SCHRÖDINGER–NEWTON EQUATIONS IN DIMENSION TWO VIA A POHOZAEV–TRUDINGER LOG-WEIGHTED INEQUALITY

DANIELE CASSANI[1] AND CRISTINA TARSI[2]

ABSTRACT. We study the following Choquard type equation in the whole plane

$$(C) \quad -\Delta u + V(x)u = (I_2 * F(x,u))f(x,u), \quad x \in \mathbb{R}^2$$

where I_2 is the Newton logarithmic kernel, V is a bounded Schrödinger potential and the nonlinearity $f(x,u)$, whose primitive in u vanishing at zero is $F(x,u)$, exhibits the highest possible growth which is of exponential type. The competition between the logarithmic kernel and the exponential nonlinearity demands for new tools. A proper function space setting is provided by a new weighted version of the Pohozaev–Trudinger inequality which enables us to prove the existence of variational, in particular finite energy solutions to (C).

1. INTRODUCTION

Consider the following class of nonlocal equations

$$(1.1) \qquad -\Delta u + V(x)u = (I_\alpha * F(u))f(u), \quad x \in \mathbb{R}^N$$

where $V \geq 0$ is the external Schrödinger potential, F is the primitive function of the nonlinearity f vanishing at zero, the kernel I_α is defined for $x \in \mathbb{R}^N \setminus \{0\}$, $N \geq 2$ by

$$I_\alpha(x) := \begin{cases} \dfrac{\frac{\Gamma((N-\alpha)/2)}{\Gamma(\alpha/2)\pi^{N/2}2^\alpha}}{|x|^{N-\alpha}}, & 0 < \alpha < N, \\[2em] \dfrac{1}{2^{N-1}\pi^{\frac{N}{2}}\Gamma(\frac{N}{2})} \log\frac{1}{|x|}, & \alpha = N, \end{cases}$$

where $\Gamma(\cdot)$ denotes Euler's Gamma function. Notice that passing from $\alpha < N$ to the limiting case $\alpha = N$ the kernel is no longer of one sign and does not decay at infinity which sets the problem in a quite different framework. By introducing the function $\phi := I_\alpha * F(u)$ one has that (1.1) is equivalent to the following system

$$(1.2) \qquad \begin{cases} -\Delta u + V(x)u = \phi f(u), \\ \\ -\Delta^{\frac{\alpha}{2}}\phi = F(u), \end{cases} \qquad x \in \mathbb{R}^N$$

which in the case $\alpha = 2$ it turns out to be the so-called Schrödinger–Poisson system which has an Hamiltonian structure and which turns out to be relevant in applications, see [5] and references therein. An extensive literature has been devoted to the higher dimensional case $N \geq 3$ and we refer to [25, 13, 14, 17, 37] for an up to date, though non exhaustive bibliography. On the contrary, just a few results are available in the planar case. Existence results in the case $\alpha < N = 2$ have been proved in [3] in the case of power-like nonlinearities and in [1] in the case of exponential growth. However, in dimension two the equivalence between (1.1) and (1.2) in the Schrödinger–Poisson case $\alpha = 2$, carries over as long as the logarithmic kernel is taken into account. Existence and qualitative properties of solutions in

the case of logarithmic kernel have been obtained in [7, 15] for power-like nonlinearities. On the other hand, the polynomial growth somehow downplays the main feature of dimension two which allows finite energy solutions to have arbitrary polynomial growth up to the exponential.

The main purpose of this paper is to tackle the problem in which one has the logarithmic kernel and the exponential growth, namely the limiting case $\alpha = N = 2$ which is in turn the Schrödinger-Poisson case. As we are going to see, the main difficulty arises in the competing presence of a too loose singular behavior of the logarithmic kernel compared with the exponential growth of the nonlinearity within the convolution, for which the problem demands for a proper function space setting. Here we develop a suitable framework in which we can prove the existence of mountain pass solutions. Let us finally mention that nonlinear terms with exponential growth outside the convolution, which cast the problem in a quite different context, have been recently considered in [2].

We will focus on the following non-autonomous problem

$$(1.3) \qquad \begin{cases} -\Delta u + V(x)u = \dfrac{1}{2\pi}\left(\log \dfrac{1}{|x|} * F(x,u)\right) f(x,u), \quad \text{in } \mathbb{R}^2 \\ u \in H^1(\mathbb{R}^2),\ u > 0 \ . \end{cases}$$

On the Schrödinger potential V we make the following assumptions:

(V_1) $V(x) \geq V_0 > 0$ in $\mathbb{R}^2$ for some $V_0 > 0$;
(V_2) $V(x)$ is a 1-periodic continuous function .

With a slight abuse of notation, we assume the nonlinearity $f(x,s) = c(x)f(s)$ where $c(x)$ is a strictly positive, 1-periodic continuous function and $f(s)$ a differentiable function whose primitive vanishing at zero is $F(s)$ and such that:

(f_1) $f(s) \geq 0$ for all $s \geq 0$, $f(s) \leq Cs^p e^{4\pi s^2}$ as $s \to +\infty$ for some $p > 0$ and $f(s) \asymp s^{q-1}$ for some $q > 2$ as $s \to 0$;

(f_2) $\exists\, C > \delta > 0$ such that $\delta \leq \dfrac{F(s)f'(s)}{f^2(s)} \leq C \quad \forall\, s > 0$;

(f_3) $\displaystyle \lim_{s \to +\infty} \dfrac{F(s)f'(s)}{f^2(s)} = 1$, or equivalently $\displaystyle \lim_{s \to +\infty} \dfrac{d}{ds}\dfrac{F(s)}{f(s)} = 0$;

(f_4) $\displaystyle \lim_{s \to +\infty} \dfrac{s^3 f(s)F(s)}{e^{8\pi s^2}} \geq \beta > \mathcal{V}$, where $\mathcal{V}$ will be explicitly given in Section 5.

Since we look for positive solutions, we may also assume $f(s) = 0$ for $s \leq 0$.

Let us make a few comments on our assumptions:

- (f_1) gives the following

$$(1.4) \qquad 0 \leq F(s) \leq C \cdot \begin{cases} s^q, & s \leq s_0 \\ s^{p-1}e^{4\pi s^2}, & s > s_0 \end{cases} \qquad \text{for some } s_0 > 1$$

  observing that $\int_{s_0}^s t^p e^{4\pi t^2}\,dt \leq s^{p-1}\int_1^s t e^{4\pi t^2}\,dt$ for any $s > 1$;

- (f_2) implies $f(s)$ is monotone increasing in s, so that $F(s) = \int_0^s f(\tau)\,d\tau \leq sf(s)$. Hence, the quantity $\dfrac{F(x,s)}{f(x,s)} = \dfrac{c(x)F(s)}{c(x)f(s)} = \dfrac{F(s)}{f(s)}$ is well defined and vanishes only at $s = 0$. Furthermore,

$$(1.5) \qquad \frac{\partial}{\partial s}\left(\frac{F(x,s)}{f(x,s)}\right) = \frac{d}{ds}\left(\frac{F(s)}{f(s)}\right) = \frac{f^2(s) - F(s)f'(,s)}{f^2(s)} \leq 1 - \delta$$

  which implies $F(x,s) \leq (1-\delta)sf(x,s)$ and this improves the previous Ambrosetti-Rabinowitz condition $F(x,s) \leq sf(x,s)$;

- (f_3) yields a fine estimate from below on the quotient $\dfrac{Ff'}{f^2}$, as $s \to +\infty$.

Indeed, for any $\varepsilon > 0$ there exists $s_\varepsilon > 0$ such that:

$$(1.6) \qquad \frac{Ff'}{f^2}(s) \geq \begin{cases} \delta s, & s \leq s_\varepsilon \\ (1-\varepsilon)s, & s > s_\varepsilon \ ; \end{cases}$$

- (f_4) is in the spirit of the the de Figueiredo–Miyagaki–Ruf condition [19] which in dimension two turns out to be a key ingredient in order to prove compactness. Loosely speaking, it plays the role of the upper bound of the energy in terms of the Sobolev constant $(1/N)S^{N/2}$ in higher dimensions. The role of condition (f_4) will be detailed in Section 5;
- Functions $F_i(s)$ satisfying our set of assumptions are for instance given by:

$$F_1(s) = \begin{cases} s^q, & s \leq s_0 \\ e^{4\pi s^2} - 1, & s > s_0 \end{cases}, \ \forall q > 2 \ F_2(s) = s^p e^{4\pi s^2}, \ \forall p > 2;$$

$$F_3(s) = \begin{cases} s^q, & s \leq s_0 \\ cs^p e^{4\pi s^2}, & s > s_0 \end{cases}, \ \forall q > 2, \, p > 1 \ .$$

Accordingly to our assumptions on the nonlinearity, we will consider functionals which are asymptotic, near the origin, to s^q, with $q > 2$. Nevertheless, for the sake of clearness, and since it has an independent interest by itself, we will state..... also in the case $q = 2$ SISTEMARE, anche nel seguito (fino a Overview) Consider the following weighted Sobolev space $H_w^1(\mathbb{R}^2)$ which is the completion of smooth compactly supported functions with respect to the norm

$$\|u\|_w^2 = \|\nabla u\|_2^2 + \|u\|_{L^2(wdx)}^2 = \int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} u^2 \log(e + |x|)dx \ .$$

Theorem 1.1. *The weighted Sobolev space $H_w^1(\mathbb{R}^2)$ embeds into the weighted Orlicz space $L_\phi(\mathbb{R}^2, \log(e + |x|)dx)$ where ϕ is the n-function $\phi(t) = e^{t^2} - 1$. More precisely, we have*

$$(1.7) \qquad \int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1 \right) \log(e + |x|)dx < \infty$$

for any $u \in H_w^1(\mathbb{R}^2)$ and any $\alpha > 0$. Moreover, the following uniform bound holds

$$(1.8) \qquad \sup_{\|u\|_w^2 \leq 1} \int_{\mathbb{R}^2} \left(e^{2\pi u^2} - 1 \right) \log(e + |x|)dx < +\infty \ .$$

In the case $q > 2$ case, the natural weighted Sobolev space turns out to be $H^1 L_w^q(\mathbb{R}^2)$, defined as the completion of smooth compactly supported functions with respect to the norm

$$\|u\|_w^2 = \|\nabla u\|_2^2 + \|u\|_{L^q(wdx)}^{2/q} = \int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} |u|^q \log(e + |x|)dx \ .$$

We furthermore let

$$\|u\|_V^2 := \int_{\mathbb{R}^2} |\nabla u|^2 + V|u|^2 \, dx,$$

and use $H_V^1(\mathbb{R}^2)$ for the set of all functions with bounded $\|\cdot\|_V$ norm. Let also $w_0(x) := \log(1 + |x|)$, and

$$\|u\|_{q,V,w_0}^2 := \|u\|_V^2 + \|u\|_{L^q(w_0 dx)}^2 = \|u\|_V^2 + \left(\int_{\mathbb{R}^2} |u|^q \log(1 + |x|)dx \right)^{\frac{2}{q}},$$

and we consider $H_V^1 L_{w_0}^q(\mathbb{R}^2)$ as the completion of smooth compactly supported functions with respect to the norm $\|\cdot\|_{q,V,w_0}$ (see (4.1)).
In the more general case $q > 2$, we prove a Trudinger type inequality similar to Theorem 1.1, see Section 3. Relying on it, we state our main result

Theorem 1.2. *Suppose the nonlinearity f satisfies (f_1)–(f_4) and that the potential V enjoys (V_1)–(V_2). Then, problem (1.3) possesses a nontrivial finite energy solution in the space $H_V^1 L_{w_0}^q(\mathbb{R}^2)$.*

Overview. Equation (1.1) has a long history, heritage of the early studies on Polarons iniziated by Fröhlich [20] and then has been revealed a good model also in completely different contexts such as plasma physics [24] and quantum gravity [29]. We refer the interested reader to the survey [25] and references therein for more on Physical aspects of the problem.

Formally, the energy associated to problem (1.3) is given by

$$I_V(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + V(x)u^2 dx - \frac{1}{4\pi} \int_{\mathbb{R}^2} \left[\log \frac{1}{|x|} * F(x,u) \right] F(x,u) dx \ .$$

In order to have the energy well defined in presence of a logarithmic kernel, the authors in [15, 34], restrict the space H^1 introducing a further constraint, eventually setting the problem in an intersection space in which the energy turns out to be well defined by the Hardy–Littlewood–Sobolev inequality. Our approach here is different, from one side we look for a proper function space setting in which such a natural constraint turns out to be automatically satisfied and on the other side, we wonder if this can be done by allowing the nonlinearity to exhibit exponential growth which is what we expect in dimension two, since the seminal work of Pohozaev [31] and Trudinger [36]. Indeed, we prove that the Sobolev space H^1 with a logarithmic weight on the L^2 mass term of the norm gives the proper function space setting in which the energy in well defined up to the natural exponential growth in the nonlinearity. Our argument throws light on the fact that, roughly speaking, as concentration phenomena in the Moser functional are controlled by the L^2 norm of the gradient whereas vanishing phenomena are controlled by the L^2 norm, here we prove that a suitable logarithmic weight in the L^2 component of the H^1 norm is enough to obtain a functional inequality which at the end yields a natural function space framework where to set up the problem. We think this result is of independent interest and that could be useful elsewhere. As pointed out also in [2, 15] an extra difficulty is given here by the lack of invariance by translations of the energy which forces to prove a priori bounds of eventually vanishing Palais–Smale sequences. Our method seems to be more natural also in this respect, as starting from any PS sequence we can prove the existence of a weak H^1-limit with no need to establish a priori bounds.

For convenience of the reader, some preliminary material is recalled in Section 2. In Section 3 we establish the fundamental embedding inequality which will provide the function space framework of Section 4 and that will be used throughout the paper. Section 5 is devoted to show the underlying mountain pass geometry for the energy functional and to prove mountain pass level estimates which in this case is a delicate matter. In Section 6 we prove compactness results by carefully analyzing the behavior of PS sequences and finally, we conclude in Section 7 the proof of Theorem 1.2.

2. Preliminaries

In this section we recall some well known results which will be used in the sequel.

Let $H_0^1(\Omega)$ be the classical Sobolev space, completion of smooth compactly supported functions with respect to the Dirichlet norm $\|\nabla \cdot\|_2$, when Ω is a bounded subset of $\mathbb{R}^N$, and with respect to the complete Sobolev norm $(\|\nabla \cdot\|_2^2 + \|\cdot\|_2^2)^{1/2}$, when the domain is unbounded and in particular for $\Omega = \mathbb{R}^N$.

If $N \geq 3$, the classical Sobolev embedding theorem reads as follows

$$(2.1) \qquad\qquad H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega) \ , \quad \text{namely} \ \ \|u\|_{2^*} \leq \frac{1}{S} \|\nabla u\|_2 \ ,$$

where $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent and the constant S in (2.1) is the best possible [35].

When $N = 2$ is the so-called Sobolev limiting case. One has the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ for all $1 \leq p < \infty$ (see also [12] for related best constants estimates), though $H_0^1(\Omega) \not\subset$

$L^\infty(\Omega)$. The maximal degree of summability for functions in $H_0^1(\Omega)$ was established independently by Pohožaev [31] and Trudinger [36] (see also [38]) and is of exponential type, in a suitable Orlicž class of functions, namely

$$(2.2) \qquad u \in H_0^1(\Omega) \implies \int_\Omega (e^{\alpha|u|^2} - 1)\, dx < \infty, \quad \forall \alpha > 0 \ .$$

Starting from the seminal work of J. Moser [26] in which a sharp version of (2.2) is established, the Pohozaev–Trudinger embedding has been further developed during the last fifty years, in particular the first extension of (2.2) to unbounded domains appears in [9] for functions with bounded Sobolev's norm in the following form

$$(2.3) \qquad \sup_{\|\nabla u\|_2 \le 1,\, \|u\|_2 \le M} \int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) dx \le C(\alpha)\|u\|_2 < \infty \ \text{ if } \alpha < 4\pi$$

Thereafter, several sharp versions have been proved and extensions in many directions for which we refer to [32, 11]. In particular, the borderline case in which $\alpha = 4\pi$ remained uncovered until Ruf in [32] established the following inequality which is sharp in the sense of Moser [26] (so-called Trudinger-Moser type inqualities):

$$(2.4) \qquad \sup_{\|\nabla u\|_2^2 + \|u\|_2^2 \le 1} \int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right)\, dx \le \widetilde{C}(\alpha) < \infty \iff \alpha \le 4\pi \ .$$

Remark 2.1. *Note that in Ruf's inequality (2.4) the constraint is defined through the complete Sobolev norm $\|\nabla \cdot\|_2^2 + \|\cdot\|_2^2$. As one may realize by Cao's result, a closer inspection of the proof reveals that Ruf's inequality still holds, at least in the subcritical case $\alpha < 4\pi$, replacing the L^2 norm with any weighted L^2 norm, provided the weight is bounded and also bounded away from the origin.*

Remark 2.2. *As an application of (2.2), consider the following functional*

$$H^1(\mathbb{R}^2) \ni u \longmapsto \int_{\mathbb{R}^2} F(u)\, dx$$

which is continuous on $H^1(\mathbb{R}^2)$, a consequence of (f_1), (1.4) and Holder's inequality. Indeed, note first that for any $t, s > 0$

$$|F(t) - F(s)| = \left| \int_s^t f(\tau) d\tau \right| \le C \left| \int_s^t \left(\tau^{q-1} + \tau^{p-1} e^{4\pi\tau^2}\right) d\tau \right|$$

so that, if $u_n \to u$ in $H^1(\mathbb{R}^2)$, as $n \to \infty$, then

$$\int_{\mathbb{R}^2} |F(u_n) - F(u)| dx \le C\|u_n - u\|_2^2 + C\int_{\mathbb{R}^2} |e^{5\pi u_n^2} - e^{5\pi u^2}| dx$$

$$\le o(1) + C\int_{\mathbb{R}^2} \left(e^{5\pi u^2} - 1\right) |e^{5\pi(u_n^2 - u^2)} - 1| dx + C\int_{\mathbb{R}^2} \left| e^{5\pi(u_n^2 - u^2)} - 1 \right| dx \longrightarrow 0$$

and the same holds for the functional $u \longmapsto \int_{\mathbb{R}^2} u f(x, u)\, dx$.

The main feature of the equation (1.1) is the nonlocal term defined through a convolution product. This turns out to be well defined in view of the following Hardy–Littlewood–Sobolev inequalities, which we state in $\mathbb{R}^N$ for any $N \ge 1$, see [23] and also [6] for the interpolation spaces approach.

Proposition 2.3 (HLS inequality). *Let $s, r > 1$ and $0 < \mu < N$ with $1/s + \mu/N + 1/r = 2$, $f \in L^s(\mathbb{R}^N)$ and $g \in L^r(\mathbb{R}^N)$. There exists a constant $C(s, N, \mu, r)$, independent of f, h, such that*

$$\int_{\mathbb{R}^N} [\frac{1}{|x|^\mu} * f(x)] g(x) \le C(s, N, \mu, r)\|f\|_s \|g\|_r.$$

Remark 2.4. *Note that the Sobolev inequality* (2.1) *is equivalent, by duality, to a special case of the HLS inequality* (2.3) *(see* [4]*). Actually, take* $\mu = N - 2$ *and* $s = r = 2^*$: *then* (2.3) *says that the inclusion* $L^{\frac{2N}{N+2}}(\mathbb{R}^N) \hookrightarrow H^{-1}(\mathbb{R}^N)$ *is continuous, and so, by duality, its counterpart,* $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.

By exploiting a limiting procedure as $\mu \to 0$, one can prove the so-called logarithmic Hardy–Littlewood–Sobolev inequality, whose main feature is the presence of a sign-changing logarithmic kernel, see [4, 10, 30].

Proposition 2.5 (Logarithmic HLS inequality). *Let* f, g *be two nonnegative functions belonging to* $L \ln L(\mathbb{R}^N)$, *such that* $\int f \log(1 + |x|) < \infty, \int g \log(1 + |x|) < \infty$ *and* $\|f\|_1 = \|g\|_1 = 1$. *There exists a constant* C_N, *independent of* f, g, *such that*

$$(2.5) \qquad 2N \int_{\mathbb{R}^N} [\log \frac{1}{|x|} * f(x)] g(x) \leq C_N + \int_{\mathbb{R}^N} f \log f \, dx + \int_{\mathbb{R}^N} g \log g \, dx \ .$$

Remark 2.6. *Let us stress the feature of the log kernel, which has variable sign, and it is unbounded both in 0 and at* $+\infty$. *This justifies the presence of the additional condition,* $f, g \in L \ln L(\mathbb{R}^N)$ *in order to have the inequality* (2.5) *well defined, and in particular that no cancellation of infinities occurs. However, this does not imply the boundedness of* $\int f \log f, \int g \log g$, *but only of the positive parts* $\int f \log_+ f, \int g \log_+ g$. *The further weight conditions* $\int f \log(1 + |x|), \int g \log(1 + |x|) < \infty$ *make both the two sides of inequality* (2.5) *finite.*

3. A log-mass weighted Pohozaev–Trudinger type inequality

This Section is devoted to prove a Pohozaev–Trudinger type inequality in the whole plane $\mathbb{R}^2$, with a logarithmic weight which appears only in the mass component of the energy. Here, the logarithmic weight plays a role only as $|x| \to +\infty$, for which we consider as prototype weight $w = \log(e + |x|)$. On the other hand, it is well known from [32, 21, 11], how the growth near zero is a key ingredient in proving Pohozaev–Trudinger type inequalities on unbounded domains, since it is strictly related to vanishing phenomena. Here we aim at proving a fundamental inequality which will provide a suitable variational setting for (1.3). Let us point out that the presence of an increasing weight prevents one to use rearrangement arguments.

3.1. **Proof of Theorem 1.1.** Let us first perform a change of variables which enables one to pass from $H^1_w(\mathbb{R}^2)$ to functions in $H^1(\mathbb{R}^2)$. Note that the inverse transformation does not turn out to be explicit and this is why we can not expect to prove directly our inequality. Let us use polar coordinates in $\mathbb{R}^2$:

$$x = (x_1, x_2) = |x|(\cos\theta, \sin\theta), \quad \text{where} \ \ |x| = \sqrt{x_1^2 + x_2^2}$$

We perform the change of variable

$$y = (y_1, y_2) = |x|\sqrt{\log(e + |x|)}(\cos\theta, \sin\theta)$$

which acts only on the radial part of any point in $\mathbb{R}^2$, equivalently

$$T(|x|) = |y|, \quad \frac{y}{|y|} = \frac{x}{|x|}, \quad |y| = |x|\sqrt{\log(e + |x|)}$$

In order to simplify the notation, set $r = |x|$ and $s = |y|$, so that the transformation becomes $s = T(r) = r\sqrt{\log(e + r)}$. Note that

$$T'(r) = \frac{2\log(e + r) + \frac{r}{e+r}}{2\sqrt{\log(e + r)}} > 0, \quad T(0) = 0, \quad \lim_{r \to +\infty} T(r) = +\infty$$

and thus T is invertible on $\mathbb{R}^2$, though the inverse map is not explicitly known.
Define

$$v(y) := u(x), \quad \text{that is,} \quad v(y) = u\left(T^{-1}(|y|)\cos\theta, T^{-1}(|y|)\sin\theta\right)$$

or, equivalently
$$u(r\cos\theta, r\sin\theta) = v\left(T(r)\cos\theta, T(r)\sin\theta\right) .$$
Then, by a direct calculation, if
$$w(r,\theta) := u(r\cos\theta, r\sin\theta) \quad \widetilde{w}(s,\theta) := v\left(s\cos\theta, s\sin\theta\right), \quad w(r,\theta) = \widetilde{w}\left(T(r),\theta\right)$$
we have
$$w_r(r,\theta) = \widetilde{w}_s\left(T(r),\theta\right)T'(r), \quad w_\theta\left(T(r),\theta\right) = \widetilde{w}_\theta\left(T(r),\theta\right)$$
so that
$$\int_{\mathbb{R}^2} |\nabla v|^2 dy_1 dy_2 = \int_0^{2\pi}\int_0^{+\infty}\left[\widetilde{w}_s^2 + \frac{\widetilde{w}_\theta^2}{s^2}\right]sds d\theta$$
$$= \int_0^{2\pi}\int_0^{+\infty}\left[\widetilde{w}_s^2(T(r),\theta) + \frac{\widetilde{w}_\theta^2(T(r),\theta)}{T^2(r)}\right]T'(r)T(r)dr d\theta$$
$$= \int_0^{2\pi}\int_0^{+\infty}\left[w_r^2(r,\theta)\cdot\frac{1}{[T'(r)]^2} + \frac{w_\theta^2(r,\theta)}{r^2}\cdot\frac{r^2}{T^2(r)}\right]T'(r)T(r)dr d\theta .$$

Now,
$$\frac{1}{[T'(r)]^2} = \frac{\log(e+r)}{\left[\log(e+r) + \frac{r}{2(e+r)}\right]^2}, \quad \frac{r^2}{T^2(r)} = \frac{1}{\log(e+r)}$$
so that
$$\frac{1}{3}\frac{r^2}{T^2(r)} < \frac{1}{[T'(r)]^2} < \frac{r^2}{T^2(r)} .$$

Then,
$$\frac{1}{3}\int_0^{2\pi}\int_0^{+\infty}\left[w_r^2 + \frac{w_\theta^2}{r^2}\right]\frac{r^2 T'(r)}{T(r)}dr d\theta \le \int_{\mathbb{R}^2}|\nabla v|^2 dy_1 dy_2$$
$$\le \int_0^{2\pi}\int_0^{+\infty}\left[w_r^2 + \frac{w_\theta^2}{r^2}\right]\frac{r^2 T'(r)}{T(r)}dr d\theta .$$

Noting that
$$\frac{r^2 T'(r)}{T(r)} = r\left[1 + \frac{r}{2(e+r)\log(e+r)}\right] \Longrightarrow r < \frac{r^2 T'(r)}{T(r)} < 2r$$
we eventually get
$$\frac{1}{3}\int_{\mathbb{R}^2}|\nabla u|^2 dx_1 dx_2 < \int_{\mathbb{R}^2}|\nabla v|^2 dy_1 dy_2 < 2\int_{\mathbb{R}^2}|\nabla u|^2 dx_1 dx_2$$
On the other hand,
$$\int_{\mathbb{R}^2}v^2 dy = \int_0^{2\pi}\int_0^\infty \widetilde{w}^2(s,\theta)sds d\theta = \int_0^{2\pi}\int_0^\infty \widetilde{w}^2(T(r),\theta)T'(r)T(r)dr d\theta$$
$$= \int_0^{2\pi}\int_0^\infty w^2(r,\theta)T'(r)T(r)dr d\theta .$$

Since
$$T'(r)T(r) = r\left[\log(e+r) + \frac{r}{2(e+r)}\right]$$
$$= r\log(e+r)\left[1 + \frac{r}{2(e+r)\log(e+r)}\right] = \frac{r^2 T'(r)}{T(r)}\log(e+r)$$
we conclude that
$$\int_{\mathbb{R}^2}v^2 dy = \int_0^{2\pi}\int_0^\infty w^2\frac{r^2 T'(r)}{T(r)}\log(e+r)dr d\theta$$

and, in turn

$$\int_{\mathbb{R}^2} u^2 \log(e + |x|)dx < \int_{\mathbb{R}^2} v^2 dy < 2\int_{\mathbb{R}^2} u^2 \log(e + |x|)dx \ .$$

Finally,

(3.1) $$\frac{1}{3}\|u\|_{w_0}^2 < \|v\|^2 = \|\nabla v\|_2^2 + \|v\|_2^2 < 2\|u\|_{w_0}^2 \ .$$

We have then proved that the map

$$\mathfrak{T}: H_w^1(\mathbb{R}^2) \rightarrow H_0^1(\mathbb{R}^2)$$
$$u \mapsto v$$

is an invertible, continuous and with continuous inverse map. Then,

$$\int_{\mathbb{R}^2}\left(e^{\alpha u^2} - 1\right)\log(e + |x|)dx = \int_0^{2\pi}\int_0^{+\infty}\left(e^{\alpha u^2(r\cos\theta, r\sin\theta)} - 1\right)\log(e + r)rdrd\theta$$

$$= \int_0^{2\pi}\int_0^{+\infty}\left(e^{\alpha v^2(T(r)\cos\theta, T(r)\sin\theta)} - 1\right)\frac{\log(e + r)r}{T'(r)T(r)}T'(r)T(r)drd\theta$$

$$\leq \int_0^{2\pi}\int_0^{+\infty}\left(e^{\alpha v^2(\rho\cos\theta, \rho\sin\theta)} - 1\right)\rho d\rho d\theta = \int_{\mathbb{R}^2}\left(e^{\alpha v^2} - 1\right)dx < +\infty$$

by [32] for any $\alpha > 0$. The uniform bound (1.8) follows directly from (3.1), as for any $u \in H_w^1$ and $\alpha \leq 2\pi$ one has

$$\int_{\mathbb{R}^2}\left(e^{2\pi u^2/\|u\|_{w_0}^2} - 1\right)\log(e + |x|)dx$$

$$= \int_0^{2\pi}\int_0^{+\infty}\left(e^{2\pi u^2(r\cos\theta, r\sin\theta)/\|u\|_{w_0}^2} - 1\right)\log(e + r)rdrd\theta$$

$$= \int_0^{2\pi}\int_0^{+\infty}\left(e^{2\pi v^2(T(r)\cos\theta, T(r)\sin\theta)/\|u\|_{w_0}^2} - 1\right)\frac{\log(e + r)r}{T'(r)T(r)}T'(r)T(r)drd\theta$$

$$\leq \int_0^{2\pi}\int_0^{+\infty}\left(e^{4\pi v^2(\rho\cos\theta, \rho\sin\theta)/\|v\|^2} - 1\right)\rho d\rho d\theta = \int_{\mathbb{R}^2}\left(e^{4\pi v^2/\|v\|^2} - 1\right)dx < C(\alpha)$$

again by [32].

A consequence of this embedding result is the continuity of the weighted Pohozaev–Trudinger functional on $H_w^1(\mathbb{R}^2)$, namely we have

Corollary 3.1. *For any $\alpha > 0$ the functional*

$$H_w^1(\mathbb{R}^2) \longrightarrow \mathbb{R}$$
$$u \longmapsto \int_{\mathbb{R}^2}\left(e^{\alpha u^2} - 1\right)\log(e + |x|)dx$$

is continuous.

Remark 3.2. *The value 2π in (1.8) is not sharp and the problem of establishing a sharp version of (1.8) in the spirit of Moser [26] remains essentially open.*

3.2. **The case $q > 2$.** Let us now consider the more general case of a growth function $F(s) \asymp s^q$ as $s \to 0$ with $q > 2$. In this case, the natural weighted Sobolev space turns out to be $H^1 L_w^q(\mathbb{R}^2)$, defined as the completion of smooth compactly supported functions with respect to the norm

$$\|u\|_w^2 = \|\nabla u\|_2^2 + \|u\|_{L^q(wdx)}^{2/q} = \int_{\mathbb{R}^2}|\nabla u|^2 dx + \int_{\mathbb{R}^2}|u|^q\log(e + |x|)dx \ .$$

Theorem 3.3. *Let $F(s)$ satisfy assumption (f_1). Then, the space $H^1 L_w^q(\mathbb{R}^2)$ embeds into the weighted Orlicz space $L_F(\mathbb{R}^2, \log(e + |x|)dx)$. More precisely,*

$$(3.2) \qquad \int_{\mathbb{R}^2} F(\alpha|u|) \log(e + |x|)dx < +\infty$$

for any $u \in H^1 L_w^q(\mathbb{R}^2)$ and any $\alpha > 0$.
Furthermore, for any $\alpha \leq 1/\sqrt{q}$ one has

$$(3.3) \qquad \sup_{\|u\|_w^2 \leq 1} \int_{\mathbb{R}^2} F(\alpha|u|) \log(e + |x|)dx < +\infty .$$

Proof. The result will follow from the following estimate

$$F(s) \leq C \begin{cases} s^q, & \text{if } s < 1 \\ s^{p-1}e^{4\pi s^2}, & \text{if } s \geq 1 \end{cases}$$

Indeed, let $u \in H^1 L_w^q$ be such that $\|u\|_w \leq 1$ and define

$$v = \begin{cases} |u|^{q/2}, & |u| < 1 \\ |u|, & |u| > 1 \end{cases} .$$

Then

$$\|v\|_w^2 = \|\nabla v\|_2^2 + \|v\|_w^2 \leq \frac{q}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx$$

$$+ \int_{|u|<1} |u|^q \log(e + |x|)dx + \int_{|u|>1} |u|^2 \log(e + |x|)dx$$

$$\leq \frac{q}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} |u|^q \log(e + |x|)dx \leq \frac{q}{2}\|u\|_w^2 \leq \frac{q}{2} .$$

Hence

$$\int_{\mathbb{R}^2} F(\alpha|u|) \log(e + |x|)dx \leq C \int_{\mathbb{R}^2} \alpha^q |u|^q \log(e + |x|)dx$$

$$+ C \int_{|u|>1} \alpha^{p-1} |u|^{p-1} e^{\alpha^2 4\pi u^2} \log(e + |x|)dx$$

$$= C\alpha^q \|u\|_{L^q(wdx)}^{2/q} + C \int_{|v|>1} \alpha^{p-1} |v|^{p-1} e^{\alpha^2 4\pi v^2} \log(e + |x|)dx$$

$$\leq C_\alpha \|u\|_{L^q(wdx)}^{2/q} + C\alpha^{p-1} \left\{ \int_{|v|>1} |v|^{\frac{\alpha^2 q}{1-\alpha^2 q}(p-1)} \right\}^{\frac{1-\alpha^2 q}{\alpha^2 q}}$$

$$\cdot \left\{ \int_{|v|>1} \left(e^{4\pi v^2/q} - 1 \right) \log(e + |x|)dx \right\}^{\alpha^2 q}$$

$$\leq C_{\alpha,q} + C_{\alpha,q,p} \left\{ \int_{\mathbb{R}^2} \left(e^{2\pi v^2/\|v\|_w^2} - 1 \right) \log(e + |x|)dx \right\}^{\alpha^2 q} \leq C_{\alpha,p,q}$$

where we have applied Theorem 1.1 and embeddings for weighted Sobolev spaces. $\square$

Clearly, a corollary similar to 3.1 holds also in this case

Corollary 3.4. *For any $\alpha > 0$ the functional*

$$H^1 L_w^q(\mathbb{R}^2) \longrightarrow \mathbb{R}$$

$$u \longmapsto \int_{\mathbb{R}^2} F(\alpha|u|) \log(e + |x|)dx$$

is continuous, where F is a function satisfying assumption (f_1).

In analogy to our case, weighted Pohozev–Trudinger inequalities allowing increasing mono-mial weights have been proved in [16, 27]. Finally, let us mention that related inequalities in Sobolev spaces with respect to *log*-weighted measures can be found in [8], where the authors consider a logarithmic weight in the full Sobolev norm.

4. The functional framework

The energy functional related to (1.3) is given by

$$I_V(u) = \frac{1}{2}\|u\|_V^2 - \mathfrak{F}(u).$$

where

$$\mathfrak{F}(u) = \frac{1}{4\pi}\int_{\mathbb{R}^2}\Big[\log\frac{1}{|x|}*F(x,u)\Big]F(x,u)\,dx,$$

and

$$\|u\|_V := \left(\int_{\mathbb{R}^2}|\nabla u|^2 + V(x)u^2 dx\right)^{1/2}.$$

Thanks to the assumptions on the potential V, $\|u\|_V$ is equivalent to the standard Sobolev norm. However, the energy functional I_V is not well defined on $H^1(\mathbb{R}^2)$ due to the nonlocal term $\mathfrak{F}(u)$. In view of the logarithmic HLS inequality (2.5), Stubbe first proposed in the unpublished paper [34] to set the problem in the intersection space adding the integrability condition for which the energy is well defined. Let $H_V^1 L_{w_0}^q(\mathbb{R}^2)$ be the completion of smooth compactly supported functions with respect to the norm

$$(4.1)\quad \|u\|_{q,V,w_0}^2 := \|u\|_V^2 + \|u\|_{L^q(w_0 dx)}^{2/q}$$

$$= \int_{\mathbb{R}^2}|\nabla u|^2 dx + \int_{\mathbb{R}^2}u^2 V(x)dx + \left\{\int_{\mathbb{R}^2}|u|^q\log(1+|x|))dx\right\}^{2/q}$$

which in the case $q = 2$ is induced by the scalar product

$$\langle u,v\rangle = \int_{\mathbb{R}^2}\nabla u\nabla v dx + \int_{\mathbb{R}^2}V(x)uv dx + \int_{\mathbb{R}^2}uv\log(1+|x|^2)dx\ .$$

Accordingly to [22, Theorem 1.11], $H_V^1 L_{w_0}^q$ is a Banach space whose dual can be characterized as follows [33, Theorem 14.9]

$$H_V^{-1}L_{w_0}^q(\mathbb{R}^2) = \left(H^1(\mathbb{R}^2)\cap L^q(\mathbb{R}^2,w_0 dx)\right)' = H^{-1}(\mathbb{R}^2) + (L^q)'(\mathbb{R}^2,w_0 dx)|_{H_V^1 L_{w_0}^q}\ .$$

Next, the Orlicz type embedding established in Section 3 will enable us to apply the loga-rithmic version of the HLS inequality in order to have the energy functional well defined on $H_V^1 L_{w_0}^q$ and sufficiently smooth for variational purposes.

In what follows we will use extensively the following elementary identity

$$\log\frac{1}{r} = \log\left(1 + \frac{1}{r}\right) - \log\left(1 + r\right)\ .$$

Furthermore, since we suppose that the nonlinearity $f(x,s)$ has the form $f(x,s) = c(x)f(s)$ where $c(x)$ is a strictly positive, 1-periodic continuous function, from now on we may reduce to the case $c(x) = 1$ (the general case immediately).

Let us introduce the following bilinear forms:

$$(u,v)\ \mapsto\ \mathfrak{B}_1(u,v) = \int_{\mathbb{R}^2}[\log(1+|x|)*u]\,v dx = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(1+|x-y|)u(x)v(y)dxdy;$$

$$(u,v)\ \mapsto\ \mathfrak{B}_2(u,v) = \int_{\mathbb{R}^2}\Big[\log(1+\frac{1}{|x|})*u\Big]\,v dx = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log\left(1 + \frac{1}{|x-y|}\right)u(x)v(y)dxdy;$$

$$(u,v)\ \mapsto\ \mathfrak{B}_0(u,v) = \int_{\mathbb{R}^2}\Big[\log\frac{1}{|x|}*u\Big]\,v dx = \mathfrak{B}_2(u,v) - \mathfrak{B}_1(u,v)\ .$$

Since $\log(1+t) \leq t$ for any $t > 0$, thanks to (2.3) with $\mu = 1, N = 2$ we have that $\mathfrak{B}_2(u,v)$ is well defined on $L^{4/3}(\mathbb{R}^2) \times L^{4/3}(\mathbb{R}^2)$, and

$$|\mathfrak{B}_2(u,v)| \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x-y|} |u|(x)|v|(y)dxdy \leq C\|u\|_{4/3}\|v\|_{4/3}$$

Since $\log(1+|x-y|) \leq \log(1+|x|+|y|) \leq \log(1+|x|) + \log(1+|y|)$ we also have

$$|\mathfrak{B}_1(u,v)| \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} [\log(1+|x|) + \log(1+|y|)] |u|(x)|v|(y)dxdy$$

$$= \|u\|_{L^1(w)}\|v\|_1 + \|u\|_1\|v\|_{L^1(w)}$$

where for simplicity we wriwe $\|\cdot\|_{L^1(w)}$ in place of $\|\cdot\|_{L^1(wdx)}$. Evaluating the bilinear forms $\mathfrak{B}_i(\cdot,\cdot)$ on the field $(F(u), F(u))$ we obtain the functionals:

$$\mathfrak{F}_1 : H^1L^q, w \to [0, +\infty), \quad \mathfrak{F}_1(u) = \mathfrak{B}_1(F(u), F(u))$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1+|x-y|)F(u(x))F(u(y))dxdy;$$

$$\mathfrak{F}_2 : H^1L_w^q \to [0, +\infty), \quad \mathfrak{F}_2(u) = \mathfrak{B}_2(F(u), F(u))$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log\left(1 + \frac{1}{|x-y|}\right) F(u(x))F(u(y))dxdy;$$

$$\mathfrak{F} : H^1 \to [0, +\infty), \mathfrak{F}(u) = \mathfrak{B}_0(F(u), F(u)) = \mathfrak{F}_2(u) - \mathfrak{F}_1(u) .$$

Reasoning as done above for the bilinear forms $\mathfrak{B}_i$, one has that $\mathfrak{F}_2(u)$ is well defined on $H_V^1 L_{w_0}^q(\mathbb{R}^2)$ (actually, on the larger Sobolev space $H^1(\mathbb{R}^2)$) by assumption (f_1) and recalling (2.2)). Whereas for $\mathfrak{F}_1(u)$, observe that the quantity $\|F(x,u)\|_{L^1(w_0dx)}$ is always finite, for any $u \in H_V^1 L_{w_0}^q$, by combining (1.4) with Theorem 1.1 and Theorem 3.3.

Remark 4.1. *Note that for instance when $q = 2$, the weight in the L^2 component of the norm is given by $V(x) + \log(1+|x|)$ for which we have $C_1[V(x) + \log(1+|x|)] \leq \log(e+|x|) \leq C_2[V(x) + \log(1+|x|)]$ for some positive constants C_i. Thus the norms involved turn out to be equivalent and this extends to $q > 2$ as the norm $\|\cdot\|_V$ is equivalent to the Sobolev norm.*

4.1. **Regularity of the energy functional.** The main goal of this section is to prove regularity of the energy functional I_V. We have the following

Lemma 4.2. *The functionals $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}$ and I_V are $\mathcal{C}^1$ on $H_V^1 L_{w_0}^q(\mathbb{R}^2)$.*

Proof. Consider $\mathfrak{F}_1$ and let $\{u_n\}$ be a sequence in $H_V^1 L_{w_0}^q(\mathbb{R}^2)$ converging to some u. Then

$$|\mathfrak{F}_1(u_n) - \mathfrak{F}_1(u)|$$

$$\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1+|x-y|) |F(u_n(x))F(u_n(y)) - F(u(x))F(u(y))| \, dxdy$$

$$\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1+|x-y|)F(u_n(x)) |F(u_n(y)) - F(u(y))| \, dxdy+$$

$$+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1+|x-y|) |F(u_n(x)) - F(u(x))| \, F(u(y))dxdy$$

$$\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1+|x|)F(u_n(x)) |F(u_n(y)) - F(u(y))| \, dxdy+$$

$$+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1+|y|) |F(u_n(y)) - F(u(y))| \, F(u_n(x))dxdy$$

$$+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1+|x|)F(u(y)) |F(u_n(x)) - F(u(x))| \, dxdy+$$

$$+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1+|y|) |F(u_n(x)) - F(u(x))| \, F(y,u(y))dxdy = I_1 + I_2 + I_3 + I_4 .$$

Now the terms I_i tend to 0, as $n \to \infty$ thanks to the continuity of the functional $\int_{\mathbb{R}^2} F(u)dx$ on H^1 and of the functional $\int_{\mathbb{R}^2} F(u)\log(1+|x|)dx$ on $H_V^1 L_{w_0}^q$, as a consequence of (1.4) and corollaries 3.1, 3.4 . For instance for I_1 we have

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1+|x|)F(u_n(x))\,|F(u_n(y)) - F(u(y))|\,dxdy =$$

$$= \left[\int_{\mathbb{R}^2} \log(1+|x|)F(u_n(x))dx \right] \cdot \left[\int_{\mathbb{R}^2} |F(u_n(y)) - F(u(y))|\,dy \right]$$

$$\leq \left[\int_{\mathbb{R}^2} \log(e+|x|)F(u(x))dx + \mathrm{o}(1) \right] \mathrm{o}(1) .$$

For any $u \in H_V^1 L_{w_0}^q$ the Gâteaux derivative of $\mathfrak{F}_1$ at $u \in H_V^1 L_{w_0}^q$ is given by

$$\mathfrak{F}_1'(u)[v] = 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1+|x-y|)F(u(x))v(y)f(u(y))dxdy, \quad v \in H^{-1} L_{w_0}^q .$$

Since

$$|\mathfrak{F}_1'(u)v| \leq 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1+|x|)F(u(x))v(y)f(u(y))dxdy$$

$$+ 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1+|y|)F(u(x))v(y)f(u(y))dxdy$$

$$\leq 2\|v\|_2 \left(\int_{\mathbb{R}^2} f^2(u)dx \right)^{1/2} \int_{\mathbb{R}^2} \log(1+|x|)F(u)dx +$$

$$+ 2 \left(\int_{\mathbb{R}^2} \log(1+|y|)v^q dy \right)^{1/q} \left(\int_{\mathbb{R}^2} \log(1+|y|)(F(u)f(u))^{\frac{q}{q-1}}dy \right)^{(q-1)/q}$$

$$\leq C(u)\|v\|_{q,V,w_0}$$

thanks to Theorems 1.1, 3.3, so that $\mathfrak{F}_1'(u) \in H_V^{-1} L_{w_0}^q$. The fact that $\mathfrak{F}_1'(u_n) \to \mathfrak{F}_1'(u)$ in $H_V^{-1} L_{w_0}^q$ if $u_n \to u$ in $H_V^1 L_{w_0}^q$ follows by similar arguments, we just sketch the proof:

$$|\mathfrak{F}_1'(u_n)v - \mathfrak{F}_1'(u)v|$$

$$= 2 \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1+|x-y|)\left[F(u_n(x))f(u_n(y)) - F(u(x))f(u(y)) \right] v(y) \right| dxdy$$

$$\leq 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1+|x-y|)\,|F(u_n(x)) - F(u(x))|\,f(u_n(y))|v(y)|dxdy +$$

$$+ 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1+|x-y|)\,|f(u_n(y)) - f(u(y))|\,F(u(x))|v(y)|dxdy$$

$$\leq 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1+|x|)|F(u_n) - F(u)|dx \int_{\mathbb{R}^2} f(u)|v|dy$$

$$+ 2 \int_{\mathbb{R}^2} |F(u_n) - F(u)|dx \int_{\mathbb{R}^2} \log(1+|y|)f(u)|v|dy +$$

$$+ 2 \int_{\mathbb{R}^2} \log(1+|x|)F(u)dx \int_{\mathbb{R}^2} |f(u_n) - f(u)|v|dy$$

$$+ 2 \int_{\mathbb{R}^2} F(u)dx \int_{\mathbb{R}^2} \log(1+|y|)|f(u_n) - f(u)||v|dy$$

We conclude by applying Holder's inequality and Theorems 1.1 and 3.3, together with Corollaries 3.1 and 3.4, which guarantee the integrals involved are continuously bounded, thus

$$|\mathfrak{F}_1'(u_n)v - \mathfrak{F}_1'(u)v| \leq C(u)\mathrm{o}_n(1)\|v\|_{q,V,w_0}$$

where $\mathrm{o}_n(1)$ tends to 0 together with $u_n \to u$ in $H_V^1 L_{w_0}^q$, as $n \to \infty$.

Consider $\mathfrak{F}_2$ and let $\{u_n\}$ be a sequence in $H^1_V L^q_{w_0}(\mathbb{R}^2)$ converging to some u and assume $u \neq 0$. Since $u_n \to u$ in $H^1(\mathbb{R}^2)$, by [28] we have

$$(4.2) \qquad \int_{\mathbb{R}^2} |F(u_n) - F(u)|^p dx \to 0, \qquad \text{for any } p > 0 .$$

Hence

$$|\mathfrak{F}_2(u_n) - \mathfrak{F}_2(u)|$$
$$\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x - y|^{-1}) \, |F(u_n(x))F(u_n(y)) - F(u(x))F(u(y))| \, dxdy$$
$$\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x - y|^{-1}) F(u_n(x)) \, |F(u_n(y)) - F(u(y))| \, dxdy +$$
$$+ \int_{\mathbb{R}^2} \log(1 + |x - y|^{-1}) \, |F(u_n(x)) - F(u(x))| \, F(u(y)) dxdy$$
$$\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x - y|} F(u_n(x)) \, |F(u_n(y)) - F(u(y))| \, dxdy +$$
$$+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x - y|} F(u(y)) \, |F(u_n(x)) - F(u(x))| \, dxdy .$$

The last two integrals tend to 0, thanks to (4.2), the continuity of the functional $\int F(u)dx$ on H^1 and the HLS inequality. In the case $u = 0$ the proof is similar.

For any $u \in H^1_V L^q_{w_0}$ the Gâteaux derivative of $\mathfrak{F}_2$ at $u \in H^1_V L^q_{w_0}$ is given by

$$\mathfrak{F}'_2(u)v = 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + |x - y|^{-1}) F(u(x))v(y)f(u(y)) dxdy .$$

Since

$$|\mathfrak{F}'_2(u)v| \leq 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x - y|} F(u(x))v(y)f(u(y)) \leq C(u)\|v\|_2 \leq C(u)\|v\|_{q,V,w_0}$$

thanks to (2.2) and HLS inequality, one has $\mathfrak{F}'_2(u) \in H^{-1}_V L^q_{w_0}$. The fact that $\mathfrak{F}'_2(u_n) \to \mathfrak{F}'_2(u)$ in $H^{-1}_V L^q_{w_0}$ if $u_n \to u$ in $H^1_V L^q_{w_0}$ is similar to previous cases.

Clearly from $\mathfrak{F} = \mathfrak{F}_2 - \mathfrak{F}_1$ one has $F \in \mathcal{C}^1$ on $H^1_V L^q_{w_0}$. $\qquad \square$

5. The variational framework

As we are going to see, the variational framework for problem (1.3) is non-standard as we will exploit the de Figueiredo–Miyagaki–Ruf type condition (f_4) to prove estimates of the mountain pass level for the energy functional I_V. Let us first establish the Mountain Pass geometry in the next

Lemma 5.1. *The energy functional I_V satisfies:*

(1) there exist $\rho, \delta_0 > 0$ such that $I_V|_{S_\rho} \geq \delta_0 > 0$ for all

$$u \in S_\rho = \{u \in H^1_V L^q_{w_0} : \|u\|_{q,V,w_0} = \rho\};$$

(2) there exists $e \in H^1_V L^q_{w_0}$ with $\|e\|_{q,V,w_0} > \rho$ such that $I_V(e) < 0$.

Proof. To shorten the notation, we assume $f(x, s) = f(s)$. From $\|u\|^2_V \leq \|u\|^2_{q,V,w_0}$, if ρ is small then the H^1 norm $\|u\|_V$ is also small. As a consequence of the logarithmic HLS inequality,

$$\mathfrak{F}(u) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \left[\log \frac{1}{|x|} * F(u)\right] F(u)dx$$
$$\leq \|F(u)\|_1 \left[C\|F(u)\|_1 + \int_{\mathbb{R}^2} F(u) \log F(u)dx - \|F(u)\|_1 \log \|F(u)\|_1\right] .$$

By (2.3) and recalling estimates (1.4), since the Sobolev norm is small enough we have that the L^1 norm of $F(u)$ can be bounded as follows

$$\|F(u)\|_1 \leq C \left(\|u\|_2^2 + \|u\|_q^q\right) \leq C\|u\|_V^2 .$$

Moreover, on one side

$$\int_{\mathbb{R}^2} F(u)\log F(u)dx \leq \int_{\mathbb{R}^2} |F(u)\log F(u)|dx \leq c_1 \int_{\mathbb{R}^2} u^r e^{4\pi^2}dx + c_2 \int_{\mathbb{R}^2} u^2 dx$$

$$\leq c_3\|u\|_{2r}^r \left(\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) dx\right)^{\frac{1}{2}} + c_2 \int_{\mathbb{R}^2} u^2 dx \leq c\|u\|_V^2$$

for some $r > 2$ and $\alpha > 4\pi$. On the other side, since $q > 2$ and $\|F(u)\|_1 \leq C\|u\|_V^2$,

$$|\|F(u)\|_1 \log \|F(u)\|_1| \leq c\|u\|_V^2 \log \|u\|_V| \leq \|u\|_V$$

provided the Sobolev norm is small and in turn we obtain

$$\mathfrak{F}(u) \leq c\|u\|_V^3$$

and thus

$$I_V(u) \geq \frac{1}{2}\|u\|_V^2 - c\|u\|_V^3 = \delta_0 > 0$$

where δ_0 depends only on ρ. This yields the first claim of the Lemma.

Let now e be a smooth function, compactly supported in a small ball, say $B_{1/4}$. Then

$$\mathfrak{F}(e) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \left[\log \frac{1}{|x|} * F(e)\right] F(e)dx$$

$$= \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log \frac{1}{|x-y|} F(e(x))F(e(y))dxdy$$

$$\geq \frac{\log 2}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} F(e(x))F(e(y))dxdy = \frac{\log 2}{4\pi} \left(\int_{\mathbb{R}^2} F(e)dx\right)^2$$

since $F(e(x))F(e(y)) \neq 0$ only for $|x|, |y| < 1/4$, which implies $|x - y| < 1/2$ and thus $|x - y|^{-1} > 2$. Similarly,

$$I_V(te) = \frac{1}{2}t^2\|e\|_V^2 - \mathfrak{F}(te) \leq \frac{1}{2}t^2\|e\|_V^2 - \frac{\log 2}{4\pi} \left(\int_{\mathbb{R}^2} F(te)dx\right)^2 \rightarrow -\infty$$

since F has exponential growth, as $t \rightarrow +\infty$. $\square$

By the Ekeland Variational Principle [18], there exists a Palais–Smale sequence (PS in the sequel) $\{u_n\} \subset H_V^1 L_{w_0}^q(\mathbb{R}^2)$ such that

$$I_V'(u_n) \rightarrow 0, \quad I_V(u_n) \rightarrow m_V,$$

where the Mountain Pass level m_V can be characterized by

(5.1) $$0 < m_V := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_V(\gamma(t))$$

where

$$\Gamma := \left\{\gamma \in \mathcal{C}^1([0,1], H_V^1 L_{w_0}^q) \; : \; \gamma(0) = 0, I_V(\gamma(1)) < 0\right\}.$$

The next energy level estimate will be crucial in the sequel, in particular in proving compactness, for which $1/2$ turns out to be a substitute of the Sobolev level $(1/N)S^{N/2}$ in higher dimensions.

Lemma 5.2. *The mountain pass level m_V satisfies*

$$m_V < \frac{1}{2}.$$

Proof. It is enough to prove that there exists a function $w \in H_V^1 L_{w_0}^q$, with $\|w\|_{q,V,w_0} = 1$, such that

$$\max_{t \geq 0} I_V(tw) < \frac{1}{2}.$$

Let us introduce the following Moser type functions with support in B_ρ

$$\overline{w}_n = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log n}, & 0 \leq |x| \leq \frac{\rho}{n}, \\[2mm] \dfrac{\log(\rho/|x|)}{\sqrt{\log n}}, & \frac{\rho}{n} \leq |x| \leq \rho, \\[2mm] 0, & |x| \geq \rho, \end{cases}$$

where ρ will be fixed later on. One has that

$$\|\overline{w}_n\|_{1,q(w)}^2 = \int_{B_\rho} |\nabla \overline{w}_n|^2 + V(x)\overline{w}_n^2 dx + \left[\int_{B_\rho} \log(1+|x|)|\overline{w}_n^q dx\right]^{2/q}$$

$$\leq \int_{\rho/n}^\rho \frac{dr}{r\log n} + V_\rho \int_0^{\rho/n} \log n \, r dr + (2\pi)^{\frac{2}{q}-1} \log n \left[\int_0^{\rho/n} \log(1+r)\, r dr\right]^{2/q}$$

$$+ V_\rho \int_{\rho/n}^\rho \frac{\log^2(\rho/r)}{\log n} r dr + \frac{(2\pi)^{\frac{2}{q}-1}}{\log n} \left[\int_{\rho/n}^\rho \log^q(\rho/r) \log(1+r) r dr\right]^{2/q}$$

$$\leq 1 + V_\rho \frac{\rho^2}{2} \frac{\log n}{n^2} + (2\pi)^{\frac{2}{q}-1} \log n \left[\int_0^{\rho/n} \log(1+r)\, r dr\right]^{2/q}$$

$$+ V_\rho \int_{\rho/n}^\rho \frac{\log^2(\rho/r)}{\log n} r dr + (2\pi)^{\frac{2}{q}-1} \frac{\log^{2/q}(1+\rho)}{\log n} \left[\int_{\rho/n}^\rho \log^q(\rho/r)\, r dr\right]^{2/q}$$

$$= 1 + V_\rho \int_{\rho/n}^\rho \frac{\log^2(\rho/r)}{\log n} r dr + (2\pi)^{\frac{2}{q}-1} \frac{\log^{2/q}(1+\rho)}{\log n} \left[\int_{\rho/n}^\rho \log^q(\rho/r)\, r dr\right]^{2/q} + o\left(\frac{1}{\log n}\right).$$

The two integral terms in the previous expression can be estimated as follows. On the one hand

$$\int \log^k(\rho/r) r dr = \frac{r^2}{2} \sum_{j=0}^k (\log(\rho/r))^{k-j} \frac{k(k-1)\cdots(k-j+1)}{2^j}$$

as well as

$$V_\rho \int_{\rho/n}^\rho \frac{\log^2(\rho/r)}{\log n} r dr = \frac{\rho^2 V_\rho}{4\log n} + o\left(\frac{1}{\log n}\right).$$

On the other hand, since q may be integer or not, a rough estimate reads as follows

$$\int_{\rho/n}^\rho \log^q(\tfrac{\rho}{r})\, r dr \ \leq \ \int_{\rho/n}^\rho \left\{\log^{[q]}(\tfrac{\rho}{r}) + \log^{[q]+1}(\tfrac{\rho}{r})\right\} r dr \ = \ \frac{\rho^2 [q]!}{2^{[q]+1}}\left[1 + \frac{[q]+1}{2}\right] + o(1)$$

so that eventually

$$1 \leq \|\overline{w}_n\|_{1,q(w)}^2 \leq 1 + \delta_n + o\left(\frac{1}{\log n}\right),$$

where

$$(5.2) \quad \delta_n = \frac{\rho^2}{4\log n}\left[V_\rho + (2\pi)^{\frac{2}{q}-1} \log^{\frac{2}{q}}(1+\rho) \frac{[q]!}{2^{[q]-1}}\left(1 + \frac{[q]+1}{2}\right)\right] + o\left(\frac{1}{\log n}\right)$$

Setting $w_n = \overline{w}_n / \sqrt{1 + \delta_n}$, we get $\|w_n\|_{q, V, w_0} \le 1$. We claim that there exists n such that

$$(5.3) \qquad \max_{t \ge 0} I_V(tw_n) < \frac{1}{2}.$$

Let us argue by contradiction and suppose this is not the case, so that for all n let $t_n > 0$ be such that

$$(5.4) \qquad \max_{t \ge 0} I_V(tw_n) = I_V(t_n w_n) \ge \frac{1}{2} .$$

Then t_n satisfies

$$\frac{d}{dt} I_V(tw_n)|_{t=t_n} = 0$$

and

$$(5.5) \qquad t_n^2 \ge \frac{1}{2\pi} \int_{\mathbb{R}^2} \left[\log \frac{1}{|x|} * F(t_n w_n) \right] t_n w_n f(t_n w_n) dx,$$

$$(5.6) \qquad t_n^2 \ge 1 + \frac{1}{2\pi} \int_{\mathbb{R}^2} \left[\log \frac{1}{|x|} * F(t_n w_n) \right] F(t_n w_n) dx .$$

Note that in (5.5) we have an inequality instead of the equality since we know $\|w_n\|_{q, V, w_0}^2 = 1$, whereas in the energy functional it appears the equivalent, though smaller norm $\|w_n\|_V$. Actually the two norms differ for a quantity which is $O(1/\log n)$.

From now on let us suppose $\rho \le 1/2$. This will simplify a few estimates, since for any $(x, y) \in \operatorname{supp} w_n \times \operatorname{supp} w_n$ we will have $|x - y| > 1$, and in turn $\log(1/|x - y|) > 0$. Let us now proceed in three steps:

Step 1. The following holds $\limsup_{n \to +\infty} t_n^2 \ge 1$.
Let us assume by contradiction that $\limsup_n t_n^2 < 1$: this implies that, up to a subsequence, there exists a positive constant δ_0 such that $t_n^2 \le 1 - \delta_0$ for n large enough. Since $\rho \le \frac{1}{2}$, for any $|x| < \rho$, the set $\{y : |x - y| > 1, |y| < \rho\}$ is empty. Recalling that the functions w_n are compactly supported in B_ρ we have

$$\int_{\mathbb{R}^2} \left[\log \frac{1}{|x|} * F(t_n w_n) \right] F(t_n w_n) dx$$

$$= \int_{B_\rho} \int_{|x-y| \le 1} \log \frac{1}{|x - y|} F(t_n w_n(x)) F(t_n w_n(y)) dx dy \ge 0$$

and thus a contradiction with (5.6).

Step 2. The following holds $\liminf_{n \to +\infty} t_n^2 \le 1$.
Let us suppose by contradiction that $\liminf_{n \to +\infty} t_n^2 > 1$. Hence, up to a subsequence, there exists a constant $\delta_0 > 0$ such that

$$t_n^2 \ge 1 + \delta_0$$

as $n \to +\infty$. Let us estimate from below the right hand side of (5.5) (taking into account the possible negative sign of the logarithmic function):

$$(5.7) \quad \int_{\mathbb{R}^2} \left[\log \frac{1}{|x|} * F(t_n w_n) \right] t_n w_n f(t_n w_n) dx$$

$$= \int_{|x| \le \frac{\rho}{n}, |y| \le \frac{\rho}{n}} \log \frac{1}{|x - y|} F(t_n w_n(x)) t_n w_n f(t_n w_n(y)) dx dy +$$

$$+ \int_{\mathbb{R}^2 \times \mathbb{R}^2 \setminus \{|x| \le \frac{\rho}{n}, |y| \le \frac{\rho}{n}\}} \log \frac{1}{|x - y|} F(t_n w_n(x)) t_n w_n f(t_n w_n(y)) dx dy = I_1 + I_2$$

Thanks to (f_4) we have for any $\varepsilon > 0$ (here we choose $\varepsilon = \beta/2$),

$$(5.8) \qquad s f(s) F(s) \geq (\beta - \varepsilon) \cdot \frac{1}{s^2} \cdot e^{8\pi s^2} = \frac{\beta}{2s^2} \cdot e^{8\pi s^2}, \qquad \text{for all } s \geq s_\varepsilon = s_\beta .$$

By the very definition of w_n and since $|x - y| < 2\rho/n < 1$, we can estimate, for n large enough, I_1 as follows

$$I_1 = \int_{B_{\rho/n}} t_n w_n f(t_n w_n) dy \int_{B_{\rho/n}} \log \frac{1}{|x - y|} F(t_n w_n) \, dx$$

$$= \int_{B_{\rho/n}} t_n \frac{\sqrt{\log n}}{\sqrt{2\pi(1 + \delta_n)}} f\left(t_n \frac{\sqrt{\log n}}{\sqrt{2\pi(1 + \delta_n)}} \right) dy$$

$$\cdot \int_{B_{\rho/n}} \log \frac{1}{|x - y|} F\left(t_n \frac{\sqrt{\log n}}{\sqrt{2\pi(1 + \delta_n)}} \right) dx$$

$$\geq 2\pi\beta \frac{e^{4t_n^2(1+\delta_n)^{-1} \log n}}{2t_n^2(1 + \delta_n)^{-1} \log n} \int_{B_{\rho/n}} dy \int_{B_{\rho/n}} \log \frac{1}{|x - y|} \, dx.$$

The last integral can be estimated as follows

$$\int_{B_{\rho/n}} dy \int_{B_{\rho/n}} \log \frac{1}{|x - y|} dx \geq \int_{B_{\rho/n}} dy \int_{B_{\rho/n}} \log \frac{n}{|2\rho|} dx = \pi^2 \left(\frac{\rho}{n} \right)^4 \log \frac{n}{2\rho}$$

As a consequence, we obtain

$$(5.9) \qquad I_1 \geq \pi^3 \rho^4 \beta \frac{e^{4(t_n^2(1+\delta_n)^{-1}-1) \log n}}{t_n^2(1 + \delta_n)^{-1} \log n} \log \frac{n}{2\rho} \geq \pi^3 \rho^4 \beta t_n^{-2} e^{4(\frac{t_n^2}{1+\delta_n}-1) \log n}$$

for any $n \geq n(\rho, \beta)$. Note that since $\rho \leq 1/2$ we have

$$I_2 \geq 0 .$$

Now, combining (5.5), (5.7) and (5.9) yields

$$(5.10) \qquad t_n^4 \geq \pi^3 \rho^4 \beta e^{4(\frac{t_n^2}{1+\delta_n}-1) \log n}$$

which is a contradiction, either if $t_n \to +\infty$ or t_n stays bounded with $t_n^2 \geq 1 + \delta_0$. The proof of Step 2 is then completed. Observe that, as a consequence of Step 1 and Step 2

$$t_n^2 \to 1 \quad \text{as } n \to +\infty .$$

Moreover, as a byproduct of (5.10), we also have for some $C > 0$,

$$e^{4(\frac{t_n^2}{1+\delta_n}-1) \log n} \leq C$$

that is

$$(5.11) \qquad \frac{t_n^2}{1 + \delta_n} \leq 1 + \frac{C}{\log n} = 1 + \mathrm{O}\left(\frac{1}{\log n} \right) .$$

Step 3. We are now in the condition of getting a contradiction and determine the quantity $\mathcal{V}$ which appears in condition (f_4). We have proved that $t_n^2 \to 1$. Moreover, we also know that $t_n^2 \geq 1$ by (5.6), since $\rho \leq 1/2$. By (5.10), recalling definition (5.2) of δ_n, we have

$$1 + \mathrm{o}(1) \geq t_n^4 \geq \pi^3 \rho^4 \beta e^{4(\frac{t_n^2}{1+\delta_n}-1) \log n} \geq \pi^3 \rho^4 \beta e^{-4\frac{\delta_n}{1+\delta_n} \log n}$$

$$\geq \pi^3 \rho^4 \beta e^{-\rho^2\left(V_\rho + (2\pi)^{\frac{2}{q}-1} \log^{\frac{2}{q}}(1+\rho) \frac{[q]!}{2^{[q]-1}} \left(1 + \frac{[q]+1}{2}\right) \right) + \mathrm{o}(1)}$$

where $V_\rho = \max_{|x| \leq \rho} V(x)$. Passing to the limit, we obtain

$$(5.12) \qquad 1 \geq \pi^3 \rho^4 \beta e^{-\rho^2\left(V_\rho + (2\pi)^{\frac{2}{q}-1} \log^{\frac{2}{q}}(1+\rho) \frac{[q]!}{2^{[q]-1}} \left(1 + \frac{[q]+1}{2}\right) \right)} .$$

Now set in assumption (f_4)

$$\mathcal{V} := \inf_{|x|\leq 1/2} \frac{1}{\pi^3} |x|^{-4} e^{|x|^2 \left(V_{1/2}+(2\pi)^{\frac{2}{q}-1} \log^{\frac{2}{q}}(1+|x|)\frac{[q]!}{2^{[q]-1}}\left(1+\frac{[q]+1}{2}\right)\right)},$$

a quantity which is actually a minimum, since the right hand function is continuous and unbounded as $|x| \to 0$. Finally, fix $\rho \in (0, 1/2]$ such that

$$\beta > \frac{1}{\pi^3}\rho^{-4}e^{\rho^2\left(V_{1/2}+(2\pi)^{\frac{2}{q}-1}\log^{\frac{2}{q}}(1+\rho)\frac{[q]!}{2^{[q]-1}}\left(1+\frac{[q]+1}{2}\right)\right)}$$

to get

$$\pi^3\rho^4\beta e^{-\rho^2\left(V_\rho+(2\pi)^{\frac{2}{q}-1}\log^{\frac{2}{q}}(1+\rho)\frac{[q]!}{2^{[q]-1}}\left(1+\frac{[q]+1}{2}\right)\right)} > 1$$

which contradicts (5.12). $\square$

6. Properties of Palais-Smale sequences

In this Section we prove that the weak limit in $H^1_V L^q_{w_0}(\mathbb{R}^2)$ of the PS sequence for I_V given by the Ekeland Variational Principle, which we know from Section 5 is at the energy level $m_V < 1/2$, is actually a weak nontrivial solution of (1.3). As we are going to see, the presence of the sign changing factor $\log(|x|)$ makes the estimates rather delicate. We start with the following Lemma in which we prove boundedness of PS sequences at any level $c < 1/2$.

Lemma 6.1. *Assume that (V_1)–(V_2) and (f_1)–(f_4) hold. Let $\{u_n\} \subset H^1_V L^q_{w_0}$ be an arbitrary PS sequence for I_V at level c, namely*

$$I_V(u_n) \to c < \frac{1}{2} \quad \text{and} \quad I'_V(u_n) \to 0 \quad \text{in } H^{-1}_V L^q_{w_0}(\mathbb{R}^2), \quad \text{as } n \to +\infty .$$

Then, the sequence u_n is bounded in $H^1(\mathbb{R}^2)$ as well as

$$\left|\int_{\mathbb{R}^2}\left[\log\frac{1}{|x|} * F(x,u_n)\right]F(x,u_n)dx\right| \leq C, \qquad \left|\int_{\mathbb{R}^2}\left[\log\frac{1}{|x|} * F(x,u_n)\right]u_nf(x,u_n)dx\right| \leq C .$$

Proof. Let $\{u_n\} \in H^1_V L^q_{w_0}(\mathbb{R}^2)$ be a PS sequence for I_V, namely as $n \to \infty$ is,

$$(6.1) \qquad \frac{1}{2}\|u_n\|_V^2 - \frac{1}{4\pi}\int_{\mathbb{R}^2}\left[\log\left(\frac{1}{|x|}\right) * F(x,u_n)\right]F(x,u_n)dx \to c$$

and

$$(6.2) \quad \left|\int_{\mathbb{R}^2}\nabla u_n\nabla v + Vu_nvdx - \frac{1}{4\pi}\int_{\mathbb{R}^2}\left[\log\left(\frac{1}{|x|}\right) * F(x,u_n)\right]vf(x,u_n)dx\right.$$

$$\left.- \frac{1}{4\pi}\int_{\mathbb{R}^2}\left[\log\left(\frac{1}{|x|}\right) * vf(x,u_n)\right]F(x,u_n)dx\right|$$

$$= \left|\int_{\mathbb{R}^2}\nabla u_n\nabla v + Vu_nvdx - \frac{1}{2\pi}\int_{\mathbb{R}^2}\left[\log\left(\frac{1}{|x|}\right) * F(x,u_n)\right]vf(x,u_n)dx\right| \leq \tau_n\|v\|_{H^{-1}L^q_w}$$

for all $v \in H^{-1}_V L^q_{w_0}(\mathbb{R}^2)$, where $\tau_n \to 0$ as $n \to +\infty$. Since $H^1(\mathbb{R}^2) \hookrightarrow H^{-1}_V L^q_{w_0}(\mathbb{R}^2)$ and $H^1_V L^q_{w_0}(\mathbb{R}^2) \hookrightarrow H^1(\mathbb{R}^2)$, we can take $v = u_n$ in (6.2), to obtain
(6.3)

$$\left|\|u_n\|_V^2 - \frac{1}{2\pi}\int_{\mathbb{R}^2}\left[\log\left(\frac{1}{|x|}\right) * F(x,u_n)\right]u_nf(x,u_n)dx\right| \leq \tau_n\|u_n\|_{H^{-1}L^q_w} \leq C\tau_n\|u_n\|_V$$

where we have also used the fact that $\|\cdot\|_V$ is an equivalent norm to the standard one in $H^1(\mathbb{R}^2)$. Another suitable choice for test function is given by

$$
v_n = \begin{cases} \dfrac{F(x, u_n)}{f(x, u_n)} = \dfrac{c(x)F(u_n)}{c(x)f(u_n)} = \dfrac{F(u_n)}{f(u_n)} & \text{if } u_n > 0 \\[2ex] (1 - \delta)u_n & \text{if } u_n \leq 0 \end{cases}
$$

where the quantity δ is the one appearing in (1.5), $F(s) \leq (1 - \delta)sf(s)$. Indeed, since $f(s) = 0$ if and only if $s = 0$, by (f_2) we have that $0 \leq v_n \leq Cu_n$ (actually, it is uniformly bounded) so that v_n is well defined and in $L^q(w_0 dx)$. Furthermore,

$$
\nabla v_n = \nabla u_n \frac{f^2(u_n) - F(u_n)f'(u_n)}{f^2(u_n)} = \nabla u_n \left(1 - \frac{F(u_n)f'(u_n)}{f^2(u_n)}\right) \ .
$$

Since the quantity $(F/f)'$ is bounded by (f_2), see also (1.5), we have

$$
|\nabla v_n|^2 \leq C|\nabla u_n|^2
$$

so that $v_n \in H_V^1 L_{w_0}^q(\mathbb{R}^2)$. Taking $v = v_n = \frac{F(u_n)}{f(u_n)}$ in (6.2) and recalling that $f(s), F(s) = 0$ for any $s \leq 0$ yields

(6.4)
$$
\left|(1 - \delta) \int_{\{u_n \leq 0\}} |\nabla u_n|^2 + \int_{\{u_n > 0\}} |\nabla u_n|^2 \left(1 - \frac{F(u_n)f'(u_n)}{f^2(u_n)}\right) dx + \int_{\mathbb{R}^2} V(x)u_n \frac{F(x, u_n)}{f(x, u_n)} dx \right.
$$
$$
\left. - \frac{1}{2\pi} \int_{\mathbb{R}^2} \left[\log\frac{1}{|x|} * F(x, u_n)\right] F(x, u_n) \right|
$$
$$
\leq \tau_n \|v_n\|_{H_V^{-1} L_{w_0}^q} \leq \tau_n \|u_n\|_V \ .
$$

Now recall (6.1), namely

$$
\frac{1}{2\pi} \int_{\mathbb{R}^2} \left[\log\left(\frac{1}{|x|}\right) * F(x, u_n)\right] F(x, u_n) dx = \|u_n\|_V^2 - 2c + o(1) \ .
$$

Only two cases may occur as $n \to +\infty$ (we are not excluding that both the two cases may appear for different subsequence of u_n):

- $\int_{\mathbb{R}^2} \left[\log\left(\frac{1}{|x|}\right) * F(x, u_n)\right] F(x, u_n) dx \leq 0$: in this case we have, directly,

$$
\|u_n\|_V^2 \leq 2c + o(1) \leq 3c
$$

  and $c \geq 0$;

- $\int_{\mathbb{R}^2} \left[\log\left(\frac{1}{|x|}\right) * F(x, u_n)\right] F(x, u_n) > 0$. In this case, combining (6.1) and (6.4) yields

$$
\int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + Vu_n^2\right) dx - 2c + o(1) - \int_{\{u_n > 0\}} |\nabla u_n|^2 \left(1 - \frac{F(u_n)f'(u_n)}{f^2(u_n)}\right) dx
$$
$$
- (1 - \delta) \int_{\{u_n \leq 0\}} |\nabla u_n|^2 dx - \int_{\mathbb{R}^2} V(x)u_n \frac{F(u_n)}{f(u_n)} dx \leq \tau_n \|u_n\|_V
$$

Since, as a consequence of (f_2), $F(x, s) \leq (1 - \delta)sf(x, s)$ (see also (1.5)) we have, as $n \to \infty$

$$
\delta \int_{\mathbb{R}^2} |\nabla u_n|^2 + Vu_n^2 dx \leq \tau_n \|u_n\|_V + 2c + o(1) \ .
$$

In conclusion we have proved that

$$
\|u_n\|_V \leq C \ .
$$

As a consequence, from (6.1) and (6.3), we also have

$$
\left|\int_{\mathbb{R}^2} \left[\log\frac{1}{|x|} * F(x, u_n)\right] F(x, u_n) dx\right| \leq C, \qquad \left|\int_{\mathbb{R}^2} \left[\log\frac{1}{|x|} * F(x, u_n)\right] u_n f(x, u_n) dx\right| \leq C,
$$

that is our thesis.

$\square$

Remark 6.2. *Let us note that, as a consequence of the previous Lemma 6.1, given any PS sequence at level c there is always another positive PS sequence at the same level. Indeed, since u_n is bounded in $H^1_V L^q_{w_0}(\mathbb{R}^2)$ as proved above, we can test $\mathcal{I}'_V(u_n)$ against $u_n^- = \max(-u_n, 0)$ obtaining (see (6.2))*

$$\left| \int_{\{u_n < 0\}} |\nabla u_n|^2 + V|u_n|^2 dx \right| \leq \tau_n C$$

Hence, the sequence $\{u_n^+\}$ is positive, and it is again a PS sequence at the same level c (since $F(s) = 0$ for $s \leq 0$).

From now on we will consider only *positive* PS sequence, even if not specified.
Differently from standard contexts in which having proved boundedness of a PS sequence brings the conclusion at hand, here it does not allow to employ standard arguments to prove the weak limit is actually a nontrivial solution to the equation. Indeed, the presence of the exponential nonlinearity together with the sign-changing behavior of the logarithmic kernel, prevents the application of standard estimates. Here comes into play the key estimate for the mountain pass level $m_V < 1/2$ established in Lemma 5.2.

Lemma 6.3. *Assume $(V_1) - (V_2)$ and $(f_1) - (f_4)$. Let $\{u_n\} \subset H^1_V L^q_{w_0}$ be a (positive) PS sequence for I_V at level $c < 1/2$. Then, for any $1 \leq \alpha < \frac{1}{2c}$ the following uniform bound holds*

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^2} [F(x, u_n)]^\alpha < \infty .$$

Proof. Without loss of generality, we consider $f(x, s) = f(s)$. By Lemma 6.1 the sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$ and we may assume $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$, $u_n \to u$ in $L^s_{loc}(\mathbb{R}^2)$ for any $1 \leq s < \infty$ and $u_n \to u$ a.e. in $\mathbb{R}^2$, with

$$\lim_{n \to +\infty} \|u_n\|_V^2 = A^2 \geq \|u\|_V^2$$

As in the proof of Lemma 6.1, we will carefully select a suitable test function v_n. Let us introduce the following auxiliary function

$$G(t) = \int_0^t \frac{\sqrt{F(s)f'(s)}}{f(s)} ds$$

which is well defined and $\mathcal{C}^1$ thanks to (f_2). Moreover, by Hölder's inequality we have

$$(6.5) \quad G^2(t) \leq \int_0^t ds \cdot \int_0^t \frac{Ff'}{f^2} ds = t \left[\int_0^t \left(\frac{Ff' - f^2}{f^2} + 1 \right) ds \right] =$$

$$= t \left[\int_0^t -\frac{d}{ds} \frac{F}{f} + t \right] = t^2 - t \frac{F(t)}{f(t)}$$

Define

$$v_n := G(u_n) ,$$

then

$$\int_{\mathbb{R}^2} |\nabla v_n|^2 dx = \int_{\mathbb{R}^2} |\nabla u_n|^2 \frac{Ff'}{f^2}(u_n) dx \leq C, \quad \int_{\mathbb{R}^2} V v_n^2 dx = \int_{\mathbb{R}^2} V G^2(u_n) dx \leq C$$

as u_n is bounded in H^1 and applying again (f_2). We aim at proving that

$$\|\nabla v_n\|_2^2 + \|\sqrt{V(x)} v_n\|_2^2 \leq 1$$

as n is large enough. First, note that as $n \to +\infty$, one has

$$0 \leq \int_{\mathbb{R}^2} V G^2(u_n) dx \leq \int_{\mathbb{R}^2} V u_n^2 dx \leq C .$$

In order to estimate the norm $\|v_n\|_V^2$ recall (6.1) and (6.4). From $\|u_n\|_V^2 \to A^2 \geq \|u\|_V^2$, we have

$$\lim_{n\to+\infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} \left[\log\left(\frac{1}{|x|}\right) * F(u_n)\right] F(u_n) dx = A^2 - 2c$$

and

$$\left| \int_{\mathbb{R}^2} |\nabla u_n|^2 \left(1 - \frac{F(u_n)f'(u_n)}{f^2(u_n)}\right) dx + \int_{\mathbb{R}^2} V(x) u_n \frac{F(u_n)}{f(u_n)} dx \right.$$
$$\left. - \frac{1}{2\pi} \int_{\mathbb{R}^2} \left[\log\frac{1}{|x|} * F(u_n)\right] F(u_n) dx \right| \to 0$$

so that

$$\int_{\mathbb{R}^2} |\nabla u_n|^2 \left(1 - \frac{F(u_n)f'(u_n)}{f^2(u_n)}\right) dx + \int_{\mathbb{R}^2} V(x) u_n \frac{F(u_n)}{f(u_n)} dx + 2c$$
$$- \int_{\mathbb{R}^2} |\nabla u_n|^2 dx - \int_{\mathbb{R}^2} V u_n^2 dx = \mathrm{o}(1)$$

and in turn

$$(6.6) \quad \|v_n\|_V^2 = \int_{R^2} |\nabla G(u_n)|^2 dx + \int_{\mathbb{R}^2} V(x) G^2(u_n) dx$$
$$= 2c + \int_{R^2} V(x) \left(u_n \frac{F(u_n)}{f(u_n)} - u_n^2 + G^2(u_n)\right) dx + \mathrm{o}(1) \leq 2c + \mathrm{o}(1) < 1$$

by (6.5), as n is large enough.

Once we have estimated the norm of v_n, let us take advantage of this to improve the exponential integrability of the original sequence u_n. By (f_3), for any $\epsilon > 0$ there exists a constant $t_\epsilon > 0$ such that

$$1 - \epsilon < \frac{\sqrt{F(t)f'(t)}}{f(t)} \leq 1 + \epsilon, \quad \text{for all } t \geq t_\epsilon .$$

Next by (f_2) we also have either $u_n(x) \leq t_\epsilon$ or $u_n(x) \geq t_\epsilon$ which implies

$$(6.7) \quad v_n \geq \int_0^{t_\epsilon} \delta dt + \int_{t_\epsilon}^{u_n} (1 - \epsilon) dt \geq \delta t_\epsilon + (1 - \epsilon)(u_n - t_\epsilon) \geq (1 - \epsilon)(u_n - t_\epsilon)$$

and thus

$$u_n \leq t_\epsilon + \frac{v_n}{1 - \epsilon}, \quad \text{for any } x \in \mathbb{R}^2 .$$

Hence (hereafter C_ϵ may change from line to line)

$$(6.8) \quad \int_{\mathbb{R}^2} [F(u_n)]^\alpha \, dx = \int_{u_n \leq t_\epsilon} [F(u_n)]^\alpha \, dx + \int_{u_n \geq t_\epsilon} [F(u_n)]^\alpha \, dx$$
$$\leq C_\epsilon \int_{u_n \leq t_\epsilon} [u_n^2]^\alpha \, dx + \int_{u_n \geq t_\epsilon} \left[F\left(t_\epsilon + \frac{v_n}{1 - \epsilon}\right)\right]^\alpha \, dx$$
$$\leq C_\epsilon \int_{u_n \leq t_\epsilon} u_n^2 dx + C \int_{u_n \geq t_\epsilon} \left(t_\epsilon + \frac{v_n}{1 - \epsilon}\right)^{\alpha(p-1)} e^{4\pi\alpha(t_\epsilon + \frac{v_n}{1-\epsilon})^2} dx$$
$$\leq C_\epsilon \|u_n\|_2^2 + C_\epsilon \int_{u_n \geq t_\epsilon} e^{4\pi\alpha(1+\epsilon)(t_\epsilon + \frac{v_n}{1-\epsilon})^2} dx$$

where, in the last line, we use the following inequality: for any $T > 0$ and for any $\epsilon > 0$ there exists $C = C_{T,\epsilon}$ such that $s^{p-1} \leq C_{T,\epsilon} e^{4\pi\epsilon s^2}$ for any $s \geq T$ (with $T = t_\epsilon$ already fixed as well

as $C = C_\epsilon$). Moreover, for any $\epsilon > 0$ there exists C_ϵ such that $(t+s)^2 \le C_\epsilon t^2 + (1+\epsilon)s^2$ for any $s, t > 0$. Then

$$(6.9) \qquad \left(t_\epsilon + \frac{v_n}{1-\epsilon}\right)^2 \le C_\epsilon t_\epsilon^2 + (1+\epsilon)\left(\frac{v_n}{1-\epsilon}\right)^2 .$$

As byproduct of (6.7), if $u_n \ge t_\epsilon$ then $v_n \ge \delta t_\epsilon$. Combining this with (6.9) and (6.8) we obtain

$$\int_{\mathbb{R}^2} [F(u_n)]^\alpha dx \le C_\epsilon \|u_n\|_2^2 + C_\epsilon \int_{u_n \ge t_\epsilon} e^{4\pi\alpha(1+\epsilon)^2 \frac{v_n^2}{(1-\epsilon)^2}} dx$$

$$\le C_\epsilon \|u_n\|_2^2 + C_\epsilon \int_{\mathbb{R}^2} e^{4\pi\alpha(1+\epsilon)^2 \frac{v_n^2}{(1-\epsilon)^2}} - 1 dx .$$

Let us now fix $0 < \epsilon < 1$ and set

$$\eta := \frac{1}{2c} - \alpha > 0, \quad \epsilon_\alpha := c^2\eta^2 = c^2\left(\frac{1}{2c} - \alpha\right)^2 < \frac{1}{4} .$$

With these choices we obtain

$$\int_{\mathbb{R}^2} [F(u_n)]^\alpha dx \le C_\alpha \|u_n\|_2^2 + C_\alpha \int_{\mathbb{R}^2} e^{4\pi\alpha \frac{(1+\epsilon_\alpha)^2}{(1-\epsilon_\alpha)^2}\|v_n\|_V^2 \frac{v_n^2}{\|v_n\|_V^2}} - 1 dx .$$

By (6.6), $\|v_n\|_V^2 \le 2c + o(1)$ as n is large enough, so that

$$\|v_n\|_V^2 \le 2c + 4c^2\eta, \quad \text{as } n \to +\infty .$$

Hence,

$$\alpha \frac{(1+\epsilon_\alpha)^2}{(1-\epsilon_\alpha)^2}\|v_n\|_V^2 \le 2c\left(\frac{1}{2c} - \eta\right)\frac{(1+c^2\eta^2)^2}{(1-c^2\eta^2)^2}(1+2c\eta) = \frac{(1+c^2\eta^2)^2}{(1-c^2\eta^2)^2}(1-4c^2\eta^2) < 1$$

since the last inequality is equivalent to

$$(1+c^2\eta^2)^2(1-4c^2\eta^2) = (1+c^4\eta^4 + 2c^2\eta^2)(1-4c^2\eta^2) < (1-c^2\eta^2)^2 = 1 + c^4\eta^4 - 2c^2\eta^2 .$$

In conclusion we have

$$\int_{\mathbb{R}^2} [F(u_n)]^\alpha dx \le C_\alpha \|u_n\|_2^2 + C_\alpha \int_{\mathbb{R}^2} e^{4\pi \frac{v_n^2}{\|v_n\|_V^2}} - 1 dx \le C_\alpha$$

by the Ruf inequality (2.4) and Remark 2.1.   □

Proposition 6.4. *Assume that conditions* (V_1)–(V_2) *and* (f_1)–(f_4) *are satisfied. Let* $\{u_n\} \subset H_V^1 L_{w_0}^q$ *be a (poisitive) PS sequence for* I_V *at level* $c < 1/2$, *weakly converging to* u *in* H^1. *If* $u \ne 0$, *then* $u \in H_V^1 L_{w_0}^q$ *and* $u_n \rightharpoonup u$ *weakly in* $H_V^1 L_{w_0}^q$. *Furthermore, as* $n \to \infty$

$$(6.10) \quad [\log|x| * F(x, u_n)] f(x, u_n) \longrightarrow [\log|x| * F(x, u)] f(x, u) \quad \text{in } L_{loc}^1(\mathbb{R}^2)$$

and u *is a weak solution to* (1.3).

Proof. Fix $\alpha \in (1, 1/2c)$ so that by (6.1) we have

$$\frac{1}{2}\|u_n\|_V^2 + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1 + |x-y|) F(u_n(x))F(u_n(y)) dx dy =$$

$$= c + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1 + \frac{1}{|x-y|}\right) F(u_n(x))F(u_n(y)) dx dy + o(1)$$

$$\le c + 1 + C_\alpha \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left[1 + \frac{1}{|x-y|^{4\frac{\alpha-1}{\alpha}}}\right] F(u_n(x))F(u_n(y)) dx dy$$

$$\le c + 1 + C_\alpha \left\{\int_{\mathbb{R}^2} [F(u_n)]^\alpha dx\right\}^2 + C_\alpha \left\{\int_{\mathbb{R}^2} F(u_n) dx\right\}^2$$

by Proposition 2.3, since $\frac{2}{\alpha} + 4\frac{\alpha-1}{\alpha}\frac{1}{2} = 2$. Hence, by Lemma 6.3,

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1 + |x - y|\right) F(u_n(x))F(u_n(y))dxdy \leq C_\alpha$$

as $n \to +\infty$, and thus also $\int_{\mathbb{R}^2 \times \mathbb{R}^2} \log\left(1 + |x - y|\right) u_n^q(x)u_n^q(y)dxdy$ is bounded. Since $u \neq 0$, by Lemma 2.1 in [15] we have

$$\int_{\mathbb{R}^2} \log(1 + |x|^2)u_n^q(x)dx \leq C \quad \text{as } n \to +\infty$$

so that $\|u_n\|_{q,V,w_0}^2$ is bounded. Up to a subsequence we have $u_n \rightharpoonup u$ in $H_V^1 L_{w_0}^q$. Moreover, recall that as $n \to +\infty$,

$$(6.11) \quad \frac{1}{2\pi}\int_{\mathbb{R}^2}\left[\log\frac{1}{|x|} * F(u_n)\right]vf(u_n)dx = \int_{\mathbb{R}^2} \nabla u \nabla v + V(x)uvdx + \mathrm{o}(1), \quad v \in H_V^{-1}L_{w_0}^q,$$

in particular for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. In order to prove that u is a weak solution of (1.3), let us suppose for the moment the following
Claim:

$$\int_{\mathbb{R}^2} \left|\log\frac{1}{|x|} * F(u_n)\right| f(u_n)u_n dx \leq C$$

of which we postpone the proof. Now we apply Lemma 2.1 in [19] to the sequence of functions

$$g(y, u_n(y)) := \left(\log\frac{1}{|x|} * F(u_n)\right)(y)f(u_n(y)),$$

restricted to any compact domain Ω: they are L^1 functions since $u_n, u \in H_V^1 L_{w_0}^q$ and, thanks to the claim, $u_n(y)g(y, u_n(y))$ is uniformly bounded in L^1. Therefore, from [19] we have

$$\left(\log\frac{1}{|x|} * F(u_n)\right)f(u_n)dx \to \left(\log\frac{1}{|x|} * F(u)\right)f(u)dx \quad \text{in } L_{loc}^1(\mathbb{R}^2)$$

as well as

$$\int_{\mathbb{R}^2}\left[\log\frac{1}{|x|} * F(u_n)\right]f(u_n)\varphi dx \to \int_{\mathbb{R}^2}\left[\log\frac{1}{|x|} * F(u)\right]f(u)\varphi dx$$

for any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, which is a dense subset of $H_V^{-1}L_{w_0}^q$. This together with (6.11) implies that u is a weak solution of (1.3).
Proof of the Claim.
The key ingredient is the uniform bound provided by Lemma 6.3. In order to simplify the notation, let us set

$$w_n(y) = \left(\log\frac{1}{|x|} * F(u_n)\right)(y)$$

By (6.3)

$$\int_{\mathbb{R}^2} w_n f(u_n)u_n dx = A^2 + \mathrm{o}(1) ,$$

where $A = \lim_{n \to +\infty} \|u_n\|_V \geq \|u\|_V$ so that

$$\int_{\mathbb{R}^2} w_n f(u_n)u_n dx > 0$$

for n large enough (note that we are assuming $u \neq 0$, that is, $A^2 > 0$). Hence,

$$0 < \int_{\mathbb{R}^2} w_n f(u_n)u_n dx = \int_{w_n>0} w_n f(u_n)u_n dx + \int_{w_n<0} w_n f(u_n)u_n dx$$

which implies

$$\int_{\mathbb{R}^2} w_n^- f(u_n)u_n dx < \int_{\mathbb{R}^2} w_n^+ f(u_n)u_n dx$$

and thus

$$\int_{\mathbb{R}^2} |w_n| f(u_n) u_n dx \leq 2 \int_{\mathbb{R}^2} w_n^+ f(u_n) u_n dx .$$

Now, we have

$$\int_{\mathbb{R}^2} w_n^+ f(u_n) u_n = \int_{y:w_n>0} f(u_n(y)) u_n(y) dy \int_{\mathbb{R}^2} \log(1 + \frac{1}{|x-y|}) F(u_n(x)) dx$$

$$- \int_{y:w_n>0} f(u_n(y)) u_n(y) dy \int_{\mathbb{R}^2} \log(1 + |x-y|) F(u_n(x)) dx$$

$$\leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(1 + \frac{1}{|x-y|}) F(u_n(x)) f(u_n(y)) u_n(y) dx dy .$$

Therefore, for any $\mu > 0$, small, there exists a constant $\delta_\mu \in (0,1)$ sufficiently small such that

$$\int_{\mathbb{R}^2} w_n^+ f(u_n) u_n dx \leq \int_{\mathbb{R}^2} dy \int_{|x-y|>\delta_\mu} \log(1 + \delta_\mu^{-1}) F(u_n(x)) f(u_n(y)) u_n(y) dx +$$

$$+ 2 \int_{\mathbb{R}^2} dy \int_{|x-y|<\delta_\mu} \frac{1}{|x-y|^\mu} F(u_n(x)) f(u_n(y)) u_n(y) dx$$

$$\leq C_\mu \int_{\mathbb{R}^2} F(u_n(x)) dx \int_{\mathbb{R}^2} f(u_n) u_n(y) dy$$

$$+ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x-y|^\mu} F(u_n(x)) f(u_n)(y) u_n(y) dx dy .$$

The first integral in the last expression is uniformly bounded, as one can see by Lemma 6.3 and Holder's inequality, recalling that $\|u_n\|_V$ is also uniformly bounded. Concerning the second term, by the HLS inequality if $2/s + \mu/2 = 1$, one has

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x-y|^\mu} F(u_n(x)) f(u_n(y)) u_n(y) dx dy \leq C(\mu) \|F(u_n)\|_s \|f(u_n) u_n\|_s .$$

Since

$$s = \frac{4}{4-\mu} \to 1 \quad \text{as} \quad \mu \to 0$$

we can choose μ small enough to apply again Lemma 6.3 and Holder's inequality, to obtain that $\|F(u_n)\|_s \|f(u_n) u_n\|_s$ stays bounded. Finally, $\int w_n^+ f(u_n) u_n$ is bounded and the same holds for $\int |w_n| f(u_n) u_n$, that is our claim. $\qquad\square$

7. Proof of Theorem 1.2

We are now ready to prove Theorem 1.2. From Lemma 5.1, the functional I_V satisfies the Mountain Pass geometry. Hence, there exists a (positive) (PS) sequence $\{u_n\} \subset H_V^1 L_{w_0}^q(\mathbb{R}^2)$ at level m_V and by Lemma 6.3, $\{u_n\}$ is bounded in H^1 and it weakly converges to some $u \in H^1$. We have that either $\{u_n\}$ is vanishing, that is for any $r > 0$

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^2} \int_{B_r(y)} u_n^2 dx = 0$$

or there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{Z}^2$ such that

$$\lim_{n\to\infty} \int_{B_r(y_n)} u_n^2 dx \geq \delta.$$

If $\{u_n\}$ is vanishing, by Lions' concentration-compactness result we have

(7.1) $u_n \to 0 \quad \text{in} \quad L^s(\mathbb{R}^2) \quad \forall s > 2,$

as $n \to \infty$. In this case it is standard to show that

$$\|F(x, u_n)\|_\gamma, \|u_n f(x, u_n)\|_\gamma \to 0$$

for some values of $\gamma > 1$ and close to 1, thanks to the improved exponential integrability given by Lemma 6.3 and the growth assumption $F(x,t) < tf(x,t)$. Hence, applying the HLS inequality we deduce, similarly to the conclusion of the proof of Proposition 6.4:

$$(7.2) \qquad \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log\left(1 + \frac{1}{|x-y|}\right) F(x, u_n(x)) F(y, u_n(y)) dx dy \to 0$$

$$(7.3) \qquad \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log\left(1 + \frac{1}{|x-y|}\right) F(x, u_n(x)) u_n(y) f(y, u_n(y)) dd x dy \to 0$$

as $n \to \infty$. Combining (7.2)–(7.3) with (6.1) and (6.11) yields

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log\left(1 + |x-y|\right) F(x, u_n(x)) \left[F(y, u_n(y)) - u_n(y) f(y, u_n(y))\right] dx dy = 2 m_V + \mathrm{o}(1)$$

so that $m_v \leq 0$, which is not possible. Therefore the vanishing case does not occur.

Now set $v_n := u_n(\cdot - y_n)$, then

$$(7.4) \qquad \int_{B_r(0)} v_n^2 dx \geq \delta \ .$$

By the periodicity assumption, I_V and I'_V are both invariant by the $\mathbb{Z}^2$-action, so that $\{v_n\}$ is still a PS sequence at level m_V. Then $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^2)$ with $v \neq 0$ by using (7.4), since $v_n \to v$ in $L^2_{loc}(\mathbb{R}^2)$. We conclude by Proposition 6.4 that $v \in H^1_V L^q_{w_0}$ is a nontrivial critical point of I_V and $I_V(v) = m_V$, which completes the proof of Theorem 1.2.

References

[1] C.O. Alves, D. Cassani, C. Tarsi and M. Yang, *Existence and concentration of ground state solutions for a critical nonlocal Schrödinger equation in* $\mathbb{R}^2$, J. Differential Equations **261** (2016), 1933–1972.

[2] C.O. Alves and G.M. Figueiredo, *Existence of positive solution for a planar Schrödinger–Poisson system with exponential growth*, J. Math. Phys. **60** (2019), 011503.

[3] L. Battaglia and J. Van Schaftingen, *Existence of groundstates for a class of nonlinear Choquard equations in the plane*, Adv. Nonlinear Stud. **17** (2017), 581–594.

[4] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser–Trudinger inequality*, Ann. Math., **138** (1993), 213–242.

[5] V. Benci and D. Fortunato, *Variational methods in nonlinear field equations. Solitary waves, hylomorphic solitons and vortices*, Springer Monographs in Mathematics, Springer, 2014.

[6] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics, **129**, Boston Academic Press, 1988.

[7] D. Bonheure, S. Cingolani and J. Van Schaftingen, *The logarithmic Choquard equation: sharp asymptotics and nondegeneracy of the groundstate*, J. Funct. Anal. **272** (2017), 5255–5281.

[8] M. Calanchi and B. Ruf, *On Trudinger–Moser type inequalities with logarithmic weights*, J. Differential Equations **258** (2015), 1967–1989.

[9] D. Cao, *Nontrivial solution of semilinear elliptic equation with critical exponent in* $\mathbb{R}^2$, Comm. Partial Differential Equation, **17** (1992), 407–435.

[10] E. Carlen and M. Loss *Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on* S^n, Geom. Funct. Anal. **2** (1992), 90–104.

[11] D. Cassani, F. Sani and C. Tarsi, *Equivalent Moser type inequalities in* $\mathbb{R}^2$ *and the zero mass case*, J. Funct. Anal. **267** (2014), 4236–4263.

[12] D. Cassani, C. Tarsi and J. Zhang, *Bounds for best constants in subcritical Sobolev embeddings*, Nonlinear Anal. **187** (2019), 438–449.

[13] D. Cassani, J. Van Schaftingen and J. Zhang, *Groundstates for Choquard type equations with Hardy–Littlewood–Sobolev lower critical exponent*, Proc. Roy. Soc. Edinburgh Sect. A **150** (2020), 1377–1400.

[14] D. Cassani and J. Zhang, *Choquard-type equations with Hardy–Littlewood–Sobolev upper-critical growth*, Adv. Nonlinear Anal. **8** (2019), 1184–1212.

[15] S. Cingolani and T. Weth *On the planar Schrödinger-Poisson system*, Ann. Inst. H. Poincaré Anal. Non Linéaire **33** (2016), 169–197.

[16] M. Dong and G. Lu, *Best constants and existence of maximizers for weighted Trudinger–Moser inequalities*, Calc. Var. Partial Differential Equations **55** (2016), 324–353.

[17] L. Du and M. Yang, *Uniqueness and nondegeneracy of solutions for a critical nonlocal equation*, Discrete Contin. Dyn. Syst. **39** (2019), 5847–5866.

[18] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. **47** (1974), 324–353.

[19] D.G. de Figueiredo, O.H. Miyagaki and B. Ruf, *Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range*, Calc. Var. Partial Differential Equations **3** (1995), 139–153.

[20] H. Fröhlich, *Theory of electrical breakdown in ionic crystal*, Proc. Roy. Soc. Ser. A **160** (1937), 230–241.

[21] M. Ishiwata, *Existence and nonexistence of maximizers for variational problems associated with Trudinger–Moser type inequalities in $\mathbb{R}^N$*, Math. Ann. **351** (2011), 781–804.

[22] A. Kufner and B. Opic, *How to define reasonably weighted Sobolev spaces*, Comment. Math. Univ. Carolin. **25** (1984), 537–554.

[23] E.H. Lieb and M. Loss *Analysis. Second edition*, Graduate Studies in Mathematics 14. American Mathematical Society, Providence, RI, 2001.

[24] E.H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Studies in Applied Mathematics **57** (1977), 93–105.

[25] V. Moroz and J. Van Schaftingen, *A guide to the Choquard equation*, J. Fixed Point Theory Appl. **19** (2017), 773–813.

[26] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1970/71), 1077-1092.

[27] V.H. Nguyen and F. Takahashi, *On a weighted Trudinger-Moser type inequality on the whole space and related maximizing problem*, Differential Integral Equations **31** (2018), 785–806.

[28] J.M. do Ó, M. de Souza, E. de Medeiros and U. Severo *An improvement for the Trudinger–Moser inequality and applications*, J. Differential Equations **256** (2014), 1317–1349.

[29] R. Penrose, *On gravity's role in quantum state reduction*, Gen. Rel. Grav. **28** (1996), 581–600.

[30] M. del Pino, J. Dolbeault, S. Filippas and A. Tertikas, *A logarithmic Hardy inequality*, J. Funct. Anal. **259** (2010), 2045–2072.

[31] S.I. Pohožaev, *The Sobolev embedding in the case $pl = n$*, Proc.Tech. Sci. Conf. on Adv. Sci., Research 1964–1965, Mathematics Section (1965), 158–170, Moskov. Ènerget. Inst., Moscow.

[32] B. Ruf, *A sharp Trudinger–Moser type inequality for unbounded domains in $\mathbb{R}^2$*, J. Funct. Anal **219** (2005), 340–367.

[33] J. Simon, *Banach, Fréchet, Hilbert and Neumann spaces*, Analysis for PDEs set. Vol. 1. Mathematics and Statistics Series, John Wiley & Sons, Inc., Hoboken, NJ (2017).

[34] J. Stubbe, *Bound states of two-dimensional Schrödinger–Newton equations*, arXiv:0807.4059v1, 2008.

[35] G. Talenti, *Best constants in Sobolev inequality*, Ann. Mat. Pura Appl., **110** (1976), 353–372.

[36] N.S. Trudinger, *On embeddings into Orlicz spaces and some applications*, J. Math. Mech. ,**17** (1967), 473–484.

[37] T. Wang and T. Yi, *Symmetry of separable functions and its applications to Choquard type equations*, Calc. Var. Partial Differential Equations **59** (2020), 23 pp.

[38] V.I. Yudovich, *Some estimates connected with integral operators and with solutions of elliptic equations*, Dokl. Akad. Nauk. SSSR **138** (1961), 805–808.

(D. Cassani)
Dip. di Scienza e Alta Tecnologia
Università degli Studi dell'Insubria
and
RISM–Riemann International School of Mathematics
Villa Toeplitz, Via G.B. Vico, 46 – 21100 Varese
Email address: daniele.cassani@uninsubria.it

(C. Tarsi)
Dipartimento di Matematica
Università degli Studi di Milano
Via C. Saldini, 50 – 20133 Milano
Email address: cristina.tarsi@unimi.it