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Essentials of Nonlocal Operators

Abstract: This preliminary chapter aims at providing some basic knowledge on nonlocal operators. Notions which are necessary to know about the fractional Laplacian and about more general nonlocal operators will be addressed. The expert users may skip this introduction.

The goal of this preliminary chapter is to bring the non-expert reader closer to the beautiful world of nonlocal operators. By no means exhaustive, this introduction gives a glance at some basic definitions, notations and well known results related to a few aspects of some nonlocal operators. With these premises, we take a look at fractional Sobolev spaces, at the fractional Laplacian and at a more general class of nonlocal operators (of which the fractional p-Laplacian is the typical representative).

0.1 Fractional Sobolev Spaces

Fractional Sobolev spaces are a classical argument in harmonic and functional analysis (see for instance [17, 23]). The last decades have seen a revival of interest in fractional Sobolev spaces, both for their mathematical importance and for their use in the study of nonlocal operators and nonlocal equations. In this section, we give an introduction to the topic and state some preliminary results, following the approach in [10] (the interested reader should check this very nice paper for the detailed argument).

To begin with, we recall the definition of a $C^{k,\alpha}$ domain. Let $k \in \boldsymbol{N}$, $\alpha \in (0,1]$ and let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set. We define

$$
\begin{aligned}
Q \ &:= \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \quad \text{s.t. } |x'| < 1, |x_n| < 1\},\\
Q_+ &:= \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \quad \text{s.t. } |x'| < 1, 0 < x_n < 1\},\\
Q_0 &:= \{x \in Q \quad \text{s.t. } x_n = 1\}.
\end{aligned}
$$

We say the domain Ω is of class $C^{k,\alpha}$ if there exists $M > 0$ such that for any $x \in \partial\Omega$ there exists a ball $B = B_r(x)$ for $r > 0$ and a isomorphism $T\colon Q \to B$ such that

$$
T \in C^{k,\alpha}(\overline{Q}), \qquad T^{-1} \in C^{k,\alpha}(\overline{B}), \qquad T(Q_+) = B \cap \Omega, \qquad T(Q_0) = B \cap \partial\Omega \quad \text{and}
$$
$$
\|T\|_{C^{k,\alpha}(\overline{Q})} + \|T^{-1}\|_{C^{k,\alpha}(\overline{B})} \leqslant M.
$$

We fix the fractional exponent $s \in (0,1)$ and the summability coefficient $p \in [1, \infty)$. Let $\Omega \subseteq \mathbb{R}^n$ be an open, possibly non-smooth domain. We define the fractional

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Sobolev space $W^{s,p}(\Omega)$ as

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \text{ s.t. } \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\}. \tag{0.1.1}$$

This space is naturally endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{p}} + \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}}, \tag{0.1.2}$$

where the second term on the right hand side

$$[u]_{W^{s,p}(\Omega)} := \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}} \tag{0.1.3}$$

is the so-called *Gagliardo semi-norm*.

We define $W_0^{s,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in norm $\| \cdot \|_{W^{s,p}(\Omega)}$. Moreover

$$W_0^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n),$$

as stated in Theorem 2.4 in [10]. In other words, the space $C_0^\infty(\mathbb{R}^n)$ of smooth functions with compact support is dense in $W^{s,p}(\mathbb{R}^n)$ (actually this happens for any $s > 0$).

We point out for $p = 2$ the particular Hilbert spaces

$$H^s(\Omega) := W^{s,2}(\Omega)$$

and

$$H_0^s(\Omega) := W_0^{s,2}(\Omega),$$

that are related to the fractional Laplacian, that we introduce in the upcoming Section 0.2.

Fractional Sobolev spaces satisfy some of the classical embeddings properties (see Chapters 2 and 5 in [10] for the proofs and more details on this argument). Let $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Then we have the following.

Proposition 0.1. *Let* $0 < s \leqslant s' < 1$ *and let* $\Omega \subseteq \mathbb{R}^n$ *be an open set. Then*

$$\|u\|_{W^{s,p}(\Omega)} \leqslant C \|u\|_{W^{s',p}(\Omega)}$$

for a suitable positive constant $C = C(n, s, p) \geqslant 1$. *In other words we have the continuous embedding*

$$W^{s',p}(\Omega) \subseteq W^{s,p}(\Omega).$$

One may wonder what happens at the limit case, when $s' = 1$. If the open set Ω is smooth with bounded boundary, then the embedding is true, as stated in the next proposition.

Proposition 0.2. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set of class $C^{0,1}$ with bounded boundary. Then*

$$\|u\|_{W^{s,p}(\Omega)} \leqslant C\|u\|_{W^{1,p}(\Omega)}$$

for a suitable positive constant $C = C(n, s, p) \geqslant 1$. In other words we have the continuous embedding

$$W^{1,p}(\Omega) \subseteq W^{s,p}(\Omega).$$

Fractional Sobolev spaces enjoy also quite a number of fractional inequalities: the Sobolev inequality is one of these. Indeed, for $p \in [1, \infty)$ and $n \geqslant sp$ we introduce the fractional Sobolev critical exponent

$$p^\star = \begin{cases} \dfrac{np}{n - sp} & \text{for } sp < n, \\ \infty & \text{for } sp = n. \end{cases}$$

Then we have the fractional counterpart of the Sobolev inequality:

Theorem 0.3. *For any $s \in (0, 1)$, $p \in (1, n/s)$ and $u \in C_0^\infty(\mathbb{R}^n)$ it holds that*

$$\|u\|_{L^{p^\star}(\mathbb{R}^n)} \leqslant C[u]_{W^{s,p}(\mathbb{R}^n)}.$$

Consequently, we have the continuous embedding

$$W^{s,p}(\mathbb{R}^n) \subseteq L^q(\mathbb{R}^n) \quad \textit{for any} \quad q \in [p, p^\star].$$

Proof. We give here a short proof, that can be found in [21] (or in [5], Theorem 3.2.1). We have that

$$|u(x)| \leqslant |u(x) - u(y)| + |u(y)|.$$

For a fixed R (that will be given later on), we integrate over the ball $B_R(x)$ and have that

$$|B_R(x)||u(x)| \leqslant \int_{B_R(x)} |u(x) - u(y)|\, dy + \int_{B_R(x)} |u(y)|\, dy = I_1 + I_2. \tag{0.1.4}$$

We apply the Hölder inequality for the exponents p and $p/(p - 1)$ in the first integral and obtain that

$$\begin{aligned} I_1 &= \int_{B_R(x)} \frac{|u(x) - u(y)|}{|x - y|^{\frac{n+sp}{p}}} |x - y|^{\frac{n+sp}{p}}\, dy \\ &\leqslant R^{\frac{n+sp}{p}} \left(\int_{B_R(x)} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dy \right)^{\frac{1}{p}} \left(\int_{B_R(x)} dy \right)^{\frac{p-1}{p}} \\ &\leqslant CR^{n+s} \left(\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, dy \right)^{\frac{1}{p}}. \end{aligned}$$

The Hölder inequality for $\frac{np}{n-sp}$ and $\frac{np}{n(p-1)+sp}$ gives in the second integral

$$I_2 \leqslant \left(\int_{B_R(x)} |u(y)|^{\frac{np}{n-sp}}\, dy\right)^{\frac{n-sp}{np}} \left(\int_{B_R(x)} dy\right)^{\frac{n(p-1)+sp}{np}}$$
$$\leqslant CR^{\frac{n(p-1)+sp}{p}} \left(\int_{\mathbb{R}^n} |u(y)|^{\frac{np}{n-sp}}\, dy\right)^{\frac{n-sp}{np}}.$$

Dividing by R^n in (0.1.4) and renaming the constants, it follows that

$$|u(x)| \leqslant CR^s \left[\left(\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dy\right)^{\frac{1}{p}} + R^{-\frac{n}{p}} \left(\int_{\mathbb{R}^n} |u(y)|^{\frac{np}{n-sp}}\, dy\right)^{\frac{n-sp}{np}}\right].$$

We take now R such that

$$\left(\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dy\right)^{\frac{1}{p}} = R^{-\frac{n}{p}} \left(\int_{\mathbb{R}^n} |u(y)|^{\frac{np}{n-sp}}\, dy\right)^{\frac{n-sp}{np}}$$

and we obtain

$$|u(x)| \leqslant C \left(\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dy\right)^{\frac{n-sp}{np}} \left(\int_{\mathbb{R}^n} |u(y)|^{\frac{np}{n-sp}}\, dy\right)^{\frac{s(n-sp)}{n^2}}.$$

Raising to the power $\frac{np}{n-sp}$ and integrating over $\mathbb{R}^n$, we get that

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{np}{n-sp}}\, dx \leqslant C \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy \left(\int_{\mathbb{R}^n} |u(y)|^{\frac{np}{n-sp}}\, dy\right)^{\frac{ps}{n}}.$$

This leads to the conclusion, namely

$$\left(\int_{\mathbb{R}^n} |u(x)|^{\frac{np}{n-sp}}\, dx\right)^{\frac{n-sp}{np}} \leqslant C \left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy\right)^{\frac{1}{p}}. \qquad \square$$

Using this fractional Sobolev inequality, one can prove the embedding $W^{s,p}(\Omega) \subseteq L^q(\Omega)$ for any $q \in [p, p^\star]$, for particular domains Ω for which a $W^{s,p}(\Omega)$ function can be extended to the whole of $\mathbb{R}^n$. These are the extension domains, defined as follows.

Definition 0.4. *For any $s \in (0,1)$ and $p \in [1,\infty)$, we say that $\Omega \subseteq \mathbb{R}^n$ is an extension domain for $W^{s,p}$ if there exists a positive constant $C = C(n,s,p,\Omega)$ such that for any $u \in W^{s,p}(\Omega)$ there exists $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$ such that $\tilde{u} = u$ in Ω and*

$$\|\tilde{u}\|_{W^{s,p}(\mathbb{R}^n)} \leqslant C\|u\|_{W^{s,p}(\Omega)}.$$

A nice example of an extension domain is any open set of class $C^{0,1}$ with bounded boundary.

We state this continuous embedding in the following theorem.

Theorem 0.5. *Let $s \in (0,1)$ and $p \in [1,\infty)$ such that $n \geqslant sp$. Let $\Omega \subseteq \mathbb{R}^n$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C = C(n,s,p,\Omega)$ such that for any $u \in W^{s,p}(\Omega)$ it holds*

$$\|u\|_{L^q(\Omega)} \leqslant C\|u\|_{W^{s,p}(\Omega)} \quad \text{for any} \quad q \in [p, p^\star].$$

In other words, we have the continuous embedding

$$W^{s,p}(\Omega) \subseteq L^q(\Omega) \quad \text{for any} \quad q \in [p, p^\star].$$

Moreover, if Ω is bounded, the embedding holds for any $q \in [1, p^\star]$.

In the case $n < sp$, we have the following embedding (see Theorem 8.2 in [10]) :

Theorem 0.6. *Let $\Omega \subseteq \mathbb{R}^n$ be an extension domain for $W^{s,p}$ with no external cups. Then for any $p \in [1,\infty)$, $s \in (0,1)$ such that $sp > n$ there exists a positive constant $C = C(n,s,p,\Omega)$ such that*

$$\|f\|_{C^{0,\alpha}(\Omega)} \leqslant C\left(\|f\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\Omega)}\right)^{\frac{1}{p}}$$

for any $u \in L^p(\Omega)$ with $\alpha := \frac{sp-n}{p}$.

0.2 The Fractional Laplacian

The fractional Laplace operator has a long history in mathematics, in particular it is well known in probability as an infinitesimal generator of Lévy processes (A detailed presentation of this aspect can be found in Chapter 7). Furthermore, this operator has numerous applications in real life models that describe a nonlocal behaviour, such as in phase transitions, anomalous diffusion, crystal dislocation, minimal surfaces, materials science, water waves and many more. As a matter of fact, Chapter 11 presents some very nice results on a nonlocal model related to crystal dislocation.

Hence, there is a rich literature on the mathematical models involving the fractional Laplacian, and different aspects of this operator can be studied. In this book, Chapters 3, 5, 6 present in a self-contained manner some very interesting aspects of the fractional Laplacian. This section gives some basic definitions and makes some preliminary observations on the fractional Laplacian. For more detailed information, the reader can see the above mentioned chapters, and i.e. [5, 22] and other references therein.

We introduce at first some useful notations. Let $n \in \boldsymbol{N}$, we denote by $\mathcal{S}$ the Schwartz space of rapidly decaying functions

$$\mathcal{S} := \left\{ f \in C^\infty(\mathbb{R}^n) \text{ s.t. for all } \alpha, \beta \in \boldsymbol{N}_0^n,\ \sup_{x\in\mathbb{R}^n} |x^\beta D^\alpha f(x)| < \infty \right\}.$$

Endowed with the family of semi-norms

$$[f]_{\mathcal{S}}^{\alpha,N} = \sup_{x\in\mathbb{R}^n}(1+|x|)^N \sum_{|\alpha|\leqslant N} |D^\alpha f(x)|,$$

where $N = 1, 2, \ldots,$ the Schwartz space is a locally convex topological space. We denote the space of tempered distributions, namely the topological dual of $\mathcal{S}$, by $\mathcal{S}'$ and use $\langle \cdot, \cdot \rangle$ for the dual pairing between $\mathcal{S}$ and $\mathcal{S}'$.

Let $s \in (0, 1)$. For any $u \in \mathcal{S}$ we define the fractional Laplacian as the singular integral

$$\begin{aligned}(-\Delta)^s u(x) &:= C(n,s)\text{P.V.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy \\ &= C(n,s)\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy,\end{aligned} \tag{0.2.1}$$

where $C(n,s)$ is a dimensional constant. The P.V. stands for “in the principal value sense” and is defined as above. The integral needs to be considered in principle values since, for $s \in \left(\frac{1}{2}, 1\right)$ the kernel $\dfrac{1}{|x-y|^{n+2s}}$ is singular in a neighborhood of x and this singularity is not integrable near x.

With a change of variables, one can also write the fractional Laplacian as

$$(-\Delta)^s u(x) = C(n,s)\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \frac{u(x)-u(x-y)}{|y|^{n+2s}}\,dy. \tag{0.2.2}$$

By putting $\tilde{y} = -y$ we have that

$$(-\Delta)^s u(x) = C(n,s)\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \frac{u(x)-u(x+\tilde{y})}{|\tilde{y}|^{n+2s}}\,dy$$

and summing this with (0.2.2), we obtain the following equivalent representation

$$(-\Delta)^s u(x) = \frac{C(n,s)}{2}\int_{\mathbb{R}^n} \frac{2u(x)-u(x-y)-u(x+y)}{|y|^{n+2s}}\,dy. \tag{0.2.3}$$

Notice that this latter formula does not require the P.V. formulation, since for u smooth enough[0.1], taking a second order Taylor expansion near the origin, the first order term vanishes by symmetry, and we are left only with the second order reminder, that makes the kernel integrable. More precisely, we have that

$$\int_{B_1} \frac{|2u(x)-u(x-y)-u(x+y)|}{|y|^{n+2s}}\,dy \leqslant C\|D^2u\|_{L^\infty(\mathbb{R}^n)}\int_{B_1}|y|^{-n-2s+2}\,dy < \infty \quad \text{and}$$
$$\int_{\mathbb{R}^n\setminus B_1} \frac{|u(x)-u(x-y)-u(x+y)|}{|y|^{n+2s}}\,dy \leqslant C\|u\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n\setminus B_1}|y|^{-n-2s}\,dy < \infty.$$

0.1 For instance, one can take $u \in L^\infty(\mathbb{R}^n)$ and locally C^2.

The fractional Laplacian is well defined for a wider class of functions. Indeed, as one may find in [22], it is enough to require that u belongs to a weighted L^1 space and is locally Lipschitz. More precisely, we define

$$L^1_s(\mathbb{R}^n) := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) \ \text{ s.t. } \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}}\, dx < \infty \right\}$$

(notice that that $L^q(\mathbb{R}^n) \subseteq L^1_s(\mathbb{R}^n)$ for any $q \in [1,\infty)$). Let $\varepsilon > 0$ be sufficiently small. Then, if u belongs to $L^1_s(\mathbb{R}^n)$ and to $C^{0,2s+\varepsilon}$ (or $C^{1,2s+\varepsilon-1}$ for $s \geqslant 1/2$) in a neighborhood of $x \in \mathbb{R}^n$, the fractional Laplacian is well defined in x as in (0.2.3). Indeed, while the fact that $u \in L^1_s(\mathbb{R}^n)$ assures that

$$\int_{\mathbb{R}^n \setminus B_1} \frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{n+2s}}\, dy < \infty,$$

if, taking for instance $s \in (0,1/2)$ and u that is $C^{0,2s+\varepsilon}$ in a neighborhood of x, one has that

$$\int_{B_1} \frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{n+2s}}\, dy \leqslant 2 \int_{B_1} |y|^{\varepsilon - n}\, dy \leqslant c(\varepsilon).$$

For $u \in \mathcal{S}$, the fractional Laplacian can be expressed as a pseudo-differential operator, as stated in the following identity:

$$(-\Delta)^s u(x) = \mathcal{F}^{-1}\left(|\xi|^{2s} \hat{u}(\xi) \right)(x). \tag{0.2.4}$$

Here, we set the usual notation for the Fourier transform and its inverse, using $x, \xi \in \mathbb{R}^n$ as the space, respectively as the frequency variable,

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx$$

and

$$\mathcal{F}^{-1}f(x) = \check{f}(x) := \int_{\mathbb{R}} f(\xi) e^{i\xi x}\, d\xi.$$

We point out that we do not account for the normalization constants in this definition. We notice here that this expression returns the classical Laplace operator for $s = 1$ (and the identity operator, for $s = 0$).

The expressions in (0.2.3) and (0.2.4) are equivalent (see [10], Proposition 3.3 for the proof of this statement). There, the dimensional constant $C(n,s)$ introduced in (0.2.1), is defined as

$$C(n,s) := \left(\int_{\mathbb{R}^n} \frac{1 - \cos(\eta_1)}{|\eta|^{n+2s}}\, d\eta \right)^{-1},$$

where η_1 is the first component of $\eta \in \mathbb{R}^n$. The explicit value of $C(n,s)$ is given by

$$C(n,s) = \frac{2^{2s} s \Gamma\left(\frac{n}{2} + s\right)}{\pi^{\frac{n}{2}} \Gamma(1-s)},$$

as it is very nicely proved in the Appendix A of Chapter 11. The interested reader can also see formula (3.1.15) and Appendix B in [5] (and other references therein) for different approaches to the computation of the constant.

At this point, relating to Section 0.1, there is an alternative definition of the fractional Hilbert space $H^s(\mathbb{R}^n)$ via Fourier transform. Let

$$\widehat{H}^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) \text{ s.t. } \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) |\hat{u}(\xi)|^2 \, d\xi < \infty \right\}.$$

Then (see Proposition 3.4 in [10]) the two spaces are equivalent, indeed

$$[u]^2_{H^s(\mathbb{R}^n)} = \frac{2}{C(n,s)} \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi.$$

Moreover, the connection between the fractional Laplacian and the fractional Hilbert space is clarified in Proposition 3.6 in [10], as in the next identity

$$[u]^2_{H^s(\mathbb{R}^n)} = \frac{2}{C(n,s)} \|(-\Delta)^{\frac{s}{2}} u\|^2_{L^2(\mathbb{R}^n)}.$$

We point out that for $u \in \mathcal{S}$, the fractional Laplacian $(-\Delta)^s u$ belongs to $C^\infty(\mathbb{R}^n)$, but $(-\Delta)^s u \notin \mathcal{S}$ (it is not true that it decays faster than any power of x). In particular, we define the linear space

$$\mathcal{P}_s := \left\{ f \in C^\infty(\mathbb{R}^n) \text{ s. t. for all } \alpha \in \boldsymbol{N}_0^n, \ \sup_{x \in \mathbb{R}^n} (1 + |x|^{n+2s}) |D^\alpha f(x)| < +\infty \right\}, \quad (0.2.5)$$

which endowed with the family of semi-norms

$$[f]^\alpha_{\mathcal{P}_s} := \sup_{x \in \mathbb{R}^n} (1 + |x|^{n+2s}) |D^\alpha f(x)|,$$

where $\alpha \in \boldsymbol{N}_0^n$, is a locally convex topological space; we denote by $\mathcal{P}'_s$ its topological dual and by $\langle \cdot, \cdot \rangle_s$ their pairing. Then one has for $u \in \mathcal{S}$ that $(-\Delta)^s u \in \mathcal{P}_s$ (see for instance, the bound (1.10) in [3]). The symmetry of the operator $(-\Delta)^s$ allows to define the fractional Laplacian in a distributional sense: for any $u \in L^1_s(\mathbb{R}^n) \subset \mathcal{P}'_s$ one defines

$$\langle (-\Delta)^s u, \varphi \rangle := \langle u, (-\Delta)^s \varphi \rangle_s \quad \text{for any} \quad \varphi \in \mathcal{S}.$$

These spaces are used in the definition of distributional solutions. Indeed, we say that $u \in L^1_s(\mathbb{R}^n)$ is a distributional solution of

$$(-\Delta)^s u = f, \quad \text{for } f \in \mathcal{S}'$$

if

$$\langle u, (-\Delta)^s v \rangle_s = \langle f, v \rangle \quad \text{for any } v \in \mathcal{S}.$$

Other type of solutions are defined for more general kernels in Section 0.3.

0.2.1 The harmonic extension

The fractional Laplacian can be obtained from a local operator acting in a space with an extra-dimension, via an extension procedure. This extension procedure was first introduced by Molčanov and Ostrovskiĭ in [20], where symmetric stable-processes are seen as traces of degenerate diffusion processes. We will follow here the approach of Caffarelli and Silvestre (see [6]), that relies on considering a local Neumann to Dirichlet operator in the half-space $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty)$. Consider for any $s \in (0,1)$ the number

$$a := 1 - 2s,$$

the function $u\colon \mathbb{R}^n \to \mathbb{R}$ and the problem in the non-divergence form

$$\begin{cases} \Delta_x U + \dfrac{a}{y}\partial_y U + \partial^2_{yy} U = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ U(x,0) = u(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{0.2.6}$$

The problem (0.2.6) can equivalently be written in the divergence form as

$$\begin{cases} \operatorname{div}(y^a \nabla U) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ U(x,0) = u(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{0.2.7}$$

Then one has for any $x \in \mathbb{R}^n$, up to constants, that

$$-\lim_{y\to 0^+} y^a \partial_y U(x,y) = (-\Delta)^s u(x). \tag{0.2.8}$$

Also, by using the change of variables $z = \left(\frac{y}{2s}\right)^{2s}$ the problem (0.2.6) is equivalent to

$$\begin{cases} \Delta_z U + z^\alpha \partial_{zz} U = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ U(x,0) = u(x) & \text{in } \mathbb{R}^n \end{cases} \tag{0.2.9}$$

for $\alpha = -2a/(1-a) = (2s-1)/s$. In this case also, for any $x \in \mathbb{R}^n$ and with the right choice of constants, one has that

$$-\partial_z U(x,0) = (-\Delta)^s u(x). \tag{0.2.10}$$

One way to prove (0.2.8) (see [6] for more details on this and for alternative proofs) is by means of the Poisson kernel

$$P(x,y) = k_s \frac{y^{1-a}}{\left(|x|^2 + y^2\right)^{\frac{n+1-a}{2}}},$$

that by convolution with u gives an explicit solution of the problem (0.2.6) as

$$U(x,y) = \int_{\mathbb{R}^n} P(x-\xi, y) u(\xi)\, d\xi.$$

Notice that k_s is chosen such that

$$\int_{\mathbb{R}^n} P(x,y)\,dx = 1.$$

One can compute then (up to constants)

$$\begin{aligned}
\lim_{y\to 0^+} y^a \frac{U(x,y)-U(x,0)}{y} &= \lim_{y\to 0^+} y^{a-1}\left[\int_{\mathbb{R}^n} P(x-\xi,y)u(\xi)\,d\xi - u(x)\right] \\
&= \lim_{y\to 0^+} y^{a-1}\int_{\mathbb{R}^n} \frac{y^{1-a}}{\left(|x-\xi|^2+y^2\right)^{\frac{n+1-a}{2}}}\,(u(\xi)-u(x))\,d\xi \\
&= \lim_{y\to 0^+}\int_{\mathbb{R}^n} \frac{u(\xi)-u(x)}{\left(|x-\xi|^2+y^2\right)^{\frac{n+1-a}{2}}}\,d\xi \\
&= \int_{\mathbb{R}^n} \frac{u(\xi)-u(x)}{|x-\xi|^{n+1-a}}\,d\xi \\
&= -(-\Delta)^{\frac{1-a}{2}}u(x),
\end{aligned}$$

for u smooth enough. Recalling that $s = \frac{1-a}{2}$, this proves formula (0.2.8).

This extension procedure is useful when one solves an equation with the fractional Laplacian on the whole $\mathbb{R}^n$: it overcomes the difficulty of dealing with a nonlocal operator, by replacing it with a local (possibly degenerate) one. For instance, a nonlinear problem of the type

$$(-\Delta)^s u(x) = f(u) \text{ in } \mathbb{R}^n$$

is translated into the system

$$\begin{cases} \operatorname{div}(y^a \nabla U) = 0 \\ -\lim\limits_{y\to 0} y^a \partial_y U = f(u), \end{cases} \tag{0.2.11}$$

where one identifies $u(x) = U(x,0)$ in a trace sense. At this point, one works with a local operator, which is of variational type. Indeed, the equation in (0.2.11) is the Euler-Lagrange equation of the functional

$$I(U) = \int_{\mathbb{R}^{n+1}_+} y^a |\nabla U|^2\,dX.$$

Here we denoted $X = (x,y) \in \mathbb{R}^{n+1}_+$. See, for instance [11, 4], where a nonlinear, nonlocal elliptic problem in the whole space $\mathbb{R}^n$ is dealt with using variational techniques related to the local extended operator.

0.2.2 Maximum Principle and Harnack inequality

In this subsection, we introduce some natural tools for the study of equations involving the fractional Laplacian: Maximum Principles and the Harnack inequality. We point out that these two type of instruments fail if one wants to apply them in the classical fashion. More precisely, we need to take into account the nonlocal character of the operator and have to require some global information on the function.

First of all, a function is s-harmonic in $x \in \mathbb{R}^n$ if $(-\Delta)^s u(x) = 0$. Of course, the class of s-harmonic functions is not trivial, one example is the one-dimensional function $u(x) = (x_+)^s = \max\{0, x\}^s$, that satisfies $(-\Delta)^s u(x) = 0$ on the half line $x > 0$ (see Theorem 3.4.1 in [5]). See also [12] for some other interesting examples of functions for which one can explicitly compute the fractional Laplacian.

We notice now that, if u has a global maximum at x_0, then by definition (0.2.3) it is easy to check that $(-\Delta)^s u(x_0) \geqslant 0$. On the other hand, this is no longer true if u merely has a local maximum at x_0. The Maximum Principle goes as follows:

Theorem 0.7. *If $(-\Delta)^s u \geqslant 0$ in B_R and $u \geqslant 0$ in $\mathbb{R}^n \setminus B_R$, then $u \geqslant 0$ in B_R. Furthermore, either $u > 0$ in B_R, or $u \equiv 0$ in $\mathbb{R}^n$.*

Proof. Suppose by contradiction that there exists $\overline{x} \in B_R$ such that $u(\overline{x}) < 0$ is a minimum in B_R. Since u is positive outside B_R, this is a global minimum. Hence for any $y \in B_{2R}$ we have that $2u(\overline{x}) - u(x - y) - u(x + y) \leqslant 0$, while for $y \in \mathbb{R}^n \setminus B_{2R}$, the inequality $|\overline{x} \pm y| \geqslant |y| - |\overline{x}| \geqslant R$ assures that $u(\overline{x} \pm y) \geqslant 0$. It yields that

$$\begin{aligned} 0 &\leqslant (-\Delta)^s u(\overline{x}) \\ &= \int_{B_{2R}} \frac{2u(\overline{x}) - u(\overline{x} - y) - u(\overline{x} + y)}{|y|^{n+2s}}\, dy \\ &\quad + \int_{\mathbb{R}^n \setminus B_{2R}} \frac{2u(\overline{x}) - u(\overline{x} - y) - u(\overline{x} + y)}{|y|^{n+2s}}\, dy \\ &\leqslant \int_{\mathbb{R}^n \setminus B_{2R}} \frac{2u(\overline{x})}{|y|^{n+2s}}\, dy \\ &= Cu(\overline{x}) R^{-2s} < 0. \end{aligned}$$

This gives a contradiction, hence $u(\overline{x}) \geqslant 0$.

Now, suppose that u is not strictly positive in B_R and there exists $x_0 \in B_R$ such that $u(x_0) = 0$. Then

$$(-\Delta)^s u(x_0) = \int_{\mathbb{R}^n} \frac{-u(x_0 - y) - u(x_0 + y)}{|y|^{n+2s}}\, dy \leqslant 0,$$

hence $(-\Delta)^s u(x_0) = 0$. Since $u \geqslant 0$ in $\mathbb{R}^n$, this happens only if $u \equiv 0$ in $\mathbb{R}^n$, and this concludes the proof. □

As said before, if a function is s-harmonic and positive only on the ball, this does not assure that the infimum and supremum on the half-ball are comparable (see [15] for a counter-example of this type). One needs some global information on the function. One simple assumption is to take the function nonnegative on the whole of $\mathbb{R}^n$. Then the Harnack inequality holds, as stated in the next Theorem.

Theorem 0.8. *Let $u\colon \mathbb{R}^n \to \mathbb{R}$ be nonnegative in $\mathbb{R}^n$ such that $(-\Delta)^s u = 0$ in B_1. Then there exists a constant $C = C(n,s) > 0$ such that*

$$\sup_{B_{1/2}} u \leqslant C \inf_{B_{1/2}} u.$$

One way to prove this Theorem is to use the harmonic extension defined in the previous Subsection 0.2.1. Namely, this result follows as the trace inequality on $\mathbb{R}^n \times \{y = 0\}$ of the Harnack inequality holding for the extended local (weighted) operator. See [6] for all the details of this proof.

Another formulation that loses the strong assumption that u should be nonnegative in $\mathbb{R}^n$ is given in the following theorem (see Theorem 2.3 in [16]):

Theorem 0.9. *There exists a positive constant c such that for any function $u\colon \mathbb{R}^n \to \mathbb{R}$ which is s-harmonic function in B_1, the following bound holds for any $x, y \in B_{1/2}$*

$$u(x) \leqslant C\left(u(y) + \int_{\mathbb{R}^n\setminus B_1} \frac{u_-(z)}{(|z|^2-1)^s|z|^n}\,dz\right).$$

Moreover, if the function u is nonnegative in B_1, then one has

$$u(x) \leqslant C\left(u(y) + \int_{\mathbb{R}^n\setminus B_1} \frac{u_-(z)}{|z|^{n+2s}}\,dz\right).$$

Here, u_- is the negative part of u, i.e. $u_-(x) = \max\{-u(x), 0\}$.

A Harnack inequality for more generals kernels is also stated further on in Subsection 0.3.1.

0.3 More General Nonlocal Operators

It is natural to continue the study of nonlocal phenomena by introducing more general type of operators. In particular, one can introduce the fractional p-Laplacian

$$(-\Delta)^s_p u(x) := \text{P.V.} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))}{|x-y|^{n+sp}}\,dy$$

(notice that for $p = 2$, one gets the fractional Laplacian defined in (0.2.1)). As a further topic, on can generalise this formula by taking instead of $|x-y|^{-n-sp}$ a different kernel. So, in this section we introduce briefly nonlocal operators obtained by means of more general kernels and make some remarks on the well-posedness of the definition. Moreover, we shortly define weak solutions and viscosity solutions, and provide a few known results on these type of solutions.

In this book, Chapter 9 present in detail some arguments related to these nonlocal operators (references therein are of guidance for the interested reader).

We define a general nonlocal operator of fractional parameter $s \in (0,1)$ and summability coefficient $p \in (1,\infty)$. Let $K\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ be a kernel that satisfies

(i) K is a measurable function

(ii) K is symmetric, i.e.

$$K(x,y) = K(y,x) \text{ for almost any } (x,y) \in \mathbb{R}^n \times \mathbb{R}^n;$$

(iii) there exists $\lambda, \Lambda \geqslant 1$ such that

$$\lambda \leqslant K(x,y)|x-y|^{n+sp} \leqslant \Lambda \text{ for almost any } (x,y) \in \mathbb{R}^n \times \mathbb{R}^n$$

for some $p > 1$.

$$\tag{0.3.1}$$

Then formally one defines for any $x \in \mathbb{R}^n$

$$\begin{aligned} \mathcal{L}_K u(x) &:= \text{P.V.} \int_{\mathbb{R}^n} |u(x)-u(y)|^{p-2}\,(u(x)-u(y))\,K(x,y)\,dy \\ &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} |u(x)-u(y)|^{p-2}\,(u(x)-u(y))\,K(x,y)\,dy \end{aligned} \tag{0.3.2}$$

where by P.V. we intend “in the principal value sense”, as defined in the last line of (0.3.2).

Let us take as an example the case $p = 2$ and a general kernel K satisfying (0.3.1) and see when $\mathcal{L}_K u(x)$ is pointwise defined. If the kernel K satisfies an additional condition of weak translation invariance, i.e.

$$K(x, x+z) = K(x, x-z) \text{ for a.e. } (x,z) \in \mathbb{R}^n \times \mathbb{R}^n \tag{0.3.3}$$

and the function u for $\gamma > 0$ is locally $C^{0,2s+\gamma}$ (or $C^{1,2s+\gamma-1}$ if $s > 1/2$) and integrable at infinity respect to the kernel K, then $\mathcal{L}_K u(x)$ is well defined for any $x \in \mathbb{R}^n$. Indeed, for $r > 0$ and $\varepsilon \in (0,r)$ we have that

$$\int_{B_r(x)\setminus B_\varepsilon(x)} (u(x)-u(y))K(x,y)\,dy = \int_{B_r(0)\setminus B_\varepsilon(0)} (u(x)-u(x+z))K(x,x+z)\,dz.$$

By the symmetry of the domain of integration and the additional property (0.3.3), we obtain

$$\begin{aligned} &\int_{B_r(x)\setminus B_\varepsilon(x)} (u(x)-u(y))K(x,y)\,dy \\ &= \frac{1}{2}\int_{B_r(0)\setminus B_\varepsilon(0)} (u(x)-u(x+z))\,K(x,x+z)\,dz \\ &\quad + \frac{1}{2}\int_{B_r(0)\setminus B_\varepsilon(0)} (u(x)-u(x-z))\,K(x,x-z)\,dz \\ &= \frac{1}{2}\int_{B_r(0)\setminus B_\varepsilon(0)} (2u(x)-u(x+z)-u(x-z))\,K(x,x-z)\,dz. \end{aligned}$$

Now, if u is in $C^{1,2s+\gamma-1}(B_r(x))$, we have that

$$\begin{aligned}|2u(x)-u(x+z)-u(x-z)| &= \left|\int_0^1 (\nabla u(x+tz)-\nabla u(x-tz))\cdot z\,dt\right| \\ &\leqslant [u]_{C^{1,2s+\gamma-1}(B_r(x))}\,|z|^{2s+\gamma}\int_0^1 (2t)^{2s+\gamma-1}\,dt \\ &\leqslant \frac{2^{2s+\gamma-1}}{2s+\gamma}\,[u]_{C^{1,2s+\gamma-1}(B_r(x))}\,|z|^{2s+\gamma}.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}&\int_{B_r(x)\setminus B_\varepsilon(x)} (u(x)-u(y))K(x,y)\,dy \\ &\leqslant \frac{2^{2s+\gamma-1}}{2s+\gamma}\,[u]_{C^{1,2s+\gamma-1}(B_r(x))}\,\Lambda\int_{B_r(0)\setminus B_\varepsilon(0)} |z|^{-n+\gamma}\,dz \leqslant \frac{c(n)}{\gamma}\,[u]_{C^{1,2s+\gamma-1}(B_r(x))}\,r^\gamma.\end{aligned}$$

Hence, for such $\gamma > 0$, the principal value exists and, moreover,

$$\left|\int_{B_r(x)} (u(x)-u(y))K(x,y)\,dy\right| \leqslant c\,[u]_{C^{1,2s+\gamma-1}(B_r(x))}\,r^\gamma.$$

0.3.1 Some remarks on weak and viscosity solutions

We give now an idea of different concepts of solutions and give some introductory properties on solutions of linear equations of the type

$$\begin{cases}\mathcal{L}_K u(x) = 0 & \text{in } \Omega \subseteq \mathbb{R}^n \\ u \text{ satisfies some "boundary condition"} & \text{in } \mathbb{R}^n\setminus\Omega,\end{cases} \tag{0.3.4}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded set. Notice at first that the boundary condition is given in the whole of the complement of Ω. This depends on the nonlocal character of the operator.

We have seen that, in the case $p = 2$, adding the weak translation invariance on K and proving sufficient regularity on u, then $\mathcal{L}_K u$ is pointwise defined. In this case, pointwise solutions of problem (0.3.4) can be considered.

The concept of pointwise solution is however reductive; in general, the boundary data is given in a trace sense or one can guarantee less regularity on the solution. We introduce two other concepts of solution, the weak and the viscosity notions.

We fix $s \in (0,1)$ and $p \in (1,\infty)$. We consider the following Dirichlet problem, with given boundary data $g \in W^{s,p}(\mathbb{R}^n)$

$$\begin{cases}\mathcal{L}_K u(x) = 0 & \text{in } \Omega \subseteq \mathbb{R}^n \\ u(x) = g(x) & \text{in } \mathbb{R}^n\setminus\Omega.\end{cases} \tag{0.3.5}$$

We recall the definition of $W^{s,p}(\mathbb{R}^n)$ as in (0.1.1)

$$W^{s,p}(\mathbb{R}^n) := \left\{ v \in L^p(\mathbb{R}^n) \text{ s.t. } \frac{|v(x) - v(y)|}{|x-y|^{n/p+s}} \in L^p(\mathbb{R}^n \times \mathbb{R}^n) \right\}$$

and we say that $v \in W_0^{s,p}(\Omega)$ if $v \in W^{s,p}(\mathbb{R}^n)$ and $v = 0$ almost everywhere in $\mathbb{R}^n \backslash \Omega$. In principle, this is a different way of defining the space $W_0^{s,p}(\Omega)$ when Ω is not a bounded Lipschitz open set (see for example the observations in Appendix B in [2]). We define the convex spaces

$$\mathcal{K}_g^{\pm} := \{ v \in W^{s,p}(\mathbb{R}^n) \text{ s.t. } (g-v)_{\pm} \in W_0^{s,p}(\Omega) \}$$

and

$$\mathcal{K}_g := \mathcal{K}_g^{+} \cap \mathcal{K}_g^{-} = \{ v \in W^{s,p}(\mathbb{R}^n) \text{ s.t. } v - g \in W_0^{s,p}(\Omega) \}.$$

The problem has a variational structure, and we introduce a functional whose minimization leads to the solution of the problem (0.3.5). For $u \in \mathcal{K}_g$ we define the functional

$$\mathcal{E}_K(u) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^p K(x,y)\, dx\, dy\,. \tag{0.3.6}$$

We have the following definition:

Definition 0.10. *Let Ω be an open set of $\mathbb{R}^n$. We say that u is a weak subsolution (supersolution) of the problem* (0.3.5) *if $u \in \mathcal{K}_g^{-}$ ($\mathcal{K}_g^{+}$) and it satisfies*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))K(x,y)\, dx\, dy \leqslant (\geqslant) 0$$

for every nonnegative $\varphi \in W_0^{s,p}(\Omega)$. Moreover, a function u is a weak solution if $u \in \mathcal{K}_g$ is both a super and a subsolution of the problem (0.3.5)*, i.e. if*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))K(x,y)\, dx\, dy = 0$$

for every nonnegative $\varphi \in W_0^{s,p}(\Omega)$.

Using the notion of weak solution introduced in definition (0.10), we have the following existence theorem.

Theorem 0.11 (Existence). *Let $s \in (0,1)$, $p \in (1,\infty)$ and $g \in W^{s,p}(\mathbb{R}^n)$. Then there exists a unique minimizer u of $\mathcal{E}_K$ over $\mathcal{K}_g$. Moreover, a function $u \in \mathcal{K}_g$ is a minimizer of $\mathcal{E}_K$ over $\mathcal{K}_g$ if and only if it is a weak solution to the problem* (0.3.5).

One can prove the existence of a unique minimizer by standard variational techniques (see Theorem 2.3 in [9] for details). We give here a sketch of the proof that a minimizer of the energy is a solution of the problem (0.3.5) and vice-versa.

Sketch of the proof. Let u be a minimizer of the functional $\mathcal{E}_k$. Consider $u + t\varphi$ to be a perturbation of u with $\varphi \in W_0^{s,p}(\Omega)$. We compute formally

$$\begin{aligned} 0 &= \frac{d}{dt}\mathcal{E}_k(u + t\varphi)\Big|_{t=0} \\ &= \frac{d}{dt}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x) + t\varphi(x) - u(y) - t\varphi(y)|^p K(x,y)\,dx\,dy\Big|_{t=0} \\ &= p\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}\,(u(x) - u(y))\,(\varphi(x) - \varphi(y))\;dx\,dy. \end{aligned}$$

This proves that u is a weak solution of (0.3.5), as introduced in Definition (0.10).

On the other hand, if u is a weak solution of (0.3.5), let $v \in \mathcal{K}_g$ ant let $\varphi = u - v \in W_0^{s,p}(\Omega)$. Then we have that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}\,((u(x) - u(y))\,(\varphi(x) - \varphi(y))\,K(x,y)\,dx\,dy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x) - u(y)|^p K(x,y)\,dx\,dy \\ &\quad - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))K(x,y)\,dx\,dy. \end{aligned}$$

Using the Young inequality, we continue

$$\begin{aligned} 0 &\geqslant \frac{1}{p}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x) - u(y)|^p K(x,y)\,dx\,dy - \frac{1}{p}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |v(x) - v(y)|^p K(x,y)\,dx\,dy \\ &= \mathcal{E}_k(u) - \mathcal{E}_k(v). \end{aligned}$$

Hence $\mathcal{E}_k(u) \leqslant \mathcal{E}_k(v)$ for any $v \in \mathcal{K}_g$ and therefore the weak solution $u \in \mathcal{K}_g$ minimizes the functional $\mathcal{E}_k$. □

In order to obtain some boundedness and regularity results, we introduce the important concept of nonlocal tail (given in [9]). The nonlocal tail takes into account the contribution of a function “coming from far”, namely it allows to have a quantitative control of the “nonlocality” of the operator. The definition goes as follows:

$$\mathrm{Tail}(v; x_0, R) := \left[R^{sp}\int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{|v(x)|^{p-1}}{|x - x_0|^{n+sp}}\right]^{\frac{1}{p-1}}. \tag{0.3.7}$$

Notice that this quantity is finite when $v \in L^q(\mathbb{R}^n)$, with $q \geqslant p - 1$ and $R > 0$.

With this in hand, we have the following local boundedness result (see Theorem 1.1 in [9] for the proof and details).

Lemma 0.12 (Local boundedness). *Let $s \in (0,1)$, $p \in (1,\infty)$ and let $u \in W^{s,p}(\mathbb{R}^n)$ be a weak solution of the problem* (0.3.5). *Let $r > 0$ such that $B_r(x_0) \subseteq \Omega$. Then*

$$\sup_{B_{r/2}(x_0)} u \leqslant \delta Tail\left(u_+; x_0, \frac{r}{2}\right) + c\delta^{-\frac{(p-1)n}{sp^2}}\left(\fint_{B_r(x_0)} u_+^p\,dx\right)^{\frac{1}{p}},$$

where $u_+ = \max\{u, 0\}$ is the positive part of u and $c = c(n, p, s, \lambda, \Lambda)$.

Here, $\delta \in (0,1]$ behaves as an interpolation parameter between local and nonlocal terms.

Using the nonlocal tail, one can state also the Harnack inequality in this general case (see Theorem 1.1 in [8] for the proof of the statement).

Theorem 0.13. *Let $u \in W^{s,p}(\mathbb{R}^n)$ be a weak solution of (0.3.5) and $u \geqslant 0$ in $B_R(x_0) \subset \Omega$. Then for any $B_r := B_r(x_0) \subset B_{\frac{R}{2}}(x_0)$ we have that*

$$\sup_{B_r} u \leqslant C \inf_{B_r} u + C\left(\frac{r}{R}\right)^{\frac{sp}{p-1}} Tail(u_-; x_0, R),$$

where $u_- = \max\{-u, 0\}$ is the negative part of u and $C = C(n, s, p, \lambda, \Lambda)$.

We point out that a Harnack inequality for nonlocal general operators in the case $p = 2$ is obtained in [1].

Viscosity solutions take into account solutions which are only continuous. The idea is to “trap” the solution, which needs to be only continuous, between two functions which are C^2 (or at least $C^{1,\gamma}$). We introduce here the notion of viscosity solution for the problem (0.3.5), as given in [7].

Definition 0.14. *Let $u\colon \mathbb{R}^n \to \mathbb{R}$ be an upper (lower) semi-continuous function on $\overline{\Omega}$. The function u is said to be a subsolution (supersolution) of $\mathcal{L}_K u = 0$ and we write $\mathcal{L}_K u \leqslant 0$ ($\mathcal{L}_K u \geqslant 0$) if the following happens. If:*

- *x is any point in Ω*
- *$N := N(x) \subset \Omega$ is a neighborhood of x*
- *φ is some $C^2(\overline{N})$ function*
- *$\varphi(x) = u(x)$*
- *$\varphi(y) > u(y)$ for any $y \in N \backslash \{x\}$*

then, setting

$$v := \begin{cases} \varphi, & in\ N \\ u, & in\ \mathbb{R}^n \backslash N \end{cases}$$

we have that $\mathcal{L}_K v \leqslant 0$ ($\mathcal{L}_K v \geqslant 0$). Moreover, u is a viscosity solution if it is both a subsolution and a supersolution.

Existence and uniqueness of viscosity solutions of problems such as (0.3.5) are established in [14]. We introduce here a Hölder regularity result for viscosity solutions of the problem (0.3.5) (see [18] for more details and Theorem 1 therein for the proof).

Theorem 0.15. *Let $s \in (0,1)$ and $p \in (1,\infty)$ (in the case $p < 2$ we require additionally that $p > 1/(1-s)$). Assume that K satisfies $K(x,y) = K(x,-y)$ and there exist $\Lambda \geqslant \lambda > 0$, $M > 0$ and $\gamma > 0$ such that*

$$\frac{\lambda}{|y|^{n+sp}} \leqslant K(x,y) \leqslant \frac{\Lambda}{|y|^{n+sp}} \quad \text{for } y \in B_2, x \in B_2$$

and

$$0 \leqslant K(x,y) \leqslant \frac{M}{|y|^{n+\gamma}} \quad \text{for } y \in \mathbb{R}^n \backslash B_{1/4}, x \in B_2,$$

Let $u \in L^\infty(\mathbb{R}^n)$ be a viscosity solution of $\mathcal{L}_K u = 0$ in B_2. Then u is Hölder continuous in B_1 and in particular there exist $\alpha = \alpha(\lambda, \Lambda, M, \gamma, p, s)$ and $C = C(\lambda, \Lambda, M, \gamma, p, s)$ such that

$$\|u\|_{C^\alpha(B_1)} \leqslant C\|u\|_{L^\infty(\mathbb{R}^n)}.$$

Of course, much remains to be said about the arguments we presented in this introduction, and about the nonlocal setting in general. The fractional Laplace operator and operators of a more general type introduced here will be studied and beautifully presented in the following Chapters 3, 5, 6, 7, 8, 9, 11. Other very interesting topics are dealt with in upcoming chapters. In Chapter 1 some bounds on heat kernels for non-symmetric nonlocal equations are obtained. Chapter 2 deals with fractional harmonic maps. In Chapter 4, nonlocal minimal surfaces are discussed. Furthermore, Chapter 10 deals with the existence of a weak solution of some fractional nonlinear problems with periodic boundary conditions.

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