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Kramers–Kronig relations and precision limits in quantum phase estimation

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Phase measurements are of paramount importance in quantum optical sensing. However, the promise of a quantum advantage, the celebrated Heisenberg scaling, is severely curtailed in the presence of noise and loss. Here we investigate systems in which phase and absorption profiles are linked by Kramers–Kronig relations and show that, in the limit of a large photon number, their use connects the uncertainties on the profiles attainable by optimal probes for loss and phase. This underlines a physical motivation for which the Heisenberg scaling for the phase is lost. Our results bear practical implications, revealing the metrological capabilities of absorption measurements in determining phase profiles.

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The purpose of quantum metrology is to establish the best possible strategy for performing the measurement of a set of parameters. This dictates the choice of the state preparation and measurement that, for a given evolution governed by those parameters, is able to deliver the most accurate estimate [1–3]. Early efforts were prompted by the possibility of improving the scaling of the precision of optical phases due to the use of quantum resources. However, this capability is hampered by the presence of loss: more sophisticated tools have been developed to investigate non-ideal scenarios in parameter estimation [4–6]. The current goals are set on achieving robust operation of quantum metrological protocols, by producing states that can operate efficiently in loss channels, as well as under more general noise models, albeit without recovering an improved scaling [7–12]. These studies assume a fixed noisy channel, with a detailed characterization available before its use for phase estimation. A different approach considers loss as a parameter to be estimated, invoking multiparameter methods [13–16], in either the independent or correlated case. While the exact expressions of the achievable precision are often cumbersome, bounds are found that enjoy simpler forms, and are achievable asymptotically for a large amount of resources [9].

When investigating material samples, the Kramers–Kronig relations (KKR) impose an inter-dependence between these parameters, as the refractive index at any given frequency can be written by means of an integral of the absorption coefficient over the frequency domain [17]. This correlates a single parameter with a continuum, and the implications for metrology have not been explored. In this Letter, we demonstrate that by employing KKR, the quantum-limited estimation of an absorption profile grants quantum-limited estimation of the dispersion profile and vice versa. In our investigations, we have made use of concepts taken from quantum function estimation, a novel approach that has been added to the toolbox of quantum metrology in recent years [18].

Kramers [19] and Kronig [20] investigated the connection between dispersion and absorption from atomic species, allowing them to establish the relation between real and imaginary parts of the complex refractive index $\tilde{n}(\omega) = n(\omega) + i\kappa(\omega)$ as a function of the optical frequency ω:

$$n(\omega) = 1 + \frac{2}{\pi}\mathcal{P}\int_0^\infty \frac{\omega'\kappa(\omega')}{\omega'^2-\omega^2}\mathrm{d}\omega',$$
$$\kappa(\omega) = -\frac{2\omega}{\pi}\mathcal{P}\int_0^\infty \frac{n(\omega')}{\omega'^2-\omega^2}\mathrm{d}\omega', \tag{1}$$

where $\mathcal{P}$ denotes the Cauchy principal value. At a more fundamental level, similar relations are found between the real and imaginary parts of the linear electric susceptibility [21], and it is now appreciated how this relation is a manifestation of causality in the linear response of a medium [17]. This has made them a precious tool in research in material science. The generality of the KKR has made it possible to find applications well outside their original scope of describing optical response, and these are now employed in acoustics and seismology [22,23], imaging problems [24,25], and coherent signal processing [26].

In optics, the relations (1) are more conveniently expressed in terms of measurable quantities. When light traverses a length l of this medium, it accumulates in a single pass a phase shift $\varphi(\omega) = (n(\omega) - 1)\omega l/c$, with respect to propagation in the vacuum. The transmission for the intensity follows the

Lambert–Beer law $\eta(\omega) = e^{-\alpha(\omega)l} = e^{-2\kappa(\omega)\omega l/c}$. Differently from $n(\omega)$ and $\kappa(\omega)$, the quantities $\varphi(\omega)$ and $\eta(\omega)$ can be accessed experimentally. The KKR can be cast in the form

$$\varphi(\omega) = -\frac{1}{2\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\log\eta(\omega')}{\omega'-\omega}\mathrm{d}\omega' = \hat{H}[\log\sqrt{\eta}](\omega),$$

$$\log\sqrt{\eta(\omega)} = -\hat{H}[\varphi(\omega)], \tag{2}$$

where $\hat{H}$ is the Hilbert transform operator. To prove these relations, we consider the first of the two equations Eq. (1), multiplying both sides by $\omega l/c$:

$$\varphi(\omega) = \frac{\omega}{\pi}\mathcal{P}\int_{0}^{\infty}\mathrm{d}\omega'\frac{\alpha(\omega')l}{\omega'^2-\omega^2}. \tag{3}$$

Since the complex refractive index satisfies $\tilde{n}(-\omega) = \tilde{n}^*(\omega)$ [17], $\kappa(\omega)$ is an odd function, making $\alpha(\omega)$ an even function. This property can be used to rewrite the integral Eq. (3) over both negative and positive frequencies; this eventually leads to the first of the equations in Eq. (2). The second is obtained by the property $\hat{H}[\hat{H}[f]] = -f$. Thus, in principle, by measuring the transmission profile $\eta(\omega)$ over the whole frequency spectrum, it is possible to recover the function $\varphi(\omega)$, and vice versa. The KKR, however, provide an unambiguous connection between phase and loss only for the minimal-phase model [27]. In the more general case, the complex transfer function should be inspected, and the presence of zeros causes further additive terms in the phase [28].

In quantum metrology, phase and loss are considered as model examples for the estimation of unitary [2] and dissipative parameters [29,30]. In ideal conditions, an optical phase can be measured with an uncertainty scaling with the number p of photons in the probe as $1/p^2$, the so-called Heisenberg limit [1], as opposed to the optimal scaling with classical light $1/p$. The estimation of phase in a lossy system reveals that, while the Heisenberg limit cannot be attained, a quantum advantage is retained as a constant factor [6]. Further, studies on the joint estimation of the two parameters have shown that simultaneous optimal estimability cannot be implemented [13,14,31]. While these works have considered only uncorrelated values, in [15], instead, a form of correlation was introduced, in that the precision limits on a parameter χ, on which both the phase $\varphi(\chi)$ and $\eta(\chi)$ depend, have been investigated. The picture offered by the KKR in the form of Eq. (2) is even more involved, as it shows how a single parameter $\varphi_0 = \varphi(\omega_0)$ at a given frequency ω_0 is related to the *function* $\eta(\omega)$. It is then convenient to study the metrological implications of the KKR Eq. (2) with the methods of the recently introduced quantum function estimation [18,32].

We define our goal as that of obtaining an estimate $\tilde{\varphi}(\omega)$ of the actual phase function $\varphi(\omega)$; for quantum metrology, the relevant figure is the error [18]

$$\delta_\varphi^2 = \mathbf{E}\left[\int_{-\infty}^{\infty}|\varphi(\omega)-\tilde{\varphi}(\omega)|^2\mathrm{d}\omega\right], \tag{4}$$

obtained as the expectation value over all the evaluations of $\tilde{\varphi}(\omega)$. Two routes are followed to obtain such a function, as shown in Fig. 1; we first consider the case of a direct measurement of the phase: $\tilde{\varphi}(\omega)$ is then estimated by assessing the values for some frequencies ω_i, and interpolating in between these points. The

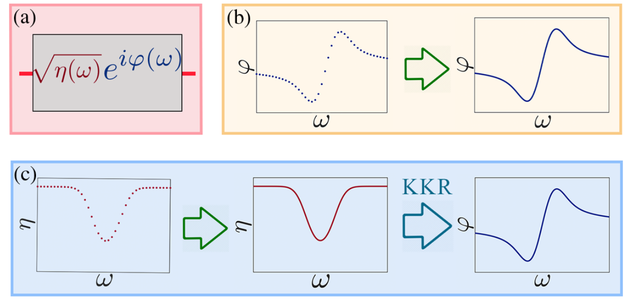

Fig. 1. Use of the Kramers–Kronig relations for phase reconstruction. (a) We aim at reconstructing the phase profile of a lossy sample, with frequency-dependent parameters $\varphi(\omega)$ and $\eta(\omega)$. For this, we can either (b) measure a set of values $\tilde{\varphi}(\omega_i)$ and interpolate them to obtain the estimated function $\tilde{\varphi}(\omega)$ or (c) use the KKR on a reconstructed transmission profile $\tilde{\eta}(\omega)$.

error Eq. (4) then originates from two contributions: the statistical uncertainties on the measured phases $\tilde{\varphi}(\omega_i)$, and the deviations due to the interpolation [18]. The ideal limit has the measured sample dense enough that the second contribution is small with respect to the statistical one, and the total error is thus well approximated by the sum of point-by-point deviations $\Delta\varphi^2(\omega) = \mathbf{E}[|\varphi(\omega)-\tilde{\varphi}(\omega)|^2]$, and hence $\delta^2 \simeq \int_{-\infty}^{\infty}\Delta\varphi^2(\omega)\mathrm{d}\omega$.

As an alternative strategy, we can obtain an estimate $\tilde{\eta}(\omega)$ of the transmission profile, and use this to derive the phase, by means of the KKR Eq. (2). The error on the phase function is then

$$\delta_\varphi^2 = \tfrac{1}{4}\mathbf{E}\left[\int_{-\infty}^{\infty}\left|\hat{H}[\log\tilde{\eta}](\omega)-\hat{H}[\log\eta](\omega)\right|^2\mathrm{d}\omega\right]. \tag{5}$$

We can simplify this expression by invoking the property that, for any square-summable function $f(\omega)$, $\int|f(\omega)|^2\mathrm{d}\omega = \int|\hat{H}[f](\omega)|^2\mathrm{d}\omega$:

$$\delta_\varphi^2 = \frac{1}{4}\mathbf{E}\left[\int_{-\infty}^{\infty}|\log\tilde{\eta}(\omega)-\log\eta(\omega)|^2\mathrm{d}\omega\right]. \tag{6}$$

If the collected sample is sufficiently dense, we can take the same approximation as above, and obtain $\delta_\varphi^2 = \int_{-\infty}^{\infty}\Delta\eta^2(\omega)/(4\eta^2(\omega))\mathrm{d}\omega$, valid when the deviation $\Delta\eta^2(\omega) = \mathbf{E}[|\tilde{\eta}(\omega)-\eta(\omega)|^2]$ is small. The adoption of the KKR thus allows an estimate for the phase function with the same error we would get from a direct phase measurement with an equivalent uncertainty:

$$\Delta\varphi_{\mathrm{eq}}^2(\omega) = \frac{\Delta\eta^2(\omega)}{4\eta^2(\omega)}. \tag{7}$$

This quantity, however, should not be interpreted as the error on the individual phase values as obtained from the function $\tilde{\varphi}(\omega)$. Optimal estimation of the transmission is obtained by using Fock state probes [30], and we allow for the same number of repetitions of the same states for every frequency. This yields to the uncertainty $\Delta\eta^2(\omega) = (1-\eta(\omega))\eta(\omega)/N$, where, for a p-photon state and M repetitions, $N = pM$. More in general the same form of the optimal precision holds also when substituting N with the average number of photons used to probe the sample at each frequency; in this regard, a more practical optimal strategy is to use half of a two-mode squeezed vacuum state [33], while single-mode squeezing offers advantages only in particular regimes [29]. Substituting this expression in Eq. (7), we obtain

$$\Delta\varphi_{\text{eq}}^2(\omega) = \frac{1-\eta(\omega)}{4\eta(\omega)N}. \quad (8)$$

This is a known bound on the ultimate quantum limit for lossy phase estimation with N photons [4,34], which is asymptotically attainable [9]. Therefore, by using the optimal states for loss estimation, we obtain an estimate $\tilde{\varphi}(\omega)$ that is, at the leading order, as good as the one we could obtain by employing the optimal states for phase estimation. This discussion, however, comes with the important caveat that the Hilbert transform of a constant function is zero. This implies that the phase profile can be reconstructed from the transmission up to a phase, which needs to be fixed up front. The boundary condition [17] that $\phi(\omega) \to 0$ for $\omega \to \infty$ may help resolve this ambiguity. While this reasoning was presented for minimal-phase models, it also applies more generally, since the additive terms in the phase—provided they are specified—will be irrelevant for the sake of analyzing the phase estimation variance.

The same reasoning can be applied, in reverse, to estimation of the transmission $\alpha(\omega)$, obtaining $\delta_\alpha^2 = 4/l^2 \int_{-\infty}^{+\infty} \Delta^2\phi(\omega)\mathrm{d}\omega$. If the phase can be estimated optimally, it will asymptotically follow the error Eq. (8), and thus we obtain $\delta_\alpha^2 = 1/l^2 \int_{-\infty}^{+\infty} \frac{1-\eta(\omega)}{N\eta(\omega)}\mathrm{d}\omega$. As expected, this is equivalent to the error obtained from a direct optimal estimation of $\eta(\omega)$, i.e., $\delta_\alpha^2 = \int_{-\infty}^{+\infty} \Delta\alpha^2(\omega)\mathrm{d}\omega$, with $\Delta\alpha^2(\omega) = \frac{\Delta^2\eta(\omega)}{l^2\eta^2(\omega)} = \frac{1-\eta(\omega)}{l^2\eta(\omega)}$, since $\alpha = -1/l \log\eta$, by definition.

Interestingly, this argument sheds a new light on the loss of Heisenberg scaling in lossy phase estimation: were it not so, the equivalent uncertainty $\Delta\alpha^2(\omega)$, and consequently, the function estimation of $\alpha(\omega)$ would show quantum enhancement in the form of a $1/p^2$ scaling of the uncertainty, although absorption is not a coherent process [2,29]. KKR are compatible with fundamental quantum metrological bounds derived without taking them into account. The KKR are established as a consequence of the physical properties of dispersive and absorptive objects, but they can also be interpreted as the fact that the complete determination of one function leads to an exhaustive knowledge of the other. Our results interpret this property as optimality in a statistical sense, in that resources are employed for the best use for one parameter as well as the other.

There may occur other mechanisms, including those related to the detectors, which can be calibrated in advance. If the overall efficiency, excluding the absorption, is $T(\omega)$, the optimal uncertainty on the transmission is $\Delta\eta^2(\omega) = (1-\eta(\omega)T(\omega))T(\omega)/(\eta(\omega)N)$. Inserting this expression into the equivalent phase uncertainty leads to $\Delta\varphi_{\text{eq}}^2(\omega) = (1-\eta(\omega)T(\omega))/(4\eta(\omega)T(\omega)N)$: the phase profile estimation is influenced by the total transmission $\eta(\omega)T(\omega)$, and the equivalent uncertainty is the optimal permitted by the total loss. The KKR are also useful to link the best precision on phase and transmission allowed by classical light. For this, we use coherent states as benchmarks, as they are optimal among classical states for both phase and loss estimation. Loss estimation with state $|\beta\rangle$ leads to the uncertainty $\Delta\eta^2(\omega) = \eta(\omega)/\beta^2$. From this expression, we derive the equivalent uncertainty Eq. (7) as $\Delta\varphi_{\text{eq}}^2(\omega) = 1/(4\eta(\omega)\beta^2)$, which is the corresponding precision for phase estimation, when loss is considered. In this case, the KKR relate the classical limits for individual parameters.

We have explored the implications of the KKR in the practical case, performing numerical calculations on a Gaussian absorption line centered around ω_0, with $\alpha(\omega) = \alpha_0 \exp(-(\omega-\omega_0)^2/(2\sigma^2))$. This is a relevant model for those media with an isolated line, or close multiplets, such as quantum memories, whenever the contribution to the phase of other absorption regions can be neglected. This is associated to a phase profile $\varphi(\omega)$ described by a Dawson function [17]. The absorption profile has parameters $\alpha_0 l = 1$, $\omega_0 = 0.5$, and $\sigma = 0.1$, and the interval [0,1] is considered. We have simulated the estimation of the functions $\tilde{\eta}(\omega)$ and $\tilde{\phi}(\omega)$ from a set of measurements of the transmission using the optimal single-photon states, with a fixed number of total resources N_{tot} divided among N_s sampled points; each of these is then estimated from $N_{\text{ev}} = N_{\text{tot}}/N_s$ events. For each of the N_s points taken on the transmittivity curve, equally spaced on the ω axis, the value of $\tilde{\eta}_i = \tilde{\eta}(\omega_i)$ is taken from a normal distribution with mean η_i and variance $\eta_i(1-\eta_i)/N_{\text{ev}}$, i.e., we assume an unbiased measurement and a sample sufficiently large to achieve an estimator for $\tilde{\eta}$ with a Gaussian distribution. A linear interpolation method is employed to connect those points, and obtain a continuous function from these discrete measurements. The error integral on the loss is approximated as a discrete sum $\delta_\eta^2 \simeq \sum_{j=1}^{N_{\text{ref}}} |\tilde{\eta}(\omega_j) - \eta(\omega_j)|^2\delta\omega$. We have defined $N_{\text{ref}} = 10000$ as the number of interpolated points, $\delta\omega = 1/N_{\text{ref}}$ as their spacing in frequency. We expect the error to reach a minimum, as a result of the trade-off between the resolution and statistical uncertainty on the measured points [18]. Indeed, for fixed N_{tot}, the lower the N_s, the higher the error from the interpolation, while increasing N_s also increases the statistical contribution to the error [32]. Further, we calculate the expected error on $\tilde{\varphi}(\omega)$ if we were able to estimate phases with a variance given by Eq. (8).

Starting from $\tilde{\eta}(\omega)$, the KKR are employed to obtain $\tilde{\varphi}(\omega)$, which is then compared to the Dawson phase profile. We used the numerical methods introduced in [35] for the evaluation of the Hilbert transform, which, for reduced intervals, introduces less numerical errors with respect to the usual approach based on the Fourier transform [17]. Given a function $f(x)$, reconstructed over the interval $[x_{\min}, x_{\max}]$, its Hilbert transform can be approximated by the sum

$$H[f](x) \approx \frac{1}{\pi}\sum_k f\left(\frac{2k+1}{2^{j+1}}\right)\log\left|\frac{2^j x - k}{2^j x - k - 1}\right|, \quad (9)$$

where j is a fixed natural, setting the accuracy of the approximation, and k is an integer satisfying $2^j x_{\min} < k < 2^j x_{\max}$, i.e., the original interval is sampled in steps of $1/2^j$; for our calculations, we used $j = 17$, as a compromise between accuracy and computing time. The results of our simulations are depicted in Figs. 2(a) and 2(b). Function estimations of $\tilde{\eta}(\omega)$ and $\tilde{\phi}(\omega)$ achieve the minimal error for the same number of sampled points N_s. For the smaller number of resources, $N_{\text{tot}} = 10^5$, the reconstruction of phase based on the KKR performs close to the limit set by Eq. (8). When the resources are increased to $N_{\text{tot}} = 10^8$, a discrepancy appears with respect to the ideal bound. We have verified that this behavior observed in Fig. 2 does originate from the setting of j: the statistics of the event is now sufficiently good to reveal systematic effects. We should observe, however, that the uncertainty limit Eq. (8) is optimistic, especially for high values of $\eta(\omega)$; while the bound is asymptotically attainable, convergence to the asymptotic bound is slow [9].

The possibility of accessing both functions without incurring a trade-off in precision suggests that, beyond metrology, quantum resources may have potential applications in the realization of

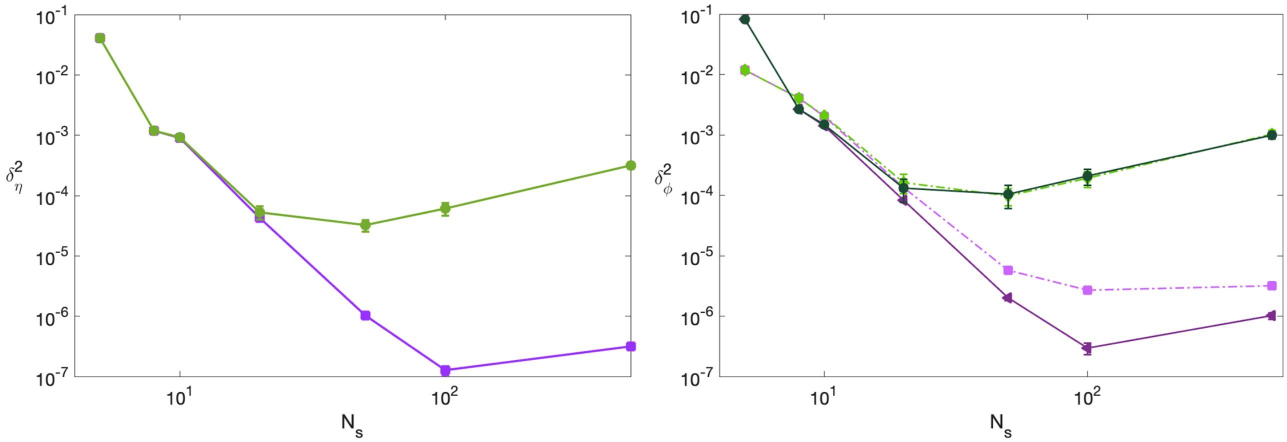

Fig. 2. Results of the simulated estimation. The error δ^2 is calculated for the transmittivity (left panel) for $N_{\mathrm{tot}} = 10^5$ (green) and $N_{\mathrm{tot}} = 10^8$ (purple). The phase errors are shown in the right panel: the error on the phase reconstructed through KKR (dashed line, green circles, for $N_{\mathrm{tot}} = 10^5$ and dashed line, pink squares, for $N_{\mathrm{tot}} = 10^8$), and the error on the phase directly estimated at the lossy bound (dark green for $N_{\mathrm{tot}} = 10^5$, dark purple for $N_{\mathrm{tot}} = 10^8$). To ascertain the uncertainties on the points, we have performed $N_{\mathrm{mc}} = 50$ numerical experiments by means of a Monte Carlo method based on bootstrapping of the simulated data. The uncertainty is taken as the standard deviation of this sample, following the procedure in real experiemnts.

Kramers–Kronig coherent receivers [26]: this communication protocol reconstructs a complex signal by means of equations similar to Eq. (2). It is not excluded (though unlikely in the face of trade-offs in joint phase-loss estimation [13]) that the precision can be further improved by more sophisticated estimation protocols that take into account KKR and retrieve information optimally from a direct joint measurement of phase and attenuation, instead of measuring one quantity and reconstructing the other. The KKR are peculiar to the optical absorption–dispersion mechanism; however, one may attempt generalizations when the parameters can be traced back to the response function to an applied field. These could then provide insight on the physical origin of the bounds on precision.

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Data Availability. Data underlying the results presented in this Letter may be obtained from the authors upon reasonable request.

REFERENCES

1. V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004).
2. V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett. **96**, 010401 (2006).
3. M. G. A. Paris, Int. J. Quantum Inf. **07**, 125 (2009).
4. B. M. Escher, R. L. de Matos Filho, and L. Davidovich, Nat. Phys. **7**, 406 (2011).
5. R. Chaves, J. B. Brask, M. Markiewicz, J. Kołodyński, and A. Acin, Phys. Rev. Lett. **111**, 120401 (2013).
6. R. Demkowicz-Dobrzański, J. Kołodyński, and M. Guţă, Nat. Commun. **3**, 1063 (2012).
7. U. Dorner, R. Demkowicz-Dobrzanski, B. J. Smith, J. S. Lundeen, W. Wasilewski, K. Banaszek, and I. A. Walmsley, Phys. Rev. Lett. **102**, 040403 (2009).
8. A. Datta, L. Zhang, N. Thomas-Peter, U. Dorner, B. J. Smith, and I. A. Walmsley, Phys. Rev. A **83**, 063836 (2011).
9. S. Knysh, V. N. Smelyanskiy, and G. A. Durkin, Phys. Rev. A **83**, 021804 (2011).
10. M. Kacprowicz, R. Demkowicz-Dobrzański, W. Wasilewski, K. Banaszek, and I. A. Walmsley, Nat. Photonics **4**, 357 (2010).
11. M. G. Genoni, S. Olivares, and M. G. A. Paris, Phys. Rev. Lett. **106**, 153603 (2011).
12. M. Sbroscia, I. Gianani, L. Mancino, E. Roccia, Z. Huang, L. Maccone, C. Macchiavello, and M. Barbieri, Phys. Rev. A **97**, 032305 (2018).
13. P. J. D. Crowley, A. Datta, M. Barbieri, and I. A. Walmsley, Phys. Rev. A **89**, 023845 (2014).
14. F. Albarelli, J. F. Friel, and A. Datta, Phys. Rev. Lett. **123**, 200503 (2019).
15. P. M. Birchall, E. J. Allen, T. M. Stace, J. L. O'Brien, J. C. F. Matthews, and H. Cable, Phys. Rev. Lett. **124**, 140501 (2020).
16. F. Albarelli and R. Demkowicz-Dobrzański, "Probe incompatibility in multiparameter noisy quantum metrology," arXiv:2104.11264 (2021).
17. V. Lucarini, J. J. Saarinen, K.-E. Peiponen, and E. M. Vartiainen, *Kramers-Kronig Relations in Optical Materials Research*, Springer series in optical sciences (Springer, 2005).
18. N. Kura and M. Ueda, Phys. Rev. Lett. **124**, 010507 (2020).
19. H. A. Kramers, *Atti del Congresso Internazionale dei Fisici* (Como, 1927), pp. 545–557.
20. R. de L. Kronig, J. Opt. Soc. Am. **12**, 547 (1926).
21. L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii, *Course of Theoretical Physics: Electrodynamics of Continuous Media*, 2nd ed. (Butterworth, 1984).
22. M. O'Donnell, E. T. Jaynes, and J. G. Miller, J. Acoust. Soc. Am. **63**, 1935 (1978).
23. V. Mikhaltsevitch, M. Lebedev, and B. Gurevich, Geophys. Res. Lett. **43**, 4986 (2016).
24. S. A. R. Horsley, M. Artoni, and G. C. La Rocca, Nat. Photonics **9**, 436 (2015).
25. Y. Baek, K. Lee, S. Shin, and Y. Park, Optica **6**, 45 (2019).
26. A. Mecozzi, C. Antonelli, and M. Shtaif, Optica **3**, 1220 (2016).
27. A. Mecozzi, Opt. Commun. **282**, 4183 (2009).
28. P. Grosse and V. Offermann, Appl. Phys. A **52**, 138 (1991).
29. A. Monras and M. G. A. Paris, Phys. Rev. Lett. **98**, 160401 (2007).
30. G. Adesso, F. Dell'Anno, S. De Siena, F. Illuminati, and L. A. M. Souza, Phys. Rev. A **79**, 040305 (2009).
31. L. O. Conlon, J. Suzuki, P. K. Lam, and S. M. Assad, npj Quantum Inf. **7**, 110 (2021).
32. I. Gianani, F. Albarelli, V. Cimini, and M. Barbieri, Phys. Rev. A **103**, 042602 (2021).
33. R. Nair, Phys. Rev. Lett. **121**, 230801 (2018).
34. J. Kołodyński and R. Demkowicz-Dobrzański, Phys. Rev. A **82**, 053804 (2010).
35. C. Zhou, L. Yang, Y. Liu, and Z. Yang, J. Comput. Appl. Math. **223**, 585 (2009).