - u_{n-1,2l+1})(u_{n+2,2l+1} - u_{n,2l+1})}{u_{n,2l+1} + u_{n-1,2l+1}} \\ & \quad + F_n^{(-)} \frac{(u_{n+1,2l+1} - u_{n-1,2l+1})(u_{n+2,2l+1} - u_{n,2l+1})}{u_{n+1,2l+1} + u_{n+2,2l+1}} = \xi_n, \end{aligned} \tag{2.55}$$

where we have taken without loss of generality $\alpha = \beta = 1$. Using the transformation (1.12b) equation (2.55) is converted into the system

$$(y_{k+1,l} - y_{k,l})(z_{k,l} - z_{k-1,l}) = \xi_{2k}(y_{k,l} + z_{k-1,l}), \tag{2.56a}$$
$$(y_{k+1,l} - y_{k,l})(z_{k+1,l} - z_{k,l}) = \xi_{2k+1}(y_{k+1,l} + z_{k+1,l}). \tag{2.56b}$$

This system is nonlinear, but if we solve (2.56b) with respect to $z_{k+1,l}$ and we substitute it along with its shift in the k direction into (2.56a) we obtain a *linear second-order ordinary difference equation* involving *only* the field $y_{k,l}$

$$\xi_{2k-1} y_{k+1,m} - (\xi_{2k} + \xi_{2k-1}) y_{k,m} + \xi_{2k} y_{k-1,m} + \xi_{2k}\xi_{2k-1} = 0. \tag{2.57}$$

We can lower the order of equation (2.57) by one using the potential transformation

$$Y_{k,l} = y_{k+1,l} - y_{k,l}. \tag{2.58}$$

Then $Y_{k,l}$ solves the equation

$$Y_{k,l} - \frac{\xi_{2k}}{\xi_{2k-1}} Y_{k-1,l} + \xi_{2k} = 0.$$

Imposing that

$$\xi_{2k} = -a_k(b_k - b_{k-1}), \qquad \xi_{2k-1} = -a_{k-1}(b_k - b_{k-1})$$

we obtain that $Y_{k,l}$ can be expressed as

$$Y_{k,l} = a_k(b_k + \alpha_l).$$

From (2.58) we have that

$$y_{k+1,l} - y_{k,l} = a_k(b_k + \alpha_l).$$

Setting

$$a_k = c_{k+1} - c_k, \qquad b_k = \frac{d_{k+1} - d_k}{c_{k+1} - c_k},$$

we have that the solution of is given by

$$y_{k,l} = \alpha_l c_k + d_k + \beta_l, \tag{2.59}$$

i.e., by equation (2.43a).

Inserting now the obtained value of $y_{k,l}$ from (2.59) into the equation (2.53) we obtain

$$V_{k+1,l} = V_{k,l} - \frac{(d_k + \alpha_{l-1}c_k - d_{k+1} - \alpha_{l-1}c_{k+1})^2}{(\alpha_l - \alpha_{l-1})(c_{k+1} - c_k)}.$$

This means that we can write down the following solution for $V_{k,l}$

$$V_{k,l} = \gamma_l - \alpha_{l-1}\frac{\alpha_{l-1}c_k + 2d_k}{\alpha_l - \alpha_{l-1}} + \frac{e_k}{\alpha_l - \alpha_{l-1}}, \tag{2.60}$$

up to a discrete integration for the function e_k

$$e_{k+1} = e_k - \frac{(d_{k+1} - d_k)^2}{c_{k+1} - c_k},$$

i.e., up to the condition (2.44a). Substituting the value of $V_{k,l}$ from (2.60) and of $y_{k,l}$ from (2.59) into equation (2.52) we have that $v_{k,l}$ is given by equation (2.43b). Plugging the obtained value of $v_{k,l}$ we can compute $w_{k,l}$ from (2.50). Finally we can compute $z_{k,l}$ from the original system (2.47) and we obtain a *single* compatibility condition given by

$$(\alpha_{l+1} - \alpha_l)\gamma_{l+1} - (\alpha_l - \alpha_{l-1})\gamma_l - \alpha_{l-1}\beta_{l-1} + \alpha_l\beta_l + \alpha_l\beta_{l-1} - \alpha_{l-1}\beta_l = 0, \tag{2.61}$$

i.e., just by (2.44b). Given this conditions all the equations in (2.47) are compatible and $z_{k,l}$ is indifferently given by solving one of the equation. E.g., solving (2.47a) we can say that $z_{k,l}$ is given by equation (2.45a). This ends the procedure of solution of the D_3 equation (1.3d). ■

Proposition 2.6. *The ${}_1D_4$ equation (1.3e) is exactly solvable. We have that the expression of the fields $y_{k,l}$ and $v_{k,l}$ is given by*

$$y_{k,l} = c_k(\alpha_l d_k + \beta_l), \tag{2.62a}$$

$$\begin{aligned} v_{k,l} = {} & c_k[(\alpha_l - \alpha_{l-1})d_k + \beta_l - \beta_{l-1}]\left\{\gamma_l + \frac{\delta_1\delta_3}{\beta_{l-1}\alpha_l - \beta_l\alpha_{l-1}}\left[\frac{\alpha_{l-1}}{c_k^2(\alpha_{l-1}d_k + \beta_{l-1})} + e_k\right]\right\} \\ & + \frac{\delta_1\delta_3}{c_k\left(\alpha_{l-1}d_k + \beta_{l-1}\right)} - \delta_2 c_k\left(\alpha_{l-1}d_k + \beta_{l-1}\right), \end{aligned} \tag{2.62b}$$

where the function functions c_k, d_k, α_l and β_l are arbitrary functions of their arguments, whereas the function e_k and γ_l are given through the discrete integrations

$$e_{k+1} - e_k = -\frac{(c_k - c_{k+1})^2}{c_k^2 c_{k+1}^2(d_{k+1} - d_k)}, \tag{2.63a}$$

$$(\beta_l\alpha_{l+1} - \beta_{l+1}\alpha_l)\gamma_{l+1} - (\beta_{l-1}\alpha_l - \beta_l\alpha_{l-1})\gamma_l = (\beta_{l-1}\alpha_l - \beta_l\alpha_{l-1})\delta_2. \tag{2.63b}$$

The fields $z_{k,l}$ and $w_{k,l}$ are then given in terms of $y_{k,l}$ and $v_{k,l}$ as

$$z_{k,l} = -\frac{\begin{bmatrix} \delta_3\left(y_{k+1,l}y_{k,l-1} - y_{k+1,l-1}y_{k,l}\right) \\ +\delta_1\delta_3 y_{k,l}(y_{k+1,l-1} - y_{k+1,l} - y_{k,l-1} + y_{k,l}) \end{bmatrix}}{\begin{bmatrix} (v_{k,l} + \delta_2 y_{k,l})(y_{k+1,l}y_{k,l-1} - y_{k+1,l-1}y_{k,l}) \\ +\delta_1\delta_3(y_{k+1,l-1} - y_{k+1,l} - y_{k,l-1} + y_{k,l}) \end{bmatrix}}, \tag{2.64a}$$

$$w_{k,l} = \delta_3 \frac{y_{k+1,l-1} - y_{k+1,l} - y_{k,l-1} + y_{k,l}}{y_{k+1,l}y_{k,l-1} - y_{k+1,l-1}y_{k,l}}. \tag{2.64b}$$

Remark 2.7. We remark that we can say that equation (2.63b) defines a discrete integration for the function γ_l since it can be expressed in the form (1.14). Defining a new function ζ_l by

$$\gamma_l = \frac{\zeta_l \delta_2}{\beta_{l-1}\alpha_l - \beta_l\alpha_{l-1}}$$

we have that ζ_l is given by the following difference equation

$$\zeta_{l+1} - \zeta_l = \alpha_l\beta_{l-1} - \alpha_{l-1}\beta_l, \tag{2.65}$$

i.e., it is given by a discrete integration. Note that (2.65) is exactly the same as (2.46).

Proof. The proof of the solution of the ${}_1D_2$ equation (1.3e) proceeds as the one outlined in Proposition 2.4. The interested reader can find the details in Appendix A. ∎

Proposition 2.8. *The ${}_2D_4$ equation (1.3f) is exactly solvable. We have that the expression of the fields $y_{k,l}$ and $v_{k,l}$ is given by*

$$y_{k,l} = c_k(\alpha_l d_k + \beta_l), \tag{2.66a}$$

$$\begin{aligned} v_{k,l} = {} & c_k[(\alpha_l - \alpha_{l-1})d_k + \beta_l - \beta_{l-1}] \\ & \times \left\{ \gamma_l - \frac{\alpha_l \delta_1\delta_3}{(\alpha_l d_k + \beta_l)c_k^2(\beta_l\alpha_{l-1} - \beta_{l-1}\alpha_l)} + \frac{\delta_3\delta_1 e_k}{\beta_l\alpha_{l-1} - \beta_{l-1}\alpha_l} \right\} \\ & + \frac{\delta_1\delta_3}{c_k(\alpha_l d_k + \beta_l)} - \delta_2 c_k(\alpha_l d_k + \beta_l), \end{aligned} \tag{2.66b}$$

where the function functions c_k, d_k, α_l and β_l are arbitrary functions of their arguments, whereas the function e_k and γ_l are given through the discrete integrations

$$e_{k+1} - e_k = \frac{(c_{k+1} - c_k)^2}{(d_{k+1} - d_k)c_k^2 c_{k+1}^2}, \tag{2.67a}$$

$$(\beta_l\alpha_{l+1} - \beta_{l+1}\alpha_l)\gamma_{l+1} - (\beta_{l-1}\alpha_l - \beta_l\alpha_{l-1})\gamma_l = (\beta_l\alpha_{l+1} - \beta_{l+1}\alpha_l)\delta_2. \tag{2.67b}$$

The fields $z_{k,l}$ and $w_{k,l}$ are then given in terms of $y_{k,l}$ and $v_{k,l}$ as

$$z_{k,l} = -\frac{1}{\delta_1} \frac{\begin{bmatrix} \delta_1\delta_3(y_{k,l-1} + y_{k+1,l} - y_{k,l} - y_{k+1,l-1}) \\ +(v_{k,l} + \delta_2 y_{k,l})(y_{k+1,l-1}y_{k,l} - y_{k+1,l}y_{k,l-1}) \end{bmatrix}}{\begin{bmatrix} y_{k+1,l-1}y_{k,l} - y_{k+1,l}y_{k,l-1} \\ +y_{k,l}(y_{k,l-1} + y_{k+1,l} - y_{k,l} - y_{k+1,l-1}) \end{bmatrix}}, \tag{2.68a}$$

$$w_{k,l} = \frac{1}{\delta_1} \frac{y_{k+1,l-1}y_{k,l} - y_{k+1,l}y_{k,l-1}}{y_{k,l-1} + y_{k+1,l} - y_{k,l} - y_{k+1,l-1}}. \tag{2.68b}$$

Remark 2.9. We remark that we can say that equation (2.63b) defines a discrete integration for the function γ_l since it can be expressed in the form (1.14). Defining a new function ζ_l by

$$\gamma_l = \frac{\zeta_l \delta_2}{\beta_{l-1}\alpha_l - \beta_l \alpha_{l-1}}$$

we have that ζ_l is given by the following difference equation

$$\zeta_{l+1} - \zeta_l = \alpha_{l+1}\beta_l - \alpha_l \beta_{l+1},$$

i.e., it is given by a discrete integration.

Proof. The proof of the solution of the ${}_2D_2$ equation (1.3f) proceeds as the one outlined in Proposition 2.4. The interested reader can find the details in Appendix A. ∎

2.3 The ${}_tH_2^\varepsilon$ and the ${}_tH_3^\varepsilon$ equations

In this subsection we construct a general solution of the ${}_tH_2^\varepsilon$ and the ${}_tH_3^\varepsilon$ equations. As we recalled in the introduction, the solution of the ${}_tH_1^\varepsilon$ through the first integrals was already presented in [26], so we will not discuss it again. Moreover we also recall that the general solution of the ${}_tH_1^\varepsilon$ equation was first found in [21, 22] without the knowledge of the first integrals. The first integrals of the ${}_tH_1^\varepsilon$ equation were first presented in [23].

The procedure we will follow will make use of the first integrals, in a similar way than in the cases presented in Section 2.2. The main difference is in the fact that the H^4 are non-autonomous only in the direction m, i.e., they depend only on the non-autonomous factors $F_m^{(\pm)}$ as given by (1.4). Therefore instead of the general transformation (1.12) we can use the simplified transformation

$$u_{n,2l} = p_{n,l}, \qquad u_{n,2l+1} = q_{n,l}. \tag{2.69}$$

Then to describe the general solution of a H^4 we only need three arbitrary functions: one in the n direction and two in the m direction.

We have then that the following propositions hold true:

Proposition 2.10. *The ${}_tH_2^\varepsilon$ equation (1.2b) is exactly solvable. If $\varepsilon \neq 0$ and the field $q_{n,l}$ do not satisfy the discrete wave equation*

$$q_{n+1,l+1} + q_{n,l} = q_{n+1,l} + q_{n,l+1}, \tag{2.70}$$

then the solution of the ${}_tH_2^\varepsilon$ equation (1.2b) is given by

$$q_{n,l} = \beta_l + \frac{\gamma_l + \zeta_l e_n + f_n}{c_n + \zeta_l}, \tag{2.71a}$$

$$p_{n,l} = \frac{\left\{\begin{gathered} (q_{n,l} - \alpha_3)q_{n+1,l-1} - (q_{n,l-1} - \alpha_3)q_{n+1,l} \\ - (\alpha_2 + \alpha_3)(q_{n,l} - q_{n,l-1}) - \varepsilon\alpha_3^2(q_{n,l} - q_{n,l-1}) \\ + \varepsilon\left[{\alpha_3}^2 + 2\alpha_3(q_{n,l-1} + \alpha_2) - (q_{n,l} - q_{n,l-1})q_{n+1,l-1}\right]q_{n+1,l} \\ + \varepsilon(\alpha_2 + q_{n,l})(\alpha_2 + q_{n,l-1})q_{n+1,l} \\ -\varepsilon q_{n+1,l-1}\left[{\alpha_3}^2 - 2(q_{n,l} + \alpha_2)\alpha_3 + (\alpha_2 + q_{n,l})(\alpha_2 + q_{n,l-1})\right] \end{gathered}\right\}}{q_{n+1,l} - q_{n,l} + q_{n,l-1} - q_{n+1,l-1}}, \tag{2.71b}$$

where c_n, ζ_l and β_l are arbitrary functions of their arguments and e_n is a solution of the equation

$$e_{n+1} - \frac{c_{n+1} - c_{n-1}}{c_n - c_{n-1}} e_n + \frac{c_{n+1} - c_n}{c_n - c_{n-1}} e_{n-1} - \alpha_2 \frac{c_{n+1} - c_{n-1}}{c_n - c_{n-1}} = 0, \tag{2.72}$$

while f_n and γ_l are given by the discrete integrations

$$f_n - f_{n-1} = e_n c_{n-1} - c_n e_{n-1}, \tag{2.73a}$$

$$\gamma_{l+1} - \gamma_l = -(\zeta_{l+1} - \zeta_l)\left(\alpha_2 + \beta_{l+1} + \beta_l + 2\alpha_3 - \frac{1}{\varepsilon}\right). \tag{2.73b}$$

If $\varepsilon = 0$, but the field $q_{n,l}$ do not satisfy the discrete wave equation (2.83) then the solution of the ${}_tH_2^\varepsilon$ equation (1.2b) is given by

$$q_{n,l} = \beta_l + \frac{\gamma_l + \zeta_0 e_n + f_n}{c_n + \zeta_0}, \tag{2.74a}$$

$$p_{n,l} = \frac{\begin{bmatrix} (q_{n,l} - \alpha_3) q_{n+1,l-1} - (q_{n,l-1} - \alpha_3) q_{n+1,l} \\ - (\alpha_2 + \alpha_3)(q_{n,l} - q_{n,l-1}) \end{bmatrix}}{q_{n+1,l} - q_{n,l} + q_{n,l-1} - q_{n+1,l-1}}, \tag{2.74b}$$

where c_n, β_l and γ_l are arbitrary functions of their arguments, ζ_0 is a constant and e_n is a solution of (2.72) and f_n is a solution of (2.73a). If the field $q_{n,l}$ satisfies the discrete wave equation (2.70) regardless of the value of the parameter ε the solution of the ${}_tH_2^\varepsilon$ equation (1.2b) is given by

$$q_{n,l} = a_n + \zeta_0, \tag{2.75a}$$

$$p_{n,l} = b_n(\beta_l + c_n), \tag{2.75b}$$

where b_n and β_l are arbitrary functions of their arguments, ζ_0 is a constant and a_n and c_n are given by the discrete integration

$$\frac{a_{n+1} - a_n + \alpha_2}{a_{n+1} - a_n - \alpha_2} = \frac{b_{n+1}}{b_n}, \tag{2.76a}$$

$$\begin{aligned} c_{n+1} - c_n = {} & \frac{(a_n + \zeta_0 - \alpha_3) b_{n+1} + b_n(\alpha_2 + \alpha_3 - \zeta_0 - a_n)}{b_n b_{n+1}} \\ & - \varepsilon \frac{\left[(\alpha_2 + \zeta_0 + a_n + \alpha_3)^2 b_{n+1} - b_n(\alpha_3 + a_n + \zeta_0)^2\right]}{b_n b_{n+1}}. \end{aligned} \tag{2.76b}$$

Remark 2.11. We remark that the function e_n can be obtained from (2.72) as the result of *two* discrete integrations. Indeed defining

$$E_n = \frac{e_{n+1} - e_n}{c_{n+1} - c_n}, \tag{2.77}$$

and substituting in (2.72) we obtain that E_n must solve the equation

$$E_n - E_{n-1} = \alpha_2 \left(\frac{1}{c_{n+1} - c_n} + \frac{1}{c_n - c_{n-1}}\right). \tag{2.78}$$

Note that the right-hand side of (2.78) is not a total difference. So the function e_n can be obtained by integrating (2.78) and subsequently integrating (2.77). This provides the value of e_n. The obtained value can be plugged in (2.73a) to give f_n after discrete integration. This reasoning shows that we can obtain the non-arbitrary functions e_n and f_n as result of a finite number of discrete integrations. Therefore we can conclude that the solution of the ${}_tH_2^\varepsilon$ equation (1.2b) in the general case is given in terms of four discrete integrations. If $\varepsilon = 0$ then the general solution is given in terms of three discrete integrations and finally in the singular case, when $q_{n,l}$ solves the discrete wave equation (2.70), we need only two discrete integrations.

Proof. We start the procedure of solution of the ${}_tH_2^\varepsilon$ equation (1.2b) by looking at the equation itself. We apply the transformation (2.69) to the ${}_tH_2^\varepsilon$ equation (1.2b) and we obtain the following system of two coupled equations

$$
\begin{aligned}
&(p_{n,l}-p_{n+1,l})(q_{n,l}-q_{n+1,l})-\alpha_2(p_{n,l}+p_{n+1,l}+q_{n,l}+q_{n+1,l})\\
&\quad+\frac{\varepsilon\alpha_2}{2}(2q_{n,l}+2\alpha_3+\alpha_2)(2q_{n+1,l}+2\alpha_3+\alpha_2)+\frac{\varepsilon\alpha_2}{2}(2\alpha_3+\alpha_2)^2+(\alpha_2+\alpha_3)^2\\
&\quad-\alpha_3^2-2\varepsilon\alpha_2\alpha_3(\alpha_2+\alpha_3)=0, && (2.79a)\\
&(q_{n,l}-q_{n+1,l})(p_{n,l+1}-p_{n+1,l+1})-\alpha_2(q_{n,l}+q_{n+1,l}+p_{n,l+1}+p_{n+1,l+1})\\
&\quad+\frac{\varepsilon\alpha_2}{2}(2q_{n,l}+2\alpha_3+\alpha_2)(2q_{n+1,l}+2\alpha_3+\alpha_2)+\frac{\varepsilon\alpha_2}{2}(2\alpha_3+\alpha_2)^2+(\alpha_2+\alpha_3)^2-\alpha_3^2\\
&\quad-2\varepsilon\alpha_2\alpha_3(\alpha_2+\alpha_3)=0. && (2.79b)
\end{aligned}
$$

We have that equation (2.79a) depends on $p_{n,l}$ and $p_{n+1,l}$ and that equation (2.79b) depends on $p_{n,l+1}$ and $p_{n+1,l+1}$. So we apply the translation operator T_l to (2.79a) to obtain two equations in terms of $p_{n,l+1}$ and $p_{n+1,l+1}$

$$
\begin{aligned}
&(p_{n,l+1}-p_{n+1,l+1})(q_{n,l+1}-q_{n+1,l+1})-\alpha_2(p_{n,l+1}+p_{n+1,l+1}+q_{n,l+1}+q_{n+1,l+1})\\
&\quad+\frac{\varepsilon\alpha_2}{2}(2q_{n,l+1}+2\alpha_3+\alpha_2)(2q_{n+1,l+1}+2\alpha_3+\alpha_2)+\frac{\varepsilon\alpha_2}{2}(2\alpha_3+\alpha_2)^2+(\alpha_2+\alpha_3)^2\\
&\quad-\alpha_3^2-2\varepsilon\alpha_2\alpha_3(\alpha_2+\alpha_3)=0, && (2.80a)\\
&(q_{n,l}-q_{n+1,l})(p_{n,l+1}-p_{n+1,l+1})-\alpha_2(q_{n,l}+q_{n+1,l}+p_{n,l+1}+p_{n+1,l+1})\\
&\quad+\frac{\varepsilon\alpha_2}{2}(2q_{n,l}+2\alpha_3+\alpha_2)(2q_{n+1,l}+2\alpha_3+\alpha_2)+\frac{\varepsilon\alpha_2}{2}(2\alpha_3+\alpha_2)^2+(\alpha_2+\alpha_3)^2-\alpha_3^2\\
&\quad-2\varepsilon\alpha_2\alpha_3(\alpha_2+\alpha_3)=0. && (2.80b)
\end{aligned}
$$

The system (2.80) is equivalent to the original system (2.79). We can solve (2.80) with respect to $p_{n,l+1}$ and $p_{n+1,l+1}$

$$
p_{n,l+1}=\frac{\left\{\begin{gathered}(q_{n,l+1}-\alpha_3)q_{n+1,l}-(q_{n,l}-\alpha_3)q_{n+1,l+1}\\ -(\alpha_2+\alpha_3)(q_{n,l+1}-q_{n,l})-\varepsilon\alpha_3^2(q_{n,l+1}-q_{n,l})\\ +\varepsilon\left[{\alpha_3}^2+2\alpha_3(q_{n,l}+\alpha_2)-(q_{n,l+1}-q_{n,l})q_{n+1,l}\right]q_{n+1,l+1}\\ +\varepsilon(\alpha_2+q_{n,l+1})(\alpha_2+q_{n,l})q_{n+1,l+1}\\ -\varepsilon q_{n+1,l}\left[{\alpha_3}^2-2(q_{n,l+1}+\alpha_2)\alpha_3+(\alpha_2+q_{n,l+1})(\alpha_2+q_{n,l})\right]\end{gathered}\right\}}{q_{n+1,l+1}-q_{n,l+1}+q_{n,l}-q_{n+1,l}}, \tag{2.81a}
$$

$$
p_{n+1,l+1}=\frac{\left\{\begin{gathered}(q_{n+1,l}-\alpha_3)q_{n,l+1}+(\alpha_3-q_{n+1,l+1})q_{n,l}\\ -(\alpha_2+\alpha_3)(q_{n+1,l}-q_{n+1,l+1})+\varepsilon\alpha_3^2(q_{n+1,l+1}-q_{n+1,l})\\ +\varepsilon q_{n,l+1}\left[(q_{n+1,l+1}-q_{n+1,l})q_{n,l}-\alpha_3^2-2\alpha_3(q_{n+1,l}+\alpha_2)\right]\\ -\varepsilon q_{n,l+1}(\alpha_2+q_{n+1,l+1})(\alpha_2+q_{n+1,l})\\ +\varepsilon q_{n,l}\left[\alpha_3^2+2(\alpha_2+q_{n+1,l+1})\alpha_3+(\alpha_2+q_{n+1,l+1})(\alpha_2+q_{n+1,l})\right]\end{gathered}\right\}}{q_{n+1,l+1}-q_{n,l+1}+q_{n,l}-q_{n+1,l}}. \tag{2.81b}
$$

We see that the right-hand sides of (2.81) are functions only of $q_{n,l}$, $q_{n+1,l}$, $q_{n,l+1}$ and $q_{n+1,l+1}$ and well defined unless $q_{n,l}$ solves the discrete wave equation (2.70), which is therefore a *singular case*. Therefore at this point the procedure of solution bifurcates into two cases. We treat them separately.

Singular case: $q_{n,l}$ solves (2.70). Let us assume that the field $q_{n,l}$ satisfies the discrete wave equation in the form (2.70). Then as discussed in the introduction the discrete wave equation is a simplest example of Darboux integrable equation and its solution is given by the discrete d'Alembert formula

$$q_{n,l} = a_n + \zeta_l, \tag{2.82}$$

where both a_n and ζ_l are arbitrary functions of their argument. Substituting (2.82) into (2.80) we obtain the compatibility condition

$$\zeta_{l+1} - \zeta_l = 0, \tag{2.83}$$

i.e., $\zeta_l = \zeta_0 = \text{const}$ and the system (2.79) is now consistent. This yield us the first part of the solution of this case (2.75a). We are therefore left with one equation for $p_{n,l}$, e.g., (2.79a). Inserting (2.82) with $\zeta_l = \zeta_0$ in (2.79a) and solving with respect to $p_{n+1,l}$ we obtain

$$\begin{aligned} p_{n+1,l} = {} & \frac{a_{n+1} - a_n + \alpha_2}{a_{n+1} - a_n - \alpha_2} p_{n,l} + \frac{\alpha_2(\alpha_2 - a_n + 2\alpha_3 - 2\zeta_0 - a_{n+1})}{\alpha_2 + a_n - a_{n+1}} \\ & + \frac{\alpha_2 \varepsilon \left[\alpha_2^2 + (2\zeta_0 + a_{n+1} + 2\alpha_3 + a_n)\alpha_2 + 2(a_{n+1} + \alpha_0 + \alpha_3)(\alpha_3 + a_n + \zeta_0)\right]}{\alpha_2 + a_n - a_{n+1}}. \end{aligned}$$

We can introduce a new function b_n through discrete integration

$$\frac{a_{n+1} - a_n + \alpha_2}{a_{n+1} - a_n - \alpha_2} = \frac{b_{n+1}}{b_n},$$

which is just formula (2.76a). Then we have that $p_{n,l}$ must solve the equation

$$\begin{aligned} \frac{p_{n+1,l}}{b_{n+1}} = {} & \frac{p_{n,l}}{b_n} + \frac{(a_n + \zeta_0 - \alpha_3)b_{n+1} + b_n(\alpha_2 + \alpha_3 - \zeta_0 - a_n)}{b_n b_{n+1}} \\ & - \varepsilon \frac{\left[(\alpha_2 + \zeta_0 + a_n + \alpha_3)^2 b_{n+1} - b_n(\alpha_3 + a_n + \zeta_0)^2\right]}{b_n b_{n+1}}. \end{aligned} \tag{2.84}$$

The solution of equation (2.84) is given by

$$p_{n,l} = b_n(\beta_l + c_n),$$

where c_n is given by the discrete integration

$$\begin{aligned} c_{n+1} - c_n = {} & \frac{(a_n + \zeta_0 - \alpha_3)b_{n+1} + b_n(\alpha_2 + \alpha_3 - \zeta_0 - a_n)}{b_n b_{n+1}} \\ & - \varepsilon \frac{\left[(\alpha_2 + \zeta_0 + a_n + \alpha_3)^2 b_{n+1} - b_n(\alpha_3 + a_n + \zeta_0)^2\right]}{b_n b_{n+1}}, \end{aligned}$$

i.e., through formula (2.76b). This yields the solution of the ${}_tH_2^\varepsilon$ equation (1.2b) when $q_{n,l}$ satisfy the discrete wave equation (2.70).

General case: $q_{n,l}$ do not solve (2.70)**.** When $q_{n,l}$ is not a solution of the discrete wave equation (2.70) the equations (2.81) are well defined. Moreover we have that (2.81a) and (2.81b) must be compatible. To impose the compatibility condition we apply T_l^{-1} to (2.81b) and we impose to the obtained expression to be equal to (2.81a). We find that $q_{n,l}$ must solve the following equation

$$\begin{aligned} & \alpha_2(q_{n-1,l+1} - q_{n-1,l} - q_{n+1,l+1} + q_{n+1,l}) + (q_{n,l} - q_{n+1,l})q_{n-1,l+1} \\ & \quad - (q_{n,l} - q_{n-1,l})q_{n+1,l+1} + q_{n,l+1}(q_{n+1,l} - q_{n-1,l}) \\ & \quad + \varepsilon\alpha_2^2(q_{n+1,l+1} - q_{n+1,l} + q_{n-1,l} - q_{n-1,l+1}) \end{aligned}$$

$$
\begin{aligned}
&+\varepsilon\alpha_2(q_{n+1,l+1}-q_{n+1,l}+q_{n-1,l}-q_{n-1,l+1})(q_{n,l+1}+2\alpha_3+q_{n,l})\\
&+\varepsilon(q_{n+1,l}-q_{n-1,l})q_{n+1,l+1}q_{n-1,l+1}+\varepsilon(q_{n,l+1}-q_{n,l}+q_{n+1,l})q_{n-1,l}q_{n-1,l+1}\\
&+\varepsilon\big[2\alpha_3q_{n+1,l}-(q_{n,l+1}+2\alpha_3)q_{n,l}\big]q_{n-1,l+1}\\
&+\varepsilon\big[(q_{n,l+1}+2\alpha_3)q_{n,l}-(2\alpha_3+q_{n+1,l})q_{n-1,l}-(q_{n,l+1}-q_{n,l})q_{n+1,l}\big]q_{n+1,l+1}\\
&-\varepsilon(2\alpha_3+q_{n,l})(q_{n+1,l}-q_{n-1,l})q_{n,l+1}=0. \qquad (2.85)
\end{aligned}
$$

This partial difference equation for $q_{n,l}$ is not defined on the square quad graph of Fig. 1, but it is defined on the six-point lattice shown in Fig. 3.

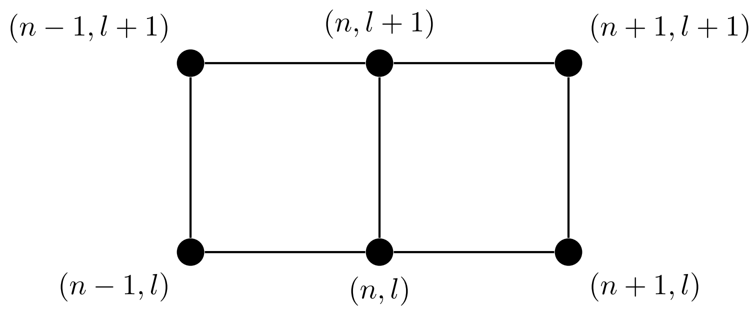

Figure 3. The six-point lattice.

In the general case we have proved that the ${}_tH_2^\varepsilon$ equation (1.2b) is equivalent to the system (2.79) which in turn is equivalent to the solution of equations (2.81a) and (2.85). However (2.81a) merely defines $p_{n,l+1}$ in terms of $q_{n,l}$ and its shifts. Therefore if we find the general solution of equation (2.85) the value of $p_{n,l}$ will follow. Applying T_l^{-1} to equation (2.80a) we obtain then the value of $p_{n,l}$ as displayed in (2.71b). To find the solution for $q_{n,l}$ solution we turn to the first integrals. Like in the case of the H^6 equations (1.3) we will find an expression for $q_{n,l}$ using the first integrals, and then we will insert it into (2.85) to reduce the number of arbitrary functions to the right one. From [26] we know that ${}_tH_2^\varepsilon$ equation (1.2b) possesses the following four-point, third-order integral in the n direction

$$
\begin{aligned}
W_1 = {}& F_m^{(+)}\frac{(u_{n+1,m}-u_{n-1,m})(u_{n+2,m}-u_{n,m})}{\varepsilon^2\alpha_2^4+4\varepsilon\alpha_2^3+\big[(8\alpha_3-2u_{n,m}-2u_{n+1,m})\varepsilon-1\big]\alpha_2^2+(u_{n,m}-u_{n+1,m})^2}\\
&-F_m^{(-)}\frac{(-u_{n+1,m}+u_{n-1,m})(u_{n,m}-u_{n+2,m})}{(-u_{n-1,m}+u_{n,m}+\alpha_2)(u_{n+1,m}+\alpha_2-u_{n+2,m})}. \qquad (2.86)
\end{aligned}
$$

We consider the equation $W_1=\xi_n$, where W_1 is given by (2.86), with $m=2l+1$

$$
\frac{(u_{n-1,2l+1}-u_{n+1,2l+1})(u_{n+2,2l+1}-u_{n,2l+1})}{(u_{n,2l+1}-u_{n-1,2l+1}+\alpha_2)(u_{n+1,2l+1}-u_{n+2,2l+1}+\alpha_2)}=\xi_n.
$$

Using the substitutions (2.69) we have

$$
\frac{(q_{n-1,l}-q_{n+1,l})\,(q_{n+2,l}-q_{n,l})}{(q_{n,l}-q_{n-1,l}+\alpha_2)\,(q_{n+1,l}-q_{n+2,l}+\alpha_2)}=\xi_n. \qquad (2.87)
$$

This equation contains only $q_{n,l}$ and its shifts. From equation (2.87) it is very simple to obtain a discrete Riccati equation. Indeed the transformation

$$
Q_{n,l}=\frac{q_{n,l}-q_{n-1,l}+\alpha_2}{q_{n+1,l}-q_{n-1,l}} \qquad (2.88)
$$

brings (2.87) into:

$$Q_{n+1,l} + \frac{1}{\xi_n Q_{n,l}} = 1 \tag{2.89}$$

which is a discrete Riccati equation. Let us assume a_n to be a particular solution of (2.89), then we express ξ_n as

$$\xi_n = \frac{1}{a_n(1-a_{n+1})}. \tag{2.90}$$

Using the standard linearization of the discrete Riccati equation

$$Q_{n,l} = a_n + \frac{1}{Z_{n,l}} \tag{2.91}$$

from (2.90) we obtain the following equation for $Z_{n,l}$

$$Z_{n+1,l} = \frac{a_n Z_{n,l} + 1}{1 - a_{n+1}}.$$

Introducing

$$a_n = \frac{b_{n-1}}{b_n + b_{n-1}} \tag{2.92}$$

we obtain

$$Z_{n+1,l} - \frac{b_{n-1}b_{n+1} + b_{n-1}b_n}{b_n b_{n+1} + b_{n-1}b_{n+1}} Z_{n,l} = \frac{b_{n-1}b_{n+1} + b_{n-1}b_n + b_n b_{n+1} + b_n^2}{b_n b_{n+1} + b_{n-1}b_{n+1}}. \tag{2.93}$$

If we assume that (2.93) can be written as a total difference, i.e.,

$$(T_n - \mathrm{Id})(d_n Z_{n,l} - c_n) = 0,$$

we obtain

$$b_n = c_{n+1} - c_n, \qquad d_n = \frac{(c_{n+1} - c_n)(c_n - c_{n-1})}{c_{n+1} - c_{n-1}}. \tag{2.94}$$

So b_n must be a total difference and therefore we can represent $Z_{n,l}$ as

$$Z_{n,l} = \frac{(c_{n+1} - c_{n-1})(c_n + \zeta_l)}{(c_{n+1} - c_n)(c_n - c_{n-1})}.$$

From (2.91) and (2.92) we obtain the form of $Q_{n,l}$

$$Q_{n,l} = \frac{(c_n - c_{n-1})(c_{n+1} + \zeta_l)}{(c_n + \zeta_l)(c_{n+1} - c_{n-1})}. \tag{2.95}$$

Introducing the value of $Q_{n,l}$ from (2.95) into (2.88) we obtain the following equation for $q_{n,l}$

$$\frac{q_{n+1,l} - q_{n-1,l}}{q_{n,l} - q_{n-1,l} + \alpha_2} = \frac{(c_n + \zeta_l)(c_{n+1} - c_{n-1})}{(c_n - c_{n-1})(c_{n+1} + \zeta_l)}.$$

Performing the transformation

$$R_{n,l} = (c_n + \zeta_l) q_{n,l} \tag{2.96}$$

we obtain the following second-order ordinary difference equation for the field $R_{n,l}$

$$R_{n+1,l} - \frac{c_{n+1}-c_{n-1}}{c_n-c_{n-1}}R_{n,l} + \frac{c_{n+1}-c_n}{c_n-c_{n-1}}R_{n-1,l} - \alpha_2(c_n+\zeta_l)\frac{c_{n+1}-c_{n-1}}{c_n-c_{n-1}} = 0. \tag{2.97}$$

Then we can represent the solutions of the equation (2.97) as

$$R_{n,l} = P_{n,l} + \zeta_l e_n + f_n, \tag{2.98}$$

where e_n and f_n are particular solutions of

$$e_{n+1} - \frac{c_{n+1}-c_{n-1}}{c_n-c_{n-1}}e_n + \frac{c_{n+1}-c_n}{c_n-c_{n-1}}e_{n-1} - \alpha_2\frac{c_{n+1}-c_{n-1}}{c_n-c_{n-1}} = 0, \tag{2.99a}$$

$$f_{n+1} - \frac{c_{n+1}-c_{n-1}}{c_n-c_{n-1}}f_n + \frac{c_{n+1}-c_n}{c_n-c_{n-1}}f_{n-1} - \alpha_2 c_n\frac{c_{n+1}-c_{n-1}}{c_n-c_{n-1}} = 0. \tag{2.99b}$$

$P_{n,l}$ will be then solve the following equation

$$P_{n+1,l} - \frac{c_{n+1}-c_{n-1}}{c_n-c_{n-1}}P_{n,l} + \frac{c_{n+1}-c_n}{c_n-c_{n-1}}P_{n-1,l} = 0. \tag{2.100}$$

The equations (2.99a) and (2.99b) are not independent. Indeed defining

$$A_n = \frac{e_n c_{n-1} - e_{n-1}c_n - f_n + f_{n-1}}{c_n - c_{n-1}}, \tag{2.101}$$

and using (2.99) it is possible to show that the function A_n lies in the kernel of the operator $T_n - \mathrm{Id}$. This implies that $A_n = A_0 = \text{const}$. We can without loss of generality assume the constant A_0 to be zero, since if we perform the transformation

$$e_n = \tilde{e}_n - A_0, \tag{2.102}$$

the equation (2.101) is mapped into

$$\frac{\tilde{e}_n c_{n-1} - \tilde{e}_{n-1}c_n - f_n + f_{n-1}}{c_n - c_{n-1}} = 0. \tag{2.103}$$

Furthermore since (2.99a) is invariant under the transformation (2.102) we can safely drop the tilde in (2.103) and assume that the functions e_n and f_n are solutions of the equations

$$e_{n+1} - \frac{c_{n+1}-c_{n-1}}{c_n-c_{n-1}}e_n + \frac{c_{n+1}-c_n}{c_n-c_{n-1}}e_{n-1} - \alpha_2\frac{c_{n+1}-c_{n-1}}{c_n-c_{n-1}} = 0, \tag{2.104a}$$

$$f_n - f_{n-1} = e_n c_{n-1} - c_n e_{n-1}. \tag{2.104b}$$

The system (2.104) is just gives the constraints expressed in formulas (2.72) and (2.73a).

Now we turn to the solution of the homogeneous equation (2.100). We can reduce (2.100) to a total difference using the potential substitution $T_{n,l} = P_{n,l} - P_{n-1,l}$

$$\frac{T_{n+1,l}}{c_{n+1}-c_n} - \frac{T_{n,l}}{c_n - c_{n-1}} = 0.$$

This clearly implies

$$\frac{P_{n,l} - P_{n-1,l}}{c_n - c_{n-1}} = \beta_l,$$

where β_l is an arbitrary function. The solution to this equation is given by[3]

$$P_{n,l} = (c_n + \zeta_l)\beta_l + \gamma_l, \tag{2.105}$$

where γ_l is an arbitrary function. Using (2.96), (2.98), (2.105) we obtain then the following expression for $q_{n,l}$

$$q_{n,l} = \beta_l + \frac{\gamma_l + \zeta_l e_n + f_n}{c_n + \zeta_l}, \tag{2.106}$$

where e_n and f_n are solutions of (2.104).

Equation (2.106) is formally the solution presented in (2.71a), but depends on three arbitrary functions in the l direction, namely ζ_l, β_l and γ_l. This means that there is a constraint between these functions, which can be recovered by plugging (2.106) into (2.85). At this point we have a second bifurcation, depending on the value of ε.

Case $\varepsilon \neq 0$. If $\varepsilon \neq 0$ inserting (2.106) into (2.85) and factorizing out the n dependent part away we are left with

$$\gamma_{l+1} - \gamma_l = -(\zeta_{l+1} - \zeta_l)\left(\alpha_2 + \beta_{l+1} + \beta_l + 2\alpha_3 - \frac{1}{\varepsilon}\right).$$

This equation tells us that the function γ_l can be expressed after a discrete integration in terms of the two arbitrary functions ζ_l and β_l. This is just the final constraint expressed in formula (2.73b). This yields the general solution of the ${}_tH_2^\varepsilon$ equation (1.2b).

Case $\varepsilon = 0$. If $\varepsilon = 0$ inserting (2.106) into (2.85) we obtain the compatibility condition $\zeta_{l+1} - \zeta_l = 0$, i.e., $\zeta_l = \zeta_0 = \text{const}$. It is easy to check that the obtained value of $q_{n,l}$ through formula (2.106) is consistent with the substitution of $\varepsilon = 0$ in (2.79). This means that in the case $\varepsilon = 0$ the value of $q_{n,l}$ is given by

$$q_{n,l} = \beta_l + \frac{\gamma_l + \zeta_0 e_n + f_n}{c_n + \zeta_0}, \tag{2.107}$$

where the functions e_n and f_n are defined implicitly and can be found by discrete integration from (2.104), i.e., from formula (2.74a). Since formula (2.71b) is not singular with respect to ε the value of $p_{n,l}$ can be recovered just by substituting $\varepsilon = 0$ and the form of $q_{n,l}$ found in (2.107). This yields equation (2.74b). This concludes the procedure of solution of the ${}_tH_2^\varepsilon$ equation (1.2b) in the case when $\varepsilon = 0$. ∎

Proposition 2.12. *The ${}_tH_3^\varepsilon$ equation (1.2c) is exactly solvable. If $\delta \neq 0$ and the field $q_{n,l}$ do not satisfy the equation*

$$q_{n+1,l+1}q_{n,l} = q_{n+1,l}q_{n,l+1} \tag{2.108}$$

then the solution of the ${}_tH_3^\varepsilon$ equation (1.2c) is given by

$$q_{n,l} = \gamma_l e_n \frac{f_n + \beta_l}{c_n + \zeta_l}, \tag{2.109a}$$

$$p_{n,l} = \frac{\left[\begin{array}{c} \alpha_2(q_{n+1,l} - q_{n+1,l-1})(\varepsilon^2 q_{n,l-1}q_{n,l} + \delta^2 {\alpha_3}^2) \\ + \delta^2\alpha_2^2\alpha_3^2(q_{n,l-1} - q_{n,l}) + \varepsilon^2 q_{n+1,l}q_{n+1,l-1}(q_{n,l-1} - q_{n,l}) \end{array}\right]}{(q_{n+1,l}q_{n,l-1} - q_{n+1,l-1}q_{n,l})\alpha_3\alpha_2}, \tag{2.109b}$$

[3] The arbitrary functions are taken in a convenient way.

where e_n, β_l and γ_l are arbitrary functions of their arguments and c_n is a solution of the equation

$$\frac{c_{n+1}-c_n}{c_n-c_{n-1}}=\frac{e_{n+1}-\alpha_2 e_n}{\alpha_2 e_n-e_{n-1}}, \tag{2.110}$$

while f_n and ζ_l are given by the discrete integrations

$$f_n-f_{n-1}=\frac{c_{n-1}-c_n}{e_n e_{n-1}}, \tag{2.111a}$$

$$\zeta_{l+1}-\zeta_l=\frac{\varepsilon^2}{\alpha_2\delta^2\alpha_3^2}\gamma_{l+1}\gamma_l(\beta_{l+1}-\beta_l). \tag{2.111b}$$

If $\varepsilon=0$, but the field $q_{n,l}$ do not satisfy the equation (A.93) then the solution of the ${}_tH_3^\varepsilon$ equation (1.2c) is given by

$$q_{n,l}=\gamma_l e_n\frac{f_n+\beta_0}{c_n+\zeta_l}, \tag{2.112a}$$

$$p_{n,l}=\frac{\varepsilon^2}{\alpha_2\alpha_3}\frac{\alpha_2(q_{n+1,l}-q_{n+1,l-1})q_{n,l-1}q_{n,l}+q_{n+1,l}q_{n+1,l-1}(q_{n,l-1}-q_{n,l})}{q_{n+1,l}q_{n,l-1}-q_{n+1,l-1}q_{n,l}}, \tag{2.112b}$$

where e_n, ζ_l and γ_l are arbitrary functions of their arguments β_0 is a constant and c_n is a solution of (2.110) and f_n is a solution of (2.111a). If the field $q_{n,l}$ satisfies the equation (2.108) regardless of the value of the parameter ε the solution of the ${}_tH_3^\varepsilon$ equation (1.2c) is given by

$$q_{n,l}=\zeta_0 a_n, \tag{2.113a}$$

$$p_{n,l}=b_n(\beta_l+c_n), \tag{2.113b}$$

where b_n and β_l are arbitrary functions of their arguments, ζ_0 is a constant and a_n and c_n are given by the discrete integration

$$\frac{\alpha_2 a_{n+1}-a_n}{a_{n+1}-\alpha_2 a_n}=\frac{b_{n+1}}{b_n}, \tag{2.114a}$$

$$c_{n+1}-c_n=\frac{\delta^2\alpha_3^2\alpha_2^2 b_n-b_{n+1}\left(\delta^2\alpha_3^2+\varepsilon^2 a_n^2\zeta_0^2\right)\alpha_2+\varepsilon^2 a_n^2\zeta_0^2 b_n}{b_n a_n\zeta_0\alpha_2\alpha_3 b_{n+1}}. \tag{2.114b}$$

Remark 2.13. We remark that the function c_n can be obtained from (2.110) as the result of *two* discrete integrations. Indeed defining

$$z_n=c_{n+1}-c_n, \tag{2.115}$$

and substituting in (2.110) we obtain that z_n must solve the equation

$$\frac{z_n}{z_{n-1}}=\frac{e_{n+1}-\alpha_2 e_n}{\alpha_2 e_n-e_{n-1}}. \tag{2.116}$$

Note that the right-hand side of (2.116) is not a total difference. So the function c_n can be obtained by integrating (2.116) and subsequently integrating (2.115). This provides the value of c_n. The obtained value can be plugged in (2.111a) to give f_n after discrete integration. This reasoning shows that we can obtain the non-arbitrary functions c_n and f_n as result of a finite number of discrete integrations. Therefore we can conclude that the solution of the ${}_tH_3^\varepsilon$ equation (1.2c) in the general case is given in terms of four discrete integrations. If $\varepsilon=0$ the general solution is given in terms of three discrete integration and finally in the singular case, when $q_{n,l}$ solves the equation (2.108), we need only two discrete integrations.

Proof. The proof of the solution of the ${}_tH_3^\varepsilon$ equation (1.2c) proceeds as the one outlined in Proposition 2.10. The interested reader can find the details in Appendix A. ∎

3 Conclusions

In this paper we presented a detailed procedure to construct the general solutions for all the H^4 and H^6 equations. As stated in the introduction, these general solutions were obtained in three different ways, but the common feature is that they can be found through some *linear or linearizable* (*discrete Riccati*) *equations*. This is the great advantage of the first integral approach over the direct one which was pursued in [21]. The Darboux integrability therefore yields extra information that it is useful to get the final result, i.e., the general solutions. Moreover linearization arises very naturally from first integrals also in the most complicated cases, whereas in the direct approach can be quite tricky, see, e.g., the examples in [21]. The linearization of the first integrals is another proof of the deep linear nature of the H^4 and H^6 equations. This result is even stronger than Darboux integrability alone, since *a priori* the first integrals do not need to define linearizable equations.

We also note that our procedure of construction of the general solution, based on the ideas from [18], is likely to be the discrete version of the procedure of linearization and solutions for continuous Darboux integrable equations presented in [46]. The preeminent rôle of the discrete Riccati equation in the solutions is reminiscent of the importance of the usual Riccati equation in the continuous case. Recall, e.g., that the first integrals of the Liouville equation [36]

$$u_{xt} = e^u,$$

which is the most famous Darboux integrable system, are Riccati equations. Many other examples of solutions presented in [46] use the reduction of higher-order differential equations to Riccati-like equation in order to obtain the solution, as we have done in the discrete case.

In this paper we constructed the general solutions of the trapezoidal H^4 and of the H^6 equations. Therefore we possess an almost complete theory about these equations ranging from the geometrical background to their analytic properties. For a discussion of the open problems in this field we refer to our previous paper [26].

A Procedure to find the general solution in the remaining cases

A.1 ${}_2D_2$ equation (1.3b)

From [26] we know that the first integrals of the ${}_2D_2$ equation (1.3b) can be different depending on the value of the parameter δ_1. For this reason we treat separately the various cases.

Case $\delta_1 \neq 0$. If $\delta_1 \neq 0$ the W_1 first integrals of the ${}_2D_2$ equation (1.3b) is given by [26]

$$\begin{aligned} W_1 = {} & F_n^{(+)} F_m^{(+)} \alpha \frac{\delta_2 + u_{n+1,m}}{\delta_2 + u_{n-1,m}} + F_n^{(+)} F_m^{(-)} \alpha \frac{[1-(1+\delta_2)\delta_1] u_{n,m} + u_{n+1,m}}{[1-(1+\delta_2)\delta_1] u_{n,m} + u_{n-1,m}} \\ & + F_n^{(-)} F_m^{(+)} \beta \frac{(u_{n+1,m} - u_{n-1,m})(u_{n,m} + \delta_2)}{1 + (u_{n,m} - 1)\delta_1} - F_n^{(-)} F_m^{(-)} \beta (u_{n+1,m} - u_{n-1,m}). \end{aligned}$$

As stated in the introduction, from the relation $W_1 = \xi_n$ this first integral defines a three-point, second-order ordinary difference equation in the n direction which depends parametrically on m. From this parametric dependence we find two different three-point non-autonomous ordinary difference equations corresponding to m even and m odd. We treat them separately.

Case $m = 2l$. If $m = 2l$ we have the following non-autonomous nonlinear ordinary difference equation

$$F_n^{(+)} \alpha \frac{\delta_2 + u_{n+1,2l}}{\delta_2 + u_{n-1,2l}} - F_n^{(-)} \beta \frac{(u_{n-1,2l} - u_{n+1,2l})(u_{n,2l} + \delta_2)}{1 + (u_{n,2l} - 1)\delta_1} = \xi_n. \tag{A.1}$$

Without loss of generality we set $\alpha = 1$ and $\beta = \delta_1$. Then making the transformation

$$u_{n,2l} = U_{n,2l} - \delta_2 \tag{A.2}$$

and putting

$$\delta = \frac{1 - \delta_1 - \delta_1\delta_2}{\delta_1} \tag{A.3}$$

equation (A.1) is mapped to

$$F_n^{(+)} \frac{U_{n+1,2l}}{U_{n-1,2l}} - F_n^{(-)} \frac{(U_{n-1,2l} - U_{n+1,2l})U_{n,2l}}{U_{n,2l} + \delta} = \xi_n. \tag{A.4}$$

From the definition (1.12a) applied to $U_{n,2l}$ instead of $u_{n,2l}$[4] we can separate again the even and the odd part in (A.4). We obtain the following system of two coupled *first-order* ordinary difference equations

$$W_{k,l} - \xi_{2k} W_{k-1,l} = 0, \tag{A.5a}$$

$$V_{k+1,l} - V_{k,l} = \xi_{2k+1} \left(1 + \frac{\delta}{W_{k,l}}\right). \tag{A.5b}$$

Putting $\xi_{2k} = a_k/a_{k-1}$ the solution to (A.5a) is given by

$$W_{k,l} = a_k \alpha_l. \tag{A.6}$$

Inserting the value of $W_{k,l}$ from (A.6) into (A.5b) we obtain

$$V_{k+1,l} - V_{k,l} = \xi_{2k+1} \left(1 + \frac{\delta}{a_k \alpha_l}\right). \tag{A.7}$$

If we define

$$\xi_{2k+1} = b_{k+1} - b_k, \qquad a_k = \frac{b_{k+1} - b_k}{c_{k+1} - c_k}, \tag{A.8}$$

then (A.7) becomes a total difference. So we obtain the following solutions for the $W_{k,l}$ and the $V_{k,l}$ fields

$$W_{k,l} = \alpha_l \frac{b_{k+1} - b_k}{c_{k+1} - c_k}, \qquad V_{k,l} = b_k + \beta_l + \delta \frac{c_k}{\alpha_l}.$$

Inverting the transformation (A.2) we obtain for the fields $w_{k,l}$ and $v_{k,l}$

$$w_{k,l} = \alpha_l \frac{b_{k+1} - b_k}{c_{k+1} - c_k} + \frac{1}{\delta_1} - 1 - \delta, \tag{A.9a}$$

$$v_{k,l} = b_k + \beta_l + \delta \frac{c_k}{\alpha_l} + \frac{1}{\delta_1} - 1 - \delta. \tag{A.9b}$$

Case $m = 2l + 1$. If $m = 2l + 1$ we have the following non-autonomous ordinary difference equation

$$F_n^{(+)} \frac{\delta\delta_1 u_{n,2l+1} + u_{n+1,2l+1}}{\delta\delta_1 u_{n,2l+1} + u_{n-1,2l+1}} + F_n^{(-)} \delta_1 (u_{n-1,2l+1} - u_{n+1,2l+1}) = \xi_n,$$

[4] We will denote the corresponding fields with capital letters.

where we already substituted δ as defined in (A.3). Using the standard transformation (1.12b) to get rid of the two-periodic factors we obtain

$$\frac{\delta\delta_1 y_{k,l} + z_{k,l}}{\delta\delta_1 y_{k,l} + z_{k-1,l}} = \xi_{2k}, \tag{A.10a}$$

$$\delta_1(y_{k,l} - y_{k+1,l}) = \xi_{2k+1}. \tag{A.10b}$$

Both equations in (A.10) are linear in $z_{k,l}$, $y_{k,l}$ and their shifts. As $\xi_{2k+1} = b_{k+1} - b_k$ we have that the solution of equation (A.10b) is given by

$$y_{k,l} = \gamma_l - \frac{b_k}{\delta_1}. \tag{A.11}$$

As $\xi_{2k} = a_k/a_{k-1}$ and $y_{k,l}$ is given by (A.11) we obtain

$$\frac{z_{k,l}}{a_k} - \frac{z_{k-1,l}}{a_{k-1}} = \left(\frac{1}{a_k} - \frac{1}{a_{k-1}}\right)(\delta b_k - \delta\delta_1\gamma_l).$$

Recalling the definition of a_k in (A.8) we represent $z_{k,l}$ as

$$z_{k,l} = \delta b_k + \frac{b_{k+1} - b_k}{c_{k+1} - c_k}(\zeta_l - \delta c_k) - \delta\delta_1\gamma_l. \tag{A.12}$$

Equations (A.9), (A.11), (A.12) provide the value of the four fields, but we have too many arbitrary functions in the l direction, namely α_l, β_l, γ_l and ζ_l. Inserting (A.9), (A.11), (A.12) into (1.3b) and separating the terms even and odd in n and m we obtain two independent equations

$$\delta_1\zeta_l + \alpha_l\delta_1^2\gamma_l - {\delta_1}^2\lambda\alpha_l + \beta_l\alpha_l\delta_1 + \delta\alpha_l\delta_1 - \alpha_l + \alpha_l\delta_1 = 0, \tag{A.13a}$$

$$\delta_1\zeta_l + \alpha_{l+1}\delta_1^2\gamma_l - \delta_1^2\lambda\alpha_{l+1} + \beta_{l+1}\alpha_{l+1}\delta_1 + \delta\alpha_{l+1}\delta_1 - \alpha_{l+1} + \alpha_{l+1}\delta_1 = 0, \tag{A.13b}$$

which allow us to reduce by two the number of independent functions in the l direction. Solving (A.13) with respect to γ_l and ζ_l we find

$$\gamma_l = -\frac{\beta_{l+1}\delta_1 - 1 - \delta_1^2\lambda + \delta\delta_1 + \delta_1}{\delta_1^2} - \frac{\alpha_l(\beta_{l+1} - \beta_l)}{(\alpha_{l+1} - \alpha_l)\delta_1}, \tag{A.14a}$$

$$\zeta_l = \alpha_l(\beta_{l+1} - \beta_l) + \frac{\alpha_l^2(\beta_{l+1} - \beta_l)}{\alpha_{l+1} - \alpha_l}. \tag{A.14b}$$

Inserting (A.14) into equations (A.9), (A.11), (A.12) we have the general solution (2.38) of the ${}_2D_2$ equation (1.3b) provided that $\delta_1 \neq 0$. Indeed the solution of the ${}_2D_2$ equation (1.3b) given by (2.38) is ill-defined if $\delta_1 = 0$. Therefore we now discuss this case separately.

Case $\boldsymbol{\delta_1 = 0}$. Following [26] we have the ${}_2D_2$ equation (1.3b) with $\delta_1 = 0$ possesses the following two-point, first-order first integral in the direction n

$$\begin{aligned} W_1^{(0,\delta_2)} &= F_n^{(+)}F_m^{(+)}\alpha(\delta_2 + u_{n+1,m})u_{n,m} - F_n^{(+)}F_m^{(-)}\alpha(u_{n+1,m} + u_{n,m}) \\ &\quad + F_n^{(-)}F_m^{(+)}\beta(\delta_2 + u_{n,m})u_{n+1,m} - F_n^{(-)}F_m^{(-)}\beta(u_{n+1,m} + u_{n,m}). \end{aligned} \tag{A.15}$$

To solve the ${}_2D_2$ equation (1.3b) with $\delta_1 = 0$ we use the first integral (A.15). Again we start separating the cases m even and odd in (A.15).

Case $\boldsymbol{m = 2l}$. If $m = 2l$ we obtain from the first integral (A.15)

$$F_n^{(+)}(\delta_2 + u_{n+1,2l})u_{n,2l} + F_n^{(-)}(\delta_2 + u_{n,2l})u_{n+1,2l} = \xi_n, \tag{A.16}$$

where we have chosen without loss of generality $\alpha = \beta = 1$. Applying the transformation (1.12a) equation (A.16) becomes the system

$$v_{k,l}(\delta_2 + w_{k,l}) = \xi_{2k}, \tag{A.17a}$$
$$v_{k+1,l}(\delta_2 + w_{k,l}) = \xi_{2k+1}. \tag{A.17b}$$

In this case the system (A.17) do not consist of purely difference equations. Indeed from (A.17a) we can derive immediately the value of the field $w_{k,l}$

$$w_{k,l} = -\delta_2 + \frac{\xi_{2k}}{v_{k,l}}. \tag{A.18}$$

Inserting (A.18) into (A.17b) we obtain that $v_{k,l}$ solves the equation

$$v_{k+1,l} - \frac{\xi_{2k+1}}{\xi_{2k}} v_{k,l} = 0. \tag{A.19}$$

Defining

$$\xi_{2k+1} = \frac{a_{k+1}}{a_k} \xi_{2k}, \tag{A.20}$$

we have that (A.19) becomes a total difference. So we have that the system (A.17) is solved by

$$v_{k,l} = a_k \alpha_l, \tag{A.21a}$$
$$w_{k,l} = -\delta_2 + \frac{\xi_{2k}}{a_k \alpha_l}. \tag{A.21b}$$

Case $m = 2l + 1$. If $m = 2l + 1$ we obtain from the first integral (A.15)

$$F_n^{(+)}(u_{n+1,2l+1} + u_{n,2l+1}) + F_n^{(-)}(u_{n+1,2l+1} + u_{n,2l+1}) = -\xi_n. \tag{A.22}$$

Applying the transformation (1.12b) equation (A.22) becomes the system

$$y_{k,l} + z_{k,l} = -\xi_{2k}, \tag{A.23a}$$
$$z_{k,l} + y_{k+1,l} = -\frac{a_{k+1}}{a_k} \xi_{2k}, \tag{A.23b}$$

where ξ_{2k+1} is given by (A.20). Equation (A.23a) is not a difference equation and can be solved to give

$$z_{k,l} = -\xi_{2k} - y_{k,l},$$

which inserted in (A.23b) gives

$$y_{k+1,l} - y_{k,l} = \left(1 - \frac{a_{k+1}}{a_k}\right) \xi_{2k}. \tag{A.24}$$

Defining

$$\xi_{2k} = -a_k \frac{b_{k+1} - b_k}{a_{k+1} - a_k}$$

equation (A.24) becomes a total difference. Therefore we can write the solution of the system (A.23) as

$$y_{k,l} = b_k + \beta_l, \tag{A.25a}$$
$$z_{k,l} = a_k \frac{b_{k+1} - b_k}{a_{k+1} - a_k} - b_k - \beta_l. \tag{A.25b}$$

In this case we have the right number of arbitrary functions in both directions. So the solution of the ${}_2D_2$ equation with $\delta_1 = 0$ is just given by combining (A.21) and (A.25), giving (2.39). It can be directly checked that (2.39) is the general solution by inserting it into the ${}_2D_2$ equation (1.3b) with $\delta_1 = 0$.

A.2 ${}_3D_2$ equation (1.3c)

From [26] we know that the ${}_3D_2$ equation (1.3c) is Darboux integrable, and that the form of the first integral depends on the value of the parameter δ. We will begin with the general case when $\delta_1 \neq 0$ and $\delta \neq 0$ and then consider the particular cases.

Case $\boldsymbol{\delta_1 \neq 0}$ and $\boldsymbol{\delta \neq 0}$. In this case we know that the W_1 first integrals of the ${}_3D_2$ equation (1.3c) is given by [26]

$$\begin{aligned} W_1 &= F_n^{(+)} F_m^{(+)} \alpha \frac{(u_{n-1,m}+\delta_2)[1+(u_{n+1,m}-1)\delta_1]}{(u_{n+1,m}+\delta_2)[1+(u_{n-1,m}-1)\delta_1]} \\ &\quad + F_n^{(+)} F_m^{(-)} \alpha \frac{u_{n,m}+(1-\delta_1-\delta_1\delta_2)u_{n-1,m}}{u_{n,m}+(1-\delta_1-\delta_1\delta_2)u_{n+1,m}} \\ &\quad + F_n^{(-)} F_m^{(+)} \beta (u_{n+1,m}-u_{n-1,m})(\delta_2+u_{n,m}) - F_n^{(-)} F_m^{(-)} \beta (u_{n+1,m}-u_{n-1,m}). \end{aligned} \tag{A.26}$$

As stated in the introduction, from the relation $W_1 = \xi_n$ this first integral defines a three-point, second-order ordinary difference equation in the n direction which depends parametrically on m. From this parametric dependence we find two different three-point non-autonomous ordinary difference equations corresponding to m even and m odd. We treat them separately.

Case $\boldsymbol{m = 2l}$. If $m = 2l$ we have the following non-autonomous nonlinear ordinary difference equation

$$F_n^{(+)} \frac{(u_{n-1,2l}+\delta_2)[1+(u_{n+1,2l}-1)\delta_1]}{(u_{n+1,2l}+\delta_2)[1+(u_{n-1,2l}-1)\delta_1]} + F_n^{(-)} (u_{n+1,2l}-u_{n-1,2l})(\delta_2+u_{n,2l}) = \xi_n, \tag{A.27}$$

where we have chosen without loss of generality $\alpha = \beta = 1$. We can apply the usual transformation (1.12a) in order to separate the even and odd part in (A.27)

$$\frac{1+(w_{k,l}-1)\delta_1}{w_{k,l}+\delta_2} = \xi_{2k} \frac{1+(w_{k-1,l}-1)\delta_1}{w_{k-1,l}+\delta_2}, \tag{A.28a}$$

$$v_{k+1,l} - v_{k,l} = \frac{\xi_{2k+1}}{(\delta_2 + w_{k,l})}. \tag{A.28b}$$

This system of equations is still nonlinear, but the equation (A.28a) is uncoupled from (A.28b). Moreover equation (A.28a) is a discrete Riccati equation which can be linearized through the Möbius transformation

$$w_{k,l} = -\delta_2 + \frac{1}{W_{k,l}}, \tag{A.29}$$

into

$$W_{k,l} - \xi_{2k} W_{k-1,l} = \frac{\delta_1}{\delta}(\xi_{2k}-1), \tag{A.30a}$$

$$v_{k+1,l} - v_{k,l} = \xi_{2k+1} W_{k,l}, \tag{A.30b}$$

where δ is given by equation (2.1). Putting $\xi_{2k} = a_k/a_{k-1}$ we have the following solution for (A.30a)

$$W_{k,l} = a_k \alpha_l - \frac{\delta_1}{\delta}. \tag{A.31}$$

Plugging (A.31) into equation (A.30b) and defining

$$\xi_{2k+1} = b_{k+1} - b_k, \qquad a_k = \frac{c_{k+1}-c_k}{b_{k+1}-b_k}, \tag{A.32}$$

we have that equation (A.30b) becomes a total difference. Then the solution of (A.30b) can be written as

$$v_{k,l} = -\frac{\delta_1}{\delta} b_k + c_k \alpha_l + \beta_l.$$

So using (A.29) we obtain the following solution for the original system (A.28):

$$w_{k,l} = \frac{\delta_1 - 1 + \delta}{\delta_1} + \frac{\delta(b_{k+1} - b_k)}{\delta\alpha_m(c_{k+1} - c_k) - (b_{k+1} - b_k)\delta_1}, \tag{A.33a}$$

$$v_{k,l} = -\frac{\delta_1}{\delta} b_k + c_k \alpha_l + \beta_l. \tag{A.33b}$$

Case $m = 2l + 1$. If $m = 2l + 1$ we have the following non-autonomous ordinary difference equation

$$F_n^{(+)} \frac{u_{n,2l+1} + \delta u_{n-1,2l+1}}{u_{n,2l+1} + \delta u_{n+1,2l+1}} - F_n^{(-)} (u_{n+1,2l+1} - u_{n-1,2l+1}) = \xi_n.$$

where without loss of generality $\alpha = \beta = 1$ and δ is given by (2.1). Solving with respect to $u_{n+1,2l+1}$ it is immediate to see that the resulting equation is linear. Then separating the even and the odd part using the transformation (1.12b) we obtain the following system of linear, first-order ordinary difference equations

$$z_{k,l} - \frac{1}{\xi_{2k}} z_{k-1,l} = \frac{1}{\delta}\left(1 - \frac{1}{\xi_{2k}}\right) y_{k,l}, \tag{A.34a}$$

$$y_{k+1} - y_k = -\xi_{2k+1}. \tag{A.34b}$$

As $\xi_{2k+1} = b_{k+1} - b_k$ we obtain the solution of equation (A.34b)

$$y_{k,l} = -b_k + \gamma_l. \tag{A.35}$$

Substituting $y_{k,l}$ given by (A.35) into equation (A.34a) being $\xi_{2k} = a_k/a_{k-1}$, we obtain

$$a_k z_{k,l} - a_{k-1} z_{k-1,l} = \frac{a_k - a_{k-1}}{\delta}(b_k - \gamma_l).$$

Then, in the usual way, we can represent the solution as

$$z_{k,l} = \frac{b_k - \gamma_l}{\delta} + \frac{b_{k+1} - b_k}{c_{k+1} - c_k}\left(\zeta_l - \frac{c_k}{\delta}\right), \tag{A.36}$$

where we have used the explicit definition of a_k given in (A.32). So we have the explicit expression for both fields $y_{k,l}$ and $z_{k,l}$.

Equations (A.33), (A.35), (A.36) provide the value of the four fields, but we have too many arbitrary functions in the l direction, namely α_l, β_l, γ_l and ζ_l. Inserting (A.33), (A.35), (A.36) into (1.3c) and separating the terms even and odd in n and m we obtain we obtain two equations

$$\zeta_l \delta^2 \alpha_l + (\beta_l - \delta_1\lambda)\delta - \delta_1\gamma_l = 0, \tag{A.37a}$$

$$\zeta_l \delta^2 \alpha_{l+1} + (\beta_{l+1} - \delta_1\lambda)\delta - \delta_1\gamma_l = 0, \tag{A.37b}$$

which allow us to reduce by two the number of independent functions in the l direction. Indeed solving (A.37) with respect to γ_l and ζ_l we find

$$\gamma_l = \frac{\delta}{\delta_1}\left(\beta_l - \lambda\delta_1 - \alpha_l \frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l}\right), \tag{A.38a}$$

$$\zeta_l = -\frac{1}{\delta}\frac{\beta_{l+1}-\beta_l}{\alpha_{l+1}-\alpha_l}. \tag{A.38b}$$

Inserting (A.38) into (A.33), (A.35), (A.36) we obtain the general solution (2.40) of the ${}_3D_2$ equation (1.3c) provided that $\delta_1 \neq 0$ and $\delta \neq 0$. It is easy to see that the solution (2.40) is ill-defined if $\delta_1 = 0$ and if $\delta = 0$. We will treat these two particular cases separately.

Case $\boldsymbol{\delta = 0}$. If $\delta = 0$ we have that δ_1 is given by equation (2.21). In this case the first integral (A.26) is singular since the coefficient of α goes to a constant. Following [26] we have that the ${}_3D_2$ equation with δ_1 given by (2.21) possesses the following first integral in the direction n

$$\begin{aligned}
W_1^{((1+\delta_2)^{-1},\delta_2)} = {}& F_n^{(+)}F_m^{(+)}\alpha\frac{u_{n+1,m}-u_{n-1,m}}{(\delta_2+u_{n+1,m})(\delta_2+u_{n-1,m})} + F_n^{(+)}F_m^{(-)}\alpha\frac{u_{n+1,m}-u_{n-1,m}}{(\delta_2+1)u_{n,m}}\\
&- F_n^{(-)}F_m^{(+)}\beta(u_{n-1,m}-u_{n+1,m})(\delta_2+u_{n,m})\\
&- F_n^{(-)}F_m^{(-)}\beta(u_{n+1,m}-u_{n-1,m}).
\end{aligned} \tag{A.39}$$

This first integral is a three-point, second-order first integral. As in the general case we consider separately the m even and odd cases.

Case $\boldsymbol{m = 2l}$. If $m = 2l$ then the first integral (A.39) becomes the following nonlinear three-point, second-order difference equation

$$Fpn\frac{u_{n+1,2l}-u_{n-1,2l}}{(\delta_2+u_{n+1,2l})(\delta_2+u_{n-1,2l})} - F_n^{(-)}(u_{n-1,2l}-u_{n+1,2l})(\delta_2+u_{n,2l}) = \xi_n,$$

where without loss of generality $\alpha = \beta = 1$. If we separate the even and the odd part using the general transformation given by (1.12a) we obtain the system

$$\frac{w_{k,l}-w_{k-1,l}}{(w_{k,l}+\delta_2)(w_{k-1,l}+\delta_2)} = \xi_{2k}, \tag{A.40a}$$

$$(v_{k+1,l}-v_{k,l})(w_{k,l}+\delta_2) = \xi_{2k+1}. \tag{A.40b}$$

This is a system of first-order nonlinear difference equations. However (A.40a) is uncoupled from (A.28b), and it is a discrete Riccati equation which can be linearized through the Möbius transformation (A.29). This linearize the system (A.40) to

$$W_{k,l} - W_{k-1,l} = \xi_{2k}, \tag{A.41a}$$

$$v_{k+1,l} - v_{k,l} = \xi_{2k+1}W_{k,l}. \tag{A.41b}$$

Defining $\xi_{2k} = a_k - a_{k-1}$ equation (A.41a) is solved by

$$W_{k,l} = a_k + \beta_l. \tag{A.42}$$

Introducing (A.42) into equation (A.41b) we have

$$v_{k+1,l} - v_{k,l} = \xi_{2k+1}(a_k + \alpha_l). \tag{A.43}$$

Equation (A.43) becomes a total difference if

$$\xi_{2k+1} = b_{k+1} - b_k, \qquad a_k = \frac{c_{k+1}-c_k}{b_{k+1}-b_k}. \tag{A.44}$$

This yields the following solution of the system (A.40)

$$v_{k,l} = c_k + b_k\alpha_l + \beta_l, \tag{A.45a}$$

$$w_{k,l} = -\delta_2 + \frac{b_{k+1} - b_k}{c_{k+1} - c_k + \alpha_l (b_{k+1} - b_k)}. \tag{A.45b}$$

Case $\boldsymbol{m = 2l + 1}$. If $m = 2l + 1$ the first integral (A.39) becomes the following nonlinear, three-point, second-order difference equation

$$F_n^{(+)} \frac{u_{n+1,2l+1} - u_{n-1,2l+1}}{(\delta_2 + 1) u_{n,2l+1}} - F_n^{(-)} (u_{n+1,2l+1} - u_{n-1,2l+1}) = \xi_n, \tag{A.46}$$

where without loss of generality $\alpha = \beta = 1$. As usual we can separate the even and odd part in n using the transformation (1.12b). This transformation brings equation (A.46) into the following linear system

$$z_{k,l} - z_{k-1,l} = (\delta_2 + 1) y_{k,l} \left(\frac{c_k - c_{k-1}}{b_k - b_{k-1}} - \frac{c_{k+1} - c_k}{b_{k+1} - b_k} \right), \tag{A.47a}$$

$$y_{k+1,l} - y_{k,l} = -b_{k+1} + b_k, \tag{A.47b}$$

where we used (A.44) and the definition $\xi_{2k+1} = a_{k+1} - a_k$. Equation (A.47b) is readily solved and gives

$$y_{k,l} = \gamma_l - b_k. \tag{A.48}$$

Inserting (A.48) into (A.47a) we obtain

$$z_{k,l} - z_{k-1,l} = (\delta_2 + 1)(\gamma_l - b_k) \left(\frac{c_k - c_{k-1}}{b_k - b_{k-1}} - \frac{c_{k+1} - c_k}{b_{k+1} - b_k} \right).$$

We can then write for $z_{k,l}$ the following expression

$$z_{k,l} = -(\delta_2 + 1) \left(\gamma_l \frac{c_{k+1} - c_k}{b_{k+1} - b_k} + d_k \right) + \zeta_l,$$

where d_k solves the equation

$$d_k - d_{k-1} = -b_k \left(\frac{c_k - c_{k-1}}{b_k - b_{k-1}} - \frac{c_{k+1} - c_k}{b_{k+1} - b_k} \right). \tag{A.49}$$

Equation (A.49) is a total difference with d_k given by

$$d_k = b_{k+1} \frac{c_{k+1} - c_k}{b_{k+1} - b_k} - c_{k+1}.$$

Therefore we have the following solution to the system (A.47)

$$y_{k,l} = \gamma_l - b_k, \tag{A.50a}$$

$$z_{k,l} = -(\delta_2 + 1) \left[(\gamma_l + b_{k+1}) \frac{c_{k+1} - c_k}{b_{k+1} - b_k} - c_{k+1} \right] + \zeta_l. \tag{A.50b}$$

Equations (A.45), (A.50) provide the value of the four fields, but we have too many arbitrary functions in the l direction, namely α_l, β_l, γ_l and ζ_l. Inserting (A.45), (A.50) into (1.3c) with δ_1 given by (2.21) and separating the terms even and odd in n and m we obtain we obtain two equations

$$\gamma_l (\delta_2 + 1) \alpha_l - \lambda + \zeta_l + \beta_l (\delta_2 + 1) = 0,$$
$$\gamma_l (\delta_2 + 1) \alpha_{l+1} - \lambda + \zeta_l + \beta_{l+1} (\delta_2 + 1) = 0.$$

Solving this compatibility condition with respect to γ_l and ζ_l we obtain

$$\gamma_l = -\frac{\beta_{l+1} - \beta_l}{\alpha_{l+1} - \alpha_l}, \tag{A.51a}$$

$$\zeta_l = (\delta_2 + 1)\frac{\beta_{l+1}\alpha_l - \alpha_{l+1}\beta_l}{\alpha_{l+1} - \alpha_l} + \lambda. \tag{A.51b}$$

Inserting then (A.51) into (A.45), (A.50) we obtain general solution (2.41) of the ${}_3D_2$ equation (1.3c) provided that $\delta = 0$.

Case $\boldsymbol{\delta_1 = 0}$. The first integral (A.26) is non-singular when $\delta_1 = 0$. Therefore the procedure of solution becomes different only when we arrive to the systems of ordinary difference equations (A.28) and (A.34). So we present the solution of the systems in this case.

Case $\boldsymbol{m = 2l}$. If $\delta_1 = 0$ the system (A.28) becomes

$$w_{k,l} + \delta_2 = \frac{w_{k-1,l} + \delta_2}{\xi_{2k}}, \tag{A.52a}$$

$$v_{k+1,l} - v_{k,l} = \frac{\xi_{2k+1}}{(\delta_2 + w_{k,l})}. \tag{A.52b}$$

The system (A.52) is nonlinear, but equation (A.52a) is uncoupled from equation (A.52a). Defining $\xi_{2k} = a_{k-1}/a_k$ equation (A.52a) is solved by

$$w_{k,l} = -\delta_2 + a_k\alpha_l. \tag{A.53}$$

Substituting $w_{k,l}$ given by (A.53) into equation (A.52b) we obtain

$$v_{k+1,l} - v_{k,l} = \frac{\xi_{2k+1}}{a_k\alpha_l}. \tag{A.54}$$

Defining

$$\xi_{2k+1} = -a_k(b_{k+1} - b_k), \tag{A.55}$$

we have that equation (A.54) is a total difference. Therefore we have the following solution of the system (A.52)

$$v_{k,l} = \beta_l + \frac{b_k}{\alpha_l}, \tag{A.56a}$$

$$w_{k,l} = -\delta_2 + a_k\alpha_l. \tag{A.56b}$$

Case $\boldsymbol{m = 2l + 1}$. If $\delta_1 = 0$ the system (2.13) becomes

$$z_{k,l} - \frac{a_k}{a_{k-1}}z_{k-1,l} = \left(\frac{a_k}{a_{k-1}} - 1\right)y_{k,l}, \tag{A.57a}$$

$$y_{k+1,l} - y_{k,l} = -a_k(b_{k+1} - b_k), \tag{A.57b}$$

where we used (A.55) and $\xi_{2k} = a_{k-1}/a_k$. The system is linear and equation (A.57b) is uncoupled from (A.57a). If we put

$$a_k = -\frac{c_{k+1} - c_k}{b_{k+1} - b_k},$$

then equation (A.57a) becomes a total difference whose solution is

$$y_{k,l} = c_k + \gamma_l. \tag{A.58}$$

Substituting $y_{k,l}$ given by (A.58) into equation (A.57a) we obtain

$$\frac{b_{k+1}-b_k}{c_{k+1}-c_k}z_{k,l}-\frac{b_k-b_{k-1}}{c_k-c_{k-1}}z_{k-1,l}=\left(\frac{b_k-b_{k-1}}{c_k-c_{k-1}}-\frac{b_{k+1}-b_k}{c_{k+1}-c_k}\right)(c_k+\gamma_l).$$

We can therefore represent the solution as

$$z_{k,l}=\frac{c_{k+1}-c_k}{b_{k+1}-b_k}(d_k+\zeta_l)-\gamma_l,$$

where d_k solves the equation

$$d_k-d_{k-1}=b_{k+1}-\frac{b_{k+1}-b_k}{c_{k+1}-c_k}c_{k+1}-b_{k-1}+\frac{b_k-b_{k-1}}{c_k-c_{k-1}}c_{k-1}. \tag{A.59}$$

Equation (A.59) is a total difference and d_k is given by

$$d_k=b_k-\frac{b_{k+1}-b_k}{c_{k+1}-c_k}c_k.$$

Therefore we have that the solution of the system (A.57) is given by

$$y_{k,l}=c_k+\gamma_l, \tag{A.60a}$$
$$z_{k,l}=\frac{c_{k+1}-c_k}{b_{k+1}-b_k}\zeta_l-\gamma_l+\frac{b_kc_{k+1}-c_kb_{k+1}}{b_{k+1}-b_k}. \tag{A.60b}$$

From equations (A.56), (A.60) we have the value of the four fields, but we have too many arbitrary functions in the l direction, namely α_l, β_l, γ_l and ζ_l. Inserting (A.56), (A.60) into (1.3c) with $\delta_1=0$ and separating the terms even and odd in n and m we obtain we obtain two equations

$$\beta_l\alpha_l-\zeta_l=0, \qquad \beta_{l+1}\alpha_{l+1}-\zeta_l=0.$$

We can solve this compatibility conditions with respect to β_l and ζ_l we obtain

$$\beta_l=\frac{\zeta_0}{\alpha_l}, \qquad \zeta_l=\zeta_0, \tag{A.61}$$

where ζ_0 is a constant.

Inserting then (A.61) into (A.56), (A.60) we obtain general solution (2.42) of the ${}_3D_2$ equation (1.3c) provided that $\delta_1=0$.

This discussion exhausts the possible cases. So for any value of the parameters we have the general solution of the ${}_3D_2$ equation (1.3c).

A.3 ${}_1D_4$ equation (1.3e)

To find the general solution we start from the ${}_1D_4$ equation (1.3e) itself. Applying the general transformation (1.12) we transform the ${}_1D_4$ into the following system

$$v_{k,l}z_{k,l}+w_{k,l}y_{k,l}+\delta_1w_{k,l}z_{k,l}+\delta_2y_{k,l}z_{k,l}+\delta_3=0, \tag{A.62a}$$
$$y_{k,l}w_{k,l+1}+z_{k,l}v_{k,l+1}+\delta_1z_{k,l}w_{k,l+1}+\delta_2y_{k,l}z_{k,l}+\delta_3=0, \tag{A.62b}$$
$$w_{k,l}y_{k+1,l}+v_{k+1,l}z_{k,l}+\delta_1w_{k,l}z_{k,l}+\delta_2z_{k,l}y_{k+1,l}+\delta_3=0, \tag{A.62c}$$
$$z_{k,l}v_{k+1,l+1}+y_{k+1,l}w_{k,l+1}+\delta_1z_{k,l}w_{k,l+1}+\delta_2z_{k,l}y_{k+1,l}+\delta_3=0. \tag{A.62d}$$

From the system (A.62) we have four different way for calculating $z_{k,l}$. This means that we have some compatibility conditions. Indeed from (A.62a) and (A.62c) we obtain the following equation for $v_{k+1,l}$

$$v_{k+1,l} = \frac{\delta_3 + w_{k,l}y_{k+1,l}}{\delta_3 + w_{k,l}y_{k,l}} v_{k,l} + \frac{(y_{k+1,l} - y_{k,l})\big(\delta_1 w_{k,l}^2 - \delta_2\delta_3\big)}{\delta_3 + w_{k,l}y_{k,l}}, \tag{A.63}$$

while from (A.62b) and (A.62d) we obtain the following equation for $v_{k+1,l+1}$

$$v_{k+1,l+1} = \frac{\delta_3 + y_{k+1,l}w_{k,l+1}}{\delta_3 + y_{k,l}w_{k,l+1}} v_{k,l+1} + \frac{(y_{k+1,l} - y_{k,l})\big(\delta_1 w_{k,l+1}^2 - \delta_2\delta_3\big)}{y_{k,l}w_{k,l+1} + \delta_3}. \tag{A.64}$$

Equations (A.63) and (A.64) give rise to a compatibility condition between $v_{k+1,l}$ and its shift in the l direction $v_{k+1,l+1}$ which is given by

$$\begin{bmatrix} (y_{k+1,l+1}y_{k,l} - y_{k+1,l}y_{k,l+1})w_{k,l+1} \\ + \delta_3(y_{k+1,l+1} + y_{k,l} - y_{k,l+1} - y_{k+1,l}) \end{bmatrix} \big(v_{k,l+1}w_{k,l+1} - \delta_2\delta_3 + \delta_1 w_{k,l+1}^2\big) = 0.$$

Discarding the trivial solution

$$v_{k,l} = -\delta_1 w_{k,l} + \frac{\delta_2\delta_3}{w_{k,l}}$$

we obtain the following value for the field $w_{k,l}$

$$w_{k,l} = \delta_3 \frac{y_{k+1,l-1} - y_{k+1,l} - y_{k,l-1} + y_{k,l}}{y_{k+1,l}y_{k,l-1} - y_{k+1,l-1}y_{k,l}}, \tag{A.65}$$

which makes (A.63) and (A.64) compatible. This gives us the first piece of the solution in (2.64b). Then we have to solve the following equation for $v_{k,l}$

$$\begin{aligned} v_{k+1,l} = {} & \frac{y_{k+1,l} - y_{k+1,l-1}}{y_{k,l} - y_{k,l-1}} v_{k,l} + \frac{\delta_1\delta_3(y_{k+1,l-1} - y_{k,l-1} - y_{k+1,l} + y_{k,l})^2}{(y_{k+1,l-1}y_{k,l} - y_{k+1,l}y_{k,l-1})(y_{k,l} - y_{k,l-1})} \\ & - \frac{\delta_2(y_{k+1,l-1}y_{k,l} - y_{k+1,l}y_{k,l-1})}{y_{k,l} - y_{k,l-1}}. \end{aligned}$$

Making the transformation

$$v_{k,l} = (y_{k,l} - y_{k,l-1})V_{k,l} + \frac{\delta_1\delta_3}{y_{k,l-1}} - \delta_2 y_{k,l-1} \tag{A.66}$$

we obtain that $V_{k,l}$ satisfies the difference equation

$$V_{k+1,l} = V_{k,l} + \frac{\delta_1\delta_3(y_{k,l-1} - y_{k+1,l-1})^2}{y_{k,l-1}y_{k+1,l-1}\big(y_{k+1,l-1}y_{k,l} - y_{k+1,l}y_{k,l-1}\big)}. \tag{A.67}$$

To go further we need to specify the form of the field $y_{k,l}$. This can be obtained from the Darboux integrability of the ${}_1D_4$ equation (1.3e). From [26] we know that the ${}_1D_4$ equation (1.3e) we have the following four-point, third-order W_1 integral

$$\begin{aligned} W_1 = {} & F_n^{(+)}F_m^{(+)}\alpha \frac{u_{n+1,m}^2\delta_1 + u_{n+1,m}u_{n+2,m} + u_{n-1,m}(u_{n,m} - u_{n+2,m}) - \delta_2\delta_3}{u_{n+1,m}(\delta_1 + u_{n,m}) - \delta_2\delta_3} \\ & + F_n^{(+)}F_m^{(-)}\alpha \frac{(u_{n,m} - u_{n+2,m} + \delta_1 u_{n+1,m})u_{n-1,m} + u_{n+1,m}u_{n+2,m}}{(u_{n,m} + \delta_1 u_{n-1,m})u_{n+1,m}} \end{aligned}$$

$$
\begin{aligned}
&+ F_n^{(-)} F_m^{(+)} \beta \frac{(u_{n+1,m}-u_{n-1,m})(u_{n+2,m}-u_{n,m})}{u_{n,m}^2\delta_1+u_{n+1,m}u_{n,m}-\delta_2\delta_3}\\
&+ F_n^{(-)} F_m^{(-)} \beta \frac{(u_{n+1,m}-u_{n-1,m})(u_{n+2,m}-u_{n,m})}{u_{n,m}(u_{n+2,m}\delta_1+u_{n+1,m})},
\end{aligned}
$$

This first integral defines as always the relation $W_1=\xi_n$ which is a third-order, four-point ordinary difference equation in the n direction depending parametrically on m. In particular when $m=2l+1$ we have the equation

$$
\begin{aligned}
&F_n^{(+)} \frac{(u_{n,2l+1}-u_{n+2,2l+1}+\delta_1 u_{n+1,2l+1})u_{n-1,2l+1}+u_{n+1,2l+1}u_{n+2,2l+1}}{(u_{n,2l+1}+\delta_1 u_{n-1,2l+1})u_{n+1,2l+1}}\\
&\quad + F_n^{(-)} \frac{(u_{n+1,2l+1}-u_{n-1,2l+1})(u_{n+2,2l+1}-u_{n,2l+1})}{u_{n,2l+1}(u_{n+2,2l+1}\delta_1+u_{n+1,2l+1})}=\xi_n.
\end{aligned}
\tag{A.68}
$$

where we have chosen without loss of generality $\alpha=\beta=1$. Using the transformation (1.12b) then (A.68) is converted into the system

$$
(y_{k,l}-y_{k+1,l}+\delta_1 z_{k,l})z_{k-1,l}+y_{k+1,l}z_{k,l}=\xi_{2k}(y_{k,l}+\delta_1 z_{k-1,l})z_{k,l}, \tag{A.69a}
$$

$$
(y_{k,l}-y_{k+1,l})(z_{k,l}-z_{k+1,l})=\xi_{2k+1}z_{k,l}(z_{k+1,l}\delta_1+y_{k+1,l}). \tag{A.69b}
$$

If we solve (A.69b) with respect to $z_{k+1,l}$ and then substitute into (A.69a) we obtain a *linear*, second-order ordinary difference equation for $y_{k,l}$

$$
\xi_{2k-1}y_{k+1,l}+(1-\xi_{2k}-\xi_{2k}\xi_{2k-1})y_{k,l}-(1-\xi_{2k})y_{k-1,l}=0. \tag{A.70}
$$

We can find the solution to this equation in a similar manner than in the case of the D_3 equation. First of all let us consider $Y_{k,l}=a_k y_{k,l}+b_k y_{k-1,l}$ such that $Y_{k+1,l}-Y_{k,l}$ is equal to the left-hand side of (A.70). To this end we define

$$
\xi_{2k}=-\frac{b_{k+1}-b_k-a_k}{a_{k+1}}, \qquad \xi_{2k-1}=\frac{a_{k+1}-a_k+b_{k+1}-b_k}{b_k}.
$$

Therefore $y_{k,l}$ must solve the following equation

$$
a_k y_{k,l}+b_k y_{k-1,l}=\alpha_l. \tag{A.71}
$$

Equation (A.71) is reduced to a total difference if we impose

$$
a_k=\frac{1}{c_k}\frac{1}{d_k-d_{k-1}}, \qquad b_k=-\frac{1}{c_{k-1}}\frac{1}{d_k-d_{k-1}}.
$$

Then the solution of (A.71) is then given by

$$
y_{k,l}=c_k(\alpha_l d_k+\beta_l). \tag{A.72}
$$

This is just equation (2.62a).

Inserting (A.72) into (A.67) we obtain

$$
\begin{aligned}
V_{k+1,l}=V_{k,l}&+\frac{\delta_1\delta_3\alpha_{l-1}}{\beta_{l-1}\alpha_l-\beta_l\alpha_{l-1}}\left[\frac{1}{(\alpha_{l-1}d_{k+1}+\beta_{l-1})c_{k+1}^2}-\frac{1}{(\alpha_{l-1}d_k+\beta_{l-1})c_k^2}\right]\\
&-\frac{\delta_1\delta_3}{\beta_{l-1}\alpha_l-\beta_l\alpha_{l-1}}\frac{(c_k-c_{k+1})^2}{c_k^2c_{k+1}^2(d_{k+1}-d_k)}.
\end{aligned}
$$

This means that the solution of $V_{k,l}$ can be represented as

$$V_{k,l} = \gamma_l + \frac{\delta_1\delta_3}{\beta_{l-1}\alpha_l - \beta_l\alpha_{l-1}}\left[\frac{\alpha_{l-1}}{c_k^2(\alpha_{l-1}d_k + \beta_{l-1})} + e_k\right], \tag{A.73}$$

where e_k is defined by the discrete integration

$$e_{k+1} - e_k = -\frac{(c_k - c_{k+1})^2}{c_k^2 c_{k+1}^2 (d_{k+1} - d_k)},$$

which is just equation (2.63a). Inserting the value of $V_{k,l}$ from (A.73) and the value of $y_{k,l}$ (A.72) into equation (A.66) we obtain equation (2.62b). From the obtained value of $v_{k,l}$ we can compute $w_{k,l}$ using (A.65). So finally we can compute $z_{k,l}$ from the original system (A.62), and we obtain a *single* compatibility condition given by

$$(\beta_l\alpha_{l+1} - \beta_{l+1}\alpha_l)\,\gamma_{l+1} - (\beta_{l-1}\alpha_l - \beta_l\alpha_{l-1})\,\gamma_l = (\beta_{l-1}\alpha_l - \beta_l\alpha_{l-1})\,\delta_2,$$

which is just equation (2.63b). Since now the system (A.62) is compatible we can use any of its equations to compute $z_{k,l}$. E.g., using equation (A.62a) and the value of $w_{k,l}$ from equation (A.65) we obtain equation (2.64b). This concludes the procedure of solution of the ${}_1D_4$ equation (1.3e).

A.4 ${}_2D_4$ equation (1.3f)

To find the general solution of the ${}_2D_4$ equation (1.3f) we start from the equation itself. Applying the general transformation (1.12) the ${}_2D_4$ equation we obtain the following system

$$v_{k,l}w_{k,l} + \delta_2 w_{k,l}y_{k,l} + \delta_1 w_{k,l}z_{k,l} + y_{k,l}z_{k,l} + \delta_3 = 0, \tag{A.74a}$$

$$v_{k,l+1}w_{k,l+1} + \delta_2 y_{k,l}w_{k,l+1} + \delta_1 z_{k,l}w_{k,l+1} + y_{k,l}z_{k,l} + \delta_3 = 0, \tag{A.74b}$$

$$w_{k,l}v_{k+1,l} + \delta_2 w_{k,l}y_{k+1,l} + \delta_1 w_{k,l}z_{k,l} + z_{k,l}y_{k+1,l} + \delta_3 = 0, \tag{A.74c}$$

$$w_{k,l+1}v_{k+1,l+1} + \delta_2 y_{k+1,l}w_{k,l+1} + \delta_1 z_{k,l}w_{k,l+1} + z_{k,l}y_{k+1,l} + \delta_3 = 0. \tag{A.74d}$$

From the system (A.74) we have four different ways to compute $z_{k,l}$. This means that we have some compatibility conditions. Indeed from (A.74a) and (A.74c) we obtain the following equation for $v_{k+1,l}$

$$v_{k+1,l} = \frac{\delta_1 w_{k,l} + y_{k+1,l}}{\delta_1 w_{k,l} + y_{k,l}} v_{k,l} - \frac{(y_{k+1,l} - y_{k,l})(\delta_2 w_{k,l}^2 \delta_1 - \delta_3)}{(\delta_1 w_{k,l} + y_{k,l})w_{k,l}}, \tag{A.75}$$

while from (A.74b) and (A.74d) we obtain the following equation for $v_{k+1,l+1}$

$$v_{k+1,l+1} = \frac{\delta_1 w_{k,l+1} + y_{k+1,l}}{\delta_1 w_{k,l+1} + y_{k,l}} v_{k,l+1} - \frac{(y_{k+1,l} - y_{k,l})(\delta_2 w_{k,l+1}^2 \delta_1 - \delta_3)}{(\delta_1 w_{k,l+1} + y_{k,l})w_{k,l+1}}. \tag{A.76}$$

Equations (A.75) and (A.76) give rise to a compatibility condition between $v_{k+1,l}$ and its shift in the l direction $v_{k+1,l+1}$ which is given by

$$\begin{bmatrix} y_{k+1,l+1}y_{k,l} + \delta_1(y_{k,l}w_{k,l+1} + y_{k+1,l+1}w_{k,l+1}) \\ - y_{k+1,l}y_{k,l+1} - \delta_1(y_{k+1,l}w_{k,l+1} - y_{k,l+1}w_{k,l+1}) \end{bmatrix} \left(v_{k,l+1}w_{k,l+1} + \delta_3 - \delta_1\delta_2 w_{k,l+1}^2\right) = 0.$$

Discarding the trivial solution

$$v_{k,l} = \delta_1\delta_2 w_{k,l} - \frac{\delta_3}{w_{k,l}}$$

we obtain the following value for the field $w_{k,l}$

$$w_{k,l} = \frac{1}{\delta_1} \frac{y_{k+1,l-1} y_{k,l} - y_{k+1,l} y_{k,l-1}}{y_{k,l-1} + y_{k+1,l} - y_{k,l} - y_{k+1,l-1}}, \tag{A.77}$$

which makes (A.75) and (A.76) compatible. This gives us the first part of our solution, as displayed in equation (2.68b). Inserting (A.77) into (A.75) we are left to solve the following equation for $v_{k,l}$

$$\begin{aligned} v_{k+1,l} = {} & \frac{y_{k+1,l} - y_{k+1,l-1}}{y_{k,l} - y_{k,l-1}} v_{k,l} - \frac{\delta_1 \delta_3 (y_{k,l-1} - y_{k+1,l-1})^2 (y_{k,l} - y_{k,l-1})}{(y_{k+1,l} y_{k,l-1} - y_{k+1,l-1} y_{k,l}) y_{k,l-1}^2} \\ & + \frac{y_{k,l-1}^2 y_{k+1,l} \delta_2 - \delta_1 \delta_3 y_{k+1,l} + \delta_1 \delta_3 y_{k+1,l-1} - \delta_2 y_{k+1,l-1} y_{k,l-1}^2}{(y_{k,l} - y_{k,l-1}) y_{k,l-1}} \\ & - \frac{\delta_2 y_{k+1,l-1} y_{k,l-1}^2 + \delta_1 \delta_3 y_{k+1,l-1} - 2\delta_3 \delta_1 y_{k,l-1}}{y_{k,l-1}^2}. \end{aligned}$$

Making the transformation

$$v_{k,l} = (y_{k,l} - y_{k,l-1}) V_{k,l} + \frac{\delta_1 \delta_3}{y_{k,l}} - \delta_2 y_{k,l} \tag{A.78}$$

we obtain that $V_{k,l}$ satisfies the difference equation

$$V_{k+1,l} = V_{k,l} - \frac{\delta_1 \delta_3 (y_{k,l} - y_{k+1,l})^2}{y_{k,l} y_{k+1,l} (y_{k+1,l-1} y_{k,l} - y_{k+1,l} y_{k,l-1})}. \tag{A.79}$$

To go further we need to specify the form of the field $y_{k,l}$. This can be obtained from the Darboux integrability of the ${}_2D_4$ equation (1.3f). From [26] we know that in the case of the ${}_2D_4$ equation (1.3f) we have the following four-point, third-order W_1 first integral

$$\begin{aligned} W_1 = {} & F_n^{(+)} F_m^{(+)} \alpha \frac{\begin{bmatrix} (u_{n,m} - u_{n+2,m} - \delta_1 \delta_2 u_{n-1,m}) u_{n+1,m}^2 \\ + u_{n+1,m} u_{n+2,m} u_{n-1,m} + \delta_3 u_{n-1,m} \end{bmatrix}}{(\delta_2 u_{n+1,m}^2 \delta_1 - \delta_3 - u_{n,m} u_{n+1,m}) u_{n-1,m}} \\ & - F_n^{(+)} F_m^{(-)} \alpha \frac{u_{n+2,m} u_{n-1,m} + (-u_{n+2,m} + u_{n,m}) u_{n+1,m} + \delta_3}{u_{n-1,m} u_{n,m} + \delta_3} \\ & - F_n^{(-)} F_m^{(+)} \beta \frac{(u_{n+1,m} - u_{n-1,m})(u_{n+2,m} - u_{n,m}) u_{n,m}}{u_{n+2,m} (\delta_2 \delta_1 {u_{n,m}}^2 - u_{n,m} u_{n+1,m} - \delta_3)} \\ & + F_n^{(-)} F_m^{(-)} \beta \frac{(u_{n+1,m} - u_{n-1,m})(u_{n+2,m} - u_{n,m})}{u_{n+1,m} u_{n+2,m} + \delta_3}. \end{aligned}$$

This first integral implies the relation $W_1 = \xi_n$ which is a third-order, four-point ordinary difference equation in the n direction depending parametrically on m. In particular if we choose the case when $m = 2l + 1$ we have the equation

$$\begin{aligned} & -F_n^{(+)} \frac{u_{n+2,2l+1} u_{n-1,2l+1} - (u_{n+2,2l+1} - u_{n,2l+1}) u_{n+1,2l+1} + \delta_3}{u_{n-1,2l+1} u_{n,2l+1} + \delta_3} \\ & \quad + F_n^{(-)} \frac{(u_{n+1,2l+1} - u_{n-1,2l+1})(u_{n+2,2l+1} - u_{n,2l+1})}{u_{n+1,2l+1} u_{n+2,2l+1} + \delta_3} = \xi_n. \end{aligned} \tag{A.80}$$

where we have chosen without loss of generality $\alpha = \beta = 1$. Using the transformation (1.12b) then equation (A.80) is converted into the system

$$(y_{k+1,l} - y_{k,l}) z_{k,l} - y_{k+1,l} z_{k+1,l} - \delta_3 = \xi_{2k} (z_{k+1,l} y_{k,l} + \delta_3), \tag{A.81a}$$

$$(y_{k+1,l} - y_{k,l})(z_{k+1,l} - z_{k,l}) = \xi_{2k+1}(y_{k+1,l}z_{k+1,l} + \delta_3). \tag{A.81b}$$

If we solve (A.81b) with respect to $z_{k+1,l}$ and substitute into (A.81a) we obtain a *linear*, second-order ordinary difference equation for $y_{k,l}$

$$\xi_{2k-1}y_{k+1,l} - (1 + \xi_{2k} - \xi_{2k}\xi_{2k-1})y_{k,l} + (1 + \xi_{2k})y_{k-1,l} = 0. \tag{A.82}$$

We can find the solution of equation (A.82) exploiting the arbitrariness of the functions ξ_{2k} and ξ_{2k+1} as in the case of the ${}_1D_4$ equation (1.3e). Let us introduce the field $Y_{k,l} = a_k y_{k,l} + b_k y_{k-1,l}$ and assume that $Y_{k+1,l} - Y_{k,l}$ equals the left-hand side of (A.82). Then we must have

$$\xi_{2k} = \frac{b_{k+1} - b_k + -a_k}{a_{k+1}}, \qquad \xi_{2k-1} = -\frac{b_{k+1} - b_k + a_{k+1} - a_k}{b_k}.$$

This implies that $y_{k,l}$ will solve the equation

$$a_k y_{k,l} + b_k y_{k-1,l} = \alpha_l. \tag{A.83}$$

If we define

$$a_k = \frac{1}{c_k}\frac{1}{d_k - d_{k-1}}, \qquad b_k = -\frac{1}{c_{k-1}}\frac{1}{d_k - d_{k-1}},$$

equation (A.83) is solved by

$$y_{k,l} = c_k(\alpha_l d_k + \beta_l). \tag{A.84}$$

This gives the second part of our solution displayed in equation (2.66a).

Inserting the value of $y_{k,l}$ from (A.84) into equation (A.79) we obtain

$$\begin{aligned} V_{k+1,l} = V_{k,l} &+ \frac{\delta_3\delta_1(c_{k+1} - c_k)^2}{(d_{k+1} - d_k)(\beta_l\alpha_{l-1} - \beta_{l-1}\alpha_l)c_k^2 c_{k+1}^2} \\ &- \frac{\alpha_l\delta_1\delta_3}{\beta_l\alpha_{l-1} - \beta_{l-1}\alpha_l}\left[\frac{1}{(\alpha_l d_{k+1} + \beta_l)c_{k+1}^2} - \frac{1}{(\alpha_l d_k + \beta_l)c_k^2}\right]. \end{aligned} \tag{A.85}$$

Therefore we can represent the solution to this equation as

$$V_{k,l} = \gamma_l - \frac{\alpha_l\delta_1\delta_3}{(\alpha_l d_k + \beta_l)c_k^2(\beta_l\alpha_{l-1} - \beta_{l-1}\alpha_l)} + \frac{\delta_3\delta_1 e_k}{\beta_l\alpha_{l-1} - \beta_{l-1}\alpha_l}, \tag{A.86}$$

where e_k is given by the discrete integration

$$e_{k+1} - e_k = \frac{(c_{k+1} - c_k)^2}{(d_{k+1} - d_k)c_k^2 c_{k+1}^2},$$

which is just equation (2.67a). Inserting the value of $V_{k,l}$ from equation (A.86) and the value of $y_{k,l}$ from equation (A.84) into equation (A.78) we obtain equation (2.66b) which is the third part of our solution. Using this value alongside with the value of $w_{k,l}$ from equation (A.65) we can compute $z_{k,l}$ from the original system (A.74). We obtain a *single* compatibility condition given by

$$(\beta_l\alpha_{l+1} - \beta_{l+1}\alpha_l)\gamma_{l+1} - (\beta_{l-1}\alpha_l - \beta_l\alpha_{l-1})\gamma_l = (\beta_l\alpha_{l+1} - \beta_{l+1}\alpha_l)\delta_2,$$

which is just equation (2.67b). This concludes the procedure of solution of the ${}_2D_4$ equation (1.3f).

A.5 ${}_tH_3^\varepsilon$ equation (1.2c)

To solve the ${}_tH_3^\varepsilon$ equation (1.2c) we start from the equation itself. We apply the transformation (2.69) to write down the ${}_tH_3^\varepsilon$ equation (1.2c) as the following system of two coupled equations

$$\begin{aligned}&\alpha_2(p_{n,l}q_{n+1,l}+p_{n+1,l}q_{n,l})-p_{n,l}q_{n,l}-p_{n+1,l}q_{n+1,l}\\&\qquad-\alpha_3\left(\alpha_2^2-1\right)\left(\delta^2+\varepsilon^2\frac{q_{n,l}q_{n+1,l}}{\alpha_3^2\alpha_2}\right)=0,\end{aligned}\tag{A.87a}$$

$$\begin{aligned}&\alpha_2(q_{n,l}p_{n+1,l+1}+q_{n+1,l}p_{n,l+1})-q_{n,l}p_{n,l+1}-q_{n+1,l}p_{n+1,l+1}\\&\qquad-\alpha_3\left(\alpha_2^2-1\right)\left(\delta^2+\varepsilon^2\frac{q_{n,l}q_{n+1,l}}{\alpha_3^2\alpha_2}\right)=0.\end{aligned}\tag{A.87b}$$

As in the case of the ${}_tH_2^\varepsilon$ equation (1.2b) we have that equation (A.87a) depends on $p_{n,l}$ and $p_{n+1,l}$ and that equation (A.87b) depends on $p_{n,l+1}$ and $p_{n+1,l+1}$. So we can apply the translation operator T_l to (A.87a) to obtain two equations in terms of $p_{n,l+1}$ and $p_{n+1,l+1}$

$$\begin{aligned}&\alpha_2(p_{n,l+1}q_{n+1,l+1}+p_{n+1,l+1}q_{n,l+1})-p_{n,l+1}q_{n,l+1}-p_{n+1,l+1}q_{n+1,l+1}\\&\qquad-\alpha_3\left(\alpha_2^2-1\right)\left(\delta^2+\varepsilon^2\frac{q_{n,l+1}q_{n+1,l+1}}{\alpha_3^2\alpha_2}\right)=0,\end{aligned}\tag{A.88a}$$

$$\begin{aligned}&\alpha_2(q_{n,l}p_{n+1,l+1}+q_{n+1,l}p_{n,l+1})-q_{n,l}p_{n,l+1}-q_{n+1,l}p_{n+1,l+1}\\&\qquad-\alpha_3\left(\alpha_2^2-1\right)\left(\delta^2+\varepsilon^2\frac{q_{n,l}q_{n+1,l}}{\alpha_3^2\alpha_2}\right)=0.\end{aligned}\tag{A.88b}$$

The system (A.88) is equivalent to the original system (A.87). Then since we can assume $\alpha_2,\alpha_3\neq 0$[5] we can solve (A.88) with respect to $p_{n,l+1}$ and $p_{n+1,l+1}$:

$$p_{n,l+1}=\frac{\left[\begin{gathered}\alpha_2(q_{n+1,l+1}-q_{n+1,l})\left(\varepsilon^2q_{n,l}q_{n,l+1}+\delta^2{\alpha_3}^2\right)\\+\delta^2\alpha_2^2\alpha_3^2(q_{n,l}-q_{n,l+1})+\varepsilon^2q_{n+1,l+1}q_{n+1,l}(q_{n,l}-q_{n,l+1})\end{gathered}\right]}{(q_{n+1,l+1}q_{n,l}-q_{n+1,l}q_{n,l+1})\alpha_3\alpha_2},\tag{A.89a}$$

$$p_{n+1,l+1}=\frac{\left[\begin{gathered}\alpha_2\left(\varepsilon^2q_{n+1,l+1}q_{n+1,l}+\delta^2{\alpha_3}^2\right)(q_{n,l}-q_{n,l+1})\\+\delta^2\alpha_2^2\alpha_3^2(q_{n+1,l+1}-q_{n+1,l})+\varepsilon^2q_{n,l}q_{n,l+1}(q_{n+1,l+1}-q_{n+1,l})\end{gathered}\right]}{(q_{n+1,l+1}q_{n,l}-q_{n+1,l}q_{n,l+1})\alpha_3\alpha_2}.\tag{A.89b}$$

We see that the right-hand sides of (A.89) are functions only of $q_{n,l}$, $q_{n+1,l}$, $q_{n,l+1}$ and $q_{n+1,l+1}$ and are well defined as long as $q_{n,l}$ is not a solution of equation (2.108), which is therefore a *singular case*. Therefore at this point the procedure of solution bifurcates into two cases. We treat them separately.

Singular case: $q_{n,l}$ solve (2.108). If $q_{n,l}$ solves equation (2.108) we first solve this equation with respect to $q_{n,l}$ and then use the system (A.87) to specify $p_{n,l}$. Indeed equation (2.108) is a trivial Darboux integrable equation, since it possesses the following two-point, first-order first integrals

$$W_1=\frac{q_{n+1,l}}{q_{n,l}},\tag{A.90a}$$

$$W_2=\frac{q_{n,l+1}}{q_{n,l}}.\tag{A.90b}$$

As remarked in the introduction the existence of a two-point, first-order first integral means that the equation is itself a first integral. Therefore the equation (2.108) can be alternatively

[5] If $\alpha_2=0$ or $\alpha_3=0$ in (A.87) we have that the system becomes trivially equivalent to $q_{n,l}=0$ and $p_{n,l}$ is left unspecified. Therefore we can discard this trivial case.

written as $(T_l - \mathrm{Id})\, W_1$ or $(T_n - \mathrm{Id})\, W_2$ with W_1 and W_2 given by (A.90). From (A.90a) we obtain

$$q_{n+1,l} = \xi_n q_{n,l} \tag{A.91}$$

being ξ_n an arbitrary function of its argument. Equation (A.91) is casted into total difference form by defining $\xi_n = a_{n+1}/a_n$, with α_n a new arbitrary function of its argument. Then we obtain that the general solution of equation (2.108) is

$$q_{n,l} = a_n \zeta_l, \tag{A.92}$$

where ζ_l is an arbitrary function of its argument.

Let us remark that equation (2.108) is the *logarithmic discrete wave equation*, since it can be mapped into the discrete wave equation (2.70) exponentiating (2.70) and then taking $q_{n,l} \to e^{q_{n,l}}$, and it is a discretization of the hyperbolic partial differential equation

$$u u_{xt} - u_x u_t = 0,$$

which is obtained from the wave equation $v_{xt} = 0$ using the transformation $v = \log u$. This fact is worth to note since being the transformation connecting (2.70) and (2.108) not bi-rational, integrability properties, in this case linearization and Darboux integrability, are not *a priori* preserved [19].

Substituting (A.92) into (A.88) we obtain the compatibility condition

$$\zeta_{l+1} - \zeta_l = 0, \tag{A.93}$$

i.e., $\zeta_l = \zeta_0 = \text{const}$ and the system (A.87) is now consistent. With this we find the first piece of the general solution in this case given by (2.113a). Therefore we are left with one equation for $p_{n,l}$, e.g., (A.87a). Therefore inserting (A.92) with $\zeta_l = \zeta_0$ in (A.87a) and solving with respect to $p_{n+1,l}$ we obtain

$$p_{n+1,l} - \frac{\alpha_2 a_{n+1} - a_n}{a_{n+1} - \alpha_2 a_n} p_{n,l} = \left(\alpha_2^2 - 1\right) \frac{\delta^2 \alpha_3^2 \alpha_2 + \varepsilon^2 a_n \zeta_0^2 a_{n+1}}{\alpha_3 \alpha_2 \zeta_0 (\alpha_2 a_n - a_{n+1})}.$$

Defining

$$\frac{\alpha_2 a_{n+1} - a_n}{a_{n+1} - \alpha_2 a_n} = \frac{b_{n+1}}{b_n}, \tag{A.94}$$

we have that $p_{n,l}$ solves the equation

$$\frac{p_{n+1,l}}{b_{n+1}} - \frac{p_{n,l}}{b_n} = \frac{\delta^2 \alpha_3^2 \alpha_2^2 b_n - b_{n+1}\left(\delta^2 \alpha_3^2 + \varepsilon^2 a_n^2 \zeta_0^2\right)\alpha_2 + \varepsilon^2 a_n^2 \zeta_0^2 b_n}{b_n a_n \zeta_0 \alpha_2 \alpha_3 b_{n+1}}. \tag{A.95}$$

Note that b_n in (A.94) is given in terms of a_n and a_{n+1} through discrete integration and it is the constraint given in equation (2.114a). Equation (A.95) is solved by

$$p_{n,l} = b_n(\beta_l + c_n),$$

where c_n is given by the discrete integration

$$c_{n+1} = c_n + \frac{\delta^2 \alpha_3^2 \alpha_2^2 b_n - b_{n+1}\left(\delta^2 \alpha_3^2 + \varepsilon^2 a_n^2 \zeta_0^2\right)\alpha_2 + \varepsilon^2 a_n^2 \zeta_0^2 b_n}{b_n a_n \zeta_0 \alpha_2 \alpha_3 b_{n+1}},$$

i.e., as in equation (2.114b). This yields the solution of the ${}_tH_3^\varepsilon$ equation (1.2c) when $q_{n,l}$ satisfy equation (2.108).

General case: $q_{n,l}$ do not solve (2.108). If the field $q_{n,l}$ do not solve (2.108) we have that we can define $p_{n,l+1}$ and $p_{n+1,l+1}$ as in (A.89a) and (A.89b) respectively. Furthermore these two equations must be compatible. The compatibility condition is obtained applying T_l^{-1} to (A.89b) and imposing to the obtained expression to be equal to (A.89a). We then find that $q_{n,l}$ must solve the following equation

$$\begin{aligned}\delta^2\alpha_2^2\alpha_3^2&\big[q_{n+1,l+1}q_{n,l} - q_{n,l}q_{n-1,l+1} + q_{n,l+1}(q_{n-1,l} - q_{n+1,l})\big]\\ &- \alpha_2\big(\varepsilon^2 q_{n,l}q_{n,l+1} + \delta^2\alpha_3^2\big)\big(q_{n+1,l+1}q_{n-1,l} - q_{n-1,l+1}q_{n+1,l}\big)\\ &+ \varepsilon^2\big[q_{n,l}q_{n+1,l+1}(q_{n-1,l} - q_{n+1,l}) - q_{n,l+1}q_{n-1,l}q_{n+1,l}\big]q_{n-1,l+1}\\ &+ \varepsilon^2 q_{n+1,l+1}q_{n+1,l}q_{n,l+1}q_{n-1,l} = 0.\end{aligned} \tag{A.96}$$

As in the case of the ${}_tH_2^\varepsilon$ equation (1.2b) the partial difference equation for $q_{n,l}$ is not defined on the square quad graph of Fig. 1, but it is defined on the six-point lattice shown in Fig. 3. Moreover once equation (A.96) is solved we can use indifferently (A.89a) or (A.89b) to obtain the value of the field $p_{n,l}$ since these two merely defines $p_{n,l+1}$ in terms of $q_{n,l}$ and its shifts. Therefore if we find the general solution of (A.96) the value of $p_{n,l}$ will follow. E.g., if we solve (A.96) applying T_l^{-1} to (A.89a) we will obtain (2.109b) which is then the first part of the general solution. To find the solution of equation (A.96) we turn to the first integrals. Like in the case of the ${}_tH_2^\varepsilon$ equation (1.2b) we will find an expression for $q_{n,l}$ using the first integrals, and then we will insert it into (A.96) to reduce the number of arbitrary functions to the right one. From [26] we know that the ${}_tH_3^\varepsilon$ equation (1.2c) possesses a four-point, third-order integral in the n direction

$$\begin{aligned}W_1 = F_m^{(+)}&\frac{(u_{n+1,m} - u_{n-1,m})(u_{n+2,m} - u_{n,m})}{\alpha_2^4\varepsilon^2\delta^2 - \alpha_2^3 u_{n+1,m}u_{n,m} + \alpha_2^2\big(u_{n,m}^2 + u_{n+1,m}^2 - 2\varepsilon^2\delta^2\big) - \alpha_2 u_{n,m}u_{n+1,m} + \varepsilon^2\delta^2}\\ &- F_m^{(-)}\frac{(u_{n+1,m} - u_{n-1,m})(u_{n+2,m} - u_{n,m})}{\alpha_2(-u_{n-1,m} + \alpha_2 u_{n,m})(-u_{n+2,m} + u_{n+1,m}\alpha_2)}.\end{aligned} \tag{A.97}$$

We consider the equation $W_1 = \xi_n/\alpha_2$[6], where W_1 is given by (A.97), with $m = 2l+1$

$$\frac{(u_{n+1,2l+1} - u_{n-1,2l+1})(u_{n+2,2l+1} - u_{n,2l+1})}{(\alpha_2 u_{n,2l+1} - u_{n-1,2l+1})(u_{n+2,2l+1} - \alpha_2 u_{n+1,2l+1})} = \xi_n.$$

Using the substitutions (2.69) we have

$$\frac{(q_{n+1,l} - q_{n-1,l})(q_{n+2,l} - q_{n,l})}{(\alpha_2 q_{n,l} - q_{n-1,l})(q_{n+2,l} - \alpha_2 q_{n+1,l})} = \xi_n. \tag{A.98}$$

This equation contains only $q_{n,l}$ and its shifts. By the transformation

$$Q_{n,l} = \frac{\alpha_2 q_{n,l} - q_{n-1,l}}{q_{n+1,l} - q_{n-1,l}} \tag{A.99}$$

equation (A.98) becomes

$$Q_{n+1,l} + \frac{1}{\xi_n Q_{n,l}} = 1, \tag{A.100}$$

which is the same discrete Riccati equation as in (2.89). This means that the solution of (A.100) is given by (2.95) with the appropriate definitions (2.90), (2.92), (2.94). We can substitute into (A.99) the solution (2.95)

$$\frac{q_{n+1,l} - q_{n-1,l}}{\alpha_2 q_{n,l} - q_{n-1,l}} = \frac{(c_n + \zeta_l)(c_{n+1} - c_{n-1})}{(c_{n+1} + \zeta_l)(c_n - c_{n-1})}$$

[6] The extra α_2 is due to the arbitrariness of ξ_n and is inserted in order to simplify the formulas.

and we obtain an equation for $q_{n,l}$. Introducing

$$P_{n,l} = (c_n + \zeta_l) q_{n,l} \tag{A.101}$$

we obtain that $P_{n,l}$ solves the equation

$$P_{n+1,l} - \alpha_2 \frac{c_{n+1} - c_{n-1}}{c_n - c_{n-1}} P_{n,l} + \frac{c_{n+1} - c_n}{c_n - c_{n-1}} P_{n-1,l} = 0. \tag{A.102}$$

Using the transformation

$$P_{n,l} = \frac{R_{n,l}}{R_{n-1,l}}, \tag{A.103}$$

we can cast equation (A.102) in discrete Riccati equation form

$$R_{n+1,l} + \frac{c_{n+1} - c_n}{c_n - c_{n-1}} \frac{1}{R_{n,l}} = \alpha_2 \frac{c_{n+1} - c_{n-1}}{c_n - c_{n-1}}. \tag{A.104}$$

Let d_n be a particular solution of equation (A.104)

$$d_{n+1} + \frac{c_{n+1} - c_n}{c_n - c_{n-1}} \frac{1}{d_n} = \alpha_2 \frac{c_{n+1} - c_{n-1}}{c_n - c_{n-1}}. \tag{A.105}$$

Assuming d_n as the new arbitrary function we can express c_n as the result of *two* discrete integrations. Indeed introducing $z_n = c_n - c_{n-1}$ in (A.105) we have

$$\frac{z_{n+1}}{z_n} = \frac{(d_{n+1} - \alpha_2) d_n}{\alpha_2 d_n - 1}. \tag{A.106}$$

Equation (A.106) represents the first discrete integration, whereas the second one is given by the definition

$$c_n - c_{n-1} = z_n. \tag{A.107}$$

Now we can linearize the discrete Riccati equation (A.104) by the transformation

$$R_{n,l} = d_n + \frac{1}{S_{n,l}} \tag{A.108}$$

and we get the following *linear* equation for $S_{n,l}$

$$S_{n+1,l} - \frac{d_n^2 (c_n - c_{n-1})}{c_{n+1} - c_n} S_{n,l} = \frac{d_n (c_n - c_{n-1})}{c_{n+1} - c_n}. \tag{A.109}$$

Defining

$$d_n = \frac{e_n}{e_{n-1}}, \tag{A.110a}$$

$$f_n - f_{n-1} = \frac{c_n - c_{n-1}}{e_n e_{n-1}}, \tag{A.110b}$$

the solution of (A.109) is

$$S_{n,l} = \frac{(f_{n-1} + \beta_l) e_{n-1}^2}{c_n - c_{n-1}}. \tag{A.111}$$

Here we have the final form of the constraint f_n, which is the same as the one given in (2.111a). Inserting (A.111) and (A.110) into (A.108) we obtain

$$R_{n,l} = \frac{e_n(f_n + \beta_l)}{e_{n-1}(f_{n-1} + \beta_l)}. \tag{A.112}$$

Inserting the definition of $R_{n,l}$ (A.103) into (A.112) we obtain

$$\frac{P_{n,l}}{e_n(f_n + \beta_l)} = \frac{P_{n-1,l}}{e_{n-1}(f_{n-1} + \beta_l)},$$

i.e.,

$$P_{n,l} = \gamma_l e_n(f_n + \beta_l). \tag{A.113}$$

Introducing (A.113) into (A.101) we obtain

$$q_{n,l} = \frac{\gamma_l e_n(f_n + \beta_l)}{c_n + \zeta_l}, \tag{A.114}$$

where f_n is defined by (A.110b), and c_n is given by (A.106) and (A.107), i.e., c_n is the solution of the equation

$$\frac{c_{n+1} - c_n}{c_n - c_{n-1}} = \frac{e_{n+1} - \alpha_2 e_n}{\alpha_2 e_n - e_{n-1}}, \tag{A.115}$$

and e_n is an arbitrary function, i.e., we have that c_n must solve equation (2.110).

Formally equation (A.114) has the form of the solution presented in formula (2.109a), but it depends on three arbitrary functions in the l direction, namely ζ_l, β_l and γ_l. Therefore there must be a constraint between these functions. This constraint can be obtained by plugging (A.114) into (A.96), but here we have another bifurcation depending on the value of parameter δ. Indeed it is easy to see that we must distinguish the cases when $\delta \neq 0$ and when $\delta = 0$.

Case $\boldsymbol{\delta \neq 0}$. Inserting (A.114) into (A.96) if $\delta \neq 0$ factorizing the n dependent part away we are left with

$$\zeta_{l+1} - \zeta_l = \frac{\varepsilon^2}{\alpha_2 \delta^2 \alpha_3^2} \gamma_{l+1} \gamma_l (\beta_{l+1} - \beta_l).$$

This equation tells us that the function ζ_l can be expressed after a discrete integration in terms of the two arbitrary functions β_l and γ_l. This condition is just (2.111b). This yields the general solution of the ${}_tH_3^\varepsilon$ equation (1.2c) when $\delta \neq 0$ and the field $q_{n,l}$ do not satisfy equation (2.108).

Case $\boldsymbol{\delta = 0}$. Inserting (A.114) into (A.96) if $\delta \neq 0$ factorizing the n dependent part the compatibility condition is just $\beta_{l+1} - \beta_l = 0$, i.e., $\beta_l = \beta_0 =$ const. It is easy to check that the obtained value of $q_{n,l}$ through formula (A.114) is consistent with the substitution of $\delta = 0$ in (A.87) and therefore that in the case $\delta = 0$ the value of $q_{n,l}$ is given by

$$q_{n,l} = \frac{\gamma_l e_n(f_n + \beta_0)}{c_n + \zeta_l},$$

where the functions c_n and f_n are defined implicitly and can be found by discrete integration from (A.110b) and (A.115) respectively. This is just equation (2.112a). Since equation (2.109b) is not singular if $\delta = 0$ we obtain that in this case the general solution for the field $p_{n,l}$ is given by substituting $\delta = 0$ into (2.109b), i.e., just by equation (2.112b) where $q_{n,l}$ is simply given by (2.112a). This yields the general solution of the ${}_tH_3^\varepsilon$ equation (1.2c) in the case when $\delta = 0$.

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