DETECTION OF DISLOCATIONS IN A 2D ANISOTROPIC ELASTIC MEDIUM

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In ricordo di Assunta, amica cara e collega
In memory of Assunta, beloved friend and colleague

Abstract. We study a model of dislocations in two-dimensional elastic media. In this model, the displacement satisfies the system of linear elasticity with mixed displacement-traction homogeneous boundary conditions in the complement of an open curve in a bounded planar domain, and has a specified jump, the slip, across the curve, while the traction is continuous there. The stiffness tensor is allowed to be anisotropic and inhomogeneous. We prove well-posedness of the direct problem in a variational setting, assuming the coefficients are Lipschitz continuous. Using unique continuation arguments, we then establish uniqueness in the inverse problem of determining the dislocation curve and the slip from a single measurement of the displacement on an open patch of the traction-free part of the boundary. Uniqueness holds when the elasticity operators admits a suitable decomposition and the curve satisfies additional geometric assumptions. This work complements the results in *Arch. Ration. Mech. Anal.*, 236(1):71–111, 2020, and in Preprint arXiv:2004.00321, which concern three-dimensional isotropic elastic media.

1. Introduction

In this work, we consider a model of dislocations in two-dimensional anisotropic and inhomogeneous elastic media. The corresponding three-dimensional model is used in geophysics (see for instance [11, 12] and references therein). We confine ourselves to the two-dimensional setting for technical reasons. We work in a bounded planar domain Ω, representing a portion of a thin elastic body, where the dislocation is located. The exposed boundary of this part of the body satisfies traction-free (homogeneous Neumann) conditions, while the hidden boundary satisfies displacement-free (homogeneous Dirichlet) conditions and it is assumed sufficiently far from the defect. The dislocation itself is represented by a bounded open curve $\mathcal{C}$, supported away from the boundary, where the displacement jumps, while the traction is continuous. The jump in the displacement $\boldsymbol{g}$ represents the relative slip of the elastic material on the two sides of the curve $\mathcal{C}$.

We study both the direct as well as the inverse dislocation problem. Uniqueness and stability in the two-dimensional homogeneous and isotropic case has been proved in [5]. In the direct or forward problem, we solve the linear elasticity system in the complement of the dislocation curve, given the elastic coefficients and slip vector along the curve (see

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Problem (5)). The coefficients are taken Lipschitz continuous. In the inverse problem, we seek to determine the location and shape of the dislocation curve and the slip vector from a given single measurement of the displacement on an open section of the exposed boundary of Ω. We refer to [4, 3] for a more in-depth discussion of the corresponding isotropic 3D model that motivates this work and its validity in the geophysical context, and for a survey of the few existing results in the literature ([10, 14]).

Under the assumption that the stiffness tensor is strongly convex and Lipschitz continuous, we prove the well-posedness of the direct problem in $H^1(\Omega \setminus \mathcal{C})$, when $\mathcal{C}$ is an orientable, Lipschitz curve and $\boldsymbol{g}$ belongs to a suitable subspace $H^{1/2}_{00}(\mathcal{C})$ of the trace space $H^{1/2}(\mathcal{C})$ (see (1)). This space has good extension properties. We utilize a harmonic lift of $\boldsymbol{g}$ to recast Problem 5 as a source problem in the whole Ω, which we solve in variational form. Uniqueness in the inverse problem is established using unique continuation results for the 2D anisotropic elasticity operator, which holds when the latter has a certain property that essentially allows to diagonalize the system [8] (see also [6]). Examples of elastic media satisfying this condition include orthotropic materials. For the inverse problem, we need to assume additionally that the curve $\mathcal{C}$ is globally a graph of a Lipschitz function with respect to an arbitrary, but fixed, coordinate system. These results complement the results in [4, 3], which pertain to the 3D isotropic case.

Well-posedness holds more generally in any space dimension, but unique continuation for the anisotropic elasticity system needed for the inverse problem is available only in the planar case. We note that the strong unique continuation holds for all 2D elasticity operators the principal symbol of which has simple characteristics, provided the coefficients are Gevrey-class [6]. This regularity is rather strong for applications. As a matter of fact, both the direct as well as the inverse problem can be addressed assuming the coefficients are piecewise Lipschitz continuous following the approach in [4], which included the important case of composite media. We confine ourselves to the case of Lipschitz coefficients to keep the article more self-contained and for ease of presentation.

The paper is organized as follows. In Section 2, we introduce some needed notation and discuss the function spaces we utilize for our work. In Section 3, we present the direct problem and prove its well-posedness Finally, in Section 4, we discuss the inverse problem and prove its uniqueness.

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2. Notation and Functional Setting

We begin by setting notation that we will use throughout. Next, we discuss the functional setting to which our results apply.

Notation (Tensors). Scalar quantities are denoted in italics, e.g. λ, μ, ν, points and vectors in bold italics, e.g. $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$, matrices and second-order tensors in bold face, e.g. $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and fourth-order tensors in blackboard face, e.g. $\mathbb{A}, \mathbb{B}, \mathbb{C}$.

The symmetric part of a second-order tensor $\mathbf{A}$ is denoted by $\widehat{\mathbf{A}} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right)$, where $\mathbf{A}^T$ is the transpose matrix. In particular, $\widehat{\nabla}\boldsymbol{u}$ represents the deformation tensor. We utilize standard notation for inner products, that is, $\boldsymbol{u} \cdot \boldsymbol{v} = \sum_i u_i v_i$, and $\mathbf{A} : \mathbf{B} = \sum_{i,j} a_{ij} b_{ij}$. $|\mathbf{A}|$ denotes the norm induced by the inner product on matrices:

$$|\mathbf{A}| = \sqrt{\mathbf{A} : \mathbf{A}}.$$

Notation (Function spaces). We follow standard notation for Lebesgue and Sobolev spaces. In particular, $H^s(\Omega)$, $s \geq 0$, denote L^2-based Sobolev spaces on Ω. $C_0^\infty(\Omega)$ is the space of smooth functions with compact support in Ω. A vector-valued function belongs to a function space if every component belongs to the same space. With slight abuse of notation, we employ the same notation for spaces of vector-valued or scalar-valued functions (e.g. for vector fields in $\mathbb{R}^2$ we write $H^1(\mathbb{R}^2)$ instead of $[H^1(\mathbb{R}^2)]^2$).

In the sequel, we will utilize function spaces on bounded Lipschitz curves. These are defined in the usual way via a partition of unity and local coordinate charts. We will also need to introduce suitable weighted spaces that have a good extension property on such curves. Let $\mathcal{C}$ be a bounded open curve, such that its closure is a Lipschitz curve. Following [7, 9], we let

$$H_{00}^{1/2}(\mathcal{C}) := \left\{ u \in H_0^{1/2}(\mathcal{C}), \delta^{-1/2} u \in L^2(\mathcal{C}) \right\}, \tag{1}$$

where $\delta(x) = \mathrm{dist}(x, \partial\mathcal{C})$, for $x \in \mathcal{C}$, and $H_0^{1/2}(\mathcal{C})$ is the closure of the space of smooth functions with compact support in $\mathcal{C}$ with respect to the $H^{1/2}(\mathcal{C})$ norm. This space is equipped with its natural norm, that is,

$$\|f\|_{H_{00}^{1/2}(\mathcal{C})} := \|f\|_{H^{1/2}(\mathcal{C})} + \|\delta^{-1/2} f\|_{L^2(\mathcal{C})},$$

We note that the distance $\delta(x)$ is comparable here with the intrinsic distance on the curve $\mathcal{C}$, given the hypotheses we made on $\mathcal{C}$. We can extend $\mathcal{C}$ to a closed Lipschitz curve $\widetilde{\mathcal{C}}$. Then $H_{00}^{1/2}(\mathcal{C})$ coincides with the space of functions $\boldsymbol{g} \in H^{1/2}(\mathcal{C})$ that can be continuously extended by zero to a function $\widetilde{\boldsymbol{g}} \in H^{1/2}(\widetilde{\mathcal{C}})$. Indeed, every elements of $H_{00}^{1/2}(\mathcal{C})$ has zero trace at the boundary of $\mathcal{C}$. Since we will consider mixed boundary conditions, given a subset $\Gamma \subset \partial\Omega$, we also introduce the space

$$H_\Gamma^1(\Omega) := \left\{ f \in H^1(\Omega) : \quad f_{|\Gamma} = 0 \right\}, \tag{2}$$

where the restriction is in trace sense. We refer to [4] for a more in-depth discussion of these spaces and the functional setting for our work.

Finally, we denote the duality pairing between a Banach space X and its dual X' with $\langle \cdot, \cdot \rangle_{(X', X)}$. When clear from the context, we will omit the explicit dependence on the spaces, writing $\langle \cdot, \cdot \rangle$.

Notation (Linear elasticity). We recall that the displacement vector $\boldsymbol{u} = [u_1, u_2]^T$ satisfies the following linear system

$$\mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \mathbf{0}, \qquad \text{in} \;\; \Omega$$

where $\mathbb{C}$ is the elasticity tensor, a fourth-order tensor describing the elastic properties of the medium. When we have a need to write the system in components, we shall use the following short-hand notation:

$$(\nabla \boldsymbol{u})_{hk} = \partial_k u_h, \qquad \text{and} \qquad (\mathrm{div} A)_i = \sum_{j=1}^{2} \partial_j A_{ij}, \quad \text{for any matrix} \;\; A$$

and

$$(\mathbb{C}A)_{ij} = \sum_{h,k=1}^{2} \mathbb{C}_{ijhk} A_{hk}.$$

3. The direct problem

We are now ready to present the direct or forward problem and prove its solvability. In the remainder of the paper, Ω is a given bounded Lipschitz domain in $\mathbb{R}^2$.

We first recall the definitions of strong convexity and strong ellipticity for the stiffness tensor $\mathbb{C}$.

Definition 3.1. *The stiffness tensor $\mathbb{C}$ is called uniformly strongly convex if, for any $x \in \Omega$, there exists a constant $c > 0$ such that*

$$\mathbb{C}(\boldsymbol{x})\widehat{A} : \widehat{A} \geq c|\widehat{A}|^2. \tag{3}$$

That is, $\mathbb{C}$ defines a positive-definite quadratic form on symmetric matrices.

Definition 3.2. *The stiffness tensor $\mathbb{C}$ is called uniformly strongly elliptic if, for any $x \in \Omega$, there exists a constant $\delta > 0$ such that for any vector $\boldsymbol{a}$ and $\boldsymbol{b}$ we have*

$$\sum_{ijhk} \mathbb{C}_{ijhk}(\boldsymbol{x}) a_i b_j a_h b_k \geq \delta |\boldsymbol{a}|^2 |\boldsymbol{b}|^2. \tag{4}$$

Remark 3.1. *It is well-known that strong convexity implies strong ellipticity.*

We continue with discussing the main assumptions we make on the dislocation curve $\mathcal{C}$ and on the stiffness tensor $\mathbb{C}$. We assume that the material is neither homogeneous nor isotropic.

Assumption 1 - elasticity tensor: The stiffness tensor $\mathbb{C} = \mathbb{C}(\boldsymbol{x})$

- satisfies minor and major symmetries, i.e.,

$$\mathbb{C}_{ijkh}(\boldsymbol{x}) = \mathbb{C}_{jikh}(\boldsymbol{x}) = \mathbb{C}_{khij}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \; 1 \leq i, j, k, h \leq 2,$$

- belongs to $C^{0,1}(\overline{\Omega})$,

- is uniformly strongly convex, Definition 3.1 above.

$\mathbb{C}$ satisfies both major and minor symmetries when the material is *hyperelastic*. Most materials in normal conditions are hyperelastic. If the material is prestressed, however, then only some symmetries of the tensor are preserved in general.

Assumption 2 - dislocation curve: The dislocation curve $\mathcal{C}$ is an open, oriented curve with Lipschitz closure

$$\overline{\mathcal{C}} \subset \Omega.$$

$\mathcal{C}$ can be extended to a closed, orientable Lipschitz curve $\widetilde{\mathcal{C}}$ satisfying

$$\widetilde{\mathcal{C}} \cap \partial\Omega = \emptyset.$$

Ω is partitioned by $\widetilde{\mathcal{C}}$ into two subsets, Ω^+ and Ω^-. We denote by Ω^+ the connected component such that $\partial\Omega^+ \cap \partial\Omega \neq \emptyset$. We choose the orientation of $\mathcal{C}$ such that the associated unit normal vector $\boldsymbol{n}$ coincides with the unit outer normal to Ω^-.

We also assume that $\partial\Omega$ is partitioned into two disjoint subsets, denoted $\partial\Omega_D$ and $\partial\Omega_N$, with $\partial\Omega_D$ the closure of an open subset in $\partial\Omega$. The direct problem consists in solving the following interface-mixed-boundary-value problem:

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \boldsymbol{0}, & \text{in } \Omega \setminus \overline{\mathcal{C}}, \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{\nu} = \boldsymbol{0}, & \text{on } \partial\Omega_N, \\ \boldsymbol{u} = \boldsymbol{0}, & \text{on } \partial\Omega_D \\ [\boldsymbol{u}]_{\mathcal{C}} = \boldsymbol{g}, \\ [(\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n}]_{\mathcal{C}} = \boldsymbol{0}, \end{cases} \tag{5}$$

where $\boldsymbol{n}$ is the normal vector induced by the orientation on $\mathcal{C}$ (see Assumption 2) and $\boldsymbol{\nu}$ is the unit outer normal vector on $\partial\Omega$. Above, $[\ ]_{\mathcal{C}}$ denotes the jump across the dislocation curve $\mathcal{C}$, defined as follows. Given a sufficiently regular function (or vector field) f on Ω, we let $f^{\pm}$ denote its restriction to $\Omega^{\pm}$ and we let $f^{\pm}_{\mathcal{C}}$ denote the restriction of $f^{\pm}$ to $\mathcal{C}$, obtained as a non-tangential limit. Then

$$[f]_{\mathcal{C}} := f^+_{\mathcal{C}} - f^-_{\mathcal{C}}.$$

The vector field $\boldsymbol{g}$ on $\mathcal{C}$ models the slip undergone by the elastic material on the two sides of the curve $\mathcal{C}$.

Since $\boldsymbol{g}$ jumps across $\mathcal{C}$, the solution cannot be in $H^1(\Omega)$. One would expect the displacement $\boldsymbol{u}$ to belongs to $H^1(\Omega \setminus \overline{\mathcal{C}})$. This condition is equivalent to having $\boldsymbol{u}^{\pm} \in H^1(\Omega^{\pm})$ and $[\boldsymbol{u}]_{\widetilde{\mathcal{C}} \setminus \mathcal{C}} = 0$ (see Remark 3.3 below). Therefore, we seek to extend $\boldsymbol{g}$ to

$$\widetilde{\boldsymbol{g}}(\boldsymbol{x}) = \begin{cases} \boldsymbol{0} & \boldsymbol{x} \in \widetilde{\mathcal{C}} \setminus \mathcal{C}, \\ \boldsymbol{g} & \boldsymbol{x} \in \mathcal{C}, \end{cases} \tag{6}$$

remaining in the trace space $H^{1/2}(\widetilde{\mathcal{C}})$. As discussed in [9], the optimal subspace of $H^{1/2}(\mathcal{C})$, where extension by zero is a bounded operator from $H^{1/2}(\mathcal{C})$ to $H^{1/2}(\widetilde{\mathcal{C}})$ is the space

$H^{1/2}_{00}(\mathcal{C})$, introduced in [7] for domains in $\mathbb{R}^n$ and defined for a curve in (1). We hence assume that

$$\boldsymbol{g} \in H^{1/2}_{00}(\mathcal{C}),$$

We can then recast Problem (5) as the following interface-mixed-boundary-value problem, where the transmission conditions are given on a closed curve:

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \boldsymbol{0}, & \text{in } \Omega \setminus \widetilde{\mathcal{C}}, \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{\nu} = \boldsymbol{0}, & \text{on } \partial\Omega_N, \\ \boldsymbol{u} = \boldsymbol{0}, & \text{on } \partial\Omega_D \\ [\boldsymbol{u}]_{\widetilde{\mathcal{C}}} = \widetilde{\boldsymbol{g}}, \\ [(\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n}]_{\widetilde{\mathcal{C}}} = \boldsymbol{0}, \end{cases} \tag{7}$$

We can prove well-posedness of the forward problem in a variational setting, thanks to the regularity and convexity of the elasticity tensor, by a suitable lifting of the jump to reduce the problem to a source problem in the whole Ω. Since the jump is concentrated on a curve, a good choice for the lifting operator is a suitable double layer potential, as, for instance, in [15]. We first recall a key result on Sobolev spaces (see e.g. [1, 2]).

Lemma 3.2. *Let $\overline{\Omega^+}$ and $\overline{\Omega^-}$ be defined as in Assumption 2 so that $\overline{\Omega} = \overline{\Omega^+} \cup \overline{\Omega^-}$. Let*

$$H_{\widetilde{\mathcal{C}}} := \{ f \in L^2(\Omega) : \ f^+ \in H^1(\Omega^+), \ f^- \in H^1(\Omega^-), \quad \textit{and} \quad [f]_{\widetilde{\mathcal{C}}} = 0 \}, \tag{8}$$

where f^+ and f^- are restrictions of f in Ω^+ and Ω^-, respectively. Then

$$H^1(\Omega) \simeq H_{\widetilde{\mathcal{C}}}.$$

Remark 3.3. *We observe that, if condition $[f]_{\widetilde{\mathcal{C}}} = 0$ in (8) is replaced by $[f]_{\widetilde{\mathcal{C}} \setminus \mathcal{C}} = 0$, i.e., we consider the space*

$$H_{\widetilde{\mathcal{C}} \setminus \overline{\mathcal{C}}} := \{ f \in L^2(\Omega) : \ f^+ \in H^1(\Omega^+), \ f^- \in H^1(\Omega^-), \quad \textit{and} \quad [f]_{\widetilde{\mathcal{C}} \setminus \mathcal{C}} = 0 \},$$

then, obviously,

$$H^1(\Omega \setminus \overline{\mathcal{C}}) \simeq H_{\widetilde{\mathcal{C}} \setminus \overline{\mathcal{C}}}. \tag{9}$$

For details, see [4].

Definition 3.3. *Let Φ be a cut-off function in $C^\infty_0(\Omega)$ such that $\Phi = 1$ in a collar neighborhood of the curve $\widetilde{\mathcal{C}}$ small enough so that it doesn't intersect $\partial\Omega$. We let*

$$\begin{aligned} \boldsymbol{u}_{\widetilde{\boldsymbol{g}}}(\boldsymbol{x}) &= \frac{\Phi(\boldsymbol{x})}{2\pi} \int_{\widetilde{\mathcal{C}}} \nabla_{\boldsymbol{y}} \left(\ln |\boldsymbol{x} - \boldsymbol{y}| \right) \cdot \boldsymbol{n}_{\boldsymbol{y}} \, \widetilde{\boldsymbol{g}}(\boldsymbol{y}) \, d\sigma(\boldsymbol{y}) \\ &= -\frac{\Phi(\boldsymbol{x})}{2\pi} \int_{\widetilde{\mathcal{C}}} \frac{(\boldsymbol{x} - \boldsymbol{y}) \cdot \boldsymbol{n}_{\boldsymbol{y}}}{|\boldsymbol{x} - \boldsymbol{y}|^2} \, \widetilde{\boldsymbol{g}}(\boldsymbol{y}) \, d\sigma(\boldsymbol{y}). \end{aligned} \tag{10}$$

The double layer potential $\boldsymbol{u}_{\widetilde{\boldsymbol{g}}}$ is the lift of the jump on the displacement via the Newtonian potential localized to a neighborhood of the curve $\widetilde{\mathcal{C}}$, so that $\boldsymbol{u}_{\widetilde{\boldsymbol{g}}}$ satisfies homogeneous boundary conditions on $\partial\Omega$.

By classical results on layer potentials (see for instance [13]), $\boldsymbol{u}^{\pm}_{\widetilde{\boldsymbol{g}}} \in H^1(\Omega^{\pm})$, since $\widetilde{\boldsymbol{g}} \in H^{1/2}(\widetilde{\mathcal{C}})$, $\Phi \in C_0^{\infty}(\Omega)$ and $\nabla_{\boldsymbol{y}} \ln|\boldsymbol{x}-\boldsymbol{y}|$, $\boldsymbol{y} \in \widetilde{\mathcal{C}}$, is a harmonic function in Ω^+ and Ω^-. Moreover, $[\boldsymbol{u}_{\widetilde{\boldsymbol{g}}}]_{\widetilde{\mathcal{C}}} = \widetilde{\boldsymbol{g}}$ thanks to the jump relations satisfied by the double layer potential, that is, in view of (6),

$$[\boldsymbol{u}_{\widetilde{\boldsymbol{g}}}]_{\widetilde{\mathcal{C}}\setminus\mathcal{C}} = \boldsymbol{0}, \qquad \text{and} \qquad [\boldsymbol{u}_{\widetilde{\boldsymbol{g}}}]_{\mathcal{C}} = \boldsymbol{g}. \tag{11}$$

Therefore, it follows that $\boldsymbol{u}_{\widetilde{\boldsymbol{g}}} \in H^1(\Omega\setminus\overline{\mathcal{C}})$ by (9).

Next, we define $\boldsymbol{w} := \boldsymbol{u}-\boldsymbol{u}_{\widetilde{\boldsymbol{g}}}$. By the linearity of (7) and Lemma 3.2, the well-posedness of (7) is reduced to finding $\boldsymbol{w} \in H^1_{\partial\Omega_D}(\Omega)$ such that

$$\int_{\Omega} \mathbb{C}\widehat{\nabla}\boldsymbol{w}\cdot\widehat{\nabla}\boldsymbol{v}\,d\boldsymbol{x} = -\int_{\Omega\setminus\widetilde{\mathcal{C}}} \mathbb{C}\widehat{\nabla}\boldsymbol{u}_{\widetilde{\boldsymbol{g}}}\cdot\widehat{\nabla}\boldsymbol{v}\,d\boldsymbol{x}, \qquad \text{for any } \; \boldsymbol{v} \in H^1_{\partial\Omega_D}(\Omega), \tag{12}$$

where $H^1_{\partial\Omega_D}(\Omega)$ is defined in (2).

Theorem 3.4. *There exists a unique solution $\boldsymbol{w} \in H^1_{\partial\Omega_D}(\Omega)$ of Problem (12).*

Proof. Let $a_{\Omega}(\boldsymbol{w},\boldsymbol{v})$ and $a_{\Omega\setminus\widetilde{\mathcal{C}}}(\boldsymbol{u}_{\widetilde{\boldsymbol{g}}},\boldsymbol{v})$ be the bilinear forms, respectively on the left-hand and right-hand sides of (12). We will show coercivity and continuity of $a_{\Omega}(\boldsymbol{w},\boldsymbol{v})$ and continuity of $a_{\Omega\setminus\widetilde{\mathcal{C}}}(\boldsymbol{u}_{\widetilde{\boldsymbol{g}}},\boldsymbol{v})$.
Continuity of $a_{\Omega}(\boldsymbol{w},\boldsymbol{v})$. By Assumption 1, there exists a constant $C>0$ such that

$$|a_{\Omega}(\boldsymbol{w},\boldsymbol{v})| \le C\|\widehat{\nabla}\boldsymbol{w}\|_{L^2(\Omega)}\|\widehat{\nabla}\boldsymbol{v}\|_{L^2(\Omega)} \le C\|\boldsymbol{w}\|_{H^1(\Omega)}\|\boldsymbol{v}\|_{H^1(\Omega)}.$$

Coercivity of $a_{\Omega}(\boldsymbol{w},\boldsymbol{v})$. By the strong convexity of the elasticity operator, Korn's and Poincaré inequalities (which hold in $H^1_{\partial\Omega_D}(\Omega)$), and by choosing $\boldsymbol{v}=\boldsymbol{w}$, there exists a constant $C>0$ such that

$$a_{\Omega}(\boldsymbol{w},\boldsymbol{w}) \ge c\|\widehat{\nabla}\boldsymbol{w}\|^2_{L^2(\Omega)} \ge C\|\nabla\boldsymbol{w}\|^2_{L^2(\Omega)} \ge C\|\boldsymbol{w}\|^2_{H^1(\Omega)}.$$

Continuity of $a_{\Omega\setminus\widetilde{\mathcal{C}}}(\boldsymbol{u}_{\widetilde{\boldsymbol{g}}},\boldsymbol{v})$. Similarly to the case of the bilinear form $a_{\Omega}(\boldsymbol{w},\boldsymbol{v})$, since $\boldsymbol{u}_{\widetilde{\boldsymbol{g}}} \in H^1(\Omega\setminus\overline{\mathcal{C}})$, there exists a constant $C>0$ such that

$$|a_{\Omega\setminus\widetilde{\mathcal{C}}}(\boldsymbol{u}_{\widetilde{\boldsymbol{g}}},\boldsymbol{v})| \le C\|\widehat{\nabla}\boldsymbol{u}_{\widetilde{\boldsymbol{g}}}\|_{L^2(\Omega\setminus\widetilde{\mathcal{C}})}\|\widehat{\nabla}\boldsymbol{v}\|_{L^2(\Omega\setminus\widetilde{\mathcal{C}})} \le C\|\boldsymbol{u}_{\widetilde{\boldsymbol{g}}}\|_{H^1(\Omega\setminus\widetilde{\mathcal{C}})}\|\boldsymbol{v}\|_{H^1(\Omega)}.$$

The assertion now follows by the Lax-Milgram Theorem. $\square$

As a consequence, we have the following result.

Corollary 3.5. *There exists a unique solution $\boldsymbol{u} \in H^1_{\partial\Omega_D}(\Omega\setminus\overline{\mathcal{C}})$ of Problem (5).*

Proof. Existence of a weak solution $\boldsymbol{u} \in H^1_{\partial\Omega_D}(\Omega \setminus \overline{\mathcal{C}})$ is guaranteed by Theorem 3.4, Equation (11), Remark 3.3, and the fact that $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{u}_{\widetilde{\boldsymbol{g}}}$. Uniqueness immediately follows by standard arguments. In fact, assume that there exist two solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in $H^1_{\partial\Omega_D}(\Omega \setminus \overline{\mathcal{C}})$. Then, their difference, $\boldsymbol{z} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, satisfies the following problem with homogeneous jump and boundary conditions

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{z}) = \boldsymbol{0} & \text{in } \Omega \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{z})\boldsymbol{\nu} = \boldsymbol{0} & \text{on } \partial\Omega \setminus \overline{\partial\Omega_D} \\ \boldsymbol{z} = \boldsymbol{0} & \text{on } \partial\Omega_D. \end{cases}$$

From Lemma 3.2, we have that $\boldsymbol{z} \in H^1_{\partial\Omega_D}(\Omega)$ and the only solution to the previous boundary value problem is $\boldsymbol{z} = \boldsymbol{0}$. Hence, $\boldsymbol{u}_1 = \boldsymbol{u}_2$. □

4. The Inverse Problem

This section is devoted to the study of the following inverse problem: *Determine uniquely the dislocation curve $\mathcal{C}$ and the slip vector $\boldsymbol{g}$ on $\mathcal{C}$ from given displacement measurements on an open section $\Sigma \subset \partial\Omega \setminus \partial\Omega_D$.*

To this end, we need to make additional assumptions on the curve $\mathcal{C}$ and on the form of the elasticity operator, in order to guarantee a unique continuation property for the elasticity system when $\mathbb{C}$ is anisotropic. In fact, unique continuation is the key tool in our proof of uniqueness for the inverse problem.

We first rewrite the elasticity system in non-divergence form, following [8]:

$$\operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \Lambda_{11}\partial_1^2\boldsymbol{u} + \Lambda_{12}\partial_1\partial_2\boldsymbol{u} + \Lambda_{22}\partial_2^2\boldsymbol{u} + R(\boldsymbol{u}) = \boldsymbol{0}, \qquad \text{a.e. in}\, \Omega,$$

where

$$\Lambda_{11} = \mathbb{C}_{i1k1}, \quad \Lambda_{22} = \mathbb{C}_{i2k2}, \quad \Lambda_{12} = \mathbb{C}_{i2k1} + \mathbb{C}^T_{i2k1},$$

and

$$R(\boldsymbol{u}) = \sum_{jhk} (\partial_j \mathbb{C}_{ijhk})\partial_k u_h.$$

Note that $\partial_j \mathbb{C}_{ijhk}$ is locally bounded, thanks to the Lipschitz regularity of the elasticity tensor, and Λ_{11} and Λ_{22} are invertible (they are positive definite), due to the strong ellipticity condition, see Assumption 1 and Remark 3.1.

Assumption 3 - further assumption on the elastic operator [8]**:** There exists a neighborhood of each point $\boldsymbol{x}_0 \in \Omega$ with the property that, at every point $\boldsymbol{x}$ in this neighborhood, the quadratic operator pencil $\Lambda_{11}p^2 + \Lambda_{12}p + \Lambda_{22}$ has at least one Lipschitz eigenvalue $\theta(\boldsymbol{x})$ with associated Lipschitz eigenvector $\boldsymbol{z}(\boldsymbol{x})$, i.e.,

$$(\Lambda_{11}\theta^2(\boldsymbol{x}) + \Lambda_{12}\theta(\boldsymbol{x}) + \Lambda_{22})\boldsymbol{z}(\boldsymbol{x}) = \boldsymbol{0},$$

such that the matrix $[\boldsymbol{z}\ \overline{\boldsymbol{z}}]$, where $\overline{\boldsymbol{z}}$ is the complex conjugate of $\boldsymbol{z}$, is non singular.

Orthotropic media are an example of anisotropic materials that satisfy this assumption. It was shown in [8] that, under Assumption 3, the system of anisotropic elasticity enjoys the unique continuation property.

Assumption 4 - further assumption on the curve $\mathcal{C}$: The curve $\mathcal{C}$ is assumed to be globally the graph of a Lipschitz function with respect to an arbitrary coordinate frame.

We can now state and prove our uniqueness result on the inverse problem.

Theorem 4.1. *Assume that Assumption 3 is satisfied. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two Lipschitz curves satisfying Assumption 2, and further satisfying Assumption 4 with respect to the same coordinate frame. Let $\boldsymbol{g}_i \in H^{1/2}_{00}(\mathcal{C}_i)$, for $i = 1, 2$, with $\mathrm{Supp}\, \boldsymbol{g}_i = \overline{\mathcal{C}}_i$, for $i = 1, 2$, and $\boldsymbol{u}_i$, for $i = 1, 2$, be the unique solution of (5) in $H^1_{\partial\Omega_D}(\Omega \setminus \overline{\mathcal{C}})$ where $\boldsymbol{g} = \boldsymbol{g}_i$ and $\mathcal{C} = \mathcal{C}_i$. If $\boldsymbol{u}_1\big|_\Sigma = \boldsymbol{u}_2\big|_\Sigma$, then $\mathcal{C}_1 = \mathcal{C}_2$ and $\boldsymbol{g}_1 = \boldsymbol{g}_2$.*

We introduce some notation used in the proof of uniqueness. Let

$$G \text{ be the connected component of } \Omega \setminus \overline{\mathcal{C}_1 \cup \mathcal{C}_2} \text{ such that } \Sigma \subset \partial G. \tag{13}$$

By definition,

$$G \subseteq \Omega \setminus \overline{\mathcal{C}_1 \cup \mathcal{C}_2}. \tag{14}$$

In addition, we define $\mathcal{G} := \partial G \setminus \partial\Omega$, which can be characterized as follows.

Lemma 4.2. *Let $\mathcal{C}_1, \mathcal{C}_2$ as in the assumptions of Theorem 4.1. Then $\mathcal{G} = \overline{\mathcal{C}_1 \cup \mathcal{C}_2}$.*

For a proof, we refer to [3, 4] both for bounded and unbounded domains.

Proof of Theorem 4.1. By contradiction, we assume that $\mathcal{C}_1 \neq \mathcal{C}_2$. The rest of the proof is divided into two parts. Let $\boldsymbol{U} := \boldsymbol{u}_1 - \boldsymbol{u}_2$. In the first part, we show that, since $\boldsymbol{U}_{|\Sigma} = \boldsymbol{0}$ and $((\mathbb{C}\widehat{\nabla}\boldsymbol{U})\boldsymbol{\nu})_{|\Sigma} = \boldsymbol{0}$ then, by the unique continuation property for anisotropic elastic system, $\boldsymbol{U} = \boldsymbol{0}$ in G, where G is defined in (13). Then, thanks to (14), in the second step, we study the cases $G = \Omega \setminus \overline{\mathcal{C}_1 \cup \mathcal{C}_2}$ and $G \subset \Omega \setminus \overline{\mathcal{C}_1 \cup \mathcal{C}_2}$ separately, showing that we get a contradiction in both cases.

Thanks to the vanishing of the conormal derivative, the traction, on $\partial\Omega_N$ in (5), and the hypothesis that $\boldsymbol{u}_1\big|_\Sigma = \boldsymbol{u}_2\big|_\Sigma$ we have that

$$\boldsymbol{U} = \boldsymbol{0}, \qquad (\mathbb{C}\widehat{\nabla}\boldsymbol{U})\boldsymbol{\nu} = \boldsymbol{0}, \quad \text{on } \Sigma.$$

Fixing a point $\boldsymbol{x}_\Sigma \in \Sigma$, we consider an open disk $B_r(\boldsymbol{x}_\Sigma)$, choosing r sufficiently small such that $B_r(\boldsymbol{x}_\Sigma) \cap \Sigma \subseteq \Sigma$. We denote $B^+_r(\boldsymbol{x}_\Sigma) := B_r(\boldsymbol{x}_\Sigma) \cap \overline{\Omega}$ and $B^-_r(\boldsymbol{x}_\Sigma) = (B^+_r(\boldsymbol{x}_\Sigma))^C$, the complementary domain. We also let

$$\widetilde{\boldsymbol{U}} := \begin{cases} \boldsymbol{U} & \text{in } B^+_r(\boldsymbol{x}_\Sigma) \\ \boldsymbol{0} & \text{in } B^-_r(\boldsymbol{x}_\Sigma) \end{cases}$$

We can assume, for simplicity, that Σ is the graph of a Lipschitz function in some coordinate system (x_1, x_2), say with respect to the x_2-axis. This is possible due to the global Lipschitz regularity of $\partial\Omega$ we assumed. We extend the stiffness tensor $\mathbb{C}$ to a Lipschitz tensor $\widetilde{\mathbb{C}}$

in $B_r(\boldsymbol{x}_\Sigma)$ as follows: for each $\boldsymbol{x} \in \Sigma$, we extend $\mathbb{C}$ in $B_r^-(\boldsymbol{x}_\Sigma)$, keeping the value $\mathbb{C}(\boldsymbol{x})$ constant along the x_2-direction. We show that $\widetilde{\boldsymbol{U}}$ is $H^1(B_r(\boldsymbol{x}_\Sigma))$. In fact, testing the elasticity system with $\boldsymbol{\varphi} \in H^1_0(B_r(\boldsymbol{x}_\Sigma))$ and then integrating by parts, we obtain

$$\int_{B_r(\boldsymbol{x}_\Sigma)} \mathrm{div}(\widetilde{\mathbb{C}}\widehat{\nabla}\widetilde{\boldsymbol{U}}) \cdot \boldsymbol{\varphi}\, d\boldsymbol{x} = -\int_{B_r^+(\boldsymbol{x}_\Sigma)} \mathbb{C}\widehat{\nabla}\boldsymbol{U} : \widehat{\nabla}\boldsymbol{\varphi}\, d\boldsymbol{x} = 0,$$

where we have used the fact that $\boldsymbol{U}$ is solution to the homogeneous elasticity system. Therefore, $\widetilde{\boldsymbol{U}}$ is a weak solution in $B_r(\boldsymbol{x}_\Sigma)$ of

$$\mathrm{div}(\widetilde{\mathbb{C}}\widehat{\nabla}\widetilde{\boldsymbol{U}}) = \boldsymbol{0}.$$

From the unique continuation property [8], it follows that $\widetilde{\boldsymbol{U}} = \boldsymbol{0}$ in $B_r(\boldsymbol{x}_\Sigma)$, that is $\boldsymbol{U} = \boldsymbol{0}$ in $B_r^+(\boldsymbol{x}_\Sigma)$. Since $B_r^+(\boldsymbol{x}_\Sigma)$, for r sufficiently small, is contained in the connected component G defined in (13), we can apply the unique continuation property repeatedly to conclude that $\boldsymbol{U} = \boldsymbol{0}$ in G. This completes the first part of the proof. Next, we distinguish two cases:

(i) $G = \Omega \setminus \overline{\mathcal{C}_1 \cup \mathcal{C}_2}$;

(ii) $G \subset \Omega \setminus \overline{\mathcal{C}_1 \cup \mathcal{C}_2}$.

We first analyze case (i). Using the hypothesis that the curves are Lipschitz and the fact that $\mathcal{C}_1 \neq \mathcal{C}_2$, without loss of generality we can assume that there exist a point $\boldsymbol{y} \in \mathcal{C}_1$ such that $\boldsymbol{y} \notin \overline{\mathcal{C}}_2$, and a ball $B_r(\boldsymbol{y})$, with r sufficiently small, that does not intersect $\mathcal{C}_2$. Consequently,

$$\boldsymbol{0} = [\boldsymbol{U}]_{B_r(\boldsymbol{y})\cap\mathcal{C}_1} = [\boldsymbol{u}_1]_{B_r(\boldsymbol{y})\cap\mathcal{C}_1} = \boldsymbol{g}_1,$$

which is a contradiction, since by hypothesis $\mathrm{supp}(\boldsymbol{g}_1) = \overline{\mathcal{C}}_1$. It follows that $\overline{\mathcal{C}}_1 = \overline{\mathcal{C}}_2$ so that

$$\boldsymbol{0} = [\boldsymbol{U}]_{\mathcal{C}_1} = [\boldsymbol{U}]_{\mathcal{C}_2} \Rightarrow [\boldsymbol{u}_1]_{\mathcal{C}_1} = [\boldsymbol{u}_2]_{\mathcal{C}_2} \Rightarrow \boldsymbol{g}_1 = \boldsymbol{g}_2.$$

Next, we analyze Case (ii). This cases arises only if $\overline{\mathcal{C}_1 \cup \mathcal{C}_2}$ contains a closed curve. We can assume without loss of generality that the complement of G in $\Omega\setminus\overline{\mathcal{C}_1 \cup \mathcal{C}_2}$ consists only of one connected component Ω^-, because if there is more than one connected component we can treat each component separately. Since by hypothesis the two curves are Lipschitz graphs with respect to an arbitrary, but common frame, by Lemma 4.2 we can also assume that $\overline{\mathcal{C}_1 \cup \mathcal{C}_2}$ is a closed Lipschitz curve, that is, $\mathcal{C}_1$ and $\mathcal{C}_2$ have common endpoints, $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, with $\boldsymbol{q}_1 \neq \boldsymbol{q}_2$, and that $\partial\Omega^- = \overline{\mathcal{C}_1 \cup \mathcal{C}_2}$. In fact, if this is not the case, any remaining part of the two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ can be reached on both of its sides with a connected path from Σ and hence treated as in Case (i). Then $G = \Omega \setminus \overline{\Omega^-} =: \Omega^+$.

From the first part of the proof, we know $\boldsymbol{U} = \boldsymbol{0}$ in Ω^+, that is, in a neighborhood of $\partial\Omega^-$. As $[(\mathbb{C}\widehat{\nabla}\boldsymbol{u}_1)\boldsymbol{n}]_{\mathcal{C}_1} = \boldsymbol{0}$ and $[(\mathbb{C}\widehat{\nabla}\boldsymbol{u}_2)\boldsymbol{n}]_{\mathcal{C}_2} = \boldsymbol{0}$ in trace sense from (5), it follows that

$$(\mathbb{C}\widehat{\nabla}\boldsymbol{U}^-)\boldsymbol{n} = \boldsymbol{0}, \tag{15}$$

in $H^{-1/2}(\partial\Omega^-)$, where $\boldsymbol{U}^-$ again denotes the restriction of $\boldsymbol{U}$ to Ω^- and $\boldsymbol{n}$ is the outward unit normal to Ω^-. Moreover, again thanks to (5), $\boldsymbol{U}^-$ satisfies

$$\mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{U}^-) = \boldsymbol{0} \quad \text{in } \Omega^-. \tag{16}$$

We conclude from (15) and (16) that $\boldsymbol{U}^-$ is in the kernel of the operator for elastostatics in $H^1(\Omega^-)$, i.e., it is a rigid motion:

$$\boldsymbol{U}^- = \mathbf{A}\boldsymbol{x} + \boldsymbol{c},$$

where $\boldsymbol{c} \in \mathbb{R}^2$ and $\mathbf{A} \in \mathbb{R}^{2\times 2}$ is a skew-symmetric matrix. We conclude the proof by showing that this rigid motion can only be the trivial one. We observe that $\boldsymbol{U}^- = [\boldsymbol{U}]_{\mathcal{C}_i} = \boldsymbol{g}_i$ on $\mathcal{C}_i$, and by hypothesis $\boldsymbol{g}_i \in H^{1/2}_{00}(\mathcal{C}_i)$, hence $\boldsymbol{g}_i(\boldsymbol{q}_i) = \mathbf{0}$ for $i = 1, 2$. This implies that

$$\mathbf{A}\boldsymbol{q}_1 + \boldsymbol{c} = \mathbf{0}, \qquad \mathbf{A}\boldsymbol{q}_2 + \boldsymbol{c} = \mathbf{0}. \tag{17}$$

Subtracting the two equations in (17) gives

$$\mathbf{A}(\boldsymbol{q}_1 - \boldsymbol{q}_2) = \mathbf{0} \tag{18}$$

Since $\mathbf{A}$ is skew, $\mathbf{A} = \begin{bmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{bmatrix}$, $a_{12} \in \mathbb{R}$. It follows from (18), given that $\boldsymbol{q}_1 \neq \boldsymbol{q}_2$, that necessarily $a_{12} = 0$, i.e., $\mathbf{A} = \mathbf{0}$. Then from (17), it also follows that $\boldsymbol{c} = \mathbf{0}$. Consequently, $\boldsymbol{U}^- = \mathbf{0}$ in Ω^- so that $[\boldsymbol{U}] = \mathbf{0}$ on $\partial\Omega^-$. In particular, $[\boldsymbol{U}]_{\mathcal{C}_1} = \mathbf{0} = [\boldsymbol{u}_1] = \boldsymbol{g}_\mathbf{1} \neq \mathbf{0}$, by the assumption that $\mathrm{Supp}(\boldsymbol{g}_i) = \overline{\mathcal{C}_i}$. We reach a contradiction and, therefore, Case (ii) does not occur. □

Remark 4.3. *The well-posedness of Problem (5) holds in $\mathbb{R}^n$ for arbitrary $n \geq 2$, and for piecewise Lipschitz coefficients that are regular on a given Lipschitz partition of Ω. In this case, Problem (5) is augmented with classical homogeneous transmission conditions at the interfaces between elements of this partition. Uniqueness of the inverse problem can be extended to this case if $n = 2$. We refer to [4] for more details.*

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