

CARLESON AND REVERSE CARLESON MEASURES ON HOMOGENEOUS SIEGEL DOMAINS

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ABSTRACT. In this paper we study Carleson and reverse Carleson measures on holomorphic function spaces on a homogeneous Siegel domain of Type II. We prove several necessary conditions and sufficient conditions in order for a measure μ to be Carleson and reverse Carleson on mixed-normed weighted Bergman spaces.

1. INTRODUCTION

Carleson measures were first introduced by L. Carleson [12, 13] in order to study the corona problem in the classical Hardy spaces on the unit disc. The study of these measures have flourished since then, and has been generalized to several different settings. From a general perspective, they can be effectively defined as follows. Given a locally compact space \mathcal{T} and a (quasi-)Banach space X of (say) continuous functions on \mathcal{T} , a positive Radon measure μ on \mathcal{T} is said to be a p -Carleson measure for X ($p \in]0, \infty[$) if there is a constant $C > 0$ such that

$$\|f\|_{L^p(\mu)} \leq C \|f\|_X$$

for every $f \in X$. Switching the roles of X and $L^p(\mu)$ in the above inequality, one then gets the definition of a reverse p -Carleson measure for X . This kind of measures was essentially introduced in a series of papers by D. H. Luecking [24, 25, 27, 28] for weighted Bergman spaces on the unit disc or more general domains, and has been later generalized to different settings by several authors (cf. [19] and the references therein).

In this paper we study Carleson and reverse Carleson measures on weighted Bergman spaces on homogeneous Siegel domain, which we are now about to introduce.

Let E be a complex hilbertian space of dimension n , F a real hilbertian space of dimension $m > 0$ and Ω a convex cone in F not containing any straight lines.¹ We also assume that Ω is (affine-)homogeneous, that is the group $G(\Omega)$ of linear transformation of F that preserve Ω acts transitively on Ω .

Let $\Phi: E \times E \rightarrow F_{\mathbb{C}}$ be a non-degenerate hermitian mapping such that $\Phi(\zeta) := \Phi(\zeta, \zeta) \in \overline{\Omega}$ for all $\zeta \in E$. Define the Siegel domain of type II associated with the cone Ω and the mapping Φ as

$$D := \{ (\zeta, z) \in E \times F_{\mathbb{C}} : \rho(\zeta, z) := \operatorname{Im} z - \Phi(\zeta) \in \Omega \}.$$

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¹We shall generally denote by $\langle \cdot, \cdot \rangle$ bilinear pairings and real scalar products, and by $\langle \cdot | \cdot \rangle$ sesquilinear pairings and complex scalar products, without specifying the involved spaces.

In the case $n = 0$, i.e. $E = \{0\}$, D is said to be of Type I, and is often called a *tubular domain* over the cone Ω . The domain D is said to be homogeneous if for every $(\zeta, z), (\zeta', z') \in D$ there is a biholomorphism φ of D such that $\varphi(\zeta, z) = (\zeta', z')$. It turns out that one may then assume φ to be affine, cf., e.g., [39, Theorem 2.3]. More precisely, D is homogeneous if and only if for every $h, h' \in \Omega$ there are $t \in GL(F)$ and $g \in GL(E)$ such that $th = h'$ and such that $t\Phi = \Phi(g \times g)$ (so that $g \times t$ preserves D), cf., e.g., [32, Propositions 2.1 and 2.2]. In particular, if D is homogeneous, then Ω is homogeneous.

The domain D is said to be symmetric if it is homogeneous and admits an involutive biholomorphism with an isolated fixed point. If D is symmetric, then Ω is symmetric, that is, homogeneous and self-dual. Conversely, if Ω is symmetric and D is a tubular domain, then D is symmetric (cf., e.g., [35, Theorem] for more details on various characterizations of symmetric Siegel domains).

The Šilov boundary of D is the set

$$bD := \{ (\zeta, z) \in E \times F_{\mathbb{C}} : \rho(\zeta, z) = 0 \},$$

and it can be identified with $E \times F$ via the mapping $(\zeta, x + i\Phi(\zeta, \zeta)) \mapsto (\zeta, x)$. In addition, bD admits a step-2 nilpotent Lie group structure, whose product can be described as follows:

$$(\zeta, x)(\zeta', x') = (\zeta + \zeta', x + x' + 2 \operatorname{Im} \Phi(\zeta, \zeta')),$$

for $(\zeta, x), (\zeta', x') \in E \times F$, see e.g. [10, Section 1.1]. We denote by \mathcal{N} the set $E \times F$ endowed with this group structure.

Observe that, by definition, ρ maps D into Ω , and that the fibres of ρ , namely the sets $bD + (0, ih)$ for $h \in \Omega$, give rise to a foliation of D . Given a function f defined on D , we shall often denote by f_h the restriction of f to $bD + (0, ih)$, interpreted as a function on \mathcal{N} for the sake of convenience, so that

$$f_h(\zeta, x) = f(\zeta, x + i\Phi(\zeta) + ih)$$

for every $h \in \Omega$ and for every $(\zeta, x) \in \mathcal{N}$. Observe that, identifying $bD + (0, ih)$ with \mathcal{N} as above for every $h \in \Omega$, we get a left action of bD on D by affine biholomorphisms.

For $p, q \in]0, \infty]$ and $\mathbf{s} \in \mathbb{R}^r$, the weighted Bergman spaces are defined as²

$$(1.1) \quad A_{\mathbf{s}}^{p,q}(D) := \left\{ f \in \operatorname{Hol}(D) : \int_{\Omega} \left(\int_{\mathcal{N}} |f_h(\zeta, x)|^p d(\zeta, x) \right)^{q/p} \Delta_{\Omega}^{q\mathbf{s}}(h) d\nu_{\Omega}(h) < \infty \right\}$$

(modification if $\max(p, q) = \infty$), where $d(\zeta, x)$ denotes a Haar measure on \mathcal{N} and ν_{Ω} denotes a positive $G(\Omega)$ -invariant measure on Ω , both fixed and unique up to a multiplicative constant. We shall often simply write $A_{\mathbf{s}}^{p,q}$ instead of $A_{\mathbf{s}}^{p,q}(D)$.

We remark that the spaces $A_{\mathbf{s}}^{p,p}$ are the weighted Bergman spaces, the unweighted case corresponding to the value $\mathbf{s} = -\mathbf{d}/p$, while the spaces $A_{\mathbf{0}}^{p,\infty}$ are the classical Hardy spaces.

We denote by $\mathcal{M}_+(D)$ the space of positive Radon measures.

Definition 1.1. Given a Banach space X of functions on D and $p \in]0, \infty[$, a measure $\mu \in \mathcal{M}_+(D)$ is said to be p -Carleson for X if there exists a constant $C > 0$ such that

$$\|f\|_{L^p(\mu)} \leq C \|f\|_X$$

for every $f \in X$. If, in addition, the canonical mapping $X \rightarrow L^p(\mu)$ is compact, then μ is said to be a compact or vanishing p -Carleson for X .

²The definitions of the rank r of Ω , of the ‘generalized power functions’ $\Delta_{\Omega}^{\mathbf{s}}$ and of $\mathbf{d} \in \mathbb{R}^r$ are deferred to Section 3.

A measure $\mu \in \mathcal{M}_+(D)$ is said to be a reverse p -Carleson measure for X if there exists $C > 0$ such that

$$\|f\|_X \leq C \|f\|_{L^p(\mu)},$$

for every $f \in X$.

Finally, a measure $\mu \in \mathcal{M}_+(D)$ is said to be a sampling, or dominant, measure for X if it is both Carleson and reverse Carleson for X , in other words, if X embeds as a closed subspace of $L^p(\mu)$.

Notice that, even though the definition of a p -Carleson and of a reverse p -Carleson measure is quite standard, the definition of a vanishing p -Carleson measure is slightly more problematic. Originally, Carleson measures were characterized requiring a suitable function F to be bounded, and vanishing Carleson measures were characterized requiring F to vanish ‘at infinity’. As noted in [34], this corresponds to the compactness of the canonical mapping $X \rightarrow L^p(\mu)$. In the literature also appear apparently different, but still equivalent, definitions of a vanishing Carleson measure (cf., e.g., [22]).

In this paper we consider the problem of obtaining necessary and sufficient conditions for the measure μ to be a p -Carleson, a vanishing p -Carleson, or a reverse p -Carleson measure for the spaces $A_{\mathbf{s}}^{p,q}$.

There exists a vast literature on Carleson measures on holomorphic function spaces, in one and several variables, and it is impossible to give a proper and complete account of it. Hence, we limit ourselves to the papers most relevant for this work, and extend our apologies to all other authors. In [12, 13, 15, 30], Carleson measures for the Hardy or Hardy–Sobolev spaces on the unit disc are studied. In [37, 40] Carleson measures for the Dirichlet and weighted Dirichlet spaces still on the unit disc are characterized. In higher dimensions, we mention [14], which deals with Hardy spaces in the unit ball, [26], which deals with Bergman spaces on general domains, [1, 2, 3, 4, 5], which deal with holomorphic Sobolev–Besov spaces on the unit ball, [22], which deals with Bergman spaces on strongly pseudoconvex domains, and [33, 9, 8], which deal with various function spaces on Siegel domains.

Carleson measures still draw a lot of attention, in connection with many problems in analysis such as operator theory, interpolation, and boundary behavior problems, just to mention a few. The results in this paper are applied in the study of Toeplitz and Cesàro operators on weighted Bergman spaces on Siegel domains of Type II, see [11].

On the other hand, reverse Carleson measures appeared at a later time, and much less is known about them. They have been studied by D. H. Luecking in the case of Bergman spaces on the unit disc [24, 28, 25, 27, 31]. See also the recent survey [19] and references therein.

Among reverse Carleson measures, those which are also Carleson measures form a particularly tractable subclass, for which better results are generally available. Such measures are often called *sampling measures*, since they can be considered as generalizations of sampling sequences. A complete characterization of sampling measures is available on the unit disc (cf. [31]), but not on more general domains. Nonetheless, the techniques developed in [24, 25, 27] to characterize dominant (or sampling) sets can be effectively extended to general weighted Bergman spaces on homogeneous Siegel domains, as already noted in [27]. A subset G of D is said to be *dominant* for $A_{\mathbf{s}}^{p,p}$ if the measure $\chi_G(\Delta_{\Omega}^{p\mathbf{s}+\mathbf{d}} \circ \rho) \cdot \mathcal{H}^{2n+2m}$. As it turns out, this notion is independent of both \mathbf{s} and p .

In this paper we consider weighted Bergman spaces on a homogeneous Siegel domain D of Type II. We provide necessary and sufficient conditions for a measure $\mu \in \mathcal{M}_+(D)$ to be a p -Carleson measure for $A_{\mathbf{s}}^{p_1,q_1}(D)$ when $p_1, q_1 \leq p$ (Theorems 5.4 and 5.9), as well as in the general case, under suitable additional assumptions (Theorems 5.5 and 5.9). We also provide a sufficient condition for a measure

to be a reverse Carleson measure (Theorem 7.14), as well as necessary conditions and sufficient conditions for a measure $\mu \in \mathcal{M}_+(D)$ to be a sampling measure for $A_s^{p,p}(D)$ (Theorems 7.3, 7.6, and 7.9).

The paper is organized as follows. In Section 2 we review our main results in the particular case $D = \mathbb{C}_+$ for the ease of the reader. In Section 3 we give the definition of homogeneous Siegel domains of Type II, while in Section 4 we give the definitions of several function spaces on D and recall some of their most relevant properties which are involved in our analysis. In Section 5, we prove necessary conditions and sufficient conditions for a measure $\mu \in \mathcal{M}_+(D)$ to be a Carleson measure for the weighted Bergman space $A_s^{p,q}$. Section 6 is devoted to an extension of a result by Hardy and Littlewood about embedding the Hardy space into a weighted Bergman space. Finally, in Section 7, we prove our results concerning reverse Carleson and sampling measures.

2. CARLESON AND REVERSE CARLESON MEASURES ON \mathbb{C}_+

In this brief section we present our main results for weighted Bergman spaces on \mathbb{C}_+ , where they become particularly simple.

In this case, for $p, q \in]0, \infty]$ and $s \in \mathbb{R}$,

$$A_s^{p,q} = \left\{ f \in \text{Hol}(\mathbb{C}_+) : \int_0^\infty y^{qs} \left(\int_{\mathbb{R}} |f(x+iy)|^p dy \right)^{q/p} \frac{dy}{y} < \infty \right\}$$

(modification when $\max(p, q) = \infty$). Then, $A_s^{p,q} = 0$ if $s < 0$ or $s = 0$ and $q < \infty$. In addition, $A_0^{p,\infty}$ is the Hardy space H^p .

Theorem 2.1. *Take $p_1, q_1, p \in]0, \infty]$ with $p < \infty$, and $s > 0$. Take a compact neighbourhood K of i in \mathbb{C}_+ , $\mu \in \mathcal{M}_+(D)$, and define*

$$M_K(\mu) : \mathbb{C}_+ \ni z \mapsto \mu(\text{Re } z + (\text{Im } z)K) \in \mathbb{R}_+.$$

Then, the following conditions are equivalent:

- (1) $A_s^{p_1, q_1}$ embeds continuously into $L^p(\mu)$;
- (2) setting $p^* := \max(1, p_1/p)'$, $q^* := \max(1, q_1/p)'$, and $s^* := -\max(1, p/p_1) - ps$,

$$\int_0^\infty y^{q^* s^*} \left(\int_{\mathbb{R}} |M_K(\mu)(x+iy)|^{p^*} dy \right)^{q^*/p^*} \frac{dy}{y} < \infty$$

(modification when $\max(p^, q^*) = \infty$).*

(Cf. Theorem 5.5 and Propositions 4.10 and 4.11 below.)

Even though Carleson measures for the Hardy spaces $A_0^{p,\infty}(\mathbb{C}_+)$ have been characterized in the literature, it is known that the problem of determining the Carleson measures for Hardy spaces on higher rank irreducible Siegel domains is highly non-trivial.

Concerning sampling measures, Theorem 7.9 becomes:

Theorem 2.2. *Take $p, s > 0$, and a p -Carleson measure μ for $A_s^{p,p}$. Assume that the support of every vague cluster point of every sequence of measures of the form $y^{ps+1}(x+y \cdot)_*(\mu)$ ($x \in \mathbb{R}, y > 0$) is a set of uniqueness for $A_{(p/q)_s}^{q,q}$ for some (fixed) $q \in]0, p[$.*

Then, μ is a p -sampling measure for $A_s^{p,p}$.

Observe that the above sufficient condition is likely to be necessary as well, *in this case*, since the corresponding assertion on the unit disc is true (cf. [31]).

Nonetheless, since the above sufficient condition need not be easily managed, other sufficient conditions can be provided.

Proposition 2.3. *Take $p, s, \varepsilon, C' > 0$, and a compact neighbourhood K of i in \mathbb{C}_+ . Then, there is a compact neighbourhood K' of i in \mathbb{C}_+ such that, for every $\mu \in \mathcal{M}_+(\mathbb{C}_+)$ such that*

$$N(\mu) := \sup_{x \in \mathbb{R}, y > 0} y^{-1-ps} \mu(x + yK') < \infty$$

and such that

$$\mathcal{H}^2(\{z \in x + yK : \mu(\operatorname{Re} z + (\operatorname{Im} z)K') \geq \varepsilon(\operatorname{Im} z)^{1+ps} N(\mu)\}) \geq C'y^2$$

for every $x \in \mathbb{R}$ and for every $y > 0$, is a p -sampling measure for $A_s^{p,p}$.

The above result is a consequence of a characterization of dominant subsets of \mathbb{C}_+ (cf. Theorem 7.3 below) which is very close to what one obtains applying Proposition 2.3 to the measure $\chi_G(\operatorname{Im} \cdot)^{ps-1}$. \mathcal{H}^2 and to some $\varepsilon \in]0, 1[$, where G is a measurable subset of \mathbb{C}_+ . Here and in what follows, we denote by \mathcal{H}^m the m -dimensional Hausdorff measure.

Proposition 2.4. *A subset G of \mathbb{C}_+ is dominant if and only if there are a constant $C > 0$ and a compact neighbourhood K of i such that*

$$\mathcal{H}^2(G \cap (x + yK)) \geq Cy^2$$

for every $x \in \mathbb{R}$ and for every $y > 0$.

Finally, the following result provides a sufficient condition for a (not necessarily Carleson) measure to be reverse Carleson (see Theorem 7.14).

Theorem 2.5. *Take $p_1, q_1, p \in]0, \infty]$ with $p < \infty$, and $s > 0$ or $s \geq 0$ if $q_1 = \infty$. Define $p^* := p_1/(p - p_1)_+$ and $q^* := q_1/(p - q_1)_+$.*

Then, there are $\delta > 0$ and a constant $C > 0$ such that every $\mu \in \mathcal{M}_+(D)$ such that

$$\left\| \left(\frac{2^{kp[s+1/p_1]}}{\mu([\delta 2^k j, \delta 2^k(j+1)] + i[\delta 2^k, \delta 2^{k+1}])} \right)_{j,k} \right\|_{\ell^{p^*, q^*}(\mathbb{Z}, \mathbb{Z})} < \infty,$$

is a p -reverse Carleson for $A_s^{p_1, q_1}$.

3. HOMOGENEOUS SIEGEL DOMAINS OF TYPE II

For the general theory of homogeneous Siegel domains of Type II and further details we refer the reader to [38, 20, 10]. Throughout the paper we adopt the notation of [10].

3.1. The structure of homogeneous cones and a description of D .

Definition 3.1. Take $r \in \mathbb{N}$ and let $A = \bigoplus_{j,k=1}^r A_{j,k}$ be a graded algebra endowed with a linear involution $*$ such that the following hold:

- $A_{j,k}A_{k',\ell} \subseteq A_{j,\ell}$ if $k = k'$ and $A_{j,k}A_{k',\ell} = \{0\}$ if $k \neq k'$;
- $A_0 = \{0\}$ and $\dim A_{j,j} = 1$ for every $j = 1, \dots, r$;
- $A_0 = \{0\}$ and $\dim A_{j,j} = 1$ for every $j = 1, \dots, r$;
- for every $j = 1, \dots, r$ there exists $e_j \in A_{j,j}$ such that the left multiplication by e_j is the identity on $A_{j,k}$ and the right multiplication by e_j is the identity on $A_{k,j}$, for every $k = 1, \dots, r$;

- if we define $\text{Tr } a = \sum_{j=1}^r \langle e'_j, a \rangle$ for every $a \in A$, where e'_j is the unique graded linear functional on A which takes the value 1 at e_j ($j = 1, \dots, r$), then the mapping $(a, b) \mapsto \text{Tr}(ab)$ is symmetric;
- for every $a, b, c \in A$, $\text{Tr}(a(bc)) = \text{Tr}((ab)c)$;
- $A_{j,k}^* = A_{k,j}$ for every $j, k = 1, \dots, r$ and $(ab)^* = b^*a^*$ for every $a, b \in A$;
- the symmetric bilinear mapping $(a, b) \mapsto \text{Tr}(a^*b)$ is positive and non-degenerate;
- setting $T := \bigoplus_{j \leq k} A_{j,k}$, one has $t(uw) = (tu)w$ and $t(uu^*) = (tu)u^*$ for every $t, u, w \in T$.

Then, we say that A is a T -algebra.

Definition 3.2. Given a T -algebra A , define

$$(3.1) \quad \begin{aligned} T_+ &:= \{ a \in T : \langle e'_j, a \rangle > 0 \quad \forall j = 1, \dots, r \}, & H(A) &:= \{ a \in A : a = a^* \}, \\ C(A) &:= \{ tt^* : t \in T_+ \}, & C'(A) &:= \{ t^*t : t \in T_+ \}. \end{aligned}$$

The cones $C(A)$ and $C'(A)$ are said to have rank r .

The following theorem was proved in [38].

Theorem 3.3. *Let A be a T -algebra. Then, the following hold:*

- (i) T_+ , endowed with the product induced by A , is a Lie group;
- (ii) $C(A)$ and $C'(A)$ are homogeneous cones, and are dual to one another with respect to the scalar product on $H(A)$;³
- (iii) $(tx)t^* = t(xt^*)$ and $(t^*x)t = t^*(xt)$ for every $t \in T_+$ and for every $x \in H(A)$;
- (iv) the mappings

$$(t, x) \mapsto txt^* \quad \text{and} \quad (t, x) \mapsto t^*xt$$

are simply transitive left and right actions of T_+ on $C(A)$ and $C'(A)$, respectively, which are dual to one another with respect to the scalar product on $H(A)$.

In addition, if Ω is a homogeneous cone in a finite-dimensional vector space F over \mathbb{R} , then, there exist a T -algebra A' and an isomorphism $\Psi: H(A') \rightarrow F$ such that $\Psi(C(A')) = \Omega$.

By the previous results, given a homogeneous cone Ω , we may select a subgroup T_+ of $G(\Omega)$ which acts simply transitively on Ω and, by transposition, on the dual cone Ω' . In order to avoid some notational inconveniences, we shall write $t \cdot h$ instead of $t(h)$ or th , and $\lambda \cdot t$ instead of $\lambda \circ t$ or λt , for every $t \in T_+$, for every $h \in \Omega$, and for every $\lambda \in \Omega'$. Then, the following hold (cf., e.g., [10, Lemma 2.9]):

- for every $t \in T_+$ there exists $g \in GL(E)$ such that $t \cdot \Phi = \Phi(g \times g)$;⁴
- for every $\mathbf{s} \in \mathbb{C}^r$ the mapping $\Delta^{\mathbf{s}}: T_+ \rightarrow \mathbb{C}^*$ defined by $\Delta^{\mathbf{s}}(t) := \prod_{j=1}^r \langle e'_j, t \rangle^{2s_j}$ is a group homomorphism;
- there is $\mathbf{b} \in \mathbb{R}^r$ such that $\Delta^{-\mathbf{b}}(t) = \det_{\mathbb{R}}(g) = |\det_{\mathbb{C}}(g)|^2$ for every $t \in T_+$ and for every $g \in GL(E)$ such that $t \cdot \Phi = \Phi(g \times g)$.

We recall that D is a tube domain, that is, $E = \{0\}$, if and only if $\mathbf{b} = \mathbf{0}$, cf. [10, Remark 2.13].

³ In other words, $C(A) = \{ x \in H(A) : \forall \lambda \in \overline{C'(A)} \setminus \{0\} \quad \text{Tr}(\lambda x) > 0 \}$.

⁴This follows from the requirement that D be homogeneous (for a suitable choice of T_+).

3.2. The generalized power and Gamma functions. Using the characters $\Delta^{\mathbf{s}}$ we define the *generalized power functions* $\Delta_{\Omega}^{\mathbf{s}}$ and $\Delta_{\Omega'}^{\mathbf{s}}$ on Ω and Ω' , respectively.

Definition 3.4. Fix base points $e_{\Omega} \in \Omega$ and $e_{\Omega'} \in \Omega'$, and define the *generalized power functions* $\Delta_{\Omega}^{\mathbf{s}}$ and $\Delta_{\Omega'}^{\mathbf{s}}$, for $\mathbf{s} \in \mathbb{C}^r$, by

$$\Delta_{\Omega}^{\mathbf{s}}(h) = \Delta^{\mathbf{s}}(t) \quad \text{and} \quad \Delta_{\Omega'}^{\mathbf{s}}(\lambda) = \Delta^{\mathbf{s}}(t')$$

where t, t' are the *unique* elements of T_+ such that $h = t \cdot e_{\Omega}$ and $\lambda = e_{\Omega'} \cdot t'$, respectively. We also set $m_{j,k} := \dim A_{j,k}$ and define

$$(3.2) \quad \mathbf{m} := \left(\sum_{k>j} m_{j,k} \right)_{j=1,\dots,r}, \quad \mathbf{m}' := \left(\sum_{k<j} m_{j,k} \right)_{j=1,\dots,r}.$$

The generalized Gamma functions on Ω and Ω' are defined for $\operatorname{Re} \mathbf{s} \in \frac{1}{2}\mathbf{m} + (\mathbb{R}_+^*)^r$ as

$$\Gamma_{\Omega}(\mathbf{s}) := \int_{\Omega} e^{-\langle e_{\Omega'}, h \rangle} \Delta_{\Omega}^{\mathbf{s}}(h) d\nu_{\Omega}(h) = c \prod_{j=1}^r \Gamma\left(s_j - \frac{m_j}{2}\right)$$

and, for $\operatorname{Re} \mathbf{s} \in \frac{1}{2}\mathbf{m}' + (\mathbb{R}_+^*)^r$,

$$\Gamma_{\Omega'}(\mathbf{s}) := \int_{\Omega'} e^{-\langle \lambda, e_{\Omega} \rangle} \Delta_{\Omega'}^{\mathbf{s}}(\lambda) d\nu_{\Omega'}(\lambda) = c \prod_{j=1}^r \Gamma\left(s_j - \frac{m'_j}{2}\right),$$

where $c > 0$ is a suitable constant.

Furthermore, we set $\mathbf{d} := -(\mathbf{1}_r + \frac{1}{2}\mathbf{m} + \frac{1}{2}\mathbf{m}')$ and define

$$(3.3) \quad \nu_{\Omega} := \Delta_{\Omega}^{\mathbf{d}} \cdot \mathcal{H}^m \quad \text{and} \quad \nu_{\Omega'} := \Delta_{\Omega'}^{\mathbf{d}} \cdot \mathcal{H}^m,$$

where \mathcal{H}^m denotes the m -dimensional Hausdorff measure on Ω and Ω' , respectively.

We point out that ν_{Ω} and $\nu_{\Omega'}$ are two $G(\Omega)$ -invariant measures on Ω and Ω' , respectively: cf. [10, Lemma 2.18] for T_+ -invariance, and argue as in the proof of [17, Proposition I.3.1] for $G(\Omega)$ -invariance.

Definition 3.5. We denote by $(I_{\Omega}^{\mathbf{s}})_{\mathbf{s} \in \mathbb{C}^r}$ the unique holomorphic family of tempered distributions on F such that $I_{\Omega}^{\mathbf{s}} = \frac{1}{\Gamma_{\Omega}(\mathbf{s})} \Delta_{\Omega}^{\mathbf{s}} \cdot \nu_{\Omega}$ for $\operatorname{Re} \mathbf{s} \in \frac{1}{2}\mathbf{m} + (\mathbb{R}_+^*)^r$ (cf. [10, Lemma 2.26, Definition 2.27, and Proposition 2.28]). We call ‘Riemann–Liouville operators’ the operators of convolution by the $I_{\Omega}^{\mathbf{s}}$.

3.3. Lattices on D . We shall make extensive use of the notion of a lattice on the domain D . We first introduce some Riemannian metrics on Ω, Ω' , and D .

Definition 3.6. We endow D with the Bergman metric, that is, with the (complete Kähler) metric \mathbf{k} defined by

$$\mathbf{k}_{(\zeta, z)} vw := \partial_v \bar{\partial}_w \log(\Delta^{\mathbf{b}+2\mathbf{d}} \circ \rho)(\zeta, z)$$

for every $(\zeta, z) \in D$ and for every $v, w \in E \times F_{\mathbb{C}}$ (cf. [10, §2.5]). We denote by $B((\zeta, z), R)$ the corresponding open ball of centre (ζ, z) and radius R . We denote by ν_D the corresponding invariant measure $(\Delta^{\mathbf{b}+2\mathbf{d}} \circ \rho) \cdot \mathcal{H}^{2n+2m}$ on D (cf., e.g., [10, Proposition 2.44]).

We endow Ω with the quotient metric induced by the submersion $\rho: D \rightarrow \Omega$, and Ω' with the Riemannian metric induced by the correspondence $\Omega \ni t \cdot e_{\Omega} \mapsto e_{\Omega'} \cdot t \in \Omega'$ (cf. [10, Definition 2.45 and Lemma 2.46]). We denote by $B_{\Omega}(h, R)$ and $B_{\Omega'}(\lambda, R)$ the corresponding open balls of centre h and λ , respectively, and radius R .

Definition 3.7. Given $\delta > 0$ and $R > 1$, we say that a family $(h_k)_{k \in K}$ of elements of Ω is a (δ, R) -lattice if the following hold:

- the balls $B_\Omega(h_k, \delta)$ are pairwise disjoint;
- the balls $B_\Omega(h_k, R\delta)$ cover D .

We define (δ, R) -lattices on Ω' in an analogous fashion.

Furthermore, we say that a family $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$ of elements of D is a (δ, R) -lattice if the following hold:

- there is a (δ, R) -lattice $(h_k)_{k \in K}$ on Ω such that $h_k := \rho(\zeta_{j,k}, z_{j,k})$ for every $j \in J$ and every $k \in K$;
- the balls $B((\zeta_{j,k}, z_{j,k}), \delta)$ are pairwise disjoint;
- the balls $B((\zeta_{j,k}, z_{j,k}), R\delta)$ cover D .

The following result is [10, Lemma 2.55], and guarantees the existence of lattices as above on D .

Lemma 3.8. *Take $\delta > 0$. Then, there is a $(\delta, 4)$ -lattice on D .*

3.4. Fourier transform on \mathcal{N} . We now recall the definition and a few properties of the Fourier transform on \mathcal{N} , cf. [10, Section 1.2] for further details.

Define $W := \{ \lambda \in F' : \exists \zeta \neq 0 \text{ such that } \langle \lambda, \text{Im } \Phi(\zeta, \cdot) \rangle = 0 \}$. Observe that, for every $\lambda \in F' \setminus W$, the quotient $\mathcal{N} / \ker \lambda$ is a $(2n+1)$ -dimensional Heisenberg group, so that the Stone–von Neumann theorem (cf., e.g., [18, Theorem 1.50]) implies that there is up to unitary equivalence a unique irreducible unitary representation π_λ of \mathcal{N} in some hilbertian space H_λ such that $\pi_\lambda(0, ix) = e^{-i\langle \lambda, x \rangle}$ for every $x \in F$. Using the Plancherel formula on the quotients $\mathcal{N} / \ker \lambda$ and integrating in λ , one may then find the Plancherel formula for \mathcal{N} (cf., e.g., [6, Section 2]). Namely, there is a constant $c > 0$ such that

$$\|f\|_{L^2(\mathcal{N})}^2 = c \int_{F' \setminus W} \|\pi_\lambda(f)\|_{\mathcal{L}^2(H_\lambda)}^2 \Delta_{\Omega'}^{-\mathbf{b}}(\lambda) \, d\lambda$$

for every $f \in L^1(\mathcal{N}) \cap L^2(\mathcal{N})$ (cf. [10, Corollary 1.17 and Proposition 2.30]), where $\mathcal{L}^2(H_\lambda)$ denotes the space of Hilbert–Schmidt endomorphisms of H_λ .

Notice, though, that $\pi_\lambda(f_h) = 0$ for almost every $\lambda \in F' \setminus (W \cup \Omega')$, for every $h \in \Omega$, and for every f in the space $A_{\mathfrak{S}}^{p,q}$ to be defined below, $p \in]0, 2]$ (cf. [10, Corollaries 1.37 and 3.3, and Proposition 3.2]). For this reason, we shall only describe π_λ for $\lambda \in \Omega'$ (‘Bargmann representation’). We define $H_\lambda := \text{Hol}(E) \cap L^2(\nu_\lambda)$, where $\nu_\lambda = e^{-2\langle \lambda, \Phi(\cdot) \rangle} \cdot \mathcal{H}^{2n}$, and

$$(3.4) \quad \pi_\lambda(\zeta, x)\psi(\omega) := e^{\langle \lambda \zeta, -ix + 2\Phi(\omega, \zeta) - \Phi(\zeta) \rangle} \psi(\omega - \zeta),$$

for every $\psi \in H_\lambda$, for every $\omega \in E$, and for every $(\zeta, x) \in \mathcal{N}$. Denote by $P_{\lambda,0}$ the self-adjoint projector of H_λ onto the space of constant functions, for every $\lambda \in \Omega'$, and define $P_{\lambda,0} = 0$ for $\lambda \in F' \setminus (W \cup \Omega')$. The introduction of $P_{\lambda,0}$ is justified by the observation that $\pi_\lambda(f_h) = \pi_\lambda(f_h)P_{\lambda,0}$ for almost every $\lambda \in F' \setminus W$, for every $h \in \Omega$, and for every f in the space $A_{\mathfrak{S}}^{p,q}$ to be defined below, $p \in]0, 2]$ (cf. [10, Corollaries 1.37 and 3.3, and Proposition 3.2]).

4. FUNCTION SPACES

In this section we introduce the functions spaces on D and \mathcal{N} that we shall consider in this paper. We refer the reader to [10] for further details and references.

Definition 4.1. Take $\mathbf{s} \in \mathbb{R}^r$ and $p, q \in]0, \infty]$. We define

$$L_{\mathbf{s}}^{p,q}(D) := \left\{ f: D \rightarrow \mathbb{C}: f \text{ is measurable, } \int_{\Omega} \left(\Delta_{\Omega}^{\mathbf{s}}(h) \|f_h\|_{L^p(\mathcal{N})} \right)^q d\nu_{\Omega}(h) < \infty \right\}$$

(modification when $q = \infty$), and $L_{\mathbf{s},0}^{p,q}(D)$ as the closure of $C_c(D)$ in $L_{\mathbf{s}}^{p,q}(D)$. Consistently with (1.1), we define

$$A_{\mathbf{s}}^{p,q}(D) = L_{\mathbf{s}}^{p,q}(D) \cap \text{Hol}(D), \quad \text{and} \quad A_{\mathbf{s},0}^{p,q}(D) = L_{\mathbf{s},0}^{p,q}(D) \cap \text{Hol}(D).$$

Notice that $L_{\mathbf{s},0}^{p,q}(D) = L_{\mathbf{s}}^{p,q}(D)$ and $A_{\mathbf{s},0}^{p,q}(D) = A_{\mathbf{s}}^{p,q}(D)$ if (and only if) $p, q < \infty$.

It is possible to characterize the values of p, q, \mathbf{s} for which $A_{\mathbf{s}}^{p,q}(D)$ is non-trivial. The following is [10, Proposition 3.5].

Proposition 4.2. Take $\mathbf{s} \in \mathbb{R}^r$ and $p, q \in]0, \infty]$. Then, $A_{\mathbf{s},0}^{p,q}(D) \neq \{0\}$ (resp. $A_{\mathbf{s}}^{p,q}(D) \neq \{0\}$) if and only if $\mathbf{s} \in \frac{1}{2q}\mathbf{m} + (\mathbb{R}_+^*)^r$ (resp. $\mathbf{s} \in \mathbb{R}_+^r$ if $q = \infty$).

Definition 4.3. For $\mathbf{s} \in \mathbb{C}^r$ we define the kernel function $B^{\mathbf{s}}$ as

$$B_{(\zeta', z')}^{\mathbf{s}}(\zeta, z) := \Delta_{\Omega}^{\mathbf{s}} \left(\frac{z - \bar{z}'}{2i} - \Phi(\zeta, \zeta') \right)$$

for every $((\zeta, z), (\zeta', z')) \in (D \times \bar{D}) \cup (\bar{D} \times D)$.

Remark 4.4. It turns out that, when $\mathbf{s} \in \frac{1}{4}\mathbf{m} + (\mathbb{R}_+^*)^r$, the reproducing kernel of $A_{\mathbf{s}}^{2,2}$ is given by

$$K_{\mathbf{s}}((\zeta, z), (\zeta', z')) := c \frac{\Gamma_{\Omega'}(2\mathbf{s} - \mathbf{b} - \mathbf{d})}{\Gamma_{\Omega}(2\mathbf{s})} B_{(\zeta', z')}^{\mathbf{d}+\mathbf{b}-2\mathbf{s}}(\zeta, z)$$

for a suitable constant $c > 0$, cf. [10, Remark 3.12].

In addition, we can also describe the *Cauchy–Szegő* kernel, that is, the reproducing kernel of the Hardy space $A_0^{2,\infty}$. Indeed, for every $(\zeta, z) \in D$ and for every $(\zeta', x') \in \mathcal{N}$, set

$$S_{(\zeta, z)}(\zeta', x') := c' (B_{(\zeta, z)}^{\mathbf{b}+\mathbf{d}})_0(\zeta', x')$$

for a suitable constant $c' > 0$, cf. [10, Lemma 5.1]. Then, for every $f \in A_0^{2,\infty}$,

$$f(\zeta, z) = \langle f_0 | S_{(\zeta, z)} \rangle$$

for every $(\zeta, z) \in D$, where $f_0 := \lim_{h \rightarrow 0} f_h$ in $L^2(\mathcal{N})$.

For $p, q \in]0, \infty]$, and two sets J and K , define

$$\ell^{p,q}(J, K) := \left\{ \lambda \in \mathbb{C}^{J \times K} : ((\lambda_{j,k})_{j \in J})_{k \in K} \in \ell^q(K; \ell^p(J)) \right\},$$

endowed with the corresponding quasi-norm, and define $\ell_0^{p,q}(J, K)$ as the closure of $\mathbb{C}^{(J \times K)}$ in $\ell^{p,q}(J, K)$.

Definition 4.5. We say that property $(L)_{\mathbf{s}, \mathbf{s}', 0}^{p,q}$ (resp. $(L)_{\mathbf{s}, \mathbf{s}'}$) holds if for every $\delta_0 > 0$ there is a $(\delta, 4)$ -lattice $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$, with $\delta \in]0, \delta_0]$, such that, defining $h_k := \rho(\zeta_{j,k}, z_{j,k})$ for every $k \in K$ and for some (hence every) $j \in J$, the mapping

$$\Psi: \lambda \mapsto \sum_{j,k} \lambda_{j,k} B_{(\zeta_{j,k}, z_{j,k})}^{\mathbf{s}' } \Delta_{\Omega}^{(\mathbf{b}+\mathbf{d})/p - \mathbf{s} - \mathbf{s}' } (h_k)$$

is well defined (with locally uniform convergence of the sum) and maps $\ell_0^{p,q}(J, K)$ into $A_{\mathbf{s},0}^{p,q}(D)$ continuously (resp. maps $\ell^{p,q}(J, K)$ into $A_{\mathbf{s}}^{p,q}(D)$ continuously).

If we may take $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$, for every $\delta_0 > 0$ as above, in such a way that the corresponding mapping Ψ is onto, then we say that property $(L')_{\mathbf{s}, \mathbf{s}', 0}^{p,q}$ (resp. $(L')_{\mathbf{s}, \mathbf{s}'}$) holds.

We shall present later (cf. Proposition 4.11) some sufficient conditions for property (L') to hold. See also [10] for a more thorough discussion and some necessary conditions.

In this paper we are also interested in a family of spaces of holomorphic functions on D , denoted by $\tilde{A}_{\mathbf{s}}^{p,q}(D)$, which is defined in connection with the boundary values of the elements of $A_{\mathbf{s}}^{p,q}(D)$. We begin by introducing some Besov-type spaces defined on the Šilov boundary bD , that we identify with \mathcal{N} .

We define a space of test functions on \mathcal{N} by setting

$$\mathcal{S}_{\Omega}(\mathcal{N}) := \{ \psi \in \mathcal{S}(\mathcal{N}) : \exists \varphi \in C_c^{\infty}(\Omega') \forall \lambda \in F' \setminus W \quad \pi_{\lambda}(\psi) = \varphi(\lambda) P_{\lambda,0} \}.$$

Then, define $\mathcal{S}_{\Omega, L}(\mathcal{N}) := \mathcal{S}(\mathcal{N}) * \mathcal{S}_{\Omega}(\mathcal{N})$, endowed with the inductive limit of the topologies induced by $\mathcal{S}(\mathcal{N})$ on its subspaces $\mathcal{S}(\mathcal{N}) * \psi$, $\psi \in \mathcal{S}_{\Omega}(\mathcal{N})$.⁵ Denote by $\mathcal{S}'_{\Omega, L}(\mathcal{N})$ the dual of $\mathcal{S}_{\Omega, L}(\mathcal{N})$. See [10, Propositions 4.2 and 4.5, and Lemma 4.14] for a proof of the following result.

Proposition 4.6. *The following hold:*

- (1) *the mapping $\mathcal{F}_{\mathcal{N}}: \varphi \mapsto [\lambda \mapsto \text{Tr}(\pi_{\lambda}(\varphi))]$ induces an isomorphism of $\mathcal{S}_{\Omega}(\mathcal{N})$ onto $C_c^{\infty}(\Omega')$;*
- (2) *for every two (δ, R) -lattices $(\lambda_k)_{k \in K}$ and $(\lambda'_{k'})_{k' \in K'}$ on Ω' , and for every two sequences $(\psi_k)_{k \in K}, (\psi'_{k'})_{k' \in K'}$ in $\mathcal{S}_{\Omega}(\mathcal{N})$ such that $\sum_{k \in K} \psi_k \geq 1$, $\sum_{k' \in K'} \psi_{k'} \geq 1$, for every $\mathbf{s} \in \mathbb{R}^r$ and for every $p, q \in]0, \infty]$, one has*

$$\left\| \Delta_{\Omega'}^{\mathbf{s}}(\lambda_{k'}) \|u * \psi'_{k'}\|_{L^p(\mathcal{N})} \right\|_{\ell^q(K')} \approx \left\| \Delta_{\Omega'}^{\mathbf{s}}(\lambda_k) \|u * \psi_k\|_{L^p(\mathcal{N})} \right\|_{\ell^q(K)},$$

for every $u \in \mathcal{S}'_{\Omega, L}(\mathcal{N})$.

Definition 4.7. Let $\mathbf{s} \in \mathbb{R}^r$, $p, q \in]0, \infty]$. Take $(\lambda_k)_{k \in K}$ and (ψ_k) as in Proposition 4.6. Then, we define $B_{p,q}^{\mathbf{s}}(\mathcal{N}, \Omega)$ as the space of $u \in \mathcal{S}'_{\Omega, L}(\mathcal{N})$ such that

$$(\Delta_{\Omega'}^{\mathbf{s}}(\lambda_k)(u * \psi_k))_k \in \ell^q(K; L^p(\mathcal{N})),$$

endowed with the corresponding topology. We denote by $\mathring{B}_{p,q}^{\mathbf{s}}(\mathcal{N}, \Omega)$ the closed subspace of $B_{p,q}^{\mathbf{s}}(\mathcal{N}, \Omega)$ consisting of the $u \in \mathcal{S}'_{\Omega, L}(\mathcal{N})$ such that

$$(\Delta_{\Omega'}^{\mathbf{s}}(\lambda_k)(u * \psi_k))_k \in \ell_0^q(K; L_0^p(\mathcal{N})).$$

See [10, Proposition 4.20 and Theorem 4.23] for a proof of the following result. Here and in what follows, we put $p' := \max(1, p)'$ for every $p \in]0, \infty]$, so that $p' = \infty$ if $p \leq 1$, and $\frac{1}{p} + \frac{1}{p'} = 1$ if $p \geq 1$.

Proposition 4.8. *Take $p, q \in]0, \infty]$ and $\mathbf{s} \in \mathbb{R}^r$. Then, the following hold:*

- (1) $\mathcal{S}_{\Omega, L}(\mathcal{N})$ *embeds continuously as a dense subspace of $\mathring{B}_{p,q}^{\mathbf{s}}(\mathcal{N}, \Omega)$;*
- (2) *the canonical sesquilinear pairings on $\mathcal{S}_{\Omega, L}(\mathcal{N}) \times \mathcal{S}'_{\Omega, L}(\mathcal{N})$ and on $\mathcal{S}'_{\Omega, L}(\mathcal{N}) \times \mathcal{S}_{\Omega, L}(\mathcal{N})$ induce unique continuous sesquilinear pairings*

$$\langle \cdot | \cdot \rangle : \mathring{B}_{p,q}^{\mathbf{s}}(\mathcal{N}, \Omega) \times B_{p',q'}^{-\mathbf{s} - (1/p-1) + (\mathbf{b} + \mathbf{d})}(\mathcal{N}, \Omega) \rightarrow \mathbb{C},$$

and

$$\langle \cdot | \cdot \rangle : B_{p,q}^{\mathbf{s}}(\mathcal{N}, \Omega) \times \mathring{B}_{p',q'}^{-\mathbf{s} - (1/p-1) + (\mathbf{b} + \mathbf{d})}(\mathcal{N}, \Omega) \rightarrow \mathbb{C},$$

respectively.

⁵It is not hard to see that this definition is equivalent to [10, Definition 4.4].

We are now able to define an extension operator from the Besov-type spaces $B_{p,q}^{-\mathbf{s}}(\mathcal{N}, \Omega)$ on the Šilov boundary \mathcal{N} to D to the function spaces $\tilde{A}_{\mathbf{s}}^{p,q}(D)$ and $\tilde{A}_{\mathbf{s},0}^{p,q}(D)$ on D .

Definition 4.9. Take $p, q \in]0, \infty]$ and $\mathbf{s} \in \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q}\mathbf{m}' + (\mathbb{R}_+^*)^r$, so that, for every $(\zeta, z) \in D$, $S_{(\zeta,z)} \in \mathring{B}_{p',q'}^{\mathbf{s} - (1/p-1)(\mathbf{b}+\mathbf{d})}(\mathcal{N}, \Omega)$ (cf. [10, Lemma 5.1]). Then, define, for every $u \in B_{p,q}^{-\mathbf{s}}(\mathcal{N}, \Omega)$ and for every $(\zeta, z) \in D$,

$$\mathcal{E}u(\zeta, z) := \langle u | S_{(\zeta,z)} \rangle,$$

so that \mathcal{E} maps $B_{p,q}^{-\mathbf{s}}(\mathcal{N}, \Omega)$ into $A_{\mathbf{s} - (\mathbf{b}+\mathbf{d})/p}^{\infty, \infty}(D)$ continuously (cf. [10, Theorem 5.2]). We define

$$\tilde{A}_{\mathbf{s}}^{p,q}(D) := \mathcal{E}(B_{p,q}^{-\mathbf{s}}(\mathcal{N}, \Omega)) \quad \text{and} \quad \tilde{A}_{\mathbf{s},0}^{p,q}(D) := \mathcal{E}(\mathring{B}_{p,q}^{-\mathbf{s}}(\mathcal{N}, \Omega)),$$

and endow both spaces with the corresponding (direct image) topology.

In addition, we define the *boundary value operator* $\mathcal{B}: A_{\mathbf{0}}^{2,\infty}(D) \rightarrow B_{2,2}^{\mathbf{0}}(\mathcal{N}, \Omega)$

$$\mathcal{B}: A_{\mathbf{0}}^{2,\infty}(D) \ni f \mapsto \lim_{h \rightarrow 0} f_h \in L^2(\mathcal{N})$$

We remark that, by [10, Corollary 1.37], \mathcal{B} is well defined and continuous, and that $\mathcal{E}\mathcal{B} = I$.

It is possible to describe the boundary values of functions in $A_{\mathbf{s}}^{p,q}(D)$, that is, the limits of f_h , as $h \rightarrow 0, h \in \Omega$, for $f \in A_{\mathbf{s}}^{p,q}(D)$, as elements of $B_{p,q}^{-\mathbf{s}}(\mathcal{N}, \Omega)$. Conversely, under suitable assumptions, every element of $B_{p,q}^{-\mathbf{s}}(\mathcal{N}, \Omega)$ is the boundary value of a unique element of $A_{\mathbf{s}}^{p,q}(D)$. Precisely, the following holds, cf. [10, Theorem 5.2, Proposition 5.4 and its proof, and Corollary 5.11].

Proposition 4.10. Take $p, q \in]0, \infty]$, and $\mathbf{s} \in \sup\left(\frac{1}{2q}\mathbf{m}, \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q}\mathbf{m}'\right) + (\mathbb{R}_+^*)^r$. Then, the following hold:

- (1) $(\mathcal{E}u)_h$ converges to u in $B_{p,q}^{-\mathbf{s}}(\mathcal{N}, \Omega)$ (resp. in $\mathcal{S}'_{\Omega,L}(\mathcal{N})$) for every $u \in \mathring{B}_{p,q}^{-\mathbf{s}}(\mathcal{N}, \Omega)$ (resp. for every $u \in B_{p,q}^{-\mathbf{s}}(\mathcal{N}, \Omega)$);
- (2) the operator \mathcal{B} induces a continuous linear mapping

$$\mathcal{B}: A_{\mathbf{s}}^{p,q}(D) \rightarrow B_{p,q}^{-\mathbf{s}}(\mathcal{N}, \Omega) \quad (\text{resp. } \mathcal{B}: A_{\mathbf{s},0}^{p,q}(D) \rightarrow \mathring{B}_{p,q}^{-\mathbf{s}}(\mathcal{N}, \Omega))$$

such that $\mathcal{E}\mathcal{B} = I$;

- (3) there are continuous inclusions

$$\mathcal{E}(\mathcal{S}_{\Omega,L}(\mathcal{N})) \subseteq A_{\mathbf{s}}^{p,q}(D) \subseteq \tilde{A}_{\mathbf{s}}^{p,q}(D) \quad (\text{resp. } \mathcal{E}(\mathcal{S}_{\Omega,L}(\mathcal{N})) \subseteq A_{\mathbf{s},0}^{p,q}(D) \subseteq \tilde{A}_{\mathbf{s},0}^{p,q}(D));$$

- (4) if, further, $\mathbf{s} \in \sup\left(\frac{1}{2q}\mathbf{m} + \left(\frac{1}{2\min(p,p')} - \frac{1}{2q}\right)_+\mathbf{m}', \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q}\mathbf{m}'\right) + (\mathbb{R}_+^*)^r$, then,

$$A_{\mathbf{s}}^{p,q}(D) = \tilde{A}_{\mathbf{s}}^{p,q}(D) \quad \text{and} \quad A_{\mathbf{s},0}^{p,q}(D) = \tilde{A}_{\mathbf{s},0}^{p,q}(D).$$

We now present some sufficient conditions for property (L'). See [10, Corollary 5.14] for a proof of the following result.

Proposition 4.11. Take $p, q \in]0, \infty]$, $\mathbf{s} \in \sup\left(\frac{1}{2q}\mathbf{m}, \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q}\mathbf{m}'\right) + (\mathbb{R}_+^*)^r$, and $\mathbf{s}' \in \frac{1}{\min(1,p)}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q}\mathbf{m}' - \left(\frac{1}{2\min(1,p)} - \frac{1}{2q}\right)_+\mathbf{m} - \mathbf{s} - (\mathbb{R}_+^*)^r$. If $A_{\mathbf{s},0}^{p,q}(D) = \tilde{A}_{\mathbf{s},0}^{p,q}(D)$ (resp. $A_{\mathbf{s}}^{p,q}(D) = \tilde{A}_{\mathbf{s}}^{p,q}(D)$), then property $(L')_{\mathbf{s},\mathbf{s}',0}^{p,q}$ (resp. $(L')_{\mathbf{s},\mathbf{s}'}$) holds.

We conclude this section with a technical lemma that will be very useful later on.

Lemma 4.12. *Take $\mathbf{s} \in \mathbb{R}^r$, $p, q \in]0, \infty]$, $t \in T_+$, and $g \in GL(E)$ such that $t \cdot \Phi = \Phi \circ (g \times g)$. Then,*

$$\|f \circ (g \times t)\|_{L_{\mathbf{s}}^{p,q}(D)} = \Delta_{\Omega}^{(\mathbf{b}+\mathbf{d})/p-\mathbf{s}}(t) \|f\|_{L_{\mathbf{s}}^{p,q}(D)}$$

for every ν_D -measurable function f on D .

Proof. Observe that

$$(f \circ (g \times t))_h(\zeta, x) = f_{t \cdot h}(g\zeta, t \cdot x)$$

for every $(\zeta, x) \in D$ and for every $h \in \Omega$, so that

$$\|(f \circ (g \times t))_h\|_{L^p(\mathcal{N})} = \Delta^{(\mathbf{b}+\mathbf{d})/p}(t) \|f_{t \cdot h}\|_{L^p(\mathcal{N})}$$

for every $h \in \Omega$, thanks to [10, Lemmas 2.9 and 2.18]. Then,

$$\|f \circ (g \times t)\|_{L_{\mathbf{s}}^{p,q}(D)} = \Delta^{(\mathbf{b}+\mathbf{d})/p-\mathbf{s}}(t) \|f\|_{L_{\mathbf{s}}^{p,q}(D)},$$

whence the result. \square

5. CARLESON MEASURES

In this section we prove our main results concerning Carleson measures for $A_{\mathbf{s}}^{p,q}$. For every $\mu \in \mathcal{M}_+(D)$ and $R > 0$, we set

$$(5.1) \quad M_R(\mu): D \ni (\zeta, z) \mapsto \mu(B((\zeta, z), R)) \in [0, \infty[.$$

Lemma 5.1. *Take $\mathbf{s} \in \mathbb{R}^r$, $p, q \in]0, \infty]$, and $\mu \in \mathcal{M}_+(D)$. Then, the following conditions are equivalent:*

- (1) *there is $R > 0$ such that $M_R(\mu) \in L_{\mathbf{s}}^{p,q}(D)$;*
- (2) *$M_R(\mu) \in L_{\mathbf{s}}^{p,q}(D)$ for every $R > 0$;*
- (3) *there is a (δ, R) -lattice $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$ on D , with $\delta > 0$ and $R > 1$, such that*

$$\left(\Delta_{\Omega}^{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p}(\rho(\zeta_{j,k}, z_{j,k})) M_{R\delta}(\mu)(\zeta_{j,k}, z_{j,k}) \right) \in \ell^{p,q}(J, K);$$

- (4) *for every (δ, R) -lattice $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$ on D , with $\delta > 0$ and $R > 1$,*

$$\left(\Delta_{\Omega}^{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p}(\rho(\zeta_{j,k}, z_{j,k})) M_{R\delta}(\mu)(\zeta_{j,k}, z_{j,k}) \right) \in \ell^{p,q}(J, K).$$

The same holds if one replaces $L_{\mathbf{s}}^{p,q}(D)$ and $\ell^{p,q}(J, K)$ with $L_{\mathbf{s},0}^{p,q}(D)$ and $\ell_0^{p,q}(J, K)$, respectively.

This extends [33, Lemmas 2.9 and 2.12], where the case in which $p = q \in [1, \infty]$, D is an irreducible symmetric tube domain, and $\mathbf{s} \in \mathbb{R}\mathbf{1}_r$, is considered.⁶

Proof. We shall only prove the first assertion. The second assertion is clear if μ has compact support, and then follows by approximation, since the equivalence of conditions (1)–(4) is quantitative by the closed graph theorem, or by the proof below.

(1) \implies (2). Take $R' > 0$. Observe that, since $\overline{B}((0, ie_{\Omega}), R')$ is compact by [10, Proposition 2.44], there is a finite family $(\zeta_j, z_j)_{j \in J}$ of elements of D such that $B((0, ie_{\Omega}), R') \subseteq \bigcup_{j \in J} B((\zeta_j, z_j), R)$. For every $j \in J$, choose $t_j \in T_+$ and $g_j \in GL(E)$ in such a way that $t_j \cdot \Phi = \Phi \circ (g_j \times g_j)$ and $t_j \cdot e_{\Omega} = \rho(\zeta_j, z_j)$. If we define

$$\varphi_j: (\zeta, z) \mapsto (\zeta_j, \operatorname{Re} z_j + i\Phi(\zeta_j)) \cdot (g_j \zeta, t_j \cdot z),$$

⁶Notice, though, that in [33, Lemmas 2.9 and 2.12] the use of the ‘pure-norm’ spaces L^p and ℓ^p allows for the use of more general lattices.

then φ_j is an affine automorphism of D and $B((\zeta, z), R') \subseteq \bigcup_{j \in J} B(\varphi_j(\zeta, z), R)$ for every $(\zeta, z) \in D$. Therefore, in order to prove that $M_{R'}(\mu) \in L_S^{p,q}(D)$, it will suffice to prove that $M_R(\mu) \circ \varphi_j \in L_S^{p,q}(D)$ for every $j \in J$. Since this follows easily from Lemma 4.12, this proves (2).

(2) \implies (4). Let $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$ be a (δ, R) -lattice on D for some $\delta > 0$ and $R > 1$. Observe that

$$\sum_{j,k} M_{R\delta}(\zeta_{j,k}, z_{j,k}) \chi_{B((\zeta_{j,k}, z_{j,k}), \delta)} \leq M_{(R+1)\delta}(\mu)(\zeta_{j,k}, z_{j,k})$$

on D , so that

$$\sum_{j,k} M_{R\delta}(\zeta_{j,k}, z_{j,k}) \chi_{B((\zeta_{j,k}, z_{j,k}), \delta)} \in L_S^{p,q}(D).$$

Now, observe that there is a constant $C_1 > 0$ such that

$$\|(\chi_{B((0, ie_\Omega), \delta)})_h\|_{L^p(\mathcal{N})} \geq C_1$$

for every $h \in B_\Omega(e_\Omega, \delta/2)$,⁷ so that, by homogeneity,

$$\|(\chi_{B((\zeta_{j,k}, z_{j,k}), \delta)})_h\|_{L^p(\mathcal{N})} \geq C_1 \Delta_\Omega^{-(\mathbf{b}+\mathbf{d})/p}(h_k)$$

for every $h \in B_\Omega(h_k, \delta/2)$ and for every $(j, k) \in J \times K$, where $h_k := \rho(\zeta_{j,k}, z_{j,k})$ for every $(j, k) \in J \times K$. It then follows that

$$\begin{aligned} & \left\| \sum_{j,k} \left(\chi_{B((\zeta_{j,k}, z_{j,k}), \delta)} \right)_h M_{R\delta}(\zeta_{j,k}, z_{j,k}) \right\|_{L^p(\mathcal{N})} \\ & \geq C_1 \left\| \left(\Delta_\Omega^{-(\mathbf{b}+\mathbf{d})/p}(h_k) M_{R\delta}(\zeta_{j,k}, z_{j,k}) \chi_{B_\Omega(h_k, \delta/2)}(h) \right)_{j,k} \right\|_{\ell^p(J \times K)} \end{aligned}$$

for every $h \in \Omega$, whence

$$\left(\Delta_\Omega^{s-(\mathbf{b}+\mathbf{d})/p}(h_k) M_{R\delta}(\zeta_{j,k}, z_{j,k}) \right) \in \ell^{p,q}(J, K)$$

thanks to [10, Corollary 2.49].

(4) \implies (3). Obvious by [10, Lemma 2.55].

(3) \implies (1). Take a (δ, R) -lattice $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$ as in the statement, define $h_k := \rho(\zeta_{j,k}, z_{j,k})$ for every $(j, k) \in J \times K$, and let us prove that

$$\left(\Delta_\Omega^{s-(\mathbf{b}+\mathbf{d})/p}(h_k) M_{R'\delta}(\zeta_{j,k}, z_{j,k}) \right) \in \ell^{p,q}(J, K)$$

for every $R' \geq R$. Indeed, for every $(j, k) \in J \times K$, define $V_{j,k}$ as the set of $(j', k') \in J \times K$ such that $d((\zeta_{j,k}, z_{j,k}), (\zeta_{j',k'}, z_{j',k'})) < (R + R')\delta$, and observe that $N := \sup_{j,k} \text{Card}(V_{j,k})$ is finite by [10,

Proposition 2.56]. In addition, by [10, Corollary 2.49] we may find a constant $C_2 > 0$ such that

$$\frac{1}{C_2} \Delta_\Omega^{s-(\mathbf{b}+\mathbf{d})/p}(h_{k'}) \leq \Delta_\Omega^{s-(\mathbf{b}+\mathbf{d})/p}(h_k) \leq C_2 \Delta_\Omega^{s-(\mathbf{b}+\mathbf{d})/p}(h_{k'})$$

for every $(j, k) \in J \times K$ and for every $(j', k') \in V_{j,k}$. Then,

$$\Delta_\Omega^{s-(\mathbf{b}+\mathbf{d})/p}(h_k) M_{R'\delta}(\zeta_{j,k}, z_{j,k}) \leq C_2 \sum_{(j', k') \in V_{j,k}} \Delta_\Omega^{s-(\mathbf{b}+\mathbf{d})/p}(h_{k'}) M_{R\delta}(\mu)(\zeta_{j',k'}, z_{j',k'})$$

⁷Observe that, since $B((0, ie_\Omega), \delta)$ is an open set, the mapping $h \mapsto \|(\chi_{B((0, ie_\Omega), \delta)})_h\|_{L^p(\mathcal{N})}$ is lower semi-continuous, and that it vanishes nowhere on the compact set $\overline{B}_\Omega(e_\Omega, \delta/2)$, thanks to [10, Lemma 2.46].

for every $(j, k) \in J \times K$. Define V'_k as the set of $k' \in K$ such that $d(h_k, h_{k'}) \leq (R + R')\delta$, so that $k' \in V'_k$ for every $(j', k') \in V_{j,k}$ and for every $j \in J$ by [10, Lemma 2.46]. In addition, $N' := \sup_k \text{Card}(V'_k)$ is finite by [10, Proposition 2.56]. Therefore,

$$\begin{aligned} & \left\| \left(\Delta_{\Omega}^{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p}(h_k) M_{R'\delta}(\zeta_{j,k}, z_{j,k}) \right)_j \right\|_{\ell^p(J)} \\ & \leq C_2 N^{\max(1,1/p)} \left\| \left(\Delta_{\Omega}^{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p}(h_{k'}) M_{R\delta}(\zeta_{j',k'}, z_{j',k'}) \right)_{j',k'} \right\|_{\ell^p(J \times V'_k)} \end{aligned}$$

for every $k \in K$, so that

$$\begin{aligned} & \left\| \left(\Delta_{\Omega}^{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p}(h_k) M_{R'\delta}(\zeta_{j,k}, z_{j,k}) \right)_{j,k} \right\|_{\ell^{p,q}(J,K)} \\ & \leq C_2 N^{\max(1,1/p)} N'^{\max(1/p,1/q)} \left\| \left(\Delta_{\Omega}^{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p}(h_{k'}) M_{R\delta}(\mu)(\zeta_{j',k'}, z_{j',k'}) \right)_{j',k'} \right\|_{\ell^{p,q}(J,K)}. \end{aligned}$$

whence our claim.

Now, let $(B_{j,k})_{j,k}$ be a Borel partition of D such that $B_{j,k} \subseteq B((\zeta_{j,k}, z_{j,k}), R\delta)$ for every $(j, k) \in J \times K$, and observe that

$$M_{\delta}(\mu) \leq \sum_{j,k} \chi_{B_{j,k}} M_{(R+1)\delta}(\zeta_{j,k}, z_{j,k})$$

on D . In addition, arguing as in the proof of [10, Theorem 3.23], we see that there is a constant $C_3 > 0$ such that

$$\|(\chi_{B((\zeta,z), R\delta)} h)\|_{L^p(\mathcal{N})} \leq C_3 \Delta_{\Omega}^{-(\mathbf{b}+\mathbf{d})/p}(\rho(\zeta, z)) \chi_{B_{\Omega}(\rho(\zeta,z), R\delta)}(h)$$

for every $(\zeta, z) \in D$ and for every $h \in \Omega$. Hence,

$$\|M_{\delta}(\mu)_h\|_{L^p(\mathcal{N})} \leq C_3 \left\| (\chi_{B(h_k, R\delta)}(h) \Delta_{\Omega}^{-(\mathbf{b}+\mathbf{d})/p}(h_k) M_{(R+1)\delta}(\zeta_{j,k}, z_{j,k}))_{j,k} \right\|_{\ell^p(J \times K)}.$$

Now, by [10, Corollary 2.49 and Proposition 2.56] there are two constants $C_4 > 0$ and $N'' \in \mathbb{N}$ such that

$$\frac{1}{C_4} \Delta_{\Omega}^{\mathbf{s}}(h) \leq \Delta_{\Omega}^{\mathbf{s}}(h_k) \leq C_4 \Delta_{\Omega}^{\mathbf{s}}(h)$$

for every $h \in B_{\Omega}(h_k, R\delta)$ and for every $k \in K$, and such that $\sum_k \chi_{B_{\Omega}(h_k, R\delta)} \leq N'' \chi_{\Omega}$. Then,

$$\|M_{\delta}(\mu)\|_{L^{p,q}(D)} \leq C_5 \left\| \left(\Delta_{\Omega}^{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p}(h_k) M_{(R+1)\delta}(\zeta_{j,k}, z_{j,k}) \right)_{j,k} \right\|_{\ell^{p,q}(J,K)},$$

where $C_5 := C_3 C_4 \nu_{\Omega}(B_{\Omega}(e_{\Omega}, R\delta))^{1/q} N''^{\max(1/p,1/q)}$. Thus, (1) follows. \square

The next proposition provides a first sufficient condition for a measure $\mu \in \mathcal{M}_+(D)$ to be a p -Carleson measure or a compact p -Carleson measure. It extends [33, Proposition 3.5 and Theorem 3.8 (i)], where the case in which $p_1 = q_1, p \in [1, \infty[, D$ is an irreducible symmetric tube domain, and $\mathbf{s} \in \mathbf{R}\mathbf{1}_r$, is considered.

We shall often use the following notation. Given $p_1, q_1, p \in]0, \infty]$, with $p < \infty$ and $\mathbf{s} \in \mathbf{R}^r$, we set⁸

$$(5.2) \quad p^* := (p_1/p)', \quad q^* := (q_1/p)', \quad \text{and} \quad \mathbf{s}^* := \max(1, p/p_1)(\mathbf{b} + \mathbf{d}) - p\mathbf{s}.$$

⁸Recall that $t' := \max(1, t)'$ for every $t \in]0, \infty]$.

Proposition 5.2. *Take $p_1, q_1, p \in]0, \infty]$, with $p < \infty$, and take $\mathbf{s} \in \mathbb{R}^r$ such that $\mathbf{s} \in \frac{1}{2q_1} \mathbf{m} + (\mathbb{R}_+^*)^r$ or $\mathbf{s} \in \mathbb{R}_+^r$ if $q_1 = \infty$. Let p^*, q^* , and \mathbf{s}^* be as in (5.2). Let $\mu \in \mathcal{M}_+(D)$, and take $R > 0$. Assume that*

$$M_R(\mu) \in L_{\mathbf{s}^*}^{p^*, q^*}(D).$$

Then, $A_{\mathbf{s}}^{p_1, q_1}(D)$ embeds continuously into $L^p(\mu)$.

If, in addition,

$$M_R(\mu) \in L_{\mathbf{s}^*, 0}^{p^*, q^*}(D),$$

then $A_{\mathbf{s}}^{p_1, q_1}(D)$ embeds compactly into $L^p(\mu)$.

Proof. Let $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$ be an $(R/4, 4)$ -lattice on D (cf. [10, Lemma 2.55]), and define $S_+ : \text{Hol}(D) \rightarrow \mathbb{C}^{J \times K}$ so that

$$(S_+ f)_{j,k} := \Delta_{\Omega}^{\mathbf{s} - (\mathbf{b} + \mathbf{d})/p_1}(h_k) \max_{\overline{B}((\zeta_{j,k}, z_{j,k}), R)} |f|$$

for every $(j, k) \in J \times K$, where $h_k := \rho(\zeta_{j,k}, z_{j,k})$, so that there is a constant $C_1 > 0$ such that

$$\frac{1}{C_1} \|f\|_{A_{\mathbf{s}}^{p_1, q_1}(D)} \leq \|S_+ f\|_{\ell^{p_1, q_1}(J, K)} \leq C_1 \|f\|_{A_{\mathbf{s}}^{p_1, q_1}(D)}$$

for every $f \in \text{Hol}(D)$, thanks to [10, Theorem 3.23]. In addition, choose a Borel partition $(B_{j,k})$ of D such that $B_{j,k} \subseteq B((\zeta_{j,k}, z_{j,k}), R)$ for every $(j, k) \in J \times K$. Then, for every $f \in A_{\mathbf{s}}^{p_1, q_1}(D)$,

$$\begin{aligned} \|f\|_{L^p(\mu)} &\leq \left\| \sum_{(j,k) \in J \times K} (S_+ f)_{j,k} \Delta_{\Omega}^{(\mathbf{b} + \mathbf{d})/p_1 - \mathbf{s}}(h_k) \chi_{B_{j,k}} \right\|_{L^p(\mu)} \\ &\leq \left\| ((S_+ f)_{j,k} \Delta_{\Omega}^{(\mathbf{b} + \mathbf{d})/p_1 - \mathbf{s}}(h_k) M_R(\mu)(\zeta_{j,k}, z_{j,k})^{1/p})_{j,k} \right\|_{\ell^p(J \times K)} \\ &\leq C_2 \|S_+ f\|_{\ell^{p_1, q_1}(J, K)} \\ &\leq C_1 C_2 \|f\|_{A_{\mathbf{s}}^{p_1, q_1}(D)} \end{aligned}$$

where $C_2 := \left\| \Delta_{\Omega}^{p[(\mathbf{b} + \mathbf{d})/p_1 - \mathbf{s}]}(h_k) M_R(\mu)(\zeta_{j,k}, z_{j,k}) \right\|_{\ell^{p^*, q^*}(J, K)}^{1/p}$. Then, the first assertion follows from Lemma 5.1.

In order to prove the second assertion, it will suffice to show that, if μ has compact support, then the canonical mapping $A_{\mathbf{s}}^{p_1, q_1}(D) \rightarrow L^p(\mu)$ is compact. Since $A_{\mathbf{s}}^{p_1, q_1}(D)$ embeds continuously, hence compactly, into the Fréchet–Montel space $\text{Hol}(D)$, which in turn embeds continuously into $L^p(\mu)$, the assertion follows easily. \square

Proposition 5.3. *Take $p_1, q_1, p \in]0, \infty]$, with $p < \infty$, and take $\mathbf{s} \in \mathbb{R}^r$ such that $\mathbf{s} \in \frac{1}{2q_1} \mathbf{m} + (\mathbb{R}_+^*)^r$ (resp. $\mathbf{s} \in \mathbb{R}_+^r$ if $q_1 = \infty$). Let $\mu \in \mathcal{M}_+(D)$ and $R > 0$. Assume that $A_{\mathbf{s}, 0}^{p_1, q_1}(D)$ (resp. $A_{\mathbf{s}}^{p_1, q_1}(D)$) embeds continuously into $L^p(\mu)$. Then,*

$$M_R(\mu) \in L_{p(\mathbf{b} + \mathbf{d})/p_1 - p\mathbf{s}}^{\infty, \infty}(D).$$

If, in addition, $A_{\mathbf{s}, 0}^{p_1, q_1}(D)$ (resp. $A_{\mathbf{s}}^{p_1, q_1}(D)$) embeds compactly into $L^p(\mu)$ and $\mathbf{s}_1 \in (\mathbb{R}_+^)^r$ if $p_1 = q_1 = \infty$, then*

$$M_R(\mu) \in L_{p(\mathbf{b} + \mathbf{d})/p_1 - p\mathbf{s}, 0}^{\infty, \infty}(D).$$

Proof. Take $\mathbf{s}' \in \mathbb{R}^r$ such that

$$\mathbf{s}' \in \frac{1}{p_1}(\mathbf{b} + \mathbf{d}) - \frac{1}{2p_1}\mathbf{m}' - (\mathbb{R}_+^*)^r \quad \text{and} \quad \mathbf{s} + \mathbf{s}' \in \frac{1}{p_1}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q_1}\mathbf{m}' - (\mathbb{R}_+^*)^r.$$

Then, $B_{(\zeta', z')}^{\mathbf{s}'}$ $\in A_{\mathbf{s}, 0}^{p_1, q_1}(D)$ (resp. $B_{(\zeta', z')}^{\mathbf{s}'}$ $\in A_{\mathbf{s}}^{p_1, q_1}(D)$) for every $(\zeta', z') \in D$, and

$$\|B_{(\zeta', z')}^{\mathbf{s}'}\|_{A_{\mathbf{s}}^{p_1, q_1}(D)} = C_1 \Delta_{\Omega}^{\mathbf{s} + \mathbf{s}' - (\mathbf{b} + \mathbf{d})/p_1}(\rho(\zeta', z'))$$

for a suitable constant $C_1 > 0$, thanks to [10, Proposition 2.41]. Hence, there is a constant $C_2 > 0$ such that

$$\|B_{(\zeta', z')}^{\mathbf{s}'}\|_{L^p(\mu)} \leq C_2 \Delta_{\Omega}^{\mathbf{s} + \mathbf{s}' - (\mathbf{b} + \mathbf{d})/p_1}(\rho(\zeta', z'))$$

for every $(\zeta', z') \in D$. In addition, there is a constant $C_3 > 0$ such that

$$\frac{1}{C_3} |B_{(\zeta', z')}^{\mathbf{s}'}| \leq |B_{(\zeta, z)}^{\mathbf{s}'}| \leq C_3 |B_{(\zeta', z')}^{\mathbf{s}'}|$$

on D , for every $(\zeta', z') \in D$ and for every $(\zeta, z) \in B((\zeta', z'), r)$, thanks to [10, Theorem 2.47]. Therefore,

$$\begin{aligned} \|B_{(\zeta', z')}^{\mathbf{s}'}\|_{L^p(\mu)} &\geq \frac{1}{C_3} \Delta_{\Omega}^{\mathbf{s}' }(\rho(\zeta', z')) \|\chi_{B((\zeta', z'), R)}\|_{L^p(\mu)} \\ &= \frac{1}{C_3} \Delta_{\Omega}^{\mathbf{s}' }(\rho(\zeta', z')) M_R(\mu)(\zeta', z')^{1/p} \end{aligned}$$

for every $(\zeta', z') \in D$. It then follows that

$$M_R(\mu)(\zeta', z') \leq C_2^p C_3^p \Delta_{\Omega}^{p[\mathbf{s}' - (\mathbf{b} + \mathbf{d})/p_1]}(\rho(\zeta', z'))$$

for every $(\zeta', z') \in D$.

Now, assume that $A_{\mathbf{s}, 0}^{p_1, q_1}(D)$ (resp. $A_{\mathbf{s}}^{p_1, q_1}(D)$) embeds compactly in $L^p(\mu)$, and that $\mathbf{s} \in (\mathbb{R}_+^*)^r$ if $p_1 = q_1 = \infty$. Define $b_{(\zeta, z)} := B_{(\zeta, z)}^{\mathbf{s}'} \Delta_{\Omega}^{(\mathbf{b} + \mathbf{d})/p_1 - \mathbf{s} - \mathbf{s}' }(\rho(\zeta, z))$ for every $(\zeta, z) \in D$, so that the family $(b_{(\zeta, z)})_{(\zeta, z) \in D}$ is uniformly bounded in $A_{\mathbf{s}}^{p_1, q_1}(D)$. In addition, since $\mathbf{s} \in \frac{\mathbf{b} + \mathbf{d}}{p_1} + (\mathbb{R}_+^*)^r$, [10, Proposition 2.41] implies that $B_{(\zeta, z)}^{\mathbf{s}'} \in A_{(\mathbf{b} + \mathbf{d})/p_1 - \mathbf{s} - \mathbf{s}', 0}^{\infty, \infty}(D)$, so that

$$\lim_{(\zeta', z') \rightarrow \infty} |b_{(\zeta', z')}(\zeta, z)| = \lim_{(\zeta', z') \rightarrow \infty} \Delta_{\Omega}^{(\mathbf{b} + \mathbf{d})/p_1 - \mathbf{s} - \mathbf{s}' }(\rho(\zeta', z')) |B_{(\zeta, z)}^{\mathbf{s}'}(\zeta', z')| = 0$$

for every $(\zeta, z) \in D$. Let b be a cluster point of $(b_{(\zeta, z)})$ in $L^p(\mu)$ for $(\zeta, z) \rightarrow \infty$, and observe that $b = 0$ since $b_{(\zeta, z)}$ converges pointwise (and also locally uniformly, thanks to [10, Theorem 2.47]) to 0 as $(\zeta, z) \rightarrow 0$. Since $A_{\mathbf{s}}^{p_1, q_1}(D)$ embeds compactly into $L^p(\mu)$, this suffices to prove that $b_{(\zeta, z)}$ converges to 0 in $L^p(\mu)$ as $(\zeta, z) \rightarrow \infty$. Therefore, the computations of the proof of the implication (1) \implies (2) show that

$$\lim_{(\zeta, z) \rightarrow \infty} \Delta_{\Omega}^{p[(\mathbf{b} + \mathbf{d})/p_1 - \mathbf{s}]}(\rho(\zeta, z)) M_R(\mu)(\zeta, z) = 0,$$

whence the result. \square

We are now ready to prove our first main result. It provides a necessary and sufficient condition for a measure $\mu \in \mathcal{M}_+(D)$ to be a p -Carleson measure for $A_{\mathbf{s}}^{p_1, q_1}$. This result is analogous to the classical characterization in the case of the unit ball, see e.g. [41, Theorem 2.25], and it extends [33, Theorem 3.5], where the case in which $p_1 = q_1, p \in [1, \infty[, D$ is an irreducible symmetric tube domain, and $\mathbf{s} \in \mathbb{R}\mathbf{1}_r$, is considered.

Theorem 5.4. *Take $p_1, q_1, p \in]0, \infty[$ with $p_1, q_1 \leq p$, and take $\mathbf{s} \in \mathbb{R}^r$ such that $\mathbf{s} \in \frac{1}{2q_1}\mathbf{m} + (\mathbb{R}_+^*)^r$. Let $\mu \in \mathcal{M}_+(D)$ and $R > 0$. Then, the following conditions are equivalent:*

- (1) $A_{\mathbf{s}}^{p_1, q_1}(D)$ embeds continuously (resp. compactly) into $L^p(\mu)$;
- (2) $M_R(\mu) \in L_{p[(\mathbf{b}+\mathbf{d})/p_1-\mathbf{s}]}^{\infty, \infty}(D)$ (resp. $M_R(\mu) \in L_{p[(\mathbf{b}+\mathbf{d})/p_1-\mathbf{s}], 0}^{\infty, \infty}(D)$).

Proof. (2) \implies (1). This follows from Proposition 5.2.

(1) \implies (2). This follows from Proposition 5.3. \square

We now deal with Carleson measures *admitting a loss*, that is, with the case $p_1, q_1 \not\leq p$. Note that when dealing with the mixed norm case, this complementary condition is not as simple as in the pure norm one. We provide a characterization of p -Carleson measures on $A_{\mathbf{s}}^{p_1, q_1}$ in two situations: when $A_{\mathbf{s}}^{p_1, q_1}$ admits a suitable atomic decomposition (Theorem 5.5); when $A_{\mathbf{s}}^{p_1, q_1}$ embeds into $A_{\mathbf{s}+(1/p-1/p_1)(\mathbf{b}+\mathbf{d})}^{p, p}$ (Theorem 5.9).

The next theorem extends [33, Theorem 3.8 (ii)], where the case in which $p_1 = q_1, p \in [1, \infty[$, D is an irreducible symmetric tube domain, and $\mathbf{s} \in \mathbb{R}\mathbf{1}_r$, is considered. Recall that property $(L)_{\mathbf{s}, \mathbf{s}', 0}^{p_1, q_1}$ is defined in Definition 4.5.

Theorem 5.5. *Take $p_1, q_1, p \in]0, \infty]$, with $p < \infty$, and $\mathbf{s} \in \frac{1}{2q_1}\mathbf{m} + (\mathbb{R}_+^*)^r$. Assume that property $(L)_{\mathbf{s}, \mathbf{s}', 0}^{p_1, q_1}$ holds for some $\mathbf{s}' \in \mathbb{R}^r$, and let $\mu \in \mathcal{M}_+(D)$. Then, the following conditions are equivalent:*

- (1) $A_{\mathbf{s}}^{p_1, q_1}(D)$ embeds continuously into $L^p(\mu)$;
- (2) $A_{\mathbf{s}, 0}^{p_1, q_1}(D)$ embeds continuously into $L^p(\mu)$;
- (3) for some (or, equivalently, every) $R > 0$, $M_R(\mu) \in L_{\mathbf{s}^*, 0}^{p^*, q^*}(D)$, where p^*, q^*, \mathbf{s}^* are as in (5.2).

If, in addition, $p < q_1$, then the following conditions are equivalent:

- (1') $A_{\mathbf{s}}^{p_1, q_1}(D)$ embeds compactly into $L^p(\mu)$;
- (2') $A_{\mathbf{s}, 0}^{p_1, q_1}(D)$ embeds compactly into $L^p(\mu)$;
- (3') for some (or, equivalently, every) $R > 0$, $M_R(\mu) \in L_{\mathbf{s}^*, 0}^{p^*, q^*}(D)$, where p^*, q^*, \mathbf{s}^* are as in (5.2).

Before we pass to the proof, we need a lemma.

Lemma 5.6. *Take $p_1, q_1, p_2, q_2 \in]0, \infty]$, $\mathbf{s} \in \mathbb{R}^r$, $R > 0$, and $\mu \in \mathcal{M}_+(D)$. Assume that $p_1 \leq p_2$ and that $q_1 \leq q_2$. Then, the following hold:*

- (1) if $M_R(\mu) \in L_{\mathbf{s}}^{p_1, q_1}(D)$, then $M_R(\mu) \in L_{\mathbf{s}+(1/p_2-1/p_1)(\mathbf{b}+\mathbf{d})}^{p_2, q_2}(D)$;
- (2) if $M_R(\mu) \in L_{\mathbf{s}, 0}^{p_1, q_1}(D)$, then $M_R(\mu) \in L_{\mathbf{s}+(1/p_2-1/p_1)(\mathbf{b}+\mathbf{d}), 0}^{p_2, q_2}(D)$;
- (3) if $q_1 < \infty$, $M_R(\mu) \in L_{\mathbf{s}}^{p_1, q_1}(D) \cap L_{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p_1, 0}^{\infty, \infty}(D)$, then $M_R(\mu) \in L_{\mathbf{s}, 0}^{p_1, q_1}(D)$.

Proof. All the assertions follow easily from Lemma 5.1 and the elementary inclusions $\ell^{p_1, q_1}(J, K) \subseteq \ell^{p_2, q_2}(J, K)$, $\ell_0^{p_1, q_1}(J, K) \subseteq \ell_0^{p_2, q_2}(J, K)$, and $\ell^{p_1, q_1}(J, K) \cap \ell_0^{\infty, \infty}(J, K) \subseteq \ell_0^{p_1, q_1}(J, K)$ (when $q_1 < \infty$) for every two sets J and K . \square

Proof of Theorem 5.5. Clearly, (1) \implies (2), while the implication (3) \implies (1) follows from Proposition 5.2. Thus, it only remains to show that (2) \implies (3).

By assumption, there is a $(\delta, 4)$ -lattice $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$ for some $\delta \leq R/4$ such that the mapping

$$\Psi: \ell_0^{p_1, q_1}(J, K) \ni \lambda \mapsto \sum_{j,k} \lambda_{j,k} B_{(\zeta_{j,k}, z_{j,k})}^{\mathbf{s}'} \Delta_{\Omega}^{(\mathbf{b}+\mathbf{d})/p_1-\mathbf{s}-\mathbf{s}'}(h_k) \in A_{\mathbf{s}, 0}^{p_1, q_1}(D)$$

is well defined and continuous, where $h_k := \rho(\zeta_{j,k}, z_{j,k})$ for every $j \in J$ and for every $k \in K$.

Now, take a probability space (X, ν) and a sequence $(r_\beta)_{\beta \in \mathbb{N}}$ of ν -measurable functions on X such that

$$\left(\bigotimes_{\beta \in B} r_\beta \right) (\nu) = \frac{1}{2^{\text{Card}(B)}} \sum_{\varepsilon \in \{-1, 1\}^B} \delta_\varepsilon$$

for every finite subset B of \mathbb{N} (cf. [21, C.1]). By Khintchine's inequality, there is a constant $C_1 > 0$ such that

$$\frac{1}{C_1} \left(\sum_{\beta \in \mathbb{N}} |a_\beta|^2 \right)^{1/2} \leq \left\| \sum_{\beta \in B} a_\beta r_\beta \right\|_{L^p(\nu)} \leq C_1 \left(\sum_{\beta \in \mathbb{N}} |a_\beta|^2 \right)^{1/2}$$

for every $(a_\beta) \in \mathbb{C}^{\mathbb{N}}$ (cf. [21, C.2]). Let $\iota: J \times K \rightarrow \mathbb{N}$ be a bijection and define $\mathbf{r} \cdot \lambda := (r_{\gamma(j,k)} \lambda_{j,k})_{j,k}$ for every $\lambda \in \mathbb{C}^{(J \times K)}$. By the assumptions and the continuity of Ψ , there is a constant $C_2 > 0$ such that, for every $\lambda \in \mathbb{C}^{(J \times K)}$,

$$\|\Psi(\mathbf{r} \cdot \lambda)\|_{L^p(\mu)} \leq C_2 \|\mathbf{r} \cdot \lambda\|_{\ell^{p_1, q_1}(J, K)} = C_2 \|\lambda\|_{\ell^{p_1, q_1}(J, K)}$$

ν -almost everywhere. Therefore, by means of Tonelli's theorem we see that

$$\left\| \left(\sum_{j,k} \left| \lambda_{j,k} B_{(\zeta_{j,k}, z_{j,k})}^{\mathbf{s}'} \right|^2 \Delta_\Omega^{2[(\mathbf{b}+\mathbf{d})/p_1 - \mathbf{s}']} (h_k) \right)^{1/2} \right\|_{L^p(\mu)} \leq C_1 C_2 \|\lambda\|_{\ell^{p_1, q_1}(J, K)}$$

for every $\lambda \in \mathbb{C}^{(J \times K)}$. Now, observe that there is $N \in \mathbb{N}$ such that $\sum_{j,k} \chi_{B((\zeta_{j,k}, z_{j,k}), R)} \leq N$ on D , thanks to [10, Proposition 2.56]. In addition, there is a constant $C_3 > 0$ such that

$$\frac{1}{C_3} \Delta_\Omega^{\mathbf{s}'} (h_k) \leq \left| B_{(\zeta_{j,k}, z_{j,k})}^{\mathbf{s}'} \right| \leq C_3 \Delta_\Omega^{\mathbf{s}'} (h_k)$$

on $B((\zeta_{j,k}, z_{j,k}), R)$ for every $(j, k) \in J \times K$, thanks to [10, Theorem 2.47]. Then,

$$\begin{aligned} & \left\| \left(\sum_{j,k} \left| \lambda_{j,k} B_{(\zeta_{j,k}, z_{j,k})}^{\mathbf{s}'} \right|^2 \Delta_\Omega^{2[(\mathbf{b}+\mathbf{d})/p_1 - \mathbf{s}']} (h_k) \right)^{1/2} \right\|_{L^p(\mu)} \\ & \geq \frac{1}{C_3} \left\| \left(\sum_{j,k} \chi_{B((\zeta_{j,k}, z_{j,k}), R)} |\lambda_{j,k}|^2 \Delta_\Omega^{2[(\mathbf{b}+\mathbf{d})/p_1 - \mathbf{s}]} (h_k) \right)^{1/2} \right\|_{L^p(\mu)} \\ & \geq \frac{N^{-(1-p/2)_+}}{C_3} \left(\sum_{j,k} |\lambda_{j,k}|^p \Delta_\Omega^{p[(\mathbf{b}+\mathbf{d})/p_1 - \mathbf{s}]} (h_k) M_R(\mu)(\zeta_{j,k}, z_{j,k}) \right)^{1/p} \end{aligned}$$

for every $\lambda \in \mathbb{C}^{(J \times K)}$. Using the natural duality between $\ell_0^{p_1/p, q_1/p}(J, K)$ and $\ell^{p^*, q^*}(J, K)$, we then see that

$$\left(\Delta_\Omega^{p[(\mathbf{b}+\mathbf{d})/p_1 - \mathbf{s}]} (h_k) M_R(\mu)(\zeta_{j,k}, z_{j,k}) \right) \in \ell^{p^*, q^*}(J, K).$$

Then, (3) follows from Lemma 5.1.

Next, we turn to the second part of the statement. The fact that (3') \implies (1') follows from Proposition 5.2, while it is obvious that (1') \implies (2'). Finally, Proposition 5.3, Lemma 5.6, and the implication (2) \implies (3) prove that (2') \implies (3'). \square

With the same techniques, but using [10, Corollary 5.14] instead of property $(L)_{\mathbf{s}, \mathbf{s}', 0}^{p_1, q_1}$, one also proves the following result. We recall that the spaces $\tilde{A}_{\mathbf{s}}^{p, q}(D)$ and $\tilde{A}_{\mathbf{s}, 0}^{p, q}(D)$ are the holomorphic extensions of the analytic-type Besov spaces $B_{p, q}^{-\mathbf{s}}(\mathcal{N}, \Omega)$ and $\mathring{B}_{p, q}^{-\mathbf{s}}(\mathcal{N}, \Omega)$, respectively, see Definition 4.9.

Proposition 5.7. *Take $p_1, q_1, p \in]0, \infty]$, with $p < \infty$, and take $\mathbf{s} \in \frac{1}{p_1}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q_1}\mathbf{m}' + (\mathbb{R}_+^*)^r$. Take $\mu \in \mathcal{M}_+(D)$ so that $\tilde{A}_{\mathbf{s}, 0}^{p_1, q_1}(D)$ embeds continuously into $L^p(\mu)$, and take p^*, q^*, \mathbf{s}^* as in (5.2). Then,*

$$M_R(\mu) \in L_{\mathbf{s}^*}^{p^*, q^*}(D)$$

for every $R > 0$.

Notice, though, that an analogue of Proposition 5.2 for the spaces $\tilde{A}_{\mathbf{s}, 0}^{p_1, q_1}(D)$ is false, in general, as the following Proposition shows. We denote by $\mathcal{L}\nu$ the Laplace transform of $\nu \in \mathcal{M}_+(\Omega)$, that is,

$$(\mathcal{L}\nu)(\lambda) = \int_{\Omega} e^{-\langle \lambda, h \rangle} d\nu(h)$$

for every $\lambda \in F'$.

Proposition 5.8. *Take $\mathbf{s} \in \frac{\mathbf{b} + \mathbf{d}}{2} + \frac{1}{4}\mathbf{m}' + (\mathbb{R}_+^*)^r$ and a bD -invariant $\mu \in \mathcal{M}_+(D)$, and let ν be the unique Radon measure ν on Ω such that $\int_D f d\mu = \int_{\Omega} \int_{\mathcal{N}} f_h(\zeta, x) d(\zeta, x) d\nu(h)$ for every $f \in C_c(D)$. Take $\mathbf{s}' \in -\frac{1}{4}\mathbf{m}' - \mathbf{s} - (\mathbb{R}_+^*)^r$. Then, the following conditions are equivalent:*

- (1) $\tilde{A}_{\mathbf{s}}^{2, 2}(D)$ embeds continuously (resp. compactly) into $L^2(\mu)$;
- (2) the function $\Delta_{\Omega'}^{2\mathbf{s}} \mathcal{L}\nu$ is bounded (resp. is bounded and vanishes at ∞) on Ω' ;
- (3) the function $h \mapsto \Delta_{\Omega}^{-\mathbf{s} - \mathbf{s}'}(h) \|\Delta^{\mathbf{s}'}(\cdot + h)\|_{L^2(\nu)}$ is bounded (resp. is bounded and vanishes at ∞) on Ω .

This extends [8, Theorem 3.1], where the case in which $\mathbf{s} = 0$, D is an irreducible symmetric tube domain, and ν is absolutely continuous with respect to Lebesgue measure, is considered.

Notice that this shows that the necessary conditions of Proposition 5.7 are not sufficient, in general. For example, when $\mathbf{s} = 0$, then a bD -invariant positive Radon measure μ on D induces a continuous embedding of the Hardy space $\tilde{A}_0^{2, 2}(D) = A_0^{2, \infty}(D)$ into $L^2(\mu)$ if and only if the measure ν defined as in the statement is integrable on Ω , while Proposition 5.7 only requires the mapping $h \mapsto \nu(B_{\Omega}(h, R))$ to be bounded for some (every) $R > 0$.

Proof. (2) \implies (1). Take $f \in \tilde{A}_{\mathbf{s}}^{2, 2}(D) \cap A_{-\mathbf{d}/2}^{2, 2}(D)$, and observe that there is a constant $c > 0$ such that, defining $\tau_f(\lambda) := \pi_{\lambda}(f_{e_{\Omega}})e^{\langle \lambda, e_{\Omega} \rangle}$ for almost every $\lambda \in \Omega'$,

$$\|f\|_{L^2(\mu)}^2 = c \int_{\Omega'} \|\tau_f(\lambda)\|_{\mathcal{L}^2(H_{\lambda})}^2 (\mathcal{L}\nu)(2\lambda) \Delta_{\Omega'}^{-\mathbf{b}}(\lambda) d\lambda,$$

arguing as in the proof of [10, Corollary 1.38]. In addition, we may choose a norm on $\tilde{A}_{\mathbf{s}}^{2, 2}(D)$ so that

$$\|f\|_{\tilde{A}_{\mathbf{s}}^{2, 2}(D)}^2 = \int_{\Omega'} \|\tau_f(\lambda)\|_{\mathcal{L}^2(H_{\lambda})}^2 \Delta_{\Omega'}^{-2\mathbf{s} - \mathbf{b}}(\lambda) d\lambda$$

thanks to [10, Lemma 2.26, Proposition 3.11 and Proposition 5.13]. Now, define

$$\mathcal{L}_{\mathbf{s}}^2(\Omega') := \left\{ \tau \in \int_{\Omega'}^{\oplus} \mathcal{L}^2(H_{\lambda}) \Delta_{\Omega'}^{-\mathbf{b} - 2\mathbf{s}}(\lambda) d\lambda : \tau = \tau P_{\cdot, 0} \right\}$$

(cf. [10, Definition 3.10]), and observe that the image of the mapping τ_f , extended to $\widetilde{A}_s^{2,2}(D)$ by continuity, is the whole of $\mathcal{L}_s^2(\Omega')$ (cf. [10, Propositions 3.15, 3.17, 5.4, 5.13, and Corollary 5.11]). Therefore, $\widetilde{A}_s^{2,2}(D)$ embeds into $L^2(\mu)$ if and only if there is a constant $C_1 > 0$ such that

$$\mathcal{L}\nu(2\lambda) \leq C_1 \Delta_{\Omega'}^{-2s}(\lambda)$$

for every $\lambda \in \Omega'$.

Now, if $\Delta_{\Omega'}^{2s} \mathcal{L}\nu$ is bounded and vanishes at infinity, then it is clear that the mapping $\tau \mapsto \Delta_{\Omega'}^{2s} \mathcal{L}\nu \tau$ is a compact endomorphism of $\mathcal{L}_s^2(\Omega')$, so that the preceding remarks show that $\widetilde{A}_s^{2,2}(D)$ embeds compactly into $L^2(\mu)$.

(1) \implies (3). Observe first that $\|B_{(\zeta,z)}^{s'+(\mathbf{b}+\mathbf{d})/2}\|_{\widetilde{A}_s^{2,2}(D)} = \Delta_{\Omega}^{s+s'}(\rho(\zeta, z))$ for every $(\zeta, z) \in D$, for a suitable choice of a norm on $\widetilde{A}_s^{2,2}(D)$ (cf. [10, Proposition 2.41 and Lemma 5.15]). In addition, there is a constant $C_2 > 0$ such that

$$\left\| \left(B_{(0,ih)}^{s'+(\mathbf{b}+\mathbf{d})/2} \right)_{h'} \right\|_{L^2(\mathcal{N})} = C_2 \Delta_{\Omega}^{s'}(h + h')$$

for every $h, h' \in \Omega$, thanks to [10, Lemma 2.39]. Therefore, (3) follows arguing as in the proof of Proposition 5.3.

(3) \implies (2). Observe that there is a constant $C_3 > 0$ such that

$$\|\Delta^{s'}(\cdot + h)\|_{L^2(\nu_t)} \leq C_3 \Delta_{\Omega}^{s+s'}(h) \quad \forall t \in T_+ \quad (\text{resp. and } \lim_{t \rightarrow \infty} \|\Delta^{s'}(\cdot + h)\|_{L^2(\nu_t)} = 0)$$

for every $h \in \Omega$, where $\nu_t = \Delta^{2s}(t)(t \cdot)_* \nu$, and that (2) is equivalent to saying that

$$\sup_{t \in t_+} \mathcal{L}\nu_t(e'_\Omega) < \infty \quad (\text{resp. } \lim_{t \rightarrow \infty} \mathcal{L}\nu_t(e'_\Omega) = 0).$$

Now, [10, Corollary 2.36] implies that

$$\Delta^{2s'}(h + h') \geq \Delta^{2s'}(2h) = 2^{2s'} \Delta^{2s'}(h)$$

for every $h \in \Omega$ and for every $h' \in \Omega \cap (h - \Omega)$, so that

$$\nu_t(\Omega \cap (h - \Omega)) \leq 2^{-2s'} \Delta^{-2s'}(h) \int_{\Omega \cap (h - \Omega)} \Delta^{2s'}(h + h') d\nu_t(h') \leq 2^{-2s'} C_3^2 \Delta^{2s}(h)$$

for every $h \in \Omega$ and for every $t \in T_+$. In addition, fix $R > 0$ so that $\langle e_{\Omega'}, h \rangle \geq 1$ for every $h \in \Omega \setminus (Re_\Omega - \Omega)$. Then,

$$\begin{aligned} \mathcal{L}\nu_t(e_{\Omega'}) &= \int_{\Omega} e^{-\langle e_{\Omega'}, h \rangle} d\nu_t(h) \\ &\leq \nu_t(\Omega \cap (Re_\Omega - \Omega)) + \sum_{k \in \mathbb{N}} e^{-2^k R} \nu_t(\Omega \cap (2^{k+1} Re_\Omega - \Omega) \setminus (2^k Re_\Omega - \Omega)) \\ &\leq 2^{-2s'} C_3^2 \left(R^{2s} + \sum_{k \in \mathbb{N}} (2^{k+1} R)^{2s} e^{-2^k R} \right) \end{aligned}$$

for every $t \in T_+$. If, in addition, $\lim_{t \rightarrow \infty} \|\Delta^{s'}(\cdot + h)\|_{L^2(\nu_t)} = 0$ for every $h \in \Omega$, then the preceding computations show that $\nu_t(\Omega \cap (2^{k+1} Re_\Omega - \Omega) \setminus (2^k Re_\Omega - \Omega)) \rightarrow 0$ for $t \rightarrow +\infty$ for every $k \in \mathbb{N}$, so that it is readily seen that $\mathcal{L}\nu_t(e_{\Omega'}) \rightarrow 0$ for $t \rightarrow \infty$. \square

We now come to the last main result of this section. It provides a necessary and sufficient condition for a measure $\mu \in \mathcal{M}_+(D)$ to be a p -Carleson measure for $A_{\mathbf{s}}^{p_1, q_1}$ also in terms of integrability properties of Bergman kernels.

Theorem 5.9. *Take $p_1, q_1, p \in]0, \infty]$, with $p < \infty$, $\mu \in \mathcal{M}_+(D)$, $\mathbf{s} \in \mathbb{R}^r$, and $\mathbf{s}' \in \mathbb{C}^r$ such that the following hold:*

- (i) $\mathbf{s} \in \frac{1}{2q_1}\mathbf{m} + (\mathbb{R}_+^*)^r$ (resp. $\mathbf{s} \in \mathbb{R}_+^r$ if $q_1 = \infty$);
- (ii) $\operatorname{Re} \mathbf{s}' \in \frac{1}{p_1}(\mathbf{b} + \mathbf{d}) - \frac{1}{2p_1}\mathbf{m}' - (\mathbb{R}_+^*)^r$ (resp. $\operatorname{Re} \mathbf{s}' \in -\mathbb{R}_+^r$ if $p_1 = \infty$);
- (iii) $\mathbf{s} + \operatorname{Re} \mathbf{s}' \in \frac{1}{p_1}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q_1}\mathbf{m}' - (\mathbb{R}_+^*)^r$ (resp. $\mathbf{s} + \operatorname{Re} \mathbf{s}' \in \frac{1}{p_1}(\mathbf{b} + \mathbf{d}) - \mathbb{R}_+^r$ if $q_1 = \infty$);
- (iv) $A_{\mathbf{s}, 0}^{p_1, q_1}(D)$ (resp. $A_{\mathbf{s}}^{p_1, q_1}(D)$) embeds continuously into $A_{\mathbf{s}''}^{p, p}(D)$, where $\mathbf{s}'' := \mathbf{s} + \left(\frac{1}{p} - \frac{1}{p_1}\right)(\mathbf{b} + \mathbf{d})$.

Then, the following conditions are equivalent for every $R > 0$:

- (1) $A_{\mathbf{s}, 0}^{p_1, q_1}(D)$ (resp. $A_{\mathbf{s}}^{p_1, q_1}(D)$) embeds continuously into $L^p(\mu)$;
- (2) there is a constant $C > 0$ such that

$$\|B_{(\zeta, z)}^{\mathbf{s}'}\|_{L^p(\mu)} \leq C \Delta^{\mathbf{s} + \operatorname{Re} \mathbf{s}' - (\mathbf{b} + \mathbf{d})/p_1}(\rho(\zeta, z))$$

for every $(\zeta, z) \in D$;

- (3) $M_R(\mu) \in L_{p[(\mathbf{b} + \mathbf{d})/p_1 - \mathbf{s}]}^{\infty, \infty}(D)$.

If, in addition, $\mathbf{s}, -\mathbf{s} - \operatorname{Re} \mathbf{s}' \in (\mathbb{R}_+^*)^r$ when $p_1 = q_1 = \infty$, then the following conditions are equivalent for every $R > 0$:

- (1') $A_{\mathbf{s}, 0}^{p_1, q_1}(D)$ (resp. $A_{\mathbf{s}}^{p_1, q_1}(D)$) embeds compactly into $L^p(\mu)$;
- (2') the function $(\zeta, z) \mapsto \Delta^{(\mathbf{b} + \mathbf{d})/p_1 - \mathbf{s} - \operatorname{Re} \mathbf{s}'}(\rho(\zeta, z)) \|B_{(\zeta, z)}^{\mathbf{s}'}\|_{L^p(\mu)}$ is bounded and vanishes at ∞ on D ;
- (3') $M_R(\mu) \in L_{p[(\mathbf{b} + \mathbf{d})/p_1 - \mathbf{s}], 0}^{\infty, \infty}(D)$.

This extends [9, Theorem 1.1], where the case in which $\mathbf{s} = 0$, $\mathbf{s}' \in \mathbb{R}\mathbf{1}_r$, $q = \infty$, and D is an irreducible symmetric tube domain, is considered.

Notice that, if (i) to (iv) hold, then (i), (ii), and (iii) hold with \mathbf{s} , p_1 , and q_1 replaced by \mathbf{s}'' , p , and p , respectively, thanks to [10, Proposition 2.41].

Conversely, if (2) implies (1) and (i), (ii), and (iii) hold with \mathbf{s} , p_1 , and q_1 replaced by \mathbf{s}'' , p , and p , respectively, then (iv) holds thanks to [10, Proposition 2.41] again.

Observe that $A_{\mathbf{0}}^{2, \infty}(D) = \tilde{A}_{\mathbf{0}}^{2, 2}(D) \subseteq \tilde{A}_{(1/p - 1/2)(\mathbf{b} + \mathbf{d})}^{p, p}(D)$ for every $p \geq 2$, thanks to [10, Proposition 4.19], so that the assumptions are satisfied whenever $\mathbf{s}'' = (1/p - 1/2)(\mathbf{b} + \mathbf{d})$ and $\tilde{A}_{\mathbf{s}''}^{p, p}(D) = A_{\mathbf{s}''}^{p, p}(D)$. This is the case, for example, when (cf. [10, Corollary 5.11])

$$\left(\frac{1}{p} - \frac{1}{2}\right)(\mathbf{b} + \mathbf{d}) \in \sup\left(\frac{1}{2p}\mathbf{m} + \left(\frac{1}{2} - \frac{1}{p}\right)\mathbf{m}', \frac{1}{p}(\mathbf{b} + \mathbf{d}) + \frac{1}{2p'}\mathbf{m}'\right) + (\mathbb{R}_+^*)^r.$$

Proof. We prove only the equivalence of conditions (1)–(3). The equivalence of conditions (1')–(3') is proved similarly, using the techniques employed in the proof of Proposition 5.3.

(1) \implies (2). This follows from [10, Proposition 2.41].

(2) \implies (3). This follows easily from [10, Theorem 2.47].

(3) \implies (1). By Theorem 5.4, $A_{\mathbf{s}}^{p, p}(D)$ embeds continuously into $L^p(\mu)$, so that the assertion follows. \square

With a similar proof, but using [10, Lemma 5.15] instead of [10, Proposition 2.41], one may prove the following result, which deals with p -Carleson measures for the spaces $\tilde{A}_s^{p_1, q_1}(D)$.

Theorem 5.10. *Take $p_1, q_1, p \in]0, \infty[$, $\mu \in \mathcal{M}_+(D)$, $\mathbf{s} \in \mathbb{R}^r$, and $\mathbf{s}' \in \mathbb{C}^r$ such that the following hold:*

- (i) $\mathbf{s} \in \frac{1}{p_1}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q_1}\mathbf{m}' + (\mathbb{R}_+^*)^r$;
- (ii) $\mathbf{s} + \operatorname{Re} \mathbf{s}' \in \frac{1}{p_1}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q_1}\mathbf{m}' - (\mathbb{R}_+^*)^r$ (resp. $\mathbf{s} + \operatorname{Re} \mathbf{s}' \in \frac{1}{p_1}(\mathbf{b} + \mathbf{d}) - \mathbb{R}_+^r$ if $q_1 = \infty$);
- (iii) $\tilde{A}_{\mathbf{s}, 0}^{p_1, q_1}(D)$ (resp. $\tilde{A}_{\mathbf{s}}^{p_1, q_1}(D)$) embeds continuously into $A_{\mathbf{s}''}^{p, p}(D)$, where $\mathbf{s}'' := \mathbf{s} + \left(\frac{1}{p} - \frac{1}{p_1}\right)(\mathbf{b} + \mathbf{d})$.

Then, the following conditions are equivalent for every $R > 0$:

- (1) $\tilde{A}_{\mathbf{s}, 0}^{p_1, q_1}(D)$ (resp. $\tilde{A}_{\mathbf{s}}^{p_1, q_1}(D)$) embeds continuously into $L^p(\mu)$;
- (2) there is a constant $C > 0$ such that

$$\|B_{(\zeta, z)}^{\mathbf{s}'}\|_{L^p(\mu)} \leq C \Delta^{\mathbf{s} + \operatorname{Re} \mathbf{s}' - (\mathbf{b} + \mathbf{d})/p_1}(\rho(\zeta, z))$$

for every $(\zeta, z) \in D$;

- (3) $M_R(\mu) \in L_{p[(\mathbf{b} + \mathbf{d})/p_1 - \mathbf{s}]}^{\infty, \infty}(D)$.

If, in addition, $\mathbf{s} + \operatorname{Re} \mathbf{s}' \in -(\mathbb{R}_+^*)^r$ when $p_1 = q_1 = \infty$, then the following conditions are equivalent for every $R > 0$:

- (1') $\tilde{A}_{\mathbf{s}, 0}^{p_1, q_1}(D)$ (resp. $\tilde{A}_{\mathbf{s}}^{p_1, q_1}(D)$) embeds compactly into $L^p(\mu)$;
- (2') the function $(\zeta, z) \mapsto \Delta^{(\mathbf{b} + \mathbf{d})/p_1 - \mathbf{s} - \operatorname{Re} \mathbf{s}'}(\rho(\zeta, z)) \|B_{(\zeta, z)}^{\mathbf{s}'}\|_{L^p(\mu)}$ is bounded and vanishes at ∞ on D ;
- (3') $M_R(\mu) \in L_{p[(\mathbf{b} + \mathbf{d})/p_1 - \mathbf{s}], 0}^{\infty, \infty}(D)$.

Notice that, by Corollary 6.3 below, if (iii) holds, then $p_1, q_1 \leq p$.

6. INCLUSION BETWEEN THE SPACES $\tilde{A}_s^{p, q}(D)$ AND $A_s^{p, q}(D)$

A classical result by Hardy and Littlewood in dimension 1 shows that the Hardy space $H^2(\mathbb{C}_+)$ embeds continuously into $A_{\frac{1}{2} - \frac{1}{p}}^{p, p}(\mathbb{C}_+)$ when $p \geq 4$. In this section we address a similar question. The main result is Theorem 6.4, in which we restrict ourselves to the setting of Siegel domains of Type I. In addition to that, we also prove some necessary conditions for the spaces A and \tilde{A} to embed into one another. For sufficient condition, see [10, Propositions 3.2, 3.7, and 4.19].

Lemma 6.1. *Take $p_1, p_2, q_1, q_2 \in]0, \infty[$ and $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^r$, and assume that $\mathbf{s}_1 \in \frac{1}{2q_1}\mathbf{m} + (\mathbb{R}_+^*)^r$ (resp. $\mathbf{s}_1 \in \mathbb{R}_+^r$ if $q_1 = \infty$) and that $A_{\mathbf{s}_1, 0}^{p_1, q_1}(D) \subseteq A_{\mathbf{s}_2}^{p_2, q_2}(D)$ (resp. $A_{\mathbf{s}_1}^{p_1, q_1}(D) \subseteq A_{\mathbf{s}_2}^{p_2, q_2}(D)$). Then,*

$$p_1 \leq p_2 \quad \text{and} \quad \mathbf{s}_2 = \mathbf{s}_1 + \left(\frac{1}{p_2} - \frac{1}{p_1}\right)(\mathbf{b} + \mathbf{d}).$$

As Theorem 6.4 below shows, it is not possible to deduce $q_1 \leq q_2$ in this generality. Cf. Corollary 6.3 below, though.

Proof. By assumption, there is a constant $C > 0$ such that

$$\|f\|_{A_{\mathbf{s}_2}^{p_2, q_2}(D)} \leq C \|f\|_{A_{\mathbf{s}_1}^{p_1, q_1}(D)}$$

for every $f \in A_{\mathbf{s}_1,0}^{p_1,q_1}(D)$ (resp. for every $f \in A_{\mathbf{s}_1}^{p_1,q_1}(D)$). We assume that C is the least possible constant for which such an inequality holds. Now, for every $t \in T_+$ choose $g_t \in GL(E)$ so that $t \cdot \Phi = \Phi \circ (g_t \times g_t)$. Then,

$$\|f \circ (g_t \times t)\|_{A_{\mathbf{s}_1}^{p_1,q_1}(D)} = \Delta^{-\mathbf{s}_1 + (\mathbf{b} + \mathbf{d})/p_1}(t) \|f\|_{A_{\mathbf{s}_1}^{p_1,q_1}(D)}$$

and

$$\|f \circ (g_t \times t)\|_{A_{\mathbf{s}_2}^{p_2,q_2}(D)} = \Delta^{-\mathbf{s}_2 + (\mathbf{b} + \mathbf{d})/p_2}(t) \|f\|_{A_{\mathbf{s}_2}^{p_2,q_2}(D)}$$

for every $f \in \text{Hol}(D)$, thanks to Lemma 4.12. Therefore, the arbitrariness of $t \in T_+$ and the minimality of C imply that

$$\mathbf{s}_2 = \mathbf{s}_1 + (1/p_2 - 1/p_1)(\mathbf{b} + \mathbf{d}).$$

Next, assume that $A_{\mathbf{s}_1,0}^{p_1,q_1}(D) \neq 0$, that is, that $\mathbf{s}_1 \in \frac{1}{2q_1}\mathbf{m} + (\mathbb{R}_+^*)^r$. Then,

$$\begin{aligned} 2^{1/p_2} \|f\|_{A_{\mathbf{s}_2}^{p_2,q_2}(D)} &= \lim_{(\zeta,x) \rightarrow \infty} \|f - f((\zeta, x + i\Phi(\zeta)) \cdot)\|_{A_{\mathbf{s}_2}^{p_2,q_2}(D)} \\ &\leq C \lim_{(\zeta,x) \rightarrow \infty} \|f - f((\zeta, x + i\Phi(\zeta)) \cdot)\|_{A_{\mathbf{s}_1}^{p_1,q_1}(D)} \\ &= 2^{1/p_1} C \|f\|_{A_{\mathbf{s}_1}^{p_1,q_1}(D)} \end{aligned}$$

for every $f \in A_{\mathbf{s}_1,0}^{p_1,q_1}(D)$, so that $1/p_1 \geq 1/p_2$, that is, $p_1 \leq p_2$, by the minimality of C .

Finally, assume that $A_{\mathbf{s}_1,0}^{p_1,q_1}(D) = 0$, so that $q_1 = \infty$ and $(s_1)_j = 0$ for some $j \in \{1, \dots, r\}$. Then, for every $\varepsilon > 0$ we may find $\mathbf{s}' \in \mathbb{R}^r$ such that $s'_j = \frac{1}{p_1}(b_j + d_j) - \frac{1}{2p_1}m'_j - \varepsilon$ and such that $B_{(\zeta,z)}^{\mathbf{s}'} \in A_{\mathbf{s}_1}^{p_1,q_1}(D)$ for every $(\zeta, z) \in D$ (cf. [10, Proposition 2.41]), so that $B_{(\zeta,z)}^{\mathbf{s}'} \in A_{\mathbf{s}_2}^{p_2,q_2}(D)$. Then, [10, Proposition 2.41] implies that $s'_j \leq \frac{1}{p_2}(b_j + d_j) - \frac{1}{2p_2}m'_j$. Since $-b_j, m'_j \geq 0$ and $-d_j > 0$, the arbitrariness of ε implies that $1/p_1 \geq 1/p_2$, that is, $p_1 \leq p_2$. \square

Lemma 6.2. *Take $p_1, p_2, q_1, q_2 \in]0, \infty]$ and $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^r$, and assume that the canonical mapping $\mathcal{S}_{\Omega,L}(\mathcal{N}) \rightarrow B_{p_2,q_2}^{\mathbf{s}_2}(\mathcal{N}, \Omega)$ induces a continuous linear mapping $\hat{B}_{p_1,q_1}^{\mathbf{s}_1}(\mathcal{N}, \Omega) \rightarrow B_{p_2,q_2}^{\mathbf{s}_2}(\mathcal{N}, \Omega)$. Then, $p_1 \leq p_2$, $q_1 \leq q_2$, and $\mathbf{s}_2 = \mathbf{s}_1 + \left(\frac{1}{p_1} - \frac{1}{p_2}\right)(\mathbf{b} + \mathbf{d})$.*

Proof. Applying the operator $u \mapsto u * I_{\Omega}^{\mathbf{s}'}$ for a suitable $\mathbf{s}' \in \mathbb{R}^r$, we may assume that $\mathbf{s}_1, \mathbf{s}_2$ are sufficiently small so as to ensure that the mapping \mathcal{E} is defined and induces isomorphisms of $\hat{B}_{p_1,q_1}^{\mathbf{s}_1}(\mathcal{N}, \Omega)$ and $B_{p_2,q_2}^{\mathbf{s}_2}(\mathcal{N}, \Omega)$ onto $A_{-\mathbf{s}_1,0}^{p_1,q_1}(D)$ and $A_{-\mathbf{s}_2}^{p_2,q_2}(D)$, respectively (cf. [10, Theorem 4.26 and Corollary 5.11]). Then, Lemma 6.1 implies that $p_1 \leq p_2$ and that $\mathbf{s}_2 = \mathbf{s}_1 + \left(\frac{1}{p_1} - \frac{1}{p_2}\right)(\mathbf{b} + \mathbf{d})$. Therefore, it only remains to prove that $q_1 \leq q_2$. Then, take $\mathbf{s}' \in \mathbb{R}^r$ and observe that [10, Lemma 5.15] implies that $B_{(\zeta,z)}^{\mathbf{s}'} \in A_{\mathbf{s}_1,0}^{p_1,q_1}(D)$ for some (or, equivalently, every) $(\zeta, z) \in D$ if and only if $\mathbf{s}_1 + \mathbf{s}' \in \frac{1}{p_1}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q_1}\mathbf{m}' - (\mathbb{R}_+^*)^r$, and that $B_{(\zeta,z)}^{\mathbf{s}'} \in A_{\mathbf{s}_2}^{p_2,q_2}(D)$ for some (or, equivalently, every) $(\zeta, z) \in D$ if and only if $\mathbf{s}_2 + \mathbf{s}' \in \frac{1}{p_2}(\mathbf{b} + \mathbf{d}) - \frac{1}{2q_2}\mathbf{m}' - (\mathbb{R}_+^*)^r$ and $q_2 < \infty$ or $\mathbf{s}_2 + \mathbf{s}' \in \frac{1}{p_2}(\mathbf{b} + \mathbf{d}) - \mathbb{R}_+^r$ and $q_2 = \infty$. Therefore, $q_1 \leq q_2$ provided that $\mathbf{m}' \neq \mathbf{0}$.

Thus, we only need to consider the case $\mathbf{m}' = \mathbf{0}$, that is, $r = m$. In this case, we may assume that $\Omega = \Omega' = (\mathbb{R}_+^*)^r$, identifying \mathbb{R}^r with its dual. Define $K = \mathbb{Z}^r$ and $t_k := \lambda_k := 2^k = (2^{k_1}, \dots, 2^{k_r})$ for every $k \in K$. Notice that the distance $(\lambda, \lambda') \mapsto \sum_{j=1}^r |\log(\lambda_j/\lambda'_j)|$ on Ω' is $G(\Omega')$ -invariant and clearly locally bi-Lipschitz equivalent to $d_{\Omega'}$, hence bi-Lipschitz equivalent to $d_{\Omega'}$ near every point, with uniform constants. Therefore, it is readily verified that (λ_k) is a (δ, R) -lattice for some $\delta > 0$ and some $R > 1$. Notice that Ω is a group under pointwise multiplication, so that we may choose

$T_+ = \Omega$ with the natural action. In this case, $\Delta^{\mathbf{s}}(t) = t^{\mathbf{s}} = \prod_{j=1}^r t_j^{s_j}$ for every $\mathbf{s} \in \mathbb{C}^r$. Now, choose a positive $\varphi \in C_c^\infty(\Omega')$ so that $\sum_{k \in K} \varphi(\cdot t_k) = 1$ on Ω' , and define $\psi_k := \mathcal{F}_{\mathcal{N}}^{-1}(\varphi(\cdot t_k))$ for every $k \in K$. Then,

$$u \mapsto \|u\|_{B_{p,q}^{\mathbf{s}}(\mathcal{N},\Omega)} := \left\| \Delta_{\Omega'}^{\mathbf{s}}(\lambda_k) \|u * \psi_k\|_{L^p(\mathcal{N})} \right\|_{\ell^q(K)}$$

is a quasi-norm which defines the topology of $B_{p,q}^{\mathbf{s}}(\mathcal{N}, \Omega)$ for every $p, q \in]0, \infty]$ and for every $\mathbf{s} \in \mathbb{R}^r$. Define $\Psi_t(u) := \Delta^{\mathbf{s}_1 - (\mathbf{b} + \mathbf{d})(1 - 1/p_1)}(t \cdot)_* u$ for every $u \in \mathcal{S}'_{\Omega, L}(\mathcal{N})$ and for every $t \in T_+$. Then, clearly

$$\|\Psi_{t_k}(u)\|_{B_{p_j, q_j}^{\mathbf{s}_j}(\mathcal{N}, \Omega)} = \|u\|_{B_{p_j, q_j}^{\mathbf{s}_j}(\mathcal{N}, \Omega)}$$

for every $j = 1, 2$, for every $k \in K$, and for every $u \in B_{p_j, q_j}^{\mathbf{s}_j}(\mathcal{N}, \Omega)$. In addition, it is readily seen that

$$\lim_{k \rightarrow \infty} \|f - \Psi_{t_k}(f)\|_{B_{p_j, q_j}^{\mathbf{s}_j}(\mathcal{N}, \Omega)} = 2^{1/q_j} \|f\|_{B_{p_j, q_j}^{\mathbf{s}_j}(\mathcal{N}, \Omega)}$$

for every $j = 1, 2$ and for every $f \in \mathcal{S}_{\Omega, L}(\mathcal{N})$. Now, by assumption there is a constant $C > 0$ such that

$$\|f\|_{B_{p_2, q_2}^{\mathbf{s}_2}(\mathcal{N}, \Omega)} \leq C \|f\|_{B_{p_1, q_1}^{\mathbf{s}_1}(\mathcal{N}, \Omega)}$$

for every $f \in \mathcal{S}_{\Omega, L}(\mathcal{N})$. Then, applying the same inequality to $f - \Psi_{t_k}(f)$ and passing to the limit for $k \rightarrow \infty$,

$$\|f\|_{B_{p_2, q_2}^{\mathbf{s}_2}(\mathcal{N}, \Omega)} \leq 2^{1/q_1 - 1/q_2} C \|f\|_{B_{p_1, q_1}^{\mathbf{s}_1}(\mathcal{N}, \Omega)}$$

for every $f \in \mathcal{S}_{\Omega, L}(\mathcal{N})$. Since we may have assumed C to be minimal, it is readily seen that $q_1 \leq q_2$. \square

Corollary 6.3. *Take $p_1, p_2, q_1, q_2 \in]0, \infty]$ and $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^r$ such that $\mathbf{s}_1 \in \frac{1}{p_1}(\mathbf{b} + \mathbf{d}) + \frac{1}{2q_1} \mathbf{m}' + (\mathbb{R}_+^*)^r$ and such that $\tilde{A}_{\mathbf{s}_1, 0}^{p_1, q_1}(D) \subseteq A_{\mathbf{s}_2}^{p_2, q_2}(D)$. Then, $p_1 \leq p_2$, $q_1 \leq q_2$, and $\mathbf{s}_2 = \mathbf{s}_1 + \left(\frac{1}{p_1} - \frac{1}{p_2}\right)(\mathbf{b} + \mathbf{d})$.*

Notice that, if $p_1 \leq p_2$, $q_1 \leq q_2$, and $\mathbf{s}_2 = \mathbf{s}_1 + \left(\frac{1}{p_1} - \frac{1}{p_2}\right)(\mathbf{b} + \mathbf{d})$, then $A_{\mathbf{s}_1}^{p_1, q_1} \subseteq A_{\mathbf{s}_2}^{p_2, q_2}$ and $\tilde{A}_{\mathbf{s}_1}^{p_1, q_1} \subseteq \tilde{A}_{\mathbf{s}_2}^{p_2, q_2}$ (cf. [10, Propositions 3.2 and 4.19]). Nonetheless, $\tilde{A}_{\mathbf{s}_1, 0}^{p_1, q_1}$ need not embed into $A_{\mathbf{s}_2}^{p_2, q_2}$.

Proof. Take $\mathbf{s}' \in \mathbb{N}_{\Omega'}$ so that $A_{\mathbf{s}_2 + \mathbf{s}'}^{p_2, q_2}(D) = \tilde{A}_{\mathbf{s}_2 + \mathbf{s}'}^{p_2, q_2}(D)$ (cf. [10, Corollary 5.11]). Observe that the mapping $f \mapsto f * I_{\Omega'}^{-\mathbf{s}'}$ induces an isomorphism of $\tilde{A}_{\mathbf{s}_1, 0}^{p_1, q_1}(D)$ onto $\tilde{A}_{\mathbf{s}_1 + \mathbf{s}', 0}^{p_1, q_1}(D)$ by [10, Proposition 5.13], and a continuous mapping of $A_{\mathbf{s}_2}^{p_2, q_2}(D)$ into $A_{\mathbf{s}_2 + \mathbf{s}'}^{p_2, q_2}(D)$ by [10, Corollary 3.27]. Thus, $\tilde{A}_{\mathbf{s}_1 + \mathbf{s}', 0}^{p_1, q_1}(D) \subseteq \tilde{A}_{\mathbf{s}_2 + \mathbf{s}'}^{p_2, q_2}(D)$, so that the conclusion follows from Lemma 6.2. \square

Theorem 6.4. *Assume that $n = 0$, and take $k \in \mathbb{N}^*$ and*

$$\mathbf{s} \in \sup \left(\frac{1}{2} \mathbf{d} + \frac{1}{4} \mathbf{m}', \frac{k-1}{2k} \mathbf{d} + \frac{1}{4k} \mathbf{m}', \frac{k-1}{2k} \mathbf{d} + \frac{1}{4k} \mathbf{m} \right) + (\mathbb{R}_+^*)^r.$$

Then, $\tilde{A}_{\mathbf{s}}^{2, 2}(D)$ embeds continuously into $A_{\mathbf{s} - (1/2 - 1/q)\mathbf{d}}^{q, q}(D)$ for every $q \in [2k, \infty]$.

This extends [9, Theorem 1.3], where the case in which $\mathbf{s} = \mathbf{0}$ and D is an irreducible symmetric tube domain is considered. Notice that [9, Theorem 1.4] provides better results when $\mathbf{s} = \mathbf{0}$, $r = 2$, and $m = 3, 4, 5, 6$.

Notice, in addition, (cf. [9, Remark 1.5]) that if $A_{\mathbf{s}_1}^{p_1, q_1}(D)$ embeds continuously into $A_{\mathbf{s}_2}^{p_2, q_2}(D)$, then also $A_{\mathbf{s}_1/k}^{kp_1, kq_1}(D)$ embeds continuously into $A_{\mathbf{s}_2/k}^{kp_2, kq_2}(D)$ for every $k \in \mathbb{N}^*$. Indeed, $f \in$

$A_{\mathbf{s}_1/k}^{kp_1, kq_1}(D)$ if and only if $f^k \in A_{\mathbf{s}_1}^{p_1, q_1}(D)$, so that $f^k \in A_{\mathbf{s}_2}^{p_2, q_2}(D)$, which is equivalent to $f \in A_{\mathbf{s}_2/k}^{kp_2, kq_2}(D)$. Unfortunately, it is not known if the weighted Bergman spaces interpolate in full generality, so that this fact cannot be used to deduce similar embedding for every $k \in [1, \infty[$.

Proof. Define $\tau: \tilde{A}_{\mathbf{s}}^{2,2}(D) \cap A_{\mathbf{0}}^{2,\infty}(D) \rightarrow L^2(\Omega', \Delta^{-2\mathbf{s}} \cdot \mathcal{H}^m)$ so that $\tau(f)(\lambda) = \mathcal{F}(f_{e_\Omega})(\lambda)e^{\langle \lambda, e_\Omega \rangle}$ for almost every $\lambda \in \Omega'$. Observe that we may choose a norm on $\tilde{A}_{\mathbf{s}}^{2,2}(D)$ such that

$$\|f\|_{\tilde{A}_{\mathbf{s}}^{2,2}(D)}^2 = \int_{\Omega'} |\tau(f)(\lambda)|^2 \Delta_{\Omega'}^{-2\mathbf{s}}(\lambda) d\lambda$$

for every $f \in \tilde{A}_{\mathbf{s}}^{2,2}(D) \cap A_{\mathbf{0}}^{2,\infty}(D)$, so that we may extend τ to the whole of $\tilde{A}_{\mathbf{s}}^{2,2}(D)$. In addition,

$$f(z) = \frac{1}{(2\pi)^m} \int_{\Omega'} \tau(f)(\lambda) e^{\langle \lambda, z \rangle} d\lambda$$

for every $z \in D$ and for every $f \in \tilde{A}_{\mathbf{s}}^{2,2}(D)$.⁹ It then follows that $\tau(f^k) = (2\pi)^{-(k-1)m} \tau(f)^{*k}$ for every $f \in \tilde{A}_{\mathbf{s}}^{2,2}(D) \cap \mathcal{E}(\mathcal{S}_{\Omega, L}(\mathcal{N}))$, where $\tau(f)^{*k}$ denotes the convolution of k functions all equal to $\tau(f)$. Now,

$$\begin{aligned} |[\tau(f)^{*k}](\lambda)| &\leq \int_{\Omega'^{k-1}} |\tau(f)(\lambda - \lambda_1) \cdots \tau(f)(\lambda_{k-2} - \lambda_{k-1})| d(\lambda_1, \dots, \lambda_{k-1}) \\ &\leq \left(\int_{\Omega'^{k-1}} (|\tau(f)|^2 \Delta_{\Omega'}^{-2\mathbf{s}})(\lambda - \lambda_1) \cdots (|\tau(f)|^2 \Delta_{\Omega'}^{-2\mathbf{s}})(\lambda_{k-2} - \lambda_{k-1}) d(\lambda_1, \dots, \lambda_{k-1}) \right)^{1/2} \\ &\quad \times \left(\int_{\Omega'^{k-1}} \Delta_{\Omega'}^{2\mathbf{s}}(\lambda - \lambda_1) \cdots \Delta_{\Omega'}^{2\mathbf{s}}(\lambda_{k-2} - \lambda_{k-1}) d(\lambda_1, \dots, \lambda_{k-1}) \right)^{1/2} \\ &= |\tau(f)|^2 \Delta_{\Omega'}^{-2\mathbf{s}} \cdot \tau(f)^{*k}(\lambda)^{1/2} (\Delta_{\Omega'}^{2\mathbf{s}})^{*k}(\lambda)^{1/2} \end{aligned}$$

for every $\lambda \in \Omega'$. In addition, [10, Corollary 2.21] shows that

$$(\Delta_{\Omega'}^{2\mathbf{s}})^{*k} = \frac{\Gamma_{\Omega'}(2\mathbf{s} - \mathbf{d})^k}{\Gamma_{\Omega'}(2k\mathbf{s} - (k-1)\mathbf{d})} \Delta_{\Omega'}^{2k\mathbf{s} - (k-1)\mathbf{d}}$$

on Ω' . Therefore, $\tau(f^k) \in L^2(\Omega', \Delta_{\Omega'}^{(k-1)\mathbf{d} - 2k\mathbf{s}} \cdot \mathcal{H}^m)$ and

$$\|\tau(f^k)\|_{L^2(\Omega', \Delta_{\Omega'}^{(k-1)\mathbf{d} - 2k\mathbf{s}} \cdot \mathcal{H}^m)} \leq \frac{\Gamma_{\Omega'}(2\mathbf{s} - \mathbf{d})^{k/2}}{(2\pi)^{(k-1)m} \Gamma_{\Omega'}(2k\mathbf{s} - (k-1)\mathbf{d})^{1/2}} \|\tau(f)\|_{L^2(\Omega', \Delta_{\Omega'}^{-2\mathbf{s}} \cdot \mathcal{H}^m)}^k.$$

Hence,

$$f^k \in \tilde{A}_{k\mathbf{s} - (k-1)\mathbf{d}/2}^{2,2}(D) = A_{k\mathbf{s} - (k-1)\mathbf{d}/2}^{2,2}(D)$$

since $k\mathbf{s} - (k-1)\mathbf{d}/2 \in \frac{1}{4}\mathbf{m} + (\mathbb{R}_+^*)^r$, so that $f \in A_{\mathbf{s} - (k-1)\mathbf{d}/(2k)}^{2k, 2k}(D)$. Hence, there is a continuous embedding $\tilde{A}_{\mathbf{s}}^{2,2}(D) \subseteq A_{\mathbf{s} - (k-1)\mathbf{d}/(2k)}^{2k, 2k}(D)$. Since there is a continuous embedding $\tilde{A}_{\mathbf{s}}^{2,2}(D) \subseteq A_{\mathbf{s} - \mathbf{d}/2}^{\infty, \infty}(D)$ by [10, Definition 5.3], the assertion follows. \square

⁹This formula is clear when $f \in \tilde{A}_{\mathbf{s}}^{2,2}(D) \cap A_{\mathbf{0}}^{2,\infty}(D)$ thanks to [10, Proposition 1.39], and then follows by continuity in the general case, thanks to [10, Propositions 2.19 and 5.4].

7. REVERSE CARLESON AND SAMPLING MEASURES FOR $A_{\mathbf{s}}^{p,p}(D)$

Recall that a measure $\mu \in \mathcal{M}_+(D)$ is p -sampling for $A_{\mathbf{s}}^{p_1, q_1}$ if this latter space embeds as a closed subspace of $L^p(\mu)$. As observe in [29], replacing μ with a suitable integral of positive measures (for instance, but not necessarily, a disintegration of μ along ρ with respect to some of its image measures), it is also possible to extend this definition to mixed-norm Lebesgue spaces. Even though we prefer to avoid such technicalities, we present here a result in this spirit, which shows how one may construct sampling measures out of lattices. See [10, Theorem 3.22] for a proof of a stronger version of the following result.

Proposition 7.1. *Take $p, q \in]0, \infty]$, $\mathbf{s} \in \mathbb{R}^r$, and $R_0 > 0$. Then, there is $\delta_0 > 0$ such that, for every (δ, R) -lattice $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$ on D , with $\delta \in]0, \delta_0]$ and $R \in]1, R_0]$, the mapping*

$$S: \text{Hol}(D) \ni f \mapsto (\Delta_{\Omega}^{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p}(\rho(\zeta_{j,k}, z_{j,k})))f(\zeta_{j,k}, z_{j,k}) \in \mathbb{C}^{J \times K}$$

induces isomorphisms of $A_{\mathbf{s}}^{p,q}$ and $A_{\mathbf{s},0}^{p,q}$ onto closed subspaces of $\ell^{p,q}(J, K)$ and $\ell_0^{p,q}(J, K)$, respectively. In addition, $A_{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p}^{\infty, \infty} \cap S^{-1}(\ell^{p,q}(J, K)) = A_{\mathbf{s}}^{p,q}$ and $A_{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p}^{\infty, \infty} \cap S^{-1}(\ell_0^{p,q}(J, K)) = A_{\mathbf{s},0}^{p,q}$.

The next result provides a necessary condition for a measure μ for which the function $M_R(\mu)$ is in some mixed-normed weighted Lebesgue space on D to be a sampling measure. It extends [28, Theorem 4.3], which deals with the case in which D is the unit disc in \mathbb{C} .

Proposition 7.2. *Take $p_1, q_1, p \in]0, \infty]$, with $p < \infty$, and $\mathbf{s} \in \frac{1}{2q_1}\mathbf{m} + (\mathbb{R}_+^*)^r$ if $q_1 < \infty$, while $\mathbf{s} \in \mathbb{R}_+^r$ if $q_1 = \infty$. Let p^*, q^* , and \mathbf{s}^* be as in (5.2). Then, for every $R, C, C' > 0$, there are $R', C'' > 0$ such that for every $\mu \in \mathcal{M}_+(D)$ such that*

$$\|f\|_{A_{\mathbf{s}}^{p_1, q_1}(D)} \leq C \|f\|_{L^p(\mu)}$$

for every $f \in A_{\mathbf{s}}^{p_1, q_1}(D)$, and such that

$$\|M_R(\mu)\|_{L_{\mathbf{s}^*}^{p^*, q^*}(D)} \leq C',$$

one has $p_1, q_1 \leq p$ and

$$M_{R'}(\mu)(\zeta, z) \geq C'' \Delta_{\Omega}^{p[\mathbf{s}-(\mathbf{b}+\mathbf{d})/p_1]}(\rho(\zeta, z))$$

for every $(\zeta, z) \in D$.

Notice that, if $\|M_R(\mu)\|_{L_{\mathbf{s}^*}^{p^*, q^*}(D)}$ is finite, then μ is a p -Carleson measure for $A_{\mathbf{s}}^{p_1, q_1}$, thanks to Proposition 5.2. Hence, a measure μ as above is indeed a p -sampling measure for $A_{\mathbf{s}}^{p_1, q_1}$.

Proof. Observe that, by inspection of the proof of Proposition 5.2, it is readily verified that for every $R' > 0$ there is a constant $C_{R'} > 0$ such that

$$\|f\|_{L^p(\mu)} \leq C_{R'} \|M_R(\mu)\|_{L_{\mathbf{s}^*}^{p^*, q^*}(D)}^{1/p} \|\chi_{B(\text{Supp}(\mu), R')} f\|_{L_{\mathbf{s}}^{p_1, q_1}(D)}$$

for every $f \in \text{Hol}(D)$ and for every positive Radon measure μ on D , where

$$B(\text{Supp}(\mu), R') := \bigcup_{(\zeta, z) \in \text{Supp}(\mu)} B((\zeta, z), R').$$

Therefore, for every $(\zeta, z) \in D$ and for every $R' > 0$,

$$\|\chi_{D \setminus B((\zeta, z), 2R')} f\|_{L^p(\mu)} \leq C_{R'} \|M_R(\mu)\|_{L_{\mathbf{s}^*}^{p^*, q^*}(D)}^{1/p} \|\chi_{D \setminus B((\zeta, z), R')} f\|_{L_{\mathbf{s}}^{p_1, q_1}(D)}$$

for every $f \in \text{Hol}(D)$ and for every positive Radon measure μ on D . Now, take $\mathbf{s}' \in \mathbb{R}^r$ so that $B_{(\zeta, z)}^{\mathbf{s}'} \in A_{\mathbb{S}}^{p_1, q_1}(D)$ for every $(\zeta, z) \in D$, and observe that there is a constant $C''' > 0$ such that

$$\|B_{(\zeta, z)}^{\mathbf{s}'}\|_{A_{\mathbb{S}}^{p_1, q_1}(D)} = C''' \Delta_{\Omega}^{\mathbf{s} + \mathbf{s}' - (\mathbf{b} + \mathbf{d})/p_1}(\rho(\zeta, z))$$

for every $(\zeta, z) \in D$ (cf. [10, Proposition 2.41]). In addition, for every $R' > 0$ there is a constant $C_{R'}^{(4)} > 0$ such that

$$\frac{1}{C_{R'}^{(4)}} \Delta_{\Omega}^{\mathbf{s}'}(\rho(\zeta, z)) \leq |B_{(\zeta, z)}^{\mathbf{s}'}(\zeta', z')| \leq C_{R'}^{(4)} \Delta_{\Omega}^{\mathbf{s}'}(\rho(\zeta, z))$$

for every $(\zeta, z), (\zeta', z') \in D$ such that $d((\zeta, z), (\zeta', z')) \leq R'$ (cf. [10, Theorem 2.47]). Therefore,

$$\begin{aligned} & (C_{2R'}^{(4)})^p \Delta_{\Omega}^{p\mathbf{s}'}(\rho(\zeta, z)) M_{2R'}(\mu)(\zeta, z) \\ & \geq \int_{B((\zeta, z), 2R')} |B_{(\zeta, z)}^{\mathbf{s}'}|^p d\mu \\ & = \|B_{(\zeta, z)}^{\mathbf{s}'}\|_{L^p(\mu)}^p - \|\chi_{D \setminus B((\zeta, z), 2R')} B_{(\zeta, z)}^{\mathbf{s}'}\|_{L^p(\mu)}^p \\ & \geq \left(\frac{C'''}{C} \Delta_{\Omega}^{\mathbf{s} + \mathbf{s}' - (\mathbf{b} + \mathbf{d})/p_1}(\rho(\zeta, z)) \right)^p - C' C_{R'}^p \|\chi_{D \setminus B((\zeta, z), R')} B_{(\zeta, z)}^{\mathbf{s}'}\|_{L_{\mathbb{S}}^{p_1, q_1}(D)}^p \end{aligned}$$

for every $(\zeta, z) \in D$, for every $R' > 0$, and for every μ as in the statement. Now, observe that, by homogeneity, setting

$$C_{R'}^{(5)} := \|\chi_{D \setminus B((0, ie_{\Omega}), R')} B_{(0, ie_{\Omega})}^{\mathbf{s}'}\|_{L_{\mathbb{S}}^{p_1, q_1}(D)},$$

one has

$$\|\chi_{D \setminus B((\zeta, z), R')} B_{(\zeta, z)}^{\mathbf{s}'}\|_{L_{\mathbb{S}}^{p_1, q_1}(D)} = C_{R'}^{(5)} \Delta_{\Omega}^{\mathbf{s} + \mathbf{s}' - (\mathbf{b} + \mathbf{d})/p_1}(\rho(\zeta, z))$$

for every $(\zeta, z) \in D$ (cf. Lemma 4.12). Therefore,

$$M_{2R'}(\mu)(\zeta, z) \geq (C_{2R'}^{(4)})^{-p} \left(\left(\frac{C'''}{C} \right)^p - C' (C_{R'} C_{R'}^{(5)})^p \right) \Delta_{\Omega}^{p[\mathbf{s} - (\mathbf{b} + \mathbf{d})/p_1]}(\rho(\zeta, z))$$

for every $(\zeta, z) \in D$ and for every $R' > 0$.

Now, observe that the function $(\Delta_{\Omega}^{p[(\mathbf{b} + \mathbf{d})/p_1 - \mathbf{s}]} \circ \rho) M_{R'}$ is both bounded from below and in $L_{\mathbb{S}^{*} + p[\mathbf{s} - (\mathbf{b} + \mathbf{d})/p_1]}^{p^*, q^*}$, thanks to Lemma 5.1. It then follows easily that $p^* = q^* = \infty$, that is, $p_1, q_1 \leq p$, in which case $\mathbf{s}^* = p[(\mathbf{b} + \mathbf{d})/p_1 - \mathbf{s}]$. The proof is complete. \square

In the next result we establish a necessary and sufficient condition in order for a ν_D -measurable set G to be a dominant (or sampling) set (recall Definition 1.1), that is, for the measure $\chi_G(\Delta_{\Omega}^{p\mathbf{s} - (\mathbf{b} + \mathbf{d})} \circ \rho) \cdot \nu_D$ to be a p -sampling measure for $A_{\mathbb{S}}^{p, p}$. It extends [24, Main Theorem], which deals with the case in which D is the unit disc in \mathbb{C} . See also [27, Theorem 1], which deals with weighted Bergman spaces on general homogeneous domains.

Theorem 7.3. *Take $p \in]0, \infty[$ and $\mathbf{s} \in \frac{1}{2p} \mathbf{m} + (\mathbb{R}_+^*)^r$. Then, for every ν_D -measurable subset G of D the following conditions are equivalent:*

(1) *there are $R, C > 0$ such that, for every $(\zeta, z) \in D$,*

$$\|\chi_{B((\zeta, z), R)}\|_{L_{\mathbb{S}}^{p, p}(D)} \leq C \|\chi_{G \cap B((\zeta, z), R)}\|_{L_{\mathbb{S}}^{p, p}(D)};$$

(2) *there exists $C' > 0$ such that, for every $f \in A_{\mathbb{S}}^{p, p}(D)$,*

$$\|f\|_{A_{\mathbb{S}}^{p, p}(D)} \leq C' \|\chi_G f\|_{L_{\mathbb{S}}^{p, p}(D)};$$

(3) there are $R', C'' > 0$ such that $\nu_D(G \cap B((\zeta, z), R')) \geq C''$ for every $(\zeta, z) \in D$.

Notice that condition (3) depends neither on \mathbf{s} , nor on p . Before we pass to the proof, we need to establish some lemmas.

Lemma 7.4. Fix $R > 0$, and define, for every $\varepsilon > 0$ and for every $f \in A_{\mathbf{s}}^{p,p}(D)$,

$$A_{f,\varepsilon} := \left\{ (\zeta, z) \in D : |f(\zeta, z)| \leq \varepsilon \left(\int_{B((\zeta,z),R)} |f|^p d\nu_D \right)^{1/p} \right\}.$$

Then, there is a constant $C > 0$ such that

$$\|\chi_{A_{f,\varepsilon}} f\|_{L_{\mathbf{s}}^{p,p}(D)} \leq C\varepsilon \|f\|_{L_{\mathbf{s}}^{p,p}(D)}$$

for every $f \in A_{\mathbf{s}}^{p,p}(D)$ and for every $\varepsilon > 0$.

This extends [28, Lemma 2], which deals with the case in which D is the unit disc in \mathbb{C} .

Proof. Observe that, by Fubini's theorem,

$$\begin{aligned} \|\chi_{A_{f,\varepsilon}} f\|_{L_{\mathbf{s}}^{p,p}(D)}^p &\leq \varepsilon^p \int_{A_{f,\varepsilon}} \int_{B((\zeta,z),R)} |f|^p d\nu_D \Delta_{\Omega}^{ps-(\mathbf{b}+\mathbf{d})}(\rho(\zeta, z)) d\nu_D(\zeta, z) \\ &\leq \varepsilon^p \int_D |f(\zeta', z')|^p \int_{B((\zeta',z'),R)} \Delta_{\Omega}^{ps-(\mathbf{b}+\mathbf{d})}(\rho(\zeta, z)) d\nu_D(\zeta, z) d\nu_D(\zeta', z'). \end{aligned}$$

Now, observe that, by homogeneity,

$$\int_{B((\zeta',z'),R)} \Delta_{\Omega}^{ps-(\mathbf{b}+\mathbf{d})}(\rho(\zeta, z)) d\nu_D(\zeta, z) = C' \Delta_{\Omega}^{ps-(\mathbf{b}+\mathbf{d})}(\rho(\zeta', z'))$$

for a suitable constant $C' > 0$. The assertion follows. \square

Lemma 7.5. Take $R, \varepsilon, p \in]0, \infty[$. Then, for every $\delta > 0$ there is $\lambda > 0$ such that, if we define

$$E_{f,\lambda}(\zeta, z) := \{ (\zeta', z') \in B((\zeta, z), R) : |f(\zeta', z')| > \lambda |f(\zeta, z)| \},$$

then

$$\nu_D(E_{f,\lambda}(\zeta, z)) \geq (1 - \delta) \nu_D(B((\zeta, z), R))$$

for every $(\zeta, z) \in D$ and for every $f \in \text{Hol}(B((\zeta, z), R))$ such that

$$|f(\zeta, z)| \geq \varepsilon \left(\int_{B((\zeta,z),R)} |f|^p d\nu_D \right)^{1/p}.$$

This extends [25, Lemma 2], which deals with the case in which D is the unit disc in \mathbb{C} .

Proof. Observe that, by homogeneity, we may reduce to proving the assertion for $(\zeta, z) = (0, ie_{\Omega})$. Then, assume by contradiction that there are $\delta > 0$ and a sequence $(f_j)_{j \in \mathbb{N}}$ of elements of $\text{Hol}(D)$ such that

$$|f_j(0, ie_{\Omega})| \geq \varepsilon \left(\int_{B((\zeta,z),R)} |f_j|^p d\nu_D \right)^{1/p}$$

and such that

$$\nu_D(E_{f_j, 2^{-j}}) < \delta \nu_D(B((0, ie_{\Omega}), R))$$

for every $j \in \mathbb{N}$. Observe that, up to multiplying each f_j by a suitable constant, we may assume that $\int_{B((\zeta, z), R)} |f_j|^p d\nu_D = 1$ for every $j \in \mathbb{N}$, so that $|f_j(0, ie_\Omega)| \geq \varepsilon$ for every $j \in \mathbb{N}$. It is then readily verified that the sequence (f_j) is bounded in $\text{Hol}(B((0, ie_\Omega), R))$, so that we may assume that it converges locally uniformly to some f . Then, take $R' \in]0, R[$ so that $\nu_D(B((0, ie_\Omega), R')) \geq \frac{\delta}{2} \nu_D(B((0, ie_\Omega), R))$, and observe that

$$\nu_D(\{ (\zeta, z) \in B((0, ie_\Omega), R') : |f_j(\zeta, z)| \leq 2^{-j} |f_j(0, ie_\Omega)| \}) \geq \frac{\delta}{2} \nu_D(B((0, ie_\Omega), R))$$

for every $j \in \mathbb{N}$. Observe that (f_j) converges uniformly to f on $B((0, ie_\Omega), R')$ (cf. [10, Proposition 2.44]), so that for every $j_0 \in \mathbb{N}$ there is $j_1 \in \mathbb{N}$ such that

$$|f_j - f| \leq 2^{-j_0}$$

on $B((0, ie_\Omega), R')$ for every $j \geq j_1$. Hence,

$$\begin{aligned} \{ (\zeta, z) \in B((0, ie_\Omega), R') : |f(\zeta, z)| \leq 2^{-j_0} + 2^{-j} |f_j(0, ie_\Omega)| \} \\ \supseteq \{ (\zeta, z) \in B((0, ie_\Omega), R') : |f_j(\zeta, z)| \leq 2^{-j} |f_j(0, ie_\Omega)| \} \end{aligned}$$

for every $j \geq j_1$, so that

$$\nu_D(\{ (\zeta, z) \in B((0, ie_\Omega), R') : |f(\zeta, z)| \leq 2^{-j_0} + 2^{-j} |f_j(0, ie_\Omega)| \}) \geq \frac{\delta}{2} \nu_D(B((0, ie_\Omega), R)).$$

By the arbitrariness of j_0 , we then see that

$$\begin{aligned} \nu_D(\{ (\zeta, z) \in B((0, ie_\Omega), R') : f(\zeta, z) = 0 \}) &= \lim_{\eta \rightarrow 0^+} \nu_D(\{ (\zeta, z) \in B((0, ie_\Omega), R') : |f(\zeta, z)| \leq \eta \}) \\ &\geq \frac{\delta}{2} \nu_D(B((0, ie_\Omega), R)). \end{aligned}$$

Then, f vanishes on a non- ν_D -negligible set, so that it vanishes identically on the connected set $B((0, ie_\Omega), R)$ by holomorphy.¹⁰ Nonetheless, $|f(0, ie_\Omega)| \geq \varepsilon$: contradiction. \square

Proof of Theorem 7.3. By means of [10, Corollary 2.49], condition (1) is readily seen to be equivalent to condition (3).

(2) \implies (1). This follows from Proposition 7.2.

(1) \implies (2). With the notation of Lemma 7.4, choose $\varepsilon > 0$ so small that

$$\|f\|_{A_{\mathbb{S}}^{p,p}(D)} \leq 2 \|\chi_{D \setminus A_{f,\varepsilon}} f\|_{L_{\mathbb{S}}^{p,p}(D)}$$

for every $f \in A_{\mathbb{S}}^{p,p}(D)$. In addition, with the notation of Lemma 7.5, choose $\lambda > 0$ so that

$$\|\chi_{B((\zeta, z), R) \setminus E_{f,\lambda}(\zeta, z)}\|_{L_{\mathbb{S}}^{p,p}(D)} \leq \frac{1}{2^{1/p} C'} \|\chi_{B((\zeta, z), R)}\|_{L_{\mathbb{S}}^{p,p}(D)}$$

for every $f \in A_{\mathbb{S}}^{p,p}(D)$, and for every $(\zeta, z) \in D \setminus A_{f,\varepsilon}$ (cf. also [10, Corollary 2.49]), so that

$$\|\chi_{G \cap E_{f,\lambda}(\zeta, z)}\|_{L_{\mathbb{S}}^{p,p}(D)} \geq \frac{1}{2^{1/p} C'} \|\chi_{B((\zeta, z), R)}\|_{L_{\mathbb{S}}^{p,p}(D)}.$$

¹⁰Observe that the Bergman distance on D is induced by a complete Riemannian metric by [10, Proposition 2.44], so that $B((0, ie_\Omega), R) = \exp_{(0, ie_\Omega)}(B(0, R))$ by the Hopf–Rinow theorem (cf., e.g., [23, the proof of Theorem 6.6]). Alternatively, every element of $B((0, ie_\Omega), R)$ may be connected to $(0, ie_\Omega)$ by a minimizing geodesic, whose image therefore lies entirely in $B((0, ie_\Omega), R)$.

Now, the definition of $E_{f,\lambda}(\zeta, z)$ implies that

$$\|\chi_{G \cap B((\zeta, z), R)} f\|_{L_{\mathbf{s}}^{p,p}(D)} \geq \|\chi_{G \cap E_{f,\lambda}(\zeta, z)} f\|_{L_{\mathbf{s}}^{p,p}(D)} \geq \frac{\lambda^p}{2^{1/p} C'} |f(\zeta, z)| \|\chi_{B((\zeta, z), R)}\|_{L_{\mathbf{s}}^{p,p}(D)}$$

for every $f \in A_{\mathbf{s}}^{p,p}(D)$, and for every $(\zeta, z) \in D \setminus A_{f,\varepsilon}$. Therefore, by means of Fubini's theorem we see that

$$\begin{aligned} \|f\|_{A_{\mathbf{s}}^{p,p}(D)}^p &\leq 2^p \|\chi_{D \setminus A_{f,\varepsilon}} f\|_{L_{\mathbf{s}}^{p,p}(D)}^p \\ &\leq \frac{2^{p+1} C'^p}{\lambda^p} \int_G |f(\zeta', z')|^p \int_{B((\zeta', z'), R)} \frac{\Delta_{\Omega}^{ps+d}(\rho(\zeta, z))}{\|\chi_{B((\zeta, z), R)}\|_{L_{\mathbf{s}}^{p,p}(D)}^p} d(\zeta, z) \Delta_{\Omega}^{ps+d}(\rho(\zeta', z')) d(\zeta', z') \end{aligned}$$

for every $f \in A_{\mathbf{s}}^{p,p}(D)$. To conclude, it suffices to observe that the function

$$(\zeta', z') \mapsto \int_{B((\zeta', z'), R)} \frac{\Delta_{\Omega}^{ps+d}(\rho(\zeta, z))}{\|\chi_{B((\zeta, z), R)}\|_{L_{\mathbf{s}}^{p,p}(D)}^p} d(\zeta, z) = \frac{\nu_D(B((\zeta', z'), R))}{\|\chi_{B((0, i\varepsilon), R)}\|_{L_{\mathbf{s}}^{p,p}(D)}^p}$$

is constant, by homogeneity (cf. Lemma 4.12). \square

Theorem 7.6. *Take $p \in]0, \infty[$, $\mathbf{s} \in \frac{1}{2p} \mathbf{m} + (\mathbb{R}_+^*)^r$, $\varepsilon, C', R_0 > 0$. Then, there are $R_1, C > 0$ such that, for every $R \in]0, R_1]$, for every $\mu \in \mathcal{M}_+(D)$ such that*

$$N(\mu) := \|M_R(\mu)\|_{L_{\mathbf{b}+d-ps}^{\infty, \infty}(D)} < \infty,$$

and such that

$$\nu_D(G_\mu \cap B((\zeta, z), R_0)) \geq C'$$

for every $(\zeta, z) \in D$, where

$$G_\mu := \left\{ (\zeta, z) \in D : \Delta_{\Omega}^{\mathbf{b}+d-ps}(\rho(\zeta, z)) M_R(\mu)(\zeta, z) \geq \varepsilon N(\mu) \right\},$$

one has

$$\|f\|_{A_{\mathbf{s}}^{p,p}(D)} \leq \frac{C R^{(2n+2m)/p}}{N(\mu)} \|f\|_{L^p(\mu)}$$

for every $f \in A_{\mathbf{s}}^{p,p}(D)$

This extends [28, Theorem 4.2], which deals with the case in which D is the unit disc in \mathbb{C} .

Before we pass to the proof, we need another result.

Proposition 7.7. *Take $p \in]0, \infty[$. Then, there are two constants $C, R_0 > 0$ such that the following hold. For every $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3 \in \mathbb{R}^r$, for every $R, R' \in]0, R_0]$, for every $\mu_1, \mu_2 \in \mathcal{M}_+(D)$ such that*

$$C_1 := \sup_{(\zeta, z) \in D} \Delta_{\Omega}^{-\mathbf{s}_1}(\rho(\zeta, z)) M_R(\mu_1)(\zeta, z) \quad \text{and} \quad C_2 := \sup_{(\zeta, z) \in D} \Delta_{\Omega}^{-\mathbf{s}_2}(\rho(\zeta, z)) M_{R+R'}(\mu_2)(\zeta, z)$$

are finite, and for every $f \in \text{Hol}(D)$,

$$\int_{d((\zeta, z), (\zeta', z')) < R} \frac{|f(\zeta, z) - f(\zeta', z')|^p}{\Delta_{\Omega}^{\mathbf{s}_1 - \mathbf{s}_3}(\rho(\zeta', z'))} d(\mu_1 \otimes \mu_2)((\zeta, z), (\zeta', z')) \leq \frac{C C_1 C_2 R^p}{R'^{p+2n+2m}} \int_D |f|^p(\Delta_{\Omega}^{\mathbf{s}_2 + \mathbf{s}_3} \circ \rho) d\nu_D,$$

This extends [28, Theorem 2.3], which deals with the case in which D is the unit disc in \mathbb{C} .

Proof. By [10, Lemmas 3.24 and 3.25], there are two constants $C_1, R_0 > 0$ such that for every $R, R' \in]0, R_0]$, for every $f \in \text{Hol}(D)$, and for every $(\zeta, z), (\zeta', z') \in D$ such that $d((\zeta, z), (\zeta', z')) \leq R$,

$$|f(\zeta, z) - f(\zeta', z')|^p \leq C' \frac{R^p}{R'^{p+2n+2m}} \int_{B((\zeta', z'), R+R')} |f|^p d\nu_D.$$

Therefore, by means of [10, Corollary 2.49], we see that there is a constant $C'' > 0$ such that

$$|f(\zeta, z) - f(\zeta', z')|^p \Delta_{\Omega}^{\mathbf{s}_3}(\rho(\zeta', z')) \leq C'' \frac{R^p}{R'^{p+2n+2m}} \int_{B((\zeta', z'), R+R')} |f|^p (\Delta_{\Omega}^{\mathbf{s}_3} \circ \rho) d\nu_D.$$

for every $R, R' \in]0, R_0]$, for every $f \in \text{Hol}(D)$, and for every $(\zeta, z), (\zeta', z') \in D$ such that $d((\zeta, z), (\zeta', z')) \leq R$. Then, integrating in (ζ, z) with respect to μ_1 ,

$$\begin{aligned} & \Delta_{\Omega}^{\mathbf{s}_3}(\rho(\zeta', z')) \int_{B((\zeta', z'), R)} |f(\zeta, z) - f(\zeta', z')|^p d\mu_1(\zeta, z) \\ & \leq C'' \frac{R^p}{R'^{p+2n+2m}} M_R(\mu_1)(\zeta', z') \int_{B((\zeta', z'), R+R')} |f|^p (\Delta_{\Omega}^{\mathbf{s}_3} \circ \rho) d\nu_D \\ & \leq C'' \frac{R^p}{R'^{p+2n+2m}} C_1 \Delta_{\Omega}^{\mathbf{s}_1}(\rho(\zeta', z')) \int_{B((\zeta', z'), R+R')} |f|^p (\Delta_{\Omega}^{\mathbf{s}_3} \circ \rho) d\nu_D \end{aligned}$$

so that, integrating in (ζ', z') with respect to the measure μ_2 ,

$$\begin{aligned} & \int_{d((\zeta, z), (\zeta', z')) < R} |f(\zeta, z) - f(\zeta', z')|^p \Delta_{\Omega}^{\mathbf{s}_3 - \mathbf{s}_1}(\rho(\zeta', z')) d(\mu_1 \otimes \mu_2)((\zeta, z), (\zeta', z')) \\ & \leq \frac{C_1 C_2 C'' R^p}{R'^{p+2n+2m}} \int_D |f|^p (\Delta_{\Omega}^{\mathbf{s}_2 + \mathbf{s}_3} \circ \rho) d\nu_D, \end{aligned}$$

whence the result. \square

Proof of Theorem 7.6. Apply Proposition 7.7 with $\mu_1 = \mu$, $\mu_2 = \nu_D$, $\mathbf{s}_1 = \mathbf{s}_3 = p\mathbf{s} - (\mathbf{b} + \mathbf{d})$, and $\mathbf{s}_2 = \mathbf{0}$. Then, we find $R_0 > 0$ and $C_1 > 0$ such that

$$\int_{d((\zeta, z), (\zeta', z')) < R} |f(\zeta, z) - f(\zeta', z')|^p d(\nu_D \otimes \mu)((\zeta, z), (\zeta', z')) \leq \frac{C_1 R^p}{R_0^{p+2n+2m}} N(\mu) \|f\|_{A_{\mathbf{s}}^{p,p}(D)}^p$$

for every $R \in]0, R_0]$ and for every $f \in \text{Hol}(D)$. Therefore,

$$\begin{aligned} & \left(\int_{d((\zeta, z), (\zeta', z')) < R} |f(\zeta, z)|^p d(\nu_D \otimes \mu)((\zeta, z), (\zeta', z')) \right)^{1/\max(1,p)} \\ & \leq \left(\int_{d((\zeta, z), (\zeta', z')) < R} |f(\zeta', z')|^p d(\nu_D \otimes \mu)((\zeta, z), (\zeta', z')) \right)^{1/\max(1,p)} \\ & \quad + \left(\frac{C_1 R^p}{R_0^{p+2n+2m}} N(\mu) \|f\|_{A_{\mathbf{s}}^{p,p}(D)}^p \right)^{1/\max(1,p)} \end{aligned}$$

for every $f \in \text{Hol}(D)$. Now, observe that Theorem 7.3 implies that there is a constant $C_2 > 0$ such that

$$\begin{aligned} \int_{d((\zeta, z), (\zeta', z')) < R} |f(\zeta, z)|^p d(\nu_D \otimes \mu)((\zeta, z), (\zeta', z')) &= \int_D |f(\zeta, z)|^p M_R(\mu)(\zeta, z) d\nu_D(\zeta, z) \\ &\geq \varepsilon N(\mu) \int_{G_\mu} |f|^p (\Delta_\Omega^{ps-(\mathbf{b}+\mathbf{d})} \circ \rho) d\nu_D \\ &\geq C_2 \varepsilon N(\mu) \|f\|_{A_{\mathbf{s}}^{p,p}(D)}^p, \end{aligned}$$

for every $f \in A_{\mathbf{s}}^{p,p}(D)$, while clearly

$$\int_{d((\zeta, z), (\zeta', z')) < R} |f(\zeta', z')|^p d(\nu_D \otimes \mu)((\zeta, z), (\zeta', z')) = \nu_D(B((0, ie_\Omega), R)) \int_D |f|^p d\mu,$$

for every $f \in A_{\mathbf{s}}^{p,p}(D)$. Therefore,

$$\begin{aligned} \left(C_2 \varepsilon N(\mu) \|f\|_{A_{\mathbf{s}}^{p,p}(D)}^p \right)^{1/\max(1,p)} &\leq \left(\nu_D(B((0, ie_\Omega), R)) \int_D |f|^p d\mu \right)^{1/\max(1,p)} \\ &\quad + \left(\frac{C_1 R^p}{R_0^{p+2n+2m}} N(\mu) \|f\|_{A_{\mathbf{s}}^{p,p}(D)}^p \right)^{1/\max(1,p)} \end{aligned}$$

for every $f \in A_{\mathbf{s}}^{p,p}(D)$. It then follows that

$$C_3 N(\mu) \|f\|_{A_{\mathbf{s}}^{p,p}(D)}^p \leq \nu_D(B((0, ie_\Omega), R)) \int_D |f|^p d\mu$$

for every $f \in A_{\mathbf{s}}^{p,p}(D)$, where

$$C_3 := \left((C_2 \varepsilon)^{1/\max(1,p)} - \left(\frac{C_1 R^p}{R_0^{p+2n+2m}} \right)^{1/\max(1,p)} \right)^{\max(1,p)}.$$

Since there is $R_1 \in]0, R_0]$ such that C_3 is well defined and > 0 whenever $R \in]0, R_1]$, the assertion follows. \square

Definition 7.8. Given $\mathbf{s} \in \mathbb{R}^r$ and $\mu \in \mathcal{M}_+(D)$, we shall denote with $W_{\mathbf{s}}(\mu)$ the vague closure of the set of measures of the form

$$\Delta_\Omega^{\mathbf{s}}(\rho(\zeta, z))(\varphi_{(\zeta, z)})_*(\mu),$$

for every $(\zeta, z) \in D$, where $\varphi_{(\zeta, z)}$ is an affine automorphism of D of the form $(\zeta', z') \mapsto (\zeta, \text{Re } z + i\Phi(\zeta)) \cdot (g\zeta', t \cdot z)$, with $t \in T_+$, $g \in GL(E)$, and $t \cdot \Phi = \Phi \circ (g \times g)$.¹¹

Theorem 7.9. Take $p, q \in]0, \infty[$, with $q < p$, $\mathbf{s} \in \mathbb{R}^r$, and $\mu \in \mathcal{M}_+(D)$ such that the following hold:

- (1) $\mathbf{s} \in \frac{1}{2p}\mathbf{m} + \frac{q}{p(p-q)}\mathbf{m}' + (\mathbb{R}_+^*)^r$;
- (2) $M_1(\mu) \in L_{\mathbf{b}+\mathbf{d}-p\mathbf{s}}^{\infty, \infty}(D)$;
- (3) the support of every element of $W_{ps-(\mathbf{b}+\mathbf{d})}(\mu)$ is a set of uniqueness for $A_{(p/q)\mathbf{s}}^{q,q}(D)$.

Then, the canonical mapping $A_{\mathbf{s}}^{p,p}(D) \rightarrow L^p(\mu)$ is an isomorphism onto its image.

¹¹Notice that $\varphi_{(\zeta, z)}$ is not uniquely determined by (ζ, z) unless $n = 0$, so that this definition may depend on the choice of the automorphisms $\varphi_{(\zeta, z)}$.

This extends the implication (b) \implies (a) in [31, Theorem 5], which deals with the case in which D is the unit disc in \mathbb{C} . Notice that, as shown in [31, Theorem 5], condition (3) holds for some $q < p$ if $A_{\mathbf{s}}^{p,p}(D)$ embeds as a closed subspace of $L^p(\mu)$ and D is the unit disc in \mathbb{C} .

Before we pass to the proof, we need some lemmas.

Lemma 7.10. *Take $p_1, q_1, p \in]0, \infty]$, with $p < \infty$, and $\mathbf{s} \in \frac{1}{2q_1}\mathbf{m} + (\mathbb{R}_+^*)^r$ and let p^*, q^* and \mathbf{s}^* be as in (5.2). Let \mathcal{M} be a set of Radon measures on D such that*

$$\sup_{\mu \in M} \|M_R(\mu)\|_{L_{\mathbf{s}^*}^{p^*, q^*}(D)} < \infty.$$

Then,

$$\lim_{\mu, \mathfrak{F}} \|f\|_{L^p(\mu)} = \|f\|_{L^p(\mu_0)}$$

for every filter \mathfrak{F} on \mathcal{M} which converges vaguely to some Radon measure μ_0 on D , and for every $f \in A_{\mathbf{s}, 0}^{p_1, q_1}(D)$.

This extends [31, Theorem 1], which deals with the case in which D is the unit disc in \mathbb{C} .

Proof. By inspection of the proof of Proposition 5.2, it is readily verified that there is a constant $C' > 0$ such that

$$\|\chi_{B((0, ie_\Omega), R')} f\|_{L^{q_2}(\mu)} \leq C' \|\chi_{B((0, ie_\Omega), R'+1)} f\|_{L_{\mathbf{s}}^{q_1, q_2}(D)}$$

for every $\mu \in M$, for every $f \in A_{\mathbf{s}, 0}^{p_1, q_1}(D)$, and for every $R' > 0$. The assertion follows easily. \square

Lemma 7.11. *Take $p_1, q_1, p \in]0, \infty]$, $\mathbf{s} \in \frac{1}{2q_1}\mathbf{m} + (\mathbb{R}_+^*)^r$ if $q_1 < \infty$ and $\mathbf{s} \in \mathbb{R}_+^r$ if $q_1 = \infty$, and $\mathbf{s}_2, \mathbf{s}_3 \in \mathbb{R}^r$. Let p^*, q^* and \mathbf{s}^* be as in (5.2). Take $\varepsilon > 0$, and define, for every $(\zeta, z) \in D$,*

$$U_\varepsilon(\zeta, z) := \left\{ f \in \text{Hol}(D) : |f(\zeta, z)| \geq \varepsilon \Delta_\Omega^{(\mathbf{b}+\mathbf{d})/p - \mathbf{s} - \mathbf{s}_2}(\rho(\zeta, z)) \|f B_{(\zeta, z)}^{\mathbf{s}_2}\|_{A_{\mathbf{s}}^{p_1, q_1}(D)} \right\}.$$

Take $\mu \in \mathcal{M}_+(D)$ such that

$$\|M_R(\mu)\|_{L_{\mathbf{s}^*}^{p^*, q^*}(D)} < \infty,$$

and such that the support of every element of $W_{p[\mathbf{s} - (\mathbf{b}+\mathbf{d})/p_1]}(\mu)$ is a set of uniqueness for $A_{\mathbf{s}_1}^{p_1, q_1}(D)$.

Then, there is a constant $C > 0$ such that

$$\|f B_{(\zeta, z)}^{\mathbf{s}_2}(\Delta_\Omega^{\mathbf{s}_3} \circ \rho)\|_{L^p(\mu')} \geq C \Delta_\Omega^{\mathbf{s}_3}(\rho(\zeta, z)) \|f B_{(\zeta, z)}^{\mathbf{s}_2}\|_{A_{\mathbf{s}}^{p_1, q_1}(D)}$$

for every $(\zeta, z) \in D$, for every $f \in U_\varepsilon(\zeta, z)$ and for every $\mu' \in W_{p[\mathbf{s} - (\mathbf{b}+\mathbf{d})/p_1]}(\mu)$.

This extends [31, Lemma 4], which deals with the case in which D is the unit disc in \mathbb{C} .

Proof. STEP I. We prove the assertion for $(\zeta, z) = (0, ie_\Omega)$. Define

$$\mu_{(\zeta, z)} := \Delta_\Omega^{p[\mathbf{s} - (\mathbf{b}+\mathbf{d})/p_1]}(\rho(\zeta, z)) (\varphi_{(\zeta, z)})_*(\mu)$$

for every $(\zeta, z) \in D$, where $\varphi_{(\zeta, z)}$ is as in Definition 7.8. Observe that it will suffice to prove the assertion with $W_{p[\mathbf{s}_1 - (\mathbf{b}+\mathbf{d})/p_1]}(\mu)$ replaced by $\{\mu_{(\zeta, z)} : (\zeta, z) \in D\}$, thanks to Lemma 7.10. Then assume, by contradiction, that there are a sequence (f_j) of elements of $U_\varepsilon(0, ie_\Omega)$, and a sequence $((\zeta_j, z_j))$ of elements of D such that

$$\|f_j B_{(0, ie_\Omega)}^{\mathbf{s}_2}\|_{A_{\mathbf{s}}^{p_1, q_1}(D)} = 1$$

for every $j \in \mathbb{N}$, while

$$\lim_{j \rightarrow \infty} \|f_j B_{(0, ie_\Omega)}^{\mathbf{s}_2}(\Delta_\Omega^{\mathbf{s}_3} \circ \rho)\|_{L^{q_2}(\mu_{(\zeta_j, z_j)})} = 0.$$

Observe that

$$\|M_R(\mu_{(\zeta', z')})\|_{L^{p_s^*, q^*}(D)} = \|M_R(\mu)\|_{L^{p_s^*, q^*}(D)}$$

for every $(\zeta', z') \in D$ (cf. Lemma 4.12), so that $W_{p[\mathbf{s}_1 - (\mathbf{b} + \mathbf{d})/p_1]}(\mu)$ is bounded, hence compact and metrizable, in the vague topology. Therefore, we may assume that $(\mu_{(\zeta_j, z_j)})$ converges vaguely to some (positive Radon) measure μ' on D . Analogously, we may assume that (f_j) converges locally uniformly to some $f \in A_S^{p_1, q_1}(D)$, so that $|f(0, ie_\Omega)| \geq \varepsilon$. Let $(\psi_k)_{k \in K}$ be a partition of the unity on D whose elements belong to $C_c(D)$, and observe that

$$\lim_{j \rightarrow \infty} \|\psi_k^{1/p} f_j\|_{L^p(\mu_{(\zeta_j, z_j)})} = \|\psi_k^{1/p} f\|_{L^p(\mu')}$$

by the previous remarks. Therefore, by Fatou's lemma,

$$\begin{aligned} 0 &= \lim_{j \rightarrow \infty} \|f_j B_{(\zeta, z)}^{\mathbf{s}_2}(\Delta_\Omega^{\mathbf{s}_3} \circ \rho)\|_{L^p(\mu_{(\zeta_j, z_j)})}^p \\ &= \lim_{j \rightarrow \infty} \sum_{k \in K} \|\psi_k^{1/p} f_j B_{(\zeta, z)}^{\mathbf{s}_2}(\Delta_\Omega^{\mathbf{s}_3} \circ \rho)\|_{L^p(\mu_{(\zeta_j, z_j)})}^p \\ &\geq \sum_{k \in K} \|\psi_k^{1/p} f B_{(\zeta, z)}^{\mathbf{s}_2}(\Delta_\Omega^{\mathbf{s}_3} \circ \rho)\|_{L^p(\mu')}^p \\ &= \|f B_{(\zeta, z)}^{\mathbf{s}_2}(\Delta_\Omega^{\mathbf{s}_3} \circ \rho)\|_{L^p(\mu')}^p. \end{aligned}$$

Since the support of $\mu' \in W_{p[\mathbf{s}_1 - (\mathbf{b} + \mathbf{d})/p_1]}(\mu)$ is a set of uniqueness for $A_S^{p_1, q_1}(D)$, this implies that $f = 0$, which is absurd, since $|f(0, ie_\Omega)| \geq \varepsilon$.

STEP II. We now prove the assertion for general $(\zeta, z) \in D$ and for $\mu' = \mu_{(\zeta', z')}$, $(\zeta', z') \in D$. Define $\psi_{(\zeta, z), (\zeta', z')} := \varphi_{(\zeta', z')} \circ \varphi_{(\zeta', z')}^{-1}(\zeta, z)$. Then, take $f \in U_\varepsilon(\zeta, z)$, and observe that $f \circ \psi_{(\zeta, z), (\zeta', z')} \in U_\varepsilon(0, ie_\Omega)$, since $B_{(\zeta, z)}^{\mathbf{s}_2} \circ \psi_{(\zeta, z), (\zeta', z')} = \Delta_\Omega^{\mathbf{s}_2}(\rho(\zeta, z)) B_{(0, ie_\Omega)}^{\mathbf{s}_2}$ (use Lemma 4.12 again). Applying STEP I to $f \circ \psi_{(\zeta, z), (\zeta', z')}$ then yields the result, by means of another application of Lemma 4.12. \square

Lemma 7.12. *Take $p \in]1, \infty[$, $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3 \in \mathbb{R}^r$, and $\mu_1, \mu_2 \in \mathcal{M}_+(D)$ such that the following hold:*

- (1) *the mapping $M_1(\mu_j) \in L_{\mathbf{b} + \mathbf{d} - \mathbf{s}_j}^{\infty, \infty}(D)$ for $j = 1, 2$;*
- (2) $\frac{1}{p} \mathbf{s}_1 + \frac{1}{p} \mathbf{s}_2 = \mathbf{b} + \mathbf{d} - \mathbf{s}_3$;
- (3) $\frac{1}{p} \mathbf{s}_1 \in \frac{1}{2p'} \mathbf{m} + \frac{1}{2p} \mathbf{m}' + (\mathbb{R}_+^*)^r$;
- (4) $\frac{1}{p} \mathbf{s}_2 \in \frac{1}{2p} \mathbf{m} + \frac{1}{2p'} \mathbf{m}' + (\mathbb{R}_+^*)^r$;

Then, the mapping

$$T: C_c(D) \ni f \mapsto \int_D f(\zeta, z) |B_{(\zeta, z)}^{\mathbf{s}_3}| d\mu_1(\zeta, z)$$

induces a continuous linear mapping of $L^p(\mu_1)$ into $L^p(\mu_2)$.

Proof. Define

$$T': C_c(D) \ni f \mapsto \int_D f(\zeta, z) |B_{(\zeta, z)}^{\mathbf{s}_3}| d\mu_2(\zeta, z).$$

Observe that our assumptions and [10, Theorem 2.47 and Corollary 2.49] imply that for every $\mathbf{s}', \mathbf{s}_5 \in \mathbb{R}^r$ there is a constant $C_1 > 0$ such that

$$T(\Delta_\Omega^{\mathbf{s}'} \circ \rho)(\zeta, z) \leq C_1 \|B_{(\zeta, z)}^{\mathbf{s}_3}\|_{A_{\mathbf{s}_1 + p' \mathbf{s}'}^{1, 1}(D)}$$

and

$$T(\Delta_\Omega^{ps_5} \circ \rho)(\zeta, z) \leq C_1 \|B_{(\zeta, z)}^{s_3}\|_{A_{s_2+ps_5}^{1,1}(D)}$$

for every $(\zeta, z) \in D$. In addition, by [10, Proposition 2.41], there are two constants $C_2, C_3 > 0$ such that

$$\|B_{(\zeta, z)}^{s_3}\|_{A_{s_1+p's'}^{1,1}(D)} = C_2 \Delta_\Omega^{s_1+s_3+p's'-(b+d)}(\rho(\zeta, z))$$

and

$$\|B_{(\zeta, z)}^{s_3}\|_{A_{s_2+ps_5}^{1,1}(D)} = C_3 \Delta_\Omega^{s_2+s_3+ps_5-(b+d)}(\rho(\zeta, z))$$

for every $(\zeta, z) \in D$ if and only if the following conditions are satisfied:

- (i) $p's' \in \frac{1}{2}\mathbf{m} - \mathbf{s}_1 + (\mathbb{R}_+^*)^r$;
- (ii) $\mathbf{s}_3 \in \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}' - (\mathbb{R}_+^*)^r$;
- (iii) $p's' \in \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}' - \mathbf{s}_1 - \mathbf{s}_3 - (\mathbb{R}_+^*)^r$;
- (iv) $ps_5 \in \frac{1}{2}\mathbf{m} - \mathbf{s}_2 + (\mathbb{R}_+^*)^r$;
- (v) $ps_5 \in \mathbf{b} + \mathbf{d} - \frac{1}{2}\mathbf{m}' - \mathbf{s}_2 - \mathbf{s}_3 - (\mathbb{R}_+^*)^r$.

In addition,

$$\mathbf{s}_1 + \mathbf{s}_3 + p's' - (\mathbf{b} + \mathbf{d}) = p's_5 \quad \text{and} \quad \mathbf{s}_2 + \mathbf{s}_3 + ps_5 - (\mathbf{b} + \mathbf{d}) = ps_4$$

if and only if

$$\mathbf{s}_5 = \mathbf{s}_4 + \frac{1}{p}(\mathbf{b} + \mathbf{d} - \mathbf{s}_2 - \mathbf{s}_3) \quad \text{and} \quad \frac{1}{p'}\mathbf{s}_1 + \frac{1}{p}\mathbf{s}_2 = \mathbf{b} + \mathbf{d} - \mathbf{s}_3.$$

If these conditions are satisfied, then conditions (iv) and (v) become:

- (iv') $ps_4 \in \mathbf{s}_3 - (\mathbf{b} + \mathbf{d}) + \frac{1}{2}\mathbf{m} + (\mathbb{R}_+^*)^r$;
- (v') $ps_4 \in -\frac{1}{2}\mathbf{m}' - (\mathbb{R}_+^*)^r$.

By our assumptions, we may take $\mathbf{s}_4, \mathbf{s}_5$ such that all the preceding conditions hold, so that the assertion follows by means of Schur's lemma (cf., e.g., [21, Lemma of I.2]). \square

Lemma 7.13. *Take $p, q \in]0, \infty[$, with $q < p$, and take $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3 \in \mathbb{R}^r$ such that the following hold:*

- (1) $\mathbf{s}_1 - \mathbf{s}_3 \in \left(\frac{1}{2q} - \frac{1}{2p}\right)\mathbf{m} + \frac{1}{2p}\mathbf{m}' + (\mathbb{R}_+^*)^r$;
- (2) $\mathbf{s}_1 + \mathbf{s}_2 - \mathbf{s}_3 \in \frac{1}{q}(\mathbf{b} + \mathbf{d}) - \frac{1}{2p}\mathbf{m} - \left(\frac{1}{2q} - \frac{1}{2p}\right)\mathbf{m}' - (\mathbb{R}_+^*)^r$.

For every $\varepsilon > 0$ and for every $f \in \text{Hol}(D)$, define

$$B_{\varepsilon, f}^{q, s_1, s_2} := \left\{ (\zeta, z) \in D : |f(\zeta, z)| \leq \varepsilon \Delta_\Omega^{(b+d)/q - s_1 - s_2}(\rho(\zeta, z)) \|f B_{(\zeta, z)}^{s_2}\|_{A_{s_1}^{q, q}(D)} \right\}.$$

Then, there is a constant $C > 0$ such that

$$\|\chi_{B_{\varepsilon, f}^{q, s_1, s_2}} f\|_{L_{s_3}^{p, p}(D)} \leq C\varepsilon \|f\|_{A_{s_3}^{p, p}(D)}$$

for every $\varepsilon > 0$ and for every $f \in A_{s_3}^{p, p}(D)$.

This extends [31, Lemma 2], which deals with the case in which D is the unit disc in \mathbb{C} .

Proof. It will suffice to prove that the operator

$$T: f \mapsto \left[(\zeta, z) \mapsto \Delta_\Omega^{\mathbf{b}+\mathbf{d}-q_2(\mathbf{s}_1+\mathbf{s}_2)}(\rho(\zeta, z)) \int_D f(\zeta', z') |B_{(\zeta, z)}^{q_2 s_2}(\zeta', z')| \Delta_\Omega^{q_2 s_1 - (\mathbf{b}+\mathbf{d})}(\rho(\zeta', z')) d\nu_D(\zeta', z') \right]$$

induces a continuous linear mapping of $L_{q_2 s_3}^{p/q, p/q}(D)$ into itself. This follows from Lemma 7.12. \square

Proof of Theorem 7.9. Set $\ell := p/q$ and $\mathbf{s}'' := \ell \mathbf{s}$. Observe that Lemma 7.11 implies that there is a constant $C' > 0$ such that

$$\|f B_{(\zeta, z)}^{\mathbf{s}'}\|_{L^q(\mu)} \geq C' \|f B_{(\zeta, z)}^{\mathbf{s}''}\|_{A_{\mathbf{s}''}^{q, q}(D)}$$

for every $(\zeta, z) \in D$ and for every $f \in \text{Hol}(D)$ such that

$$|f(\zeta, z)| > \varepsilon \Delta_{\Omega}^{(\mathbf{b}+\mathbf{d})/q - \mathbf{s}'' - \mathbf{s}'}(\rho(\zeta, z)) \|f B_{(\zeta, z)}^{\mathbf{s}'}\|_{A_{\mathbf{s}''}^{q, q}(D)},$$

that is, for every $f \in \text{Hol}(D)$ and for every $(\zeta, z) \in D \setminus B_{f, \varepsilon}^{q, \mathbf{s}'', \mathbf{s}'}$, with the notation of Lemma 7.13. In addition, by [10, Proposition 3.2], there is a constant $C'' > 0$ such that

$$\|f\|_{A_{\mathbf{s}'' - (\mathbf{b}+\mathbf{d})/q}^{\infty, \infty}(D)} \leq C'' \|f\|_{A_{\mathbf{s}''}^{q, q}(D)}$$

for every $f \in A_{\mathbf{s}''}^{q, q}(D)$, so that

$$\|f B_{(\zeta, z)}^{\mathbf{s}'}\|_{L^q(\mu)} \geq \frac{C'}{C''} \Delta_{\Omega}^{\mathbf{s}' + \mathbf{s}'' - (\mathbf{b}+\mathbf{d})/q}(\rho(\zeta, z)) |f(\zeta, z)|$$

for every $f \in \text{Hol}(D)$ and for every $(\zeta, z) \in D \setminus B_{f, \varepsilon}^{q, \mathbf{s}'', \mathbf{s}'}$. By Lemma 7.13, we may choose ε so small that

$$\|f\|_{A_{\mathbf{s}}^{p, p}(D)} \leq 2 \|\chi_{D \setminus B_{f, \varepsilon}^{q, \mathbf{s}'', \mathbf{s}'}} f\|_{L_{\mathbf{s}}^{p, p}(D)}$$

for every $f \in A_{\mathbf{s}}^{p, p}(D)$, provided that

$$\mathbf{s}' + \mathbf{s}'' - \mathbf{s} \in \frac{1}{q}(\mathbf{b} + \mathbf{d}) - \frac{1}{2p}\mathbf{m} - \left(\frac{1}{2q} - \frac{1}{2p}\right)\mathbf{m}' - (\mathbb{R}_+^*)^r.$$

Therefore,

$$\|f\|_{A_{\mathbf{s}}^{p, p}(D)} \leq \frac{2}{C' C''} \left\| (\zeta, z) \mapsto \|f B_{(\zeta, z)}^{\mathbf{s}'}\|_{L^q(\mu)} \right\|_{L_{\mathbf{s} - \mathbf{s}' - \mathbf{s}'' + (\mathbf{b}+\mathbf{d})/q}^{p, p}(D)}$$

for every $f \in A_{\mathbf{s}}^{p, p}(D)$. Hence, it will suffice to show that the linear mapping

$$T: f \mapsto \left[(\zeta, z) \mapsto \int_D f |B_{(\zeta, z)}^{q \mathbf{s}'}| d\mu \right]$$

induces a continuous linear mapping from $L^{p/q}(\mu)$ into $L^{p/q}((\Delta_{\Omega}^{p[\mathbf{s} - \mathbf{s}' - \mathbf{s}'' + (1/q - 1/p)(\mathbf{b}+\mathbf{d})]} \circ \rho) \cdot \nu_D)$. By Lemma 7.12, this is the case if the following conditions are satisfied:

- (i) $\frac{p}{\ell} \mathbf{s} \in \frac{1}{2\ell} \mathbf{m} + \frac{1}{2\ell} \mathbf{m}' + (\mathbb{R}_+^*)^r$;
- (ii) $q \mathbf{s}' + \frac{p}{\ell} \mathbf{s} \in \mathbf{b} + \mathbf{d} - \frac{1}{2\ell} \mathbf{m} + \frac{1}{2\ell} \mathbf{m}' - (\mathbb{R}_+^*)^r$.

This is the case if \mathbf{s}' is sufficiently small. \square

Finally, we provide a sufficient condition for a measure $\mu \in \mathcal{M}_+(D)$ to be a reverse Carleson, but not necessarily Carleson, measure for the weighted Bergman space $A_{\mathbf{s}}^{p, q}(D)$. A necessary condition (for sampling measures only) has already been provided in Proposition 7.2.

Theorem 7.14. *Take $p_1, q_1, p \in]0, \infty]$ with $p < \infty$, $R_0 > 1$, and $\mathbf{s} \in \frac{1}{2q_1} \mathbf{m} + (\mathbb{R}_+^*)^r$ if $q_1 < \infty$, while $\mathbf{s} \in \mathbb{R}_+^r$ if $q_1 = \infty$. Define $p^* := p_1/(p - p_1)_+$ and $q^* := q_1/(p - q_1)_+$.*

Then, there are $\delta_0 > 0$ and a constant $C > 0$ such that the following hold. For every (δ, R) -lattice $(\zeta_{j,k}, z_{j,k})_{j \in J, k \in K}$ on D , with $\delta \in]0, \delta_0]$ and $R \in]1, R_0]$, for every Borel partition $(B_{j,k})_{(j,k) \in J \times K}$ of D such that

$$B((\zeta_{j,k}, z_{j,k}), \delta) \subseteq B_{j,k} \subseteq B((\zeta_{j,k}, z_{j,k}), R\delta)$$

for every $(j, k) \in J \times K$, and for every $\mu \in \mathcal{M}_+(D)$ such that

$$C' := \left\| \left(\frac{\Delta_{\Omega}^{p[s-(\mathbf{b}+\mathbf{d})/p_1]}(h_k)}{\mu(B_{j,k})} \right)_{j,k} \right\|_{\ell^{p^*,q^*}(J,K)} < \infty,$$

one has

$$\|f\|_{A_{\mathbf{s}}^{p_1,q_1}(D)} \leq C C'^{1/p} \delta^{(2n+m)/p_1+m/q_1} \|f\|_{L^p(\mu)}$$

for every $f \in A_{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p_1}^{\infty,\infty}(D)$, where $h_k := \rho(\zeta_{j,k}, z_{j,k})$ for every $(j, k) \in J \times K$.

Remark 7.15. We point out that the condition $C' < \infty$ is sufficient, but far from being necessary. As a matter of fact, using Theorem 7.3 one may easily construct reverse Carleson measures which vanish on any given compact subset of D . Moreover, the condition $f \in A_{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p_1}^{\infty,\infty}(D)$ can be relaxed (in the spirit of [10, Theorem 3.23]) but not eliminated, as observed in [24].

Proof. By [10, Theorem 3.23], we may take $\delta_0 > 0$ and $C_1 > 0$ so that, if we define

$$S_- : \text{Hol}(D) \ni f \mapsto \left(\Delta_{\Omega}^{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p_1}(h_k) \min_{\overline{B}((\zeta_{j,k}, z_{j,k}), R\delta)} |f| \right)_{j,k} \in \mathbb{R}_+^{J \times K},$$

then

$$\frac{1}{C_1} \|f\|_{A_{\mathbf{s}}^{p_1,q_1}(D)} \leq \delta^{(2n+m)/p_1+m/q_1} \|S_- f\|_{\ell^{p_1,q_1}(J,K)} \leq C_1 \|f\|_{A_{\mathbf{s}}^{p_1,q_1}(D)}$$

for every $f \in A_{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p_1}^{\infty,\infty}(D)$ and for every lattice as in the statement. Then, take $f \in A_{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p_1}^{\infty,\infty}(D)$, and observe that, by Hölder's inequality,

$$\begin{aligned} & \left\| \left(\frac{\Delta_{\Omega}^{p[s-(\mathbf{b}+\mathbf{d})/p_1]}(h_k)}{\mu(B_{j,k})} \right)_{j,k} \right\|_{\ell^{p^*,q^*}(J,K)}^{1/p} \|f\|_{L^{q_2}(\mu)} \\ & \geq \left\| \left(\frac{\Delta_{\Omega}^{\mathbf{s}-(\mathbf{b}+\mathbf{d})/p_1}(h_k)}{\mu(B_{j,k})^{1/p}} \right)_{j,k} \right\|_{\ell^{p^*,p q^*}(J,K)} \left\| (\Delta_{\Omega}^{(\mathbf{b}+\mathbf{d})/p_1-\mathbf{s}}(h_k) \mu(B_{j,k})^{1/p} (S_- f)_{j,k})_{j,k} \right\|_{\ell^p(J \times K)} \\ & \geq \|S_- f\|_{\ell^{p_1,q_1}(J,K)} \\ & \geq \frac{1}{C_1 \delta^{(2n+m)/p_1+m/q_1}} \|f\|_{A_{\mathbf{s}}^{p_1,q_1}(D)}, \end{aligned}$$

whence the result. \square

We conclude this section with some rather ‘pathological’ examples of non-Carleson reverse Carleson measures for the spaces $A_{\mathbf{s}}^{p,q}(\mathbb{C}_+)$ which fail to satisfy the preceding sufficient condition.

Remark 7.16. Take $q \in]0, \infty]$ and $p \in]0, \infty[$. In addition, fix $a < b$ in \mathbb{R} and $\nu \in \mathcal{M}_+(\mathbb{R}_+^*)$ such that $\nu([0, 1]) = \infty$. Define

$$\mu := (\chi_{[a,b]} \cdot \mathcal{H}^1) \otimes \nu,$$

and observe that μ is a positive Radon measure on $\mathbb{C}_+ \cong \mathbb{R} \times \mathbb{R}_+^*$. Take a non-zero $f \in H^q$, and let us prove that $\|f\|_{L^p(\mu)} = \infty$. Indeed, the function

$$f^* : \mathbb{R} \ni x \mapsto \lim_{y \rightarrow 0^+} f(x + iy) \in \mathbb{C}$$

is well-defined and non-zero almost everywhere (cf. [16, Corollary to Theorem 11.1]). Hence, by Fatou's lemma,

$$\liminf_{y \rightarrow 0^+} \int_a^b |f(x + iy)|^p dx \geq \int_a^b |f^*(x)|^p dx > 0,$$

so that clearly

$$\int_{\mathbb{C}_+} |f|^p d\mu = +\infty.$$

By the arbitrariness of f , it is clear that μ is (trivially) reverse p -Carleson for H^q .

In order to extend the preceding example to the spaces $A_s^{p,q}(\mathbb{C}_+)$ for $s > 0$, we need the following lemma.

Lemma 7.17. *Take a non-zero $u \in \mathcal{S}'(\mathbb{R})$. If the Fourier transform of u is supported in a half-line, then $\text{Supp}(u) = \mathbb{R}$.*

Proof. Up to replacing u with $e^{i\xi \cdot} u(\varepsilon \cdot)$ for some $\xi \in \mathbb{R}$ and some $\varepsilon \in \{-1, 1\}$, we may assume that the Fourier transform of u is supported in \mathbb{R}_+ . Now, take a non-zero $g \in \mathcal{S}(\mathbb{R})$ whose Fourier transform is supported in \mathbb{R}_+ , and observe that gu is non-zero, and has a Fourier transform supported in \mathbb{R}_+ , thanks to [36, Theorems XIII and XIV of Chapter VI]. Now, fix $\varphi \in C_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \varphi(x) dx = 1$ and $\text{Supp}(\varphi) \subseteq [-1, 1]$, and define $\varphi_j := 2^j \varphi(2^j \cdot)$ for every $j \in \mathbb{N}$. Then, it is easily verified that $(gu) * \varphi_j$ belongs to $\mathcal{S}(\mathbb{R})$ and that its Fourier transform is supported in \mathbb{R}_+ for every $j \in \mathbb{N}$. In addition, $(gu) * \varphi_j \rightarrow gu$ in $\mathcal{S}'(\mathbb{R})$, so that $(gu) * \varphi_j \neq 0$ if j is sufficiently large. Therefore, [16, Corollary to Theorem 11.1, and Theorem 11.9] imply that $(gu) * \varphi_j$ is non-zero almost everywhere if j is sufficiently large. Now, assume by contradiction that $\text{Supp}(u) \neq \mathbb{R}$, and take $a, b \in \mathbb{R}$ such that $a < b$ and u vanishes on $]a, b[$. Then, clearly $(gu) * \varphi_j$ vanishes on $]a + 2^{-j}, b - 2^{-j}[$, which is not empty if $b - a > 2^{1-j}$, that is, if j is sufficiently large: contradiction. \square

Remark 7.18. Take $p_1, q_1 \in]0, \infty]$ and $s, p \in]0, \infty[$. In addition, fix $a < b$ in \mathbb{R} and $\nu \in \mathcal{M}_+(\mathbb{R}_+^*)$ such that $\nu([0, 1]) = \infty$. Define

$$\mu := (\chi_{[a,b]} \cdot \mathcal{H}^1) \otimes \nu,$$

and observe that μ is a positive Radon measure on $\mathbb{C}_+ \cong \mathbb{R} \times \mathbb{R}_+^*$. Take a non-zero $f \in A_s^{p_1, q_1}$, and let us prove that $\|f\|_{L^q(\mu)} = \infty$. Take $(g^{(\varepsilon)})$ as in [10, Lemma 1.22], and observe that $g^{(\varepsilon)} f \in A_{s+(1/p_1-1)_+}^{1,1}$ for every $\varepsilon > 0$ (cf. [10, Proposition 3.2]). Hence, [7, Theorem 1.7] implies that $(g^{(\varepsilon)} f)_y$ converges to some non-zero f_0 in $\mathcal{S}'(\mathbb{R})$, for $y \rightarrow 0^+$, and that the support of the Fourier transform of f_0 is contained in \mathbb{R}_+ . Therefore, Lemma 7.17 implies that $\text{Supp}(f_0) = \mathbb{R}$, so that clearly

$$0 < \liminf_{y \rightarrow 0^+} \int_a^b |(g^{(\varepsilon)} f)(x + iy)|^q dx \leq \liminf_{y \rightarrow 0^+} \int_a^b |f(x + iy)|^q dx.$$

Hence,

$$\int_{\mathbb{C}_+} |f|^q d\mu = +\infty.$$

By the arbitrariness of f , it is clear that μ is (trivially) reverse p -Carleson for $A_s^{p_1, q_1}$.

8. DECLARATIONS

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