TESI DI DOTTORATO DI RICERCA IN CO-TUTELA

Università degli studi di Milano
Corso di Scienze matematiche
Codice R32 - Dottorati
Ciclo XXXIV

Université de Poitiers

Dipartimento di Matematica
"Federigo Enriques"

Linear systems on irreducible holomorphic symplectic manifolds

Settore MAT/03

Dottorando:
Simone Novario
Matricola: R12353

Direttore di tesi:
Lambertus Van Geemen

Direttore di tesi:
Samuel Boissière

Coordinatore corso dottorato:
Vieri Mastropietro

Anno Accademico 2020/2021

Thèse

Pour l'obtention du Grade de

Docteur de l'Université de Poitiers
Faculté des Sciences Fondamentales et Appliqués
(Diplôme National - Arrêté du 25 mai 2016)

École Doctorale Sciences et Ingénierie des Systèmes, Mathématiques, Informatique (ED SISMI)

Cotutelle : Università degli studi di Milano (Italie)

Spécialité :
Mathématiques

Présentée par :
Simone Novario

Linear systems on irreducible holomorphic symplectic manifolds

---

Directeurs de thèse :
Samuel Boissière, Lambertus Van Geemen

Sintesi

In questa tesi studiamo alcuni sistemi lineari completi associati a divisori di schemi di Hilbert di 2 punti su una superficie K3 proiettiva complessa con gruppo di Picard di rango 1, e le mappe razionali indotte. Queste varietà sono chiamate quadrati di Hilbert su superfici K3 generiche, e sono esempi di varietà irriducibili olomorfe simplettiche (varietà IHS).

Nella prima parte della tesi, usando la teoria dei reticoli, gli operatori di Nakajima e il modello di Lehn–Sorger, diamo una base per il sottospazio vettoriale dell'anello di coomologia singolare a coefficienti razionali generato dalle classi di Hodge razionali di tipo $(2,2)$ sul quadrato di Hilbert di una qualsiasi superficie K3 proiettiva. In seguito sfruttiamo un teorema di Qin e Wang insieme a un risultato di Ellingsrud, Göttsche e Lehn per ottenere una base del reticolo delle classi di Hodge integrali di tipo $(2,2)$ sul quadrato di Hilbert di una qualsiasi superficie K3 proiettiva.

Nella seconda parte della tesi studiamo il problema seguente: se X è il quadrato di Hilbert di una superficie K3 generica che ammette un divisore ampio D con $q_X(D) = 2$, dove q_X è la forma quadratica di Beauville–Bogomolov–Fujiki, descrivere geometricamente la mappa razionale indotta dal sistema lineare completo $|D|$. Il risultato principale della tesi mostra che tale X, tranne nel caso del quadrato di Hilbert di una superficie quartica generica di $\mathbb{P}^3$, è una doppia EPW sestica, cioè il ricoprimento doppio di una EPW sestica, una ipersuperficie normale di $\mathbb{P}^5$, ramificato nel suo luogo singolare. Inoltre la mappa razionale indotta da $|D|$ coincide proprio con tale ricoprimento doppio. Gli strumenti principali per ottenere questo risultato sono la descrizione del reticolo delle classi integrali di Hodge di tipo $(2,2)$ della prima parte della tesi e l'esistenza di un'involuzione anti-simplettica su tali varietà per un teorema di Boissière, Cattaneo, Nieper-Wißkirchen e Sarti.

Abstract

In this thesis we study some complete linear systems associated to divisors of Hilbert schemes of 2 points on complex projective K3 surfaces with Picard group of rank 1, together with the rational maps induced. We call these varieties Hilbert squares of generic K3 surfaces, and they are examples of irreducible holomorphic symplectic (IHS) manifold.

In the first part of the thesis, using lattice theory, Nakajima operators and the model of Lehn–Sorger, we give a basis for the subvector space of the singular cohomology ring with rational coefficients generated by rational Hodge classes of type $(2,2)$ on the Hilbert square of any projective K3 surface. We then exploit a theorem by Qin and Wang together with a result by Ellingsrud, Göttsche and Lehn to obtain a basis of the lattice of integral Hodge classes of type $(2,2)$ on the Hilbert square of any projective K3 surface.

In the second part of the thesis we study the following problem: if X is the Hilbert square of a generic K3 surface admitting an ample divisor D with $q_X(D) = 2$, where q_X is the Beauville–Bogomolov–Fujiki form, describe geometrically the rational map induced by the complete linear system $|D|$. The main result of the thesis shows that such an X, except on the case of the Hilbert square of a generic quartic surface of $\mathbb{P}^3$, is a double EPW sextic, i.e., the double cover of an EPW sextic, a normal hypersurface of $\mathbb{P}^5$, ramified over its singular locus. Moreover, the rational map induced by $|D|$ is a morphism and coincides exactly with this double covering. The main tools to obtain this result are the description of integral Hodge classes of type $(2,2)$ of the first part of the thesis and the existence of an anti-symplectic involution on such varieties due to a theorem by Boissière, Cattaneo, Nieper-Wißkirchen and Sarti.

Keywords: Algebraic geometry; Hilbert schemes; Irreducible holomorphic symplectic manifolds; Hodge classes; Automorphism group; Algebraic cycles; Linear systems

Résumé

Dans cette thèse, nous étudions certains systèmes linéaires complets associés aux diviseurs des schémas de Hilbert de 2 points sur des surfaces K3 projectives complexes avec groupe de Picard de rang 1, et les fonctions rationnelles induites. Ces variétés sont appelées carrés de Hilbert sur des surfaces K3 génériques, et sont un exemple de variété symplectique holomorphe irréductible (variété IHS).

Dans la première partie de la thèse, en utilisant la théorie des réseaux, les opérateurs de Nakajima et le modèle de Lehn–Sorger, nous donnons une base pour le sous-espace vectoriel de l'anneau de cohomologie singulière à coefficients rationnels engendré par les classes de Hodge rationnels de type $(2,2)$ sur le carré de Hilbert de toute surface K3 projective. Nous exploitons ensuite un théorème de Qin et Wang ainsi qu'un résultat de Ellingsrud, Göttsche et Lehn pour obtenir une base du réseau des classes de Hodge intégraux de type $(2,2)$ sur le carré de Hilbert d'une surface K3 projective quelconque.

Dans la deuxième partie de la thèse, nous étudions le problème suivant : si X est le carré de Hilbert d'une surface K3 générique tel que X admet un diviseur ample D avec $q_X(D) = 2$, où q_X est la forme quadratique de Beauville–Bogomolov–Fujiki, on veut décrire géométriquement la fonction rationnelle induite par le système linéaire complet $|D|$. Le résultat principal de la thèse montre qu'une telle X, sauf dans le cas du carré de Hilbert d'une surface quartique générique de $\mathbb{P}^3$, est une double sextique EPW, c'est-à-dire le revêtement double d'une sextique EPW, une hypersurface normale de $\mathbb{P}^5$, ramifié sur son lieu singulier. En plus la fonction rationnelle induite par $|D|$ est exactement ce revêtement double. Les outils principaux pour obtenir ce résultat sont la description des classes de Hodge intégraux de type $(2,2)$ de la première partie de la thèse et l'existence d'une involution anti-symplectique sur de telles variétés par un théorème de Boissière, Cattaneo, Nieper-Wißkirchen et Sarti.

Mots clés: Géométrie algébrique, Variétés symplectiques, Schémas de Hilbert, Groupes d'automorphismes, Cycles algébriques.

Acknowledgements

First of all, I would like to thank my advisor in Poitiers, Samuel Boissière, for having proposed me this nice subject of thesis, for all the help he gave me during this three years, for the enthusiasm and the passion he showed and shared with me. I thank my advisor in Milano, Bert van Geemen, for the opportunity he gave me, for the time he dedicated to me every time I needed and for his support. I deeply thank Pietro Beri, for all the conversations and discussions we had (and we are still having) almost every day during these three years. I thank Alessandra Sarti, and all the people working at LMA, for the warm welcome I received. I also want to thank Michela Artebani, Alice Garbagnati and Ángel David Ríos Ortiz for the interest they showed in my work and for the help they gave me. I thank Simon Brandhorst for some useful explanations and clarifications on the software Sage and on the code of Appendix A. I also thank the two referees of the thesis for the remarks they shared with me and all the Professors who have accepted to be part of the thesis committee.

I thank all the friends who I had the chance to know in Poitiers: Irene, Pietro, Camille, Paolo, Carlos, Antoine, Romain, Rachad, Meghdad, Nasim. I thank all my fellow PhD students in Milano, who shared with me this beautiful experience. I thank Stefania Leonardi, for the great help he gave me with all the non-mathematical and technical part of these three years.

I was economically supported by the Università Italo-Francese (Bando Vinci 2019, Capitolo II: Contributi di mobilità per tesi di Dottorato in cotutela): I would like to express my gratitude.

"I never quite realized...how beautiful this world is..."
From "NieR: Automata"

A mio zio Roberto

Contents

Introduction (English)

A classical result in the theory of complex projective K3 surfaces says that if a K3 surface admits an ample divisor D with $D^2 = 2$ with respect to the intersection product, then the complete linear system $|D|$ is basepoint free and the morphism that it induces is a double cover of the plane $\mathbb{P}^2$ ramified on a sextic curve. See [SD74] for details.

It is natural to study a similar problem for a projective IHS (Irreducible Holomorphic Symplectic) manifold of dimension $2n$ with $n \geq 2$, a sort of higher dimensional generalization of K3 surfaces. By definition an IHS manifold X is a compact simply connected Kähler manifold with $H^{2,0}(X) \cong \mathbb{C} \cdot \sigma_X$ generated by a closed nowhere vanishing holomorphic $(2,0)$-form σ_X, called symplectic form. The complex dimension of an IHS manifold is necessarily even: the only IHS manifolds of dimension 2 are K3 surfaces, as shown by the Enriques–Kodaira classification of compact complex surfaces. The interest in these varieties has been increasing thanks to the Beauville–Bogomolov decomposition theorem, see [Bea83b]: up to a finite cover, any compact Kähler manifold with trivial first Chern class is the product of complex tori, irreducible Calabi–Yau manifolds and IHS manifolds. These varieties have been studied in several areas of mathematics and physics, for instance in differential geometry, string theory and mirror symmetry. Our approach is the one of complex algebraic geometry. In this thesis we focus on Hilbert schemes of 2 points on a K3 surface, also known as Hilbert squares of K3 surfaces, the first example of IHS manifold other than K3 surfaces to be found, see [Fuj83]. We denote by $S^{[2]}$ the Hilbert square of a K3 surface S. In particular we consider *generic* K3 surfaces, i.e., projective K3 surfaces whose Picard group is generated by the class of an ample divisor. We say that a generic K3 surface has degree $2t$ if its Picard group is generated by an ample divisor H with $H^2 = 2t$ with respect to the intersection form, where t is a non-zero positive integer. We denote a generic K3 surface of degree $2t$ by S_{2t} and its Hilbert square by $S_{2t}^{[2]}$. An important tool in the study of IHS manifolds is given by lattice theory: indeed the second cohomology group $H^2(X, \mathbb{Z})$ of an IHS manifold X admits an integral quadratic form q_X, which is known as *Beauville–Bogomolov–Fujiki* (BBF) *form*. This coincides with the intersection form in the case of K3 surfaces. We denote by $(\cdot\, , \cdot)$ the bilinear form induced by q_X. If $X = S^{[2]}$ is the Hilbert square of a K3 surface S, then $H^2(X, \mathbb{Z})$ together with q_X is a lattice of rank $\mathrm{rk}(H^2(X, \mathbb{Z})) = 23$ isomorphic to $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle$, where U is the hyperbolic plane, $E_8(-1)$ is the unique negative-definite, even, unimodular lattice of rank 8, and $\langle -2 \rangle$ is a one-rank lattice whose quadratic form takes value -2 on the generator which can be identified with the line bundle $\delta \in \mathrm{Pic}(X)$ such that $2\delta = [E]$ is the class of the exceptional divisor of the Hilbert–Chow morphism $S^{[2]} \to S^{(2)}$, where $S^{(2)}$ is the quotient of $S \times S$ by the involution $(p,q) \mapsto (q,p)$, with $p,q \in S$. Since the Picard group of an IHS manifold, which is isomorphic to its Néron–Severi group, embeds in the second integral cohomology group, the line bundle δ can be seen as an element in $H^2(S^{[2]},\mathbb{Z})$, which admits a decomposition of the following form:

$$H^2(S^{[2]},\mathbb{Z}) = H^2(S,\mathbb{Z}) \oplus \mathbb{Z}\delta.$$

Similarly we have $\mathrm{Pic}(S^{[2]}) = \mathrm{Pic}(S) \oplus \mathbb{Z}\delta$. This implies that a line bundle on S induces a line bundle on $S^{[2]}$. If S_{2t} is a generic K3 surface with $\mathrm{Pic}(S_{2t}) = \mathbb{Z}H$, $H^2 = 2t$, we denote by $h \in \mathrm{Pic}(S_{2t}^{[2]})$ the line bundle induced by H in $X := S_{2t}^{[2]}$. In particular we have $\mathrm{Pic}(X) = \mathbb{Z}h \oplus \mathbb{Z}\delta$, with $q_X(h) = 2t$, $q_X(\delta) = -2$ and $(h,\delta) = 0$.

The first problem that we study in this thesis is to determine the lattice of integral Hodge classes of type $(2,2)$ on the Hilbert square of a K3 surface S, which is defined as

$$H^{2,2}(S^{[2]},\mathbb{Z}) := H^4(S^{[2]},\mathbb{Z}) \cap H^{2,2}(S^{[2]}),$$

with integral bilinear form given by the cup product. Hodge classes are usually studied in the context of the so-called *Hodge conjecture*: this states that given a smooth complex projective variety Y, the subspace of $H^{2k}(Y,\mathbb{Q})$ generated by algebraic cycles, i.e., classes which are obtained as fundamental cohomological classes $[Z]$ of subvarieties $Z \subset Y$, coincides with $H^{k,k}(Y,\mathbb{Q})$, which is the set $H^{2k}(Y,\mathbb{Q}) \cap H^{k,k}(Y)$ of rational Hodge classes of type (k,k). The first important result of this thesis is the following: using the theory of Nakajima operators, see [Nak97], [Leh99], we find a basis of the lattice $H^{2,2}(S_{2t}^{[2]},\mathbb{Z})$, where S_{2t} is a generic K3 surface of degree $2t$, cf. Theorem 3.3.17 and Corollary 3.4.11.

Theorem A. *Let $X = S_{2t}^{[2]}$ be the Hilbert square of a generic projective K3 surface S_{2t} of degree $2t$, and let $h \in \mathrm{Pic}(X)$ be the line bundle induced by the ample generator of $\mathrm{Pic}(S_{2t})$. Then, denoting by $c_2(X) \in H^4(X,\mathbb{Z})$ the second Chern class of X, we have*

$$H^{2,2}(X,\mathbb{Z}) = \mathbb{Z}h^2 \oplus \mathbb{Z}\frac{h^2 - h\delta}{2} \oplus \mathbb{Z}\left(\frac{1}{8}\delta^2 + \frac{1}{24}c_2(X)\right) \oplus \mathbb{Z}\delta^2,$$

where $\delta \in \mathrm{Pic}(X)$ is the line bundle such that 2δ is the class of the exceptional divisor of the Hilbert–Chow morphism $S_{2t}^{[2]} \to S_{2t}^{(2)}$. Moreover, $H^{2,2}(X,\mathbb{Z})$ is an odd lattice of discriminant $\mathrm{disc}(H^{2,2}(X,\mathbb{Z})) = 84t^3$, and the Gram matrix in the basis given above is the following:

$$\begin{pmatrix} 12t^2 & 6t^2 & 2t & -4t \\ 6t^2 & t(3t-1) & t & -2t \\ 2t & t & 1 & -1 \\ -4t & -2t & -1 & 12 \end{pmatrix}.$$

We also find a basis of the lattice $H^{2,2}(S^{[2]},\mathbb{Z})$ when S is any projective K3 surface with known Picard group, cf. Theorem 3.4.12. We now present the statement, without explaining the notation of Nakajima operators, which will be introduced in Chapter 3.

Theorem B. *Let S be a projective K3 surface with Picard group of rank* $\mathrm{rk}(\mathrm{Pic}(S)) = r$. *Let* $\{b_1, \dots, b_r\}$ *be a basis of* $\mathrm{Pic}(S)$. *Then:*

(i) $\mathrm{rk}(H^{2,2}(S^{[2]}, \mathbb{Z})) = \frac{(r+1)r}{2} + r + 2$.

(ii) A basis of $H^{2,2}(S^{[2]}, \mathbb{Z})$ *is given by the following elements:*

- $\mathfrak{q}_2(b_i)|0\rangle$, *for* $i = 1, \dots, r$,
- $\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle$, *where* $1 \in H^0(S, \mathbb{Z})$ *is the unit and* $x \in H^4(S, \mathbb{Z})$ *is the class of a point.*
- $\frac{1}{2}\left(\mathfrak{q}_1(b_i)^2 - \mathfrak{q}_2(b_i)\right)|0\rangle$, *for* $i = 1, \dots, r$,
- $\mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)|0\rangle$, *for* $1 \leq i < j \leq r$,
- δ^2, *where* $\delta \in \mathrm{Pic}(S^{[2]})$ *is the line bundle such that* 2δ *is the class of the exceptional divisor of the Hilbert–Chow morphism* $S^{[2]} \to S^{(2)}$.

Equivalently, the following is a basis of $H^{2,2}(S^{[2]}, \mathbb{Z})$:

$$\left\{ b_i b_j, \frac{b_i^2 - b_i\delta}{2}, \frac{1}{8}\delta^2 + \frac{1}{24}c_2(S^{[2]}), \delta^2 \right\}_{1 \leq i \leq j \leq r,}$$

where $c_2(S^{[2]}) \in H^4(S^{[2]}, \mathbb{Z})$ *is the second Chern class of* $S^{[2]}$. *Moreover,* $H^{2,2}(S^{[2]}, \mathbb{Z})$ *is an odd lattice.*

Suppose now that $X = S_{2t}^{[2]}$ is the Hilbert square of a generic K3 surface admitting an ample divisor D with $q_X(D) = 2$. This setting generalises to Hilbert squares of generic K3 surfaces the one studied by Saint–Donat in [SD74], and presented in the first lines. As a consequence of the Hirzebruch–Riemann–Roch theorem we obtain $\dim(H^0(X, \mathcal{O}_X(D))) = 6$, cf. Theorem 3.1.9, hence the image of the map induced by the complete linear system $|D|$ is contained in $\mathbb{P}^5$. The main problem of this thesis is the following.

Problem. *Describe the base locus of the complete linear system* $|D|$. *Describe geometrically the map* $\varphi_{|D|} : X \dashrightarrow \mathbb{P}^5$ *induced by* $|D|$.

By a result obtained by Boissière, Cattaneo, Nieper-Wißkirchen and Sarti in [BCNWS16], there exists an anti-symplectic involution ι which generates the group $\mathrm{Aut}(X)$ of biregular automorphisms on X, and ι is such that $\iota^*[D] = [D]$ in the Néron–Severi group $\mathrm{NS}(X)$. Here anti-symplectic means that $\iota^*\sigma_X = -\sigma_X$, where $\sigma_X \in H^0(X, \Omega_X^2)$ is a symplectic form of X. The paper [O'G08b], where O'Grady gives a classification, up to deformation equivalence, of *numerical* $K3^{[2]}$, will play an important role in this thesis, in particular in Chapter 5. A numerical $K3^{[2]}$ is by definition an IHS manifold M which admits an isomorphism of abelian groups

$$\psi : H^2(M, \mathbb{Z}) \to H^2(S^{[2]}, \mathbb{Z})$$

such that

$$\int_M \alpha^4 = \int_{S^{[2]}} \psi(\alpha)^4 \qquad \text{for every } \alpha \in H^2(M, \mathbb{Z}),$$

where S is a K3 surface. In particular he showed that a numerical $K3^{[2]}$ is deformation equivalent to one of the following.

(i) An IHS manifold Z of dimension 4 carrying an anti-symplectic involution $\iota : Z \to Z$ such that the quotient $Z/\langle\iota\rangle$ is isomorphic to an EPW sextic $Y \subset \mathbb{P}^5$, hence Z is a so-called *double EPW sextic.*

(ii) An IHS manifold Z of dimension 4 admitting a rational map $f : Z \dashrightarrow \mathbb{P}^5$ which is birational onto its image Y, with $6 \leq \deg(Y) \leq 12$.

Here an EPW sextic is a normal hypersurface of $\mathbb{P}^5$ of degree 6, first studied by Eisenbud, Popescu and Walters in [EPW01], and a double EPW sextic is a double cover of an EPW sextic ramified in its singular locus. O'Grady showed that a double EPW sextic is an IHS fourfold of $K3^{[2]}$-type, see [O'G06]. The similarity between our problem and the one studied by O'Grady in [O'G08b] is given by the fact that O'Grady proved the following: a numerical $K3^{[2]}$ is deformation equivalent to an IHS manifold Z of $K3^{[2]}$-type such that $\mathrm{Pic}(Z)$ is generated by the class of an ample divisor $H \in \mathrm{Pic}(Z)$ with $q_Z(H) = 2$, hence the complete linear system $|H|$ induces a rational map $\varphi_{|H|} : Z \dashrightarrow \mathbb{P}^5$.

Our strategy is to follow [O'G08b], using the anti-symplectic involution which generates $\mathrm{Aut}(X)$ given by [BCNWS16] in order to get as much information as possible on the geometry of the complete linear system. In the case studied by O'Grady, an important result is the so-called *irreducibility property* of $|H|$, where $H \in \mathrm{Div}(Z)$ is the ample divisor introduced above: if $D_1, D_2 \in |H|$ are two distinct divisors, then $D_1 \cap D_2$ is a reduced and irreducible surface. The proof of this property is quite easy, for details see [O'G08b, Proposition 4.1]. In our setting, we will show the following similar statement, cf. Theorem 4.6.5.

Theorem C. *Let X be the Hilbert square $S_{2t}^{[2]}$ of a generic K3 surface S_{2t} of degree $2t$. Let $h \in \mathrm{Pic}(X)$ be the line bundle induced by the ample generator of $\mathrm{Pic}(S_{2t})$. Suppose that X admits an ample divisor D with $q_X(D) = 2$. Let $D_1, D_2 \in |D|$ be two distinct divisors.*

(i) If $t = 2$, then the surface $D_1 \cap D_2$ can be reducible. If so, this surface has two irreducible components A and B, whose fundamental cohomological classes in $H^{2,2}(X, \mathbb{Z})$ are the following:

$$[A] = \tfrac{1}{2}h^2 - \tfrac{1}{4}\delta^2 - \tfrac{1}{2}h\delta - \tfrac{1}{12}c_2(X) \in H^{2,2}(X, \mathbb{Z}),$$
$$[B] = \tfrac{1}{2}h^2 + \tfrac{5}{4}\delta^2 - \tfrac{3}{2}h\delta + \tfrac{1}{12}c_2(X) \in H^{2,2}(X, \mathbb{Z}).$$

(ii) If $t \neq 2$, then $D_1 \cap D_2$ is a reduced and irreducible surface.

Hence the irreducibility property is not true when X is the Hilbert square of a generic K3 surface of degree 4, i.e., when $t = 2$: note that $[A]+[B] = h^2-2h\delta+\delta^2$, which coincides with the square, with respect of the cup product, of the class $h - \delta$ of the ample divisor $D \in \mathrm{Div}(S_4)$ with $q_X(D) = 2$. When it holds, the irreducibility property is much harder to prove in our case than in the one studied by O'Grady: the key point of the proof of Theorem C is the explicit description of the lattice $H^{2,2}(X, \mathbb{Z})$ of Hodge classes of type $(2, 2)$ on the Hilbert square of a generic K3 surface given by Theorem A. Once having obtained Theorem C, we can follow the same strategy used by O'Grady in [O'G08b] to study the rational map $\varphi_{|D|} : S_{2t}^{[2]} \dashrightarrow \mathbb{P}^5$ when $t \neq 2$, where S_{2t} is a generic K3 surface of degree $2t$. Let X be the Hilbert square of S_{2t} such that X admits an ample divisor D with $q_X(D) = 2$, so that there is an anti-symplectic involution ι on X.

One of the most interesting facts shown in this thesis is that the existence of this anti-symplectic involution simplifies the solution of the Problem stated above a lot, compared to the one of O'Grady: for instance, this implies that $\varphi_{|D|}$ is finite of *even* degree on its image, cf. Theorem 4.5.11 and Corollary 4.5.12, while in the case of O'Grady this is not necessarily true, which makes the problem be more complicated to study. Let $F := \mathrm{Fix}(\iota)$ be the locus of points on X fixed by ι. Beauville showed in [Bea11] that F is a Lagrangian submanifold, hence it is a smooth surface in our case, in particular its fundamental cohomological class belongs to the lattice $H^{2,2}(X, \mathbb{Z})$ of integral Hodge classes. We now state the main result of this thesis, cf. Theorem 4.5.11 and Theorem 4.6.5.

Theorem D. *Let $X = S_{2t}^{[2]}$ be the Hilbert square of a generic K3 surface S_{2t} of degree $2t$ such that X admits an ample divisor D with $q_X(D) = 2$. Suppose that $t \neq 2$. Let ι be the anti-symplectic involution which generates* $\mathrm{Aut}(X)$ *and let $F = \mathrm{Fix}(\iota)$ be the fixed locus. Then*

$$[F] = 5D^2 - \frac{1}{3}c_2(X) \in H^{2,2}(X, \mathbb{Z}),$$

where $[F]$ is the fundamental cohomological class of F in $H^{2,2}(X, \mathbb{Z})$. Moreover, $\varphi_{|D|} : X \to Y \subset \mathbb{P}^5$ is a morphism, and it is a double cover of an EPW sextic $Y \subset \mathbb{P}^5$, in particular $Y = X/\langle\iota\rangle$ and X is a double EPW sextic.

This solves the Problem given above. Note that $t = 2$ is not considered in the statement of Theorem D. This is exactly the only value for which the irreducibility property of $|D|$ does not hold, as shown by Theorem C. In this case the variety $X := S_4^{[2]}$ is the Hilbert square of a smooth complex quartic surface of $\mathbb{P}^3$, and the morphism $\varphi_{|D|} : X \to Y \subset \mathbb{P}^5$ is finite of degree 6: its image Y is isomorphic to the Grassmannian $\mathbb{G}(1, \mathbb{P}^3)$ of lines in $\mathbb{P}^3$, which is a quadric in $\mathbb{P}^5$. This was studied in detail in [Bea83a], see also [BCNWS16], cf. Section 4.4.1.

The thesis is organised as follows. In Chapter 1 we recall some definitions and results in Complex Algebraic Geometry. First of all we introduce basic notions of positivity in Algebraic Geometry, like Weil and Cartier divisors on a complex variety, the class group and the Picard group, ampleness and nefness of divisors, bigness and pseudoeffectiveness of divisors, the notion of complete linear system associated to a divisor and the rational map induced by a complete linear system. Then we define the so-called Gysin homomorphism, which can be seen as a push-forward map between singular cohomology groups induced by a morphism between complex manifolds. We introduce topological Chern classes of a complex vector bundle over a complex manifold, and we state the Grothendieck–Riemann–Roch theorem. Then main results in lattice theory are presented, and we define Pell equations and Pell-type equations. Finally we give some useful properties of double covers.

In Chapter 2 we give the definition of IHS manifold, together with some examples: we deal with K3 surfaces, recalling some important results, we see details of the construction of the Hilbert square of a K3 surfaces and we introduce double EPW sextics. We then summarize the main notions on deformation theory in the context of IHS manifolds and we state some of the most important properties and results concerning this family of varieties, for instance the local and global Torelli theorems, the surjectivity of the period map and the construction of the Mukai flop, which is a fundamental example of birational map between IHS manifolds. We then introduce the birational Kähler cone and the moving cone of an IHS manifold, and we briefly present the pseudoeffective cone of cycles on IHS manifolds of $K3^{[n]}$-type. We conclude the chapter by stating, without proofs, some useful results on IHS manifolds.

In Chapter 3 we begin by introducing Nakajima operators, following [Nak97] and [Leh99]. We recall the Lehn–Sorger model presented in [LS03], which we use to compute cup products in the ring $H^*(S^{[2]},\mathbb{Z})$, where we denote by S a K3 surface, in terms of Nakajima operators. Then, using a result by Ellingsrud, Göttsche and Lehn in [EGL01], we obtain an explicit description of the second Chern class $c_2(S^{[2]})$ in $H^{2,2}(S^{[2]},\mathbb{Z})$, which we exploit to prove Theorem A and Theorem B. We conclude the chapter with results on integral Hodge classes of type $(3,3)$ on Hilbert squares of K3 surfaces.

In Chapter 4 we introduce the main problem of the thesis, stated above. First of all we recall the Bayer–Macrì theorem, which describes the nef cone, the moving cone and the pseudo-effective cone of the Hilbert square of a generic K3 surface, cf. Theorem 4.1.1. We then show that the Hilbert square of a generic K3 surface is a *Mori dream space*, whose definition is given in Section 4.2. Following [BCNWS16] and [DM19], we present the groups of regular automorphisms and birational automorphisms on $S_{2t}^{[2]}$, cf. Theorem 4.3.1 and Theorem 4.3.2. We then describe $S_4^{[2]}$ and $S_{10}^{[2]}$ geometrically, following [Bea83a] and [O'G05] respectively, and we see some important properties of the locus of points on an IHS manifold fixed by an anti-symplectic involution. Let $X = S_{2t}^{[2]}$ be the Hilbert square of a generic K3 surface S_{2t} such that X admits an ample divisor D with $q_X(D) = 2$: we show that the rational map $\varphi_{|D|}$ induced by the complete linear system $|D|$ factors through the quotient $\pi : X \to X/\langle\iota\rangle$, where we denote by ι the anti-symplectic involution which generates $\mathrm{Aut}(X)$, and we compute the fundamental cohomological class in $H^{2,2}(X,\mathbb{Z})$ of $F = \mathrm{Fix}(\iota)$, the locus of points on X fixed by ι, cf. Theorem 4.5.11. We then finally prove Theorem C, and we briefly discuss how to approach the more general case of a smooth birational model X of the Hilbert square $S_{2t}^{[2]}$ of a generic K3 surface such that X admits an ample divisor D with $q_X(D) = 2$.

In Chapter 5, following the strategy developed by O'Grady in [O'G08b], with remarkable simplifications, we prove Theorem D.

We conclude with Chapter 6, where we present open problems concerning the topics studied in the thesis.

Introduction (Français)

Un résultat classique dans la théorie des surfaces K3 projectives complexes dit que si une surface K3 admet un diviseur ample D avec $D^2 = 2$ par rapport au produit d'intersection, alors le système linéaire complet $|D|$ a lieu de base vide et le morphisme qu'il induit est un revêtementt double du plan $\mathbb{P}^2$ ramifié sur une courbe sextique. Voir [SD74] pour plus de détails.

Il est naturel d'étudier un problème similaire pour une variété symplectique holomorphe irréductible (IHS) projective de dimension $2n$ avec $n \geq 2$, une sorte de généralisation en dimension supérieure des surfaces K3. Par définition, une variété X est IHS si elle est une variété lisse de Kähler compacte simplement connexe avec $H^{2,0}(X) \cong \mathbb{C} \cdot \sigma_X$ engendré par une forme fermée holomorphe $(2,0)$ jamais nulle, appelée forme symplectique. La dimension complexe d'une variété IHS est nécessairement paire : les seules variétés IHS de dimension 2 sont des surfaces K3, comme montré par la classification de Enriques–Kodaira des surfaces complexes compactes. L'intérêt pour ces variétés a augmenté grâce au théorème de décomposition de Beauville–Bogomolov, voir [Bea83b] : à revêtement fini près, toute variété de Kähler compacte avec première classe de Chern triviale est le produit de tores complexes, de variétés Calabi–Yau irréductibles et de variétés IHS irréductibles. Ces variétés ont été étudiées dans plusieurs domaines des mathématiques et de la physique, par exemple en géométrie différentielle, en théorie des cordes et en symétrie miroir. Notre approche est celle de la géométrie algébrique complexe. Dans cette thèse, nous nous concentrons sur les schémas de Hilbert de 2 points sur une surface K3, également connus comme carrés de Hilbert de surfaces K3, qui est le premier exemple de variété IHS de dimension supérieure à 2 à avoir été trouvé, voir [Fuj83]. Si S est une surface K3, nous désignons par $S^{[2]}$ son carré de Hilbert. En particulier, nous considérons des surfaces K3 *génériques*, c'est-à-dire des surfaces K3 projectives dont le groupe de Picard est engendré par la classe d'un diviseur ample. Une surface K3 générique a degré $2t$ si son groupe de Picard est engendré par la classe d'un diviseur ample H avec $H^2 = 2t$ par rapport à la forme d'intersection, où t est un entier positif non nul. Nous désignons par S_{2t} une surface K3 générique de degré $2t$ et son carré de Hilbert par $S_{2t}^{[2]}$. La théorie des réseaux constitue un outil important pour l'étude des variétés IHS : en effet, si X est une variété IHS, son deuxième groupe de cohomologie $H^2(X, \mathbb{Z})$ admet une forme quadratique intégrale q_X, la *forme de Beauville–Bogomolov–Fujiki* (BBF). Celle-ci coïncide avec la forme d'intersection dans le cas des surfaces K3. Nous désignons par $(\cdot\,, \cdot)$ la forme bilinéaire induite par q_X. Si $X = S^{[2]}$ est le carré de Hilbert d'une surface K3, alors $H^2(X, \mathbb{Z})$ et q_X donnent lieu à un réseau de rang $\mathrm{rk}(H^2(X, \mathbb{Z})) = 23$ isomorphe à $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle$, où U est le plan hyperbolique, $E_8(-1)$

est l'unique réseau unimodulaire, pair et défini-négatif de rang 8, et $\langle -2\rangle$ est un réseau de rang 1 dont la forme quadratique prend la valeur -2 sur le générateur. Celui-ci peut être identifié avec le fibré en droites $\delta \in \mathrm{Pic}(X)$ tel que $2\delta = [E]$ est la classe du diviseur exceptionnel du morphisme de Hilbert–Chow $S^{[2]} \to S^{(2)}$, où $S^{(2)}$ est le quotient de $S \times S$ sur l'involution $(p,q) \mapsto (q,p)$, avec $p, q \in S$. Puisque le groupe de Picard d'une variété IHS, qui est isomorphe à son groupe de Néron–Severi, se plonge dans le deuxième groupe de cohomologie intégrale, le fibré en droites δ peut être vu comme un élément dans $H^2(S^{[2]},\mathbb{Z})$, qui admet une décomposition de la forme suivante :

$$H^2(S^{[2]},\mathbb{Z}) = H^2(S,\mathbb{Z}) \oplus \mathbb{Z}\delta.$$

De la même manière, nous avons $\mathrm{Pic}(S^{[2]}) = \mathrm{Pic}(S) \oplus \mathbb{Z}\delta$. Ceci montre qu'un fibré en droites sur S induit un fibré en droites sur $S^{[2]}$. Si S_{2t} est une surface K3 générique avec $\mathrm{Pic}(S_{2t}) = \mathbb{Z}H$, $H^2 = 2t$, nous désignons par $h \in \mathrm{Pic}(S_{2t}^{[2]})$ le fibré en droites induit par H dans $X := S_{2t}^{[2]}$. En particulier, nous avons $\mathrm{Pic}(X) = \mathbb{Z}h \oplus \mathbb{Z}\delta$, avec $q_X(h) = 2t$, $q_X(\delta) = -2$ et $(h,\delta) = 0$.

Soit S une surface K3: le premier problème que nous étudions dans cette thèse est de déterminer le réseau des classes de Hodge intégrales de type $(2,2)$ sur le carré de Hilbert de S, qui est défini comme

$$H^{2,2}(S^{[2]},\mathbb{Z}) := H^4(S^{[2]},\mathbb{Z}) \cap H^{2,2}(S^{[2]}),$$

où la forme bilinéaire intégrale est donnée par le produit cup. Les classes de Hodge sont généralement étudiés dans le contexte de la conjecture de *Hodge*: celle-ci dit que, étant donné une variété projective complexe lisse Y, le sous-espace de $H^{2k}(Y,\mathbb{Q})$ engendré par les cycles algébriques, c'est-à-dire les classes qui sont obtenues comme classes fondamentales en cohomologie $[Z]$ des sous-variétés $Z \subset Y$, coïncide avec $H^{k,k}(Y,\mathbb{Q})$, qui est l'ensemble $H^{2k}(Y,\mathbb{Q}) \cap H^{k,k}(Y)$ des classes de Hodge rationnels de type (k,k). Le premier résultat important de cette thèse est le suivant : en utilisant la théorie des opérateurs de Nakajima, voir [Nak97], [Leh99], nous trouvons une base du réseau $H^{2,2}(S_{2t}^{[2]},\mathbb{Z})$, où S_{2t} est une surface K3 générique de degré $2t$, cf. Theorem 3.3.17 and Corollary 3.4.11.

Théorème A. *Soit $X = S_{2t}^{[2]}$ le carré de Hilbert d'une surface K3 projective générique S_{2t} de degré $2t$, et soit $h \in \mathrm{Pic}(X)$ le fibré en droites induit par le générateur de $\mathrm{Pic}(S_{2t})$. Alors, si $c_2(X) \in H^4(X,\mathbb{Z})$ est la deuxième classe de Chern de X, nous avons*

$$H^{2,2}(X,\mathbb{Z}) = \mathbb{Z}h^2 \oplus \mathbb{Z}\frac{h^2 - h\delta}{2} \oplus \mathbb{Z}\left(\frac{1}{8}\delta^2 + \frac{1}{24}c_2(X)\right) \oplus \mathbb{Z}\delta^2,$$

où $\delta \in \mathrm{Pic}(X)$ est le fibré en droites tel que 2δ est la classe du diviseur exceptionnel du morphisme de Hilbert–Chow $S_{2t}^{[2]} \to S_{2t}^{(2)}$. En plus, $H^{2,2}(X,\mathbb{Z})$ est un réseau impair de discriminant $\mathrm{disc}(H^{2,2}(X,\mathbb{Z})) = 84t^3$, et la matrice de Gram dans la base donnée dessus est:

$$\begin{pmatrix} 12t^2 & 6t^2 & 2t & -4t \\ 6t^2 & t(3t-1) & t & -2t \\ 2t & t & 1 & -1 \\ -4t & -2t & -1 & 12 \end{pmatrix}.$$

Nous trouvons également une base du réseau $H^{2,2}(S^{[2]},\mathbb{Z})$ lorsque S est une surface projective K3 quelconque, cf. Theorem 3.4.12. Nous présentons maintenant l'énoncé, sans expliquer la notation des opérateurs de Nakajima, qui sera introduite dans le Chapitre 3.

Théorème B. *Soit S une surface projective K3 avec un groupe de Picard de rang $\mathrm{rk}(\mathrm{Pic}(S)) = r$. Soit $\{b_1,\dots,b_r\}$ une base de $\mathrm{Pic}(S)$. Alors :*

(i) $\mathrm{rk}(H^{2,2}(S^{[2]},\mathbb{Z})) = \frac{(r+1)r}{2} + r + 2$.

(ii) Une base de $H^{2,2}(S^{[2]},\mathbb{Z})$ est donnée par les éléments suivants :

- $\mathfrak{q}_2(b_i)|0\rangle$, *où* $i = 1,\dots,r$,
- $\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle$, *où* $1 \in H^0(S,\mathbb{Z})$ *est l'unité et* $x \in H^4(S,\mathbb{Z})$ *est la classe d'un point.*
- $\frac{1}{2}\left(\mathfrak{q}_1(b_i)^2 - \mathfrak{q}_2(b_i)\right)|0\rangle$, *où* $i = 1,\dots,r$,
- $\mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)|0\rangle$, *où* $1 \le i < j \le r$,
- δ^2, *où* $\delta \in \mathrm{Pic}(X)$ *est le fibré en droites tel que* 2δ *est la classe du diviseur exceptionnel du morphisme de Hilbert–Chow* $S^{[2]} \to S^{(2)}$.

De façon équivalente, une base pour $H^{2,2}(S^{[2]},\mathbb{Z})$ est donnée par:

$$\left\{b_ib_j, \frac{b_i^2 - b_i\delta}{2}, \frac{1}{8}\left(\delta^2 + \frac{2}{5}q_X^\vee\right), \delta^2\right\}_{1\le i\le j\le r.}$$

En plus, $H^{2,2}(S^{[2]},\mathbb{Z})$ est un réseau impair.

Supposons maintenant que $X = S_{2t}^{[2]}$ est le carré de Hilbert d'une surface K3 générique qui admet un diviseur ample D avec $q_X(D) = 2$. Il s'agit d'une généralisation aux carrés de Hilbert de surfaces K3 génériques du problème étudié par Saint–Donat dans [SD74] et présenté avant. Comme conséquence du théorème de Hirzebruch–Riemann–Roch on obtient $\dim(H^0(X,\mathcal{O}_X(D))) = 6$, cf. Theorem 3.1.9, donc l'image de la fonction induite par le système linéaire complet $|D|$ est contenue dans $\mathbb{P}^5$. Le problème principal de cette thèse est le suivant.

Problem. *Décrire le lieu de base du système linéaire complet $|D|$. Décrire géométriquement la fonction $\varphi_{|D|} : X \dashrightarrow \mathbb{P}^5$ induite par $|D|$.*

Par un résultat obtenu par Boissière, Cattaneo, Nieper-Wißkirchen et Sarti dans [BCNWS16], il existe une involution anti-symplectique ι qui engendre le groupe $\mathrm{Aut}(X)$ des automorphismes biréguliers sur X, et ι est telle que $\iota^*[D] = [D]$ dans le groupe de Néron–Severi $\mathrm{NS}(X)$. Ici, anti-symplectique signifie que $\iota^*\sigma_X = -\sigma_X$, où $\sigma_X \in H^0(X,\Omega_X^2)$ est une forme symplectique de X. L'article [O'G08b], où O'Grady donne une classification, à équivalence par déformation près, des $K3^{[2]}$ *numériques*, jouera un rôle important dans cette thèse, en particulier dans le Chapitre 5. Une $K3^{[2]}$ numérique M est par définition une variété IHS qui admet un isomorphisme de groupes abéliens

$$\psi : H^2(M,\mathbb{Z}) \to H^2(S^{[2]},\mathbb{Z})$$

tel que
$$\int_M \alpha^4 = \int_{S^{[2]}} \psi(\alpha)^4 \qquad \text{pour tout } \alpha \in H^2(M,\mathbb{Z}),$$
où S est une surface K3. En particulier, il a montré qu'une $K3^{[2]}$ numérique est équivalente par déformation à l'une des suivantes.

(i) Une variété IHS, que nous désignons par Z, de dimension 4 qui admet une involution anti-symplectique $\iota : Z \to Z$ telle que le quotient $Z/\langle\iota\rangle$ est isomorphe à une sextique EPW, donc Z est une *double sextique EPW.*

(ii) Une variété IHS, que nous désignons par Z, de dimension 4 qui admet une fonction rationnelle $f : Z \dashrightarrow \mathbb{P}^5$ qui est birationnelle sur son image Y, avec $6 \leq \deg(Y) \leq 12$.

Ici, une sextique EPW est une hypersurface normale de $\mathbb{P}^5$ de degré 6, étudiée pour la première fois par Eisenbud, Popescu et Walters dans [EPW01], et une double sextique EPW est un revêtement double d'une sextique EPW ramifié dans son lieu singulier. O'Grady a montré qu'une double sextique EPW est une variété IHS de dimension 4 de type $K3^{[2]}$, voir [O'G06]. La similarité entre notre problème et celui étudié par O'Grady dans [O'G08b] est donnée par le fait que O'Grady a prouvé le suivant : une $K3^{[2]}$ numérique est équivalent par déformation à une variété IHS de type $K3^{[2]}$, que nous désignons par Z, tel que $\mathrm{Pic}(Z)$ est engendré par la classe d'un diviseur ample $H \in \mathrm{Pic}(Z)$ avec $q_Z(H) = 2$, donc le système linéaire complet $|H|$ induit une fonction rationnelle $\varphi_{|H|} : Z \dashrightarrow \mathbb{P}^5$.

Notre stratégie consiste à suivre [O'G08b], en utilisant l'involution anti-symplectique ι qui engendre $\mathrm{Aut}(X)$ afin d'obtenir le plus d'informations possible sur la géométrie du système linéaire complet. Dans le cas étudié par O'Grady, un résultat important est la *propriété d'irréductibilité* de $|H|$, où $H \in \mathrm{Div}(Z)$ est le diviseur ample introduit ci-dessus : si $D_1, D_2 \in |H|$ sont deux diviseurs distincts, alors $D_1 \cap D_2$ est une surface réduite et irréductible. La preuve de cette propriété est assez facile, voir [O'G08b, Proposition 4.1]. Dans notre cadre, nous montrerons l'énoncé similaire suivant, cf. Theorem 4.6.5.

Théorème C. *Soit X le carré de Hilbert $S_{2t}^{[2]}$ d'une surface K3 générique S_{2t} de degré $2t$. Soit $h \in \mathrm{Pic}(X)$ le fibré en droites induit par le générateur de $\mathrm{Pic}(S_{2t})$. Supposons que X admet un diviseur ample D avec $q_X(D) = 2$. Soit $D_1, D_2 \in |D|$ deux diviseurs distincts.*

(i) Si $t = 2$, alors la surface $D_1 \cap D_2$ peut être réductible. Si oui, cette surface a deux composantes irréductibles A et B, dont les classes fondamentales en cohomologie dans $H^{2,2}(X,\mathbb{Z})$ sont les suivantes :
$$[A] = \tfrac{1}{2}h^2 - \tfrac{1}{4}\delta^2 - \tfrac{1}{2}h\delta - \tfrac{1}{12}c_2(X) \in H^{2,2}(X,\mathbb{Z}),$$
$$[B] = \tfrac{1}{2}h^2 + \tfrac{5}{4}\delta^2 - \tfrac{3}{2}h\delta + \tfrac{1}{12}c_2(X) \in H^{2,2}(X,\mathbb{Z}).$$

(ii) Si $t \neq 2$, alors $D_1 \cap D_2$ est une surface réduite et irréductible.

La propriété d'irréductibilité n'est pas vraie si X est le carré de Hilbert d'une surface K3 générique de degré 4, c'est-à-dire si $t = 2$: nous remarquons que $[A] + [B] = h^2 - 2h\delta + \delta^2$, qui coïncide avec le carré, par rapport au produit cup, de la classe $h - \delta$ du diviseur ample $D \in \mathrm{Div}(S_4)$ avec $q_X(D) = 2$. Lorsqu'elle est vraie, la propriété d'irréductibilité est beaucoup plus difficile à prouver dans notre cas que dans celui étudié par O'Grady : le point clé de la preuve du Théorème C est la description explicite du réseau $H^{2,2}(X, \mathbb{Z})$ de classes de Hodge de type $(2,2)$ sur le carré de Hilbert d'une surface K3 générique donné par le Théorème A. Après avoir obtenu le Théorème C, nous pouvons suivre la même stratégie utilisée par O'Grady dans [O'G08b] pour étudier la fonction rationnelle $\varphi_{|D|} : S_{2t}^{[2]} \dashrightarrow \mathbb{P}^5$ quand $t \neq 2$, où S_{2t} est une surface K3 générique de degré $2t$. Soit X le carré de Hilbert de S_{2t} tel que X admet un diviseur ample D avec $q_X(D) = 2$, de sorte que X admet une involution anti-symplectique ι. Un des faits les plus intéressants montrés dans cette thèse est que l'existence de cette involution anti-symplectique simplifie beaucoup la solution du problème énoncé ci-dessus, par rapport à celle de O'Grady : par exemple, cela implique que $\varphi_{|D|}$ est fini de degré *pair* sur son image, cf. Theorem 4.5.11 et Corollary 4.5.12, alors que dans le cas de O'Grady ce n'est pas nécessairement vrai, ce qui rend le problème plus compliqué à étudier. Nous désignons par $F := \mathrm{Fix}(\iota)$ le lieu des points sur X fixés par ι. Beauville a montré dans [Bea11] que F est une sous-variété lagrangienne, donc F est une surface lisse dans notre cas, en particulier sa classe fondamentale en cohomologie appartient à $H^{2,2}(X, \mathbb{Z})$. Le résultat principal de cette thèse est le suivant, cf. Theorem 4.5.11 et Theorem 4.6.5.

Théorème D. *Soit $X = S_{2t}^{[2]}$ le carré de Hilbert d'une surface K3 générique S_{2t} de degré $2t$ telle que X admet un diviseur ample D avec $q_X(D) = 2$. Supposons que $t \neq 2$. Soit ι l'involution anti-symplectique qui engendre* $\mathrm{Aut}(X)$ *et soit $F = \mathrm{Fix}(\iota)$ le lieu fixe. Alors*

$$[F] = 5D^2 - \frac{1}{3}c_2(X) \in H^{2,2}(X, \mathbb{Z}),$$

où $[F]$ est la classe fondamentale en cohomologie de F dans $H^{2,2}(X, \mathbb{Z})$. De plus, $\varphi_{|D|} : X \to Y \subset \mathbb{P}^5$ est un morphisme, et c'est un revêtement double d'une sextique EPW, en particulier $Y = X/\langle \iota \rangle$ et X est une double sextique EPW.

Ceci résout le problème donné ci-dessus. Nous remarquons que $t = 2$ n'est pas considéré dans l'énoncé du Théorème D. C'est exactement la seule valeur pour laquelle la propriété d'irréductibilité de $|D|$ n'est pas vrai, comme montré par le Théorème C. Dans ce cas, la variété $X := S_4^{[2]}$ est le carré de Hilbert d'une surface quartique complexe lisse de $\mathbb{P}^3$, et le morphisme $\varphi_{|D|} : X \to Y \subset \mathbb{P}^5$ est fini de degré 6 : son image Y est isomorphe à la Grassmannien $\mathbb{G}(1, \mathbb{P}^3)$ des droites dans $\mathbb{P}^3$, qui est une quadrique dans $\mathbb{P}^5$. Ceci a été étudié en détail dans [Bea83a], voir aussi [BCNWS16], cf. Section 4.4.1.

La thèse est organisée comme suit. Dans le Chapitre 1 nous rappelons quelques définitions et résultats de Géométrie Algébrique Complexe. Tout d'abord, nous introduisons les notions de base de positivité en Géométrie Algébrique, comme les diviseurs de Weil et de Cartier sur une variété complexe, le groupe des classes de diviseurs et le groupe de Picard, diviseurs amples et nef, diviseurs big et pseudo-effectifs, la notion de système linéaire complet associé à un diviseur et la fonction rationnelle induite par un système linéaire complet. Nous définissons ensuite l'homomorphisme de Gysin, qui peut être vu comme un poussé en avant entre groupes de cohomologie singuliers induit par un morphisme entre des variétés complexes. Nous introduisons les classes de Chern topologiques d'un fibré vectoriel complexe sur une variété complexe, et nous énonçons le théorème de Grothendieck–Riemann–Roch. On présente ensuite les principaux résultats de la théorie des réseaux et on définit les équations de Pell et les équations de type Pell. Enfin, nous donnons quelques propriétés utiles des revêtements doubles.

Dans le Chapitre 2, nous donnons la définition de variété IHS, ainsi que quelques exemples : nous traitons des surfaces K3, en rappelant quelques résultats importants, nous voyons les détails de la construction du carré de Hilbert d'une surface K3 et nous introduisons les doubles sextiques EPW. Nous résumons ensuite les notions principales sur la théorie de la déformation dans le contexte des variétés IHS et nous énonçons certaines des propriétés et résultats les plus importants concernant cette famille de variétés, par exemple les théorèmes de Torelli, local et global, la surjectivité de la fonction des périodes et la construction du flop de Mukai, qui est un exemple fondamental de fonction birationnelle entre les variétés IHS. Nous introduisons ensuite le cône birationnel de Kähler et le cône mobile d'une variété IHS, et nous présentons brièvement le cône pseudo-effectif des cycles sur les variétés IHS de type $K3^{[n]}$. Nous concluons le chapitre en énonçant, sans preuves, quelques résultats utiles sur les variétés IHS.

Dans le Chapitre 3, nous introduisons les opérateurs de Nakajima, voir [Nak97] et [Leh99]. Nous rappelons le modèle de Lehn–Sorger présenté dans [LS03], que nous utilisons pour calculer les produits cup dans l'anneau $H^*(S^{[2]}, \mathbb{Z})$, où nous désignons par S une surface K3, en termes d'opérateurs de Nakajima. Ensuite, en utilisant un résultat d'Ellingsrud, Göttsche et Lehn dans [EGL01], nous obtenons une description explicite de la deuxième classe de Chern $c_2(S^{[2]})$ dans $H^{2,2}(S^{[2]}, \mathbb{Z})$, que nous utilisons pour prouver Théorème A et Théorème B. Nous concluons le chapitre avec les classes de Hodge intégrales de type $(3, 3)$ sur les carrés de Hilbert des surfaces K3.

Dans le Chapitre 4 nous introduisons le problème principal de la thèse, énoncé ci-dessus. Tout d'abord nous rappelons le théorème de Bayer–Macrì, qui décrit le cône nef, le cône mobile et le cône pseudo-effective du carré de Hilbert d'une surface K3 générique S_{2t}, cf. Theorem 4.1.1. Nous montrons ensuite que le carré de Hilbert d'une surface K3 générique est un *Mori dream space*, dont la définition est donnée dans la Section 4.2. Nous présentons les groupes d'automorphismes réguliers et d'automorphismes birationnels sur le carré de Hilbert $S_{2t}^{[2]}$, cf. Theorem 4.3.1 et Theorem 4.3.2, étudiés dans [BCNWS16] et [DM19]. Nous décrivons ensuite géométriquement $S_4^{[2]}$ et $S_{10}^{[2]}$, en suivant respectivement [Bea83a] et [O'G05], et nous voyons quelques propriétés importantes du lieu des points fixés par une involution anti-symplectique sur une variété IHS. Soit X le carré de Hilbert d'une surface K3 générique S_{2t} telle que X admet un diviseur ample D avec $q_X(D) = 2$: nous montrons que $\varphi_{|D|}$, la fonction rationnelle induite par le système linéaire complet $|D|$, factorise par le quotient $\pi : X \to X/\langle\iota\rangle$, où nous désignons par ι l'involution anti-symplectique qui engendre $\mathrm{Aut}(X)$, et nous calculons la classe fondamentale en cohomologie du lieu fixe $F = \mathrm{Fix}(\iota)$ de ι dans $H^{2,2}(X, \mathbb{Z})$, cf. Theorem 4.5.11. Nous prouvons enfin Théorème C, et nous discutons brièvement de la manière d'aborder le cas plus général du modèle birationnel X du carré de Hilbert $S_{2t}^{[2]}$ d'une surface K3 générique tel que X admet un diviseur ample D avec $q_X(D) = 2$.

Dans le Chapitre 5, en suivant la stratégie développée par O'Grady dans [O'G08b], avec des simplifications remarquables, nous prouvons Théorème D.

Nous concluons avec le Chapitre 6, où nous présentons des problèmes ouverts concernant les sujets étudiés dans la thèse.

Chapter 1

Generalities on Complex Algebraic Geometry

In this chapter we recall general definitions and results in Complex Algebraic Geometry that we need in this thesis. In Section 1.1 positivity in Complex Algebraic Geometry is studied: we introduce Weil and Cartier divisors, the class group and the Picard group, ampleness and nefness of divisors and line bundles, complete linear systems, bigness and pseudoeffectiveness of divisors and line bundles. In Section 1.2 we define the Gysin homomorphism and we recall the projection formula. In Section 1.3 we introduce topological Chern classes of a vector bundle over a complex manifold, and we recall the Grothendieck–Riemann–Roch theorem. In Section 1.4 we discuss basics on Lattice theory. In Section 1.5 we define Pell equations and Pell-type equations. In Section 1.6 we deal with double covers and their main properties.

1.1 Positivity in Algebraic Geometry

We call *scheme* a separated algebraic scheme of finite type over $\mathbb{C}$ and *variety* a reduced and irreducible scheme. In this section we recall some classical definitions and results of positivity in algebraic geometry. We will give statements for complex varieties, even if they hold more generally for schemes. The main references for this section are [Ful13], [Har13] and [Laz17].

1.1.1 Weil divisors and class group

We begin with the definition of *Weil divisor*.

Definition 1.1.1. Let X be a variety over $\mathbb{C}$. A *prime divisor* on X is a closed subvariety of codimension 1. A *Weil divisor* is an element of the free abelian group generated by prime divisors, denoted by $\mathrm{WDiv}(X)$. A Weil divisor is of the form

$$D = \sum_i d_i Z_i,$$

where the sum is finite, the Z_i's are prime divisors and $d_i \in \mathbb{Z}$. We say that D is *effective*, and we write $D \geq 0$, if $d_i \geq 0$ for every i. A *Weil $\mathbb{Q}$-divisor* is an element of the $\mathbb{Q}$-vector space

$$\mathrm{WDiv}_{\mathbb{Q}}(X) := \mathrm{WDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

A *Weil* $\mathbb{R}$*-divisor* is an element of the $\mathbb{R}$-vector space

$$\mathrm{WDiv}_{\mathbb{R}}(X) := \mathrm{WDiv}(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Suppose that X is a *normal* variety over $\mathbb{C}$. If $Z \in \mathrm{WDiv}(X)$ is a prime divisor, then the local ring $\mathcal{O}_{X,Z}$ is a DVR whose fraction field is $\mathcal{M}_X(X)$. Thus we can define a homomorphism

$$\mathrm{ord}_Z : \mathcal{M}_X(X)^* \to \mathbb{Z}$$

as follows: if $f = a/b$, with $a, b \in \mathcal{O}_{X,Z}$, then $\mathrm{ord}_Z(f) := v(a) - v(b)$, where v is the valuation of the ring $\mathcal{O}_{X,Z}$. If X is not normal, $\mathcal{O}_{X,Z}$ is not necessarily a DVR: in this case, if $a \in \mathcal{O}_{X,Z}$, we set

$$\mathrm{ord}_Z(a) := \mathrm{length}_{\mathcal{O}_{X,Z}} \left(\mathcal{O}_{X,Z} \big/ (a) \right),$$

and if $f \in \mathcal{M}_X(X)^*$ with $f = a/b$ for some $a, b \in \mathcal{O}_{X,Z}$, then

$$\mathrm{ord}_Z(f) := \mathrm{ord}_Z(a) - \mathrm{ord}_Z(b).$$

The two definitions are equivalent for normal varieties, see [Ful13, Appendix A]. If $\mathrm{ord}_Z(f) > 0$ we say that f has a *zero* along Z of order $\mathrm{ord}_Z(f)$, if $\mathrm{ord}_Z(f) < 0$ we say that f has a *pole* of order $-\mathrm{ord}_Z(f)$. We can now associate to a non-zero rational function a Weil divisor.

Definition 1.1.2. Let X be a variety over $\mathbb{C}$, and $f \in \mathcal{M}_X(X)^*$. Then the *Weil divisor associated to* f is

$$\mathrm{div}(f) := \sum \mathrm{ord}_Z(f) \cdot Z, \tag{1.1.1}$$

where the sum is over all the prime divisors. A Weil divisor $D \in \mathrm{WDiv}(X)$ is *principal* if $D = \mathrm{div}(f)$ for some $f \in \mathcal{M}_X(X)^*$. We denote by $\mathrm{PDiv}(X)$ the subgroup of $\mathrm{WDiv}(X)$ of principal divisors. We say that two Weil divisors $D, D' \in \mathrm{WDiv}(X)$ are *linearly equivalent* if $D - D' \in \mathrm{PDiv}(X)$.

The definition of divisor associated to a non-zero rational function is well-posed since the sum in (1.1.1) is always finite, see [Har13, Lemma II.6.1].

Definition 1.1.3. Let X be a variety over $\mathbb{C}$. The *class group* of X is the quotient group

$$\mathrm{Cl}(X) := \mathrm{WDiv}(X) \big/ \mathrm{PDiv}(X). \tag{1.1.2}$$

Let X be a normal variety. Then a Weil divisor $D \in \mathrm{WDiv}(X)$ defines a coherent sheaf $\mathcal{O}_X(D)$ on X given locally by

$$\Gamma(U, \mathcal{O}_X(D)) = \{ f \in \mathcal{M}_X(X) \mid (D + \mathrm{div}(f))|_U \geq 0 \} \cup \{0\}. \tag{1.1.3}$$

This is a rank one reflexive sheaf, but in general this is not an invertible sheaf.

1.1.2 Cartier divisors and Picard group

We now introduce Cartier divisors on a variety over $\mathbb{C}$.

Definition 1.1.4. Let X be a variety over $\mathbb{C}$. A *Cartier divisor* on X is a global section of the quotient sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$. We denote by $\mathrm{Div}(X)$ the group of Cartier divisors on X, i.e.,

$$\mathrm{Div}(X) = \Gamma\left(X, \mathcal{M}_X^*/\mathcal{O}_X^*\right).$$

Equivalently, $D \in \mathrm{Div}(X)$ is given by $\{(U_i, f_i)\}_{i\in I}$, where $\{U_i\}_{i\in I}$ is an open covering of X and $f_i \in \Gamma(U_i, \mathcal{M}_X^*)$ such that $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$. The equation f_i is called a *local equation* for D on U_i. We say that a Cartier divisor $D = \{(U_i, f_i)\}_{i\in I}$ is *effective*, and we write $D \geq 0$, if $f_i \in \mathcal{O}_X(U_i)$ for every $i \in I$. A Cartier divisor is *principal* if $D = (X, f)$ for some $f \in \mathcal{M}_X(X)^*$. We denote by $\mathrm{PDiv}(X)$ the subgroup of $\mathrm{Div}(X)$ of principal divisors. Two Cartier divisors $D, D' \in \mathrm{Div}(X)$ are *linearly equivalent* if $D - D' \in \mathrm{PDiv}(X)$. We call *Cartier $\mathbb{Q}$-divisor* an element of the $\mathbb{Q}$-vector space

$$\mathrm{Div}_{\mathbb{Q}}(X) := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

A *Cartier $\mathbb{R}$-divisor* is an element of the $\mathbb{R}$-vector space

$$\mathrm{Div}_{\mathbb{R}}(X) := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

If X is a projective variety over $\mathbb{C}$ one can show that

$$\mathrm{Div}(X) \Big/ \mathrm{PDiv}(X) \cong \mathrm{Pic}(X), \tag{1.1.4}$$

where $\mathrm{Pic}(X)$ is the group of invertible sheaves of X up to isomorphisms, called *Picard group*. See [Nak63, p.301] for a proof of isomorphism (1.1.4) for complex projective varieties, and [Har13, Proposition II.6.15] for the case of integral schemes.

Let X be a variety over $\mathbb{C}$. A Cartier divisor $D = \{(U_i, f_i)\}_{i\in I}$ defines a coherent sheaf $\mathcal{O}_X(D)$ defined locally as the $\mathcal{O}_X$-submodule of $\mathcal{M}_X^*$ generated over U_i by f_i^{-1}. We have the following fundamental proposition.

Proposition 1.1.5 (Proposition II.6 in [Har13])**.** *Let X be a variety over $\mathbb{C}$. Then:*

(i) For any Cartier divisor D, the sheaf $\mathcal{O}_X(D)$ is invertible. Moreover, the map $D \mapsto \mathcal{O}_X(D)$ gives a $1:1$ correspondence between Cartier divisors on X and invertible subsheaves of $\mathcal{M}_X$.

(ii) $\mathcal{O}_X(D_1 - D_2) \cong \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1}$.

(iii) $D_1 \sim D_2$ if and only if $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$ as abstract invertible sheaves.

1.1.3 Weil and Cartier divisors, cycle map

It is natural to wonder if there is a relation between Weil and Cartier divisors on a variety X over $\mathbb{C}$. There is a homomorphism of groups called *cycle map*, defined by

$$\mathrm{Div}(X) \to \mathrm{WDiv}(X), \qquad D = \{(U_i, f_i)\}_{i\in I} \mapsto \sum_Z \mathrm{ord}_Z(f_i) \cdot Z,$$

where the sum is taken over all the prime divisors on X and we consider an $i \in I$ such that $Z \cap U_i \neq \emptyset$. The cycle map is in general neither injective nor surjective. It is injective if X is normal, see [Ful13, Example 2.1.1], so Cartier divisors on a normal variety are Weil divisors which are locally everywhere represented by a single equation. Suppose that X is locally factorial, i.e., $\mathcal{O}_{X,x}$ is a UFD for every $x \in X$. Recall that a UFD is normal, see [Eis13, Proposition 4.10]. Then the cycle map is an isomorphism, see again [Ful13, Example 2.1.1]. Moreover, the cycle map sends principal divisors to principal divisors. Then for locally factorial varieties we have an isomorphism

$$\mathrm{Cl}(X) \cong \mathrm{Pic}(X),$$

in particular this holds for smooth projective varieties. Moreover, one can show that a Weil divisor $D \in \mathrm{WDiv}(X)$ is Cartier if and only if the sheaf $\mathcal{O}_X(D)$ given in (1.1.3) is invertible. If so, the sheaves $\mathcal{O}_X(D)$ given in (1.1.3) and in Section 1.1.2 coincide: this happens for instance for smooth projective varieties.

1.1.4 Globally generated invertible sheaves

In this section we introduce the notion of globally generated sheaf on a smooth projective variety. See [Har13] and [Laz17] for more general statements.

Definition 1.1.6. Let X be a smooth projective variety and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on X. Then we say that $\mathcal{F}$ is *globally generated*, or *generated by its global sections*, if there exist $\{s_i\}_{i \in I}$, with $s_i \in \Gamma(X, \mathcal{F})$ such that $\{s_{i,x}\}_{i \in I}$ generate $\mathcal{F}_x$ as a $\mathcal{O}_{X,x}$-module for every $x \in X$.

Recall that if $\mathcal{F}$ is a coherent sheaf on X, by [Har13, Theorem II.5.19] the vector space $\Gamma(X, \mathcal{F})$ is finitely generated. Note that the set of points $x \in X$ such that $\mathcal{F}$ is globally generated is the complement of the support of the cokernel of the evaluation map

$$\mathrm{ev} : \Gamma(X, \mathcal{F}) \otimes \mathcal{O}_X \to \mathcal{F},$$

thus this is an open subset. Hence $\mathcal{F}$ is globally generated if and only if the evaluation map is surjective. Since the set of closed points is dense in X, it suffices to check global generation at every closed point $x \in X$. Then similarly to the proof of [Har13, Proposition III.5.3], by Nakayama's lemma this is equivalent to the surjectivity of

$$\mathrm{ev}_x : \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F} \otimes \mathbb{C}(x)),$$

where $\mathbb{C}(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ is the residue field of the point $x \in X$, and $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$. Thus, if $\mathcal{L}$ is an invertible sheaf on X, then $\mathcal{L}$ is globally generated if and only if for each closed point $x \in X$ there exists a global section $s \in \Gamma(X, \mathcal{L})$ which does not vanish on the point x, i.e., $\mathrm{ev}_x(s) \neq 0$ in $\Gamma(X, \mathcal{L} \otimes \mathbb{C}(x)) \cong \mathbb{C}(x)$. From now on we will write $s(x)$ for $\mathrm{ev}_x(s)$. One can cover X with a finite number of open subsets, thanks to the Noetherianity (which holds since X is projective), where the property above holds, so an invertible sheaf $\mathcal{L}$ is globally generated if and only if for every $x \in X$ there exists a global section $s \in \Gamma(X, \mathcal{L})$ which does not vanish on x.

Example 1.1.7. The sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$ is globally generated by the global sections $x_0, \ldots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, where $x_0, \ldots, x_n$ are the homogeneous coordinates of $\mathbb{P}^n$.

Given a smooth projective variety X, the following theorem gives a relation between morphisms $\varphi : X \to \mathbb{P}^n$ and globally generated sections of an invertible sheaf on X.

Theorem 1.1.8 (Theorem II.7.1 in [Har13])**.** *Let X be a smooth projective variety over $\mathbb{C}$.*

(i) If $\varphi : X \to \mathbb{P}^n$ is a morphism, then $\varphi^*(\mathcal{O}_{\mathbb{P}^n}(1))$ is an invertible sheaf generated by the global sections $s_i = \varphi^*(x_i)$ for $i = 0, 1, \dots, n$.

(ii) If $\mathcal{L}$ is an invertible sheaf on X globally generated by $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$, then there exists a unique morphism $\varphi : X \to \mathbb{P}^n$ such that $\mathcal{L} \cong \varphi^*(\mathcal{O}_{\mathbb{P}^n}(1))$, and $s_i = \varphi^*(x_i)$ under this isomorphism.

1.1.5 Ampleness and nefness

In this section X will be a smooth projective variety. Given two divisors $D, D' \in \mathrm{Div}(X)$, we say that they are *numerically equivalent*, and we write $D \equiv D'$, if

$$D \cdot C = D' \cdot C \qquad \text{for every curve } C \subseteq X,$$

where $\cdot$ denotes the intersection product, see [Deb13, §1.1] for details.

Definition 1.1.9. Let X be a smooth projective variety. Then we define the *Néron–Severi group* of X as the quotient group

$$\mathrm{NS}(X) := {\mathrm{Div}(X)}/_{\equiv}.$$

Sometimes the Néron–Severi group is defined as the quotient group

$$\mathrm{NS}(X) := \mathrm{Pic}(X) \big/ \mathrm{Pic}^0(X),$$

where $\mathrm{Pic}^0(X) := H^1(X, \mathcal{O}_X) \big/ H^1(X, \mathbb{Z})$. The two definitions agree up to some torsion, see [Laz17, Remark 1.1.21], in particular they coincide when one works with rational or real coefficient. Moreover, if $H^2(X, \mathbb{Z})_f$ denotes the torsion free quotient group of $H^2(X, \mathbb{Z})$, we have

$$\mathrm{NS}(X) = H^2(X, \mathbb{Z})_f \cap H^{1,1}(X).$$

We can now define very ampleness and ampleness for line bundles and divisors.

Definition 1.1.10. Let X be a smooth projective variety and let $L \in \mathrm{Pic}(X)$ be a line bundle.

(i) We say that L is *very ample* if there exists a closed embedding $i : X \hookrightarrow \mathbb{P}^n$ such that $i^*\mathcal{O}_{\mathbb{P}^n}(1) \cong L$.

(ii) We say that L is *ample* if for every coherent sheaf $\mathcal{F}$ on X there exists an integer $n_0 > 0$ such that $\mathcal{F} \otimes L^{\otimes n}$ is globally generated for every $n \geq n_0$.

We say that a divisor $D \in \mathrm{Div}(X)$ is very ample, respectively ample, if the line bundle $\mathcal{O}_X(D)$ is so.

We recall the following theorem by Cartan, Grothendieck and Serre, see [Laz17, Theorem 1.2.6].

Theorem 1.1.11 (Cartan–Grothendieck–Serre)**.** *Let X be a smooth projective variety and $L \in \mathrm{Pic}(X)$ be a line bundle. The following are equivalent.*

(i) L is ample.

(ii) Given a coherent sheaf $\mathcal{F}$ on X, there exists a positive integer $m_1 = m_1(\mathcal{F})$ with the following property:

$$H^i(X, \mathcal{F} \otimes L^{\otimes m}) = 0 \qquad \textit{for all } i > 0,\ m \geq m_1(\mathcal{F}).$$

(iii) There is a positive integer $m_3 > 0$ such that $L^{\otimes m}$ is very ample for every $m \geq m_3$.

We will often use the characterisation of ampleness given by Theorem 1.1.11, Item *(iii)*. The following theorem gives a necessary and sufficient condition for the ampleness of a line bundle, see [Laz17, Theorem 1.2.23].

Theorem 1.1.12 (Nakai–Moishezon criterion)**.** *Let X be a smooth projective variety and $L \in \mathrm{Pic}\,(X)$ be a line bundle. Then L is ample if and only if*

$$\int_V c_1(L)^{\dim V} > 0$$

for every positive dimensional subvariety $V \subseteq X$. Equivalently a Cartier divisor $D \in \mathrm{Div}\,(X)$ is ample if and only if

$$D^{\dim V} \cdot V > 0$$

for every positive dimensional subvariety $V \subseteq X$.

Note that by the Nakai–Moishezon criterion the ampleness of a divisor D depends only on its numerical class in $\mathrm{NS}(X)$, so ampleness can be defined in $\mathrm{NS}(X)$. We now introduce nefness of line bundles and divisors.

Definition 1.1.13. Let X be a smooth projective variety. We say that a line bundle $L \in \mathrm{Pic}(X)$ is *nef*, or *numerically effective*, if

$$\int_C c_1(L) \geq 0$$

for every curve $C \subseteq X$. Similarly, a Cartier divisor D on X is *nef* if

$$D \cdot C \geq 0$$

for all curves $C \subseteq X$.

Note that the nefness of a Cartier divisor depends only on its numerical equivalence class in $\mathrm{NS}(X)$, so the notion of nefness can be defined in $\mathrm{NS}(X)$. The following theorem by Kleiman shows that nef line bundles are limits of ample line bundles, see [Laz17, Theorem 1.4.9].

Theorem 1.1.14 (Kleiman)**.** *Let X be a smooth projective variety. A line bundle $L \in \mathrm{Pic}(X)$ is nef if and only if*

$$\int_V c_1(L)^{\dim V} \geq 0$$

for every positive dimensional subvariety $V \subseteq X$. Similarly a divisor $D \in \mathrm{Div}(X)$ is nef if and only if

$$D^{\dim V} \cdot V \geq 0$$

for every positive dimensional subvariety $V \subseteq X$.

While nefness of a line bundle L, or of a divisor D, can be checked by definition only by computing $\int_C c_1(L)$, respectively $D \cdot C$, for every curve $C \subseteq X$, the same does not work for ampleness. An example of a smooth projective variety X admitting a non-ample divisor D with $D \cdot C > 0$ for every curve $C \subset X$ is given in [Har06, §1.10].

We now recall the following useful proposition, see [Laz17, Proposition 1.2.13, Corollary 1.2.28, Example 1.4.4].

Proposition 1.1.15. *Let X be a smooth projective variety and let $L \in \mathrm{Pic}(X)$ be a line bundle on X.*

(i) Let $f : Y \to X$ be a proper morphism.

- *If L is nef, then f^*L is nef.*
- *If L is ample and f is finite, then f^*L is ample.*

In particular, restrictions of nef bundles to closed subschemes are nef and restrictions of ample bundles to closed subschemes are ample.

(ii) Let $f : Y \to X$ be a proper surjective morphism.

- *If f^*L is nef then L is nef.*
- *If f^*L is ample and f is finite then L is ample.*

Recall than a *cone* $\mathcal{C}$ in a vector space V over $\mathbb{R}$ is a subset such that for every $x \in \mathcal{C}$ the element αx belongs to $\mathcal{C}$ for every $\alpha \in \mathbb{R}_{\geq 0}$. We conclude this section by introducing the *ample cone* and the *nef cone* of a smooth projective variety X.

Definition 1.1.16. The *ample cone* of a smooth projective variety X is the open convex cone

$$\mathrm{Amp}(X) \subseteq \mathrm{NS}(X) \otimes \mathbb{R}$$

generated by classes of ample divisors. The *nef cone* is the convex cone

$$\mathrm{Nef}(X) \subseteq \mathrm{NS}(X) \otimes \mathbb{R}$$

generated by classes of nef divisors.

As a consequence of the Kleiman's theorem we have the following result, see [Laz17, Theorem 1.4.23].

Theorem 1.1.17. *Let X be a smooth projective variety.*

(i) The nef cone of X is the closure of the ample cone of X:

$$\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}.$$

(ii) The ample cone of X is the interior of the nef cone of X:

$$\mathrm{Amp}(X) = (\mathrm{Nef}(X))^{\circ}.$$

We conclude this section by recalling the *Kodaira vanishing theorem*, see [Laz17, Theorem 4.2.1].

Theorem 1.1.18 (Kodaira vanishing theorem)**.** *Let X be a smooth projective variety and let $A \in \mathrm{Div}(X)$ be an ample divisor on X. Then*

$$H^i(X, \mathcal{O}_X(\mathcal{K}_X + A)) = 0 \ \text{for } i > 0\,,$$

where $\mathcal{K}_X$ is a canonical divisor of X.

1.1.6 Complete linear systems

In this section we introduce complete linear systems associated to a divisor on a smooth projective variety. This will be a fundamental definition for the second part of this thesis. As before we work over the complex field $\mathbb{C}$.

Let X be a smooth projective variety and $\mathcal{L}$ be an invertible sheaf on X. Given a non-zero section $s \in \Gamma(X, \mathcal{L})$ of $\mathcal{L}$, we recall the construction of the effective divisor $D = (s)_0$, called *divisor of zeros of s*. Over any open set $U \subseteq X$ where $\mathcal{L}$ is trivial, let $\varphi : \mathcal{L}|_U \xrightarrow{\sim} \mathcal{O}_U$ be an isomorphism. Then $\varphi(s) \in \Gamma(U, \mathcal{O}_U)$. As U ranges over a covering of X, the collection $\{U, \varphi(s)\}$ determines an effective Cartier divisor D on X. Indeed, φ is determined up to multiplication by an element of $\Gamma(U, \mathcal{O}_U^*)$, so we get a well defined Cartier divisor. We can now state the following result, see [Har13, Proposition II.7.7].

Proposition 1.1.19. *Let X be a smooth projective variety. Let $D \in \mathrm{Div}(X)$ be a divisor on X and let $\mathcal{L} = \mathcal{O}_X(D)$ be the corresponding invertible sheaf. Then:*

- *(i) For each non-zero $s \in \Gamma(X, \mathcal{L})$, the divisor of zeros $(s)_0$ is an effective divisor linearly equivalent to D.*
- *(ii) Every effective divisor linearly equivalent to D is of the form $(s)_0$ for some $s \in \Gamma(X, \mathcal{L})$.*
- *(iii) Two sections $s, s' \in \Gamma(X, \mathcal{L})$ have the same divisor of zeros if and only if there is a $\lambda \in \mathbb{C}^*$ such that $s' = \lambda s$.*

From now on we will denote the $\mathbb{C}$-vector space of global sections of an invertible sheaf $\mathcal{L}$ on X by $H^0(X, \mathcal{L})$. Note that by Proposition 1.1.19, a divisor $D \in \mathrm{Div}(X)$ of a smooth projective variety is effective if and only if $H^0(X, \mathcal{O}_X(D)) \neq 0$. We can now define complete linear systems.

Definition 1.1.20. We call *complete linear system* on a smooth projective variety X the set of all effective divisors linearly equivalent to some given divisor $D \in \mathrm{Div}(X)$. We denote it by $|D|$. A *linear system* δ on X is a subset of a complete linear system $|D|$ which is a linear subspace for the projective space structure of $|D|$, so δ corresponds to a sub-vector space $V \subseteq H^0(X, \mathcal{O}_X(D))$, where

$$V = \{s \in H^0(X, \mathcal{O}_X(D)) \,|\, (s)_0 \in \delta\} \cup \{0\}.$$

Note that a complete linear system $|D|$ can be equal to the empty set. Moreover, by Proposition 1.1.19 we have the following isomorphism:

$$|D| \cong \mathbb{P}(H^0(X, \mathcal{O}_X(D))).$$

Similarly a linear system δ is such that $\delta \cong \mathbb{P}(V)$ for some sub-vector space $V \subseteq H^0(X, \mathcal{O}_X(D))$. Recall that the *support* of a divisor $D \in \mathrm{Div}(X)$ on a smooth projective variety, denoted by $\mathrm{Supp}(D)$, is by definition the union of the prime divisors appearing in $D = \sum_i n_i D_i$, where D is seen as a Weil divisor. Equivalently, it is the union of all prime divisors Z of X such that a local equation for D, seen as a Cartier divisor, in the local ring $\mathcal{O}_{X,Z}$ is not a unit.

Definition 1.1.21. Let X be a smooth projective variety, and $D \in \mathrm{Div}(X)$. The *base locus* of the complete linear system $|D|$ is the subset of X defined as

$$\mathrm{Bs}|D| := \{x \in X \mid s(x) = 0 \text{ for every } s \in H^0(X, \mathcal{O}_X(D))\}.$$

Equivalently,

$$\mathrm{Bs}|D| = \bigcap_{D' \in |D|} \mathrm{Supp}(D').$$

We say that $|D|$ is *basepoint free* if $\mathrm{Bs}|D| = \emptyset$. The divisor D is *basepoint free* if $|D|$ is basepoint free. We say that D is *semiample* if nD is basepoint free for some $n > 0$.

Similar definitions can be given for linear systems δ on a smooth projective variety X. We now associate to a complete linear system $|D|$ a rational map

$$\varphi_{|D|} : X \dashrightarrow \mathbb{P}\left(H^0(X, \mathcal{O}_X(D))^\vee\right).$$

Let $x \in X \setminus \mathrm{Bs}|D|$, then $\varphi_{|D|}(x)$ is by definition the hyperplane in $H^0(X, \mathcal{O}_X(D))$ consisting of those sections vanishing at $x \in X$. Alternatively, $\varphi_{|D|}$ can be described as follows. Let $\{s_0, \ldots, s_n\}$ be a basis of the complex vector space $H^0(X, \mathcal{O}_X(D))$. If $D = \{(U_\alpha, f_\alpha)\}_{\alpha \in \mathcal{A}}$, then the global sections s_i can be represented as collections $\{(U_\alpha, s_{i,\alpha})\}_{\alpha \in \mathcal{A}}$, where $s_{i,\alpha} : U_\alpha \to \mathbb{C}$ is a holomorphic map and

$$s_{i,\alpha}(x) = \frac{f_\alpha}{f_\beta}(x) s_{i,\beta}(x) \qquad \text{for every } x \in U_\alpha \cap U_\beta.$$

Then, if $x \in U_\alpha \setminus \mathrm{Bs}|D|$,

$$\varphi_{|D|}(x) = (s_{0,\alpha}(x) : \cdots : s_{n,\alpha}(x)). \tag{1.1.5}$$

Note that the definition of $\varphi_{|D|}$ in (1.1.5) is well posed. Indeed, if $x \in U_\alpha \cap U_\beta$, we have

$$\begin{aligned} (s_{0,\alpha}(x) : \cdots : s_{n,\alpha}(x)) &= \left(\tfrac{f_\alpha}{f_\beta}(x) s_{0,\beta}(x) : \cdots : \tfrac{f_\alpha}{f_\beta}(x) s_{n,\beta}(x)\right) \\ &= (s_{0,\beta}(x) : \cdots : s_{n,\beta}(x)). \end{aligned}$$

In a similar way one can associate to a linear system δ on a smooth projective variety X a rational map φ_δ. We now recall the definitions of *fixed part* and *movable part* of a complete linear system.

Definition 1.1.22. Let X be a smooth projective variety and $D \in \mathrm{Div}(X)$ be an effective divisor. The *fixed part* of the complete linear system $|D|$ is the greatest effective divisor F such that $D' - F \geq 0$ for every $D' \in |D|$. The *movable part* of $|D|$ is the complete linear system $|M| := |D| - F$.

Thus the fixed part of $|D|$ is the component of codimension 1 of the base locus $\mathrm{Bs}|D|$. Note that the rational map $\varphi_{|D|}$ can be extended to the fixed part of $|D|$, except for the points which belong to the non-divisorial component of the base locus $\mathrm{Bs}|D|$ of $|D|$, and this extension coincides with the rational map $\varphi_{|M|}$. Indeed, assume that the fixed part F of $|D|$ is given by the effective divisor $\{(U_\alpha, g_\alpha)\}_{\alpha\in\mathcal{A}}$ with $g_\alpha \in \mathcal{O}_X(U_\alpha)$. Then, since $|M| = |D| - F$, we have $s_{i,\alpha} = g_\alpha \cdot s'_{i,\alpha}$ for $i = 0, 1, \ldots, n$, where $s'_i = \{(U_\alpha, s'_{i,\alpha})\}_{\alpha\in\mathcal{A}}$ and $\{s'_0, \ldots, s'_N\}$ is a basis of $H^0(X, \mathcal{O}_X(M))$. Note that by definition of fixed and movable part of the complete linear system $|D|$, for every $x \in \mathrm{Supp}(F)$ there exists a global section in $H^0(X, \mathcal{O}_X(M))$ which does not vanish on x. Moreover, the rational maps $\varphi_{|M|}$ and $\varphi_{|D|}$ coincide out of $\mathrm{Supp}(F)$, since $g_\alpha(x) \neq 0$ for $x \in X\backslash\mathrm{Supp}(F)$, so that $(g_\alpha(x)\cdot s'_{0,\alpha}(x) : \cdots : g_\alpha(x)\cdot s'_{N,\alpha}(x))$ and $(s'_{0,\alpha}(x) : \cdots : s'_{N,\alpha}(x))$ coincide as points in the projective space. Hence the map defined by

$$x \mapsto (s'_{0,\alpha}(x) : \cdots : s'_{N,\alpha}(x))$$

for $x \in U_\alpha$ has indeterminacy locus of codimension ≥ 2, in particular this coincides with $\varphi_{|M|}$, where $|M|$ is the movable part of $|D|$. Thus, when we study the rational map $\varphi_{|D|}$ induced by a complete linear system $|D|$, we always consider the rational map $\varphi_{|M|}$ induced by the movable part.

In general, in order to study geometric properties of a smooth projective variety, it is useful to describe morphisms $\varphi : X \to \mathbb{P}^n$ in some projective space. Complete linear systems can be seen as a tool to construct rational maps. First of all, one needs to determine the base locus of a complete linear system $|D|$ in order to find the indeterminacy locus of $\varphi_{|D|}$. Then, one has to describe geometrically $\varphi_{|D|}$. This is the problem that we will study in Chapter 4 and in Chapter 5 for Hilbert squares of generic K3 surfaces.

1.1.7 Bigness and pseudoeffectiveness

In this section we introduce big line bundles and big divisors, and we define the big cone and the pseudoeffective cone on a smooth projective variety over $\mathbb{C}$. For more general statements see [Laz17, §2.2].

Definition 1.1.23. Let X be a smooth projective variety, and let $D \in \mathrm{Div}(X)$ be a divisor. The *Iitaka dimension* of D is defined as

$$\kappa(D) := \begin{cases} \max\limits_{m\geq 1}\{\dim \mathrm{Im}(\varphi_{|mD|})\} & \text{if } |mD| \neq \emptyset \text{ for some } m > 0, \\ -\infty & \text{if } |mD| = \emptyset \text{ for every } m > 0. \end{cases}$$

The Iitaka dimension of a line bundle $L \in \mathrm{Pic}(X)$ is defined as $\kappa(D)$, where $D \in \mathrm{Div}(X)$ is such that $L = \mathcal{O}_X(D)$. If $\mathcal{K}_X$ is a canonical divisor of X, we say that $\kappa(\mathcal{K}_X)$ is the *Kodaira dimension* of X.

Definition 1.1.24. Let X be a smooth projective variety, and let $L \in \mathrm{Pic}(X)$. We say that L is *big* if $\kappa(L) = \dim(X)$. Similarly, a divisor $D \in \mathrm{Div}(X)$ is big if $\kappa(D) = \dim(X)$.

We state the following lemma by Kodaira, see [Laz17, Lemma 2.2.6].

Proposition 1.1.25 (Kodaira's lemma)**.** *Let X be a smooth projective variety, and consider $D \in \mathrm{Div}(X)$ a big Cartier divisor and $F \in \mathrm{Div}(X)$ an effective Cartier divisor. Then*

$$H^0(X, \mathcal{O}_X(mD - F)) \neq 0$$

for $m \gg 0$.

A consequence of Kodaira's lemma is the following characterization of big divisors, see [Laz17, Corollary 2.2.7].

Corollary 1.1.26. *Let X be a smooth projective variety and $D \in \mathrm{Div}(X)$ be a divisor. The following are equivalent.*

(i) *D is big.*

(ii) *For any ample divisor A on X, there exists a positive integer $m > 0$ and an effective divisor N on X such that $mD \sim A + N$.*

(iii) *For some ample divisor A on X, there exists a positive integer $m > 0$ and an effective divisor N on X such that $mD \sim A + N$.*

(iv) *There exists an ample divisor A, a positive integer $m > 0$, and an effective divisor N such that $mD \equiv A + N$.*

Note that by Corollary 1.1.26, (iv), the bigness of a divisor D on a smooth projective variety X depends only on the numerical equivalence class of D, so bigness can be defined in $\mathrm{NS}(X)$.

The following is a useful criterion to determine the bigness of nef divisors.

Theorem 1.1.27 (Theorem 2.2.16 in [Laz17])**.** *Let X be a smooth projective variety of dimension n and $D \in \mathrm{Div}(X)$ be a nef divisor on X. Then D is big if and only if $D^n > 0$.*

We now state the *Kawamata–Shokurov basepoint freeness theorem* for smooth projective varieties. See [KMM87, Theorem 3.1.1] for a more general statement.

Theorem 1.1.28 (Kawamata–Shokurov basepoint freeness theorem)**.** *Let X be a smooth projective variety and $L \in \mathrm{Div}(X)$ be a nef divisor such that $L - \mathcal{K}_X$ is nef and big, where $\mathcal{K}_X$ is a canonical divisor of X. Then $|mL|$ is basepoint free for $m \gg 0$.*

We recall the *Kawamata–Viehweg vanishing theorem* for smooth projective varieties, see [Laz17, Theorem 4.3.1] for a more general statement.

Theorem 1.1.29 (Kawamata–Viehweg vanishing theorem)**.** *Let X be a smooth projective variety and $D \in \mathrm{Div}(X)$ a nef and big Cartier divisor on X. Then*

$$H^i(X, \mathcal{O}_X(\mathcal{K}_X + D)) = 0 \qquad \textit{for every } i > 0.$$

We can now introduce the big cone and the pseudoeffective cone of a smooth projective variety X.

Definition 1.1.30. Let X be a smooth projective variety. The *big cone*

$$\mathrm{Big}(X) \subseteq \mathrm{NS}(X) \otimes \mathbb{R}$$

is the convex cone generated by classes of big divisors. The *pseudoeffective cone*

$$\overline{\mathrm{Eff}}(X) \subseteq \mathrm{NS}(X) \otimes \mathbb{R}$$

is the closure of the convex cone generated by classes of effective divisors. A divisor $D \in \mathrm{Div}(X)$ is *pseudoeffective* if its numerical class lies in $\overline{\mathrm{Eff}}(X)$.

We conclude this section with the following theorem.

Theorem 1.1.31 (Theorem 2.2.26 in [Laz17])**.** *Let X be a smooth projective variety. Then the closure of the big cone coincide with the pseudoeffective cone*

$$\overline{\mathrm{Big}(X)} = \overline{\mathrm{Eff}}(X),$$

and the big cone coincides with the interior of the pseudoeffective cone

$$\mathrm{Big}(X) = \left(\overline{\mathrm{Eff}}(X)\right)^{\circ}.$$

1.2 Gysin homomorphism

In this section we work with compact complex manifolds, even if definitions and results presented hold in a more general context. Let $f : X \to Y$ be a morphism between compact complex manifolds. There is an induced pullback morphism between singular cohomology groups

$$f^* : H^k(Y, \mathbb{Z}) \to H^k(X, \mathbb{Z}),$$

see [Voi02, §7.3.2]. We now define the *Gysin homomorphism*.

Definition 1.2.1. Let $f : X \to Y$ be a morphism between compact complex manifolds. Suppose that $n = \dim_{\mathbb{C}} X$, $m = \dim_{\mathbb{C}} Y$ and let $r := m - n$, where r can be negative. Then for every $k \geq 0$ we define the *Gysin homomorphism*

$$f_* : H^k(X, \mathbb{Q}) \to H^{k+2r}(Y, \mathbb{Q})$$

by setting

$$f_*(\alpha) := PD^{-1} f_* PD(\alpha) \in H^{k+2r}(Y, \mathbb{Q}),$$

where we denote by PD the Poincaré duality and f_* in the right-hand side is the pushforward map in singular homology.

See [Bre13, IV, §4] for details on the definition of pushforward in singular homology. We will use the same symbol f_* for the pushforward in homology and the Gysin homomorphism, which can be distinguished by the context. Recall that the pullback f^* is compatible with the cup product, i.e.,

$$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta) \qquad \text{for every } \alpha, \beta \in H^*(Y, \mathbb{Z}).$$

Moreover, if $\dim(X) = \dim(Y)$ and f is a finite morphism, the following holds:

$$f_* \circ f^* = \deg(f) \cdot \mathrm{Id} : H^k(Y, \mathbb{Z}) \to H^k(Y, \mathbb{Z}), \tag{1.2.1}$$

hence $f_* \circ f^*(\alpha) = \deg(\alpha) \cdot \alpha$ for every $\alpha \in H^k(Y, \mathbb{Z})$. See [Voi02, §7.3.2] for details. Thus we have the following result, see [Deb13, Proposition 1.10] for an equivalent statement in the case of intersection of Cartier divisors.

Proposition 1.2.2. *Let $f : X \to Y$ be a finite surjective morphism between compact complex manifolds X and Y. Let $D_1, \dots, D_r$ be Cartier divisors on Y with $r \geq \dim(X)$. Then we have*

$$\int_X c_1(\mathcal{O}_X(f^*D_1)) \cup \cdots \cup c_1(\mathcal{O}_X(f^*D_r)) = \deg(f) \int_Y c_1(\mathcal{O}_Y(D_1)) \cup \cdots \cup c_1(\mathcal{O}_Y(D_r)).$$

We conclude this section by recalling the *projection formula*. Let $f : X \to Y$ be a morphism between compact complex manifolds. By [Bre13, p.241], the following holds in the homology groups:

$$f_*(\alpha) \cap \beta = f_*(\alpha \cap f^*(\beta)), \tag{1.2.2}$$

where $\alpha \in H_*(X, \mathbb{Z})$, $\beta \in H^*(Y, \mathbb{Z})$, the map f_* is the push-forward in homology and $\cap$ is the cap product. Recall that the Poincaré duality is defined by

$$PD : H^k(Y, \mathbb{Z}) \to H_{\dim_{\mathbb{R}}(Y)-k}(Y, \mathbb{Z}), \qquad \alpha \mapsto [Y] \cap \alpha,$$

where $[Y]$ is the fundamental class of Y. We can now show the projection formula in cohomology.

Proposition 1.2.3 (Projection formula)**.** *Let X and Y be compact complex manifolds and consider a morphism $f : X \to Y$. Then*

$$\alpha \cup f_*(\beta) = f_*(f^*(\alpha) \cup \beta),$$

where f_* is the Gysin homomorphism, $\alpha \in H^p(Y, \mathbb{Z})$, and $\beta \in H^q(X, \mathbb{Z})$.

1.3 Chern classes

Let $E \to X$ be a complex vector bundle of rank r over a compact complex manifold X. In this section we introduce Chern classes and the Todd class of the vector bundle E, and we state the Grothendieck–Riemann–Roch theorem. We begin by defining the *k-th topological Chern class* $c_k(E) \in H^{2k}(X, \mathbb{Z})$ of E and the *total Chern class*

$$c(E) := c_0(E) + c_1(E) + \cdots \in H^*(X, \mathbb{Z}).$$

Chern classes satisfy the following axioms.

1. If $\mathrm{rk}(E) = 1$, then $c(E) = 1 + c_1(E)$.
2. (Naturality). If $f : Y \to X$ is a continuous map between topological spaces and f^*E is the pullback vector bundle, then $c_k(f^*E) = f^*c_k(E)$.
3. (Whitney sum formula). The total Chern class of the direct sum $E \oplus F$ of two complex vector bundles E, F on X is

   $$c(E \oplus F) = c(E) \cup c(F),$$

   where $\cup$ denotes the cup product, hence the k-th topological Chern class of $E \oplus F$ is

   $$c_k(E \oplus F) = \sum_{i=0}^{k} c_i(E) \cup c_{k-i}(F).$$

Alternatively, let $\pi : \mathbb{P}(E) \to X$ be the projectivization of the rank r complex vector bundle E, and let $\mathcal{O}_{\mathbb{P}(E)}(1)$ be the Serre line bundle over $\mathbb{P}(E)$. We denote by $\xi \in H^2(\mathbb{P}(E), \mathbb{Z})$ the Euler class of $\mathcal{O}_{\mathbb{P}(E)}(1)$. By the Leray–Hirsch theorem, see [Hat05, Theorem 4D.1], the ring $H^*(\mathbb{P}(E), \mathbb{Z})$ has a structure of free module over $H^*(X, \mathbb{Z})$ with basis $1, \xi, \dots, \xi^{r-1}$. Then the topological Chern classes $c_k(E) \in H^{2k}(X, \mathbb{Z})$ are defined by the following relation:

$$\sum_{k=0}^{r} (-1)^k \pi^* c_k(E) \cup \xi^{r-k} = 0.$$

Topological Chern classes obtained in this way satisfy the axioms given above, see [Voi02, Théorème 11.23]. Moreover, these axioms uniquely characterise topological Chern classes of a complex vector bundle. The proof of the uniqueness uses the *splitting principle*, which says that for every complex vector bundle $E \to X$ there exists a continuous map $\phi : Y \to X$ such that the pullback maps $\phi^* : H^l(X, \mathbb{Z}) \to H^l(Y, \mathbb{Z})$ are injective and $\phi^* E$ is a direct sum of line bundles, see [Voi02, Lemme 11.24]. If we denote by $c_t(E)$ the *Chern polynomial*

$$c_t(E) := c_o(E) + c_1(E)t + c_2(E)t^2 + \cdots \in H^*(X, \mathbb{Z})[t],$$

then by the splitting principle we have $\phi^* E \cong \oplus_i L_i$ for some line bundles L_i over Y, thus

$$\phi^* c_t(E) = \prod_i (1 + t c_1(L_i)),$$

and this relation determines $c_t(E)$ since ϕ^* is injective on $H^*(X, \mathbb{Z})$. This shows that $c_i(E) = 0$ for $i > \mathrm{rk}(E)$ and that for every complex vector bundle we can write

$$c_t(E) = \prod_{i=1}^{r} (1 + \alpha_i t),$$

where the α_i's are formal variables such that the k-th elementary symmetric polynomial in the α_i's is exactly the Chern class $c_k(E)$. The α_i's are called *Chern roots* of E. We then define the *exponential Chern character*

$$\mathrm{ch}(E) = \sum_{i=1}^{r} \exp(\alpha_i) \in H^*(X, \mathbb{Z}) \otimes \mathbb{Q} \cong H^*(X, \mathbb{Q}),$$

where

$$\exp(x) = 1 + x + \frac{1}{2}x^2 + \dots,$$

and the *Todd class* of E by

$$\mathrm{td}(E) = \prod_{i=1}^{r} \frac{\alpha_i}{1 - \exp(-\alpha_i)} \in H^*(X, \mathbb{Q}),$$

where

$$\frac{x}{1 - \exp(-x)} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \dots.$$

These are symmetric expressions on the α_i, so these can be expressed as polynomials in the $c_i(E)$ with rational coefficients. One can show that

$$\begin{aligned}\mathrm{ch}(E) &= r + c_1 + \tfrac{1}{2}(c_1^2 - 2c_2) + \tfrac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) \\ &\quad + \tfrac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) + \dots\end{aligned}$$

and

$$\begin{aligned}\mathrm{td}(E) &= 1+\tfrac{1}{2}c_1+\tfrac{1}{12}(c_1^2+c_2)+\tfrac{1}{24}c_1c_2\\ &\quad -\tfrac{1}{720}(c_1^4-4c_1^2c_2-3c_2^2-c_1c_3+c_4)+\dots\end{aligned}$$

where $c_i = c_i(E)$.

Remark 1.3.1. If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles on X, then the Chern roots of E are the union of the Chern roots of E' and E'', hence we have

$$\mathrm{ch}(E) = \mathrm{ch}(E') + \mathrm{ch}(E''), \qquad \mathrm{td}(E) = \mathrm{td}(E') \cdot \mathrm{td}(E'').$$

1.3.1 Grothendieck–Riemann–Roch theorem

In this section we state the *Grothendieck–Riemann–Roch theorem*, following [AH59, Theorem 1], [HA62] and [Ful13, Theorem 15.2], the last reference in the case of Chow groups.

Let X be a scheme. We first recall the constructions of the Grothendieck group of vector bundles on X, denoted by $K(X)$, and of the Grothendieck group of coherent sheaves on X, denoted by $K(\mathrm{Coh}(X))$. We follow [Ful13, §15.1]. The Grothendieck group of vector bundles $K(X)$ over X is the group of formal finite sums $\sum_i a_i[E_i]$, where $a_i \in \mathbb{Z}$ and the $E_i's$ are vector bundles over X, modulo the relation

$$[E] = [E'] + [E'']$$

for every exact sequence $0 \to E' \to E \to E'' \to 0$. Note that $\sum_{i=1}^r [E_i] = 0$ for every exact sequence

$$0 \to E_1 \to E_2 \to \cdots \to E_r \to 0.$$

The tensor product of vector bundles induces a structure of ring on $K(X)$:

$$[E] \cdot [F] := [E \otimes F].$$

Moreover, if $f : Y \to X$ is a morphism, there is an induced homomorphism

$$f^! : K(X) \to K(Y)$$

given by $f^!([E]) := [f^*E]$, where f^*E is the pullback bundle. The Grothendieck group of coherent sheaves $K(\mathrm{Coh}(X))$ on X is defined as the free abelian group generated by isomorphism classes $[\mathcal{F}]$ of coherent sheaves on X, modulo the relation

$$[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$$

for each exact sequence of coherent sheaves on X of the form

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0.$$

The tensor product induces a structure of $K(X)$-module on $K(\mathrm{Coh}(X))$:

$$K(X) \otimes K(\mathrm{Coh}(X)) \to K(\mathrm{Coh}(X)), \qquad [E] \cdot [\mathcal{F}] := [E \otimes_{\mathcal{O}_X} \mathcal{F}].$$

If $f : X \to Y$ is a proper morphism, there is a homomorphism

$$f_! : K(\mathrm{Coh}(X)) \to K(\mathrm{Coh}(Y))$$

defined by $f_!([\mathcal{F}]) := \sum_{i\geq 0}(-1)^1[R^i f_*\mathcal{F}]$, where $R^i f_*\mathcal{F}$ is the Grothendieck's higher direct image sheaf, see [Har13, § III.8] for details. The properness of the morphism f is required to guarantee that $R^i f_*\mathcal{F}$ is coherent when $\mathcal{F}$ is coherent, see [GD66, § III.3.2.1]. Given $\mathcal{F}$ a coherent sheaf on X, we denote by $\mathcal{F}^* := \mathrm{Hom}(\mathcal{F}, \mathcal{O}_X)$ the classical dual of $\mathcal{F}$. In view of Chapter 3, it is useful to give another definition of the dual of $\mathcal{F}$, see [HL10, §1.1] for details.

Definition 1.3.2. Let X be a scheme and $\mathcal{F} \in \mathrm{Coh}(X)$ be a coherent sheaf. The *dual* of $\mathcal{F}$ is

$$\mathcal{F}^\vee := \sum_i (-1)^i \mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X).$$

For any scheme X there is a canonical duality homomorphism

$$K(X) \to K(\mathrm{Coh}(X))$$

which maps a vector bundle to its sheaf of sections. When X is non singular, this duality map is an isomorphism, since for every coherent sheaf $\mathcal{F}$ on a non singular X there is a finite resolution by locally free sheaves

$$0 \to E_n \to E_{n-1} \to \cdots \to E_1 \to E_0 \to \mathcal{F} \to 0,$$

and the inverse homomorphism is given by

$$K(\mathrm{Coh}(X)) \to K(X), \qquad [\mathcal{F}] \mapsto \sum_{i=0}^n (-1)^i [E_i].$$

See [Ful13, B.8.3] for details. Since we will always work with smooth projective varieties, we can identify $K(X)$ with $K(\mathrm{Coh}(X))$: given a proper morphism $f : X \to Y$ between smooth projective varieties, the induced homomorphism $f_!$ given above will be considered as defined on the Grothendieck groups of vector bundles.

From now on X will be a smooth projective variety. By Remark 1.3.1 the Chern character of a vector bundle is additive on exact sequences, so we obtain a homomorphism

$$\mathrm{ch} : K(X) \to H^*(X, \mathbb{Q}), \qquad [E] \mapsto \mathrm{ch}(E).$$

Recall that the *Euler characteristic* of a vector bundle E over X is defined as

$$\mathcal{X}(X, E) := \sum_{i\geq 0} (-1)^i \dim(H^i(X, E)).$$

Note that the definition is well posed, since $H^i(X, E)$ is a finite dimensional vector space by a result of Serre, see [Har13, Theorem III.5.2]. Moreover, for every exact sequence of vector bundles of the form

$$0 \to E' \to E \to E'' \to 0$$

we have $\mathcal{X}(X, E) = \mathcal{X}(X, E') + \mathcal{X}(X, E'')$, hence we obtain a homomorphism

$$\mathcal{X} : K(X) \to \mathbb{Z}, \qquad [E] \mapsto \mathcal{X}(X, E).$$

We recall the *Hirzebruch–Riemann–Roch* theorem, see [HBS66].

Theorem 1.3.3 (Hirzebruch–Riemann–Roch)**.** *Let X be a smooth projective variety and E a vector bundle over X. Denote by $\mathcal{T}_X$ the tangent bundle of X. Then*

$$\mathcal{X}(X,E) = \int_X \mathrm{ch}(E) \cup \mathrm{td}(\mathcal{T}_X).$$

Hence we obtain the following homomorphism:

$$K(X) \to H^*(X,\mathbb{Q}), \qquad [E] \mapsto \mathrm{ch}(E) \cup \mathrm{td}(\mathcal{T}_X).$$

We can now state the Grothendieck–Riemann–Roch theorem.

Theorem 1.3.4. *Let $f : X \to Y$ be a proper morphism of smooth projective varieties. Then for all $\alpha \in K(X)$ we have*

$$\mathrm{ch}(f_!\alpha) \cup \mathrm{td}(\mathcal{T}_Y) = f_*(\mathrm{ch}(\alpha) \cup \mathrm{td}(\mathcal{T}_X)) \in H^*(Y,\mathbb{Q}),$$

where f_* is the Gysin homomorphism extended $\mathbb{Q}$-linearly and $\mathcal{T}_X$ is the tangent bundle of X.

1.4 Lattices

In this section we recall the most important definitions and results of lattice theory that we need in the next chapters. General references on lattice theory used are [Nik80] and [CS13], see also [Men19, §2].

Definition 1.4.1. A *lattice* L is a free $\mathbb{Z}$-module of finite rank together with a symmetric bilinear form $b : L \times L \to \mathbb{Z}$. We denote by $q : L \to \mathbb{Z}$ the quadratic form $q(x) := b(x,x)$.

Let L be a lattice of rank n, and let $\mathcal{B} := \{e_1, \ldots, e_n\}$ be a $\mathbb{Z}$-basis of L. The *Gram matrix* of L associated to $\mathcal{B}$ is the $n \times n$ symmetric matrix

$$\begin{pmatrix} b(e_1,e_1) & \cdots & b(e_1,e_n) \\ \vdots & \ddots & \vdots \\ b(e_n,e_1) & \cdots & b(e_n,e_n) \end{pmatrix}.$$

We say that a lattice L of rank n is:

- *non-degenerate* if for any non-zero $l \in L$ there exists $l' \in L$ such that $b(l,l') \neq 0$, equivalently, $\det(G) \neq 0$ if G is a Gram matrix of L;
- *even* if $b(l,l) \in 2\mathbb{Z}$ for every $l \in L$, in particular L is even if and only if there is a $\mathbb{Z}$-basis $\{e_1, \ldots, e_n\}$ such that $b(e_i,e_i) \in 2\mathbb{Z}$ for $i = 1, \ldots, n$;
- *odd* if it is not even.

The *determinant* of a lattice L is the determinant of a Gram matrix G of the lattice. This does not depend on the choice of the Gram matrix: if G and G' are two Gram matrices associated to two distinct $\mathbb{Z}$-basis of L, then $G' = S^t G S$, where S is an invertible matrix with integer entries, so $\det(S) = \pm 1$ and $\det(G') = \det(G)$. A lattice L is *unimodular* if $\det(L) = \pm 1$. A *sublattice* of a lattice L is a free submodule $L' \subseteq L$ together with the symmetric bilinear form

$b' := b|_{L'\times L'}$. The *divisibility* of an element $l \in L$ in a lattice L is the positive generator of the ideal

$$\{b(l,m) \,|\, m \in L\} \subset \mathbb{Z}.$$

A sublattice $L' \subseteq L$ is *primitive* if L/L' is a free module.

Example 1.4.2. Let L be a lattice, and $S \subseteq L$ be a subset. Then the *orthogonal* of S in L, defined as

$$S^\perp := \{l \in L \,|\, b(l,s) = 0 \text{ for every } s \in S\},$$

is a primitive sublattice of L.

The *direct sum* of two lattices L_1 and L_2 is the lattice $L_1 \oplus L_2$ whose bilinear form is

$$b(v_1 + v_2, w_1 + w_2) := b_1(v_1, w_1) + b_2(v_2, w_2)$$

for every $v_1, w_1 \in L_1$ and $v_2, w_2 \in L_2$, where b_1 and b_2 are the bilinear forms of L_1 and L_2 respectively. Note that, if M is a sublattice of L, then

$$M \oplus M^\perp \subseteq L$$

is a sublattice of maximal rank, i.e., $\mathrm{rk}(M) + \mathrm{rk}(M^\perp) = \mathrm{rk}(L)$.

For a lattice L of rank n we write $L_\mathbb{R} := L \otimes_\mathbb{Z} \mathbb{R}$ and the bilinear form b is extended $\mathbb{R}$-bilinearly to $L_\mathbb{R}$, similarly q is extended to $L_\mathbb{R}$. If the lattice is non-degenerate, the quadratic form q on $L_\mathbb{R} \cong \mathbb{R}^n$ admits an orthonormal basis by Sylvester's theorem, i.e., there is an $\mathbb{R}$-basis $\{f_1, \ldots, f_n\}$ of $L_\mathbb{R}$ such that

$$q\left(\sum_{i=1}^n x_i f_i\right) = \epsilon_1 x_1^2 + \cdots + \epsilon_n x_n^2 \qquad \text{with } \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}.$$

After a permutation of the vectors of the basis $\{f_1, \ldots, f_n\}$ we can assume that $\epsilon_i = 1$ for $i = 1, \ldots, l_{(+)}$ and $\epsilon_i = -1$ for $i = l_{(+)} + 1, \ldots, n$ for some $l_{(+)} \in \{0, \ldots, n\}$. If $l_{(-)} := n - l_{(+)}$, the *signature* of L is the pair of integers $(l_{(+)}, l_{(-)})$. A non-degenerate lattice is *positive definite* if $l_{(-)} = 0$, similarly it is *negative definite* if $l_{(+)} = 0$, while it is *indefinite* if $l_{(+)}, l_{(-)} \neq 0$. We now give some examples of lattices which will appear in the next chapters.

Example 1.4.3. If k is a non-zero integer, let $\langle k \rangle$ be the rank one lattice $L = \mathbb{Z}e$ with bilinear form $b(e,e) = k$.

Example 1.4.4. If L is a lattice, for every non-zero integer k we denote by $L(k)$ the lattice obtained by taking the same $\mathbb{Z}$-module and bilinear form $b_{(k)}$ defined as

$$b_{(k)}(v,w) := kb(v,w)$$

for every $v, w \in L$.

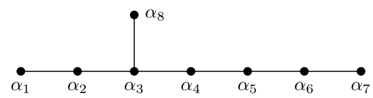
Example 1.4.5. Let U be the *hyperbolic lattice*, i.e., the unique unimodular lattice of rank 2 and signature $(1,1)$. Its Gram matrix is the following:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.4.1}$$

Example 1.4.6. Let E_8 be the even unimodular lattice of signature $(8,0)$ whose Gram matrix is the following:

$$\begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & -1 \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & -1 & & & & & 2 \end{pmatrix}.$$

Equivalently, E_8 is represented by the following Dynkin diagram



,

where $\{\alpha_1, \dots, \alpha_8\}$ is a $\mathbb{Z}$-basis of E_8 and the bilinear form is described as follows:

- $b(\alpha_i, \alpha_i) = 2$ for every $i = 1, \dots, 8$;
- $b(\alpha_i, \alpha_j) = 0$ if the nodes α_i and α_j in the diagram are not linked;
- $b(\alpha_i, \alpha_j) = -1$ if the nodes α_i and α_j in the diagram are linked.

Example 1.4.7. Let $E_8(-1)$ be the lattice obtained by multiplying by -1 the Gram matrix of E_8, i.e., the lattice whose Gram matrix is the following:

$$\begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & 1 \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & \\ & & 1 & & & & & -2 \end{pmatrix}. \tag{1.4.2}$$

It is an even unimodular lattice of signature $(0,8)$.

If L and L' are two lattices with bilinear forms b and b' respectively, we call *morphism of lattices* $\varphi : L \to L'$ a morphism of $\mathbb{Z}$-modules such that for every $l_1, l_2 \in L$ we have $b(l_1, l_2) = b'(\varphi(l_1), \varphi(l_2))$. Note that morphisms between two non-degenerate lattices are injective. We say that a lattice *embeds primitively* in a lattice L' if there is a morphism $\varphi : L \to L'$ such that $\varphi(L)$ is a primitive sublattice of L'. An *isometry* is a bijective morphism of lattices. The group of isometries of a lattice to itself is denoted by $O(L)$. We now recall the *Smith normal form* of a matrix with integer entries, see [Smi61], which we will use for matrices representing morphisms of lattices.

Proposition 1.4.8 (Smith normal form)**.** *Let $f : L_1 \to L_2$ be a morphism of lattices. Then there exists $\mathcal{B}_1$ and $\mathcal{B}_2$ basis of L_1 and L_2 such that the matrix* *which represents f is of the form*

$$\begin{pmatrix} \lambda_1 & 0 & \dots & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_r & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix}.$$

1.4.1 Discriminant group and primitive embeddings

From now on, L will be a non-degenerate lattice. A fundamental tool in lattice theory is the discriminant group associated to a lattice L. In order to define it, we need to introduce the *dual* of a lattice L, which is $L^\vee := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. Consider the following morphism of lattices

$$\phi : L \hookrightarrow L^\vee, \qquad v \mapsto b(v, \cdot\,).$$

Since the bilinear form b is non-degenerate, ϕ is injective. We then obtain an isomorphism

$$\phi_{\mathbb{Q}} : L \otimes \mathbb{Q} \xrightarrow{\sim} L^\vee \otimes \mathbb{Q}.$$

The restriction of $\phi_{\mathbb{Q}}^{-1}$ to $L^\vee$ gives an embedding $L^\vee \hookrightarrow L \otimes \mathbb{Q}$, which characterizes the dual $L^\vee$ as

$$L^\vee = \{u \in L \otimes_{\mathbb{Z}} \mathbb{Q} \,|\, b(u, v) \in \mathbb{Z} \text{ for every } l \in L\}.$$

We now see how to obtain a basis for the dual $L^\vee$. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of L and let M be the Gram matrix associated to $\mathcal{B}$. If $\mathcal{B}^\vee = \{v_1^\vee, \dots, v_n^\vee\}$ is the dual basis of $\mathcal{B}$, then the matrix which represents ϕ in the basis $\mathcal{B}$ and $\mathcal{B}'$ is $\mathrm{mat}_{\mathcal{B},\mathcal{B}^\vee}(\phi) = M$. Since M is also the matrix of $\phi_{\mathbb{Q}}$ we have

$$\mathrm{mat}_{\mathcal{B}^\vee,\mathcal{B}}(\phi_{\mathbb{Q}}^{-1}) = M^{-1}.$$

Moreover, $\phi_{\mathbb{Q}}^{-1}$ represents the embedding $L^\vee \hookrightarrow L \otimes \mathbb{Q}$, so the columns of M^{-1} give a basis of $L^\vee$. Note that if we use the Smith normal form of the morphism $\phi : L \to L^\vee$ of Proposition 1.4.8, we obtain the following.

Lemma 1.4.9. *Let L be a non-degenerate lattice. Then there exists a basis $\{v_1, \dots, v_n\}$ of L and non-zero integers $\lambda_1, \dots \lambda_n \in \mathbb{Z}$ such that $\{\frac{v_1}{\lambda_1}, \dots, \frac{v_n}{\lambda_n}\}$ is a basis of $L^\vee \subseteq L \otimes \mathbb{Q}$.*

Since $L \subseteq L^\vee$ is a subgroup of maximal rank, the quotient

$$A_L := L^\vee / L$$

is a finite group: we call it the *discriminant group* of L. We denote by $\mathrm{discr}(L)$ the order of the discriminant group: this coincides with $|\det(G)|$, where G is a Gram matrix of L. We see that if $A_L = \{0\}$, then L is unimodular. We say that the lattice L is *p-elementary* if

$$A_L \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{\oplus k}$$

for a prime number p and a non-negative integer k. The *length* $l(A_L)$ of the discriminant group A_L is the minimal number of generators of A_L. In general the dual $L^\vee$ is not a lattice according to our definition: the bilinear form $b_\mathbb{Q}$ obtained on $L^\vee$ by extending $\mathbb{Q}$-bilinearly the bilinear form b of L can take non-integer values. Note that, for every $x_1, x_2 \in L^\vee$ and $l_1, l_2 \in L$ we have

$$\begin{aligned} b_\mathbb{Q}(x_1 + l_1, x_2 + l_2) &= b_\mathbb{Q}(x_1, x_2) + b_\mathbb{Q}(x_1, l_2) + b_\mathbb{Q}(l_1, x_2) + b_\mathbb{Q}(l_1, l_2) \\ &\equiv b_\mathbb{Q}(x_1, x_2) \pmod{\mathbb{Z}}. \end{aligned}$$

Hence A_L is equipped with a so-called *finite bilinear form*

$$b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}, \qquad (\bar{x}, \bar{y}) \mapsto \overline{b_\mathbb{Q}(x, y)}.$$

Moreover, the $\mathbb{Q}$-extension of the quadratic form $q : L \to \mathbb{Z}$ induces a quadratic form on A_L modulo $\mathbb{Z}$:

$$q_L : A_L \to \mathbb{Q}/\mathbb{Z}, \qquad \bar{x} \mapsto \overline{q_\mathbb{Q}(x)}.$$

If L is an even lattice, we can say more: for every $x \in L^\vee$ and $l \in L$ we have

$$q_\mathbb{Q}(x + l) = q_\mathbb{Q}(x) + q_\mathbb{Q}(l) + 2b_\mathbb{Q}(x, l) \equiv q_\mathbb{Q}(x) \pmod{2\mathbb{Z}}$$

Thus, if L is even, A_L is equipped with a so-called *finite quadratic form*

$$q_L : A_L \to \mathbb{Q}/2\mathbb{Z}, \qquad \bar{x} \mapsto \overline{q_\mathbb{Q}(x)}.$$

Both the finite bilinear form and the finite quadratic form of A_L can be represented by a matrix: if $\{x_i\}_i$ is a system of independent generators of A_L, i.e., a basis of A_L, then:

- the matrix $M_{b_L} = (a_{i,j})$ with $a_{i,j} = b_L(x_i, x_j) \in \mathbb{Q}/\mathbb{Z}$ represents the finite bilinear form b_L;
- the matrix $M_{q_L} = (a_{i,j})$ with

  $$a_{i,j} = \begin{cases} b_L(x_i, x_j) \in \mathbb{Q}/\mathbb{Z} & \text{if } i \neq j \\ q_L(x_i) \in \mathbb{Q}/2\mathbb{Z} & \text{if } i = j \end{cases}$$

  represents the finite quadratic form q_L.

We conclude this part by stating the following theorem on primitive embeddings. The original result was obtained by Nikulin, see [Nik80, Theorem 1.14.4], but we state it in the weaker form given in [Huy16, Theorem 14.1.12].

Theorem 1.4.10 (Theorem 14.1.12 in [Huy16])**.** *Let L be an even, unimodular lattice of signature $(l_{(+)}, l_{(-)})$ and L' be an even lattice of signature $(l'_{(+)}, l'_{(-)})$. If $l'_{(+)} < l_{(+)}$, $l'_{(-)} < l_{(-)}$ and*

$$l(A_{L'}) + 2 \leq \mathrm{rk}(L) - \mathrm{rk}(L'),$$

then there exists a primitive embedding $L' \hookrightarrow L$ which is unique up to isometries of L.

1.4.2 Overlattices

Let L and R be two lattices such that $L \subseteq R$ and $\mathrm{rk}(L) = \mathrm{rk}(R)$. We say that R is an *overlattice* of L. The discriminant group A_L of a lattice L plays an important role in the study of the overlattices of L. Note that if R is an overlattice of L, then L has finite index in R. We have the following lemma.

Lemma 1.4.11. *Let L be a lattice and $R \supseteq L$ be an overlattice. Then*

$$[R : L]^2 = \frac{\mathrm{discr}(L)}{\mathrm{discr}(R)} = \frac{|A_L|}{|A_R|}.$$

Proof. Consider the following inclusions

$$L \to R \to R^\vee \to L^\vee,$$

where the composition is the canonical inclusion of L in its dual $L^\vee$. Let $\mathcal{B}_L$ and $\mathcal{B}_R$ be two basis of L and R respectively, and G_L and G_R be the Gram matrices associated. Let W be the matrix which represents the inclusion $L \hookrightarrow R$ in the basis $\mathcal{B}_L$ and $\mathcal{B}_R$. Then the transposed matrix W^t represents the inclusion $R^\vee \hookrightarrow L^\vee$, so $G_L = W^t G_R W$. Since $|\det(W)|$ is equal to the index $[R : L]$, and $|\det(G_R)|$ and $|\det(G_L)|$ are by definition the discriminants of R and L respectively, we have

$$[R : L]^2 = \frac{\mathrm{discr}(L)}{\mathrm{discr}(R)} = \frac{|A_L|}{|A_R|},$$

as we wanted. □

Let L be a lattice. We say that a subgroup $G \subseteq A_L$ is *isotropic* if

$$b_L(g, g') \equiv 0 \pmod{\mathbb{Z}}$$

for every $g, g' \in G$. The following result gives a relation between the overlattices of L and the isotropic subgroups of A_L.

Proposition 1.4.12 (Proposition 1.4.1, Item (a), in [Nik80])**.** *Let L be a lattice with discriminant group A_L. For every overlattice $R \supseteq L$, let H_R be the subgroup $H_R := R/L \subseteq A_L$. Then the following is a bijection.*

$$\begin{array}{ccc} \{\textit{overlattices of } L\} & \leftrightarrow & \{\textit{isotropic subgroups of } A_L\}, \\ R & \mapsto & H_R. \end{array}$$

Proof. We refer to [Nik80, Proposition 1.4.1,(a)]. Let $\pi : L^\vee \to L^\vee/L = A_L$ be the natural projection. There is a bijection between the set of groups R such that $L \subseteq R \subseteq L^\vee$ and the set of subgroups of A_L, obtained by sending R to $H_R := \pi(R)$. Now, $\pi(R)$ is isotropic if and only if $b_L(x, y) \equiv 0$ for every $x, y \in R$, i.e., $b_{\mathbb{Q}}(x, y) \in \mathbb{Z}$ for every $x, y \in R$. This holds if and only if R is a lattice. We conclude that the bijection above gives a bijection between the overlattices of L and the isotropic subgroups of A_L. □

Proposition 1.4.12 says that in order to determine all the overlattices of a lattice L, it suffices to find the subgroups of the discriminant group A_L which are isotropic. Since A_L is a finite group, this shows that a lattice has a finite number of overlattices.

1.5 Pell equations and Pell type equations

We give a brief overview of Pell equations and Pell-type equations, which will play an important role in Chapter 4 and in Chapter 5.

Definition 1.5.1. A *Pell equation* is a diophantine equation of the form

$$x^2 - dy^2 = 1,$$

where $d \in \mathbb{Z}_{>0}$ is a positive integer and x, y are variables. We denote an equation of this form by $P_d(1)$. A *Pell-type equation* is a diophantine equation of the form

$$x^2 - dy^2 = n,$$

where $d \in \mathbb{Z}_{>0}$ is a positive integer, $n \in \mathbb{Z} \setminus \{0\}$ is a non-zero integer and x, y are variables. We denote an equation of this form by $P_d(n)$.

We are interested in integral solutions of Pell equations and Pell-type equations. Note that if the integer d is a square $d = c^2$, $c \in \mathbb{Z}$, the only solutions of the Pell equation $P_d(1)$ are $(x, y) = (\pm 1, 0)$, and the Pell-type equation $P_d(n)$ can be written as $(x + cy)(x - cy) = n$, so it can be easily solved. From now on, we will assume that d is not a square. In this case the Pell-type equation $P_d(n)$ can be written as

$$(x + y\sqrt{d})(x - y\sqrt{d}) = n$$

in the ring $\mathbb{Z}[\sqrt{d}]$ defined as

$$\mathbb{Z}[\sqrt{d}\,] := \mathbb{Z}[x] \Big/ (x^2 - d)\,,$$

where $\mathbb{Z}[x]$ is the ring of polynomials in the variable x with integer coefficients and $\sqrt{d}$ is the class

$$\sqrt{d} := [x + (x^2 - d)] \in \mathbb{Z}[x]/(x^2 - d).$$

More concretely, $\mathbb{Z}[\sqrt{d}]$ can be seen as the set

$$\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} \,|\, a, b \in \mathbb{Z}\}$$

with sum and product given by

$$\begin{aligned} (a + b\sqrt{d}) + (e + f\sqrt{d}) \quad &:= \quad (a + e) + (b + f)\sqrt{d}, \\ (a + b\sqrt{d}) \cdot (e + f\sqrt{d}) \quad &:= \quad (ae + bfd) + (af + be)\sqrt{d}. \end{aligned}$$

Definition 1.5.2. The *conjugate* of $z = x + y\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ is defined as

$$\bar{z} := x - y\sqrt{d} \in \mathbb{Z}[\sqrt{d}],$$

and its *norm* is defined as

$$N(z) := z\bar{z} = x^2 - dy^2 \in \mathbb{Z}.$$

With this notation we can rewrite the Pell equation $P_d(1)$ and the Pell-type equation $P_d(n)$ respectively as

$$N(z) = 1, \qquad N(z) = n,$$

where $z = x + y\sqrt{d}$.

Definition 1.5.3. Given a Pell equation $P_d(1)$ or a Pell-type equation $P_d(n)$, two solutions (X, Y) and (X', Y') are said to be *equivalent* if

$$\frac{XX' - dYY'}{n} \in \mathbb{Z}, \qquad \frac{XY' - X'Y}{n} \in \mathbb{Z}.$$

Note that two solutions (X, Y) and (X', Y') of a Pell-type equation $P_d(n)$ are equivalent if $\frac{z_1 \cdot \bar{z}_2}{n} \in \mathbb{Z}[\sqrt{d}]$, where $z_1 = X + Y\sqrt{d}$ and $z_2 = X' + Y'\sqrt{d}$.

Definition 1.5.4. The *fundamental* solution (X, Y) in an equivalence class of solutions is the one with smallest non-negative Y if such a solution is unique in its class. Otherwise, there are two solutions (X, Y), $(-X, Y)$, which are said to be *conjugated*, with smallest non-negative Y: the fundamental solution is (X, Y) with $X > 0$.

One can show that the Pell equation $P_d(1)$ is always solvable and all its solutions are equivalent. If $z_0 = a + b\sqrt{d}$ is the fundamental solution, then all the other solutions are of the form

$$z = \pm z_0^m, \qquad m \in \mathbb{Z}_{>0}.$$

Similarly, if (X, Y) is a fundamental solution of a Pell-type equation $P_d(n)$, all other solutions (X', Y') in the same equivalence class are of the form

$$\begin{cases} X' = aX + dbY, \\ Y' = bX + aY, \end{cases}$$

where (a, b) is a solution of $P_d(1)$. Equivalently, if z_0 is a fundamental solution of $P_d(n)$ and $\tilde{z}_0$ is the fundamental solution of $P_d(1)$, all the other solutions equivalent to z_0 are of the form

$$z = \pm \tilde{z}_0^m \cdot z_0, \qquad m \in \mathbb{Z}_{>0}.$$

We now recall the definition of *positive* and *minimal* solution of a Pell-type equation.

Definition 1.5.5. Let $P_d(n)$ be a Pell-type equation. A solution (X, Y) is *positive* if $X, Y > 0$. The *minimal* solution of $P_d(n)$ is the positive solution with smallest X.

Note that a minimal solution is in particular a fundamental solution.

Remark 1.5.6. The minimal solution of the Pell equation $P_t(1)$ coincides with the square of the minimal solution of the negative Pell equation $P_t(-1)$, if this exists, i.e., if $a + b\sqrt{t} \in \mathbb{Z}[\sqrt{t}]$ is the minimal solution of $N(z) = -1$ and $c + d\sqrt{t} \in \mathbb{Z}[\sqrt{t}]$ is the minimal solution of $N(z) = 1$, then

$$c + d\sqrt{t} = (a + b\sqrt{t})^2 = a^2 + b^2 + 2ab\sqrt{t}\,,$$

hence

$$a^2 + tb^2 = c, \qquad d = 2ab. \tag{1.5.1}$$

We present the following useful result by Perron, [Per13, p.106-109], see also [Yok94, Proposition 1], which we will use in the proof of Proposition 4.6.3.

Proposition 1.5.7. *For any positive square-free integer $t \neq 2$, at most only one of the following three equations is solvable in integers:*

$$x^2 - ty^2 = -1, \qquad x^2 - ty^2 = 2, \qquad x^2 - ty^2 = -2.$$

We conclude this section with the following elementary result, which will be useful in the proof of Theorem 4.6.5, one of the main theorems of this thesis.

Proposition 1.5.8. *Let (a, b) be the minimal solution of the Pell-type equation $P_t(-1)$. Then b is odd.*

Proof. Suppose that b is even. Then, since $a^2 - tb^2 = -1$, we have that $a^2 + 1$ is divisible by 4, i.e., $a^2 + 1 = 4X$ for some $X \in \mathbb{Z}$.

- If a is even, i.e., $a = 2Y$ for some $Y \in \mathbb{Z}$, then $4Y^2 + 1 = 4X$, which is not possible.
- If a is odd, i.e., $a = 2Y + 1$ for some $Y \in \mathbb{Z}$, then $4Y^2 + 1 + 4Y + 1 = 4X$, which gives $4(X - Y^2 - Y) = 2$, which is not possible.

We conclude that b is odd. □

1.6 Double covers

Double covers will play an important role in Chapter 4 and in Chapter 5. In this section a variety will be reduced and irreducible. We recall the definition of double cover of a variety, and we give useful results from [BHPVdV15, §I.16, I.17] in the case of double covers between smooth varieties.

Definition 1.6.1. Let X and Y be varieties. A morphism $f : X \to Y$ is a *double cover* if it is finite of degree 2, i.e., $f_*\mathcal{O}_X$ has rank 2, and there is an involution $\iota : X \to X$ such that $f \circ \iota = f$ and $Y \cong X/\iota$.

Note that the morphism f in Definition 1.6.1 is not necessarily flat. We recall the following remark, see [DK19, Remark 2.4] for details.

Remark 1.6.2. Let $f : X \to Y$ be a finite map of degree 2 between *normal* varieties. Then the existence of an involution $\iota : X \to X$ such that $Y \cong X/\iota$ is guaranteed, i.e., f is a double cover.

If $f : X \to Y$ is a double cover, we call *branch locus* the set B of points $y \in Y$ such that the cardinality of $f^{-1}(y)$ is 1, and *ramification locus* the set R of points $x \in X$ such that $f(x)$ is a point in the ramification locus. If B is empty, we say that f is *unbranched*, or *unramified*, or *étale*, otherwise we say that f is *branched*, or *ramified*. If X and Y are smooth varieties, the morphism f is flat by [Har13, Exercise III.9.3]. Hence by Zariski–Nagata purity theorem, see [Zar58] and [Nag58], the branch locus is either empty or of pure codimension 1. In the second case, B and R are divisors, called *branch divisor* and *ramification divisor* respectively. The following Hurwitz formula holds, see [BHPVdV15, §1.16]:

$$\omega_X = \pi^*(\omega_Y) \otimes \mathcal{O}_X(R).$$

We now show how to obtain a double cover from a complex manifold Y which is connected and which admits an effective divisor whose class in $\mathrm{Pic}(Y)$ is a multiple of a primitive line bundle. See [BHPVdV15, §1.17] for a more general discussion. Let Y be a connected complex manifold, and suppose that there exists a divisor $B \in \mathrm{Div}(X)$ either effective or zero such that

$$\mathcal{O}_Y(B) = \mathcal{L}^{\otimes 2}$$

for some primitive line bundle $\mathcal{L}$ on Y. Assume that there is a global section $s \in H^0(Y, \mathcal{O}_Y(B))$ vanishing exactly along B. If $B = 0$, let s be the constant function 1. We denote by L the total space of $\mathcal{L}$, by $p : L \to X$ the projection and by $t \in H^0(L, p^*\mathcal{L})$ the tautological section, i.e., $t(l) = (l, l)$ for every $l \in L$. Then the zero divisor of $p^*s - t^2$ defines an analytic subspace X in L. If $B \neq 0$ and B is reduced, X is an irreducible normal analytic subspace of L, and $\pi := p|_X$ is a covering of degree n branched along B. If $\mathrm{Pic}(Y)$ is torsion free, then B uniquely determines $\mathcal{L}$. Moreover, X has at most singularities over singular points of B. In particular, if B is reduced and smooth, then also X is smooth. If $B = 0$, then X is connected when $\mathcal{L}$ is exactly of order 2 in $\mathrm{Pic}(Y)$: the double cover is unramified and determined by the torsion bundle $\mathcal{L}$. We conclude with the following two useful results, see [BHPVdV15, Lemma I.17.1, Lemma I.17.2] for more general statements.

Lemma 1.6.3. *Let $\pi : X \to Y$ be a double cover of complex manifolds branched along a smooth divisor B and determined by $\mathcal{L}$, where $\mathcal{L}^{\otimes 2} = \mathcal{O}_Y(B)$. Let B_1 be the reduced divisor $\pi^{-1}(B)$ on X. Then:*

(i) $\mathcal{O}_X(B_1) = \pi^*\mathcal{L}$.

(ii) $\pi^*B = 2B_1$.

(iii) $\omega_X = \pi^*(\omega_Y \otimes \mathcal{L})$.

Lemma 1.6.4. *Let $\pi : X \to Y$ be as in Lemma 1.6.3. Then*

$$\pi_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{L}^{-1}.$$

Chapter 2

Generalities on IHS manifolds

In this Chapter we introduce varieties which will be studied in this thesis, the *IHS manifolds*. In Section 2.1 we briefly recall K3 surfaces, which will turn out to be the only example in dimension 2 of IHS manifold, and we also define Hodge structures, which will play an important role in Chapter 3. We then introduce IHS manifolds in Section 2.2, focusing in particular on the construction of Hilbert squares of K3 surfaces. Moreover, we recall the Torelli theorems for IHS manifolds, the construction of elementary Mukai flops, the definition of birational Kähler cone. We also define double EPW sextics, which are examples of IHS fourfolds: they will be crucial in Chapter 5. We conclude by collecting some useful results on IHS manifolds in Section 2.3.

2.1 K3 surfaces

We give the definition of K3 surface.

Definition 2.1.1. A *K3 surface* is a smooth compact complex surface S whose canonical bundle is trivial and such that $H^1(S, \mathcal{O}_S) = 0$.

By a result of Siu, see [Siu83], every K3 surface is a Kähler manifold. Before discussing other properties of K3 surfaces, we give some examples.

Example 2.1.2. Let $S_4 \subset \mathbb{P}^3$ be a smooth quartic surface. By the adjunction formula the canonical bundle of S_4 is given by

$$\omega_{S_4} = (\omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(4))|_{S_4} = \mathcal{O}_{S_4}.$$

Moreover, by the Lefschetz hyperplane theorem, see [Voi02, §13], the surface $S_4 \subset \mathbb{P}^3$ is simply connected, which implies $H^1(S_4, \mathcal{O}_{S_4}) = 0$. Thus a smooth quartic surface in $\mathbb{P}^3$ is a K3 surface.

Example 2.1.3. The only other complete intersections which are K3 surfaces are $S_{2,3} \subset \mathbb{P}^4$ and $S_{2,2,2} \subset \mathbb{P}^5$, respectively the intersection of a quadric and a cubic hypersurface in $\mathbb{P}^4$ and the intersection of three quadrics in $\mathbb{P}^5$. See [Bea96, Example VIII.8] for details.

Example 2.1.4. Let $\pi : S_2 \to \mathbb{P}^2$ be a double cover ramified along a smooth curve $C_6 \subset \mathbb{P}^2$ of degree 6. By results seen in Section 1.6, the canonical bundle of S_2 is

$$\omega_{S_2} = \pi^*(\omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(3)) = \mathcal{O}_{S_2},$$

and we have

$$\begin{aligned} H^1(S_2, \mathcal{O}_{S_2}) &= H^1(\mathbb{P}^2, \pi_*(\mathcal{O}_{S_2})) \\ &= H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)) \\ &= 0, \end{aligned}$$

hence S_2 is a K3 surface.

Example 2.1.5. Let A be an abelian surface and consider the involution

$$\iota : A \to A, \qquad x \mapsto -x.$$

There are 16 points $p_1, \dots, p_{16} \in A$ which are fixed by ι. If $\beta : A' \to A$ is the blow-up of A in $p_1, \dots, p_{16}$, then ι gives rise to an involution ι' on A' which fixes the exceptional divisors $E_i = \beta^{-1}(p_i)$. One can show that the quotient variety $S := A'/\langle \iota' \rangle$ is a K3 surface, called *Kummer surface*. For further details on this construction, see [Bea96, Example VIII.10].

Kodaira in [Kod64] has shown that any K3 surface is a deformation of a smooth quartic surface in $\mathbb{P}^3$. In particular all the K3 surfaces are diffeomorphic. Hence a K3 surface is simply connected. Applying the Noether formula

$$\mathcal{X}(\mathcal{O}_S) = \frac{1}{12}(\mathcal{K}_S^2 + e(S))$$

in the case of a K3 surface S, where $\mathcal{K}_S$ is a canonical divisor and $e(S)$ is the Euler characteristic of S, we obtain

$$e(S) = 12 \cdot \mathcal{X}(\mathcal{O}_S) = 24,$$

hence the second Betti number of S is $b_2(S) = 22$. Then the Hodge diamond of a K3 surface is the following:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1\,. & & \end{array}$$

Since $H^1(S, \mathcal{O}_S) = 0$, the Picard group $\mathrm{Pic}(S)$ is isomorphic to the Néron–Severi group $\mathrm{NS}(S)$ and it embeds in $H^2(S, \mathbb{Z})$. This implies in particular that

$$\mathrm{Pic}(S) \cong H^{1,1}(S) \cap H^2(S, \mathbb{Z}).$$

Combining the Noether formula and the Riemann–Roch theorem, one obtains the following.

Theorem 2.1.6 (Riemann–Roch for K3 surfaces)**.** *Let S be a K3 surface and $L \in \mathrm{Pic}(S)$. Then*

$$\mathcal{X}(L) = 2 + \frac{1}{2} c_1(L)^2.$$

If L is ample, then $\mathcal{X}(L) = \dim(H^0(S, L))$.

In particular, if $C \subset S$ is an irreducible curve, then its arithmetic genus is

$$p_a(C) = 1 + \frac{1}{2}C^2.$$

By the Universal Coefficient Theorem $H^2(S, \mathbb{Z})$ is torsion free. Moreover, $H^2(S, \mathbb{Z})$ has a lattice structure with product given by the intersection pairing. One can show the following isometry of lattices:

$$H^2(S, \mathbb{Z}) \cong \Lambda_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2},$$

where U and $E_8(-1)$ are the lattices defined respectively in Example 1.4.5 and in Example 1.4.7. See [BHPVdV15, §VIII.3] for details. Hence $H^2(S, \mathbb{Z})$ has a structure of even unimodular lattice of signature $(3, 19)$.

2.1.1 Complete linear systems on K3 surfaces

In this section we state two important results in the theory of complete linear systems induced by divisors on K3 surfaces. The first one characterises the base locus of complete linear systems induced by big and nef divisors on K3 surfaces. See [May72] for the case of complex K3 surfaces and [SD74, Proposition 8.1] for the case of ample divisors on a K3 surface over an arbitrary algebraically closed field of characteristic $\neq 2$.

Theorem 2.1.7 (Mayer, Saint–Donat)**.** *Let X be a K3 surface with a big and nef line bundle $H \in \mathrm{Pic}(X)$. Then H has base points if and only if $H = mE + C$, where $m \geq 2$, E is a smooth elliptic curve, and C is a smooth rational curve, such that $(E, C) = 1$. In this case, the base locus of H is exactly C.*

Rieß obtained in [Rie18] a partial generalization of this result for IHS manifolds of dimension greater than 2, cf. Theorem 4.6.1. The following theorem gives an explicit geometrical description of K3 surfaces admitting an ample divisor D with $D^2 = 2$, where the square is taken with respect to the intersection form. A generalization of this result to the case of the Hilbert square of a K3 surface S with $\mathrm{Pic}(S)$ of rank 1 will be the aim of Chapter 4 and Chapter 5.

Theorem 2.1.8 (Saint–Donat)**.** *Let X be a projective K3 surface and suppose that there exists an ample divisor D with $D^2 = 2$. Then $\varphi_{|D|} : X \to \mathbb{P}^2$ is a double cover of $\mathbb{P}^2$ ramified over a smooth sextic curve.*

2.1.2 Hodge structures

In this section we recall the basic definitions and results on (pure) Hodge structures, which we will use in Chapter 3. We discuss Hodge structures after having introduced K3 surfaces since we will see some definitions and results linked to the theory of K3 surfaces. We refer to [Huy16, §3]. We deal with integral and rational Hodge structures: V will denote either a free $\mathbb{Z}$-module of finite rank or a finite-dimensional vector space over $\mathbb{Q}$, and $V_{\mathbb{C}}$ will be the $\mathbb{C}$-vector space obtained by scalar extension. Since V is defined over $\mathbb{Z}$ or $\mathbb{Q}$, which are subrings of $\mathbb{R}$, the complex vector space $V_{\mathbb{C}}$ admits a real structure, the complex conjugation $v \mapsto \overline{v}$.

Definition 2.1.9. Let V be a free $\mathbb{Z}$-module of finite rank or a finite-dimensional $\mathbb{Q}$-vector space. A *Hodge structure of weight* $n \in \mathbb{Z}$ on V is given by a direct sum decomposition of the complex vector space $V_{\mathbb{C}}$

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$.

Example 2.1.10. (i) If X is a complex projective manifold, then the torsion free quotient group $H^n(X,\mathbb{Z})_f$ of the singular cohomology group $H^n(X,\mathbb{Z})$ is an integral Hodge structure of weight n, and the rational cohomology group $H^n(X,\mathbb{Q})$ is a rational Hodge structure of weight n. The direct sum decomposition is the standard Hodge decomposition

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X).$$

Here, $H^{p,q}(X)$ could either be viewed as the space of de Rham classes of bidegree (p,q) or as the Dolbeault cohomology $H^q(X,\Omega_X^p)$.

(ii) Let X be a compact complex manifold of *even* dimension $2n$. We denote by $H^k(X,\mathbb{Z})[2n]$ the k-th shifted cohomology group, which is an integral Hodge structure of weight $k-2n$ with the following Hodge decomposition:

$$H^k(X,\mathbb{C})[2n] = \bigoplus_{p+q=k-2n} H^{p,q}(X)[2n],$$

where $p,q \in \{-n, 1-n, \ldots, k-n\}$ and

$$H^{p,q}(X)[2n] = H^{p+n,q+n}(X).$$

Similarly we define the rational Hodge structure $H^k(X,\mathbb{Q})[2n]$ of weight $k-2n$. The same construction can be performed with an integral or rational Hodge structure V of weight n, obtaining the shifted Hodge structure $V[2k]$ of weight $n-2k$.

For a Hodge structure V of *even* weight $n=2k$ the intersection $V \cap V^{k,k}$ is called the space of *Hodge classes* in V.

Definition 2.1.11. Let V be an integral Hodge structure (respectively rational Hodge structure) of weight n. Then a *sub-Hodge structure* V' of V is given by a $\mathbb{Z}$-submodule $V' \subset V$ (respectively $\mathbb{Q}$-linear subspace) such that the Hodge structure on V induces a Hodge structure on V', i.e.,

$$V'_{\mathbb{C}} = \bigoplus_{p+q=n} (V'_{\mathbb{C}} \cap V^{p,q}).$$

We say that a sub-Hodge structure $V' \subset V$ of an integral Hodge structure V is *primitive* if V/V' is torsion free.

Note that the space of all Hodge classes is a sub-Hodge structure.

Example 2.1.12. Most of the standard linear algebra constructions have analogues in Hodge theory.

(i) The *direct sum* $V \oplus W$ of two Hodge structures V and W of the same weight n is a Hodge structure of weight n by setting

$$(V \oplus W)^{p,q} = V^{p,q} \oplus W^{p,q}.$$

(ii) Let V and W be Hodge structures of weight n and m respectively. The *tensor product* $V \otimes W$ is a Hodge structure of weight $n+m$ by setting

$$(V \otimes W)^{p,q} = \bigoplus V^{p_1,q_1} \otimes W^{p_2,q_2},$$

where the sum is over all pairs of t-uples $(p_1, q_1), (p_2, q_2)$ with $p_1 + p_2 = p$ and $q_1 + q_2 = q$. Note that if p is given, then necessarily $q = n - p$.

(iii) For a Hodge structure V of weight n, the *dual* $V^* := \mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Z})$, or $\mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$ if V is rational, is a Hodge structure of weight $-n$ by setting

$$(V^*)^{-p,-q} := \mathrm{Hom}_{\mathbb{C}}(V^{p,q}, \mathbb{C}) \subset \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C}) = V_{\mathbb{C}}^*.$$

Since by the universal coefficient theorem there is an isomorphism

$$H_n(X, \mathbb{C}) \xrightarrow{\sim} \mathrm{Hom}(H^n(X, \mathbb{C}), \mathbb{C}),$$

the torsion free quotient group $H_n(X, \mathbb{Z})_f$ of the singular homology group $H_n(X, \mathbb{Z})$ is a Hodge structure of weight $-n$, and $H_n(X, \mathbb{Q})$ is a rational Hodge structure of weight $-n$.

(iv) If V is a Hodge structure of weight n, then the *symmetric product* $\mathrm{Sym}^k V$ is a Hodge structure of weight kn, where $(\mathrm{Sym}^k V)^{p,q}$ is the sum of all $\bigotimes \mathrm{Sym}^{k_i}(V^{p_i,q_i})$ with $\sum k_i = k$ and $\sum k_i p_i = p$. For instance, if $n = 1$ and $k = 2$ we have

$$\begin{aligned} (\mathrm{Sym}^2 V)^{2,0} &= \mathrm{Sym}^2 V^{1,0}, \\ (\mathrm{Sym}^2 V)^{1,1} &= V^{1,0} \otimes V^{0,1}, \\ (\mathrm{Sym}^2 V)^{0,2} &= \mathrm{Sym}^2 V^{0,1}. \end{aligned}$$

Similarly one defines the Hodge structure of the *exterior product* $\bigwedge^k V$.

We now define the notion of morphism between Hodge structures.

Definition 2.1.13. Let V and W be Hodge structures of weight n and m respectively. A *morphism of weight* k from V to W is a $\mathbb{Z}$-linear, or $\mathbb{Q}$-linear, map $f : V \to W$ such that its $\mathbb{C}$-linear extension satisfies

$$f(V^{p,q}) \subset W^{p+k,q+k}.$$

Note that for a non-trivial f of weight k, if V has weight n then we have necessarily $m = n + 2k$.

Remark 2.1.14. A morphism $f : V \to W$ of weight k can be seen as a morphism $f : V \to W[2k]$ of weight zero. Then we have

$$\mathrm{Hom}_k(V, W) = \mathrm{Hom}_0(V, W[2k]),$$

where $\mathrm{Hom}_k(V, W)$ is the space of Hodge morphisms of weight k. Note that

$$\mathrm{Hom}_k(V, W) = (V^* \otimes W) \cap (V^* \otimes W)^{k,k},$$

which is the space of Hodge classes of the Hodge structure $V^* \otimes W$, which has weight $2k$. In particular $\mathrm{Hom}_k(\mathbb{Z}, V) = V \cap V^{k,k}$ is the space of Hodge classes in V.

We now introduce the Hodge structures of *K3 type*, a family of Hodge structures of weight 2.

Definition 2.1.15. A Hodge structure of *K3 type* is a (rational or integral) Hodge structure V of weight 2 with

$$\dim_{\mathbb{C}}(V^{2,0}) = 1 \text{ and } V^{p,q} = 0 \text{ for } |p - q| > 2.$$

Hodge structures of K3 type have clearly a link with K3 surfaces: the cohomology groups $H^2(S, \mathbb{Z})$ and $H^2(S, \mathbb{Q})$ of a complex K3 surface S are Hodge structures of K3 type. Any Hodge structure of K3 type contains two natural sub-Hodge structures: the sub-Hodge structure of all Hodge classes $V^{1,1} \cap V$ and the *transcendental lattice* or *transcendental part*, which we now define.

Definition 2.1.16. Let V be an integral or rational Hodge structure of K3 type. Then the *transcendental lattice* or *transcendental part* T is the minimal primitive sub-Hodge structure

$$T \subset V \text{ with } V^{2,0} = T^{2,0} \subset T_{\mathbb{C}}.$$

The primitivity, i.e., the condition that V/T is torsion free, has to be added for integral Hodge structures to obtain the existence of the minimal sub-Hodge structure with the property given above. The transcendental lattice T is again of K3 type. Note that if $V = H^2(S, \mathbb{Z})$ with S a complex K3 surface, then

$$V^{1,1} \cap V = H^{1,1}(S) \cap H^2(S, \mathbb{Z}) \cong \mathrm{NS}(S) \cong \mathrm{Pic}(S),$$

and we denote the transcendental lattice T by

$$T(S) \subset H^2(S, \mathbb{Z}).$$

In this case, there exists another characterization of $T(S)$.

Lemma 2.1.17 (Lemma 3.3.1 in [Huy16])**.** *The transcendental lattice $T(S)$ of a complex K3 surface S is the orthogonal complement of the Néron–Severi group in the lattice $H^2(S, \mathbb{Z})$:*

$$T(S) = \mathrm{NS}(S)^{\perp}.$$

Proof. We follow [Huy16, Lemma 3.3.1]. We write $N := \mathrm{NS}(S)$ and $T := T(S)$. By the properties of the intersection pairing on a K3 surface, we have

$$H^{2,0}(S) \oplus H^{0,2}(S) \perp H^{1,1}(S),$$

where the orthogonality is with respect to the intersection pairing. Then every integral class which is orthogonal to T is in particular orthogonal to $H^{2,0}(S)$, hence it is of type $(1, 1)$. Since for a K3 surface we have the isomorphism

$$N \cong H^2(S, \mathbb{Z}) \cap H^{1,1}(S), \tag{2.1.1}$$

we obtain
$$T^{\perp} \subseteq N. \tag{2.1.2}$$
By (2.1.1) we see that $H^{2,0}(S) \subseteq N_{\mathbb{C}}^{\perp}$. Since by definition of transcendental lattice we have $T^{2,0} = H^{2,0}(S)$ and T is the minimal primitive sub-Hodge structure of $H^2(S,\mathbb{Z})$ with $H^{2,0}(S) = T^{2,0}$ we obtain $T \subseteq N^{\perp}$, hence taking the orthogonal complements
$$(N^{\perp})^{\perp} \subseteq T^{\perp}. \tag{2.1.3}$$
Clearly we have
$$N \subseteq (N^{\perp})^{\perp}, \tag{2.1.4}$$
so (2.1.2), (2.1.3) and (2.1.4) give
$$T^{\perp} \subseteq N \subseteq (N^{\perp})^{\perp} \subseteq T^{\perp}. \tag{2.1.5}$$
Since the inclusions in (2.1.5) are equalities, we have $T \subseteq (T^{\perp})^{\perp} = N^{\perp}$. If we show that $T = (T^{\perp})^{\perp}$, we are done. Note that $T_{\mathbb{R}}$ always contains the positive plane $(T^{2,0} \oplus T^{0,2}) \cap T_{\mathbb{R}}$. If T is non-degenerate, then we get $T = (T^{\perp})^{\perp}$, otherwise T has exactly one isotropic direction, i.e., there is exactly one element in an orthonormal basis of $T_{\mathbb{R}}$ whose square with respect to the bilinear form is 0, since $H^2(S,\mathbb{Z})$ has signature $(3,19)$. Consider this second case. After diagonalizing the intersection form on $H^2(S,\mathbb{R})$ to $(1,1,1,-1,\ldots,-1)$, we can assume that $T_{\mathbb{R}}$ has basis either $\{e_1, e_2, e_3+e_4\}$ or $\{e_1, e_2, e_3+e_4, e_5, \ldots, e_n\}$ for $n \geq 5$, hence $T_{\mathbb{R}}^{\perp}$ has basis $\{e_3+e_4, e_5, \ldots, e_{22}\}$ or $\{e_3+e_4, e_{n+1}, \ldots, e_{22}\}$, respectively. Then $T_{\mathbb{R}} = (T_{\mathbb{R}}^{\perp})^{\perp}$, which implies $T = (T^{\perp})^{\perp}$. We conclude that $T = N^{\perp}$, as we wanted. □

Note that in the proof of Lemma 2.1.17 we have obtained following equality of $\mathbb{C}$-vector spaces for a K3 surface S:
$$(H^{1,1}(S))^{\perp} = H^{2,0}(S) \oplus H^{0,2}(S),$$
where the orthogonality is taken with respect to the $\mathbb{C}$-extension of the intersection form. We conclude this section with the following result, which will be fundamental in Section 3.3.4.

Lemma 2.1.18 (Lemma 3.3.3 in [Huy16])**.** *Let T be an integral or rational Hodge structure of K3 type such that there is no proper (primitive) sub-Hodge structure $0 \neq T' \subset T$ of K3 type. If $a : T \to T$ is any endomorphism of the Hodge structure with $a = 0$ on $T^{2,0}$, then $a = 0$. Similarly, if $a = \mathrm{id}$ on $T^{2,0}$, then $a = \mathrm{id}$.*

Proof. Suppose that a is zero on $T^{2,0}$. By assumption, $T' := \mathrm{Ker}(a) \subset T$ is a Hodge structure with $T'^{2,0} \neq 0$ and T/T' is torsion free. Hence $T' = T$ and so $a = 0$. If $a = \mathrm{id}$ on $T^{2,0}$, repeat the argument with the endomorphsim $a - \mathrm{id}$. □

2.1.3 Torelli theorem and surjectivity of the period map

In this section we briefly recall the Torelli theorem for K3 surfaces and the surjectivity of the period map.

Let S be a K3 surface. Recall the isomorphism of lattices
$$H^2(S,\mathbb{Z}) \cong \Lambda_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

We define a *marking* on a K3 surface S as an isometry $\eta : H^2(S,\mathbb{Z}) \to \Lambda_{K3}$, and we call *marked K3 surface* a couple (S,η) given by a K3 surface S and a marking η on S. Two marked K3 surfaces (S_1,η_1) and (S_2,η_2) are *isomorphic* if there exists an isomorphism $f : S_1 \to S_2$ such that $\eta_2 = \eta_1 \circ f^*$. Let S_1 and S_2 be two K3 surfaces: we say that an isometry $H^2(S_1,\mathbb{Z}) \to H^2(S_2,\mathbb{Z})$ is a *Hodge isometry* if the isometry is an isomorphism of Hodge structures. Recall that a closed two-form ω on a K3 surface S is called *Kähler form* if it is the negative imaginary part of a Hermitian metric $h = g - i\omega$. A class in $H^{1,1}(S,\mathbb{R})$ is called a *Kähler class* if it can be represented by a Kähler form. A fundamental result in the theory of K3 surfaces is the following *Torelli theorem*, see [BR75].

Theorem 2.1.19 (Torelli theorem for K3 surfaces)**.** *Let S_1 and S_2 be two K3 surfaces. Then S_1 and S_2 are isomorphic if and only if there exists a Hodge isometry $H^2(S_1,\mathbb{Z}) \to H^2(S_2,\mathbb{Z})$. Moreover, if $f : H^2(S_1,\mathbb{Z}) \to H^2(S_2,\mathbb{Z})$ is a Hodge isometry, then there exists an isomorphism $\tilde{f} : S_2 \to S_1$ such that $\tilde{f}^* = f$ if and only if f sends a Kähler class on S_1 to a Kähler class on S_2. If such $\tilde{f}$ exists, it is unique.*

Let now $\mathcal{M}_{\Lambda_{K3}}$ be the moduli space of marked K3 surfaces modulo isomorphisms, and let

$$\Omega := \{[x] \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) \,|\, q_{\Lambda_{K3}}(x) = 0,\ (x,\bar{x})_{\Lambda_{K3}} > 0\},$$

where $q_{\Lambda_{K3}}$ and $(\cdot\,,\cdot)_{\Lambda_{K3}}$ are the quadratic form and the bilinear form of the lattice Λ_{K3}. Note that Ω is an open subset of a quadric in $\mathbb{P}(\Lambda_{K3} \otimes \mathbb{C})$. Denote by $\eta_{\mathbb{C}} : H^2(X,\mathbb{C}) \to \Lambda_{K3} \otimes \mathbb{C}$ be the $\mathbb{C}$-linear extension of a marking η. We define the *period map* as follows:

$$\mathcal{P} : \mathcal{M}_{K3} \to \Omega, \qquad [(X,\eta)] \mapsto \eta_{\mathbb{C}}(H^{2,0}(X)).$$

The following result is known as the *surjectivity of the period map*, see [Kul77], [Tod80], [Siu81].

Theorem 2.1.20 (Surjectivity of the period map)**.** *The period map $\mathcal{P}$ defined above is surjective.*

2.2 IHS manifolds

2.2.1 Definition of IHS manifold

Let X be a complex manifold. We say that a holomorphic 2-form $\sigma \in H^0(X,\Omega_X^2)$ on X is *non-degenerate* if the induced skew-symmetric pairing $\mathcal{T}_X \times \mathcal{T}_X \to \mathcal{O}_X$ is non-degenerate at every point $x \in X$. Note that if X admits a non-degenerate 2-form σ, then the complex dimension of X is even: for the skew-symmetry we have $\sigma_x(v,w) = -\sigma_x(w,v)$ for every $x \in X$ and $v,w \in \mathcal{T}_xX$, so the matrix which represents the map is anti-symmetric, and the non-degeneracy implies that this matrix is invertible, but there are no matrix of odd order with non-zero determinant which are anti-symmetric. Moreover, if X is a complex variety which admits a non-degenerate form and $\dim(X) = 2n$, then σ^n is a nowhere vanishing section of the canonical bundle ω_X, which is then trivial. We say that a holomorphic 2-form $\sigma \in H^0(X,\Omega_X^2)$ is a *symplectic structure* if σ is closed and non-degenerate.

Definition 2.2.1. An *irreducible holomorphic symplectic* (IHS) *manifold* is a simply connected compact complex Kähler manifold X such that $H^0(X, \Omega_X^2)$ is generated by a symplectic structure.

The definition of IHS manifold is a generalization to higher dimensions of the definition of K3 surface, the only example of IHS manifold of dimension 2, as shown by the Enriques–Kodaira classification of compact complex surfaces. There is another possible generalization of K3 surfaces, leading to the definition of *Calabi–Yau variety*, which is a compact complex manifold X with trivial canonical bundle and $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, \dots, \dim(X) - 1$. The interest in IHS manifolds has been increasing thanks to the following Beauville–Bogomolov decomposition theorem, see [Bea83b, Théorème 2].

Theorem 2.2.2 (Beauville–Bogomolov decomposition)**.** *Let X be a compact Kähler manifold with $c_1(X)_\mathbb{R} = 0$. Then there exists a finite étale cover $\tilde{X}$ of X such that*

$$\tilde{X} = T \times \prod_i C_i \times \prod_j Y_j,$$

where T is a complex torus, C_i is an irreducible Calabi–Yau variety for every i and Y_j is an IHS manifold for every j.

If X is an IHS manifold, then the $\mathbb{C}$-vector space $H^0(X, \Omega_X^p)$ is zero if p is odd, and it is generated by $\sigma^{\frac{p}{2}}$ if $0 \leq p \leq \dim(X)$ is even, where σ is the symplectic form, see [Bea83b, Proposition 3]. This implies in particular that the Picard group $\mathrm{Pic}(X)$ is isomorphic to the Néron–Severi group $\mathrm{NS}(X)$, and this embeds in the second cohomology group $H^2(X, \mathbb{Z})$.

We now introduce a quadratic form on the $\mathbb{C}$-vector space $H^2(X, \mathbb{C})$, see [GHJ12, Definition 22.10].

Definition 2.2.3. Let X be an IHS manifold with complex dimension $2n$, and denote by $\sigma \in H^{2,0}(X)$ a symplectic form such that $\int_X (\sigma\overline{\sigma})^n = 1$. We define the *Beauville–Bogomolov form* $\tilde{q}_X$ on $H^2(X, \mathbb{C})$ as follows. If $\alpha \in H^2(X, \mathbb{C})$ is of the form $\alpha = \lambda\sigma + \beta + \mu\overline{\sigma} \in H^2(X, \mathbb{C})$, with $\beta \in H^{1,1}(X)$, then

$$\tilde{q}_X(\alpha) := \lambda\mu + \frac{n}{2} \int_X \beta^2 (\sigma\overline{\sigma})^{n-1}.$$

The form $\tilde{q}_X$ gives rise to a non-degenerate integral quadratic form q_X on the second singular cohomology group $H^2(X, \mathbb{Z})$, which is torsion free by the Universal Coefficient Theorem. This result is due to Beauville, Bogomolov and Fujiki, see [Bea83b] and [Fuj87].

Theorem 2.2.4 (Beauville–Bogomolov–Fujiki form)**.** *Let X be an IHS manifold of dimension $2n$. Then there exists an integral indivisible quadratic form*

$$q_X : H^2(X, \mathbb{Z}) \to \mathbb{Z}$$

and $c_X \in \mathbb{Q}_{>0}$ such that

$$\int_X \alpha^{2n} = c_X \frac{(2n)!}{n! 2^n} q_X(\alpha)^n \qquad \text{for all } \alpha \in H^2(X, \mathbb{Z}).$$

The quadratic form q_X is called Beauville–Bogomolov–Fujiki (BBF) form, and c_X is called Fujiki constant of X. Moreover, $(H^2(X, \mathbb{Z}), q_X)$ is a lattice of signature $(3, b_2(X) - 3)$, where $b_2(X)$ is the second Betti number of X, and q_X is a multiple of the form $\tilde{q}_X$ given in Definition 2.2.3.

2.2.2 Hilbert square of a smooth complex surface

The first example of IHS manifold of dimension greater than 2 that we discuss is given by *Hilbert squares of K3 surfaces.* We recall the general construction of the Hilbert square of a smooth complex surface S, denoted by $S^{[2]}$.

Let $\Delta \subset S^2$ be the diagonal, i.e., $\Delta = \{(p,p) \in S^2 \,|\, p \in S\}$, and denote by $\beta : \mathrm{Bl}_\Delta(S^2) \to S^2$ the blow-up in Δ. Denote by $N \subset \mathrm{Bl}_\Delta(S^2)$ the exceptional divisor of β. Consider the following diagrams:

$$\begin{array}{ccc} \mathrm{Bl}_\Delta(S^2) & \xrightarrow{\tilde\pi} & S^{[2]} \\ \big\downarrow{\scriptstyle\beta} & & \big\downarrow{\scriptstyle HC} \\ S^2 & \xrightarrow{\pi} & S^{(2)}, \end{array} \qquad\qquad \begin{array}{ccc} N & \xrightarrow{\tilde\pi} & E \\ \big\downarrow{\scriptstyle\beta} & & \big\downarrow{\scriptstyle HC} \\ \Delta & \xrightarrow{\pi} & \mathrm{Sing}(S^{(2)}), \end{array} \tag{2.2.1}$$

where π is the quotient with respect to the involution of S^2 given by $\iota : S^2 \to S^2$, $(p,q) \overset{\iota}{\mapsto} (q,p)$, the morphism $\tilde\pi$ is a double cover with ramification divisor N and branch divisor E, and HC is the so-called Hilbert–Chow morphism, i.e., the blow-up of $S^{(2)}$ in its singular locus. We show explicitly that the diagrams are commutative, that $S^{(2)}$ is singular in $\pi(\Delta)$, and that every singularity is locally the vertex of a cone. Moreover, we show that $S^{[2]}$ is smooth. Note that this construction can be performed for any smooth complex surface S.

Let

$$\iota : S \times S \longrightarrow S \times S\,, \qquad (p,q) \mapsto (q,p)$$

be the involution which interchanges the two factors. For $p, q \in S$, let (z_1, z_2) and (w_1, w_2) be local coordinates near these two points, considering the structure of complex manifold of S. Then

$$u_1 := z_1 + w_1, \qquad u_2 := z_2 + w_2\,, \qquad v_1 := z_1 - w_1, \qquad v_2 := z_2 - w_2$$

are local coordinates near $(p,q) \in S^2$, and the first two are invariant with respect to the involution ι, whereas the last two are anti-invariant. Locally the image of (p,q) in $S^{(2)} := S^2/\iota$ is thus isomorphic to the subvariety of $\mathbb{C}^2 \times \mathbb{C}^3 \cong \mathbb{C}^5$ with coordinates

$$u_1, u_2, u_3 := v_1^2, u_4 := v_1 v_2, u_5 := v_2^2 \tag{2.2.2}$$

defined by the quadratic equation

$$u_3 u_5 = u_4^2.$$

Hence locally near the image of a point $(p,p) \in S^2$, the variety $S^{(2)}$ is isomorphic to $\mathbb{C}^2 \times Q$ where Q is the subvariety of $\mathbb{C}^3$ defined by $u_3 u_5 = u_4^2$, which is a cone. Since the cone Q is singular in the point $(0,0,0)$, the variety $S^{(2)}$ is singular exactly in the image of the diagonal $\Delta := \{(p,p) \in S^2 \,|\, p \in S\}$.

We blow up $S^{(2)}$ in its singular locus to obtain a smooth variety. To see that this gives a smooth fourfold we just need to check that the blow-up of Q in the vertex $O := (0,0,0)$ is smooth. Equivalently, we check that the strict transform of Q in the blow-up of $\mathbb{C}^3$ in O is smooth. This blow-up is the subvariety of $\mathbb{C}^3 \times \mathbb{P}^2$, with coordinates $((u_3, u_4, u_5), (z_3 : z_4 : z_5))$, defined by the equations

$$u_i z_j - u_j z_i = 0, \qquad 3 \le i, j \le 5. \tag{2.2.3}$$

On the open subset where $z_5 = 1$ these equations reduce to

$$u_3 = u_5 z_3, \qquad u_4 = u_5 z_4, \tag{2.2.4}$$

so this open subset is isomorphic to $\mathbb{C}^2 \times \mathbb{C} \times \mathbb{C}^2$ with coordinates $(u_1, u_2, u_5, z_3, z_4)$. Substituting the equations (2.2.4) in $u_3 u_5 - u_4^2 = 0$ we obtain the (reducible) equation $u_5^2(z_3 - z_4^2) = 0$. The equation $u_5 = 0$ implies $u_3 = u_4 = 0$ which defines the exceptional divisor $O \times \mathbb{P}^2$ of the blow up. Hence the strict transform of Q is defined by $z_3 - z_4^2 = 0$, so this is smooth. On this open subset one has the coordinates (u_1, u_2, u_5, z_4), note that $z_3 = z_4^2$. Using $u_5 = v_2^2$, $u_4 = v_1 v_2$ from (2.2.2) and $u_4 = u_5 z_4$ from (2.2.4), this open set is the closure of the image of the rational map

$$\mathbb{C}^2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \times \mathbb{C}^2, \qquad (u_1, u_2, v_1, v_2) \mapsto (u_1, u_2, v_2^2, v_1/v_2).$$

Similarly one shows the smoothness of the strict transform of Q in the other coordinates patches.

Another way to proceed is to first blow up $S \times S$ in Δ to obtain a fourfold $\mathrm{Bl}_\Delta(S^2)$. The involution ι of S^2 lifts to an involution $\tilde{\iota}$ on $\mathrm{Bl}_\Delta(S^2)$: the quotient $\mathrm{Bl}_\Delta(S^2)/\tilde{\iota}$ is smooth by Chevalley–Shephard–Todd theorem, see [ST54] and [Che55], being the fixed locus E of $\tilde{\iota}$ of codimension 1 and $\tilde{\iota}$ a quasi-reflection. Moreover, $\mathrm{Bl}_\Delta(S^2)/\tilde{\iota}$ coincides with the blow-up of S^2/ι in its singular locus. To see this, we remark that since Δ is locally defined by $v_1 = v_2 = 0$, the blow-up $\mathrm{Bl}_\Delta(S^2)$ is locally defined by the subset of $\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{P}^1$ with coordinates $((u_1, u_2, v_1, v_2), (w_1 : w_2))$ given by

$$v_1 w_2 - v_2 w_1 = 0. \tag{2.2.5}$$

The involution ι lifts to the involution $\tilde{\iota}$ induced by

$$\begin{array}{rcl} \tilde{\iota} : \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{P}^1 & \to & \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{P}^1, \\ ((u_1, u_2, v_1, v_2), (w_1 : w_2)) & \mapsto & ((u_1, u_2, -v_1, -v_2), (w_1 : w_2)), \end{array}$$

since it must coincide with ι for points with $(v_1, v_2) \neq (0, 0)$ and it must preserve the equation (2.2.5) defining the blow-up. On the open subset determined by the equation $w_2 = 1$, which gives $v_1 = v_2 w_1$, we find that the blow-up is isomorphic to $\mathbb{C}^4$ with coordinates (u_1, u_2, v_2, w_1) and the involution $\tilde{\iota}$ is the change of sign of v_2. Thus the invariants for this action are generated by u_1, u_2, v_2^2, w_1, so the quotient is again isomorphic to $\mathbb{C}^4$. The main point is that the fixed point set has codimension one after blowing up, and this fixed point set is exactly the exceptional divisor, defined by $(v_1, v_2) = (0, 0)$, but locally in the patch $w_2 = 1$ by $v_2 = 0$. As $v_1 = v_2 w_1$, we see that this open set of the quotient of the blow-up is the closure of the image of the rational map:

$$\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C} \times \mathbb{C}, \qquad (u_1, u_2, v_1, v_2) \mapsto (u_1, u_2, v_2^2, v_1/v_2).$$

In local coordinates we thus see that $\mathrm{Bl}_\Delta(S^2)/\tilde{\iota}$ is isomorphic to the blow-up of S^2/ι in its singular locus. This gives the commutativity of (2.2.1).
The points of $S^{[2]}$ are the zero-dimensional subschemes of S of length 2. We introduce the following notation for points $x \in S^{[2]}$:

- $x = p + q$, where $p, q \in S$ are distinct points.

- $x = (p, t)$, where $t \in \mathbb{P}^1$ is a tangent direction through a point $p \in S$.

We will use this notation in the next sections. Fujiki showed in [Fuj83] that the Hilbert square of a K3 surface is an IHS manifold.

Theorem 2.2.5 (Fujiki)**.** *Let S be a K3 surface. Then the Hilbert square $S^{[2]}$ of S is an IHS manifold.*

It was the first example of IHS manifold of dimension greater than 2 to be found. We conclude this section by recalling the following useful result, see [NW04, Remark 4.13].

Theorem 2.2.6. *Let S be a K3 surface and $X = S^{[2]}$. Then*

$$\int_X c_2(X)^2 = 828, \qquad \int_X c_4(X) = 324.$$

2.2.3 Hilbert schemes of n points on a K3 surface

Hilbert squares of K3 surfaces are an example of the more general construction of the *Hilbert scheme of n points on a K3 surface S*, which is the scheme which parametrises zero-dimensional closed subschemes of length n of S. We denote it by $S^{[n]}$. By a theorem of Fogarty, $S^{[n]}$ is a $2n$-dimensional irreducible smooth variety, see [Fog68]. If S is a projective K3 surface, $S^{[n]}$ is projective by a result of Grothendieck, see [Gro61]. Let $S^{(n)}$ be the quotient of $S^n = S \times \cdots \times S$ by the symmetric group of n elements, so that $S^{(n)}$ is the variety of 0-cycles of degree n. Then the *Hilbert–Chow morphism* $\rho : S^{[n]} \to S^{(n)}$ is defined as follows: a point $[\xi] \in S^{[n]}$ is mapped to the cycle $\sum_x l(\mathcal{O}_{\xi,x}) \cdot x$, see [Ive06]. The singular locus of $S^{(n)}$ is the so-called diagonal, i.e., the set of cycles $p_1 + \cdots + p_n$ such that there exist distinct i and j with $p_i = p_j$. Then the Hilbert–Chow morphism is a desingularization of $S^{(n)}$: the pre-image of the diagonal is an irreducible divisor E on $S^{[n]}$. The Hilbert scheme of n points on a K3 surface is an IHS manifold by the following result by Beauville, see [Bea83b, Théorème 3].

Theorem 2.2.7 (Beauville)**.** *Let S be a K3 surface. Then $S^{[n]}$ is an IHS manifold of dimension $2n$ for every $n \geq 2$.*

Let S be a K3 surface and $n \geq 2$. There exists a primitive class $\delta \in \mathrm{Pic}(S^{[n]})$ such that $2\delta = [E]$. Moreover, there exists a primitive embedding of lattices

$$i : H^2(S, \mathbb{Z}) \hookrightarrow H^2(S^{[n]}, \mathbb{Z})$$

such that $H^2(S^{[n]}, \mathbb{Z}) = i(H^2(S, \mathbb{Z})) \oplus \delta$, and $q_{S^{[n]}}(\delta) = -2(n-1)$, hence there is an isomorphism

$$H^2(S^{[n]}, \mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1) \rangle.$$

Similarly we have

$$\mathrm{Pic}(S^{[n]}) = i(\mathrm{Pic}(S)) \oplus \mathbb{Z}\delta.$$

See [Bea83b, §6] for details. The Fujiki constant of Hilbert schemes of n points on a K3 surface was computed by Beauville in [Bea83b, §9].

Proposition 2.2.8 (Beauville)**.** *Let $X = S^{[n]}$ be the Hilbert scheme of n points on a K3 surface S. Denote by c_X the Fujiki constant of X of Theorem* 2.2.4*. Then $c_X = 1$.*

In particular, for K3 surfaces the BBF form coincides with the intersection form. We recall the following useful result on the cohomology ring of the Hilbert scheme of n points on a K3 surface.

Theorem 2.2.9 (Theorem 1 in [Mar07]). *Let S be a projective K3 surface. Then $H^*(S^{[n]}, \mathbb{Z})$ is torsion free for every $n \geq 1$.*

We have seen in Section 2.1 that the Hodge diamond of a K3 surface S is given by

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1\,. & & \end{array}$$

The following result by Göttsche, see [Göt90], shows that the Hodge numbers of the Hilbert scheme $S^{[n]}$ for $n \geq 2$ are determined by the Hodge numbers of the K3 surface S.

Theorem 2.2.10 (Göttsche). *Let S be a smooth projective complex surface. Then*

$$\sum_{\substack{n \geq 0 \\ 0 \leq s,t \leq 2n}} (-1)^{s+t} h^{s,t}(S^{[n]}) x^{s-n} y^{t-n} q^n = \prod_{n=1}^{\infty} \frac{\prod_{s+t \text{ odd}} (1 - x^{s-1} y^{t-1} q^n)^{h^{s,t}(S)}}{\prod_{s+t \text{ even}} (1 - x^{s-1} y^{t-1} q^n)^{h^{s,t}(S)}},$$

where x, y, q are variables, and $h^{p,q}(S^{[n]}) := \dim(H^{p,q}(S^{[n]}))$.

If S is a K3 surface, expanding the expression of Theorem 2.2.10 up to $n = 2$ we obtain that the Hodge diamond of $S^{[2]}$ is given by the following:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 1 & & 21 & & 1 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 1 & & 21 & & 232 & & 21 & & 1 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 1 & & 21 & & 1 & & \\ & & & 0 & & 0 & & & \\ & & & & 1\,. & & & & \end{array}$$

Similarly one can compute the Hodge diamond of $S^{[n]}$ for $n > 2$.

2.2.4 Other examples of IHS manifolds

We briefly present the other known examples of IHS manifolds up to deformation equivalence, cf. Section 2.2.6.

- Let A be an abelian surface. Consider the map

$$A^n = A \times \cdots \times A \to A$$

induced by the group law on A. Since this is commutative, we get an induced morphism $A^{(n)} \to A$, where $A^{(n)}$ is the quotient of A^n on the n-th symmetric group. Let $\Sigma_n : A^{[n]} \to A$ be the composition of this map with the Hilbert–Chow morphism. If $n \geq 1$, we call $K_n(A) := \Sigma_{n+1}^{-1}(0)$ the *n-th generalised Kummer variety.* In [Bea83b, §7] Beauville has shown that $K_n(A)$ is an IHS manifold of dimension $2n$ with second Betti number $b_2 = 7$.

- O'Grady obtained an example of an IHS manifold of dimension 10 and second Betti number $b_2 = 24$ in [O'G99].

- O'Grady obtained an example of an IHS manifold of dimension 6 and second Betti number $b_2 = 8$ in [O'G03].

We do not discuss further details on these examples since in this thesis we will deal only with Hilbert schemes of n points on a K3 surface.

2.2.5 Birational Kähler cone and movable cone

In this section we give the definitions of the birational Kähler cone and of the movable cone of an IHS manifold X and some useful results on this topic. We refer to [Huy99], [Huy03], [Bou04] and [Mar11].

Recall that a closed two-form ω on a complex manifold is called *Kähler form* if it is the negative imaginary part of a Hermitian metric $h = g - i\omega$. A complex manifold which admits a Kähler form is called *Kähler manifold.* Projective complex manifolds are Kähler manifolds by the Kodaira embedding theorem, see [GH78, I, §4] for details. Let X be an IHS manifold. A class in $H^{1,1}(X, \mathbb{R})$ is called a *Kähler class* if it can be represented by a Kähler form. Since a positive linear combination of Kähler forms is again a Kähler form, the set of all Kähler classes in $H^{1,1}(X, \mathbb{R})$ describes a convex cone.

Definition 2.2.11. Let X be an IHS manifold. The *Kähler cone*

$$\mathcal{K}_X \subseteq H^{1,1}(X, \mathbb{R})$$

is the open convex cone of all Kähler classes $[\omega] \in H^{1,1}(X, \mathbb{R})$. The *positive cone* of X is the connected component

$$\mathcal{C}_X \subseteq H^{1,1}(X, \mathbb{R})$$

of the open set $\{\alpha \in H^{1,1}(X, \mathbb{R}) \,|\, q_X(\alpha) > 0\}$ that contains $\mathcal{K}_X$.

By the Kodaira embedding theorem we have the equality

$$\mathrm{Amp}(X) = \mathcal{K}_X \cap \mathrm{NS}(X)_{\mathbb{R}}.$$

The following is a projectivity criterion for IHS manifolds given by Huybrechts, see [Huy99, Theorem 3.11].

Theorem 2.2.12 (Projectivity criterion for IHS manifolds)**.** *Let X be an IHS manifold. Then X is projective if and only if there exists a line bundle L on X with $q_X(c_1(L)) > 0$.*

The following result characterises ampleness and nefness for line bundles on complex projective manifolds with trivial canonical bundle.

Proposition 2.2.13 (Proposition 6.3 in [Huy99])**.** *Let X be a projective manifold with $\mathcal{K}_X \cong \mathcal{O}_X$. Then a line bundle L on X is ample if and only if $\int_X c_1(L)^{\dim(X)} > 0$ and $\int_C c_1(L) > 0$ for all curves $C \subseteq X$.*

We have the following two corollaries when the manifold considered is an IHS manifold. We denote as usual by $(\,\cdot\,,\,\cdot\,)$ the BBF bilinear form.

Corollary 2.2.14 (Corollary 6.4 in [Huy99])**.** *Let X be a projective IHS manifold of dimension $2n$ and let L be a line bundle on X. Then*

- *L is ample if and only if $c_1(L) \in \mathcal{C}_X$ and $\int_C c_1(L) > 0$ for all curves $C \subseteq X$.*
- *L is nef if and only if $c_1(L) \in \overline{\mathcal{C}}_X$ and $\int_C c_1(L) \geq 0$ for all curves $C \subseteq X$.*

Corollary 2.2.15 (Corollary 6.5 in [Huy99])**.** *Let X be an IHS manifold and let L be a line bundle on X. Then L is ample if and only if $c_1(L)$ satisfies the following:*

(i) $(c_1(L), \cdot\,)$ *is positive on* $\mathrm{Amp}(X)$.

(ii) If $M \in \mathrm{Pic}(X)$ *such that* $q_X(c_1(M), \cdot\,)$ *is positive on* $\mathrm{Amp}(X)$*, then* $(c_1(M), c_1(L)) > 0$.

Similarly L is nef if and only if $c_1(L)$ satisfies the following:

(i) $(c_1(L), \cdot\,)$ *is non-negative on* $\mathrm{Nef}(X)$.

(ii) If $M \in \mathrm{Pic}(X)$ *such that* $q_X(c_1(M), \cdot\,)$ *is non-negative on* $\mathrm{Nef}(X)$*, then* $(c_1(M), c_1(L)) \geq 0$.

We can now define the birational Kähler cone of an IHS manifold, which will play an important role in Section 4.3.

Definition 2.2.16. Let X be an IHS manifold. Then the *birational Kähler cone* of X is defined as

$$\mathcal{BK}_X := \bigcup_f f^*(\mathcal{K}_{X'}) \subseteq H^{1,1}(X, \mathbb{R}),$$

where f ranges through all birational maps from X to an IHS manifold X'.

We recall that a variety F of dimension n is *uniruled* if there is a variety F' of dimension $n-1$ and a dominant rational map $\phi : F' \times \mathbb{P}^1 \dashrightarrow F$, i.e., the image of ϕ is dense in F. A divisor of an IHS manifold is *uniruled* if it is a prime divisor which is a uniruled subvariety. The following proposition gives us examples of uniruled divisors on an IHS manifold.

Proposition 2.2.17 (Proposition 4.7 in [Bou04])**.** *Let X be an IHS manifold. Every irreducible effective divisor $D \in \mathrm{Div}(X)$ with $q_X(D) < 0$ is uniruled.*

We have the following characterisation of the closure of the birational Kähler cone of an IHS manifold.

Proposition 2.2.18 (Proposition 4.2 in [Huy03])**.** *Let X be an IHS manifold. Then $\alpha \in H^{1,1}(X, \mathbb{R})$ is in the closure $\overline{\mathcal{BK}}_X$ of the birational Kähler cone $\mathcal{BK}_X$ if and only if $\alpha \in \overline{\mathcal{C}}_X$ and $(\alpha, [D]) \geq 0$ for all uniruled divisors $D \subset X$.*

The following is a useful corollary, see for instance [Rie18, Corollary 2.3].

Corollary 2.2.19. *Let X be an IHS manifold, $E \in \mathrm{Pic}(X)$ an effective divisor and $H \in \overline{\mathcal{BK}}_X$. Then $(H, E) \geq 0$. In particular this applies for all nef line bundles $H \in \mathrm{Pic}(X)$.*

We now recall the definition of movable cone of a smooth projective variety. See also Definition 1.1.22.

Definition 2.2.20. Let X be a smooth projective variety. An effective divisor $D \in \mathrm{Div}(X)$ is *movable* if there exists an integer $k > 0$ such that $|kD|$ has no fixed components. A line bundle $L \in \mathrm{Pic}(X)$ is *movable* if it is the class of a movable divisor. The *movable cone*, or *moving cone*, of X, is the convex cone

$$\mathrm{Mov}(X) \subseteq \mathrm{NS}(X) \otimes \mathbb{R}$$

generated by movable classes.

The following corollary of Proposition 2.2.18 gives a relation between the birational Kähler cone and the movable cone of an IHS manifold, see for instance [Rie18, Corollary 2.6].

Corollary 2.2.21. *Let X be an IHS manifold. Then*

$$\overline{\mathrm{Mov}(X)} = \overline{\mathcal{BK}}_X \cap \mathrm{NS}(X)_{\mathbb{R}}.$$

If X is a projective IHS manifold, the closure of the movable cone admits a wall-and-chamber decomposition, see [Mar11, §5.2] for details.

Theorem 2.2.22 (Markman)**.** *Let X be an IHS manifold. Then*

$$\overline{\mathrm{Mov}(X)} = \bigcup_f f^*\mathrm{Nef}(X'),$$

where the union is taken over all non-isomorphic IHS birational models with birational maps $f : X \dashrightarrow X'$.

2.2.6 Deformation theory

In this section we recall the most important definitions and results, without proofs, from deformation theory. We mainly follow [Bea83b], [Huy99] and [GHJ12, §22].

Definition 2.2.23. Let X be a compact complex manifold. Then a *deformation* of X is given by a smooth proper morphism $\mathcal{X} \to S$, where $\mathcal{X}$ and S are connected complex spaces, and by an isomorphism $X \cong \mathcal{X}_0$, where $0 \in S$ is a distinguished point and $\mathcal{X}_0$ is the fiber over $0 \in S$. An *infinitesimal deformation* of X is a deformation with base space $S = \mathrm{Spec}(\mathbb{C}[\epsilon])$, where $\mathbb{C}[\epsilon]$ is the algebra of dual numbers $\mathbb{C}[\epsilon] = \mathbb{C}[x]/(x^2)$.

We will write $\mathcal{X} \to (S, 0)$ to denote a deformation $\mathcal{X} \to S$ with distinguished point $0 \in S$. We now introduce the notion of deformation equivalence.

Definition 2.2.24. Let X_1 and X_2 be two compact complex manifolds. We say that X_1 and X_2 are *deformation equivalent* if there exist connected complex spaces $\mathcal{X}$ and S, a smooth proper morphism $\mathcal{X} \to S$ and two points $t_1, t_2 \in S$ such that $X_1 \cong \mathcal{X}_{t_1}$ and $X_2 \cong \mathcal{X}_{t_2}$.

It is natural to wonder if a deformation of an IHS manifold is still an IHS manifold.

Proposition 2.2.25 (Beauville, [Bea83b])**.** *Let $\mathcal{X} \to S$ be a deformation of an IHS manifold $X \cong \mathcal{X}_0$. Then for t close to $0 \in S$, the fibre $\mathcal{X}_t$ is an IHS manifold.*

An example of deformation equivalent IHS manifolds is given by birational IHS manifolds, see [Huy99, Theorem 4.6] for details.

Proposition 2.2.26 (Huybrechts)**.** *Let X and X' be birational IHS manifolds. Then X and X' are deformation equivalent.*

We now define a *universal deformation*, which will play an important role in the Torelli theorems of Section 2.2.7.

Definition 2.2.27. Let X be a compact complex manifold. A deformation $\mathcal{X} \to (S, 0)$ of X is called *universal* if any other deformation $\mathcal{X}' \to (S', 0')$ is isomorphic to the pullback under a uniquely determined morphism $\varphi : S' \to S$ with $\varphi(0') = 0$.

If a universal deformation exists, this is unique up to isomorphism, and we denote it by $\mathcal{X} \to \mathrm{Def}(X)$, with distinguished point $0 \in \mathrm{Def}(X)$. The following result by Kuranishi gives a sufficient condition for the existence of a universal deformation, see [Kur62] and [Kur65] for details.

Theorem 2.2.28 (Kuranishi)**.** *If X is a compact complex manifold such that $H^0(X, \mathcal{T}_X) = 0$, where $\mathcal{T}_X$ is the tangent bundle of X, then there exists a universal deformation of X. Moreover, the universal deformation is universal for any of its fibers.*

If X is an IHS manifold, $H^0(X, \mathcal{T}_X) = H^0(X, \Omega^1_X) = 0$, hence there exists a universal deformation $\mathcal{X} \to \mathrm{Def}(X)$. Moreover, $\mathrm{Def}(X)$ is smooth and has dimension $h^1(X, \mathcal{T}_X) = h^1(X, \Omega_X) = h^{1,1}(X)$, see [Bog78], [Tia87], [Ran92], [Tod89] and [Kaw92] for details.

2.2.7 Local and global Torelli theorem

In this section we recall the most important definitions and results leading to the local Torelli theorem and to the global Torelli theorem for IHS manifolds. We will also recall a Hodge theoretic version of the global Torelli theorem, which will be useful in Chapter 4. We refer to [Bea83b], [Huy99], [Huy02], [Huy03], [Mar11], [GHJ12].

We will denote by Λ a lattice of signature $(3, b-3)$, where b represents the second Betti number $b_2(X)$ of an IHS manifold X. Note that for every IHS manifold X we have $b_2(X) \geq 3$ by [Gua01, Corollary 1]. We begin with the definition of *marked IHS manifold*, see [Huy99, §1.15].

Definition 2.2.29. Let Λ be a lattice of signature $(3, b-3)$. A *marked IHS manifold* is a pair (X, η), with X an IHS manifold and $\eta : H^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda$ an isometry of lattices, where $H^2(X, \mathbb{Z})$ is equipped with the BBF form. The isometry η is called *marking*. Two marked IHS manifolds (X_1, η_1) and (X_2, η_2) are said to be *isomorphic* if there exists an isomorphism $f : X_1 \to X_2$ such that $\eta_2 = \eta_1 \circ f^*$.

If X is a K3 surface, the abstract lattice Λ of Definition 2.2.29 is given by

$$\Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2},$$

and if X is an IHS manifold of $K3^{[n]}$-type, with $n \geq 2$, the abstract lattice Λ is given by

$$\Lambda_{K3^{[n]}} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1) \rangle,$$

see Example 1.4.3, Example 1.4.5 and Example 1.4.7.

Let (X, η) be a marked IHS manifold, and let $\pi : \mathcal{X} \to \mathrm{Def}(X)$ be the universal deformation introduced in Section 2.2.6. The marking η can be used to construct a family of markings $\{F_b : H^2(\mathcal{X}_b, \mathbb{Z}) \to \Lambda\}_{b \in \mathrm{Def}(X)}$ such that $F_0 = \eta$, see [Kod06, Theorem 2.4] for details.

Definition 2.2.30. Let (X, η) be a marked IHS manifold. The *local period map* is defined as

$$\mathcal{P} : \mathrm{Def}(X) \to \mathbb{P}(\Lambda \otimes \mathbb{C}), \qquad b \mapsto [F_{b,\mathbb{C}}(H^{2,0}(\mathcal{X}_b))],$$

where $F_{b,\mathbb{C}} : H^2(\mathcal{X}_b, \mathbb{C}) \to \Lambda \otimes \mathbb{C}$ denotes the $\mathbb{C}$-linear extension of F_b, the marking given above. The *period domain* is defined as

$$\Omega := \{[x] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \,|\, q_\Lambda(x) = 0,\, (x, \bar{x})_\Lambda > 0\},$$

where q_Λ and $(\cdot\,,\cdot)_\Lambda$ are the quadratic form and the bilinear form of Λ.

The period domain Ω is an open subvariety of a quadric hypersurface of $\mathbb{P}(\Lambda \otimes \mathbb{C})$. Note that by Definition 2.2.3 and Theorem 2.2.4 the image of the local period map $\mathcal{P}$ is contained in Ω.

Definition 2.2.31. Let (X, η) be a marked IHS manifold and

$$\mathcal{P} : \mathrm{Def}(X) \to \mathbb{P}(\Lambda \otimes \mathbb{C})$$

be the local period map. The *period point* of (X, η) is $\mathcal{P}(0)$.

We can now state the *local Torelli theorem* for IHS manifolds by Beauville, see [Bea83b, §8, Theorem 5, Item (b)].

Theorem 2.2.32 (Local Torelli theorem)**.** *Let (X, η) be a marked IHS manifold. The local period map $\mathcal{P} : \mathrm{Def}(X) \to \Omega$ is a local isomorphism.*

We now introduce *parallel transport operators* and *monodromy operators*, which will lead us to the global Torelli theorem.

Definition 2.2.33. Let X, X_1 and X_2 be IHS manifolds. An isomorphism $f : H^*(X_1, \mathbb{Z}) \to H^*(X_2, \mathbb{Z})$ is said to be a *parallel transport operator* if there exist a smooth and proper family $\pi : \mathcal{X} \to B$ of IHS manifolds over an analytic base B, points $b_i \in B$, isomorphisms $\psi_i : X_i \to \mathcal{X}_{b_i}$ for $i = 1, 2$, and a continuous path $\gamma : [0, 1] \to B$ such that:

- $\gamma(0) = b_1$,
- $\gamma(1) = b_2$,
- the parallel transport in the local system $R\pi_*\mathbb{Z}$ along γ induces the homomorphism
$$\psi_{2*} \circ f \circ \psi_1^* : H^*(\mathcal{X}_{b_1}, \mathbb{Z}) \to H^*(\mathcal{X}_{b_2}, \mathbb{Z}).$$

An isomorphism $g : H^k(X_1, \mathbb{Z}) \to H^k(X_2, \mathbb{Z})$ is a *parallel transport operator* if it is the k-th graded summand of a parallel transport operator f as above. An automorphism $f : H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ is a *monodromy operator* if it is a parallel transport operator.

Every parallel transport operator is a lattice isometry. Moreover, the composition of two parallel transport operators is a parallel transport operator. The set of monodromy operators gives a subgroup $\mathrm{Mon}^2(X) \subseteq \mathrm{O}(H^2(X, \mathbb{Z}))$, called *monodromy group*. We now define the moduli space $\mathcal{M}_\Lambda$ of marked IHS manifolds associated to the lattice Λ.

Definition 2.2.34. Let Λ be a lattice of signature $(3, b-3)$. The moduli space of marked IHS manifolds associated to Λ, denoted by $\mathcal{M}_\Lambda$, is the set of marked IHS manifolds (X, η), where $\eta : H^2(X, \mathbb{Z}) \to \Lambda$ is a marking on Λ, modulo isomorphisms of marked IHS manifolds.

Using the local Torelli theorem, we obtain the following, see [Huy99, §1.18].

Theorem 2.2.35. *Let Λ be a lattice of signature $(3, b-3)$, and suppose that the moduli space $\mathcal{M}_\Lambda$ of marked IHS manifolds associated to Λ is not empty. Then the moduli space $\mathcal{M}_\Lambda$ is a smooth non-Hausdorff complex manifold.*

In particular, one can show that the moduli space $\mathcal{M}_{\Lambda_{K3}}$ has dimension 20 and the moduli space $\mathcal{M}_{\Lambda_{K3^{[n]}}}$ has dimension 21 for $n \geq 2$. Two marked IHS manifolds (X_1, η_1) and (X_2, η_2) are deformation equivalent if and only if their classes are in the same connected component of $\mathcal{M}_\Lambda$. This happens if and only if $\eta_2^{-1} \circ \eta_1$ is a parallel transport operator by [Mar11, Lemma 7.5].

Glueing together all the local period maps, we obtain a global period map
$$\mathcal{P} : \mathcal{M}_\Lambda \to \Omega.$$

We can now state the following result, known as the *surjectivity of the period map*, see [Huy99, Theorem 8.1].

Theorem 2.2.36 (Surjectivity period map)**.** *Let Λ be a lattice of signature $(3, b-3)$ and suppose that the moduli space $\mathcal{M}_\Lambda$ is not empty. Denote by $\mathcal{M}_\Lambda^\circ$ a connected component of $\mathcal{M}_\Lambda$. Then the restriction of the global period map $\mathcal{P} : \mathcal{M}_\Lambda \to \Omega$ to $\mathcal{M}_\Lambda^\circ$ is surjective.*

We can now state the global Torelli theorem for marked IHS manifolds: for the first part of the statement see [Huy99, Theorem 4.3], for the second part see [Ver13, Theorem 1.16].

Theorem 2.2.37 (Global Torelli theorem)**.** *Let Λ be a lattice of signature $(3, b-3)$, and consider the moduli space $\mathcal{M}_\Lambda$. If (X_1, η_1) and (X_2, η_2) are two inseparable points in $\mathcal{M}_\Lambda$, then X_1 and X_2 are bimeromorphic.*
Let $\mathcal{M}_\Lambda^\circ$ be a connected component of $\mathcal{M}_\Lambda$, and $\mathcal{P}_\circ$ be the restriction of the global period map to $\mathcal{M}_\Lambda^\circ$. Then for every $p \in \Omega$ the fiber $\mathcal{P}_\circ^{-1}(p)$ consists of pairwise inseparable points.

Markman in [Mar11] gave a Hodge-theoretic version of the global Torelli theorem, which we will use in Section 4.3.

Theorem 2.2.38 (Hodge-theoretic global Torelli theorem)**.** *Let X and Y be two IHS manifolds which are deformation equivalent.*

1. *X and Y are bimeromorphic if and only if there exists a parallel transport operator $f : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ which is an isomorphism of Hodge structures.*
2. *Let $f : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ be a parallel transport operator, which is an isomorphism of integral Hodge structures. There exists an isomorphism $\tilde{f} : X \to Y$ such that $f = \tilde{f}_*$ if and only if f maps a Kähler class on X to a Kähler class on Y.*

We conclude this section with the following useful result used in the proof of Theorem 2.2.38, see [Huy03] and [Mar11, §3]. Let X_1 and X_2 be two IHS manifolds of complex dimension $2n$, and suppose that they are birational. Denote by $Z \subset X_1 \times X_2$ the closure of the graph of a birational map, and by $[Z]$ the fundamental cohomological class of Z in $X_1 \times X_2$. Consider the following homomorphism

$$[Z]_* : H^*(X_1,\mathbb{Z}) \to H^*(X_2,\mathbb{Z}), \qquad \alpha \mapsto \pi_{2*}(\pi_1^*(\alpha) \cup [Z]), \tag{2.2.6}$$

where π_1 and π_2 are the projections of $X_1 \times X_2$ on X_1 and X_2 respectively.

Theorem 2.2.39 (Corollary 2.7 in [Huy03])**.** *Keep notation as above. Then the Hodge structures of X_1 and X_2 are isomorphic. In particular, Hodge and Betti numbers of X_1 and X_2 coincide, and $H^*(X_1,\mathbb{Z}) \cong H^*(X_2,\mathbb{Z})$ as graded rings. Moreover, there exists an effective cycle $\Gamma := Z + \sum Y_j$ in $X_1 \times X_2$ of pure dimension $2n$, with the following properties.*

(i) The homomorphism $[\Gamma]_* : H^*(X_1,\mathbb{Z}) \to H^*(X_2,\mathbb{Z})$ given by

$$[\Gamma]_*(\alpha) = \pi_{2*}(\pi_1^*(\alpha) \cup [\Gamma])$$

is a parallel transport operator.

(ii) The image $\pi_i(Y_j)$ has codimension ≥ 2 in X_i for all j. In particular, $[\Gamma]_*$ and $[Z]_*$ defined in (2.2.6) coincide on $H^2(X_1,\mathbb{Z})$.

(iii) The isomorphism $[\Gamma]_*$ is compatible with the BBF forms, i.e.,

$$q_{X_2}([\Gamma]_*(\alpha)) = q_{X_1}(\alpha) \qquad \text{for every } \alpha \in H^2(X_1,\mathbb{Z}).$$

2.2.8 Mukai flops and birational maps

In this section we recall the construction of the Mukai flop, introduced in [Muk84]. We follow [GHJ12, Example 21.7] and [O'G10, §1.3].

Let X be an IHS manifold of dimension $2n$, and suppose that there exists a submanifold $P \subset X$ with $P \cong \mathbb{P}^n$. Let $\sigma \in H^0(X, \Omega_X^2)$ be a symplectic form. Consider the following two short exact sequences, one the dual of the other:

$$0 \to \mathcal{T}_P \hookrightarrow \mathcal{T}_X|_P \to \mathcal{N}_P|_X \to 0,$$
$$0 \to \mathcal{N}_P^*|_X \hookrightarrow \Omega_X|_P \to \Omega_P \to 0,$$

where $\mathcal{N}_P^*|_X$ is the dual vector bundle $\mathrm{Hom}(\mathcal{N}_P|_X, \mathcal{O}_P)$. Since $\mathbb{P}^n$ does not admit any regular 2-form, the restriction of σ to P, denoted by $\sigma_P : \mathcal{T}_P \to \Omega_P$, is trivial. Since σ is non-degenerate, we have an isomorphism $\sigma|_P : \mathcal{T}_X|_P \xrightarrow{\sim} \Omega_X|_P$, and the following composition is trivial:

$$\mathcal{T}_P \hookrightarrow \mathcal{T}_X|_P \xrightarrow{\sim} \Omega_X|_P \to \Omega_P.$$

Hence $\mathcal{T}_P$ is contained in the kernel of the surjective map $\mathcal{T}_X|_P \to \Omega_P$. For dimensional reasons, $\mathcal{T}_P$ must coincide with the kernel of the previous surjective map, hence we obtain the following isomorphism:

$$\mathcal{N}_P|_X \cong \Omega_P.$$

Let $\beta : \hat{X} \to X$ be the blow-up of X in P and denote by E the exceptional divisor. By [Voi02, §3.3.3] we have $E \cong \mathbb{P}(\Omega_P)$. Consider now the Euler sequence given in [Har13, Theorem II.8.13]:

$$0 \to \Omega_P \to \mathcal{O}_P(-1)^{\oplus n+1} \to \mathcal{O}_P \to 0.$$

One can show that $\mathcal{O}_P(-1)^{\oplus n+1} \cong \underline{V} \otimes \mathcal{O}_P(-1)$, where $\underline{V}$ is the trivial vector bundle of rank $n+1$ of $\mathbb{P}(V)$, with V a $\mathbb{C}$-vector space of dimension $n+1$. A way to see this is working on fibers over $l \in P$. Indeed, an element of the fiber $\mathcal{O}_P(-1)$ is of the form (l, z), where $z \in \mathbb{C}^{n+1}$ lies on l, i.e., if $l = (l_0 : \cdots : l_n)$ is a fixed representation of l, then $z = (z_0, \ldots, z_n)$ with $z_i = \lambda l_i$ for some $\lambda \in \mathbb{C}$, so we can identify elements in $\mathcal{O}_P(-1)$ with elements $\lambda \in \mathbb{C}$. An element of the fiber $(\underline{V})_l = (\mathbb{P}(V) \times V)_l$ is of the form $(l, (\alpha_0, \ldots, \alpha_n))$, where $(\alpha_0, \ldots, \alpha_n) \in \mathbb{C}^{n+1}$. Hence the isomorphism of vector bundles $\underline{V} \otimes \mathcal{O}_P(-1) \cong \mathcal{O}_P(-1)^{\oplus n+1}$ is induced by

$$(\underline{V} \otimes \mathcal{O}_P(-1))_l \to (\mathcal{O}_P(-1))^{\oplus n+1}, \qquad (\alpha_0, \ldots, \alpha_n) \otimes \lambda \mapsto (\lambda\alpha_0, \ldots, \lambda\alpha_n).$$

The inclusion $\Omega_P \hookrightarrow \underline{V} \otimes \mathcal{O}_P(-1)$ implies the following:

$$\mathbb{P}(\Omega_P) \hookrightarrow \mathbb{P}(\underline{V} \otimes \mathcal{O}_P(-1)) \cong P \times P^*,$$

where P^* is the dual of P. Then $E \subset P \times P^*$ is isomorphic to the incidence variety

$$\{(x, H) \in P \times P^* \,|\, x \in H\} \subset P \times P^*,$$

we now sketch the proof, working again on fibers over $l \in P$. Consider the map

$$(\mathcal{O}_P(-1)^{\oplus n+1})_l \to (\mathcal{O}_P)_l, \qquad (\lambda_0, \ldots, \lambda_n) \mapsto \lambda_0 l_0 + \cdots + \lambda_n l_n.$$

Since $\Omega_P = \ker(\mathcal{O}_P(-1)^{\oplus n+1} \to \mathcal{O}_P)$, an element in $(\Omega_P)_l$ can be seen as $\lambda = (\lambda_0, \ldots, \lambda_n) \in \mathbb{C}^{n+1}$ such that $\lambda_0 l_0 + \cdots + \lambda_n l_n = 0$, i.e., it is a linear function $\varphi_\lambda : V \to \mathbb{C}$ such that $\varphi_\lambda|_l \equiv 0$. Passing to $\mathbb{P}(\Omega_P) \subset \mathbb{P}(V) \times \mathbb{P}(V^*) = P \times P^*$, we see that $E = \mathbb{P}(\Omega_P) \subset P \times P^*$ is the incidence variety of pairs (l, H) of lines $l \subset V$ and hyperplanes $H \subset V$ such that $l \subset H$:

$$E = \mathbb{P}(\Omega_P) = \{((l_0 : \cdots : l_n), (\lambda_0 : \cdots : \lambda_n)) \in P \times P^* \,|\, \lambda_0 l_0 + \cdots + \lambda_n l_n = 0\}.$$

In particular $E \in |\mathcal{O}(1,1)|$, hence adjunction formula gives the following isomorphisms:

$$\begin{aligned}
\omega_E &\cong (\omega_{P\times P^*} \otimes \mathcal{O}(E))|_E &&\cong \mathcal{O}(-n,-n)|_E,\\
\omega_E &\cong (\omega_{\hat{X}} \otimes \mathcal{O}(E))|_E &&\cong (\beta^*\omega_X \otimes \mathcal{O}((n-1)E))|_E \otimes \mathcal{O}(E)|_E\\
& && \cong \beta^*(\omega_X|_P) \otimes \mathcal{O}(nE)|_E,\\
\omega_P &\cong \omega_X|_P \otimes \det(\mathcal{N}_P|_X) &&\cong \omega_X|_P \otimes \det(\Omega_P)\\
& &&\cong \omega_X|_P \otimes \omega_P.
\end{aligned}$$

Thus $\omega_X|_P$ is trivial and $\mathcal{O}(E)|_E \cong \mathcal{O}(-1,-1)|_E$. By a result of Nakano and Fujiki, see [FN71], there exists a blow-down $\beta' : \hat{X} \to X'$, where X' is a complex manifold of dimension $2n$, and E is the exceptional divisor of β'. Moreover, the restriction of β' to E coincide with the projection $E \to P^* \subset X'$. Note that X' admits a symplectic form σ' with $H^0(X', \Omega^2_{X'}) \cong \mathbb{C} \cdot \sigma'$: the restriction of σ to $X \setminus P$ can be extended on X' to a 2-form σ' which is non-degenerate, otherwise the zero locus of $(\sigma')^{2n}$, which is a divisor, would be in $P^* \subset X'$, which has codimension at least two. Moreover, X' is simply connected. We call X' the *elementary Mukai flop* of X in P. In general the elementary Mukai flop of a Kähler manifold X can be non Kähler, thus the elementary Mukai flop X' of an IHS manifold X is an IHS manifold if X' is Kähler.

Elementary Mukai flops are the building blocks of birational maps between projective IHS manifolds of dimension 4, as shown by the following theorem, see [WW03].

Theorem 2.2.40 (Wierzba–Wiśniewski). *Let X and X' be two projective IHS manifolds of dimension 4, and let $f : X \dashrightarrow X'$ be a birational map. Then there exists a sequence of elementary Mukai flops*

$$f_1 : X = X_0 \dashrightarrow X_1, \dots f_n : X_{n-1} \dashrightarrow X_n := X$$

with $f = f_n \circ \cdots \circ f_1$.

2.2.9 Pseudoeffective cone of cycles on IHS manifolds of $K3^{[n]}$-type

In this section we recall the definitions of numerical equivalence of cycles and of pseudoeffective cone of numerical cycles. We will give some properties that we will use in Chapter 4. For more details on adequate equivalence relations on cycles, see [Sam58], [And04], [Har13, Appendix A], [Ful13].

Definition 2.2.41. Let X be a variety over $\mathbb{C}$. A *k-cycle* on X is an element of the free abelian group generated by k-dimensional closed subvarieties of X. A k-cycle is of the form

$$\sum_i n_i[V_i],$$

where the sum is finite, $n_i \in \mathbb{Z}$ and the V_i's are closed k-dimensional subvarieties. We denote by $Z_k(X)$ the free abelian group generated by k-dimensional subvarieties of X. If $V \subset X$ is a k-dimensional closed subvariety of X, we denote by $[V]$ the corresponding element in $Z_k(X)$.

Let $Z_*(X)$ be the direct sum of the groups $Z_k(X)$ for $k = 0, 1, \dots, \dim(X)$. Suppose that X is a smooth projective variety over $\mathbb{C}$. Then one can introduce an *adequate equivalence relation* on $Z_*(X)$, which is a family of equivalence relations satisfying some conditions, see [Sam58, $(\mathrm{RA_I})$-$(\mathrm{RA_{IV}})$]. Examples of adequate equivalence relations are the rational equivalence, which gives rise to Chow groups, see [Ful13], the homological equivalence, which gives rise to singular homology groups, and by Poincaré duality to singular cohomology groups, and numerical equivalence. We say that $\alpha, \beta \in Z_k(X)$ are *numerically equivalent*, and we write $\alpha \equiv \beta$, if $\deg(\alpha \cap \gamma) = \deg(\beta \cap \gamma)$ for every cycle $\gamma \in Z_{\dim(X)-k}(X)$, where the degree map is defined as follows: for $\alpha \in Z_k(X)$ and $\gamma \in Z_{\dim(X)-k}(X)$ closed subvarieties, $\deg(\alpha \cap \gamma)$ is the number of points of intersection of α and γ counted with multiplicity, then we extend linearly on all the other cycles. We denote by $N_k(X)$ the quotient group

$$N_k(X) := Z_k(X) /_{\equiv}.$$

The rational equivalence is the finest adequate equivalence relation, while the numerical equivalence is the coarsest equivalence relation, see [And04] for details. We can now define the pseudoeffective cone of k-cycles.

Definition 2.2.42. Let X be a smooth projective variety over $\mathbb{C}$. We define the *pseudoeffective cone* of k-cycles

$$\overline{\mathrm{Eff}}_k(X) \subseteq N_k(X) \otimes \mathbb{R}$$

as the closure of the convex cone generated by classes of k-dimensional subvarieties of X. The interior of $\overline{\mathrm{Eff}}_k(X)$ is called the *big cone* of k-cycles.

If $\pi : X \to Y$ is a proper morphism of smooth projective varieties, from the definition of pushforward of a cycle, see [Ful13, §1.4], we obtain the following.

Proposition 2.2.43. *Let $\pi : X \to Y$ be a proper morphism of smooth projective varieties. Then $\pi_*(\overline{\mathrm{Eff}}_k(X)) \subseteq \overline{\mathrm{Eff}}_k(Y)$. If π is surjective, then equality holds both for pseudoeffective cones and big cones.*

If $\pi : X \to Y$ is a flat morphism of relative dimension d between smooth projective varieties, then the pullback of cycles defined in [Ful13, §1.7] induces a pullback between numerical equivalence classes with the following property, see [Ful13, Example 19.2.3] and [FL17, Remark 2.4].

Proposition 2.2.44 (Proposition 8.1.2 [Ful13])**.** *Let $\pi : X \to Y$ be a flat morphism of relative dimension d of smooth projective varieties. Then the flat pullback of cycles induces a pullback*

$$\pi^* : N_k(Y) \to N_{k+d}(X).$$

Moreover, $\pi^*(\overline{\mathrm{Eff}}_k(Y)) \subseteq \overline{\mathrm{Eff}}_{k+d}(X)$.

Suppose now that X is a smooth projective variety over $\mathbb{C}$ such that the homological equivalence and the numerical equivalence coincide. Then the pseudoeffective cone of numerical classes of codimension k can be seen as a cone in $H^{k,k}(X, \mathbb{R})$, i.e.,

$$\overline{\mathrm{Eff}}_{\dim(X)-k}(X) \subseteq H^{k,k}(X) \cap H^{2k}(X, \mathbb{R}).$$

This coincides with the closure of the cone generated by fundamental cohomological classes of subvarieties of complex dimension $\dim(X) - k$. This holds for Hilbert schemes $S^{[n]}$ of smooth projective varieties by [Ara06, Corollary 7.5] and for IHS manifolds of $K3^{[n]}$-type by [CM13, Theorem 1.1].

Theorem 2.2.45 (Charles–Markman)**.** *Let X be an IHS manifold deformation equivalent to the Hilbert scheme of n points on a K3 surface. Then the homological equivalence and the numerical equivalence coincide, so*

$$\overline{\mathrm{Eff}}_{2n-k}(X) \subseteq H^{k,k}(X) \cap H^{2k}(X, \mathbb{R})$$

coincides with the closure of the cone generated by fundamental cohomological classes of subvarieties of complex dimension $2n-k$, with $0 \leq k \leq 2n$.

2.2.10 Double EPW sextics

In this section we recall the definition of double EPW sextic, which is an example of IHS fourfold of $K3^{[2]}$-type. This is obtained as a double cover of an EPW sextic, an hypersurface of $\mathbb{P}^5$ of degree 6, first studied by Eisenbud, Popescu and Walters in [EPW01, Example 9.3]. We begin with the definition of EPW sextic. We follow [O'G08a], [O'G13], [O'G12], see also [Fer12], [DK19] and [Ber20].

Let V_6 be a $\mathbb{C}$-vector space of dimension 6. We fix a volume form on $\bigwedge^6 V_6$:

$$\mathrm{vol} : \bigwedge^6 V_6 \xrightarrow{\sim} \mathbb{C}.$$

This induces a symplectic form on $\bigwedge^3 V_6$ given by:

$$\omega : \left(\bigwedge^3 V_6\right) \times \left(\bigwedge^3 V_6\right) \to \mathbb{C}, \qquad (u, v) \mapsto \mathrm{vol}(u \wedge v).$$

Since ω is symplectic, the induced map

$$\bigwedge^3 V_6 \to \left(\bigwedge^3 V_6\right)^*, \qquad v \mapsto \omega(v, \cdot\,)$$

is an isomorphism, where $\left(\bigwedge^3 V_6\right)^* = \mathrm{Hom}(\bigwedge^3 V_6, \mathbb{C})$ is the dual vector space. Given two subvector spaces $U_1, U_2 \subseteq \bigwedge^3 V_6$, we define the linear map

$$\omega_{U_1,U_2} : U_1 \to U_2^*, \qquad u \mapsto \omega(u, \cdot\,)|_{U_2},$$

given by the composition of the inclusion $U_1 \hookrightarrow \bigwedge^3 V_6$ with the dual of the inclusion $U_2 \hookrightarrow \bigwedge^3 V_6$.

Definition 2.2.46. An *isotropic subspace* $i : I \hookrightarrow \bigwedge^3 V_6$ is a vector subspace of $\bigwedge^3 V_6$ on which the restriction of the symplectic form is zero. A *Lagrangian subspace* $i : A \hookrightarrow \bigwedge^3 V_6$ is an isotropic subspace whose dimension is maximal, or equivalently $\omega_{A,A}$ induces an exact sequence

$$0 \to A \xrightarrow{i} \bigwedge^3 V_6 \xrightarrow{i^* \circ \omega_{A,A}} A^* \to 0. \tag{2.2.7}$$

In the exact sequence (2.2.7) we have denoted by i^* the restriction map

$$i^* : \left(\bigwedge^3 V_6\right)^* \to A^*, \qquad f \mapsto f|_A,$$

whose kernel is the *annihilator of* A, i.e.,

$$\operatorname{Ann}(A) := \{f \in \left(\bigwedge^3 V_6\right)^* \mid f(v) = 0 \text{ for every } v \in A\}.$$

Note that $\left((\bigwedge^3 V_6)/A\right)^* \cong \operatorname{Ann}(A)$. Moreover, we see that A is a maximal isotropic subspace if and only if $\operatorname{Ann}(A) = A^*$. Then the two definitions of Lagrangian subspace given above are equivalent.

We can always find a basis of V_6 such that the matrix associated to ω is of the form

$$\left(\begin{array}{c|c} 0 & I_{10} \\ \hline -I_{10} & 0 \end{array}\right),$$

where I_{10} is the 10-dimensional identity matrix. Then a Lagrangian subspace has always dimension 10.
We denote by $Gr(10, \bigwedge^3 V_6)$ the Grassmannian of 10-dimensional vector subspaces of $\bigwedge^3 V_6$.

Definition 2.2.47. The *symplectic Grassmannian* $\mathbb{LG}(\bigwedge^3 V_6)$ is the subset of $Gr(10, \bigwedge^3 V_6)$ given by Lagrangian subspaces with respect to the volume form in $\bigwedge^3 V_6$.

Since two volume forms differ by a non-zero constant, $\mathbb{LG}(\bigwedge^3 V_6)$ does not depend on the choice of the volume form. Moreover, it is well known that $\mathbb{LG}(\bigwedge^3 V_6)$ is a smooth subvariety of $Gr(10, \bigwedge^3 V_6)$ of dimension 55, see for instance [O'G12, Corollary 1.2].

For each non-zero $v \in V_6$ we can consider the Lagrangian subspace

$$F_v := v \wedge \bigwedge^2 V_6.$$

Since for any $\lambda \in \mathbb{C}^*$ the subspaces F_v and $F_{\lambda v}$ coincide, we can define a vector bundle $F \subset \mathcal{O}_{\mathbb{P}(V_6)} \otimes \bigwedge^3 V_6$ on $\mathbb{P}(V_6)$ whose fiber on $[v]$ is F_v. Moreover, given $v \in V_6 \setminus \{0\}$, we can fix a decomposition $V_6 \cong \mathbb{C}v \oplus V_5$ for some 5-dimensional subvector space $V_5 \subset V_6$, which induces a decomposition $\bigwedge^3 V_6 \cong \bigwedge^3 V_5 \oplus F_v$, and every element of F_v can be written in the form $v \wedge \eta$ for some $\eta \in \bigwedge^2 V_5$. We then have an induced isomorphism of vector spaces

$$\rho_v : F_v \to \bigwedge^2 V_5, \qquad v \wedge \eta \mapsto \eta.$$

Hence F is a vector bundle of rank 10. This is in particular a *Lagrangian subbundle* of $\mathcal{O}_{\mathbb{P}(V_6)} \otimes \bigwedge^3 V_6$, i.e., a vector bundle such that every fiber is a Lagrangian subspace of $\bigwedge^3 V_6$. The isomorphisms ρ_v yield an isomorphism

$$F \cong \mathcal{S} \otimes \bigwedge^2 \mathcal{Q},$$

where $\mathcal{S}$ is the tautological subbundle on $\mathbb{P}(V_6)$ and $\mathcal{Q}$ is the tautological quotient bundle, which appear in the following exact sequence of vector bundles:

$$0 \to \mathcal{S} \to \mathcal{O}_{\mathbb{P}(V_6)} \otimes V_6 \to \mathcal{Q} \to 0. \tag{2.2.8}$$

Let $A \in \mathbb{LG}(\bigwedge^3 V_6)$ be a Lagrangian subspace in $\bigwedge^3 V_6$. Note that the exact sequence (2.2.7) gives a canonical identification

$$\left(\bigwedge^3 V_6\right)^* / A \cong A^*.$$

Let

$$\lambda_A : F \to \mathcal{O}_{\mathbb{P}(V)} \otimes A^* \tag{2.2.9}$$

be the inclusion $F \hookrightarrow \mathcal{O}_{\mathbb{P}(V)} \otimes \bigwedge^3 V_6$ followed by the projection modulo A. Note that λ_A is a map of vector bundles of equal rank 10, so we can give the following definition.

Definition 2.2.48. Consider the map

$$\det(\lambda_A) : \det(F) \to \det(\mathcal{O}_{\mathbb{P}(V_6)} \otimes A^*)$$

induced by map (2.2.9). We set

$$Y_A := Z(\det(\lambda_A)),$$

the zero locus of the determinant of λ_A. This is a subscheme of $\mathbb{P}(V_6)$.

When Y_A is not the whole space $\mathbb{P}(V_6)$, it is a hypersurface of degree 6. We show it in the following lemma, see [O'G06, §1] and [Fer12, §1] for details.

Lemma 2.2.49. *Keep notation as above. Then there is an isomorphism $\det(F) \cong \mathcal{O}_{\mathbb{P}(V_6)}(-6)$, so that Y_A is a sextic hypersurface when it is not the whole space. In particular, Y_A is never empty.*

Proof. We follow [Fer12, §1] and [Ber20, Lemma 4.7]. Consider the exact sequence (2.2.8), obtained tensorising by $\mathcal{O}_{\mathbb{P}(V)}(-1)$ the dual of the Euler exact sequence, see [Har13, Theorem II.8.13]. Let $h \in \mathrm{Pic}(\mathbb{P}(V_6))$ be a hyperplane class. Then $c(\mathcal{S}) = 1 - h$ and $c_1(\mathcal{S}) = -h$. Moreover, we have

$$c(\mathcal{S}) \cdot c(\mathcal{Q}) = c(\mathcal{O}_{\mathbb{P}(V_6)} \otimes V_6) = 1,$$

which implies $c_1(\mathcal{S}) + c_1(\mathcal{Q}) = 0$, hence $c_1(\mathcal{Q}) = h$. Now, $\mathcal{Q}$ is a vector bundle of rank 5, so if $\alpha_1, \dots, \alpha_5$ are its Chern roots, we have $c_1(\mathcal{Q}) = h = \alpha_1 + \dots + \alpha_5$. The Chern roots of $\bigwedge^2 \mathcal{Q}$ are $(\alpha_i + \alpha_j)_{i<j}$, hence we have

$$c_1\left(\bigwedge^2 \mathcal{Q}\right) = 4\alpha_1 + \dots + 4\alpha_5 = 4h.$$

We conclude that

$$c_1(F) = c_1\left(\bigwedge^2 \mathcal{Q}\right) + \mathrm{rk}(F) c_1(\mathcal{L}) = -6h,$$

which implies $\det(F) \cong \mathcal{O}_{\mathbb{P}(V_6)}(-6)$. Moreover, note that λ_A is a section of the line bundle

$$\mathrm{Hom}(F, \mathcal{O}_{\mathbb{P}(V_6)}) \otimes A^* \cong F^* \otimes (\mathcal{O}_{\mathbb{P}(V_6)} \otimes A^*) \cong F^* \otimes A^*,$$

so $Y_A = Z(\det(\lambda_A)) = Z(\det(F^*))$ is non-empty and it is a sextic. □

We can now give the definition of EPW sextic.

Definition 2.2.50. Keep notation as above. When $Y_A \neq \mathbb{P}(V_6)$, the sextic hypersurface Y_A is called *Eisenbud–Popescu–Walter* (EPW) *sextic.*

The following result gives us a sufficient condition to have that Y_A is an EPW sextic.

Proposition 2.2.51 (Corollary 1.5 in [O'G12])**.** *Take $A \in \mathbb{LG}(\bigwedge^3 V_6)$. If $\mathbb{P}(A) \cap Gr(3, V_6) = \emptyset$, then Y_A is an EPW sextic.*

This result makes us give the following definition.

Definition 2.2.52. We call Θ_A the locus of the classes of *decomposable vectors* in $\mathbb{P}(A)$, i.e.,

$$\Theta_A := \mathbb{P}(A) \cap Gr(3, V_6).$$

We can associate to $A \in \mathbb{LG}(\bigwedge^3 V_6)$ a stratification of $\mathbb{P}(V_6)$: for every $k \geq 0$ we set

$$Y_A^{\geq k} := \{[v] \in \mathbb{P}(V_6) \ | \dim(F_v \cap A) \geq k\}.$$

Set-theoretically we have the equalities $\mathbb{P}(V_6) = Y_A^{\geq 0}$ and $Y_A = Y_A^{\geq 1}$. Since A is a Lagrangian subspace, $Y_A^{\geq k}$ can be seen as the locus of points at which the map $\lambda_A : F \to \mathcal{O}_{\mathbb{P}(V_6)} \otimes A^*$ has corank at least k. Using this description, we can define a scheme structure for $Y_A^{\geq k}$ for every $k \geq 0$: this is the one given by the vanishing of the determinants of the $(11-k) \times (11-k)$ minors of a matrix representing λ_A. This structure corresponds to that of $\mathbb{P}(V_6)$ and Y_A for $k = 0, 1$. We also define

$$Y_A^k := \{[v] \in \mathbb{P}(V_6) \,|\, \dim(F_v \cap A) = k\}.$$

O'Grady obtained the following explicit description of the singular locus of an EPW sextic Y_A.

Proposition 2.2.53 (Corollary 1.5 in [O'G12])**.** *If $Y_A \neq \mathbb{P}(V_6)$, we have*

$$\mathrm{Sing}(Y_A) = Y_A^{\geq 2} \cup \left(\bigcup_{W \in \Theta_A} \mathbb{P}(W) \right).$$

We will consider Lagrangian subspaces A such that $\Theta_A = \emptyset$. In this case, $\mathrm{Sing}(Y_A) = Y_A^{\geq 2}$ and $Y_A^3 = Y_A^{\geq 3}$, see [O'G13, Claim 3.7].

Proposition 2.2.54 (Proposition 1.9 in [O'G12])**.** *Let $A \in \mathbb{LG}(\bigwedge^3 V_6)$. Suppose that $[v] \in Y_A^2$ and that $A \cap F_v$ does not contain a non-zero decomposable element. Then Y_A^2 is smooth and 2-dimensional in a neighborhood of $[v]$.*

We denote by $\mathcal{R}$ the *first Lagrangian cointersection sheaf*, i.e., the cokernel of the restriction of λ_A to Y_A. Then O'Grady obtained the following result, see [O'G06, §4] and [Ber20, Theorem 4.30] for details.

Theorem 2.2.55 (O'Grady)**.** *Consider $A \in \mathbb{LG}(\bigwedge^3 V_6)$ such that $\Theta_A = \emptyset$. Then there is a unique double cover $f_A : X_A \to Y_A$ with branch locus $Y_A^{\geq 2}$ such that*

$$(f_A)_* \mathcal{O}_{X_A} \cong \mathcal{O}_{Y_A} \oplus \mathcal{R}(-3),$$

where $\mathcal{R}$ is the first Lagrangian cointersection sheaf. The variety X_A is normal, and it is smooth away from $f_A^{-1}(Y_A^3)$. In particular, if $Y_A^3 = \emptyset$, then X_A is smooth.

Definition 2.2.56. We call *double EPW sextic* the double cover X_A, and ι_A the associated covering involution.

We are interested in double EPW sextics since, for a general $A \in \mathbb{LG}(\bigwedge^3 V_6)$, the variety X_A is smooth and it is an IHS manifold of $K3^{[2]}$-type. First of all, we have to explain what a general $A \in \mathbb{LG}(\bigwedge^3 V_6)$ is.

Definition 2.2.57. We denote by

$$\Sigma := \{A \in \mathbb{LG}(\bigwedge^3 V_6) \,|\, \mathbb{P}(A) \cap Gr(3, V_6) \neq \emptyset\}$$

the set of Lagrangian subspaces which contain a non-zero decomposable vector. We denote by

$$\Delta := \{A \in \mathbb{LG}(\bigwedge^3 V_6) \,|\, Y_A^{\geq 3} \neq \emptyset\}$$

the set of Lagrangian subspaces whose associated third stratum is not empty.

The subsets Σ and Δ are distinct irreducible divisors in $\mathbb{LG}(\bigwedge^3 V_6)$, see [O'G12, Proposition 2.1] and [O'G13, Proposition 2.2].

Definition 2.2.58. An element $A \in \mathbb{LG}(\bigwedge^3 V_6)$ is *generic* if $A \notin \Delta \cup \Sigma$. We set

$$\mathbb{LG}(\bigwedge^3 V_6)^0 = \mathbb{LG}(\bigwedge^3 V_6) \setminus (\Delta \cup \Sigma).$$

We can state the following important result, see [O'G06, Proposition 2.8], [Fer12, Proposition 1.10], [O'G13, Theorem 4.25], [Ber20, Lemma 4.41] and [Ber20, Lemma 4.42].

Theorem 2.2.59. *Let $A \in \mathbb{LG}(\bigwedge^3 V_6)^0$. Then*

(i) X_A is an IHS manifold of $K3^{[2]}$-type.

(ii) The branch locus $Y_A^{\geq 2} = \mathrm{Sing}(Y_A)$ of the double cover $f_A : X_A \to Y_A$ is a smooth surface of degree 40 in $\mathbb{P}(V_6)$ with

$$2K_{Y_A^{\geq 2}} = \mathcal{O}_{Y_A^{\geq 2}}(6),$$

where $K_{Y_A^{\geq 2}}$ is its canonical bundle, in particular it is a surface of general type.

(iii) If $D_A := f_A^* \mathcal{O}_{Y_A}(1)$, then D_A is ample with $q_{X_A}(D_A) = 2$, where q_{X_A} is the BBF form of X_A.

(iv) The covering involution ι_A associated to f_A acts in cohomology as the inverse of the reflection with respect to $\mathbb{Z}D_A$. In particular ι_A is non-symplectic.

2.3 Some useful tools on IHS manifolds

In this section we collect some useful tools and results on IHS manifolds that we will use in next sections. We begin with the *divisorial Zariski decomposition* for a class of an $\mathbb{R}$-divisor in $H^{1,1}(X,\mathbb{R})$, where X is an IHS manifold. See [Bou04] for details.

Proposition 2.3.1. *Let X be an IHS manifold and $\alpha \in H^{1,1}(X,\mathbb{R})$. Then there exists a divisorial Zariski decomposition of the form $\alpha = Z(\alpha) + N(\alpha)$, where $Z(\alpha) \in \overline{\mathrm{Mov}(X)}$ and $N(\alpha)$ is the class of an $\mathbb{R}$-uniruled divisor.*

We now give two useful results concerning the basepoint freeness of some divisors on IHS manifolds of $K3^{[2]}$-type.

Lemma 2.3.2 (Lemma 2.23 in [Rie20])**.** *Consider a K3 surface S, a line bundle $H \in \mathrm{Pic}(S)$ and the associated line bundle $h \in \mathrm{Pic}(S^{[2]})$. If H is basepoint free then h is also basepoint free.*

Theorem 2.3.3 (Corollary 1.1 in [Mat17])**.** *Let X be a $K3^{[n]}$-type IHS manifold and $0 \neq L \in \mathrm{Pic}(X)$ be a primitive nef line bundle with $q_X(L) = 0$. Then $\dim H^0(X,L) = n+1$ and $|L|$ induces a Lagrangian fibration $\varphi_{|L|} : X \to \mathbb{P}^n$. In particular L is basepoint free.*

We conclude with the following two results which will be useful in the proof of Proposition 4.6.3.

Proposition 2.3.4 (Proposition 4.2 (ii) in [Bou04])**.** *Let X be an IHS manifold, and let $E, F \in \mathrm{Pic}(X)$ be effective divisors with no common component. Then we have $(E,F) \geq 0$, where $(\,\cdot\,,\,\cdot\,)$ is the BBF bilinear form.*

Lemma 2.3.5 (Lemma 3.7 in [Mar13], Lemma 3.5 [Rie18])**.** *Let X be an IHS manifold. Suppose that $D \in \mathrm{Pic}(X)$ is an irreducible and reduced divisor with $q_X(D) < 0$. Let $(\,\cdot\,,\,\cdot\,)$ be the BBF bilinear form and*

$$\mathrm{div}(D) := \gcd\{(\alpha, D) \mid \alpha \in H^2(X,\mathbb{Z})\}.$$

Then $q_X(D) \mid 2\,\mathrm{div}(D)$. In particular

$$-\frac{1}{2} q_X(D) \leq \mathrm{div}(D).$$

Chapter 3

Hodge classes of Hilbert squares of K3 surfaces

In this chapter we study rational and integral Hodge classes on the Hilbert square of a projective K3 surface S. If X is a complex projective manifold of dimension n, we have seen in Section 2.1.2 that $H^i(X,\mathbb{Z})_f$ and $H^i(X,\mathbb{Q})$ are Hodge structures of weight i, where $H^i(X,\mathbb{Z})_f$ denotes the torsion free quotient group of the cohomology group $H^i(X,\mathbb{Z})$. The even part $\oplus_{k\geq 0}H^{2k}(X,\mathbb{Q})$ of the cohomology ring $H^*(X,\mathbb{Q})$ contains all the *algebraic cycles*, i.e., classes which are obtained as fundamental cohomological classes $[Z]$ of subvarieties $Z\subset X$. It is natural to wonder if algebraic cycles of degree $2k$ generate the subspace of *rational Hodge classes* of type (k,k), which is by definition the subspace

$$H^{2k}(X,\mathbb{Q})\cap H^{k,k}(X).$$

Similarly we call *integral Hodge classes* of type (k,k) the classes contained in

$$H^{2k}(X,\mathbb{Z})\cap H^{k,k}(X).$$

This problem is known as the *Hodge conjecture*, and can be stated as follows.

Conjecture (Hodge conjecture)**.** *Let X be a smooth complex projective variety. Then the subspace of $H^{2k}(X,\mathbb{Q})$ generated by algebraic cycles coincides with the space of rational Hodge classes, i.e.,*

$$H^{2k}(X,\mathbb{Q})\cap H^{k,k}(X)=\langle [Z]\mid Z\subset X\rangle_{\mathbb{Q}}.$$

The same problem can be studied for integral Hodge classes. As a consequence of the Lefschetz theorem on $(1,1)$-classes, see [Gri79], the Hodge conjecture is true for classes of type $(1,1)$: any element of $H^2(X,\mathbb{Z})\cap H^{1,1}(X)$ is the cohomology class of a divisor on X. The Hodge conjecture holds for Hodge classes of type $(n-1,n-1)$ by the hard Lefschetz theorem, see [Voi02, Théorème 6.25]. In particular the Hodge conjecture is true when X has dimension at most 3. The Hodge conjecture is in general a very difficult problem. It holds for K3 surfaces, but it is still an open problem for products $S\times\cdots\times S$, where S is a K3 surface, see [Huy16, §3.1.2]. See also [Sch10], [Bus19] and [Huy19] for results in the case of products $S\times S$ of two K3 surfaces.

Our interest for Hodge classes on Hilbert squares of K3 surfaces does not have as a goal the study of the Hodge conjecture: our aim is to obtain an explicit description of the lattice $H^{2,2}(S^{[2]},\mathbb{Z})$ of integral Hodge classes of type $(2,2)$ for a *generic K3 surface* S, where by generic K3 surface we mean the following.

Definition 3.0.1. A *generic K3 surface* S is a projective K3 surface whose Picard group $\mathrm{Pic}(S)$ is generated by the class of an ample divisor. A generic K3 surface S has *degree* $2t$ if $\mathrm{Pic}(S) \cong \mathbb{Z}H$ with $H^2 = 2t$, $t \geq 1$.

This explicit description of the lattice $H^{2,2}(S^{[2]},\mathbb{Z})$ that we will obtain will be used in Chapter 4 to determine the irreducibility of some surfaces obtained as intersections of supports of effective divisors in a complete linear system $|D|$, with $D \in \mathrm{Div}(S^{[2]})$.

The chapter is organised as follows. In Section 3.1 we will recall the description of the intersection pairing on $H^4(X,\mathbb{Q})$, where X is an IHS manifold of $K3^{[2]}$-type, following [O'G08b, §2, §3]. In Section 3.2 Nakajima operators will be introduced, following [Nak97], [Leh99], then we will recall the so-called Ellingsrud–Göttsche–Lehn (EGL) formula and the Qin–Wang theorem which describes the lattice $H^4(S^{[2]},\mathbb{Z})$, where S is a K3 surface. In Section 3.3 we will study the model by Lehn and Sorger, see [LS03], with particular emphasis on the case of Hilbert squares of K3 surfaces: we will discuss in detail how to use the model to compute cup products on the cohomology ring $H^*(S^{[2]},\mathbb{Z})$. Moreover, we will show how to combine all these results to obtain a first description of the lattice $H^{2,2}(S^{[2]},\mathbb{Z})$ of integral Hodge classes of type $(2,2)$ on Hilbert squares of generic K3 surfaces in terms of Nakajima operators. In Section 3.4 we will use the EGL-formula to describe the second Chern class $c_2(S^{[2]})$ of the Hilbert square of a K3 surface in terms of Nakajima operators. We will use this to give a more geometrical and explicit description of $H^{2,2}(S^{[2]},\mathbb{Z})$, for S a generic K3 surface. This will be crucial in Chapter 4. We will also describe explicitly the lattice $H^{2,2}(S^{[2]},\mathbb{Z})$ for any projective K3 surface S, knowing its Picard group $\mathrm{Pic}(S)$. We will conclude with Section 3.5, where rational and integral Hodge classes of type $(3,3)$ on Hilbert squares of K3 surfaces will be discussed. We remark that from now on the definition of variety is slightly different from the one used in Chapter 1 and in Chapter 2: now a variety is not required to be necessarily irreducible.

3.1 Intersection pairing on $H^4(X,\mathbb{Q})$

Let X be a projective IHS manifold of dimension 4 of $K3^{[2]}$-type. In this section we recall the general construction of the intersection pairing on $H^4(X,\mathbb{Q})$, and we show that this can be expressed in terms of the BBF form on $H^2(X,\mathbb{Q})$.

First of all, we state the following corollary of Verbitsky's results in [Ver96], obtained by Guan in [Gua01], see also [O'G10, Corollary 2.5]. We denote by $b_i(X)$ the i-th Betti number of X.

Proposition 3.1.1 (Guan)**.** *Let X be an IHS manifold of dimension 4. Then $b_2(X) \leq 23$. If equality holds then $b_3(X) = 0$ and moreover the map*

$$\mathrm{Sym}^2 H^2(X,\mathbb{Q}) \to H^4(X,\mathbb{Q}) \tag{3.1.1}$$

induced by the cup product is an isomorphism. This happens when X is an IHS fourfold of $K3^{[2]}$-type.

Since X is a compact complex manifold of dimension $\dim_{\mathbb{C}}(X) = 4$, the singular cohomology group $H^4(X, \mathbb{Z})$ has an intersection pairing

$$\langle \cdot\,, \cdot \rangle : H^4(X, \mathbb{Z}) \times H^4(X, \mathbb{Z}) \to \mathbb{Z}, \qquad \langle \alpha, \beta \rangle := \epsilon\left((\alpha \cup \beta) \cap [X]\right),$$

where

- $[X] \in H_8(X, \mathbb{Z})$ is the fundamental class of X,
- $\cup : H^4(X, \mathbb{Z}) \otimes H^4(X, \mathbb{Z}) \to H^8(X, \mathbb{Z})$ is the cup product,
- $\cap : H^8(X, \mathbb{Z}) \otimes H_8(X, \mathbb{Z}) \to H_0(X, \mathbb{Z})$ is the cap product,
- $\epsilon : H_0(X, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$ is the usual isomorphism between the zeroth homology group and $\mathbb{Z}$, see [Bre13, Theorem IV.2.1] for details.

Equivalently, the isomorphism $H^8(X, \mathbb{Z}) \cong \mathbb{Z}$ identifies the unit with an integral volume form ω on X, which is in particular an orientable manifold, being a complex manifold, hence

$$\langle \alpha, \beta \rangle = \int_X \alpha \cup \beta. \tag{3.1.2}$$

Note that $\int_X \alpha \cup \beta \in \mathbb{Z}$ since $\alpha \cup \beta = k\omega$ for some $k \in \mathbb{Z}$. Moreover, $\int_X \omega \in \mathbb{Z}$ since ω is integral. See [Bre13], [GH78] and [Hat05] for details on cap product and cup product. From now on we will usually write $\alpha\beta$ to denote the cup product $\alpha \cup \beta$. Since $\dim_{\mathbb{R}}(X) = 8$, again by Poincaré duality $\langle \cdot\,, \cdot \rangle$ is unimodular, i.e., if $\{\alpha_1, \ldots, \alpha_N\}$ is a basis of $H^4(X, \mathbb{Z})$, then the matrix $A = (a_{i,j})_{i,j}$ with $a_{i,j} := \langle \alpha_i, \alpha_j \rangle$ has determinant $\det A = \pm 1$. Moreover, A is symmetric.

The following result of Hodge theory explains how the cup product on the ring $H^*(X, \mathbb{C})$ for X a complex Kähler manifold behaves with respect to the Hodge decomposition. We state it for projective complex manifolds.

Proposition 3.1.2 (Corollaire 6.15 in [Voi02])**.** *Let X be a projective complex manifold. The cup product*

$$H^k(X, \mathbb{C}) \otimes H^l(X, \mathbb{C}) \to H^{k+l}(X, \mathbb{C})$$

is bigraded for the bigraduation given by the Hodge decomposition.

Let now X be an IHS fourfold of $K3^{[2]}$-type and let

$$\langle \cdot\,, \cdot \rangle_{\mathbb{C}} : H^4(X, \mathbb{C}) \times H^4(X, \mathbb{C}) \to \mathbb{C}$$

be the $\mathbb{C}$-bilinear extension of $\langle \cdot\,, \cdot \rangle$. The following classical result of Hodge theory can be obtained from Proposition 3.1.2.

Corollary 3.1.3. *Let X be an IHS fourfold of $K3^{[2]}$-type and $\alpha, \beta \in H^4(X, \mathbb{C})$. Suppose that α and β have bidegree (p, q) and (p', q') respectively with respect to the Hodge structure on $H^4(X, \mathbb{C})$. If $(p + p', q + q') \neq (4, 4)$ then $\langle \alpha, \beta \rangle_{\mathbb{C}} = 0$.*

Proof. We have $h^{4,4}(X) = 1$ and $h^{p,q}(X) = 0$ if either p or q is strictly bigger than 4 since X is a projective complex manifold of dimension 4. Assume that $\alpha, \beta \in H^4(X, \mathbb{C})$ are two classes which have bidegree (p, q) and (p', q') respectively. By Proposition 3.1.2 the cup product $\alpha\beta \in H^8(X, \mathbb{C})$ has bidegree $(p+p', q+q')$, so it is zero if $(p + p', q + q') \neq 0$. By (3.1.2) this implies $\langle \alpha, \beta \rangle_{\mathbb{C}} = 0$. □

By Proposition 3.1.1 we have an isomorphism $\mathrm{Sym}^2 H^2(X,\mathbb{Q}) \cong H^4(X,\mathbb{Q})$. A similar isomorphism does not hold when we pass to integer coefficients by the following result obtained by Boissière, Nieper-Wißkrichen and Sarti, see also [Kap16a, Proposition 2.2].

Proposition 3.1.4 (Proposition 3 in [BNWS13])**.** *Let X be an IHS fourfold of $K3^{[2]}$-type. Then*

$$\frac{H^4(X,\mathbb{Z})}{\mathrm{Sym}^2 H^2(X,\mathbb{Z})} \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{\oplus 23} \oplus \left(\frac{\mathbb{Z}}{5\mathbb{Z}}\right),$$

where by $\mathrm{Sym}^2 H^2(X,\mathbb{Z})$ we mean its image in $H^4(X,\mathbb{Z})$ under the map induced by the cup product.

We extend by $\mathbb{Q}$-bilinearlity the intersection pairing $\langle\,\cdot\,,\,\cdot\,\rangle$, obtaining

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{Q}} : H^4(X,\mathbb{Q}) \times H^4(X,\mathbb{Q}) \to \mathbb{Q},$$

so that $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{Q}}$ gives a $\mathbb{Q}$-valued intersection pairing on $\mathrm{Sym}^2 H^2(X,\mathbb{Q})$. From now on we will use the notation $\langle\,\cdot\,,\,\cdot\,\rangle$ for $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{Q}}$. We then have the following relation between $\langle\,\cdot\,,\,\cdot\,\rangle$ and the $\mathbb{Q}$-extension of the BBF form on $H^2(X,\mathbb{Q})$.

Proposition 3.1.5 (Remark 2.1 in [O'G08b])**.** *Let X be an IHS fourfold of $K3^{[2]}$-type. The intersection pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{Q}}$ defined above on*

$$\mathrm{Sym}^2 H^2(X,\mathbb{Q}) \cong H^4(X,\mathbb{Q})$$

is the bilinear form on $\mathrm{Sym}^2 H^2(X,\mathbb{Q})$ given by

$$\langle \alpha_1\alpha_2, \alpha_3\alpha_4\rangle = (\alpha_1,\alpha_2)(\alpha_3,\alpha_4) + (\alpha_1,\alpha_3)(\alpha_2,\alpha_4) + (\alpha_1,\alpha_4)(\alpha_2,\alpha_3) \quad (3.1.3)$$

for every $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in H^2(X,\mathbb{Q})$, where $(\,\cdot\,,\,\cdot\,)$ denotes the BBF form on $H^2(X,\mathbb{Q})$.

Proof. By Theorem 2.2.4 we have

$$\int_X \alpha_i^4 = 3(\alpha_i,\alpha_i)^2, \qquad 1 \le i \le 4. \quad (3.1.4)$$

We have also

$$\int_X (\alpha_i+\alpha_j)^4 = 3(\alpha_i+\alpha_j, \alpha_i+\alpha_j)^2, \qquad 1 \le i < j \le 4.$$

Expanding the powers and using the equalities (3.1.4) we obtain

$$\begin{aligned} 4\int_X \alpha_i^3\alpha_j + 4\int_X \alpha_i\alpha_j^3 + 6\int_X \alpha_i^2\alpha_j^2 \;&=\; 12(\alpha_i,\alpha_j)^2 + 6(\alpha_i,\alpha_i)(\alpha_j,\alpha_j) \\ &\quad +12(\alpha_i,\alpha_i)(\alpha_i,\alpha_j) + 12(\alpha_i,\alpha_j)(\alpha_j,\alpha_j) \end{aligned} \quad (3.1.5)$$

for $1 \le i < j \le 4$. Similarly, expanding the powers in

$$\int_X (\alpha_i+\alpha_j+\alpha_k)^4 = 3(\alpha_i+\alpha_j+\alpha_k, \alpha_i+\alpha_j+\alpha_k)^2, \qquad 1 \le i < j < k \le 4,$$

using (3.1.4) and (3.1.5) we have

$$\begin{aligned} 12\int_X \alpha_i^2\alpha_j\alpha_k + 12\int_X \alpha_i\alpha_j^2\alpha_k + 12\int_X \alpha_i\alpha_j\alpha_k^2 \quad = \quad & 12(\alpha_i,\alpha_i)(\alpha_j,\alpha_k) \\ & +12(\alpha_j,\alpha_j)(\alpha_i,\alpha_k) \\ & +12(\alpha_i,\alpha_j)(\alpha_k,\alpha_k) \\ & +24(\alpha_i,\alpha_j)(\alpha_i,\alpha_k) \\ & +24(\alpha_i,\alpha_j)(\alpha_j,\alpha_k) \\ & +24(\alpha_i,\alpha_k)(\alpha_j,\alpha_k). \end{aligned} \tag{3.1.6}$$

Finally, expanding the powers in

$$\int_X (\alpha_1+\alpha_2+\alpha_3+\alpha_4)^2 = 3(\alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\alpha_4)^2,$$

and using the equalities (3.1.4), (3.1.5), (3.1.6) we obtain

$$24\int_X \alpha_1\alpha_2\alpha_3\alpha_4 = 24(\alpha_1,\alpha_2)(\alpha_3,\alpha_4) + 24(\alpha_1,\alpha_3)(\alpha_2,\alpha_4) + 24(\alpha_1,\alpha_4)(\alpha_2,\alpha_3),$$

hence

$$\int_X \alpha_1\alpha_2\alpha_3\alpha_4 = (\alpha_1,\alpha_2)(\alpha_3,\alpha_4) + (\alpha_1,\alpha_3)(\alpha_2,\alpha_4) + (\alpha_1,\alpha_4)(\alpha_2,\alpha_3).$$

By Proposition 3.1.1, we have $H^4(X,\mathbb{Q}) \cong \mathrm{Sym}^2 H^2(X,\mathbb{Q})$, and the isomorphism is induced by the cup product. This, together with (3.1.2), implies that

$$\int_X \alpha_1\alpha_2\alpha_3\alpha_4 = \langle \alpha_1\alpha_2, \alpha_3\alpha_4 \rangle,$$

as we wanted.

□

Let q_X be the BBF quadratic form on X. Let $\{e_1,\dots,e_{23}\}$ be a basis of $H^2(X,\mathbb{Q})$ and $\{e_1^\vee,\dots,e_{23}^\vee\}$ be the dual basis in $H^2(X,\mathbb{Q})^\vee$, i.e.,

$$e_i^\vee(e_j) = \delta_{i,j}.$$

Then we have

$$q_X = \sum_{i,j} g_{i,j} e_i^\vee \otimes e_j^\vee$$

where $g_{i,j} := (e_i, e_j)$, in particular $(g_{i,j})$ is a symmetric matrix. The dual of q_X is then an element in $H^4(X,\mathbb{Q})$ of the form

$$q_X^\vee = \sum_{i,j} m_{i,j} e_i e_j, \tag{3.1.7}$$

where $(m_{i,j}) = (g_{i,j})^{-1}$ and e_ie_j denotes the cup product $e_i \cup e_j$. The following proposition shows how to compute the product $\langle q_X^\vee, \alpha\rangle$ for every $\alpha \in H^4(X,\mathbb{Q})$. See [O'G08b, Proposition 2.2] for a more general statement.

Proposition 3.1.6. *Let X be an IHS fourfold of $K3^{[2]}$-type. Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be the bilinear form on $H^4(X,\mathbb{Q}) \cong \mathrm{Sym}^2 H^2(X,\mathbb{Q})$ described in Proposition* 3.1.5. *Then $\langle\,\cdot\,,\,\cdot\,\rangle$ is non-degenerate and*

$$\begin{aligned} \langle q_X^\vee, \alpha\beta\rangle &= 25(\alpha,\beta) \qquad \text{for all } \alpha,\beta \in H^2(X,\mathbb{Q}),\\ \langle q_X^\vee, q_X^\vee\rangle &= 23\cdot 25. \end{aligned}$$

Proof. We consider the inclusion $H^2(X,\mathbb{Q}) \hookrightarrow H^2(X,\mathbb{C})$ and we extend the BBF form $\mathbb{C}$-bilinearly to $H^2(X,\mathbb{C})$. Let $\{e_1,\dots,e_{23}\}$ be an orthonormal basis of $H^2(X,\mathbb{C})$ with respect to the BBF form. Then by (3.1.7) and by definition of the coefficients $g_{i,j}$ given above we have

$$q_X^\vee = \sum_{i=1}^{23} e_i e_i.$$

Let $\alpha,\beta \in H^2(X,\mathbb{Q})$. We first show that $\langle q_X^\vee, \alpha\beta\rangle = 25(\alpha,\beta)$. If we consider α and β as elements of $H^2(X,\mathbb{C})$, we can write

$$\alpha = \sum_{i=1}^{23} \alpha_i e_i, \qquad \beta = \sum_{i=1}^{23} \beta_j e_j, \tag{3.1.8}$$

for some $\alpha_i,\beta_j \in \mathbb{C}$. By linearity, it suffices to show it when $\alpha = e_i$ and $\beta = e_j$ for $i,j \in \{1,\dots,23\}$. By Proposition 3.1.5 we have

$$\begin{aligned} \langle q_X^\vee, e_i e_j\rangle &= \langle \sum_{k=1}^{23} e_k e_k, e_i e_j\rangle \\ &= \sum_{k=1}^{23} \langle e_k e_k, e_i e_j\rangle \\ &= \sum_{k=1}^{23} \left((e_k,e_k)(e_i,e_j) + 2(e_k,e_i)(e_k,e_j)\right). \end{aligned}$$

If $i \neq j$, we obtain

$$\langle q_X^\vee, e_i e_j\rangle = 0,$$

which is equal to (e_i,e_j). Otherwise, if $i=j$ we have

$$\langle q_X^\vee, e_i e_i\rangle = 25(e_i,e_i).$$

Similarly, we show that $\langle q_X^\vee, q_X^\vee\rangle = 23\cdot 25$. For every $i \in \{1,\dots,23\}$ we have

$$\langle e_i e_i, \sum_{j=1}^{23} e_j e_j\rangle = \langle e_i e_i, \sum_{j\neq i, j=1}^{23} e_j e_j\rangle + 3(e_i,e_i)^2 = 22+3 = 25,$$

hence

$$\langle q_X^\vee, q_X^\vee\rangle = \langle \sum_{i=1}^{23} e_i e_i, \sum_{j=1}^{23} e_j e_j\rangle = 23\cdot 25,$$

as we wanted. $\square$

We obtain the following corollary.

Corollary 3.1.7. *Let X be an IHS fourfold of $K3^{[2]}$-type. Then $q_X^\vee \in H^{2,2}(X,\mathbb{Q})$, i.e., $q_X^\vee$ is a rational Hodge class of X of type $(2,2)$.*

Proof. Let $\sigma \in H^{2,0}(X)$ be a symplectic form such that $\int_X (\sigma\overline{\sigma})^2 = 1$. We have seen in Theorem 2.2.4 that the BBF form is a scalar multiple of the following bilinear form, which we call b: given $\alpha_1, \alpha_2 \in H^2(X,\mathbb{C})$ with

$$\alpha_1 = \lambda_1\sigma + \beta_1 + \mu_1\overline{\sigma}, \qquad \alpha_2 = \lambda_2\sigma + \beta_2 + \mu_2\overline{\sigma},$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$ and $\beta_1, \beta_2 \in H^{1,1}(X)$, then

$$b(\alpha_1,\alpha_2) = \frac{1}{2}(\lambda_1\mu_2 + \lambda_2\mu_1) + \int_X \beta_1\beta_2\sigma\overline{\sigma}. \tag{3.1.9}$$

Assume that

$$q_X^\vee = (q_X^\vee)^{4,0} + (q_X^\vee)^{0,4} + (q_X^\vee)^{3,1} + (q_X^\vee)^{1,3} + (q_X^\vee)^{2,2} \in H^4(X,\mathbb{Q}), \tag{3.1.10}$$

where $(q_X^\vee)^{i,j} \in H^{i,j}(X)$ is the component of type (i,j) of $q_X^\vee$ in $H^4(X,\mathbb{C})$ for the Hodge decomposition. By Proposition 3.1.6 we have

$$\langle q_X^\vee, \sigma^2\rangle = 25(\sigma,\sigma) = 0, \tag{3.1.11}$$

where the last equality comes from (3.1.9). Since $\dim_{\mathbb{C}}(H^{0,4}(X)) = 1$, we have $(q_X^\vee)^{0,4} = \mu\overline{\sigma}$ for some $\mu \in \mathbb{C}$. By Corollary 3.1.3 and (3.1.10) we have

$$\begin{aligned} \langle q_X^\vee, \sigma^2\rangle &= \langle (q_X^\vee)^{0,4}, \sigma^2\rangle \\ &= \langle \mu\overline{\sigma}^2, \sigma^2\rangle \\ &= 2\mu(\sigma,\overline{\sigma})^2. \end{aligned} \tag{3.1.12}$$

Since $(\sigma,\overline{\sigma}) \neq 0$, by (3.1.11) and (3.1.12) we obtain $(q_X^\vee)^{0,4} = 0$. Similarly from $\langle q_X^\vee, \overline{\sigma}^2\rangle = 0$ we get $(q_X^\vee)^{4,0} = 0$. Suppose now that $\{\alpha_1,\ldots,\alpha_{23}\}$ is an orthonormal basis of $H^2(X,\mathbb{C})$ such that $\{\alpha_1,\ldots,\alpha_{21}\} \subset H^{1,1}(X)$. We have seen in Section 2.2.3 that $\dim_{\mathbb{C}}(H^{3,1}(X)) = 21$, so a basis of $H^{3,1}(X)$ is given by $\{\sigma\alpha_1,\ldots,\sigma\alpha_{21}\}$, hence we can write

$$(q_X^\vee)^{3,1} = \sum_{i=1}^{21} x_i \cdot \sigma\alpha_i \tag{3.1.13}$$

for some $x_i \in \mathbb{C}$. By Proposition 3.1.6 we have for $j = 1,\ldots,21$ that

$$\langle q_X^\vee, \overline{\sigma}\alpha_j\rangle = 25(\overline{\sigma},\alpha_j) = 0 \tag{3.1.14}$$

where the last equality comes from (3.1.9). Using Corollary 3.1.3, from (3.1.10) and (3.1.13) we have

$$\begin{aligned} \langle q_X^\vee, \overline{\sigma}\alpha_j\rangle &= \sum_{i=1}^{21} x_i\langle \sigma\alpha_i, \overline{\sigma}\alpha_j\rangle \\ &= x_j(\sigma,\overline{\sigma}). \end{aligned} \tag{3.1.15}$$

Since $(\sigma,\overline{\sigma}) \neq 0$, for $j = 1,\ldots,23$ we obtain from (3.1.14) and (3.1.15) that $x_j = 0$, hence $(q_X^\vee)^{3,1} = 0$. Similarly $(q_X^\vee)^{1,3} = 0$. We then conclude that $q_X^\vee \in H^{2,2}(X,\mathbb{Q})$. □

We can say more on $q_X^\vee \in H^{2,2}(X,\mathbb{Q})$. O'Grady in [O'G08b, §3] has shown that it is a rational multiple of $c_2(X)$, the second Chern class of the tangent bundle of X. We state the precise result in the following proposition.

Proposition 3.1.8 (O'Grady)**.** *Let X be a projective IHS fourfold of $K3^{[2]}$-type. Then*

$$\frac{6}{5}q_X^\vee = c_2(X) \in H^{2,2}(X,\mathbb{Z}).$$

Moreover, $\frac{2}{5}q_X^\vee \in H^{2,2}(X,\mathbb{Z})$ is an integral Hodge class of X of type $(2,2)$.

Proof. If Z is an IHS manifold of dimension $2n$ and α is a class which is of bidegree $(2,2)$ on all small deformations of Z and which is contained in the image of $\mathrm{Sym}^2 H^2(Z,\mathbb{Q}) \to H^4(Z,\mathbb{Q})$, Fujiki showed in [Fuj87] that there exists a constant $c_\alpha \in \mathbb{Q}$ such that

$$\int_Z \alpha\beta^{2(n-1)} = c_\alpha q_Z(\beta)^{n-1} \qquad \text{for every } \beta \in H^2(Z,\mathbb{Q}),$$

which is a multiple of $q_Z(\beta)$, see also [Huy99, §1.11]. Hence α is a rational multiple of $q_Z^\vee$. We can apply this argument in our case with $\alpha = c_2(X)$, since by [NW02, §2, Remark 5] rational Chern classes of an IHS manifold are topological invariants, so we have $c_2(X) = aq_X^\vee$ for some $a \in \mathbb{Q}$. Let $h \in \mathrm{Pic}(X)$ be an ample class. By [Miy87, Theorem 1] and Proposition 3.1.6 we have

$$0 \leq \langle c_2(X), h^2\rangle = \langle aq_X^\vee, h^2\rangle = 50a,$$

so $a \geq 0$. By Theorem 2.2.6 we have $c_2(X)^2 = 828$, hence

$$828 = c_2(X)^2 = \langle aq_X^\vee, aq_X^\vee\rangle = a^2 23 \cdot 25.$$

Since $a \geq 0$, we conclude that $a = \frac{6}{5}$ and

$$c_2(X) = \frac{6}{5}q_X^\vee \in H^{2,2}(X,\mathbb{Z}).$$

Recall that the lattice $(H^2(X,\mathbb{Z}), q_X)$ is isomorphic to $E_8(-1)^{\oplus 2} \oplus U^{\oplus 3} \oplus \langle -2\rangle$, so it has discriminant $\mathrm{disc}(H^2(X,\mathbb{Z})) = 2$. We see from (3.1.7) that this implies $2q_X^\vee \in H^{2,2}(X,\mathbb{Z})$, hence

$$H^{2,2}(X,\mathbb{Z}) \ni 2c_2(X) - 2q_X^\vee = \frac{2}{5}q_X^\vee.$$

□

We conclude this section with the following Riemann–Roch formula for IHS manifolds of $K3^{[2]}$-type, see [O'G10, Formula (2.2.7)].

Theorem 3.1.9. *Let X be an IHS manifold deformation equivalent to the Hilbert square of a K3 surface. Let $L \in \mathrm{Pic}(X)$ be a line bundle on X. Then the Euler characteristic of L is*

$$\mathcal{X}(L) = \frac{1}{8}(q_X(L)+4)(q_X(L)+6). \tag{3.1.16}$$

In particular, if L is ample, then $\mathcal{X}(L) = h^0(X,L) = \dim(H^0(X,L))$.

Proof. By Theorem 1.3.3 we have

$$\mathcal{X}(L) = \int_X \mathrm{ch}(L) \cup \mathrm{td}(\mathcal{T}_X),$$

where $\mathrm{ch}(L)$ is the exponential Chern character of L and $\mathrm{td}(\mathcal{T}_X)$ is the Todd class of X introduced in Section 1.3. We have

$$\mathrm{ch}(L) = 1 + c_1(L) + \frac{1}{2}c_1(L)^2 + \frac{1}{6}c_1(L)^3 + \frac{1}{24}c_1(L)^4 \tag{3.1.17}$$

and, since all odd Chern classes of X vanish,

$$\mathrm{td}(\mathcal{T}_X) = 1 + \frac{1}{12}c_2(X) - \frac{1}{720}(-3c_2(X)^2 + c_4(X)). \tag{3.1.18}$$

Note that in (3.1.17) and (3.1.18) powers are taken with respect to the cup product. Let $h \in \mathrm{Div}(X)$ such that $L = \mathcal{O}_X(h)$. From now on we write h instead of $c_1(\mathcal{O}_X(h))$. We get

$$\mathcal{X}(L) = \int_X \left(1 + h + \frac{1}{2}h^2 + \frac{1}{6}h^3 + \frac{1}{24}h^4\right)\left(1 + \frac{1}{12}c_2 - \frac{1}{720}(-3c_2^2 + c_4)\right). \tag{3.1.19}$$

From Theorem 2.2.6 we have $c_2(X)^2 = 828$ and $c_4(X) = 324$. Moreover by Proposition 3.1.8 we have

$$c_2(X) = \frac{6}{5}q_X^\vee \in H^4(X, \mathbb{Q}). \tag{3.1.20}$$

By Theorem 2.2.4 and Proposition 2.2.8 we obtain

$$\int_X h^4 = \frac{4!}{2! \cdot 2^2} q_X(h)^2. \tag{3.1.21}$$

Then by (3.1.19), (3.1.20) and (3.1.21) we get

$$\begin{aligned}
\mathcal{X}(L) &= \int_X \left(\frac{1}{240}c_2^2 - \frac{1}{720}c_4\right) + \int_X \left(1 + h + \frac{1}{2}h^2 + \frac{1}{6}h^3 + \frac{1}{24}h^4\right)\left(1 + \frac{1}{12}c_2\right) \\
&= 3 + \frac{1}{24}\left(\int_X h^4\right) + \frac{1}{24}\left(\int_X c_2(X)h^2\right) \\
&= 3 + \tfrac{1}{24} \cdot 3 \cdot q_X(h)^2 + \tfrac{1}{24} \cdot \tfrac{6}{5} \cdot 25 \cdot q_X(h) \\
&= \tfrac{1}{8}(q_X(L) + 4)(q_X(L) + 6),
\end{aligned}$$

where in the third equality we have used Proposition 3.1.6. We conclude that:

$$\begin{aligned}
\mathcal{X}(L) &= 3 + \frac{1}{24} \cdot 3 \cdot q_X(L)^2 + \frac{1}{24} \cdot \frac{6}{5} \cdot 25 \cdot q_X(L) \\
&= \tfrac{1}{8}(q_X(L) + 4)(q_X(L) + 6).
\end{aligned}$$

Thus we obtain (3.1.16), as we wanted. Now, if L is ample, since $\omega_X \cong \mathcal{O}_X$, using Theorem 1.1.18 we have that $h^0(X, L) = \mathcal{X}(L)$, which shows the second part of the statement. □

3.2 Nakajima operators

In this section we introduce the *Nakajima operators*. We first recall the most important definitions and results from [Nak97], [Leh99], [Boi05], [BNW07], [BNWS13]. We will then see how the theory of Nakajima operators applies in the case of the Hilbert square X of a K3 surface, with the computation of an integral basis of $H^2(X,\mathbb{Z})$ and $H^4(X,\mathbb{Z})$ given by Qin and Wang in [QW05].

3.2.1 Basic definitions and results

Let S be a smooth irreducible complex projective surface. For any integer $n \geq 0$ let $S^{[n]}$ be the Hilbert scheme of n points on S. The idea is to study the singular cohomology groups with rational coefficients of $S^{[n]}$ for every $n \geq 0$ taken all together. We define

$$\mathbb{H}_n^S := \bigoplus_{i=0}^{4n} H^i(S^{[n]},\mathbb{Q}), \qquad \mathbb{H}^S := \bigoplus_{n\geq 0} \mathbb{H}_n^S.$$

The unit of the $\mathbb{Q}$-algebra (with cup-product) $\mathbb{H}_0^S \cong \mathbb{Q}$ is called *vacuum vector* and it is denoted by $|0\rangle$. The unit in $\mathbb{H}^S$ for the cup product is given by

$$|1\rangle := \sum_{n\geq 0} 1_{S^{[n]}}.$$

The space $\mathbb{H}^S$ is double graded by (n,i): we say that n is the *conformal weight* and i is the *cohomological degree*, denoted by $|\cdot|$. Let $\mathfrak{f} \in \mathrm{End}(\mathbb{H}^S)$ be a linear operator. We say that $\mathfrak{f}$ is *homogeneous of bidegree* (ν,ι) if for any n we have $\mathfrak{f}\left(H^i(S^{[n]},\mathbb{Q})\right) \subset H^{i+\iota}(S^{[n+\nu]},\mathbb{Q})$. The *commutator* of two homogeneous operators $\mathfrak{f},\mathfrak{g} \in \mathrm{End}(\mathbb{H}^S)$ is defined by:

$$[\mathfrak{f},\mathfrak{g}] := \mathfrak{f}\circ\mathfrak{g} - (-1)^{|\mathfrak{f}|\cdot|\mathfrak{g}|}\mathfrak{g}\circ\mathfrak{f}.$$

We can define an intersection pairing $\langle\cdot,\cdot\rangle$ on $\mathbb{H}^S$ generalizing the construction of $\langle\cdot,\cdot\rangle$ given in Section 3.1. First of all, fix an integer $n \geq 0$, and let $\alpha,\beta \in \mathbb{H}_n^S$. We set

$$\langle\cdot,\cdot\rangle : \mathbb{H}_n^S \times \mathbb{H}_n^S \to \mathbb{Q}, \qquad \langle\alpha,\beta\rangle := \int_{S^{[n]}} \alpha \cup \beta.$$

In particular we have $\langle\alpha,\beta\rangle = 0$ if $|\alpha| + |\beta| \neq 4n$. Then $\langle\cdot,\cdot\rangle$ extends naturally to a non-degenerate graded-symmetric bilinear form on $\mathbb{H}^S$, which we denote again by $\langle\cdot,\cdot\rangle$. If $\mathfrak{f} \in \mathrm{End}(\mathbb{H}^S)$ is a homogeneous operator, we define the *adjoint operator* $\mathfrak{f}^\dagger$ as the homogeneous operator characterised by the relation

$$\langle\mathfrak{f}(\alpha),\beta\rangle = (-1)^{|\mathfrak{f}|\cdot|\alpha|}\langle\alpha,\mathfrak{f}^\dagger(\beta)\rangle.$$

In order to define Nakajima operators, we have to define $S^{[n,n+k]}$, an irreducible subvariety of $S^{[n]}\times S\times S^{[n+k]}$, for any integers $n \geq 0$, $k > 0$. For this construction we follow [Leh99, §1.2]. We denote by $X^{[n,n+k]} \subset S^{[n]} \times S^{[n+k]}$ the uniquely determined closed subscheme with the property that any morphism

$$f = (f_1, f_2) : T \to S^{[n]} \times S^{[n+k]}$$

from an arbitrary variety T factors through $X^{[n,n+k]}$ if and only if the following holds:

$$(f_1 \times \mathrm{id}_S)^{-1}(\Xi_n^S) \subset (f_2 \times \mathrm{id}_S)^{-1}(\Xi_{n+k}^S),$$

where $\Xi_n^S \subset S^{[n]} \times S$ is the universal family of subschemes parametrized by $S^{[n]}$. Closed points in $X^{[n,n+k]}$ corresponds to pairs (ξ, ξ') of subschemes with $\xi \subseteq \xi'$. Then one obtains a morphism similar to the Hilbert–Chow morphism:

$$\rho : X^{[n,n+k]} \to \mathrm{Sym}^k S.$$

We set $X_0^{[n,n+k]} := \rho^{-1}(\Delta)$, where $\Delta \subset \mathrm{Sym}^k S$ is the small diagonal, and we consider on $X_0^{[n,n+k]}$ the reduced induced subscheme structure. Set-theoretically we can identify $X_0^{[n,n+k]}$ with the following subset of $S^{[n]} \times S \times S^{[n+k]}$:

$$X_0^{[n,n+k]} := \{(\xi, x, \xi') \mid \xi \subseteq \xi' \text{ and } \mathrm{Supp}(\mathcal{I}_\xi/\mathcal{I}_{\xi'}) = x\}, \tag{3.2.1}$$

where $\mathcal{I}_\xi$ is the ideal sheaf of ξ. We define $S^{[n,n+k]}$ as

$$S^{[n,n+k]} := \overline{\{(\xi, x, \xi') \in X_0^{[n,n+k]} \mid l(\xi_x) = 0\}},$$

where the closure is the Zariski closure. Then $S^{[n,n+k]}$ is an irreducible subvariety of $S^{[n]} \times S \times S^{[n+k]}$ of dimension $2n + k + 1$ by [Leh99, Lemma 1.1].

Consider the following diagram:

$$\begin{tikzcd} & S^{[n]} \times S \times S^{[n+k]} \arrow[dl, "\varphi"'] \arrow[d, "\rho"] \arrow[dr, "\psi"] & \\ S^{[n]} & S & S^{[n+k]}, \end{tikzcd} \tag{3.2.2}$$

where φ, ρ and ψ are the projections respectively on $S^{[n]}$, S and $S^{[n+k]}$. We can now define the *Nakajima operators*.

Definition 3.2.1. Let S be a smooth irreducible complex projective surface. We define the *Nakajima creation operators* (known also as *Heisenberg operators*)

$$\mathfrak{q}_k : H^*(S, \mathbb{Q}) \to \mathrm{End}(\mathbb{H}^S), \qquad k \geq 0$$

in the following way: for $\alpha \in H^*(S, \mathbb{Q})$ and $x \in H^*(S^{[n]}, \mathbb{Q})$ we set

$$\mathfrak{q}_k(\alpha)(x) := \psi_* \left(PD^{-1} \left[S^{[n,n+k]} \right] \cdot \varphi^*(x) \cdot \rho^*(\alpha) \right), \tag{3.2.3}$$

where $\left[S^{[n,n+k]}\right] \in H_{4n+2k+2}(S^{[n]} \times S \times S^{[n+k]}, \mathbb{Q})$ is the fundamental class of $S^{[n,n+k]}$, PD is the Poincaré duality, the dot is the cup product, and ψ_* is the Gysin homomorphism given in Definition 1.2.1. The *Nakajima annihilation operators* are defined as

$$\mathfrak{q}_k(\alpha) := (-1)^{-k} \mathfrak{q}_{-k}(\alpha)^\dagger \qquad \text{for all } k < 0.$$

By convention, $\mathfrak{q}_0 = 0$. Note that the unit $|1\rangle \in \mathbb{H}^S$ can be represented in terms of the Nakajima operators as

$$|1\rangle = \sum_{n \geq 0} 1_{S^{[n]}} = \exp(\mathfrak{q}_1(1_S))|0\rangle,$$

which gives

$$1_{S^{[n]}} = \frac{1}{n!} \mathfrak{q}_1(1)^n |0\rangle. \tag{3.2.4}$$

The following commutation formula was obtained by Nakajima in [Nak97].

Theorem 3.2.2 (Nakajima). *The operators $\mathfrak{q}_i$ satisfy the following commutation formula:*

$$[\mathfrak{q}_i(\alpha), \mathfrak{q}_j(\beta)] = i \cdot \delta_{i+j,0} \cdot \int_S \alpha\beta \cdot \mathrm{id}_{\mathbb{H}^S},$$

where δ is the Kronecker delta.

3.2.2 The boundary operator

We now want to define a derivative for a linear operator $\mathfrak{f} \in \mathrm{End}(\mathbb{H}^S)$. In order to do that, we introduce the *boundary operator.* Fix a positive integer $n > 0$ and let $\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0\}$ be a partition of n, i.e., an s-uple of ordered positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_s = n$. Consider the Hilbert–Chow morphism

$$\rho : S^{[n]} \to \mathrm{Sym}^n S.$$

We set

$$\mathrm{Sym}^n_\lambda S := \{\alpha \in \mathrm{Sym}^n S \mid \alpha = \sum_{1 \leq i \leq s} \lambda_i x_i, \;\; x_i \in S \text{ pairwise distinct}\}$$

for a fixed partition λ. By a theorem of Briançon, see [Bri77], $S^{[n]}_\lambda := \rho^{-1}(\mathrm{Sym}^n_\lambda S)$ is irreducible of dimension $n + s$. For example, $S^{[n]}_{(1,1,\dots,1)}$ is the open subset of $S^{[n]}$ which corresponds to the configuration space of unordered n-tuples of pairwise distinct points: it is the only stratum which is open. The only stratum of codimension 1 is $S^{[n]}_{(2,1,\dots,1)}$.

Definition 3.2.3. The *boundary* of $S^{[n]}$ for $n \geq 2$ is the irreducible divisor

$$\partial S^{[n]} := \bigcup_{\lambda \neq (1,1,\dots,1)} S^{[n]}_\lambda = \overline{S^{[n]}_{(2,1,\dots,1)}}.$$

Remark 3.2.4. Clearly for $n = 1$ the boundary $\partial S^{[n]}$ is the empty set.

There exists another description of the divisor $\partial S^{[n]}$ in sheaf theoretic terms. Consider the *universal family* of subschemes

$$\Xi_n \subset S^{[n]} \times S.$$

Let $p : \Xi_n \to S^{[n]}$ be the projection on the first factor. Since p is a flat morphism of finite degree n, we have that

$$\mathcal{O}^{[n]}_S := p_*(\mathcal{O}_{\Xi_n}) \in \mathrm{Coh}(S^{[n]})$$

is a locally free sheaf of rank n.

Lemma 3.2.5 (Lemma 3.7 in [Leh99]). *Keep notation as above. Then we have*

$$[\partial S^{[n]}] = -2c_1(\mathcal{O}^{[n]}_S).$$

We can now define the *boundary operator.*

Definition 3.2.6. The *boundary operator* $\mathfrak{d} : \mathbb{H}^S \to \mathbb{H}^S$ is the homogeneous linear map of bidegree $(0, 2)$ given by

$$\mathfrak{d}(x) := c_1(\mathcal{O}_S^{[n]}) \cdot x = -\frac{1}{2}[\partial S^{[n]}] \cdot x \qquad \text{for all } x \in H^*(S^{[n]}).$$

For any endomorphism $\mathfrak{f} \in \operatorname{End}(\mathbb{H}^S)$ we define the *derivative* of $\mathfrak{f}$ as

$$\mathfrak{f}' := [\mathfrak{d}, \mathfrak{f}] = \mathfrak{d} \circ \mathfrak{f} - \mathfrak{f} \circ \mathfrak{d}.$$

We denote by $\mathfrak{f}^{(n)}$ the higher derivatives.

Note that as a consequence of the Jacobi identity, $\mathfrak{f} \mapsto \mathfrak{f}'$ is a derivation, i.e., for any $\mathfrak{a}, \mathfrak{b} \in \operatorname{End}(\mathbb{H}^S)$ the Leibniz rule holds:

$$(\mathfrak{a}\mathfrak{b})' = \mathfrak{a}'\mathfrak{b} + \mathfrak{a}\mathfrak{b}' \qquad \text{and} \qquad [\mathfrak{a}, \mathfrak{b}]' = [\mathfrak{a}', \mathfrak{b}] + [\mathfrak{a}, \mathfrak{b}'].$$

The following relation is part of the main result of [Leh99].

Theorem 3.2.7 (Theorem 3.10 in [Leh99])**.** *Let S be a smooth irreducible complex projective surface and K be the canonical class of X. Then the following holds for all $n, m \in \mathbb{Z}$ and $\alpha, \beta \in H^*(S, \mathbb{Q})$:*

$$[\mathfrak{q}_n'(\alpha), \mathfrak{q}_m(\beta)] = -nm \cdot \left\{ \mathfrak{q}_{n+m}(\alpha\beta) + \frac{|n| - 1}{2} \delta_{n+m,0} \cdot \int_S K\alpha\beta \cdot \mathrm{id}_{\mathbb{H}} \right\},$$

where δ is the Kronecker delta. In particular, for any integers n, m such that $n + m \neq 0$ and cohomology classes $\alpha, \beta \in H^*(S, \mathbb{Q})$ we have

$$[\mathfrak{q}_n'(\alpha), \mathfrak{q}_m(\beta)] = -nm \cdot \mathfrak{q}_{n+m}(\alpha\beta).$$

3.2.3 Ellingsrud–Göttsche–Lehn formula

In this section we state the *Ellingsrud–Göttsche–Lehn* (EGL) *formula*. This will be a fundamental tool for our computation of the lattice $H^{2,2}(S^{[2]}, \mathbb{Z})$ of the Hilbert square of a generic K3 surface S.

Let $\Xi_n^S \subset S^{[n]} \times S$ be the universal family of subschemes parametrized by $S^{[n]}$. Then by [Leh99, Theorem 1.9], we have that $S^{[n,n+1]}$ is a smooth irreducible variety which is isomorphic to the blow-up of $S^{[n]} \times S$ along Ξ_n^S, i.e., $S^{[n,n+1]} \cong \mathrm{Bl}_{\Xi_n^S}(S^{[n]} \times S)$, where Ξ_n^S is the universal family. We get the following diagram

$$\begin{tikzcd}
 & S^{[n,n+1]} \arrow[r, "\psi"] \arrow[d, "\sigma"] \arrow[ddl, bend right, "\varphi"'] \arrow[ddr, bend left, "\rho"] & S^{[n+1]} \\
 & S^{[n]} \times S \arrow[dl, "p"] \arrow[dr, "q"'] & \\
S^{[n]} & \Xi_n^S \arrow[u, hook, "\iota"] & S,
\end{tikzcd} \tag{3.2.5}$$

where σ is the blow-up in Ξ_n^S, the morphism ι is the inclusion and the other maps are the projections. If N is the exceptional divisor of σ, let $\mathcal{L} := \mathcal{O}_{S^{[n,n+1]}}(-N)$. We denote by $\mathcal{T}_n$ the tangent bundle $\mathcal{T}S^{[n]}$ and by ω_S the canonical bundle of S.

Recall that given a smooth irreducible projective variety X and $F \in \mathrm{Coh}(X)$, by Definition 1.3.2 the dual is

$$F^\vee := \sum_i (-1)^i \mathcal{E}xt^i(F, \mathcal{O}_X).$$

We can now state the Ellingsrud–Göttsche–Lehn formula.

Proposition 3.2.8 (Proposition 2.3 in [EGL01])**.** *Keep notation as above. The following relation holds in $K(S^{[n,n+1]})$:*

$$\begin{aligned} \psi^! \mathcal{T}_{n+1} &= \varphi^! \mathcal{T}_n + \mathcal{L} - \mathcal{L} \cdot \sigma^!(\mathcal{O}_{\Xi_n}^\vee) + \mathcal{L}^\vee \cdot \rho^! \omega_S^\vee \\ & \quad - \mathcal{L}^\vee \cdot \sigma^!(\mathcal{O}_{\Xi_n}) \cdot \rho^! \omega_S^\vee - \rho^!(\mathcal{O}_S - \mathcal{T}_S + \omega_S^\vee). \end{aligned}$$

We will use the EGL-formula to compute the second Chern class $c_2(X)$ of the Hilbert square $X = S^{[2]}$ of a K3 surface in terms of Nakajima operators.

3.2.4 Hilbert squares of K3 surfaces and Nakajima operators

We apply the general theory on Nakajima operators to the case of Hilbert squares of projective K3 surfaces. Let S be a projective K3 surface. Let $\{\alpha_i\}_{i=1,\dots,22}$ be an integral basis of $H^2(S,\mathbb{Z})$, denote by $1 \in H^0(S,\mathbb{Z})$ the unit and by $x \in H^4(S,\mathbb{Z})$ the class of a point. The following theorem by Qin and Wang describes the integral basis of of $H^2(S^{[2]},\mathbb{Z})$ and $H^4(S^{[2]},\mathbb{Z})$ in terms of the Nakajima operators. We state it as presented in [BNWS13, p.17], for a more general statement see [QW05, Theorem 5.4, Remark 5.6].

Theorem 3.2.9 (Qin–Wang)**.** *Let S be a projective K3 surface and $X = S^{[2]}$ be its Hilbert square. Let $\{\alpha_i\}_{i=1,\dots,22}$ be an integral basis of $H^2(S,\mathbb{Z})$. Denote by $1 \in H^0(S,\mathbb{Z})$ the unit and by $x \in H^4(S,\mathbb{Z})$ the class of a point.*

(i) An integral basis of $H^2(X,\mathbb{Z})$ in terms of Nakajima operators is given by

$$\frac{1}{2}\mathfrak{q}_2(1)|0\rangle, \qquad \mathfrak{q}_1(1)\mathfrak{q}_1(\alpha_i)|0\rangle,$$

with $i = 1, \dots, 22$.

(ii) An integral basis of $H^4(X,\mathbb{Z})$ in terms of Nakajima operators is given by

$$\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \qquad \mathfrak{q}_2(\alpha_i)|0\rangle, \qquad \mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle \text{ with } i<j,$$
$$\mathfrak{m}_{1,1}(\alpha_i)|0\rangle := \tfrac{1}{2}\left(\mathfrak{q}_1(\alpha_i)^2 - \mathfrak{q}_2(\alpha_i)\right)|0\rangle,$$

with $i, j = 1, \dots, 22$.

Remark 3.2.10. If $X = S^{[2]}$ is the Hilbert square of a projective K3 surface, then by Definition 3.2.1 with $k = 2$, $n = 0$, we have $\frac{1}{2}\mathfrak{q}_2(1)|0\rangle = \delta$, and if $\alpha \in H^2(S,\mathbb{Z})$, then by Definition 3.2.1 with $n = k = 1$ we obtain that α seen as an element of $H^2(S^{[2]},\mathbb{Z})$ is represented by $\mathfrak{q}_1(1)\mathfrak{q}_1(\alpha)|0\rangle$.

Let S be a K3 surface and $\tau_2 : S \to S \times S$ be the diagonal embedding. We denote the Gysin homomorphism followed by the Künneth isomorphism by $\tau_{2*} : H^*(S,\mathbb{Z}) \to H^*(S,\mathbb{Z}) \otimes H^*(S,\mathbb{Z})$. We can write

$$\tau_{2*}1 = \sum_{i,j} \mu_{i,j}\alpha_i \otimes \alpha_j + a(1 \otimes x) + b(x \otimes 1) \qquad (3.2.6)$$

for some $\mu_{i,j} \in \mathbb{Z}$ such that $\mu_{i,j} = \mu_{j,i}$, $i,j \in 1,\dots,22$, and some $a,b \in \mathbb{Z}$. We want to explicitly determine the $\mu_{i,j}$'s. We have seen in Section 2.1 that $H^2(S,\mathbb{Z}) \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$. We will use the basis $\{\alpha_1,\dots,\alpha_{22}\}$ of $H^2(S,\mathbb{Z})$ obtained as follows: let $\{\alpha_1,\dots,\alpha_8\}$ and $\{\alpha_9,\dots,\alpha_{16}\}$ be the basis of the two copies of $E_8(-1)$ with the Gram matrix given in Example 1.4.7 and $\{\alpha_{17},\alpha_{18}\}$, $\{\alpha_{19},\alpha_{20}\}$, $\{\alpha_{21},\alpha_{22}\}$ be the basis of the three copies of U with Gram matrix given in Example 1.4.5. The values of the $\mu_{i,j}$'s are given by the following lemma.

Lemma 3.2.11. *Let $X = S^{[2]}$ be the Hilbert square of a K3 surface S. Assume that $\{\alpha_1,\dots,\alpha_{22}\}$ is the basis of the lattice $H^2(S,\mathbb{Z})$ constructed above. Then*

$$\tau_{2*}1 = \sum_{i,j} \mu_{i,j}\alpha_i \otimes \alpha_j + 1 \otimes x + x \otimes 1,$$

where the $\mu_{i,j}$'s are represented in Table 3.1 (we write down only the $\mu_{i,j}$'s which are non zero and such that $i \leq j$):

$\mu_{1,1} = -4$	$\mu_{1,2} = -7$	$\mu_{1,3} = -10$	$\mu_{1,4} = -8$	$\mu_{1,5} = -6$
$\mu_{1,6} = -4$	$\mu_{1,7} = -2$	$\mu_{1,8} = -5$	$\mu_{2,2} = -14$	$\mu_{2,3} = -20$
$\mu_{2,4} = -16$	$\mu_{2,5} = -12$	$\mu_{2,6} = -8$	$\mu_{2,7} = -4$	$\mu_{2,8} = -10$
$\mu_{3,3} = -30$	$\mu_{3,4} = -24$	$\mu_{3,5} = -18$	$\mu_{3,6} = -12$	$\mu_{3,7} = -6$
$\mu_{3,8} = -15$	$\mu_{4,4} = -20$	$\mu_{4,5} = -15$	$\mu_{4,6} = -10$	$\mu_{4,7} = -5$
$\mu_{4,8} = -12$	$\mu_{5,5} = -12$	$\mu_{5,6} = -8$	$\mu_{5,7} = -4$	$\mu_{5,8} = -9$
$\mu_{6,6} = -6$	$\mu_{6,7} = -3$	$\mu_{6,8} = -6$	$\mu_{7,7} = -2$	$\mu_{7,8} = -3$
$\mu_{8,8} = -8$	$\mu_{9,9} = -4$	$\mu_{9,10} = -7$	$\mu_{9,11} = -10$	$\mu_{9,12} = -8$
$\mu_{9,13} = -6$	$\mu_{9,14} = -4$	$\mu_{9,15} = -2$	$\mu_{9,16} = -5$	$\mu_{10,10} = -14$
$\mu_{10,11} = -20$	$\mu_{10,12} = -16$	$\mu_{10,13} = -12$	$\mu_{10,14} = -8$	$\mu_{10,15} = -4$
$\mu_{10,16} = -10$	$\mu_{11,11} = -30$	$\mu_{11,12} = -24$	$\mu_{11,13} = -18$	$\mu_{11,14} = -12$
$\mu_{11,15} = -6$	$\mu_{11,16} = -15$	$\mu_{12,12} = -20$	$\mu_{12,13} = -15$	$\mu_{12,14} = -10$
$\mu_{12,15} = -5$	$\mu_{12,16} = -12$	$\mu_{13,13} = -12$	$\mu_{13,14} = -8$	$\mu_{13,15} = -4$
$\mu_{13,16} = -9$	$\mu_{14,14} = -6$	$\mu_{14,15} = -3$	$\mu_{14,16} = -6$	$\mu_{15,15} = -2$
$\mu_{15,16} = -3$	$\mu_{16,16} = -8$	$\mu_{17,18} = 1$	$\mu_{19,20} = 1$	$\mu_{21,22} = 1$

Table 3.1: The $\mu_{i,j}$'s.

Proof. Let $\langle \cdot , \cdot \rangle$ be the intersection pairing of $\mathbb{H}^S$ seen in Section 3.2.1. Since the map τ_{2*} is the adjoint of the cup product, we have the relation

$$\langle \tau_{2*}1, \alpha_k \otimes \alpha_l \rangle = \int_S \alpha_k\alpha_l,$$

which gives, together with $\tau_{2*}1 = \sum_{i,j} \mu_{i,j}\alpha_i \otimes \alpha_j + a(1 \otimes x) + b(x \otimes 1)$, the following system:

$$\sum_{i,j} \mu_{i,j} \int_S \alpha_i\alpha_k \int_S \alpha_j\alpha_l = \int_S \alpha_k\alpha_l. \tag{3.2.7}$$

From (3.2.7), with the help of a computer, we can compute the coefficients $\mu_{i,j}$. The solution of the system is given in Table 3.1, where we have written down only the $\mu_{i,j}$'s which are non zero and such that $i \leq j$, since $\mu_{i,j} = \mu_{j,i}$. Similarly, from the relations

$$\langle \tau_{2*}1, 1 \otimes x \rangle = \int_S x = 1, \qquad \langle \tau_{2*}1, x \otimes 1 \rangle = \int_S x = 1,$$

we obtain $a = b = 1$. □

3.3 Lehn–Sorger model

In this section we introduce the algebraic model developed by Lehn and Sorger in [LS03], see also [HHT12] and [Kap16a]. Using this model, one can find a basis of the cohomology ring $H^*(S^{[n]}, \mathbb{Q})$ in terms of the Nakajima operators starting from a basis of $H^*(S, \mathbb{Q})$. We will show in detail the case $n = 2$. Moreover, the model will give us tools to compute explicitly the cup product between elements in $H^*(S^{[n]}, \mathbb{Q})$. As an example we will describe the cup product between elements in $H^2(S^{[2]}, \mathbb{Z})$ in Lemma 3.3.9.

3.3.1 The algebraic model: graded Frobenius algebras

We begin with the definition of graded Frobenius algebra.

Definition 3.3.1. A *graded Frobenius algebra* of degree d is a finite dimensional graded vector space

$$A = \bigoplus_{i=-d}^{d} A^i$$

endowed with a graded commutative and associative multiplication $A \otimes A \to A$ of degree d, i.e., $\deg(ab) = \deg(a) + \deg(b) + d$, and unit element 1, necessarily of degree $-d$, together with a linear form

$$T : A \to \mathbb{Q}$$

of degree $-d$ such that the induced symmetric bilinear form $\langle a, b \rangle := T(ab)$ is non-degenerate and of degree 0.

Since $1 \cdot 1 = 1$, we see that d must be an even number. Note that the degree convention is taken in such a way that A is centered around degree 0. The degree of an element A will be denoted by $|a|$, as seen in Section 3.2.1. A typical example is $A := H^*(X, \mathbb{Q})[2n]$, the shifted cohomology ring of a compact complex manifold X of even dimension $2n$, see Example 2.1.10, (ii).

From now on, we consider $A := H^*(S, \mathbb{Q})[2]$, where S is a projective K3 surface. The linear form $T : A \to \mathbb{Q}$ is

$$T(\alpha) := -\int_S \alpha, \tag{3.3.1}$$

and the induced bilinear form $\langle \cdot, \cdot \rangle$ is then

$$\langle \alpha, \beta \rangle = T(\alpha\beta) = -\int_S \alpha\beta,$$

where as always $\alpha\beta$ denotes the cup product $\alpha \cup \beta$. Note that the bilinear form is just the intersection pairing already seen in Section 3.2.1 with a change of sign.

If A is a graded Frobenius algebra, then $A^{\otimes n}$ has an induced structure of Frobenius algebra. As remarked in [HHT12], if $A = H^*(S, \mathbb{Q})[2]$ with S a projective K3 surface, then A has only graded pieces of even degree, so the general construction of Lehn–Sorger simplifies. In this case, the multiplication induced on $A^{\otimes n}$ is

$$(a_1 \otimes \cdots \otimes a_n) \cdot (b_1 \otimes \cdots \otimes b_n) = (a_1 b_1) \otimes \cdots \otimes (a_n b_n), \tag{3.3.2}$$

and the linear form is

$$T : A^{\otimes n} \to \mathbb{Q}, \qquad a_1 \otimes \cdots \otimes a_n \mapsto T(a_1)T(a_2)\dots T(a_n).$$

Denote by S_n the symmetric group of order n. Then S_n acts on $A^{\otimes n}$ as

$$\pi(a_1 \otimes \cdots \otimes a_n) := a_{\pi^{-1}(1)} \otimes \cdots \otimes a_{\pi^{-1}(n)}.$$

Given a partition $n = n_1 + \cdots + n_k$ with $n_1, \dots, n_k \in \mathbb{Z}_{>0}$, we have a generalized multiplication map $A^{\otimes n} \to A^{\otimes k}$ defined by

$$a_1 \otimes \cdots \otimes a_n \mapsto (a_1 \dots a_{n_1}) \otimes \cdots \otimes (a_{n_1+\cdots+n_{k-1}+1} \dots a_{n_1+\cdots+n_k}). \tag{3.3.3}$$

Given a finite set $I \subset \{1, \dots, n\}$, let $A^{\otimes I}$ denote the tensor power with factors indexed by elements of I. Given a surjection $\phi : I \to J$, there is an induced multiplication

$$\phi^* : A^{\otimes I} \to A^{\otimes J} \tag{3.3.4}$$

defined as above. Let

$$\phi_* : A^{\otimes J} \to A^{\otimes I}$$

denote the *adjoint* of ϕ^*, i.e.,

$$\langle \phi^* a, b \rangle = \langle a, \phi_* b \rangle$$

for $a \in A^{\otimes I}$ and $b \in A^{\otimes J}$. If I and J are finite sets, we can use the same construction: it suffices to identify I and J with the finite sets $\{1, 2, \dots, |I|\}$ and $\{1, 2, \dots, |J|\}$ respectively. Consider the multiplication of A, i.e., the cup product $\cup : A \otimes A \to A$. We denote by Δ_* its adjoint, so we have the composite

$$A \xrightarrow{\Delta_*} A \otimes A \to A. \tag{3.3.5}$$

Let

$$e := e(A) \tag{3.3.6}$$

denote the image of 1 under the composed map.

Remark 3.3.2 (Remark 3.1 in [HHT12])**.** Let Δ_S denote the fundamental cohomological class of the diagonal in $H^*(S \times S, \mathbb{Z}) \cong H^*(S, \mathbb{Z}) \otimes H^*(S, \mathbb{Z})$. Let $\{e_1, \dots, e_{24}\}$ be a homogeneous basis of $H^*(S, \mathbb{Q})$ with dual basis $\{e_1^\vee, \dots, e_{24}^\vee\}$, i.e.,

$$\int_S e_i e_j^\vee = \delta_{i,j},$$

see also [MS16, Theorem 11.10]. By [MS16, Theorem 11.11] we have

$$\Delta_S = \sum_{j=1}^{24} e_j \otimes e_j^\vee.$$

Let $\alpha, \beta \in H^*(S, \mathbb{Z})$: we can write $\alpha = \sum_i \alpha_i e_i$ and $\beta = \sum_i \beta_k e_k$ for some $\alpha_k, \beta_k \in \mathbb{Q}$. We obtain

$$\begin{aligned}
\langle \Delta_S, \alpha \otimes \beta \rangle &= \langle \sum_{j=1}^{24} e_j \otimes e_j^\vee, \alpha \otimes \beta \rangle \\
&= \sum_{j=1}^{24} T(e_j \alpha) T(e_j^\vee \beta) \\
&= \sum_{j=1}^{24} \left(-\int_S e_j \alpha \right) \left(-\int_S e_j^\vee \sum_{k=1}^{24} \beta_k e_k \right) \\
&= \int_S \alpha \left(\sum_{j=1}^{24} \beta_j e_j \right) \\
&= \int_S \alpha\beta,
\end{aligned}$$

while by definition of Δ_* we have

$$\begin{aligned}
\langle \Delta_* 1, \alpha \otimes \beta \rangle &= \langle 1, \alpha\beta \rangle \\
&= T(\alpha\beta) \\
&= -\int_S \alpha\beta.
\end{aligned}$$

Hence we obtain

$$\Delta_* 1 = -\Delta_S.$$

Note that in the same way, if $\langle \cdot\,, \cdot \rangle$ is the intersection pairing on $\mathbb{H}^S$ seen in Section 3.2.1 and τ_{2*} is the map given in Section 3.2.4, then we have

$$\Delta_* = -\tau_{2*}, \qquad \tau_{2*} 1 = \Delta_S.$$

See also [Kap16b, Remark 13.6].

Let n be a fixed positive integer, $\pi \in S_n$ be a permutation and $\langle \pi \rangle \subset S_n$ be the subgroup generated by π. If $[n] := \{1, \dots, n\}$, denote by $\langle \pi \rangle \setminus [n]$ the set of orbits. We set

$$A\{S_n\} := \bigoplus_{\pi \in S_n} A^{\otimes \langle \pi \rangle \setminus [n]} \cdot \pi.$$

The grading of an element $a \cdot \pi$ is $|a \cdot \pi| := |a|$.

Example 3.3.3. If $n = 2$, we have $S_2 = \{\mathrm{id}, (1\,2)\}$ and

$$\langle \mathrm{id} \rangle \setminus [2] = \{\{1\}, \{2\}\}, \qquad \langle (1\,2) \rangle \setminus [2] = \{\{1, 2\}\},$$

hence we obtain

$$A\{S_2\} = A^{\otimes 2}\mathrm{id} \oplus A(1\,2).$$

Similarly for $n = 3$ we have

$$A\{S_3\} = A^{\otimes 3}\mathrm{id} \oplus A^{\otimes 2}(1\,2) \oplus A^{\otimes 2}(1\,3) \oplus A^{\otimes 2}(2\,3) \oplus A(1\,2\,3) \oplus A(1\,3\,2).$$

Let $\sigma \in S_n$. There is a bijection

$$\sigma : \langle\pi\rangle \setminus [n] \to \langle\sigma\pi\sigma^{-1}\rangle \setminus [n], \qquad x \mapsto \sigma x$$

and an isomorphism

$$\tilde{\sigma} : A\{S_n\} \to A\{S_n\}, \qquad a\pi \mapsto \sigma^*(a)\sigma\pi\sigma^{-1}. \tag{3.3.7}$$

Thus we obtain an action of the symmetric group S_n on $A\{S_n\}$. We denote by

$$A^{[n]} := (A\{S_n\})^{S_n}$$

the subspace of invariants.

3.3.2 Description of the canonical isomorphism

We can now state the main theorem of [LS03].

Theorem 3.3.4 (Theorem 3.2 in [LS03]). *Let S be a projective K3 surface. Then there is a canonical isomorphism of graded rings*

$$(H^*(S,\mathbb{Q})[2])^{[n]} \xrightarrow{\sim} H^*(S^{[n]},\mathbb{Q})[2n].$$

The structure of graded Frobenius algebra of $H^*(S^{[n]},\mathbb{Q})[2n]$ is obtained by setting

$$T(a) := (-1)^n \int_{S^{[n]}} a \qquad \text{for all } a \in H^*(S^{[n]},\mathbb{Q}).$$

Note that the sign convention agrees with the one taken for $H^*(S,\mathbb{Q})[2]$ in (3.3.1) if $n = 1$. We do not prove the theorem. We will describe the product which gives a ring structure to $(H^*(S,\mathbb{Q})[2])^{[n]}$ in Section 3.3.3. Our aim now is to see explicitly how to obtain a basis of $H^*(S^{[n]},\mathbb{Q})[2n]$ starting from a basis of $H^*(S,\mathbb{Q})[2]$.

We introduce the space

$$\mathcal{V}(A) := \mathrm{Sym}^*(A \otimes t^{-1}\mathbb{Q}[t^{-1}]),$$

called the *bosonic Fock space* modelled on the graded vector space A. Then $\mathcal{V}(A)$ is bigraded by degree and weight, where an element $a \otimes t^{-m} \in A \otimes t^{-m}$ has degree $|a|$ and weight m. The component of $\mathcal{V}(A)$ of constant weight n is the graded vector space

$$\mathcal{V}(A)_n \cong \bigoplus_{||\alpha||=n} \bigotimes_i \mathrm{Sym}^{\alpha_i} A, \tag{3.3.8}$$

where the direct sum is taken over all the possible partitions $\alpha = (1^{\alpha_1}, 2^{\alpha_2}, \dots)$ of n, and

$$||\alpha|| := \alpha_1 \cdot 1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3 + \dots.$$

We now fix $\pi \in S_n$. Let $f : \{1, 2, \dots N\} \to \langle\pi\rangle \setminus [n]$ be an enumeration of the orbits of $\pi \in S_n$. We denote by l_i the length of the i-th orbit, i.e., $l_i := |f(i)|$. We define

$$\Phi' : A^{\otimes N} \to \mathcal{V}(A)$$

by setting

$$a_1 \otimes \cdots \otimes a_N \overset{\Phi'}{\longmapsto} \frac{1}{n!}(a_1 \otimes t^{-l_1}) \ldots (a_N \otimes t^{-l_N}). \tag{3.3.9}$$

Let

$$\Phi : \bigoplus_{n \geq 0} A\{S_n\} \to \mathcal{V}(A)$$

be defined on each summand $A^{\otimes \langle \pi \rangle \backslash [n]} \cdot \pi$ by the composition

$$A^{\otimes \langle \pi \rangle \backslash [n]} \cdot \pi \xrightarrow{\tilde{f}^{-1}} A^{\otimes N} \xrightarrow{\Phi'} \mathcal{V}(A),$$

where $\tilde{f}^{-1}$ denotes the identification of $A^{\otimes \langle \pi \rangle \backslash [n]} \cdot \pi$ with $A^{\otimes N}$ through the enumeration f. The first step to obtain the isomorphism of Theorem 3.3.4 is given by the following.

Proposition 3.3.5 (Proposition 2.11 in [LS03])**.** Φ *induces an isomorphism of graded vector spaces*

$$A^{[n]} \to \mathcal{V}(A)_n.$$

We now give the second part of the isomorphism of Theorem 3.3.4. We have to explain how to obtain an element of $H^*(S^{[n]}, \mathbb{Q})[2n]$ starting from an element of $\mathcal{V}(A)_n$.

Theorem 3.3.6 (Theorem 3.6 in [LS03])**.** *There is an isomorphism of graded vector spaces*

$$\Psi : \mathcal{V}(A) \to \bigoplus_{n \geq 0} H^*(S^{[n]}, \mathbb{Q})[2n]$$

given by

$$(a_1 \otimes t^{-n_1}) \ldots (a_s \otimes t^{-n_s}) \mapsto \mathfrak{q}_{n_1}(a_1) \ldots \mathfrak{q}_{n_s}(a_s)|0\rangle. \tag{3.3.10}$$

The notation in [LS03] is different from the one we use: in [LS03] the Nakajima operator $\mathfrak{q}_{n_i}$ is denoted by $\mathfrak{p}_{-n_i}$ and the vacuum vector $|0\rangle$ is denoted by $\mathbf{1}$. The Nakajima operators in the shifted cohomology are defined in the same way as seen in Section 3.2.1 (we only change the cohomological degrees).

Combining (3.3.9) and (3.3.10) we can obtain a basis of $H^*(S^{[n]}, \mathbb{Q})[2n]$ from a basis of $H^*(S, \mathbb{Q})[2]$. Note that in order to use correctly the isomorphism of Theorem 3.3.4, if $A := H^*(S, \mathbb{Q})[2]$, we have to work with elements of $A\{S_n\}$ which are invariant for the action of S_n. This will be especially important for the computations of the cup product in Section 3.3.3. The basis of $H^*(S^{[n]}, \mathbb{Q})[2n]$ obtained is clearly also a basis of $H^*(S^{[n]}, \mathbb{Q})$ with the standard grading. As an example, we show in detail the case $n = 2$.

Example 3.3.7. Let $n = 2$. By Theorem 3.3.4 there is an isomorphism of graded vector spaces

$$(H^*(S, \mathbb{Q})[2])^{[2]} \xrightarrow{\sim} H^*(S^{[2]}, \mathbb{Q})[4]. \tag{3.3.11}$$

Let $A = H^*(S, \mathbb{Q})[2]$. By Proposition 3.3.5 and isomorphism (3.3.8) we have $A^{[2]} \cong \mathrm{Sym}^2 A \oplus A$. We now impose the isomorphism between the components of degree -2 in (3.3.11), obtaining

$$H^2(S^{[2]}, \mathbb{Q}) \cong H^0(S, \mathbb{Q}) \oplus \left(H^0(S, \mathbb{Q}) \otimes H^2(S, \mathbb{Q})\right),$$

we omit the shifting of the cohomology groups. Let $1 \in H^0(S,\mathbb{Q})$ be the unit and $\{\alpha_1,\dots,\alpha_{22}\}$ be a basis of $H^2(S,\mathbb{Q})$. With the notation used above, we describe the map Φ' from $A^{\otimes 2}$ and A to $\mathcal{V}(A)$. By Example 3.3.3 we have $A\{S_2\} = A^{\otimes 2}\mathrm{id} \oplus A(1\,2)$, and from (3.3.9) we have

$$\begin{aligned} \Phi' :\ & A^{\otimes 2} \to \mathcal{V}(A) \\ & a_1 \otimes a_2 \mapsto \tfrac{1}{2}(a_1 \otimes t^{-1})(a_2 \otimes t^{-1}) \end{aligned} \qquad \begin{aligned} \Phi' :\ & A \to \mathcal{V}(A) \\ & a \mapsto \tfrac{1}{2} a \otimes t^{-2}. \end{aligned} \tag{3.3.12}$$

We now describe Ψ on elements of the form $\frac{1}{2}(a_1 \otimes t^{-1})(a_2 \otimes t^{-1})$ and $\frac{1}{2}a \otimes t^{-2}$: from (3.3.10) we have

$$\tfrac{1}{2}(a_1 \otimes t^{-1})(a_2 \otimes t^{-1}) \overset{\Psi}{\longmapsto} \tfrac{1}{2}\mathfrak{q}_1(a_1)\mathfrak{q}_1(a_2)|0\rangle, \quad \tfrac{1}{2}a \otimes t^{-2} \overset{\Psi}{\longmapsto} \tfrac{1}{2}\mathfrak{q}_2(a)|0\rangle. \tag{3.3.13}$$

Note that Theorem 3.2.2 implies that $\mathfrak{q}_1(a_1)\mathfrak{q}_1(a_2)|0\rangle = \mathfrak{q}_1(a_2)\mathfrak{q}_1(a_1)|0\rangle$ for all $a_1, a_2 \in A$. As remarked above, we have to consider elements of $A\{S_2\}$ which are invariant for the action of S_2. Thus the images of 1 and $1 \otimes \alpha_i + \alpha_i \otimes 1$ under the composition $\Psi \circ \Phi'$ will give us a basis of $H^2(S^{[2]},\mathbb{Q})$. Combining (3.3.12) and (3.3.13) we have

$$1 \overset{\Phi'}{\longmapsto} \frac{1}{2} 1 \otimes t^{-2} \overset{\Psi}{\longmapsto} \frac{1}{2}\mathfrak{q}_2(1)|0\rangle,$$

and

$$1 \otimes \alpha_i + \alpha_i \otimes 1 \overset{\Phi'}{\longmapsto} (1 \otimes t^{-1})(\alpha_i \otimes t^{-1}) \overset{\Psi}{\longmapsto} \mathfrak{q}_1(1)\mathfrak{q}_1(\alpha_i)|0\rangle.$$

We conclude that a basis of $H^2(S^{[2]},\mathbb{Q})$ is given by

$$\left\{ \frac{1}{2}\mathfrak{q}_2(1)|0\rangle, \mathfrak{q}_1(1)\mathfrak{q}_1(\alpha_i)|0\rangle \right\},$$

which is consistent with Theorem 3.2.9. Note that this basis corresponds by Remark 3.2.10 to

$$\{\delta, \alpha_1, \dots, \alpha_{22}\},$$

where $\alpha_1, \dots \alpha_{22}$ are now seen as elements of $H^2(S^{[2]},\mathbb{Q})$. This shows that $H^2(S^{[2]},\mathbb{Q}) \cong H^2(S,\mathbb{Q}) \oplus \mathbb{Q}\delta$.

We now give a basis for $H^4(S^{[2]},\mathbb{Q})$. Imposing the isomorphism between the components of degree 0 in (3.3.11) we have

$$H^4(S^{[2]},\mathbb{Q}) \cong H^2(S,\mathbb{Q}) \oplus \big(H^0(S,\mathbb{Q}) \otimes H^4(S,\mathbb{Q})\big) \oplus \mathrm{Sym}^2(H^2(S,\mathbb{Q})). \tag{3.3.14}$$

Let $x \in H^4(S,\mathbb{Q})$ be the class of a point. With the same procedure seen above, a basis of $H^4(S^{[2]},\mathbb{Q})$ is obtained as follows:

$$\begin{aligned} &\alpha_i \overset{\Phi'}{\longmapsto} \tfrac{1}{2}\alpha_i \otimes t^{-2} \overset{\Psi}{\longmapsto} \tfrac{1}{2}\mathfrak{q}_2(\alpha_i)|0\rangle, \\ &1 \otimes x + x \otimes 1 \overset{\Phi'}{\longmapsto} (1 \otimes t^{-1})(x \otimes t^{-1}) \overset{\Psi}{\longmapsto} \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \\ &\alpha_i \otimes \alpha_j + \alpha_j \otimes \alpha_i \overset{\Phi'}{\longmapsto} (\alpha_i \otimes t^{-1})(\alpha_j \otimes t^{-1}) \overset{\Psi}{\longmapsto} \mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle \quad \text{if } i < j, \\ &\alpha_i \otimes \alpha_i \overset{\Phi'}{\longmapsto} \tfrac{1}{2}(\alpha_i \otimes t^{-1})(\alpha_i \otimes t^{-1}) \overset{\Psi}{\longmapsto} \tfrac{1}{2}\mathfrak{q}_1(\alpha_i)^2|0\rangle. \end{aligned}$$

Thus we have the basis

$$\left\{ \frac{1}{2}\mathfrak{q}_2(\alpha_i)|0\rangle, \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle, \frac{1}{2}\mathfrak{q}_1(\alpha_i)^2|0\rangle \right\},$$

where $i, j \in \{1, \dots, 22\}$ and $i \leq j$. Equivalently, a basis of $H^4(S^{[2]}, \mathbb{Q})$ is given by

$$\{\mathfrak{q}_2(\alpha_i)|0\rangle, \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle, \mathfrak{m}_{1,1}(\alpha_i)|0\rangle\},$$

where $i, j \in \{1, \dots, 22\}$, $i < j$ and $\mathfrak{m}_{1,1}(\alpha_i)|0\rangle := \frac{1}{2}\left(\mathfrak{q}_1(\alpha_i)^2 - \mathfrak{q}_2(\alpha_i)\right)|0\rangle$. This is consistent with Theorem 3.2.9.

3.3.3 The cup product

In this section we describe the product of $(H^*(S, \mathbb{Q})[2])^{[n]}$ defined in [LS03] to give the structure of ring. This corresponds by the isomorphism of graded rings

$$(H^*(S, \mathbb{Q})[2])^{[n]} \xrightarrow{\sim} H^*(S^{[n]}, \mathbb{Q})[2n]$$

of Theorem 3.3.4 to the cup product on $H^*(S^{[n]}, \mathbb{Q})[2n]$. As an example, we will use the product of $(H^*(S, \mathbb{Q})[2])^{[n]}$ to compute explicitly the cup product between elements in $H^2(S^{[2]}, \mathbb{Z})$ in Lemma 3.3.9. See [HHT12] for the case $n = 3$.

We set $A := H^*(S, \mathbb{Q})[2]$. The idea is to define a product on $A\{S_n\}$ such that $A^{[n]}$ becomes a commutative subring of $A\{S_n\}$. We follow [LS03, §2], see also [Kap16a, §1].

Let $H \subset K$ be subgroups of S_n and let $H \setminus [n]$ and $K \setminus [n]$ be the sets of orbits. We have a surjection $H \setminus [n] \twoheadrightarrow K \setminus [n]$, then by (3.3.3) and (3.3.4) (identify $H \setminus [n]$ and $K \setminus [n]$ with the sets $\{1, \dots, |H \setminus [n]|\}$ and $\{1, \dots, |K \setminus [n]|\}$) we have a multiplication map

$$f^{H,K} : A^{\otimes H\setminus[n]} \to A^{\otimes K\setminus[n]}.$$

Let

$$f_{K,H} : A^{\otimes K\setminus[n]} \to A^{\otimes H\setminus[n]}$$

be the adjoint of $f^{H,K}$. From now on, if $H = \langle\pi\rangle$ is the subgroup generated by the permutation π, we omit $\langle - \rangle$ in the notation.

We now define the *graph defect map*. Let $H \subset S_n$ be a subgroup and $B \subset [n] = \{1, \dots, n\}$ be an H-stable subset. Denote by $H \setminus B$ the orbit space for the induced action. If $\pi, \rho \in S_n$ and $\langle\pi, \rho\rangle$ is the subgroup of S_n generated by π and ρ, the *graph defect*

$$g(\pi, \rho) : \langle\pi, \rho\rangle \setminus [n] \to \mathbb{Z}_{\geq 0}$$

is the map given by

$$g(\pi, \rho)(B) := \frac{1}{2}\left(|B| + 2 - |\pi \setminus B| - |\rho \setminus B| - |\pi\rho \setminus B|\right).$$

Another notation is the following, adopted in [Kap16a]: if $p_\pi : \pi\setminus[n] \to \langle\pi, \rho\rangle\setminus[n]$ is the natural surjection of orbit spaces, then $g(\pi, \rho) : \langle\pi, \rho\rangle \setminus [n] \to \mathbb{Z}_{\geq 0}$ is given by

$$g(\pi, \rho)(B) = \frac{1}{2}\left(|B| + 2 - |p_\pi^{-1}(\{B\})| - |p_\rho^{-1}(\{B\})| - |p_{\pi\rho}^{-1}(\{B\})|\right).$$

Note that $g(\pi, \rho)$ takes value in $\mathbb{Z}_{\geq 0}$ by [LS03, Lemma 2.7].

Let e be the element of A defined in (3.3.6). For any numerical function $\nu : I \to \mathbb{Z}_{\geq 0}$, we define

$$e^\nu := \otimes_{i\in I} e^{\nu(i)} \in A^{\otimes I}.$$

For $\pi, \rho \in S_n$, we set

$$m_{\pi,\rho} : A^{\otimes \pi\backslash[n]} \otimes A^{\otimes \rho\backslash[n]} \to A^{\otimes \pi\rho\backslash[n]},$$
$$m_{\pi,\rho}(a \otimes b) := f_{\langle\pi,\rho\rangle,\pi\rho}\left(f^{\pi,\langle\pi,\rho\rangle}(a) \cdot f^{\rho,\langle\pi,\rho\rangle}(b) \cdot e^{g(\pi,\rho)}\right),$$

where the dot is the multiplication given in (3.3.2), i.e., it is the cup product on each tensor factor.

We can now define the product in $A\{S_n\}$.

Definition 3.3.8. Keep notation as above. The product on $A\{S_n\}$ is given by

$$A\{S_n\} \times A\{S_n\} \xrightarrow{\cdot} A\{S_n\}, \qquad a\pi \cdot b\rho := m_{\pi,\rho}(a \otimes b)\pi\rho.$$

The notation used in [LS03] for the product of two permutations is from right to left, i.e., it is such that, for example, in S_3 we have $(1\,2\,3) = (1\,3)(1\,2)$. By [LS03, Proposition 2.13] this product gives a (non-commutative) ring structure on $A\{S_n\}$, and the ring structure is preserved by the conjugation action of S_n on $A\{S_n\}$ induced by (3.3.7). Hence $A^{[n]}$ is a subring of $A\{S_n\}$. Moreover, by [LS03, Proposition 2.15] it is contained in the centre of $A\{S_n\}$, so $A^{[n]}$ is a commutative subring. By Theorem 3.3.4 this product on $A^{[n]}$ corresponds to the cup product on $H^*(S^{[n]}, \mathbb{Q})[2n]$. As an example, we compute the cup product between elements in $H^2(S^{[2]}, \mathbb{Z})$, obtaining the following lemma, see also [BNWS13, p.18].

Lemma 3.3.9. *Let $X = S^{[2]}$ be the Hilbert square of a projective K3 surface S. Let $\{\alpha_1, \dots, \alpha_{22}\}$ be the basis of the lattice $H^2(S, \mathbb{Z})$ used in Lemma* 3.2.11. *Denote by $1 \in H^0(S, \mathbb{Z})$ the unit and by $x \in H^4(S, \mathbb{Z})$ the class of a point. Then the following equalities hold in $H^4(X, \mathbb{Z})$.*

(i) For every $\alpha \in H^2(S, \mathbb{Z})$ we have

$$\frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cup \mathfrak{q}_1(1)\mathfrak{q}_1(\alpha)|0\rangle = \mathfrak{q}_2(\alpha)|0\rangle.$$

(ii) For every $\alpha, \beta \in H^2(S, \mathbb{Z})$ we have

$$\mathfrak{q}_1(1)\mathfrak{q}_1(\alpha)|0\rangle \cup \mathfrak{q}_1(1)\mathfrak{q}_1(\beta)|0\rangle = \left(\int_S \alpha\beta\right)\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle + \mathfrak{q}_1(\alpha)\mathfrak{q}_1(\beta)|0\rangle.$$

(iii) If $\mu_{i,j}$, with $i, j = 1, \dots, 22$, are the coefficients computed in Lemma 3.2.11, *then*

$$\frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cup \frac{1}{2}\mathfrak{q}_2(1)|0\rangle = -\sum_{i<j}\mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle - \frac{1}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle.$$

Proof. Let $A := H^*(S, \mathbb{Q})[2]$. By Example 3.3.3 we have

$$A\{S_2\} = A^{\otimes 2}\mathrm{id} \oplus A(1\,2).$$

(i) Let $\pi = (1\,2)$ and $\rho = \mathrm{id}$. Then $\langle\pi,\rho\rangle \setminus [2] = (1\,2) \setminus [2] = \{\{1,2\}\}$ and $\mathrm{id} \setminus [2] = \{\{1\},\{2\}\}$, so

$$g(\pi,\rho)(\{1,2\}) = \frac{1}{2}(2 + 2 - 1 - 2 - 1) = 0,$$

hence $e^{g(\pi,\rho)} = 1$ in the definition of the product of $A\{S_2\}$. Moreover, $A^{\otimes(1\,2)\setminus[2]}(1\,2)$ and $A^{\otimes \mathrm{id}\setminus[2]}\mathrm{id}$ can be identified respectively with A and $A^{\otimes 2}$. We then have

$$\begin{aligned} f^{\mathrm{id},\langle\pi,\rho\rangle} &: A^{\otimes 2} \to A, \quad a_1 \otimes a_2 \mapsto a_1 a_2, \\ f^{(1\,2),(1\,2)} &: A \to A, \quad a \mapsto a, \end{aligned}$$

i.e., $f^{(1\,2),(1\,2)} = \mathrm{id}_A = f_{(1\,2),(1\,2)}$. Thus

$$a(1\,2) \cdot (b \otimes c)(\mathrm{id}) = abc(1\,2),$$

where $a(1\,2) \in A(1\,2)$ and $(b \otimes c)(\mathrm{id}) \in A^{\otimes 2}\mathrm{id}$. We want to compute $\frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cup \mathfrak{q}_1(1)\mathfrak{q}_1(\alpha)|0\rangle$ for every $\alpha \in H^2(S,\mathbb{Z})$ by computing the corresponding product in $A^{[2]}$, using Theorem 3.3.4. As already remarked, we need to take elements in $A\{S_2\}$ which are invariant for the action of S_2. By (3.3.12) and (3.3.13) we have

$$1 \xmapsto{\Psi\circ\Phi'} \frac{1}{2}\mathfrak{q}_2(1)|0\rangle, \qquad \alpha \otimes 1 + 1 \otimes \alpha \xmapsto{\Psi\circ\Phi'} \mathfrak{q}_1(1)\mathfrak{q}_1(\alpha)|0\rangle.$$

Note that we have used the equality $\mathfrak{q}_1(1)\mathfrak{q}_1(\alpha)|0\rangle = \mathfrak{q}_1(\alpha)\mathfrak{q}_1(1)|0\rangle$ given by Theorem 3.2.2. Since

$$1(1\,2) \cdot (\alpha \otimes 1 + 1 \otimes \alpha)(\mathrm{id}) = 2\alpha(1\,2), \qquad \alpha \xmapsto{\Psi\circ\Phi'} \frac{1}{2}\mathfrak{q}_2(\alpha)|0\rangle,$$

we conclude that

$$\frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cup \mathfrak{q}_1(1)\mathfrak{q}_1(\alpha)|0\rangle = \mathfrak{q}_2(\alpha)|0\rangle.$$

(ii) Let $\pi = \rho = \mathrm{id}$. Since $\langle\pi,\rho\rangle \setminus [2] = \mathrm{id} \setminus [2] = \{\{1\},\{2\}\}$, we have for $i \in \{1,2\}$

$$g(\pi,\rho)(\{i\}) = \frac{1}{2}(1 + 2 - 1 - 1 - 1) = 0,$$

so $e^{g(\pi,\rho)} = 1 \otimes 1$ in the definition of the product of $A\{S_2\}$. Clearly $f^{\mathrm{id},\mathrm{id}} = f_{\mathrm{id},\mathrm{id}} = \mathrm{id}_{A^{\otimes 2}}$ and we have

$$(a \otimes b)(\mathrm{id}) \cdot (c \otimes d)(\mathrm{id}) = (ac \otimes bd)(\mathrm{id})$$

for every $a, b, c, d \in A$. As before, in order to compute products in $A^{[2]}$ we consider elements in $A\{S_2\}$ which are invariant for the action of S_2. We want to compute $\mathfrak{q}_1(1)\mathfrak{q}_1(\alpha)|0\rangle \cup \mathfrak{q}_1(1)\mathfrak{q}_1(\beta)|0\rangle$ for every $\alpha, \beta \in H^2(S,\mathbb{Z})$. We have

$$\alpha \otimes 1 + 1 \otimes \alpha \xmapsto{\Psi\circ\Phi'} \mathfrak{q}_1(1)\mathfrak{q}_1(\alpha)|0\rangle, \qquad \beta \otimes 1 + 1 \otimes \beta \xmapsto{\Psi\circ\Phi'} \mathfrak{q}_1(1)\mathfrak{q}_1(\beta)|0\rangle,$$

and

$$(\alpha\otimes 1+1\otimes\alpha)(\mathrm{id})\cdot(\beta\otimes 1+1\otimes\beta)(\mathrm{id}) = (\alpha\beta\otimes 1+\alpha\otimes\beta+\beta\otimes\alpha+1\otimes\alpha\beta)(\mathrm{id}). \tag{3.3.15}$$

If $x \in H^4(S,\mathbb{Z})$ is the class of a point, we have

$$\alpha\beta \otimes 1 + 1 \otimes \alpha\beta \xmapsto{\Psi\circ\Phi'} \left(\int_S \alpha\beta\right) \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle,$$
$$\alpha \otimes \beta + \beta \otimes \alpha \xmapsto{\Psi\circ\Phi'} \mathfrak{q}_1(\alpha)\mathfrak{q}_1(\beta)|0\rangle.$$

We conclude that

$$\mathfrak{q}_1(1)\mathfrak{q}_1(\alpha)|0\rangle \cup \mathfrak{q}_1(1)\mathfrak{q}_1(\beta)|0\rangle = \left(\int_S \alpha\beta\right) \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle + \mathfrak{q}_1(\alpha)\mathfrak{q}_1(\beta)|0\rangle.$$

(iii) Let $\pi = \rho = (1\,2)$. Since $\langle\pi,\rho\rangle \setminus [2] = (1\,2) \setminus [2] = \{\{1,2\}\}$, we have

$$g(\pi,\rho)(\{1,2\}) = \frac{1}{2}(2+2-1-1-2) = 0,$$

hence $e^{g(\pi,\rho)} = 1$ in the definition of the product of $A\{S_2\}$. Note that

$$f^{\mathrm{id},(1\,2)} : A^{\otimes 2} \to A, \qquad a_1 \otimes a_2 \mapsto a$$

is the multiplication map of A, so its adjoint $f_{(1\,2),\mathrm{id}}$ is the map Δ_* given in (3.3.5) and in Remark 3.3.2. We then have for every $a, b \in A$

$$a(1\,2) \cdot b(1\,2) = \Delta_*(ab)(\mathrm{id}).$$

We want to compute $\frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cup \frac{1}{2}\mathfrak{q}_2(1)|0\rangle$. As we have already seen,

$$1 \xmapsto{\Psi\circ\Phi'} \frac{1}{2}\mathfrak{q}_2(1)|0\rangle,$$

and

$$1(1\,2) \cdot 1(1\,2) = \Delta_*(1)(\mathrm{id}).$$

By Remark 3.3.2 we have $\Delta_* = -\tau_{2*}1$, and by Lemma 3.2.11

$$\tau_{2*}1 = \sum_{i,j} \mu_{i,j}\alpha_i \otimes \alpha_j + 1 \otimes x + x \otimes 1,$$

where $x \in H^4(S,\mathbb{Z})$ is the class of a point, $\{\alpha_1, \ldots \alpha_{22}\}$ is the basis of the lattice $H^2(S,\mathbb{Z})$ given in Lemma 3.2.11 and Table 3.1 gives the $\mu_{i,j}$'s. Since

$$\alpha_i \otimes \alpha_j + \alpha_i \otimes \alpha_j \xmapsto{\Psi\circ\Phi'} \mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle,$$
$$1 \otimes x + x \otimes 1 \xmapsto{\Psi\circ\Phi'} \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle,$$

we conclude that

$$\frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cup \frac{1}{2}\mathfrak{q}_2(1)|0\rangle = -\sum_{i<j} \mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle - \frac{1}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle.$$

□

Note that in the proof the fact that we work with elements of $A\{S_2\}$ which are invariant for the action of S_2 is important: for instance, in the product $\mathfrak{q}_1(1)\mathfrak{q}_1(\alpha)|0\rangle \cup \mathfrak{q}_1(1)\mathfrak{q}_1(\beta)|0\rangle$, if we take only the non invariant elements $1 \otimes \alpha$ and $1 \otimes \beta$ in $A\{S_2\}$, we have

$$(1 \otimes \alpha) \cdot (1 \otimes \beta) = (1 \otimes \alpha\beta) \xmapsto{\Psi \circ \Phi'} \frac{1}{2} \left(\int_S \alpha\beta \right) \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle,$$

while as we have seen

$$\mathfrak{q}_1(1)\mathfrak{q}_1(\alpha)|0\rangle \cup \mathfrak{q}_1(1)\mathfrak{q}_1(\beta)|0\rangle = \left(\int_S \alpha\beta \right) \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle + \mathfrak{q}_1(\alpha)\mathfrak{q}_1(\beta)|0\rangle.$$

Remark 3.3.10. The result for the product $\frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cup \frac{1}{2}\mathfrak{q}_2(1)|0\rangle$ given in [BNWS13, p.18] is not correct: we need to change the sign of the right-hand side. The map Δ_* in [BNWS13, p.18] corresponds to τ_{2*} in our notation: in the article the cohomology ring taken is $H^*(S, \mathbb{Q})$, without the shifting used in the Lehn–Sorger model, which gives the change of sign of the intersection pairing on $H^*(S, \mathbb{Q})[2]$, as seen above.

3.3.4 Hodge structures on the Lehn–Sorger model

Let S be a projective K3 surface. The rational cohomology groups $H^{2i}(S, \mathbb{Q})$ and $H^{2j}(S^{[n]}, \mathbb{Q})$, where $i, j \in \mathbb{Z}_{\geq 0}$, are Hodge structures of weight $2i$ and $2j$ respectively, as seen in Section 2.1.2. We have the following Hodge decompositions:

$$H^{2i}(S, \mathbb{C}) = \bigoplus_{p+q=2i} H^{p,q}(S), \qquad H^{2j}(S^{[n]}, \mathbb{C}) = \bigoplus_{r+s=2j} H^{r,s}(S^{[n]}), \tag{3.3.16}$$

where $p, q \in \{0, 1, \ldots, i\}$ and $r, s \in \{0, 1, \ldots, j\}$. Consider now the shifted cohomology groups $H^{2i}(S, \mathbb{Q})[2]$ and $H^{2j}(S^{[n]}, \mathbb{Q})[2n]$: as seen in Example 2.1.10, (ii), these are Hodge structures of weight $2i - 2$ and $2j - 2n$ respectively, with the following Hodge decompositions

$$\begin{aligned} H^{2i}(S, \mathbb{C})[2] &= \bigoplus_{p+q=2i-2} H^{p,q}(S)[2], \\ H^{2j}(S^{[n]}, \mathbb{Q})[2n] &= \bigoplus_{r+s=2j-2n} H^{r,s}(S^{[n]})[2n], \end{aligned}$$

where $p, q \in \{-1, 0, \ldots, i-1\}$ and $r, s \in \{-n, 1-n, \ldots, j-n\}$, and

$$H^{p,q}(S)[2] = H^{p+1,q+1}(S), \qquad H^{r,s}(S^{[n]})[2] = H^{r+n,s+n}(S^{[n]}).$$

The aim of this section is to see how the isomorphism of graded rings given by Theorem 3.3.4 behaves with respect to the Hodge structures. First of all, we need to describe the Hodge structure taken on $A^{[n]}$, where $A := H^*(S, \mathbb{Q})[2]$. Recall that by Proposition 3.3.5 we have

$$A^{[n]} \cong \bigoplus_{||\alpha||=n} \bigotimes_i \mathrm{Sym}^{\alpha_i} A.$$

Remark 3.3.11. Let $(A^{[n]})^{2i}$ be the component of $A^{[n]}$ of degree $2i-2n$. By Example 2.1.12, $(i),(ii),(iv)$ the Hodge structures $H^{2j}(S,\mathbb{Q})[2]$ seen above give rise to a Hodge structure on $(A^{[n]})^{2i}$. Since the weights of the Hodge structures considered depend only on the (shifted) cohomological degrees, we have that $(A^{[n]})^{2i}$ is a Hodge structure of weight $2i-2n$. Note that $H^{2i}(S^{[n]},\mathbb{Q})[2n]$ is a Hodge structure of weight $2i-2n$ and that we have $(A^{[n]})^{2i} \cong H^{2i}(S^{[n]},\mathbb{Q})[2n]$ by Theorem 3.3.4.

It is natural to wonder if the isomorphism $(A^{[n]})^{2i} \cong H^{2i}(S^{[n]},\mathbb{Q})[2n]$ maps a cycle of bidegree (p,q) to a class of bidegree (p,q), i.e., if it is an isomorphism of Hodge structures of weight 0.

We have seen that the isomorphism $(A^{[n]})^{2i} \cong H^{2i}(S^{[n]},\mathbb{Q})[2n]$ describes the elements in $H^{2i}(S^{[n]},\mathbb{Q})[2n]$ in terms of the Nakajima operators. Recall the diagram given in (3.2.2). We extend Definition 3.2.1 to complex coefficients, i.e., to

$$\mathfrak{q}_k : H^*(S,\mathbb{C}) \to \mathrm{End}(\mathbb{H}^S \otimes \mathbb{C}), \qquad k \geq 0,$$

by setting

$$\mathfrak{q}_k(\alpha)(x) := \psi_*(PD^{-1}[S^{[n,n+k]}] \cdot \varphi^*(x) \cdot \rho^*(\alpha))$$

for every $\alpha \in H^*(X,\mathbb{C})$ and $x \in H^*(S^{[n]},\mathbb{C})$, so we consider the cup product and the Gysin homomorphism with complex coefficients. Suppose that $\alpha \in H^{r,s}(S)$ and $x \in H^{p,q}(S^{[n]})$. We want to determine the bidegree of $\mathfrak{q}_k(\alpha)(x)$. Note that we are working with the standard gradings for the moment. The following proposition will be very useful, see [Voi02, §7.3.2].

Proposition 3.3.12. *Let $f : X \to Y$ be a morphism between two complex projective manifolds. Let $n = \dim_{\mathbb{C}}(X)$, $m = \dim_{\mathbb{C}}(Y)$ and $r = m-n$, where r can be negative.*

(i) The pullback map

$$f^* : H^k(Y,\mathbb{Q}) \to H^k(X,\mathbb{Q})$$

is a morphism of Hodge structures of weight 0 for every $k \geq 0$.

(ii) The Gysin homomorphism

$$f_* : H^k(X,\mathbb{Q}) \to H^{k+2r}(Y,\mathbb{Q})$$

is a morphism of Hodge structures of weight r for every $k \geq 0$.

We then obtain the following lemma.

Lemma 3.3.13. *Let S be a projective K3 surface. Consider $\alpha \in H^{r,s}(S)$ and $x \in H^{p,q}(S^{[n]})$ for some integer $n > 0$. Then $\mathfrak{q}_k(\alpha)(x)$ is a cohomology class of bidegree $(k-1+p+r, k-1+q+s)$ in $S^{[n+k]}$, i.e.,*

$$\mathfrak{q}_k(\alpha)(x) \in H^{k-1+p+r,k-1+q+s}(S^{[n+k]}).$$

Proof. Since $S^{[n,n+k]}$ is a subvariety of $S^{[n]} \times S \times S^{[n+k]}$ of complex dimension $2n+k+1$, we have that $PD^{-1}[S^{[n,n+k]}]$ is a Hodge class, i.e.,

$$PD^{-1}[S^{[n,n+k]}] \in H^{2n+k+1,2n+k+1}(S^{[n]} \times S \times S^{[n+k]},\mathbb{Q}),$$

in particular it belongs to $H^{2n+k+1,2n+k+1}(S^{[n]} \times S \times S^{[n+k]})$. Consider diagram (3.2.2). By Proposition 3.3.12, (i), we have

$$\varphi^*(x) \in H^{p,q}(S^{[n]} \times S \times S^{[n+k]}), \qquad \rho^*(\alpha) \in H^{r,s}(S^{[n]} \times S \times S^{[n+k]}).$$

Hence by Proposition 3.1.2 we have

$$PD^{-1}[S^{[n,n+k]}] \cdot \varphi^*(x) \cdot \rho^*(\alpha) \in H^{2n+k+1+p+r,2n+k+1+q+s}(S^{[n]} \times S \times S^{[n+k]}),$$

and by Proposition 3.3.12, (ii) we obtain

$$\psi_* \left(PD^{-1}[S^{[n,n+k]}] \cdot \varphi^*(x) \cdot \rho^*(\alpha) \right) \in H^{k-1+p+r,k-1+q+s}(S^{[n+k]}),$$

i.e., $\mathfrak{q}_k(\alpha)(x) \in H^{k-1+p+r,k-1+q+s}(S^{[n+k]})$. $\square$

As a consequence we obtain the following proposition.

Proposition 3.3.14. *Let S be a projective K3 surface. Fix integers $n > 0$, $N > 0$ and $l_1, \ldots, l_N > 0$ such that $\sum_{i=1}^N l_i = n$. If $a_i \in H^{p_i,q_i}(S)$ for $i \leq i \leq N$, then*

$$\mathfrak{q}_{l_1}(a_1) \ldots \mathfrak{q}_{l_N}(a_N)|0\rangle \in H^{n+P-N,n+Q-N}(S^{[n]}),$$

where $P = \sum_{i=1}^N p_i$ and $Q = \sum_{i=1}^N q_i$.

Proof. First of all, we apply Lemma 3.3.13 with $x = |0\rangle$ and $\alpha = a_N$, obtaining

$$\mathfrak{q}_{l_N}(a_N)|0\rangle \in H^{l_N-1+p_N,l_N-1+q_N}(S^{[l_N]}).$$

Then we apply again Lemma 3.3.13 with $x = \mathfrak{q}_{l_N}(a_N)|0\rangle$ and $\alpha = a_{N-1}$, obtaining that $\mathfrak{q}_{l_{N-1}}(a_{N-1})\mathfrak{q}_{l_N}(a_N)|0\rangle$ is a cohomology class of $S^{[l_N+l_{N-1}]}$ of bidegree $(l_N + l_{N-1} + p_N + p_{N-1} - 2, l_N + l_{N-1} + q_N + q_{N-1} - 2)$. Iterating the procedure, we have

$$\mathfrak{q}_{l_1}(a_1) \ldots \mathfrak{q}_{l_N}(a_N)|0\rangle \in H^{\sum_{i=1}^N (l_i+p_i)-N,\, \sum_{i=1}^N (l_i+q_i)-N}(S^{[\sum_{i=1}^N l_i]}),$$

i.e.,

$$\mathfrak{q}_{l_1}(a_1) \ldots \mathfrak{q}_{l_N}(a_N)|0\rangle \in H^{n+P-N,n+Q-N}(S^{[n]}),$$

where $P := \sum_{i=1}^N p_i$ and $Q := \sum_{i=1}^N q_i$. $\square$

We now explicitly state the following result, which is well known to experts and implicitly given in [LS03].

Theorem 3.3.15. *Let S be a projective K3 surface. Let $A := H^*(S, \mathbb{Q})[2]$ and consider the isomorphism $(A^{[n]})^{2i} \cong H^{2i}(S^{[n]}, \mathbb{Q})[2n]$ induced by Theorem 3.3.4 on the components of (shifted) cohomological degree $2i - 2n$. Take on $(A^{[n]})^{2i}$ the Hodge structure of weight $2i - 2n$ of Remark 3.3.11 and on $H^{2i}(S^{[n]}, \mathbb{Q})[2n]$ the Hodge structure of weight $2i - 2n$ induced by shifted cohomology. Then*

$$(A^{[n]})^{2i} \xrightarrow{\sim} H^{2i}(S^{[n]}, \mathbb{Q})[2n]$$

is an isomorphism of Hodge structures of weight 0.

Proof. Since $A^{[n]} \cong \bigoplus_{||\alpha||=n} \bigotimes_i \mathrm{Sym}^{\alpha_i} A$ and $A^{[n]}$ is the subset of elements of $A\{S_n\} = \bigoplus_{\pi \in S_n} A^{\otimes \langle \pi \rangle \backslash [n]} \cdot \pi$ which are invariant for the action of S_n, it is enough to show that the composition $\Psi \circ \Phi'$ described in Section 3.3.2 maps an element $a_1 \otimes \cdots \otimes a_N \in A^{\otimes \langle \pi \rangle \backslash [n]}$, with $a_i \in H^{p_i,q_i}(S)[2]$, in an element of $H^*(S^{[n]}, \mathbb{C})[2n]$ of bidegree $(\sum_{i=1}^N p_i, \sum_{i=1}^N q_i)$. Seen as elements of the cohomology ring with standard grading, we have $\alpha_i \in H^{p_i+1,q_i+1}(S)$. By (3.3.9) and (3.3.10) we have

$$a_1 \otimes \cdots \otimes a_N \xmapsto{\Psi \circ \Phi'} \mathfrak{q}_{l_1}(a_1) \ldots \mathfrak{q}_{l_N}(a_N)|0\rangle,$$

where l_i is the length of the i-th orbit of $\pi \in S_n$. By Proposition 3.3.14 we have

$$\mathfrak{q}_{l_1}(a_1) \ldots \mathfrak{q}_{l_N}(a_N)|0\rangle \in H^{n+P,n+Q}(S^{[n]}),$$

where $P = \sum_{i=1}^N p_i$, $Q = \sum_{i=1}^N q_i$. If we see $\mathfrak{q}_{l_1}(a_1) \ldots \mathfrak{q}_{l_n}(a_N)|0\rangle$ as an element of the shifted cohomology group $H^{2n+P+Q}(S^{[n]}, \mathbb{C})[2n]$, then this has bidegree $(n + P - n, n + Q - n) = (P, Q)$, i.e.,

$$\mathfrak{q}_{l_1}(a_1) \ldots \mathfrak{q}_{l_N}(a_N)|0\rangle \in H^{P,Q}(S^{[n]})[2n],$$

as we wanted. □

3.3.5 Hodge classes on Hilbert squares of K3 surfaces

Let S be a projective K3 surface and consider its Hilbert square $S^{[2]}$. In this section we give a basis for the vector space $H^{2,2}(S^{[2]}, \mathbb{Q})$ of rational $(2,2)$-Hodge classes and a basis for the lattice $H^{2,2}(S^{[2]}, \mathbb{Z})$ of integral $(2,2)$-Hodge classes. We begin with the rational case: the idea is to use Theorem 3.3.15.

Theorem 3.3.16. *Let S be a projective K3 surface with Picard group of rank* $\mathrm{rk}(\mathrm{Pic}(S)) = r$*. Let $\{b_1, \ldots, b_r\}$ be a basis of* $\mathrm{Pic}(S)$*. Then:*

(i) $\dim(H^{2,2}(S^{[2]}, \mathbb{Q})) = \frac{(r+1)r}{2} + r + 2$.

(ii) A basis of $H^{2,2}(S^{[2]}, \mathbb{Q})$ is given by the following elements:

- $\frac{1}{2}\mathfrak{q}_2(b_i)|0\rangle$ *for* $i = 1, \ldots, r$.
- $\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle$, *where* $1 \in H^0(S, \mathbb{Q})$ *is the unit and* $x \in H^4(S, \mathbb{Q})$ *is the class of a point.*
- $\frac{1}{2}\mathfrak{q}_1(b_i)^2|0\rangle$ *for* $i = 1, \ldots, r$.
- $\mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)|0\rangle$ *for* $1 \leq i < j \leq r$.
- $-\sum_{i<j} \mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle - \dfrac{1}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle$,
  where $\{\alpha_1, \ldots, \alpha_{22}\}$ is the basis of $H^2(S, \mathbb{Z})$ used in Lemma 3.2.11 *and the $\mu_{i,j}$'s are given in Table* 3.1.

Proof. As seen in Example 3.3.7, we have an isomorphism

$$H^4(S^{[2]}, \mathbb{Q}) \cong H^2(S, \mathbb{Q}) \oplus \left(H^0(S, \mathbb{Q}) \otimes H^4(S, \mathbb{Q})\right) \oplus \mathrm{Sym}^2(H^2(S, \mathbb{Q})), \quad (3.3.17)$$

we omit the shiftings of the cohomology groups. By Theorem 3.3.15, this is an isomorphism of Hodge structures of weight 0. The Hodge classes of $S^{[2]}$ of bidegree $(2,2)$ have bidegree $(0,0)$ in the shifted cohomology, so we look for the components of (shifted) bidegree $(0,0)$ in the right-hand side of (3.3.17). Recall that $H^2(S,\mathbb{Q})$ can be decomposed as

$$H^2(S,\mathbb{Q}) \cong \mathrm{NS}(S)_{\mathbb{Q}} \oplus T(S)_{\mathbb{Q}}, \tag{3.3.18}$$

where $\mathrm{NS}(S)$ is the Néron-Severi group of S and $T(S) = (\mathrm{NS}(S))^{\perp}$ is the transcendental lattice.
The first summand of (3.3.17) has as component of bidegree $(0,0)$ the $\mathbb{Q}$-vector space $\mathrm{NS}(S)_{\mathbb{Q}}[2]$. Since $\mathrm{NS}(S) \cong \mathrm{Pic}(S)$ and $\mathrm{rk}(\mathrm{Pic}(S)) = r$ by assumption, the component of bidegree $(0,0)$ of $H^2(S,\mathbb{Q})[2]$ has dimension r. By (3.3.9) and (3.3.10) we have

$$b_i \xmapsto{\Psi\circ\Phi'} \frac{1}{2}\mathfrak{q}_2(b_i)|0\rangle,$$

so $\frac{1}{2}\mathfrak{q}_2(b_i)|0\rangle$ is in a basis of $H^{2,2}(S^{[2]},\mathbb{Q})$ for $i = 1,\dots,r$.
The second summand of (3.3.17) is $H^0(S,\mathbb{Q})[2] \otimes H^4(S,\mathbb{Q})[2]$, which is a vector space over $\mathbb{Q}$ of dimension 1. This is generated by $1 \otimes x$, which is an element of bidegree $(0,0)$. Since

$$1 \otimes x + x \otimes 1 \xmapsto{\Psi\circ\Phi'} \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle,$$

the element $\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle$ is in a basis of $H^{2,2}(S^{[2]},\mathbb{Q})$.
Consider $\mathrm{Sym}^2(H^2(S,\mathbb{Q})[2])$, the third summand of (3.3.17). Using (3.3.18), we can decompose it as

$$\begin{aligned}\mathrm{Sym}^2(H^2(S,\mathbb{Q})[2]) \quad &\cong \quad \mathrm{Sym}^2(\mathrm{Pic}(S)_{\mathbb{Q}}[2]) \oplus \mathrm{Sym}^2(T(S)_{\mathbb{Q}}[2]) \\ &\qquad \oplus(\mathrm{Pic}(S)_{\mathbb{Q}}[2] \otimes T(S)_{\mathbb{Q}}[2]).\end{aligned}$$

By assumption $\mathrm{rk}(\mathrm{Pic}(S)) = r$, so $\mathrm{Sym}^2(\mathrm{Pic}(S)_{\mathbb{Q}}[2])$, whose elements have all bidegree $(0,0)$, has dimension $\frac{(r+1)r}{2}$ as $\mathbb{Q}$-vector space. By (3.3.9) and (3.3.10) we have:

$$b_i \otimes b_i \xmapsto{\Psi\circ\Phi'} \tfrac{1}{2}\mathfrak{q}_1(b_i)^2|0\rangle$$
$$b_i \otimes b_j + b_j \otimes b_i \xmapsto{\Psi\circ\Phi'} \mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)|0\rangle$$

for $i,j \in \{1,\dots,r\}$ and $i < j$. Then the elements $\frac{1}{2}\mathfrak{q}_1(b_i)^2|0\rangle$, for $i = 1,\dots,r$, and $\mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)|0\rangle$, for $1 \le i < j \le r$, are in a basis of $H^{2,2}(S^{[2]},\mathbb{Q})$. Note that $\mathrm{Pic}(S)_{\mathbb{Q}}[2] \otimes T(S)_{\mathbb{Q}}[2]$ does not contain any element of bidegree $(0,0)$. It remains to determine

$$(\mathrm{Sym}^2(T(S)_{\mathbb{Q}}[2]))^{0,0} \cap (\mathrm{Sym}^2(T(S)_{\mathbb{Q}}[2])).$$

Consider $T(S)_{\mathbb{Q}} \otimes T(S)_{\mathbb{Q}}$ with the standard grading. We know that $T(S)_{\mathbb{Q}}$ is the minimal sub-Hodge structure of $H^2(S,\mathbb{Q})$ with $H^{2,0}(S) = T(S)^{2,0}_{\mathbb{Q}}$, in particular $T(S)_{\mathbb{Q}}$ is a Hodge structure of weight 2. By Example 2.1.12, (iii), the dual $T(S)^*_{\mathbb{Q}} = \mathrm{Hom}(T(S)_{\mathbb{Q}},\mathbb{Q})$ is a Hodge structure of weight -2, and there is an isomorphism of Hodge structures of weight -2 from $T(S)_{\mathbb{Q}}$ to $T(S)^*_{\mathbb{Q}}$. This implies that

$$(T(S)_{\mathbb{Q}} \otimes T(S)_{\mathbb{Q}})^{2,2} \xrightarrow{\sim} (T(S)^*_{\mathbb{Q}} \otimes T(S)_{\mathbb{Q}})^{0,0},$$

and by Remark 2.1.14 we have

$$\mathrm{Hom}_0(T(S)_{\mathbb{Q}}, T(S)_{\mathbb{Q}}) \cong (T(S)^*_{\mathbb{Q}} \otimes T(S)_{\mathbb{Q}}) \cap (T(S)^*_{\mathbb{Q}} \otimes T(S)_{\mathbb{Q}})^{0,0},$$

where $\mathrm{Hom}_0(T(S)_{\mathbb{Q}}, T(S)_{\mathbb{Q}})$ denotes the space of Hodge endomorphisms on $T(S)_{\mathbb{Q}}$ of weight 0. Since S is a K3 surface we have $T(S)_{\mathbb{Q}}^{2,0} \cong H^{2,0}(S) \cong \mathbb{C} \cdot \sigma_S$, where σ_S is a symplectic form on S. Then for every $\varphi \in \mathrm{Hom}_0(T(S)_{\mathbb{Q}}, T(S)_{\mathbb{Q}})$ the $\mathbb{C}$-linear extension gives a map $\varphi^{2,0}$ of $\mathbb{C}$-vector spaces

$$\begin{array}{ccc} T(S)_{\mathbb{Q}}^{2,0} & \xrightarrow{\varphi^{2,0}} & T(S)_{\mathbb{Q}}^{2,0} \\ \downarrow\wr & & \downarrow\wr \\ \mathbb{C} \cdot \sigma_S & \xrightarrow{\varphi^{2,0}} & \mathbb{C} \cdot \sigma_S. \end{array}$$

By Lemma 2.1.18 the map φ is uniquely determined by $\varphi^{2,0}$, so we have

$$\mathrm{Hom}_0(T(S)_{\mathbb{Q}}, T(S)_{\mathbb{Q}}) \cong \mathbb{Q} \cdot \mathrm{id}$$

Passing to the shifted cohomology groups, this implies that the $\mathbb{Q}$-vector space $(\mathrm{Sym}^2(T(S)_{\mathbb{Q}}[2]))^{0,0} \cap (\mathrm{Sym}^2(T(S)_{\mathbb{Q}}[2]))$ has dimension 1. We now describe the element induced by its generator on $H^4(S^{[2]}, \mathbb{Q})$. Let $\{\beta_{r+1}, \ldots, \beta_{22}\}$ be an orthogonal basis of $T(S)_{\mathbb{Q}}$ with respect to the intersection form, and let $\{\beta_{r+1}^\vee, \ldots, \beta_{22}^\vee\}$ be the basis of $T(S)_{\mathbb{Q}}^*$ given by

$$\beta_i^\vee := (\beta_i, \cdot\,) \in \mathrm{Hom}(T(S)_{\mathbb{Q}}, \mathbb{Q}) \cong T(S)_{\mathbb{Q}}^*$$

for $i \in \{r+1, \ldots, 22\}$. Then

$$\mathrm{id} = \sum_{i=r+1}^{22} \frac{1}{(\beta_i, \beta_i)} \beta_i^\vee \otimes \beta_i \in T(S)_{\mathbb{Q}}^* \otimes T(S)_{\mathbb{Q}},$$

since for every $k \in \{r+1, \ldots, 22\}$ we have

$$\begin{aligned} \left(\sum_{i=r+1}^{22} \frac{1}{(\beta_i, \beta_i)} \beta_i^\vee \otimes \beta_i \right)(\beta_k) &= \sum_{i=r+1}^{22} \frac{1}{(\beta_i, \beta_i)} (\beta_i, \beta_k) \cdot \beta_i \\ &= \beta_k. \end{aligned}$$

Note that $(\beta_i, \beta_i) \neq 0$ since $\{\beta_{r+1}, \ldots, \beta_{22}\}$ is an orthogonal basis and the intersection form on $H^2(S, \mathbb{Q})$ is non-degenerate. Since $T(S)_{\mathbb{Q}} \cong T(S)_{\mathbb{Q}}^*$ by the map $\beta_i \mapsto \beta_i^\vee$, we have that the identity, seen as element in $T(S)_{\mathbb{Q}} \otimes T(S)_{\mathbb{Q}}$, is

$$\mathrm{id} = \sum_{i=r+1}^{22} \frac{1}{(\beta_i, \beta_i)} \beta_i \otimes \beta_i \in T(S)_{\mathbb{Q}} \otimes T(S)_{\mathbb{Q}}.$$

We see that id is invariant for the action of the symmetric group S_2 on $A\{S_2\}$, where $A := H^*(S, \mathbb{Q})[2]$, so we obtain the following element of $H^4(S^{[2]}, \mathbb{Q})[4]$:

$$\mathrm{id} \xmapsto{\Psi \circ \Phi'} \frac{1}{2} \sum_{i=r+1}^{22} \frac{1}{(\beta_i, \beta_i)} \mathfrak{q}_1(\beta_i)^2 |0\rangle. \tag{3.3.19}$$

Hence the last element of the basis of $H^{2,2}(S^{[2]}, \mathbb{Q})$ is (3.3.19). For later discussions, we prefer to substitute this element in the basis of $H^{2,2}(S^{[2]}, \mathbb{Q})$ obtained with

$$-\sum_{i<j} \mu_{i,j} \mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle - \frac{1}{2} \sum_i \mathfrak{q}_1(\alpha_i)^2 |0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle.$$

We show why this is possible. Let $\{\beta_1, \dots, \beta_r\}$ be a orthogonal basis of $\mathrm{Pic}(S)_{\mathbb{Q}}$. Then $\{\beta_1, \dots, \beta_{22}\}$ is a orthogonal basis for $H^2(S, \mathbb{Q})$, and

$$\Psi \circ \Phi'(\mathrm{id}) + \frac{1}{2}\sum_{i=1}^{r} \frac{1}{(\beta_i, \beta_i)} \mathfrak{q}_1(\beta_i)^2|0\rangle = \frac{1}{2}\sum_{i=1}^{22} \frac{1}{(\beta_i, \beta_i)} \mathfrak{q}_1(\beta_i)^2|0\rangle. \tag{3.3.20}$$

The orthogonal basis $\{\beta_1, \dots, \beta_r\}$ of $\mathrm{Pic}(S)_{\mathbb{Q}}$ is obtained with the Gram–Schmidt procedure from the basis $\{b_1, \dots, b_r\}$ of $\mathrm{Pic}(S)$, so β_k, for every $k = 1, \dots, r$, is a rational linear combination of $b_1, \dots, b_r$. Recall that, by definition, Nakajima operators are linear, and $\mathfrak{q}_1(\alpha)\mathfrak{q}_1(\beta)|0\rangle = \mathfrak{q}_1(\beta)\mathfrak{q}_1(\alpha)|0\rangle$ by Theorem 3.2.2 for every $\alpha, \beta \in H^2(S, \mathbb{Q})$. Then $\mathfrak{q}_1(\beta_k)^2|0\rangle$, for $k = 1, \dots, r$, is a rational linear combination of $\frac{1}{2}\mathfrak{q}_1(b_i)^2|0\rangle$ and $\mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)|0\rangle$, where $i, j \in \{1, \dots, r\}$ and $i < j$, so (3.3.20) is a rational linear combination of elements of the basis of $H^{2,2}(S^{[2]}, \mathbb{Q})$ obtained before. Moreover, repeating the proof of Lemma 3.2.11 with the orthogonal basis $\{\beta_1, \dots, \beta_{22}\}$ of $H^2(S, \mathbb{Q})$, we obtain

$$\tau_{2*}1 = \sum_{i=1}^{22} \frac{1}{(\beta_i, \beta_i)} \beta_i \otimes \beta_i + 1 \otimes x + x \otimes 1,$$

where τ_{2*} is the map seen in Section 3.2.4. With the same argument of the proof of Lemma 3.3.9, (iii), with the basis $\{\beta_1, \dots, \beta_{22}\}$, we have

$$\delta^2 = -\sum_{i=1}^{22} \frac{1}{(\beta_i, \beta_i)} \mathfrak{q}_1(\beta_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \tag{3.3.21}$$

where $\delta \in \mathrm{Pic}(S^{[2]})$ is the class such that 2δ is the class of the excpetional divisor of the Hilbert–Chow morphism. By Lemma 3.3.9, (iii) we have

$$\delta^2 = -\sum_{i<j} \mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle - \frac{1}{2}\sum_{i} \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle. \tag{3.3.22}$$

From (3.3.21) and (3.3.22) we see that the following set

$$\begin{aligned}
&\tfrac{1}{2}\mathfrak{q}_2(b_i)|0\rangle \qquad \text{for } i = 1, \dots, r.\\
&\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle,\\
&\tfrac{1}{2}\mathfrak{q}_1(b_i)^2|0\rangle \qquad \text{for } i = 1, \dots, r.\\
&\mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)|0\rangle \qquad \text{for } 1 \le i < j \le r.\\
&-\sum_{i<j} \mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle - \frac{1}{2}\sum_{i} \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle,
\end{aligned} \tag{3.3.23}$$

is a set of linearly independent elements. Moreover, this generates $H^{2,2}(S^{[2]}, \mathbb{Q})$ by construction. We conclude that $\dim(H^{2,2}(S^{[2]}, \mathbb{Q})) = \frac{(r+1)r}{2} + r + 2$ and (3.3.23) is a basis of the $\mathbb{Q}$-vector space $H^{2,2}(S^{[2]}, \mathbb{Q})$. □

We can combine Theorem 3.3.16 with Theorem 3.2.9 in order to study the lattice $H^{2,2}(S^{[2]}, \mathbb{Z})$ of integral Hodge classes of type $(2, 2)$ when S is a generic K3 surface.

Theorem 3.3.17. *Let S be a generic K3 surface and $h \in \mathrm{Pic}(S)$ be the ample generator of* $\mathrm{Pic}(S)$*. Then a basis of the lattice $H^{2,2}(S^{[2]},\mathbb{Z})$ of integral Hodge classes of type $(2,2)$ is given by the following elements:*

$$\mathfrak{q}_2(h)|0\rangle, \qquad \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \qquad \tfrac{1}{2}\left(\mathfrak{q}_1(h)^2 - \mathfrak{q}_2(h)\right)|0\rangle,$$
$$-\sum_{i<j}\mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle - \frac{1}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle,$$

where $x \in H^4(S,\mathbb{Z})$ is the class of a point, $1 \in H^0(S,\mathbb{Z})$ is the unit, $\{\alpha_1,\dots,\alpha_{22}\}$ is the basis of $H^2(S,\mathbb{Z})$ used in Lemma 3.2.11 *and the $\mu'_{i,j}s$ are the integers given by Table* 3.1*.*

Proof. By Proposition 3.3.16 we have $\dim\left(H^{2,2}(S^{[2]},\mathbb{Q})\right) = 4$. Moreover, by Theorem 2.2.9, the lattice $H^4(S^{[2]},\mathbb{Z})$ is torsion-free. Hence $H^{2,2}(S^{[2]},\mathbb{Z})$ is a lattice of rank 4. After a slight modification of the basis of $H^{2,2}(S^{[2]},\mathbb{Q})$ given by Theorem 3.3.16, we obtain the following basis for $H^{2,2}(S^{[2]},\mathbb{Q})$:

$$\begin{gathered}\mathfrak{q}_2(h)|0\rangle, \qquad \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \qquad \tfrac{1}{2}\left(\mathfrak{q}_1(h)^2 - q_2(h)\right)|0\rangle,\\ -\sum_{i<j}\mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle - \frac{1}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle.\end{gathered} \tag{3.3.24}$$

In order to prove the theorem, we give a basis of the lattice $H^4(S^{[2]},\mathbb{Z})$ which contains the elements in (3.3.24). Since h is an ample line bundle on S, we have $h^2 = 2t$ with respect to the intersection form on S for some integer $t > 0$. By Theorem 1.4.10 the Picard group $\mathrm{Pic}(S) \cong \mathbb{Z}h$ can be primitively embedded in $H^2(S,\mathbb{Z})$ in a unique way, up to isometries. We can assume that such an embedding is obtained by mapping h to $\alpha_{17} + t\alpha_{18}$. We will identify h with $\alpha_{17} + t\alpha_{18}$. We show that the following is a basis of $H^4(S^{[2]},\mathbb{Z})$:

$$\begin{gathered}\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle\\ \mathfrak{q}_2(\beta_i)|0\rangle \text{ for } i = 1,\dots,22,\\ \mathfrak{q}_1(\beta_i)\mathfrak{q}_1(\beta_j)|0\rangle \qquad \text{for } i<j \text{ and } (i,j) \neq (21,22),\\ \tfrac{1}{2}\left(\mathfrak{q}_1(\beta_i)^2 - \mathfrak{q}_2(\beta_i)\right)|0\rangle \qquad \text{for } i = 1,\dots,22,\\ -\sum_{i<j}\mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle - \frac{1}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle,\end{gathered} \tag{3.3.25}$$

where $\beta_i := \alpha_i$ if $i \neq 17$ and $\beta_{17} := \alpha_{17} + t\alpha_{18} = h$. We call L the sublattice of $H^4(S^{[2]},\mathbb{Z})$ whose basis is (3.3.25). By Table 3.1 we have $\mu_{21,22} = 1$, so the elements in (3.3.25) are linearly independent in $H^4(S^{[2]},\mathbb{Q})$, which implies that $\mathrm{rk}(L) = \mathrm{rk}(H^4(S^{[2]},\mathbb{Z}))$. We want to show that $L = H^4(S^{[2]},\mathbb{Z})$. Clearly $\{\beta_1,\dots,\beta_{22}\}$ is a basis of $H^2(S,\mathbb{Z})$, so by Theorem 3.2.9 the following is a basis of $H^4(S^{[2]},\mathbb{Z})$:

$$\begin{gathered}\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \qquad \mathfrak{q}_2(\beta_i)|0\rangle, \qquad \mathfrak{q}_1(\beta_i)\mathfrak{q}_1(\beta_j)|0\rangle,\\ \tfrac{1}{2}\left(\mathfrak{q}_1(\beta_i)^2 - \mathfrak{q}_2(\beta_i)\right)|0\rangle,\end{gathered} \tag{3.3.26}$$

where $i,j \in \{1,\dots,22\}$ and $i<j$. Note that substituting $\mathfrak{q}_1(\beta_{21})\mathfrak{q}_1(\beta_{22})|0\rangle$ in (3.3.26) with $-\sum \mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle - \frac{1}{2}\sum \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle$ we obtain (3.3.25). The following elements are in L:

- $\mathfrak{q}_1(\alpha_{17})\mathfrak{q}_1(\alpha_i)|0\rangle$ for $i \geq 19$, since
$$\begin{aligned}\mathfrak{q}_1(\beta_{17})\mathfrak{q}_1(\beta_i)|0\rangle &= \mathfrak{q}_1(\alpha_{17}+t\alpha_{18})\mathfrak{q}_1(\alpha_i)|0\rangle \\ &= \mathfrak{q}_1(\alpha_{17})\mathfrak{q}_1(\alpha_i)|0\rangle + t\mathfrak{q}_1(\alpha_{18})\mathfrak{q}_1(\alpha_i)|0\rangle,\end{aligned}$$
and $\mathfrak{q}_1(\beta_{17})\mathfrak{q}_1(\beta_i)|0\rangle$ and $\mathfrak{q}_1(\alpha_{18})\mathfrak{q}_1(\alpha_i)|0\rangle$ are in (3.3.25) for $i \geq 19$.
- $\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_{17})|0\rangle$ for $i \leq 16$, since
$$\begin{aligned}\mathfrak{q}_1(\beta_i)\mathfrak{q}_1(\beta_{17})|0\rangle &= \mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_{17}+t\alpha_{18})|0\rangle \\ &= \mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_{17})|0\rangle + t\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_{18})|0\rangle,\end{aligned}$$
and $\mathfrak{q}_1(\beta_i)\mathfrak{q}_1(\beta_{17})|0\rangle$ and $\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_{18})|0\rangle$ are in (3.3.25) for $i \leq 16$.
- $\mathfrak{q}_1(\alpha_{17})\mathfrak{q}_1(\alpha_{18})|0\rangle$. Indeed, $\mathfrak{q}_1(\beta_{17})\mathfrak{q}_1(\beta_{18})|0\rangle$ is in (3.3.25), and
$$\begin{aligned}\mathfrak{q}_1(\beta_{17})\mathfrak{q}_1(\beta_{18})|0\rangle &= \mathfrak{q}_1(\alpha_{17}+t\alpha_{18})\mathfrak{q}_1(\alpha_{18})|0\rangle \\ &= \mathfrak{q}_1(\alpha_{17})\mathfrak{q}_1(\alpha_{18})|0\rangle + t\mathfrak{q}_1(\alpha_{18})^2|0\rangle.\end{aligned}$$
Moreover, $\frac{1}{2}(\mathfrak{q}_1(\beta_{18})^2 - \mathfrak{q}_2(\beta_{18}))|0\rangle$ and $\mathfrak{q}_2(\beta_{18})|0\rangle$ are in (3.3.25), hence
$$t\mathfrak{q}_1(\alpha_{18})^2|0\rangle = 2t\cdot\frac{1}{2}\left(\mathfrak{q}_1(\beta_{18})^2 - \mathfrak{q}_2(\beta_{18})\right)|0\rangle + t\mathfrak{q}_2(\beta_{18})|0\rangle$$
is in L, so $\mathfrak{q}_1(\alpha_{17})\mathfrak{q}_1(\alpha_{18})|0\rangle$ is the difference of two elements of L.
- $\mathfrak{q}_2(\alpha_{17})|0\rangle$, since $\mathfrak{q}_2(\beta_{17})|0\rangle = \mathfrak{q}_2(\alpha_{17}+t\alpha_{18})|0\rangle = \mathfrak{q}_2(\alpha_{17})|0\rangle + t\mathfrak{q}_2(\alpha_{18})|0\rangle$ and $\mathfrak{q}_2(\alpha_{18})|0\rangle$ is in (3.3.25).
- $\frac{1}{2}(\mathfrak{q}_1(\alpha_{17})^2 - \mathfrak{q}_2(\alpha_{17}))|0\rangle$, since $\frac{1}{2}(\mathfrak{q}_1(\beta_{17})^2 - \mathfrak{q}_2(\beta_{17}))|0\rangle$ is in (3.3.25) and
$$\begin{aligned}\tfrac{1}{2}\left(\mathfrak{q}_1(\alpha_{17})^2 - \mathfrak{q}_2(\alpha_{17})\right)|0\rangle &= \tfrac{1}{2}\left(\mathfrak{q}_1(\beta_{17})^2 - \mathfrak{q}_2(\beta_{17})\right)|0\rangle - t\mathfrak{q}_1(\alpha_{17})\mathfrak{q}_1(\alpha_{18})|0\rangle \\ &\quad -\tfrac{t^2}{2}\left(\mathfrak{q}_1(\alpha_{18})^2 - \mathfrak{q}_2(\alpha_{18})\right)|0\rangle - \tfrac{t^2-t}{2}\mathfrak{q}_2(\alpha_{18})|0\rangle\end{aligned}$$
is an integral linear combination of elements in L: note that $\frac{t^2-t}{2}$ is an integer for every $t \geq 1$.

From Table 3.1, the integer $\mu_{i,i}$ is even for every $i = 1, \ldots, 22$. Hence the following is an element in L:

$$\begin{aligned}\mu_{21,22}\mathfrak{q}_1(\alpha_{21})\mathfrak{q}_1(\alpha_{22})|0\rangle &= -\sum_{i<j}\mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle - \frac{1}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle \\ &\quad + \left(\sum_i \frac{\mu_{i,i}}{2}\cdot(\mathfrak{q}_1(\alpha_i)^2 - \mathfrak{q}_2(\alpha_i))|0\rangle\right) \\ &\quad + \sum_i \frac{\mu_{i,i}}{2}\mathfrak{q}_2(\alpha_i)|0\rangle \\ &\quad + \sum_{\substack{i<j \\ (i,j)\neq(21,22)}} \mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle \\ &\quad + \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle.\end{aligned}$$

Since $\mu_{21,22} = 1$, we have $\mathfrak{q}_1(\alpha_{21})\mathfrak{q}_1(\alpha_{22})|0\rangle \in L$. Hence every element in (3.3.24) is in L. We conclude that $L = H^4(S^{[2]}, \mathbb{Z})$ and

$$\mathfrak{q}_2(h)|0\rangle, \qquad \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \qquad \tfrac{1}{2}\left(\mathfrak{q}_1(h)^2 - \mathfrak{q}_2(h)\right)|0\rangle,$$
$$-\sum_{i<j}\mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle - \frac{1}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle$$

is a basis of $H^{2,2}(S^{[2]}, \mathbb{Z})$, as we wanted. □

3.4 Second Chern class of the Hilbert square of a K3 surface

Let S be a generic K3 surface and $X := S^{[2]}$ be its Hilbert square. We have seen that Theorem 3.3.17 gives a basis of the lattice $H^{2,2}(X, \mathbb{Z})$ in terms of Nakajima operators. In this section we want to obtain a description of this basis which does not depend on Nakajima operators. Let $q_X^\vee \in H^{2,2}(X, \mathbb{Q})$ be the dual of the BBF form introduced in Section 3.1. As always we denote by $\delta \in \mathrm{Pic}(X)$ the class such that 2δ is the class of the exceptional divisor of the Hilbert–Chow morphism $S^{[2]} \to S^{(2)}$. First of all, we give another basis of $H^{2,2}(X, \mathbb{Q})$.

Proposition 3.4.1. *Let $X = S^{[2]}$ be the Hilbert square of a generic K3 surface and let $h \in \mathrm{Pic}(X)$ be the class induced by the ample generator of $\mathrm{Pic}(S)$. Then*

$$\{h^2, h\delta, \delta^2, \frac{2}{5}q_X^\vee\}$$

is a basis of the $\mathbb{Q}$-vector space $H^{2,2}(X, \mathbb{Q})$.

Proof. We denote by h both the ample generator of $\mathrm{Pic}(S)$ and the line bundle that this induces on X. We have $h^2 = 2t$ for some integer $t > 0$ with respect to the intersection form on S. By Remark 3.2.10 we have $\delta = \frac{1}{2}\mathfrak{q}_2(1)|0\rangle$ and $h = \mathfrak{q}_1(1)\mathfrak{q}_1(h)|0\rangle$, hence applying Lemma 3.3.9 we obtain

$$\begin{gathered} h^2 = 2t\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle + \mathfrak{q}_1(h)^2|0\rangle, \qquad h\delta = \mathfrak{q}_2(h)|0\rangle, \\ \delta^2 = -\sum_{i<j}\mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle - \frac{1}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \end{gathered} \tag{3.4.1}$$

where $1 \in H^0(S, \mathbb{Z})$ is the unit, $x \in H^4(S, \mathbb{Z})$ is the class of a point, $\{\alpha_1, \ldots, \alpha_{22}\}$ is the basis of $H^2(S, \mathbb{Z})$ used in Lemma 3.2.11 and the $\mu_{i,j}$'s are the integers given in Table 3.1. By Theorem 1.4.10, the lattice $\mathrm{Pic}(S) \cong \mathbb{Z}h$ can be primitively embedded in a unique way, up to isometries, in $H^2(S, \mathbb{Z})$: we can assume that such embedding is obtained by mapping h to $\alpha_{17} + t\alpha_{18}$. Then by (3.4.1) we have that h^2, $h\delta$ and δ^2 are linearly independent in $H^{2,2}(S^{[2]}, \mathbb{Q})$. Suppose that

$$\frac{2}{5}q_X^\vee = xh^2 + yh\delta + z\delta^2 \in H^{2,2}(X, \mathbb{Q}) \tag{3.4.2}$$

for some $x, y, z \in \mathbb{Q}$. By Proposition 3.1.6 we have

$$\langle \frac{2}{5}q_X^\vee, h^2\rangle = 20t, \qquad \langle \frac{2}{5}q_X^\vee, h\delta\rangle = 0, \qquad \langle \frac{2}{5}q_X^\vee, \delta^2\rangle = -20. \tag{3.4.3}$$

From (3.4.2) and (3.4.3) we obtain, using Proposition 3.1.5,

$$x = \frac{5}{4t}, \qquad y = 0, \qquad z = -\frac{5}{4}.$$

Again by Proposition 3.1.6 we have $\langle \frac{2}{5}q_X^\vee, \frac{2}{5}q_X^\vee \rangle = 92$, while

$$\langle \frac{5}{4t}h^2 - \frac{5}{4}\delta^2, \frac{5}{4t}h^2 - \frac{5}{4}\delta^2 \rangle = 50.$$

We get a contradiction, hence $\{h^2, h\delta, \delta^2, \frac{2}{5}q_X^\vee\}$ is a set of linearly independent elements in $H^{2,2}(X,\mathbb{Q})$. By Theorem 3.3.16 the $\mathbb{Q}$-vector space $H^{2,2}(X,\mathbb{Q})$ has dimension 4, so $\{h^2, h\delta, \delta^2, \frac{2}{5}q_X^\vee\}$ is a basis for $H^{2,2}(X,\mathbb{Q})$. □

3.4.1 Example: basis of $H^{2,2}(S_2^{[2]},\mathbb{Z})$

Let S be a generic K3 surface and $X = S^{[2]}$ be its Hilbert square. Let $h \in \mathrm{Pic}(X)$ be the line bundle induced by the ample generator of $\mathrm{Pic}(S)$. By Theorem 3.3.16 the element $\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle$ is in a basis of $H^{2,2}(X,\mathbb{Q})$, where as usual $x \in H^4(S,\mathbb{Z})$ is the class of a point. We wonder how this element is described in terms of $h^2, h\delta, \delta^2, \frac{2}{5}q_X^\vee \in H^{2,2}(X,\mathbb{Q})$. This information, together with Lemma 3.3.9 and Theorem 3.3.17, will give us a basis of the lattice $H^{2,2}(X,\mathbb{Z})$ in terms of some rational linear combinations of the elements

$$h^2, h\delta, \delta^2, \frac{2}{5}q_X^\vee \in H^{2,2}(X,\mathbb{Q}).$$

It is possible to conjecture such a basis by finding it explicitly when $X = S_2^{[2]}$ is the Hilbert square of the generic K3 surface of degree 2, i.e., $q_X(h) = 2$. We first need to recall the following general fact. Let S be a generic K3 surface with $\mathrm{Pic}(S) \cong \mathbb{Z}h$, and $h^2 = 2t > 0$ for some integer $t > 0$. Suppose that the Pell-type equation $P_{4t}(5)$ is solvable: this implies the existence of a (-10)-class $\rho \in H^2(X,\mathbb{Z})$ of divisibility 2, i.e., $q_X(\rho) = -10$ and $\mathrm{div}(\rho) = 2$ is the positive generator of the ideal

$$\{(\rho, x) \mid x \in H^2(X,\mathbb{Z})\}.$$

Geometrically this means that there exists a 2-dimensional subvariety $P \subset X$ such that $P \cong \mathbb{P}^2$, cf. Theorem 4.1 and Theorem 2.2.40. Denoting by $[P]$ the fundamental cohomological class of P in X, by [HT09, §5] and [Bak15, p.17] we have that $[P] \in H^{2,2}(X,\mathbb{Z})$ is equal to

$$[P] = \frac{1}{24}(3\rho^2 + c_2(X)) \in H^{2,2}(X,\mathbb{Z}).$$

Since $c_2(X) = \frac{6}{5}q_X^\vee$ by Proposition 3.1.8, and $\rho = 2b_5h - a_5\delta$, where (a_5, b_5) is the minimal solution of the Pell-type equation $P_{4t}(5)$, we obtain

$$[P] = \frac{1}{2}b_5^2h^2 + \frac{1}{8}a_5^2\delta^2 - \frac{1}{2}a_5b_5h\delta + \frac{1}{20}q_X^\vee \in H^{2,2}(X,\mathbb{Z}). \tag{3.4.4}$$

In Example 3.4.2 we will use some Sage packages, see [SDWSDJ+20], in particular *sage.modules.free_ quadratic_ module_ integer_ symmetric*, to define lattices, and *sage.modules.torsion_ quadratic_ module*, to obtain an explicit description of the finite bilinear form associated to a discriminant group. Both the packages have been developed by Simon Brandhorst, with a contribution by Paolo Menegatti in the first one.

Example 3.4.2. Let $X := S_2^{[2]}$, where S_2 is the double cover of the plane $\mathbb{P}^2$ ramified over a smooth sextic curve with $\mathrm{Pic}(S_2) \cong \mathbb{Z}h$ and $h^2 = 2$. Denote by $h \in \mathrm{Pic}(X)$ also the class induced by h on X. The Pell-type equation $P_4(5)$ has minimal solution $(a_5, b_5) = (3, 1)$, so $\rho = 2h - 3\delta \in H^2(X, \mathbb{Z})$ is a (-10)-class of divisibility 2. By the discussion above, there is a 2-dimensional subvariety $P \subset X$ and by (3.4.4) we obtain the following integral Hodge class of type $(2, 2)$:

$$[P] = \frac{1}{2}h^2 + \frac{9}{8}\delta^2 - \frac{3}{2}h\delta + \frac{1}{20}q_X^\vee \in H^{2,2}(X, \mathbb{Z}).$$

By (3.4.1) we have

$$\frac{h^2 - h\delta}{2} = \frac{1}{2}\left(\mathfrak{q}_1(h)^2 - \mathfrak{q}_2(h)\right)|0\rangle + \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle \in H^{2,2}(X, \mathbb{Z}),$$

which is indivisible in $H^{2,2}(X, \mathbb{Z})$ by Theorem 3.3.17. Moreover, the following is an element in $H^{2,2}(X, \mathbb{Z})$:

$$[P] - \delta^2 + h\delta - \frac{h^2 - h\delta}{2} = \frac{1}{8}\left(\delta^2 + \frac{2}{5}q_X^\vee\right) \in H^{2,2}(X, \mathbb{Z}).$$

Thus the following is a sublattice of $H^{2,2}(X, \mathbb{Z})$:

$$L := \mathbb{Z}h^2 \oplus \mathbb{Z}\frac{h^2 - h\delta}{2} \oplus \mathbb{Z}\frac{1}{8}\left(\delta^2 + \frac{2}{5}q_X^\vee\right) \oplus \mathbb{Z}\delta^2 \subseteq H^{2,2}(X, \mathbb{Z}). \tag{3.4.5}$$

Note that $\mathrm{rk}(L) = 4$ by Proposition 3.4.1. In order to obtain $H^{2,2}(X, \mathbb{Z})$, we look for the overlattices of L. There is a bijection, by Proposition 1.4.12, between overlattices of L and isotropic subgroups of the discriminant group $A_L := L^\vee/L$ of L. The Gram matrix of the lattice L in the basis $\{h^2, \frac{h^2-h\delta}{2}, \frac{1}{8}\left(\delta^2 + \frac{2}{5}q_X^\vee\right), \delta^2\}$ is the following (recall that the bilinear form is the restriction of $\langle \cdot, \cdot \rangle$ to $L \times L$):

$$\begin{pmatrix} 12 & 6 & 2 & -4 \\ 6 & 2 & 1 & -2 \\ 2 & 1 & 1 & -1 \\ -4 & -2 & -1 & 12 \end{pmatrix}.$$

Since $|\det(L)| = 2^2 \cdot 3 \cdot 7$, by Lemma 1.4.11 an overlattice R of L strictly bigger than L is such that $[R : L] = 2$. Using a Sage program, we find a basis $\{v_1, v_2, v_3, v_4\}$ of A_L such that the finite bilinear form is represented by the following Gram matrix with values in $\mathbb{Q}/\mathbb{Z}$:

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{3}{7} \end{pmatrix}. \tag{3.4.6}$$

We denote by w_1, w_2, w_3, w_4 the images of $h^2, \frac{h^2-h\delta}{2}, \frac{1}{8}\left(\delta^2 + \frac{2}{5}q_X^\vee\right), \delta^2$ respectively under the embedding $L \hookrightarrow L^\vee$. A Sage program gives us the following equivalences modulo L hold:

$$\begin{aligned} v_1 &\equiv \tfrac{1}{2}w_2 + \tfrac{1}{2}w_4, \\ v_2 &\equiv \tfrac{1}{2}w_1 + \tfrac{1}{2}w_2 + \tfrac{1}{2}w_4, \\ v_3 &\equiv \tfrac{1}{3}w_1 + \tfrac{2}{3}w_3 + \tfrac{1}{3}w_4, \\ v_4 &\equiv \tfrac{1}{7}w_1 + \tfrac{2}{7}w_3 + \tfrac{4}{7}w_4. \end{aligned} \tag{3.4.7}$$

We have remarked that an overlattice R of L which strictly contains L is such that $[R : L] = 2$, so R corresponds to an isotropic subgroup of A_L whose cardinality is 2. We see from (3.4.6) that the only isotropic subgroup of A_L of cardinality 2 is the one generated by $v_1 + v_2$, and from (3.4.7) we have

$$v_1 + v_2 \equiv \frac{1}{2}w_1 \pmod{L}.$$

We conclude that the only overlattice of L is the one generated by L and $\frac{h^2}{2}$. By (3.4.1) we have

$$\frac{h^2}{2} = \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle + \frac{1}{2}\mathfrak{q}_1(h)^2|0\rangle \in H^{2,2}(X, \mathbb{Q}),$$

which is not an element of $H^{2,2}(X, \mathbb{Z})$ by Theorem 3.3.17. We conclude that $H^{2,2}(X, \mathbb{Z}) = L$, i.e.,

$$H^{2,2}(X, \mathbb{Z}) = \mathbb{Z}h^2 \oplus \mathbb{Z}\frac{h^2 - h\delta}{2} \oplus \mathbb{Z}\frac{1}{8}\left(\delta^2 + \frac{2}{5}q_X^\vee\right) \oplus \mathbb{Z}\delta^2.$$

Example 3.4.2 makes us conjecture the following: if S is a generic K3 surface, $X = S^{[2]}$ and $h \in \mathrm{Pic}(X)$ is the class induced by the ample generator of $\mathrm{Pic}(S)$, then

$$H^{2,2}(X, \mathbb{Z}) = \mathbb{Z}h^2 \oplus \mathbb{Z}\frac{h^2 - h\delta}{2} \oplus \mathbb{Z}\frac{1}{8}\left(\delta^2 + \frac{2}{5}q_X^\vee\right) \oplus \mathbb{Z}\delta^2. \tag{3.4.8}$$

We will show this conjecture in Corollary 3.4.11. Note that using Lemma 3.3.9 we can represent h^2, $\frac{h^2 - h\delta}{2}$ and δ^2 in terms of Nakajima operators. In order to obtain (3.4.8) from the description of $H^{2,2}(X, \mathbb{Z})$ given in Theorem 3.3.17 we need to express $\frac{2}{5}q_X^\vee \in H^{2,2}(X, \mathbb{Z})$ in terms of Nakajima operators. Recall that $\frac{6}{5}q_X^\vee = c_2(X)$ by Proposition 3.1.8, so we will look for a representation of $c_2(X) \in H^{2,2}(X, \mathbb{Z})$ in terms of Nakajima operators. This is the aim of the next section.

3.4.2 EGL formula and $c_2(S^{[2]})$

Let S be a projective K3 surface. In this section we will use the EGL formula given in Proposition 3.2.8 to describe $c_2(S^{[2]})$, the second Chern class of $S^{[2]}$, in terms of Nakajima operators.

We denote by $\mathcal{T}_2 := \mathcal{T}_{S^{[2]}}$ the tangent bundle of $S^{[2]}$. From now on, we will denote by $\cdot$ the cup product. We define the following operator on the cohomology ring $H^*(S^{[2]}, \mathbb{Q})$:

$$\mathrm{ch}(\mathcal{T}_2) : H^*(S^{[2]}, \mathbb{Q}) \to H^*(S^{[2]}, \mathbb{Q}), \qquad x \mapsto \mathrm{ch}(\mathcal{T}_2) \cdot x.$$

By the general construction of $S^{[n,n+k]}$ seen in Section 3.2.1, if $\Delta \subset S \times S$ is the diagonal, we have $S^{[1,2]} \cong \mathrm{Bl}_\Delta(S^2)$. Diagram (3.2.5) for $n = 1$ gives

$$\begin{tikzcd}
 & S^{[1,2]} \arrow[r, "\psi"] \arrow[d, "\sigma"] \arrow[ddl, bend right, "\varphi"'] \arrow[ddr, bend left, "\rho"] & S^{[2]} \\
 & S \times S \arrow[dl, "p"'] \arrow[dr, "q"] & \\
S & \Delta \arrow[u, hook, "\iota"] & S.
\end{tikzcd} \tag{3.4.9}$$

Note that the morphisms φ, ρ and ψ appearing in diagram (3.4.9) correspond to the morphisms φ, ρ and ψ of diagram (3.2.2), with $n = k = 1$, precomposed with the inclusion of $S^{[1,2]}$ in $S \times S \times S^{[2]}$. With this notation, the definition of the Nakajima operator $\mathfrak{q}_1$ is the same of (3.2.3) without the component $PD^{-1}[S^{[1,2]}]$, i.e., for $\alpha, x \in H^*(S)$ we have

$$\mathfrak{q}_1(\alpha)(x) = \psi_* \left(\varphi^*(x) \cdot \rho^*(\alpha)\right).$$

By properties of cup and cap product, the latter denoted by $\cap$, the following equality holds in $H^*(S^{[1,2]})$ for every $\alpha, x \in H^*(S)$:

$$\varphi^*(x) \cdot \rho^*(\alpha) = PD^{-1}\left([S^{[1,2]}] \cap \varphi^*(x) \cdot \rho^*(\alpha)\right).$$

Then we have

$$\begin{aligned}
\mathrm{ch}(\mathcal{T}_2) \cdot \mathfrak{q}_1(\alpha)(x) &= \mathrm{ch}(\mathcal{T}_2) \cdot \psi_* \left(\varphi^*(x) \cdot \rho^*(\alpha)\right) \\
&= \mathrm{ch}(\mathcal{T}_2) \cdot PD^{-1}\psi_* \left([S^{[1,2]}] \cap \varphi^*(x) \cdot \rho^*(\alpha)\right) \\
&= PD^{-1}\psi_* \left([S^{[1,2]}] \cap \psi^*(\mathrm{ch}(\mathcal{T}_2)) \cdot \varphi^*(x) \cdot \rho^*(\alpha)\right) \\
&= PD^{-1}\psi_* \left([S^{[1,2]}] \cap \mathrm{ch}(\psi^!\mathcal{T}_2) \cdot \varphi^*(x) \cdot \rho^*(\alpha)\right),
\end{aligned}$$

where ψ_* in the first equality is the Gysin homomorphism, while in the other equalities is the pushforward in homology, and the third equality comes from the projection formula. By Proposition 3.2.8 applied with $n = 1$ and ω_S trivial, we then obtain:

$$\begin{aligned}
\mathrm{ch}(\mathcal{T}_2) \cdot \mathfrak{q}_1(\alpha)(x) = \; & PD^{-1}\psi_* \left([S^{[1,2]}] \cap \varphi^*(\mathrm{ch}(\mathcal{T}_S) \cdot x) \cdot \rho^*(\alpha)\right) \\
& + PD^{-1}\psi_* \left([S^{[1,2]}] \cap \mathrm{ch}(\mathcal{L}) \cdot \varphi^*(x) \cdot \rho^*(\alpha)\right) \\
& - PD^{-1}\psi_* \left([S^{[1,2]}] \cap \mathrm{ch}(\mathcal{L}) \cdot \sigma^*(\mathrm{ch}(\mathcal{O}_\Delta^\vee)) \cdot \varphi^*(x) \cdot \rho^*(\alpha)\right) \\
& + PD^{-1}\psi_* \left([S^{[1,2]}] \cap \mathrm{ch}(\mathcal{L}^\vee) \cdot \varphi^*(x) \cdot \rho^*(\alpha)\right) \\
& - PD^{-1}\psi_* \left([S^{[1,2]}] \cap \mathrm{ch}(\mathcal{L}^\vee) \cdot \sigma^*(\mathrm{ch}(\mathcal{O}_\Delta)) \cdot \varphi^*(x) \cdot \rho^*(\alpha)\right) \\
& - PD^{-1}\psi_* \left([S^{[1,2]}] \cap \varphi^*(x) \cdot \rho^*(\mathrm{ch}(2\mathcal{O}_S - \mathcal{T}_S))\right),
\end{aligned} \tag{3.4.10}$$

where $\mathcal{L} := \mathcal{O}_{S^{[1,2]}}(-N)$ and N is the exceptional divisor of the blowing up $\sigma : \mathrm{Bl}_\Delta(S^2) \to S^2$. If we set $x := \mathfrak{q}_1(1)|0\rangle$ and $\alpha := 1$, we can use formula (3.4.10) to compute $c_2(S^{[2]})$ in terms of Nakajima operators. Recall that here the dual of $F \in K(X)$ for X a smooth irreducible projective variety is

$$F^\vee := \sum_i (-1)^i \mathcal{E}xt^i(F, \mathcal{O}_X),$$

while we will denote by $F^* := \mathrm{Hom}(F, \mathcal{O}_X)$ the classical dual. We recall the following three useful general results.

Proposition 3.4.3 (Proposition III.6.3 in [Har13]). *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{G}$ be a sheaf of $\mathcal{O}_X$-modules. Then:*

(i) $\mathcal{E}xt^0(\mathcal{O}_X, \mathcal{G}) = \mathcal{G}$.

(ii) $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{G}) = 0$ *for* $i > 0$.

Proposition 3.4.4 (Proposition III.6.7 in [Har13])**.** *Let $\mathcal{L}$ be a locally free sheaf of finite rank, and let $\mathcal{L}^* = Hom(\mathcal{L}, \mathcal{O}_X)$ be its classical dual. Then, if $\mathcal{F}$ and $\mathcal{G}$ are two sheaves, we have*

$$\mathcal{E}xt^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{L}^* \otimes \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{L}^*.$$

Lemma 3.4.5 (Lemma 1 in [Sch])**.** *Let X be a nonsingular algebraic variety, $\mathcal{L}$ be a line bundle on X and $Z \subseteq X$ be a locally complete intersection of codimension $r \geq 2$. Let $\mathcal{L}'$ be the line bundle $\det N_{Z|X} \otimes i^*\mathcal{L}$ on Z, where $i : Z \hookrightarrow X$ is the inclusion embedding. For every $q \geq 0$ we have*

$$\mathcal{E}xt^q(\mathcal{O}_Z, \mathcal{L}) \cong \mathcal{E}xt^q(\mathcal{O}_Z, \mathcal{O}_X) \otimes \mathcal{L} \cong \begin{cases} \mathcal{L}' & \text{if } q = r \\ 0 & \text{otherwise} \end{cases}$$

We now study the duals appearing in the right-hand side of (3.4.10).

Lemma 3.4.6. *Keep notation as above. Then:*

(i) The dual $\mathcal{L}^\vee$ is isomorphic to the dual $\mathcal{L}^* = \mathrm{Hom}(\mathcal{L}, \mathcal{O}_{S^{[1,2]}})$.

(ii) $\mathcal{O}_\Delta^\vee = \mathcal{O}_\Delta$.

Proof. (i) By Proposition 3.4.4 we have

$$\mathcal{E}xt^i(\mathcal{L}, \mathcal{O}_{S^{[1,2]}}) \cong \mathcal{E}xt^i(\mathcal{O}_{S^{[1,2]}}, \mathcal{O}_{S^{[1,2]}}) \otimes \mathcal{L}^*,$$

and by Proposition 3.4.3 we have

$$\mathcal{E}xt^i(\mathcal{O}_{S^{[1,2]}}, \mathcal{O}_{S^{[1,2]}}) = \begin{cases} 0 & \text{if } i > 0, \\ \mathcal{O}_{S^{[1,2]}} & \text{if } i = 0. \end{cases}$$

We conclude that $\sum_i (-1)^i \mathcal{E}xt^i(\mathcal{L}, \mathcal{O}_{S^{[1,2]}}) \cong \mathcal{L}^*$.

(ii) We apply Lemma 3.4.5 with $X = S \times S$, $Z = \Delta$ and $\mathcal{L} = \mathcal{O}_{S\times S}$. We have $\mathcal{E}xt^i(\mathcal{O}_\Delta, \mathcal{O}_{S\times S}) = 0$ if $i \neq 2$ and $\mathcal{E}xt^2(\mathcal{O}_\Delta, \mathcal{O}_{S\times S}) = \det N_{\Delta|S\times S}$. Moreover, $N_{\Delta|S\times S} = \mathcal{T}_S$ and $\det \mathcal{T}_S \cong \mathcal{O}_S$ since S is a K3 surface. We conclude that $\sum_i (-1)^1 \mathcal{E}xt^i(\mathcal{O}_\Delta, \mathcal{O}_{S\times S}) = \mathcal{O}_\Delta$, as we wanted. □

We now compute the Chern character of $\mathcal{O}_\Delta$, where $\mathcal{O}_\Delta := i_*\mathcal{O}_\Delta$ with $i : \Delta \hookrightarrow S \times S$ the inclusion map.

Lemma 3.4.7. *Let S be a K3 surface and $\Delta \subset S \times S$ be the diagonal. Denote by $[\Delta] \in H^4(S \times S, \mathbb{Z})$ the fundamental cohomological class of Δ in $S \times S$. Then we have*

$$\mathrm{ch}(\mathcal{O}_\Delta) = [\Delta] - 2y,$$

where $y \in H^8(S \times S, \mathbb{Q})$ is the class of a point in $S \times S$.

Proof. This follows from the Grothendieck–Riemann–Roch theorem. Indeed, let $i : \Delta \hookrightarrow S \times S$ be the inclusion. By Theorem 1.3.4 we have

$$\begin{aligned} \mathrm{ch}(\mathcal{O}_\Delta) &= i_* \left(\mathrm{ch}(\mathcal{O}_\Delta) \cdot \mathrm{td}(S)\right) \cdot \mathrm{td}(S \times S)^{-1} \\ &= i_* \left(\mathrm{td}(S) \cdot i^*\mathrm{td}(S \times S)^{-1}\right) \\ &= i_* \left(\mathrm{td}(S)^{-1}\right), \end{aligned}$$

where i_* is the Gysin homomorphism. Let $x \in H^4(S,\mathbb{Q})$ be the class of a point of S. The Todd class of a K3 surface is $\mathrm{td}(S) = 1 + 2x$, hence we obtain

$$\mathrm{ch}(\mathcal{O}_\Delta) = [\Delta] - 2y.$$

$\square$

Consider formula (3.4.10). We introduce the following notation:

$$\begin{aligned}
L1 &:= \mathrm{ch}(\mathcal{T}_2)\cdot \mathfrak{q}_1(\alpha)(x),\\
R1 &:= PD^{-1}\psi_*\left([S^{[1,2]}] \cap \varphi^*(\mathrm{ch}(\mathcal{T}_S)\cdot x)\cdot \rho^*(\alpha)\right),\\
R2 &:= PD^{-1}\psi_*\left([S^{[1,2]}] \cap \mathrm{ch}(\mathcal{L})\cdot \varphi^*(x)\cdot \rho^*(\alpha)\right),\\
R3 &:= PD^{-1}\psi_*\left([S^{[1,2]}] \cap \mathrm{ch}(\mathcal{L})\cdot \sigma^*(\mathrm{ch}(\mathcal{O}_\Delta^\vee))\cdot \varphi^*(x)\cdot \rho^*(\alpha)\right),\\
R4 &:= PD^{-1}\psi_*\left([S^{[1,2]}] \cap \mathrm{ch}(\mathcal{L}^\vee)\cdot \varphi^*(x)\cdot \rho^*(\alpha)\right),\\
R5 &:= PD^{-1}\psi_*\left([S^{[1,2]}] \cap \mathrm{ch}(\mathcal{L}^\vee)\cdot \sigma^*(\mathrm{ch}(\mathcal{O}_\Delta))\cdot \varphi^*(x)\cdot \rho^*(\alpha)\right),\\
R6 &:= PD^{-1}\psi_*\left([S^{[1,2]}] \cap \varphi^*(x)\cdot \rho^*(\mathrm{ch}(2\mathcal{O}_S - \mathcal{T}_S)\cdot x)\right).
\end{aligned}$$

Recall that we have taken $x = \mathfrak{q}_1(1)|0\rangle$ and $\alpha = 1$. We can compute $c_2(S^{[2]})$ in terms of Nakajima operators.

Proposition 3.4.8. *Let S be a projective K3 surface. Then the second Chern class $c_2(S^{[2]}) \in H^{2,2}(S^{[2]},\mathbb{Z})$ of $S^{[2]}$ in terms of Nakajima operators is*

$$c_2(S^{[2]}) = 27\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle + 3\sum_{i<j}\mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle + \frac{3}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle,$$

where $1 \in H^0(S,\mathbb{Z})$ is the unit and $x \in H^4(S,\mathbb{Z})$ is the class of a point, $\{\alpha_1,\ldots,\alpha_{22}\}$ is the basis of the lattice $H^2(S,\mathbb{Z})$ used in Lemma 3.2.11 *and the $\mu_{i,j}$'s are given in Table* 3.1.

Note that by Table 3.1 the integers $\mu_{i,i}$ are all even, so the expression given above for $c_2(S^{[2]})$ is really an element of $H^{2,2}(S^{[2]},\mathbb{Z})$.

Proof. We make some computations on $L1, R1, \ldots, R6$ introduced above.

- We have $L1 = \mathrm{ch}(\mathcal{T}_2)\cdot\mathfrak{q}_1(\alpha)(x) = \mathrm{ch}(\mathcal{T}_2)\cdot\mathfrak{q}_1(1)\mathfrak{q}_1(1)|0\rangle$. The exponential Chern character of $\mathcal{T}_2$ is, by Section 1.3,

  $$\mathrm{ch}(\mathcal{T}_2) = 4 - c_2(S^{[2]}) + \frac{1}{12}c_2(S^{[2]})^2 - \frac{1}{6}c_4(S^{[2]}),$$

  where $c_i(S^{[2]}) = c_i(\mathcal{T}_2)$ for $i \geq 0$, and (3.2.4) gives

  $$\mathfrak{q}_1(1)\mathfrak{q}_1(1)|0\rangle = 2\cdot 1_{S^{[2]}}.$$

  Thus we obtain

  $$L1 = 8\cdot 1_{S^{[2]}} - 2c_2(S^{[2]}) + \frac{1}{6}c_2(S^{[2]})^2 - \frac{1}{3}c_4(S^{[2]}).$$

- Since S is a K3 surface we have $c_1(S) = 0$ and $c_2(S) = 24x$, where $x \in H^4(S, \mathbb{Z})$ is the class of a point on S. Hence $\mathrm{ch}(\mathcal{T}_S) = 2 - 24x$ and we obtain
$$\begin{aligned} R1 &= 2\mathfrak{q}_1(1)\mathfrak{q}_1(1)|0\rangle - 24\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle \\ &= 4 \cdot 1_{S^{[2]}} - 24\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \end{aligned}$$
where the second equality comes from (3.2.4).

- Let $d := c_1(\mathcal{L})$. Then we have
$$\mathrm{ch}(\mathcal{L}) = \sum_{\nu \geq 0} \frac{1}{\nu!} d^\nu.$$
Now, [Leh99, Lemma 3.9] implies that the cycle $[S^{[1,2]}] \cap d^\nu$ induces the operator $\mathfrak{q}_1^{(\nu)}$, as observed in the proof of [Leh99, Lemma 4.2], hence we obtain
$$\begin{aligned} R2 &= \sum_{\nu \geq 0} \frac{1}{\nu!} \mathfrak{q}_1^{(\nu)}(\alpha) \cdot x \\ &= \sum_{\nu \geq 0} \frac{1}{\nu!} \mathfrak{q}_1^{(\nu)}(1)\mathfrak{q}_1(1)|0\rangle. \end{aligned}$$
We now compute $\mathfrak{q}_1^{(\nu)}(1)\mathfrak{q}_1(1)|0\rangle$ for every $\nu \geq 0$. If $\nu = 0$, we have
$$\mathfrak{q}_1^{(0)}(1)\mathfrak{q}_1(1)|0\rangle = \mathfrak{q}_1(1)\mathfrak{q}_1(1)|0\rangle = 2 \cdot 1_{S^{[2]}} \in H^0(S^{[2]}, \mathbb{Z}).$$
If $\nu = 1$, by Theorem 3.2.7 we have
$$\mathfrak{q}_1'(1)\mathfrak{q}_1(1)|0\rangle = -\mathfrak{q}_2(1)|0\rangle \in H^2(S^{[2]}, \mathbb{Z}). \tag{3.4.11}$$
If $\nu = 2$, we have
$$\begin{aligned} \mathfrak{q}_1^{(2)}(1)\mathfrak{q}_1(1)|0\rangle &= \left(\partial\mathfrak{q}_1' - \mathfrak{q}_1'\partial\right)(1)\mathfrak{q}_1(1)|0\rangle \\ &= \partial\mathfrak{q}_1'(1)\mathfrak{q}_1(1)|0\rangle - \mathfrak{q}_1'\partial(1)\mathfrak{q}_1(1)|0\rangle. \end{aligned}$$
The boundary of S is empty by Remark 3.2.4, so $\mathfrak{q}_1'\partial(1)\mathfrak{q}_1(1)|0\rangle = 0$. Moreover, using (3.4.11), we get
$$\partial\mathfrak{q}_1'(1)\mathfrak{q}_1(1)|0\rangle = -\partial\mathfrak{q}_2(1)|0\rangle,$$
and by Definition 3.2.6 we obtain
$$\mathfrak{q}_1^{(2)}(1)\mathfrak{q}_1(1)|0\rangle = \frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \mathfrak{q}_2(1)|0\rangle \in H^4(S^{[2]}, \mathbb{Z}).$$
Similarly for $\nu = 3$ and $\nu = 4$ we obtain the following:
$$\begin{aligned} &\mathfrak{q}_1^{(3)}(1)\mathfrak{q}_1(1)|0\rangle = -\tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \mathfrak{q}_2(1)|0\rangle \in H^6(S^{[2]}, \mathbb{Z}), \\ &\mathfrak{q}_1^{(4)}(1)\mathfrak{q}_1(1)|0\rangle = \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \mathfrak{q}_2(1)|0\rangle \in H^8(S^{[2]}, \mathbb{Z}). \end{aligned}$$
If $\nu \geq 5$, we obtain an element in $H^{2\nu}(S^{[2]}, \mathbb{Z}) = 0$. We conclude that
$$\begin{aligned} R2 = \; & 2 \cdot 1_{S^{[2]}} - \mathfrak{q}_2(1)|0\rangle + \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \\ & -\tfrac{1}{3}\left(\tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle\right) \\ & +\tfrac{1}{12}\left(\tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle\right). \end{aligned}$$

- As before, we set $d := c_1(\mathcal{L})$. By Lemma 3.4.6 we have $\mathrm{ch}(\mathcal{O}_\Delta) = \mathrm{ch}(\mathcal{O}_\Delta^\vee)$, and by Lemma 3.4.7 we have $\mathrm{ch}(\mathcal{O}_\Delta) = [\Delta] - 2y$, where $y \in H^8(S \times S, \mathbb{Z})$ is the class of a point. Moreover, Lemma 3.2.11 gives

$$[\Delta] = \sum_{i,j} \mu_{i,j} \alpha_i \otimes \alpha_j + 1 \otimes x + x \otimes 1,$$

where $\{\alpha_1, \dots, \alpha_{22}\}$ is the basis of $H^2(S, \mathbb{Z})$ used in Lemma 3.2.11 and the $\mu_{i,j}$'s are given in Table 3.1. Recall diagram (3.4.9)

$$\begin{tikzcd}
 & S^{[1,2]} \arrow[r, "\psi"] \arrow[d, "\sigma"] \arrow[ddl, bend right, "\varphi"'] \arrow[ddr, bend left, "\rho"] & S^{[2]} \\
 & S \times S \arrow[dl, "p"] \arrow[dr, "q"'] & \\
S & \Delta \arrow[u, hook, "\iota"] & S,
\end{tikzcd}$$

where p and q are the two projections, σ is the blowing up of $S \times S$ in the diagonal and $\varphi = p \circ \sigma$, $\rho = q \circ \sigma$. By the Künneth theorem we have $y = x \otimes x$. Then

$$\begin{aligned}
\sigma^*(\mathrm{ch}(\mathcal{O}_\Delta)) &= \sigma^* \left(\sum_{i,j} \mu_{i,j} \alpha_i \otimes \alpha_j + 1 \otimes x + x \otimes 1 - 2(x \otimes x) \right) \\
&= \sum_{i,j} \mu_{i,j} \varphi^*(\alpha_i) \cdot \rho^*(\alpha_j) \\
&\quad + \varphi^*(1) \cdot \rho^*(x) + \varphi^*(x) \cdot \rho^*(1) - 2 \left(\varphi^*(x) \cdot \rho^*(x) \right).
\end{aligned}$$

Proceeding as for $R2$ we get

$$\begin{aligned}
R3 &= \sum_{i,j} \mu_{i,j} \sum_{\nu \geq 0} \frac{1}{\nu!} \mathfrak{q}_1^{(\nu)}(\alpha_i) \mathfrak{q}_1(\alpha_j) |0\rangle \\
&\quad + \sum_{\nu \geq 0} \frac{1}{\nu!} \mathfrak{q}_1^{(\nu)}(x) \mathfrak{q}_1(1) |0\rangle \\
&\quad + \sum_{\nu \geq 0} \frac{1}{\nu!} \mathfrak{q}_1^{(\nu)}(1) \mathfrak{q}_1(x) |0\rangle \\
&\quad - 2 \sum_{\nu \geq 0} \frac{1}{\nu!} \mathfrak{q}_1^{(\nu)}(x) \mathfrak{q}_1(x) |0\rangle.
\end{aligned} \tag{3.4.12}$$

We call $R3_{\nu=i}$ the component of $R3$ obtained by putting $\nu = i$ in (3.4.12) for $i \geq 0$. Using the commutativity rule given by Theorem 3.2.2 we have

$$R3_{\nu=0} = \sum_{i,j} \mu_{i,j} \mathfrak{q}_1(\alpha_i) \mathfrak{q}_1(\alpha_j) |0\rangle + 2\mathfrak{q}_1(1) \mathfrak{q}_1(x) |0\rangle - 2\mathfrak{q}_1(x) \mathfrak{q}_1(x) |0\rangle.$$

Note that $\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle \in H^4(S^{[2]}, \mathbb{Z})$ and $\mathfrak{q}_1(x)\mathfrak{q}_1(x)|0\rangle \in H^8(S^{[2]}, \mathbb{Z})$. If $\nu \geq 3$ we obtain elements in $H^i(S^{[2]}, \mathbb{Q})$ with $i \geq 10$, so these are equal to zero. We do not compute explicitly $R3_{\nu=1}$: we will see that this is not necessary. If $\nu = 2$, using Definition 3.2.6 we obtain, after some computations,

$$\begin{aligned} R3_{\nu=2} &= \sum_{i,j} \mu_{i,j} \frac{1}{2}\left(\frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \frac{1}{2}\mathfrak{q}_2(1)|0\rangle\right) \cdot \mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle \\ &\quad + \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \end{aligned}$$

which is an element of $H^8(S^{[2]},\mathbb{Z})$. Note that we have not written down the element

$$-\left(\frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \frac{1}{2}\mathfrak{q}_2(1)|0\rangle\right) \cdot \mathfrak{q}_1(x)\mathfrak{q}_1(x)|0\rangle$$

since it belongs to $H^{12}(S^{[2]},\mathbb{Z})$, hence it is zero. We conclude that

$$\begin{aligned} R3 &= \sum_{i,j} \mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle \\ &\quad +2\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle \\ &\quad -2\mathfrak{q}_1(x)\mathfrak{q}_1(x)|0\rangle \\ &\quad +R3_{\nu=1} \\ &\quad +\tfrac{1}{2}\sum_{i,j} \mu_{i,j}\frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle \\ &\quad +\tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle. \end{aligned}$$

- By Lemma 3.4.6 the dual $\mathcal{L}^\vee$ is isomorphic to $\mathrm{Hom}(\mathcal{L}, \mathcal{O}_{S^{[1,2]}})$, which is the classic dual. Hence if $d := c_1(\mathcal{L})$ we have

$$\mathrm{ch}(\mathcal{L}^\vee) = \sum_{\nu\geq 0} \frac{(-1)^\nu}{\nu!} d^\nu.$$

  Thus $R4$ is computed in the same way as $R2$, with a change of sign for the components obtained when $\nu = 1$ and $\nu = 3$. We obtain

$$\begin{aligned} R4 &= 2\cdot 1_{S^{[2]}} + \mathfrak{q}_2(1)|0\rangle + \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \\ &\quad +\tfrac{1}{3}\left(\tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle\right) \\ &\quad +\tfrac{1}{12}\left(\tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle\right). \end{aligned}$$

- By Lemma 3.4.6 we have $\mathcal{L}^\vee = \mathrm{Hom}(\mathcal{L}, \mathcal{O}_{S^{[1,2]}})$ and $\mathcal{O}_\Delta^\vee = \mathcal{O}_\Delta$, so $R5$ is computed in the same way as $R3$, with a change of sign for the component $R3_{\nu=1}$, so we obtain

$$\begin{aligned} R5 &= \sum_{i,j} \mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle \\ &\quad +2\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle \\ &\quad -2\mathfrak{q}_1(x)\mathfrak{q}_1(x)|0\rangle \\ &\quad -R3_{\nu=1} \\ &\quad +\tfrac{1}{2}\sum_{i,j} \mu_{i,j}\frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle \\ &\quad +\tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle. \end{aligned}$$

- Since S is a K3 surface, we have

$$\begin{aligned} \mathrm{ch}(2\mathcal{O}_S - \mathcal{T}_S) &= 2 - \mathrm{ch}(\mathcal{T}_S) \\ &= 2 - 2 + c_2(S) \\ &= 24x, \end{aligned}$$

where $x \in H^4(S,\mathbb{Z})$ is the class of a point. Then

$$R6 = 24\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle,$$

where we have used $\mathfrak{q}_1(x)\mathfrak{q}_1(1)|0\rangle = \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle$ from Theorem 3.2.2.

Thus formula (3.4.10) with $x = \mathfrak{q}_1(1)|0\rangle$ and $\alpha = 1$ gives

$$\begin{aligned} L1 = \ & 8 \cdot 1_{S^{[2]}} \\ & -52\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle + 2\left(\tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle\right) - 2\sum_{i,j}\mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle \\ & +\tfrac{1}{6}\left(\tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle\right) \\ & +4\mathfrak{q}_1(x)\mathfrak{q}_1(x)|0\rangle \\ & -\sum_{i,j}\mu_{i,j}\frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle \\ & -2\left(\tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle\right), \end{aligned} \tag{3.4.13}$$

where

$$L1 = 8 \cdot 1_S^{[2]} - 2c_2(S^{[2]}) + \frac{1}{6}c_2(S^{[2]})^2 - \frac{1}{3}c_4(S^{[2]}). \tag{3.4.14}$$

We now impose equalities between elements belonging to $H^4(S^{[2]},\mathbb{Z})$ in the right-hand side of (3.4.13) and (3.4.14). We obtain

$$c_2(S^{[2]}) = 26\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle - \frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \frac{1}{2}\mathfrak{q}_2(1)|0\rangle + \sum_{i,j}\mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle.$$

Using the commutativity rule of Theorem 3.2.2 and Lemma 3.3.9, (iii), we get

$$c_2(S^{[2]}) = 27\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle + 3\sum_{i<j}\mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle + \frac{3}{2}\sum_{i}\mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle,$$

and we are done. □

3.4.3 Another description of $H^{2,2}(S^{[2]},\mathbb{Z})$

Using Proposition 3.4.8 we can finally prove the following theorem.

Theorem 3.4.9. *Let S be a projective K3 surface and $X = S^{[2]}$ be its Hilbert square. Consider $q_X^\vee \in H^{2,2}(X,\mathbb{Q})$ the dual of the BBF form. Then*

$$\frac{2}{5}q_X^\vee = 9\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle + \sum_{i<j}\mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle + \frac{1}{2}\sum_{i}\mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle \in H^{2,2}(X,\mathbb{Z}), \tag{3.4.15}$$

where $1 \in H^0(S,\mathbb{Z})$ *is the unit,* $x \in H^4(S,\mathbb{Z})$ *is the class of a point,* $\{\alpha_1,\dots,\alpha_{22}\}$ *is the basis of* $H^2(S,\mathbb{Z})$ *used in Lemma* 3.2.11 *and the* $\mu_{i,j}$*'s are the integers given in Table* 3.1*. Moreover,* $\frac{2}{5}q_X^\vee$ *is indivisible in* $H^{2,2}(X,\mathbb{Z})$ *and*

$$\frac{1}{8}\left(\delta^2+\frac{2}{5}q_X^\vee\right)=\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle \in H^{2,2}(X,\mathbb{Z}). \tag{3.4.16}$$

Proof. By Proposition 3.1.8 we have $\frac{6}{5}q_X^\vee = c_2(X)$, so Proposition 3.4.8 implies (3.4.15). Moreover, taking the basis of $H^4(X,\mathbb{Z})$ given by Theorem 3.2.9, we see that $\frac{2}{5}q_X^\vee$ is indivisible in $H^4(X,\mathbb{Z})$, i.e., there is no $\alpha \in H^4(X,\mathbb{Z})$ such that $n\alpha = \frac{2}{5}q_X^\vee$ for some integer $n \in \mathbb{Z}_{>1}$: it suffices to find some $\mu_{i,j}$ which are coprime with 9, the coefficient of $\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle$ in (3.4.15), by Table 3.1. This implies that $\frac{2}{5}q_X^\vee$ is indivisible also in $H^{2,2}(X,\mathbb{Z})$. Recall that by Lemma 3.3.9 we have

$$\delta^2=-\sum_{i<j}\mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle-\frac{1}{2}\sum_i\mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle-\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \tag{3.4.17}$$

thus from (3.4.15) and (3.4.17) we obtain

$$\delta^2+\frac{2}{5}q_X^\vee = 8\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle \in H^{2,2}(X,\mathbb{Z}),$$

which implies

$$\frac{1}{8}\left(\delta^2+\frac{2}{5}q_X^\vee\right)=\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle \in H^{2,2}(X,\mathbb{Z}).$$

□

Note that (3.4.16) is consistent with (3.4.13) and (3.4.14) in the proof of Proposition 3.4.8. Indeed, imposing the equality between the elements belonging to $H^8(S^{[2]},\mathbb{Z})$ in the right-hand side of (3.4.13) and (3.4.14) we have

$$\begin{aligned}\tfrac{1}{6}c_2(S^{[2]})^2-\tfrac{1}{3}c_4(S^{[2]}) \quad = \quad & \tfrac{1}{6}\left(\tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle\cdot\tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle\cdot\tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle\cdot\tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle\right)\\ &+4\mathfrak{q}_1(x)\mathfrak{q}_1(x)|0\rangle\\ &-\sum_{i,j}\mu_{i,j}\frac{1}{2}\mathfrak{q}_2(1)|0\rangle\cdot\frac{1}{2}\mathfrak{q}_2(1)|0\rangle\cdot\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle\\ &-2\left(\tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle\cdot\tfrac{1}{2}\mathfrak{q}_2(1)|0\rangle\cdot\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle\right).\end{aligned} \tag{3.4.18}$$

Let $z \in H^8(S^{[2]},\mathbb{Z})$ be the class of a point of $S^{[2]}$. By Theorem 2.2.6 we have $c_2(S^{[2]})^2 = 828z$ and $c_4(S^{[2]}) = 324z$, so the left-hand side of (3.4.18) is $30z$. The first term in the right-hand side is $\frac{1}{6}\delta^4$, which is equal to $2z$ as a consequence of Theorem 2.2.4, being $q_X(\delta)=-2$. The second term is $4z$. By Lemma 3.3.9, (ii), we have

$$\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle = \mathfrak{q}_1(1)\mathfrak{q}_1(\alpha_i)|0\rangle\cdot\mathfrak{q}_1(1)\mathfrak{q}_1(\alpha_j)|0\rangle-\left(\int_S\alpha_i\alpha_j\right)\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle,$$

Moreover, by Theorem 3.4.9 we have

$$\frac{1}{2}\mathfrak{q}_2(1)|0\rangle\cdot\frac{1}{2}\mathfrak{q}_2(1)|0\rangle\cdot\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle=\langle\delta^2,\frac{1}{8}\left(\delta^2+\frac{2}{5}q_X^\vee\right)\rangle z=-z, \tag{3.4.19}$$

thus we get

$$\frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle = \left(\langle \delta^2, \alpha_i\alpha_j\rangle + \int_S \alpha_i\alpha_j\right) z. \qquad (3.4.20)$$

Using (3.4.20) and Table 3.1 it is possible to show that

$$-\sum_{i,j} \mu_{i,j} \frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \frac{1}{2}\mathfrak{q}_2(1)|0\rangle \cdot \mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle = 22z.$$

By (3.4.19) the last term of (3.4.18) is $2z$. We conclude that the right-hand side of (3.4.18) is $30z$, which is equal to $\frac{1}{6}c_2(S^{[2]})^2 - \frac{1}{3}c_4(S^{[2]})$ as seen above.

Theorem 3.4.9 is consistent with the computation of $\mathrm{ch}(S^{[2]})$ given by the Maude program in [BNW07, §11]. See [CDE+02] for documentation about Maude. The output of the program is given in Figure 3.4.1.

```
==========================================
rewrite in HILB : CHT(2) .
rewrites: 14750 in 0ms cpu (18ms real) (~ rewrites/second)
result End:
        2 < I 1 > . < I 1 > . |> +
        -1 < K 2 > . |> +
        1/2 < (K . K) 2 > . |> +
        1/4 << K (1 ; 1) >> . |> +
        1/4 < I 1 > . < (K . K) 1 > . |> +
        1/4 < (K . K) 1 > . < I 1 > . |> +
        5/8 << e (1 ; 1) >> . |> +
        -1/2 < I 1 > . < K 1 > . |> +
        -1/2 < I 1 > . < e 1 > . |> +
        -1/2 < K 1 > . < I 1 > . |> +
        -1/2 < e 1 > . < I 1 > . |> +
        -3/2 << I (1 ; 1) >> . |> + -5/12 << (K . K) (1 ; 1) >> . |>
==========================================
```

Figure 3.4.1: Output of $\mathrm{ch}(S^{[2]})$

Without going into details, in Figure 3.4.1 the canonical bundle of S is represented by K. In our case K is trivial, and the program gives

$$\mathrm{ch}(S^{[2]}) = 2\mathfrak{q}_1(1)\mathfrak{q}_1(1)|0\rangle + \frac{5}{8}\mathfrak{q}_{11}(e)|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(e)|0\rangle - \frac{3}{2}\mathfrak{q}_{11}(1)|0\rangle,$$

where $e = 24x$ is the Euler class of S, with $x \in H^4(S,\mathbb{Z})$ the class of a point, and by definition

$$\begin{aligned} \mathfrak{q}_{11}(\alpha) &= (\mathfrak{q}_1 \circ \mathfrak{q}_1)\tau_{2*}\alpha \\ &= \sum_i \mathfrak{q}_1(\alpha_1^i)\mathfrak{q}_1(\alpha_2^i), \end{aligned}$$

where $\tau_{2*}(\alpha) = \sum_i \alpha_1^i \otimes \alpha_2^i \in H^*(S)\otimes H^*(S)$ and $\tau_{2*} : H^*(S) \to H^*(S)\otimes H^*(S)$

is the map defined in Section 3.2.4. In particular we have

$$\begin{aligned}
c_2(S^{[2]}) &= 24\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle + \tfrac{3}{2}(\mathfrak{q}_1 \circ \mathfrak{q}_1)(\tau_{2*}(1)) \\
&= 24\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle + \tfrac{3}{2}(\mathfrak{q}_1 \circ \mathfrak{q}_1)\left(\sum_{i,j} \mu_{i,j}\alpha_i \otimes \alpha_j + 1 \otimes x + x \otimes 1\right) \\
&= 24\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle + 3\sum_{i<j} \mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle + \frac{3}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle \\
&\quad +3\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle \\
&= 27\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle + 3\sum_{i<j} \mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle + \frac{3}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle,
\end{aligned}$$

where as usual $\{\alpha_1, \dots, \alpha_{22}\}$ is the basis of $H^2(S, \mathbb{Z})$ used in Lemma 3.2.11 and the $\mu_{i,j}$'s are given in Table 3.1. Note that the second equality comes from equation (3.2.6). Then we have

$$\frac{2}{5}q_X^\vee = 9\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle + \sum_{i<j} \mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle + \frac{1}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle,$$

which is (3.4.15).

Remark 3.4.10. If S is a projective K3 surface, the lattice $H^{2,2}(S^{[2]}, \mathbb{Z})$ is always an odd lattice: this follows from the product

$$\langle \frac{1}{8}\left(\delta^2 + \frac{2}{5}q_X^\vee\right), \frac{1}{8}\left(\delta^2 + \frac{2}{5}q_X^\vee\right)\rangle = 1.$$

Let now S be a generic K3 surface. We denote by h both the class which generates $\mathrm{Pic}(S)$ and the class induced on X. Theorem 3.4.9 can be used, together with Lemma 3.3.9 and Theorem 3.3.17, to obtain the following description of $H^{2,2}(S^{[2]}, \mathbb{Z})$ which does not depend on Nakajima operators, proving the conjecture made in (3.4.8).

Corollary 3.4.11. *Let S be a generic K3 surface of degree $2t$ and $X = S^{[2]}$ be its Hilbert square. Let $h \in \mathrm{Pic}(X)$ be the class induced by the ample generator of $\mathrm{Pic}(S)$. Then*

$$H^{2,2}(X, \mathbb{Z}) = \mathbb{Z}h^2 \oplus \mathbb{Z}\frac{h^2 - h\delta}{2} \oplus \mathbb{Z}\frac{1}{8}\left(\delta^2 + \frac{2}{5}q_X^\vee\right) \oplus \mathbb{Z}\delta^2.$$

Moreover, $H^{2,2}(X, \mathbb{Z})$ is an odd lattice of discriminant $\mathrm{disc}(H^{2,2}(X, \mathbb{Z})) = 84t^3$, *and the Gram matrix in the basis given above is the following:*

$$\begin{pmatrix} 12t^2 & 6t^2 & 2t & -4t \\ 6t^2 & t(3t-1) & t & -2t \\ 2t & t & 1 & -1 \\ -4t & -2t & -1 & 12 \end{pmatrix}.$$

Proof. We denote by h both the class which generates $\mathrm{Pic}(S)$ and the class induced on X. By Theorem 3.3.17 the following is a basis for the lattice $H^{2,2}(X, \mathbb{Z})$:

$$\{\mathfrak{q}_2(h)|0\rangle, \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \frac{1}{2}\left(\mathfrak{q}_2(h) - \mathfrak{q}_1(h)^2\right)|0\rangle, \delta^2\}. \tag{3.4.21}$$

Note that the following is another basis of $H^{2,2}(X,\mathbb{Z})$:

$$\{2t\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle + \mathfrak{q}_1(h)^2|0\rangle, \tfrac{1}{2}\left(\mathfrak{q}_2(h) - \mathfrak{q}_1(h)^2\right)|0\rangle + t\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle,$$
$$\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \delta^2\}, \tag{3.4.22}$$

where $q_X(h) = 2t$ for some integer $t > 0$. Indeed, every element in (3.4.21) is an integral linear combination of elements in (3.4.22). By Lemma 3.3.9, the equalities in (3.4.1) and Theorem 3.4.9, the basis (3.4.22) is equal to

$$\{h^2, \frac{h^2 - h\delta}{2}, \frac{1}{8}\left(\delta^2 + \frac{2}{5}q_X^\vee\right), \delta^2\},$$

as we wanted. By Remark 3.4.10 the lattice $H^{2,2}(X,\mathbb{Z})$ is odd, and the Gram matrix is easily computed using Proposition 3.1.5 and Proposition 3.1.6. □

3.4.4 $H^{2,2}(S^{[2]},\mathbb{Z})$ for any projective K3 surface S

Let S be a projective K3 surface. In Theorem 3.3.16 we have given a basis of the vector space $H^{2,2}(S^{[2]},\mathbb{Q})$ for any projective K3 surface S. Then we have described in Theorem 3.3.17 a basis of the lattice $H^{2,2}(S^{[2]},\mathbb{Z})$ when S has Picard group of rank $r = 1$. We now present a basis of the lattice $H^{2,2}(S^{[2]},\mathbb{Z})$ for any projective K3 surface S with Picard group of rank r, where $1 \le r \le 20$. We will use results obtained in Section 3.4.2 and in Section 3.4.3.

Theorem 3.4.12. *Let S be a projective K3 surface with Picard group of rank* $\mathrm{rk}(\mathrm{Pic}(S)) = r$. *Let $\{b_1, \dots, b_r\}$ be a basis of* $\mathrm{Pic}(S)$. *Then:*

(i) $\mathrm{rk}(H^{2,2}(S^{[2]},\mathbb{Z})) = \frac{(r+1)r}{2} + r + 2$.

(ii) A basis of $H^{2,2}(S^{[2]},\mathbb{Z})$ is given by the following elements:

- $\mathfrak{q}_2(b_i)|0\rangle$, *for* $i = 1, \dots, r$,
- $\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle$, *where $1 \in H^0(S,\mathbb{Z})$ is the unit and $x \in H^4(S,\mathbb{Z})$ is the class of a point.*
- $\frac{1}{2}\left(\mathfrak{q}_1(b_i)^2 - \mathfrak{q}_2(b_i)\right)|0\rangle$, *for* $i = 1, \dots, r$,
- $\mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)|0\rangle$, *for* $1 \le i < j \le r$,
- $-\sum_{i<j} \mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle - \frac{1}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle$, *where* $\{\alpha_1, \dots, \alpha_{22}\}$ *is the basis of $H^2(S,\mathbb{Z})$ used in Lemma* 3.2.11 *and the* $\mu_{i,j}$*'s are given in Table* 3.1.

Equivalently, the following is a basis of $H^{2,2}(S^{[2]},\mathbb{Z})$:

$$\left\{b_ib_j, \frac{b_i^2 - b_i\delta}{2}, \frac{1}{8}\left(\delta^2 + \frac{2}{5}q_X^\vee\right), \delta^2\right\}_{1\le i\le j\le r.} \tag{3.4.23}$$

Moreover, $H^{2,2}(S^{[2]},\mathbb{Z})$ is an odd lattice.

Proof. By Theorem 3.3.16 we have $\dim(H^{2,2}(S^{[2]},\mathbb{Q})) = \frac{(r+1)r}{2} + r + 2$. Since by Theorem 2.2.9 the cohomology groups $H^i(S^{[2]},\mathbb{Z})$ are torsion free for $i \ge 0$, we obtain that $\mathrm{rk}(H^{2,2}(S^{[2]},\mathbb{Z})) = \frac{(r+1)r}{2} + r + 2$. After a slight modification of the basis given in Theorem 3.3.16, we have that the following is a basis of the $\mathbb{Q}$-vector space $H^{2,2}(S^{[2]},\mathbb{Q})$:

$$\begin{aligned}
&\mathfrak{q}_2(b_i)|0\rangle \text{ for } i = 1,\dots,r\\
&\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle\\
&\tfrac{1}{2}\left(\mathfrak{q}_1(b_i)^2 - \mathfrak{q}_2(b_i)\right)|0\rangle \text{ for } i = 1,\dots,r\\
&\mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)|0\rangle \text{ for } 1 \le i < j \le r\\
&-\sum_{i<j}\mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle - \frac{1}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle.
\end{aligned} \tag{3.4.24}$$

In order to prove the theorem, we look for a sublattice L of $H^4(S^{[2]},\mathbb{Z})$ of maximal rank such that

$$L \cap H^{2,2}(S^{[2]},\mathbb{Q}) = H^{2,2}(S^{[2]},\mathbb{Z})$$

and such that a basis of L contains the elements in (3.4.24). Since the Picard group $\mathrm{Pic}(S)$ of S can be primitively embedded in $H^2(S,\mathbb{Z})$, there exists a basis of $H^2(S,\mathbb{Z})$ of the form

$$\{b_1,\dots,b_r,b_{r+1},\dots,b_{22}\}$$

for some $b_{r+1},\dots,b_{22} \in H^2(S,\mathbb{Z})$. By Theorem 3.2.9 the following is a basis of $H^4(S^{[2]},\mathbb{Z})$:

$$\begin{gathered}
\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \qquad \mathfrak{q}_2(b_i)|0\rangle, \qquad \mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)|0\rangle,\\
\tfrac{1}{2}\left(\mathfrak{q}_1(b_i)^2 - \mathfrak{q}_2(b_i)\right)|0\rangle,
\end{gathered} \tag{3.4.25}$$

where $i,j \in \{1,\dots,22\}$ and $i < j$. Recall that by Lemma 3.3.9 we have

$$\delta^2 = -\sum_{i<j}\mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle - \frac{1}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \tag{3.4.26}$$

where $\{\alpha_1,\dots,\alpha_{22}\}$ is the basis of the lattice $H^2(S,\mathbb{Z})$ used in Lemma 3.2.11 and the $\mu_{i,j}$'s are the integers given in Table 3.1. Using the same procedure of Lemma 3.2.11 and Lemma 3.3.9 with the basis $\{b_1,\dots,b_{22}\}$, we obtain

$$\delta^2 = -\sum_{i<j}\sigma_{i,j}\mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)|0\rangle - \frac{1}{2}\sum_i \sigma_{i,i}\mathfrak{q}_1(b_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle \tag{3.4.27}$$

for some integers $\sigma_{i,j}$. Then the $\mu_{i,j}$'s are associated to the basis $\{\alpha_1,\dots,\alpha_{22}\}$ by the description of δ^2 in (3.4.26), and the $\sigma_{i,j}$'s are associated to the basis $\{b_1,\dots,b_{22}\}$ by the description of δ^2 in (3.4.27). We show that there exist positive integers l and k with $l < k$ and $k \ge r+1$ such that $\sigma_{l,k} \ne 0$.

Suppose by contradiction that $\sigma_{i,j} = 0$ for every (i,j) such that $i < j$ and $j \ge r+1$. Then (3.4.27) becomes

$$\delta^2 = -\sum_{1\le i<j\le r}\sigma_{i,j}\mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)|0\rangle - \frac{1}{2}\sum_{i=1}^{22}\sigma_{i,i}\mathfrak{q}_1(b_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle. \tag{3.4.28}$$

Consider the transcendental lattice $T(S)$, which is, by Lemma 2.1.17, the orthogonal to the Néron–Severi group $\mathrm{NS}(S) \cong \mathrm{Pic}(S)$. Let $x \in T(S)$. Since $H^2(S,\mathbb{Z})$ is a unimodular lattice, there exists $y \in H^2(S,\mathbb{Z})$ such that $\int_S xy = 1$. By Proposition 3.1.5 we have

$$\langle \delta^2, xy \rangle = -2. \tag{3.4.29}$$

By (3.4.28), Lemma 3.3.9 and Theorem 3.4.15 we have

$$\begin{aligned} \langle \delta^2, xy \rangle \;=\; & \langle -\sum_{1\le i<j\le r} \sigma_{i,j}\left[b_ib_j - \left(\int_S b_ib_j\right)\left(\frac{1}{8}\delta^2 + \frac{1}{20}q_X^\vee\right)\right] \\ & -\tfrac{1}{2}\sum_{i=1}^{22}\sigma_{i,i}\left[b_i^2 - \left(\int_S b_i^2\right)\left(\frac{1}{8}\delta^2 + \frac{1}{20}q_X^\vee\right)\right] \\ & -\left(\tfrac{1}{8}\delta^2 + \tfrac{1}{20}q_X^\vee\right), xy\rangle. \end{aligned} \tag{3.4.30}$$

Since $x \in T(S)$ we have

$$\int_S b_i x = 0 \qquad \text{for } i = 1, \ldots, r,$$

hence the right-hand side of (3.4.30) is equal to

$$\begin{aligned} & -\sum_{1\le i<j\le r}\sigma_{i,j}\left[\int_S b_ib_j + \frac{1}{4}\int_S b_ib_j - \frac{5}{4}\int_S b_ib_j\right] \\ & -\tfrac{1}{2}\sum_{i=1}^{r}\sigma_{i,i}\left[\int_S b_i^2 + \frac{1}{4}\int_S b_i^2 - \frac{5}{4}\int_S b_i^2\right] \\ & -\tfrac{1}{2}\sum_{i=r+1}^{22}\sigma_{i,i}\left[\int_S b_i^2 + 2\int_S b_ix\int_S b_iy + \frac{1}{4}\int_S b_i^2 - \frac{5}{4}\int_S b_i^2\right]. \end{aligned}$$

Note that

$$\int_S b_ib_j + \frac{1}{4}\int_S b_ib_j - \frac{5}{4}\int_S b_ib_j = 0, \qquad \int_S b_i^2 + \frac{1}{4}\int_S b_i^2 - \frac{5}{4}\int_S b_i^2 = 0,$$

hence we finally obtain

$$\langle \delta^2, xy \rangle = -\sum_{i=r+1}^{22}\sigma_{i,i}\int_S b_ix\int_S b_iy - 1. \tag{3.4.31}$$

Thus (3.4.29) and (3.4.31) imply

$$\sum_{i=r+1}^{22}\sigma_{i,i}\int_S b_ix\int_S b_iy = 1. \tag{3.4.32}$$

The $\sigma_{i,i}$'s are all even, otherwise by Theorem 3.2.9 the element δ^2 in (3.4.27) would not be integral, since a basis of $H^4(S^{[2]},\mathbb{Z})$ is given by (3.4.25). Hence the left-hand side of (3.4.32) is even, so it cannot be equal to 1. We get a contradiction, so there exist l, k positive integers with $l < k$ and $k \ge r+1$ such that $\sigma_{l,k} \neq 0$. Let now L be the sublattice of $H^4(S^{[2]}, \mathbb{Z})$ with the following basis:

$$
\begin{array}{ll}
\text{(i)} & \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \\
\text{(ii)} & \mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)|0\rangle \ \text{ with } 1 \le i < j \le 22 \ \text{ and } (i,j) \neq (l,k), \\
\text{(iii)} & \delta^2 = -\sum_{i<j} \sigma_{i,j}\mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)|0\rangle - \frac{1}{2}\sum_i \sigma_{i,i}\mathfrak{q}_1(b_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle, \\
\text{(iv)} & \mathfrak{q}_2(b_i)|0\rangle \ \text{ for } i = 1, \ldots, 22, \\
\text{(v)} & \frac{1}{2}\left(\mathfrak{q}_1(b_i)^2 - \mathfrak{q}_2(b_i)\right)|0\rangle \ \text{ for } i = 1, \ldots, 22.
\end{array}
\tag{3.4.33}
$$

Recall that by (3.4.27) the element in (iii) is also equal to

$$-\sum_{i<j} \mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle - \frac{1}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle.$$

Since $\sigma_{l,k} \neq 0$, the elements in (3.4.33) give a basis for $H^4(S^{[2]}, \mathbb{Q})$, thus L is a sublattice of $H^4(S^{[2]}, \mathbb{Z})$ of maximal rank. If $\sigma_{l,k} = \pm 1$, then $\mathfrak{q}_1(b_l)\mathfrak{q}_1(b_k)|0\rangle$ can be obtained as an integral linear combination of elements in (3.4.33), hence every element in the basis (3.4.25) of $H^4(S^{[2]}, \mathbb{Z})$ is in L, so $L = H^4(S^{[2]}, \mathbb{Z})$ and (3.4.24) is a basis of $H^{2,2}(S^{[2]}, \mathbb{Z})$. If $\sigma_{l,k} \neq \pm 1$, then $L \neq H^4(S^{[2]}, \mathbb{Z})$. More precisely, we have

$$\frac{H^4(S^{[2]}, \mathbb{Z})}{L} \cong \frac{\mathbb{Z}}{|\sigma_{l,k}|\mathbb{Z}} \tag{3.4.34}$$

generated by $\mathfrak{q}_1(b_l)\mathfrak{q}_1(b_k)|0\rangle$. We show that

$$L \cap H^{2,2}(S^{[2]}, \mathbb{Q}) = H^{2,2}(S^{[2]}, \mathbb{Z}).$$

The inclusion $L \cap H^{2,2}(S^{[2]}, \mathbb{Q}) \subseteq H^{2,2}(S^{[2]}, \mathbb{Z})$ is clear. We now prove the inclusion $L \cap H^{2,2}(S^{[2]}, \mathbb{Q}) \supseteq H^{2,2}(S^{[2]}, \mathbb{Z})$ by showing that if $z \notin L \cap H^{2,2}(S^{[2]}, \mathbb{Q})$ then $z \notin H^{2,2}(S^{[2]}, \mathbb{Z})$. If $z \notin H^{2,2}(S^{[2]}, \mathbb{Q})$ we are done. Suppose now that $z \notin L$. Clearly we have

$$H^4(S^{[2]}, \mathbb{Z}) \cap H^{2,2}(S^{[2]}, \mathbb{Q}) = H^{2,2}(S^{[2]}, \mathbb{Z}).$$

Since the quotient in (3.4.34) is generated by $\mathfrak{q}_1(b_l)\mathfrak{q}_1(b_k)|0\rangle$, it suffices to show that $\mathfrak{q}_1(b_l)\mathfrak{q}_1(b_k)|0\rangle \notin H^{2,2}(S^{[2]}, \mathbb{Q})$ to get the inclusion. Suppose by contradiction that $\mathfrak{q}_1(b_l)\mathfrak{q}_1(b_k)|0\rangle \in H^{2,2}(S^{[2]}, \mathbb{Q})$. Hence $q_1(b_l)\mathfrak{q}_1(b_k)|0\rangle$ is a rational linear combination of elements in (3.4.24). Recall that the last element in (3.4.24) can be written as

$$\delta^2 = -\sum_{i<j} \sigma_{i,j}\mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)|0\rangle - \frac{1}{2}\sum_i \sigma_{i,i}\mathfrak{q}_1(b_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle.$$

Since $\mathfrak{q}_1(b_l)\mathfrak{q}_1(b_k)|0\rangle$ appears only in δ^2 among the elements in (3.4.24), we see that $\mathfrak{q}_1(b_l)\mathfrak{q}_1(b_k)|0\rangle \in H^{2,2}(S^{[2]}, \mathbb{Q})$ only if

$$\sigma_{i,j} = 0 \text{ for } i \le j, j \ge r+1 \text{ and } (i,j) \neq (l,k). \tag{3.4.35}$$

Let again $x \in T(S)$ and $y \in H^2(S,\mathbb{Z})$ such that $\int_S xy = 1$. Similarly to (3.4.31) we have

$$\begin{aligned}\langle \delta^2, xy\rangle &= -\sum_{\substack{i<j\\ j\geq r+1}} \sigma_{i,j}\left(\int_S b_i x\int_S b_j y + \int_S b_j x \int_S b_i y\right)\\ &\quad -\tfrac{1}{2}\sum_{i=r+1}^{22}\sigma_{i,i}\left(2\int_S b_i x\int_S b_i y\right) - 1,\end{aligned} \tag{3.4.36}$$

which implies by (3.4.29) the following:

$$\sum_{\substack{i<j\\ j\geq r+1}} \sigma_{i,j}\left(\int_S b_i x\int_S b_j y + \int_S b_j x\int_S b_i y\right) + \sum_{i=r+1}^{22}\sigma_{i,i}\left(\int_S b_i x\int_S b_i y\right) = 1.$$

Thus we see that (3.4.35) is not true, otherwise (3.4.36) becomes

$$\sigma_{l,k}\left(\int_S b_l x\int_S b_k y + \int_S b_k x\int_S b_l y\right) = 1,$$

which is false since by assumption $\sigma_{l,k} \neq \pm 1$. Hence there exists $(\bar{l},\bar{k}) \neq (l,k)$ with $\bar{l} \leq \bar{k}$ and $\bar{k} \geq r+1$ such that $\sigma_{\bar{l},\bar{k}} \neq 0$. As remarked above, this shows that $\mathfrak{q}_1(b_l)\mathfrak{q}_1(b_k)|0\rangle \notin H^{2,2}(S^{[2]},\mathbb{Q})$. Then

$$L \cap H^{2,2}(S^{[2]},\mathbb{Q}) = H^{2,2}(S^{[2]},\mathbb{Z}).$$

We conclude that (3.4.24) is a basis of $H^{2,2}(S^{[2]},\mathbb{Z})$. To show that (3.4.23) is a basis of $H^{2,2}(S^{[2]},\mathbb{Z})$, we remark that by Lemma 3.3.9 and Theorem 3.4.9 the following equalities hold for elements in the basis (3.4.24):

$$\begin{aligned}&\mathfrak{q}_2(b_i)|0\rangle = b_i\delta,\\ &\mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle = \tfrac{1}{8}\left(\delta^2 + \tfrac{2}{5}q_X^\vee\right),\\ &\tfrac{1}{2}\left(\mathfrak{q}_1(b_i)^2 - \mathfrak{q}_2(b_i)\right)|0\rangle = \tfrac{1}{2}\left(b_i^2 - \int_S b_i^2\left(\frac{1}{8}\delta^2 + \frac{1}{20}q_X^\vee\right) - b_i\delta\right),\\ &\mathfrak{q}_1(b_i)\mathfrak{q}_1(b_j)|0\rangle = b_ib_j - \int_S b_ib_j\left(\frac{1}{8}\delta^2 + \frac{1}{20}q_X^\vee\right),\\ &-\sum_{i<j}\mu_{i,j}\mathfrak{q}_1(\alpha_i)\mathfrak{q}_1(\alpha_j)|0\rangle - \frac{1}{2}\sum_i \mu_{i,i}\mathfrak{q}_1(\alpha_i)^2|0\rangle - \mathfrak{q}_1(1)\mathfrak{q}_1(x)|0\rangle = \delta^2.\end{aligned} \tag{3.4.37}$$

It is now easy to see that every element in (3.4.37) is an integral linear combination of elements in (3.4.23), which all belong to $H^{2,2}(S^{[2]},\mathbb{Z})$. Hence (3.4.23) is a basis of $H^{2,2}(S^{[2]},\mathbb{Z})$, which is odd by Remark 3.4.10. □

3.5 Hodge classes of type $(3,3)$ on Hilbert squares of K3 surfaces

Let S be a projective K3 surface. In this section we study rational and integral Hodge classes of type $(3,3)$ on the Hilbert square $S^{[2]}$.

3.5.1 BBF form on second homology group

Before discussing $H^{3,3}(S^{[2]},\mathbb{Q})$ and $H^{3,3}(S^{[2]},\mathbb{Z})$, where S is a K3 surface, we recall a useful correspondence between primitive elements in $H^2(X,\mathbb{Z})$ and primitive elements in $H_2(X,\mathbb{Z})_f$, where X is an IHS manifold and $H_2(X,\mathbb{Z})_f$ is the torsion free quotient group of the homology group $H_2(X,\mathbb{Z})$. We follow [HT01].

Let X be an IHS manifold. Since $H^2(X,\mathbb{Z})$ is torsion free, the universal coefficient theorem gives an isomorphism

$$h : H^2(X,\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}(H_2(X,\mathbb{Z})_f,\mathbb{Z}), \qquad h([f])([x]) := f(x) \tag{3.5.1}$$

induced by evaluating a cochain on a chain. By [Hat05, Pag.239, Formula (∗)] the isomorphism h can also be described by

$$h(v)(R) = \epsilon(R \cap v), \tag{3.5.2}$$

where $\epsilon : H_0(X,\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$ is the isomorphism given in [Bre13, Theorem IV.2.1]. Then h induces the following isomorphism:

$$h^\vee : \mathrm{Hom}\left(\mathrm{Hom}(H_2(X,\mathbb{Z})_f,\mathbb{Z}),\mathbb{Z}\right) \xrightarrow{\sim} \mathrm{Hom}(H^2(X,\mathbb{Z}),\mathbb{Z}), \qquad h^\vee(g) := g \circ h.$$

Moreover, since $H_2(X,\mathbb{Z})_f$ is free and finitely generated, the following evaluation map gives an isomorphism between $H_2(X,\mathbb{Z})_f$ and its double dual

$$e : H_2(X,\mathbb{Z})_f \xrightarrow{\sim} \mathrm{Hom}\left(\mathrm{Hom}(H_2(X,\mathbb{Z})_f,\mathbb{Z}),\mathbb{Z}\right),$$

where $R \in H_2(X,\mathbb{Z})_f$ is mapped to e_R, described as follows:

$$e_R : \mathrm{Hom}(H_2(X,\mathbb{Z})_f,\mathbb{Z}) \to \mathbb{Z}, \qquad \varphi \mapsto \varphi(R).$$

Then, for every $R \in H_2(X,\mathbb{Z})_f$ we have $h^\vee(e_R) = e_R \circ h$, and the following holds for every $v \in H^2(X,\mathbb{Z})$:

$$\begin{aligned} e_R \circ h(v) \;=\; e_R(h(v)) \;&=\; h(v)(R) \\ &=\; \epsilon(R \cap v), \end{aligned}$$

where the last equality comes from (3.5.2). We conclude that there is an isomorphism

$$\psi : H_2(X,\mathbb{Z})_f \xrightarrow{\sim} \mathrm{Hom}(H^2(X,\mathbb{Z}),\mathbb{Z}), \qquad R \mapsto \{\psi_R : v \mapsto \epsilon(R \cap v)\}. \tag{3.5.3}$$

This shows that $H_2(X,\mathbb{Z})_f$ can be equipped with the BBF form, denoted as usual by $(\cdot\,,\cdot)$, with values in $\mathbb{Q}$ of the dual lattice $\mathrm{Hom}(H^2(X,\mathbb{Z}),\mathbb{Z})$, denoted also by $(H^2(X,\mathbb{Z}))^\vee$. We can now prove the following.

Proposition 3.5.1 (Hassett–Tschinkel). *Let X be an IHS manifold and denote by $H_2(X,\mathbb{Z})_f$ the torsion free quotient group of $H_2(X,\mathbb{Z})$. Then there is a correspondence between primitive elements in $H^2(X,\mathbb{Z})$ and primitive elements in $H_2(X,\mathbb{Z})_f$. In particular:*

(i) For every primitive element $R \in H_2(X,\mathbb{Z})_f$ there exists a unique class $\omega \in H^2(X,\mathbb{Q})$ such that

$$\epsilon(R \cap v) = (\omega, v) \qquad \text{for every } v \in H^2(X,\mathbb{Z}).$$

The primitive $\rho \in H^2(X,\mathbb{Z})$ associated to R is the primitive element such that $c\rho = \omega$ for some $c \in \mathbb{Q}_{>0}$.

(ii) For every primitive element $\rho \in H^2(X,\mathbb{Z})$ of divisibility $\mathrm{div}(\rho) = d$ *in $H^2(X,\mathbb{Z})$, there exists a unique primitive $R \in H_2(X,\mathbb{Z})_f$ such that*

$$d \cdot \epsilon(R \cap v) = (\rho, v) \qquad \textit{for every } v \in H^2(X,\mathbb{Z}).$$

Proof. (i) Let $\psi : H_2(X,\mathbb{Z})_f \xrightarrow{\sim} (H^2(X,\mathbb{Z}))^\vee$ be the isomorphism seen in (3.5.3). Let $\{\alpha_i\}$ be a basis of the lattice $H^2(X,\mathbb{Z})$ obtained by applying Lemma 1.4.9, and denote by $\alpha_i^\vee \in H^2(X,\mathbb{Z})^\vee$ the element $\alpha_i^\vee := (\alpha_i, \cdot)$. Again by Lemma 1.4.9 there is a basis of $(H^2(X,\mathbb{Z}))^\vee$ of the form $\{\frac{1}{\lambda_i}\alpha_i^\vee\}$ for some non-zero integers $\lambda_i \in \mathbb{Z}$. Let $R \in H_2(X,\mathbb{Z})_f$ be a primitive element. Then $\psi_R = \sum_i x_i\alpha_i^\vee$ for some $x_i \in \mathbb{Q}$. We show that the primitive $\omega \in H^2(X,\mathbb{Q})$ is $\omega := \sum_i x_i\alpha_i$. Indeed:

$$\begin{aligned}(\omega, v) &= (\textstyle\sum_i x_i\alpha_i, v) &&= \textstyle\sum_i x_i\alpha_i^\vee(v)\\ &&&= \psi_R(v)\\ &&&= \epsilon(R \cap v)\end{aligned}$$

for every $v \in H^2(X,\mathbb{Z})$. Note that the isomorphism (3.5.3) gives the uniqueness of such an $\omega \in H^2(X,\mathbb{Q})$. Then $\rho \in H^2(X,\mathbb{Z})$ is the primitive element such that $c\rho = \omega$ for a certain $c \in \mathbb{Q}_{>0}$.

(ii) Let $\rho \in H^2(X,\mathbb{Z})$ be primitive with $\mathrm{div}(\rho) = d$. Then

$$\frac{1}{d}\rho \in (H^2(X,\mathbb{Z}))^\vee$$

is primitive. We take

$$R := (h^\vee \circ e)^{-1}\left(\frac{1}{d}\rho\right) \in H_2(X,\mathbb{Z})_f.$$

Then $R \in H_2(X,\mathbb{Z})_f$ is primitive by construction and by (i) we have

$$\epsilon(R \cap v) = \frac{1}{d}(v, \rho)$$

for every $v \in H^2(X,\ Z)$. Hence

$$d \cdot \epsilon(R \cap v) = (v, \rho) \qquad \text{for every } v \in H^2(X,\mathbb{Z}).$$

□

Example 3.5.2. Let S be a K3 surface and $X := S^{[n]}$ be the Hilbert scheme of n points on S. By Theorem 2.2.9 and by the universal coefficient theorem, $H_2(X,\mathbb{Z})$ is free. Let $2\delta \in H^2(X,\mathbb{Z})$ be the class of the exceptional divisor of the Hilbert–Chow morphism. We have seen above that $H_2(X,\mathbb{Z})$ can be identified with the dual $(H^2(X,\mathbb{Z}))^\vee$, so we can embed it in $H^2(X,\mathbb{Q})$. We show that the primitive element $\delta^\vee$ in $H_2(X,\mathbb{Z})$ associated to δ, seen as an element in $H^2(X,\mathbb{Q}) \cong H^2(X,\mathbb{Z}) \otimes \mathbb{Q}$, is

$$\delta^\vee = \frac{1}{2(n-1)}\delta, \qquad (\delta^\vee, \delta^\vee) = -\frac{1}{2(n-1)}\,.$$

Using notations of Proposition 3.5.1, we have $d = 2(n-1)$ and

$$2(n-1)\delta^\vee \cap v = (\delta, v)$$

for every $v \in H^2(X,\mathbb{Z})$. In particular, taking $v = \delta$, since $(\delta,\delta) = -2(n-1)$, we obtain $\delta^\vee \cap \delta = -1$. Now, we know that there exists a unique $\omega \in H^2(X,\mathbb{Q})$ such that $\delta^\vee \cap v = (\omega, v)$ for every $v \in H^2(X,\mathbb{Z})$. In particular $\delta^\vee \cap \delta = (\omega,\delta) = -1$. Moreover, $(x,\delta) = 0$ for every $x \in H^2(S,\mathbb{Z})$, hence $\omega = \alpha\delta$ for some positive α. Since $(\alpha\delta,\delta) = -1$ we obtain $\alpha = \frac{1}{2(n-1)}$.

Example 3.5.3. Let S be a generic K3 surface and let $X := S^{[2]}$ be its Hilbert square. We denote by h both the ample generator of $\mathrm{Pic}(S)$ and the line bundle induced on X. We have $h^2 = 2t$ for some $t \geq 1$. We show that $h^\vee = h$ as elements in $H^2(X,\mathbb{Q})$, similarly to Example 3.5.2.
Let $\{\alpha_1,\dots,\alpha_{22}\}$ be the basis of $H^2(S,\mathbb{Z})$ used in Lemma 3.2.11. Then by Theorem 1.4.10 we have that $\mathrm{Pic}(S)$ can be primitively embedded in $H^2(S,\mathbb{Z})$ in a unique way up to isometries, and we can suppose that such an embedding identifies h with $\alpha_{17} + t\alpha_{18}$. Clearly

$$\{\alpha_1,\dots,\alpha_{16}, h, \alpha_{18},\dots,\alpha_{22}\}$$

is another basis of $H^2(S,\mathbb{Z})$. Since $(h,\alpha_{18}) = 1$, we have $\mathrm{div}(h) = 1$. Let

$$h^\vee = a_1\alpha_1 + \dots, a_{16}\alpha_{16} + a_{17}h + \dots, +a_{22}\alpha_{22} + a_{23}\delta,$$

with $a_i \in \mathbb{Q}$. Since $\mathrm{div}(h) = 1$, we have $h^\vee \cap v = (v,h)$ for every $v \in H^2(X,\mathbb{Z})$.

- If $v = \alpha_{18}$, we obtain $h^\vee \cap \alpha_{18} = a_{17}$ and $(\alpha_{18},h) = 1$, hence $a_{17} = 1$.
- If $v = h$, we obtain $h^\vee \cap h = a_{17}\cdot 2t + a_{18}$ and $(h,h) = 2t$. Since $a_{17} = 1$, we get $a_{18} = 0$.
- If $v = \delta$, we obtain $h^\vee \cap \delta = -2a_{23}$ and $(\delta,h) = 0$, hence $a_{23} = 0$.
- If $v = x_i$ for $i = 1,\dots,16,19,\dots,22$, the system obtained by imposing $h^\vee \cap \alpha_i = (h,\alpha_i) = 0$ gives $\alpha_i = 0$.

We conclude that $h^\vee = h$.

Remark 3.5.4. Taking the dual of a primitive element of $H^2(X,\mathbb{Z})$ is not linear, i.e., if $\sum_i a_i x_i \in H^2(X,\mathbb{Z})$ is primitive, with $a_i \in \mathbb{Z}$, $x_i \in H^2(X,\mathbb{Z})$, then $(\sum_i a_i x_i)^\vee \neq \sum_i a_i x_i^\vee$. For example, if $X = S_4^{[2]}$ is the Hilbert square of a smooth quartic surface $S_4 \subset \mathbb{P}^3$, then a simple computation shows that $(h-\delta)^\vee = h^\vee - 2\delta^\vee \in H_2(X,\mathbb{Z})$.

Recall that $H_2(X,\mathbb{Z})_f$ and $H_2(X,\mathbb{Q})$ are Hodge structures of weight -2 by Example 2.1.12, (iii). Similarly to the case of cup product of Proposition 3.1.2 one can show that the cap product

$$H^2(X,\mathbb{Z}) \otimes H_2(X,\mathbb{Z})_f \to H_0(X,\mathbb{Z})$$

is a morphism of Hodge structures of weight 0, see [PS08] for details. This shows that, if $R \in H_2(X,\mathbb{Z})_f$ is primitive of type $(-1,-1)$, the corresponding

$\rho \in H^2(X, \mathbb{Z})$ is primitive of type $(1,1)$, and viceversa. Similarly, the Poincaré duality

$$PD : H^6(X, \mathbb{Z}) \xrightarrow{\sim} H_2(X, \mathbb{Z})_f$$

is an isomorphism of Hodge structures of weight -4. Thus Proposition 3.5.1 implies the following.

Corollary 3.5.5. *Let S be a K3 surface. Suppose that $\{\alpha_1, \ldots, \alpha_r\}$ is a basis of* $\mathrm{Pic}(S^{[2]}) \cong \mathrm{NS}(S^{[2]})$*. Then a basis of the $\mathbb{Z}$-module $H^{3,3}(S^{[2]}, \mathbb{Z})$ is given by*

$$\{cl(\alpha_1^\vee), \ldots, cl(\alpha_r^\vee)\},$$

where $\alpha_i^\vee \in H_2(S^{[2]}, \mathbb{Z})$ is the primitive element associated to $\alpha_i \in H^2(S^{[2]}, \mathbb{Z})$ by Proposition 3.5.1*, and $cl(\alpha_i^\vee)$ is the inverse of the Poincaré dual of $\alpha_i^\vee$. Moreover, this is a basis also for the $\mathbb{Q}$-vector space $H^{3,3}(S^{[2]}, \mathbb{Q})$.*

Proof. First of all, by Theorem 2.2.9 and by the universal coefficient theorem, $H_2(S^{[2]}, \mathbb{Z})$ is free. Let $\{\alpha_1, \ldots, \alpha_r\}$ be a basis of $\mathrm{Pic}(S^{[2]})$. As remarked above, the correspondence of Proposition 3.5.1 is such that $\alpha_i^\vee \in H_2(S^{[2]}, \mathbb{Z})$ is of type $(-1,-1)$, and viceversa every primitive element of $H_2(S^{[2]}, \mathbb{Z})$ of type $(-1,-1)$ is associated to an element of type $(1,1)$ in $H^2(S^{[2]}, \mathbb{Z})$. Then $\{\alpha_1^\vee, \ldots, \alpha_r^\vee\}$ is a basis of

$$H_{-1,-1}(S^{[2]}, \mathbb{Z}) := H_2(S^{[2]}, \mathbb{Z}) \cap H_{-1,-1}(S^{[2]}),$$

where $H_{-1,-1}(S^{[2]})$ is the component of type $(-1,-1)$ of $H_2(S^{[2]}, \mathbb{C})$. Moreover, as remarked above, the Poincaré duality

$$PD : H^6(S^{[2]}, \mathbb{Z}) \xrightarrow{\sim} H_2(S^{[2]}, \mathbb{Z})$$

is an isomorphism of Hodge structures of weight -4, so we have

$$H^{3,3}(S^{[2]}, \mathbb{Z}) \xrightarrow{\sim} H_{-1,-1}(S^{[2]}, \mathbb{Z}).$$

We conclude that $\{cl(\alpha_1^\vee), \ldots, cl(\alpha_r^\vee)\}$ is a basis of $H^{3,3}(S^{[2]}, \mathbb{Z})$. Clearly this is a basis also for the $\mathbb{Q}$-vector space $H^{3,3}(S^{[2]}, \mathbb{Q})$, since $H^{3,3}(S^{[2]}, \mathbb{Z})$ is free. □

3.5.2 The case of Hilbert squares of generic K3 surfaces

Corollary 3.5.5 describes the space $H^{3,3}(S^{[2]}, \mathbb{Q})$ of rational Hodge classes of type $(3,3)$ and the lattice $H^{3,3}(S^{[2]}, \mathbb{Z})$ of integral Hodge classes of type $(3,3)$ on the Hilbert square of a K3 surface S. If S is a generic K3 surface, we can be more precise.

Theorem 3.5.6. *Let S be a generic K3 surface of degree $2t$, and $h \in \mathrm{Pic}(S^{[2]})$ be the class induced by the ample generator of* $\mathrm{Pic}(S)$*. Then:*

(i) $H^{1,1}(S^{[2]}, \mathbb{Z}) = \mathbb{Z}h \oplus \mathbb{Z}\delta$, where $\delta \in \mathrm{Pic}(S^{[2]})$ is as usual the line bundle such that 2δ is the class of the exceptional divisor of the Hilbert–Chow morphism $S^{[2]} \to S^{(2)}$.

(ii) Let $h^\vee, \delta^\vee \in H_2(S^{[2]}, \mathbb{Z})$ be the primitive classes in the second homology group which correspond to $h, \delta \in H^2(S^{[2]}, \mathbb{Z})$ by Proposition 3.5.1*, and let*

$$h_6^\vee, \delta_6^\vee \in H^6(S^{[2]}, \mathbb{Z})$$

be $PD^{-1}(h^\vee)$ and $PD^{-1}(\delta^\vee)$ respectively. Then

$$H^{3,3}(S^{[2]},\mathbb{Z}) \cong \mathbb{Z}h_6^\vee \oplus \mathbb{Z}\delta_6^\vee.$$

Moreover,

$$h_6^\vee = \frac{1}{6t}h^3, \qquad \delta_6^\vee = \frac{1}{4t}h^2\delta$$

and the following equalities hold in $H^{3,3}(S^{[2]},\mathbb{Z})$:

$$\delta^3 = -\frac{3}{t}h^2\delta, \qquad h\delta^2 = -\frac{1}{3t}h^3, \qquad q_X^\vee h = \frac{25}{6t}h^3, \qquad q_X^\vee\delta = \frac{25}{2t}h^2\delta.$$

Proof. (i) is obvious by assumption. We now show (ii). The cup product with h^2 gives an isomorphism between $H^{1,1}(S^{[2]},\mathbb{Q})$ and $H^{3,3}(S^{[2]},\mathbb{Q})$. Then $H^{3,3}(S^{[2]},\mathbb{Q})$ is 2-dimensional and it is generated by h^3 and $h^2\delta$. We now represent δ^3, $h\delta^2, q_X^\vee h$ and $q_X^\vee\delta$ in terms of this basis. If $\delta^3 = xh^3 + yh^2\delta$ with $x, y \in \mathbb{Q}$, using Proposition 3.1.5 we have

$$\begin{cases} h\delta^3 = 0 = x\langle h^2,h^2\rangle + y\langle h^2,h\delta\rangle = 12t^2x, \\ \delta^4 = 3\cdot(-2)^2 = x\langle h^2,h\delta\rangle + y\langle h^2,\delta^2\rangle = -4ty, \end{cases}$$

which gives $x = 0$ and $y = -\frac{3}{t}$, hence

$$\delta^3 = -\frac{3}{t}h^2\delta. \tag{3.5.4}$$

Similarly we obtain

$$h\delta^2 = -\frac{1}{3t}h^3, \qquad q_X^\vee h = \frac{25}{6t}h^3, \qquad q_X^\vee\delta = \frac{25}{2t}h^2\delta. \tag{3.5.5}$$

Consider now $H^{3,3}(S^{[2]},\mathbb{Z})$. Since by Theorem 2.2.9 the cohomology group $H^6(S^{[2]},\mathbb{Z})$ is torsion free, and $H^{3,3}(S^{[2]},\mathbb{Q})$ has dimension 2 as already seen above, we have $\mathrm{rk}(H^{3,3}(S^{[2]},\mathbb{Z})) = 2$. Let $h^\vee, \delta^\vee \in H_2(S^{[2]},\mathbb{Z})$ be the primitive classes which correspond, by Proposition 3.5.1, respectively to the primitive classes $h, \delta \in H^2(S^{[2]},\mathbb{Z})$. By Example 3.5.2 and Example 3.5.3 we have $h^\vee = h$ and $\delta^\vee = \frac{\delta}{2}$. Then $h_6^\vee := PD^{-1}(h^\vee)$ and $\delta_6^\vee := PD^{-1}(\delta^\vee)$ give a basis of $H^{3,3}(S^{[2]},\mathbb{Z})$ by Corollary 3.5.5, so we have

$$H^{3,3}(S^{[2]},\mathbb{Z}) \cong \mathbb{Z}h_6^\vee \oplus \mathbb{Z}\delta_6^\vee.$$

Moreover, if $\cdot$ denotes the cup product and $\cap$ the cap product, we have

$$\int_{S^{[2]}} h_6^\vee \cdot x = \epsilon(h^\vee \cap x), \qquad \int_{S^{[2]}} \delta_6^\vee \cdot x = \epsilon(\delta^\vee \cap x) \tag{3.5.6}$$

for every $x \in H^2(S^{[2]},\mathbb{Z})$, see [Hat05, p.249]. By Proposition 3.5.1 we have

$$\epsilon(h^\vee \cap h) = (h,h) = 2t, \qquad \epsilon(h^\vee \cap \delta) = (h,\delta) = 0. \tag{3.5.7}$$

If we write $h_6^\vee = \alpha h^3 + \beta h^2\delta$ for some $\alpha, \beta \in \mathbb{Q}$, by Proposition 3.1.5 we obtain

$$\begin{aligned} \int_{S^{[2]}} h_6^\vee \cdot h &= \alpha\langle h^2,h^2\rangle + \beta\langle h^2,h\delta\rangle = 12\alpha t^2, \\ \int_{S^{[2]}} h_6^\vee \cdot \delta &= \alpha\langle h^2,h\delta\rangle + \beta\langle h^2,\delta^2\rangle = -4\beta t. \end{aligned} \tag{3.5.8}$$

Then (3.5.6), (3.5.7) and (3.5.8) give $\alpha = \frac{1}{6t}$ and $\beta = 0$, hence

$$h_6^\vee = \frac{1}{6t} h^3.$$

Similarly we obtain

$$\delta_6^\vee = \frac{1}{4t} h^2 \delta. \tag{3.5.9}$$

and we are done □

Note that Proposition 3.5.1 and Theorem 3.5.6 give two equivalent methods to compute the intersection product between an element $x \in H^{1,1}(S^{[2]}, \mathbb{Z})$ and an element $y \in H_{-1,-1}(S^{[2]}, \mathbb{Z})$, where S is a generic K3 surface. If we use Proposition 3.5.1, we see x and y as elements in $H^2(S^{[2]}, \mathbb{Q})$, and we compute the intersection product as (x, y), where $(\cdot\,,\,\cdot)$ denotes the BBF bilinear form extended $\mathbb{Q}$-bilinearly to $H^2(S^{[2]}, \mathbb{Q})$, otherwise we take $y_6^\vee \in H^{3,3}(S^{[2]}, \mathbb{Z})$, the element which correspond to y by Theorem 3.5.6, (ii), and we compute the product $\langle x, y_6^\vee \rangle$ using the bilinear form $\langle \cdot\,,\,\cdot \rangle$ of Proposition 3.1.5. In the second method, if $y_6^\vee = \alpha \frac{1}{6t} h^3 + \beta \frac{1}{4t} h^2 \delta$ for some $\alpha, \beta \in \mathbb{Z}$, then

$$\langle x, y_6^\vee \rangle = \frac{\alpha}{6t} \langle xh, h^2 \rangle + \frac{\beta}{4t} \langle xh, h\delta \rangle.$$

Theorem 3.5.6 will be used in the proof of Proposition 5.1.3 in the analysis of the rational map induced by the complete linear system associated to an ample divisor D of a smooth birational model X of the Hilbert square $S^{[2]}$ of a generic K3 surface S, with $q_X(D) = 2$.

Chapter 4

Complete linear systems on IHS manifolds

In this chapter we begin to study the main problem of the second part of this thesis. Let X be the Hilbert square of a generic K3 surface S_{2t} of degree $2t$, and suppose that there exists an ample divisor $D \in \mathrm{Div}(X)$ with $q_X(D) = 2$. We want to describe geometrically the map induced by $|D|$, the complete linear system associated to D. This is a generalization of a problem studied by Saint–Donat in [SD74]: if S is a K3 surface which admits an ample divisor D with $D^2 = 2$ with respect to the intersection product, then the complete linear system $|D|$ is basepoint free and the morphism that it induces is a double cover of $\mathbb{P}^2$ ramified on a sextic curve.

In Section 4.1 we recall an application of the Bayer–Macrì theorem which describes the nef cone and the movable cone of the Hilbert square of a generic K3 surface S_{2t} of degree $2t$. In Section 4.2 we show that $S_{2t}^{[2]}$ and all its smooth birational models are Mori Dream Spaces. In Section 4.3 we recall the descriptions of the group $\mathrm{Aut}(X)$ of regular automorphisms on X and of the group $\mathrm{Bir}(X)$ of birational automorphisms on X, where $X = S_{2t}^{[2]}$, obtained respectively in [BCNWS16] and [DM19], and we prepare the setting of the main problem. In Section 4.4 we describe what happens in the cases $t = 2$ and $t = 5$, already analysed respectively by Beauville and O'Grady. The case $t = 5$ is interesting, since there is a big and nef divisor D with $q_X(D) = 2$ which is not ample. In Section 4.5, starting from a result in [CGM19], we compute the fundamental cohomological class in $H^{2,2}(X, \mathbb{Z})$ of the fixed locus $\mathrm{Fix}(\iota)$, where X is the Hilbert square of a generic K3 surface which admits an ample divisor D with $q_X(D) = 2$ and ι is the anti-symplectic involution which fixes the class of D and generates $\mathrm{Aut}(X)$. We also show that the map induced by the complete linear system $|D|$ factors through the quotient $\pi : X \to X/\langle\iota\rangle$. In Section 4.6 we first recall a result by Rieß, which characterises the divisorial base component of the complete linear system $|D|$ given by a big and nef divisor on an IHS manifold of $K3^{[n]}$-type. We conclude by proving Theorem 4.6.5, which says that in our case, if $t \neq 2$, then the surface $D_1 \cap D_2$ is reduced and irreducible for every $D_1, D_2 \in |D|$ distinct divisors. This will be crucial for Chapter 5. We conclude with Section 4.6.3, where we show that, except for this last theorem, the results of this section hold also when X is a smooth birational model of the Hilbert square of a generic K3 surface S_{2t} such that X admits an ample divisor D with $q_X(D) = 2$.

4.1 Bayer–Macrì theorem for IHS fourfolds

As in Chapter 3, a *generic K3 surface of degree* $2t$ will be a projective K3 surface S_{2t} with $\mathrm{Pic}(S_{2t}) = \mathbb{Z}H$ and $H^2 = 2t$, $t \geq 1$. The following fundamental theorem by Bayer and Macrì describes the movable cone, the nef cone and the pseudoeffective cone of the Hilbert square $S_{2t}^{[2]}$ of a generic K3 surface S_{2t}, see [BM14, Proposition 13.1].

Theorem 4.1.1 (Bayer–Macrì)**.** *Let S_{2t} be a generic K3 surface, $H \in \mathrm{Pic}(S_{2t})$ be the ample generator of $\mathrm{Pic}(S_{2t})$, where $H^2 = 2t$, $t \geq 1$, and let $h \in \mathrm{Pic}(S_{2t}^{[2]})$ be the class induced by H. Then the cones of classes of divisors on $X := S_{2t}^{[2]}$ can be described as follows:*

1. *The extremal rays of the closure of the movable cone $\mathrm{Mov}(X)$ are spanned by h and $h - \mu_t\delta$, where:*
   - *if t is a perfect square, $\mu_t = \sqrt{t}$;*
   - *if t is not a perfect square and (c, d) is the minimal solution of the Pell equation $P_t(1)$, then $\mu_t = t \cdot \frac{d}{c}$.*
2. *The extremal rays of the nef cone $\mathrm{Nef}(X)$ are spanned by h and $h - \nu_t\delta$, where:*
   - *if the equation $P_{4t}(5)$ is not solvable, $\nu_t = \mu_t$;*
   - *if the equation $P_{4t}(5)$ is solvable and (a_5, b_5) is its minimal solution, $\nu_t = 2t \cdot \frac{b_5}{a_5}$.*
3. *The extremal rays of the pseudoeffective cone $\overline{\mathrm{Eff}}(X)$ are spanned by δ and $h - \omega_t\delta$, where:*
   - *if t is a perfect square, $\omega_t = \sqrt{t}$;*
   - *if t is not a perfect square, $\omega_t = \frac{c}{d}$, where (c, d) is the minimal solution of $P_t(1)$.*

Note that the theorem implies that the closure of the movable cone $\overline{\mathrm{Mov}(X)}$ is the dual of the pseudoeffective cone $\overline{\mathrm{Eff}}(X)$ with respect to the BBF form. This holds for every IHS manifold, as shown in the following proposition.

Proposition 4.1.2 (Proposition 4.4 in [Bou04])**.** *Let X be an IHS manifold. Then $(\overline{\mathrm{Eff}}(X))^\vee = \overline{\mathrm{Mov}(X)}$, where the dual is taken with respect to the BBF form.*

Proof. Suppose that $x \in (\overline{\mathrm{Eff}}(X))^\vee$, so $(x, y) \geq 0$ for every $y \in \overline{\mathrm{Eff}}(X)$. In particular $(x, y) \geq 0$ for every y which is a class of a uniruled divisor, hence by Proposition 2.2.18 we have $x \in \overline{\mathcal{BK}}_X \cap \mathrm{Pic}(X)_\mathbb{R}$, where $\overline{\mathcal{BK}}_X$ is the closure of the birational Kähler cone of X. Moreover, by Corollary 2.2.21 we have $\overline{\mathrm{Mov}(X)} = \overline{\mathcal{BK}}_X \cap \mathrm{Pic}(X)_\mathbb{R}$, hence $(\overline{\mathrm{Eff}}(X))^\vee \subseteq \overline{\mathrm{Mov}(X)}$.

Let now $x \in \overline{\mathrm{Mov}(X)}$. We show that $(x, y) \geq 0$ for every $y \in \overline{\mathrm{Eff}}(X)$. Applying Proposition 2.3.1 we can write

$$y = y_1 + y_2 \in H^{1,1}(X, \mathbb{R}),$$

where $y_1 \in \overline{\mathrm{Mov}(X)}$ and y_2 is the class of a uniruled divisor. Then $(x, y_2) \geq 0$ by Proposition 2.2.18 and Corollary 2.2.21. If we show that $(x, y_1) \geq 0$ we are done. Suppose first that at least one between x and y_1 is big, so in the interior part of $\overline{\mathrm{Eff}}(X)$. Without loss of generality we can assume that x is big. By Proposition 1.1.25 we have $H^0(X, mx) \neq 0$ for $m \gg 0$, so by Corollary 2.2.19 we have $(mx, y_1) \geq 0$, which implies $(x, y_1) \geq 0$. Suppose now that both x and y_1 are not big. Since they are both in $\overline{\mathrm{Mov}(X)}$, they must be nef for some smooth birational model of X. Hence by Theorem 1.1.27 and Theorem 2.2.4 we have $q_X(x) = q_X(y_1) = 0$. Thus,

$$\begin{aligned} q_X(x + y_1) &= q_X(x) + q_X(y_1) + 2(x, y_1) \\ &= 2(x, y_1), \end{aligned}$$

and $q_X(x + y_1) \geq 0$ since $x + y_1 \in \overline{\mathrm{Mov}(X)}$, which is contained in the closure of the positive cone $\overline{\mathcal{C}}_X$. Then $(x, y_1) \geq 0$. We conclude that $(x, y) \geq 0$ for every $y \in \overline{\mathrm{Eff}}(X)$, so $x \in (\overline{\mathrm{Eff}}(X))^\vee$. □

4.2 Hilbert squares of K3 surfaces and Mori dream spaces

In this section we show that if X is the Hilbert square of a generic K3 surface, then X is a *Mori dream space*. We recall the definition of *small $\mathbb{Q}$-factorial modification*, see [HK00, Definition 1.8], and the definition of *Mori dream space*, see [HK00, Definition 1.10]. Recall that a normal variety X is *$\mathbb{Q}$-factorial* if for every Weil divisor $D \in \mathrm{WDiv}(X)$ there exists $n \in \mathbb{Z}_{>0}$ such that $nD \in \mathrm{Div}(X)$ is a Cartier divisor.

Definition 4.2.1. Let X be a normal and $\mathbb{Q}$-factorial projective variety. A *small $\mathbb{Q}$-factorial modification* of X is a birational map $g : X \dashrightarrow \tilde{X}$, where $\tilde{X}$ is normal, projective and $\mathbb{Q}$-factorial, and g is an isomorphism in codimension 1.

Definition 4.2.2. Let X be a normal and $\mathbb{Q}$-factorial projective variety. We say that X is a *Mori dream space* if the following properties hold:

(i) The Picard group $\mathrm{Pic}(X)$ is finitely generated, equivalently, $h^1(\mathcal{O}_X) = 0$;

(ii) The nef cone $\mathrm{Nef}(X) \subseteq H^{1,1}(X, \mathbb{R})$ is generated by the classes of finitely many semiample divisors, see Definition 1.1.21;

(iii) There is a finite collection of small $\mathbb{Q}$-factorial modifications $g_i : X \dashrightarrow X_i$, for $i = 1, \ldots, r$, such that every X_i satisfies condition (ii), and

$$\mathrm{Mov}(X) = \bigcup_{i=1}^{r} g_i^*(\mathrm{Nef}(X_i)).$$

We can state the following result.

Theorem 4.2.3. *Let $X = S_{2t}^{[2]}$, where S_{2t} is a generic K3 surface. Then X is a Mori dream space. The same holds for any IHS manifold X' birational to X. Moreover,* Mov(X) *is closed.*

Proof. Condition (i) of Definition 4.2.2 is satisfied by the Hilbert square of a K3 surface by definition of IHS manifold. We now show that condition (ii) is verified. By Theorem 2.2.22, the closure of $\mathrm{Mov}(X)$ is divided into chambers which represent the nef cones of smooth birational models of X. Thus, if we show that every element in $\overline{\mathrm{Mov}(X)}$ is semiample for some birational model of X, then $\mathrm{Nef}(X')$, which has two extremal rays, is in particular generated by the classes of finitely many semiample divisors for every smooth birational model X' of X, which is (ii). If $x \in \overline{\mathrm{Mov}(X)}$ is in the interior part of $\overline{\mathrm{Mov}(X)}$, then the class x is ample for some birational model X' of X, so nx is very ample for $n \gg 0$ if seen as a class of X', hence $x \in \mathrm{Nef}(X')$ is semiample. Consider now a class in the boundary of a chamber of $\overline{\mathrm{Mov}(X)}$. Suppose that this class is on the extremal ray of $\overline{\mathrm{Mov}(X)}$ generated by $h \in \mathrm{Pic}(X)$, the line bundle induced by the ample generator of $\mathrm{Pic}(S_{2t})$. Since by Lemma 2.3.2 we have that h is basepoint free, hence semiample, every class on this extremal ray is semiample for the IHS manifold X.

If t is a perfect square, by the Bayer–Macrì theorem the other extremal ray of $\overline{\mathrm{Mov}(X)}$ is generated by a primitive element $n \in \mathrm{Pic}(X)$ with $q_X(n) = 0$. This element n is basepoint free, hence semiample, by Theorem 2.3.3. In particular $\mathrm{Mov}(X) = \overline{\mathrm{Mov}(X)}$ is closed. In order to show that elements in the interior of $\mathrm{Mov}(X)$ are semiample, one proceeds exactly as in the next case, so we now suppose that t is not a perfect square.

If t is not a perfect square, by Bayer–Macrì theorem $\overline{\mathrm{Mov}(X)}$ is strictly contained in the interior of the positive cone, i.e., for every $l \in \overline{\mathrm{Mov}(X)}$ we have $q_X(l) > 0$. In particular this holds for the primitive classes which generate the extremal rays of the chambers of $\overline{\mathrm{Mov}(X)}$. Each chamber represents the nef cone of a birational model of X, hence these primitive classes are nef for some birational model. Since both the top-self intersection and the BBF quadratic form of a nef class D are non-negative, see for instance Proposition 2.2.13, Corollary 2.2.14 and Corollary 2.2.15, by Theorem 2.2.4 we have that $D^4 > 0$ if and only if $q_X(D) > 0$. Hence by Theorem 1.1.27 we have that D is big. Thus the extremal rays of the chambers of $\overline{\mathrm{Mov}(X)}$ are generated by big and nef primitive classes for a birational model of X. We conclude that these are semiample for a birational model of X by Theorem 1.1.28. This shows that condition (ii) is verified for every birational model of X. Note that also in this case $\mathrm{Mov}(X) = \overline{\mathrm{Mov}(X)}$ is closed. Now Theorem 2.2.22 shows that condition (iii) is verified. This concludes the proof. □

4.3 Regular and birational automorphisms on $S_{2t}^{[2]}$

Let $X = S_{2t}^{[2]}$ be the Hilbert square of a generic K3 surface S_{2t}. We recall the description of the group of biregular automorphisms $\mathrm{Aut}(X)$ and of the group of birational automorphisms $\mathrm{Bir}(X)$ given respectively in [BCNWS16] and [DM19].

Theorem 4.3.1 (Proposition 4.3, Proposition 5.1, Lemma 5.3, Theorem 5.5 in [BCNWS16])**.** *Let S_{2t} be a generic K3 surface of degree $2t$, with $t \geq 1$. Let $h \in \mathrm{Pic}(S_{2t}^{[2]})$ be the line bundle induced by the ample generator of* $\mathrm{Pic}(S_{2t})$.

1. *If $t = 1$, then S_2 is the double cover of $\mathbb{P}^2$ branched along a smooth sextic curve and if ι is the covering involution then*

$$\mathrm{Aut}(S_2^{[2]}) = \{\mathrm{id}_{S_2^{[2]}}, \iota^{[2]}\} \cong \mathbb{Z}/2\mathbb{Z}.$$

2. *If $t \geq 2$, then $X := S_{2t}^{[2]}$ admits a non-trivial automorphism if and only if one of the following equivalent conditions is satisfied:*
   - *t is not a square, the Pell-type equation $P_{4t}(5)$ has no solution and the negative Pell equation $P_t(-1)$ has a solution.*
   - *There exists an ample class $D \in \mathrm{NS}(S_{2t}^{[2]})$ such that $q_X(D) = 2$.*

   Moreover, if this is the case, the class D is unique, the automorphism ι is unique and it is a non-natural anti-symplectic involution. Its action on $\mathrm{NS}(S_{2t}^{[2]})$ is the reflection in the span of the class D of square 2 represented in the basis $\{h, -\delta\}$ by the matrix

$$\begin{pmatrix} c & -d \\ td & -c \end{pmatrix}, \tag{4.3.1}$$

   i.e.,

$$\iota^*(xh - y\delta) = (cx - dy)h - (tdx - cy)\delta,$$

   where (c, d) is the minimal solution of the Pell equation $P_t(1)$,

Theorem 4.3.2 (Proposition B.3 in [DM19])**.** *Let S_{2t} be a generic K3 surface with $\mathrm{Pic}(S_{2t}) = \mathbb{Z}H$ and $H^2 = 2t$, $t \geq 1$. Let $h \in \mathrm{Pic}(S_{2t}^{[2]})$ be the line bundle induced by H. Then the group $\mathrm{Aut}(S_{2t}^{[2]})$ is trivial and the group $\mathrm{Bir}(S_{2t}^{[2]})$ is not trivial if and only if $t > 1$, and either $t = 5$ or $5 \nmid t$, and both equations $P_t(-1)$ and $P_{4t}(5)$ are solvable, in which case $\mathrm{Bir}(S_{2t}^{[2]}) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by a non-natural non-regular anti-symplectic involution ι. Moreover, the action of ι on $\mathrm{NS}(S_{2t}^{[2]})$ is a non-trivial isometry, in particular it is the reflection in the span of the class D with $q_X(D) = 2$ represented in the basis $\{h, -\delta\}$ by the matrix in* (4.3.1)*, where (c, d) is the minimal solution of the Pell equation $P_t(1)$.*

Remark 4.3.3. Let S_{2t} be a generic K3 surface with $\mathrm{Pic}(S_{2t}) \cong \mathbb{Z}H$, $H^2 = 2t$, and let $X := S_{2t}^{[2]}$. Suppose that there exists a non natural $\iota \in \mathrm{Bir}(X)$. Then the action of ι on $\mathrm{NS}(X)$ preserves the closure of the movable cone $\overline{\mathrm{Mov}(X)}$, exchanging the two extremal rays. Moreover, the class D of Theorem 4.3.2 is the unique class in the moving cone of X with $q_X(D) = 2$. For further details, see [Mar11, Lemma 6.22] and [BC20]

Let $X = S_{2t}^{[2]}$ be the Hilbert square of a generic K3 surface of degree $2t$ with $t > 1$, and suppose that the Pell-type equation $P_t(-1)$ is solvable, so that there exists a unique divisor $D \in \mathrm{Div}(X)$ with $q_X(D) = 2$ in the moving cone of X. We now briefly discuss three possible cases that one can study, depending on the solvability of the Pell-type equation $P_{4t}(5)$. See [DM19, Appendix B], for more details and [BC20] for a discussion on $\mathrm{Aut}(S_{2t}^{[n]})$ and $\mathrm{Bir}(S_{2t}^{[n]})$, where $S_{2t}^{[n]}$ is a generic K3 surface and $n \geq 2$. As above, we denote by $h \in \mathrm{Pic}(X)$ the line bundle induced by the ample generator of $\mathrm{Pic}(S_{2t})$. For a divisor $D \in \mathrm{Div}(X)$, we will denote by D also its class in $\mathrm{Pic}(X)$.

- **Case 1.** Suppose that $P_t(-1)$ is solvable and $P_{4t}(5)$ is not solvable. Consider $X := S_{2t}^{[2]}$. Then by Theorem 4.3.1 and Theorem 4.3.2 we have $\mathrm{Aut}(X) \equiv \mathrm{Bir}(X) \cong \langle \iota \rangle$, where ι is an anti-symplectic involution that fixes the class of the ample divisor D with $q_X(D) = 2$, i.e., $\iota^* D \cong D$. Explicitly, $D = bh - a\delta$, where (a, b) is the minimal solution of the negative Pell equation $P_t(-1)$. As an example we represent the nef cone $\mathrm{Nef}(S_4^{[2]})$ for $t = 2$ in Figure 4.3.1. By Theorem 4.1.1 the nef cone and the movable cone of X coincide. The only primitive (-2)-classes are δ and $\iota^*\delta$. The extremal rays of the nef cone, which we call R_1 and R_2, are the orthogonal respectively to δ and $\iota^*\delta$, i.e., every integral class $x \in \mathrm{NS}(X)$ on R_1 is such that $(x, \delta) = 0$, similarly every $y \in \mathrm{NS}(X)$ on R_2 is such that $(y, \iota^*\delta) = 0$. By Remark 4.3.3 the action of ι on $\mathrm{NS}(S_{2t}^{[2]})$ preserves the nef cone, exchanging the extremal rays R_1 and R_2, and fixes the ray R_D which passes through the ample class D.

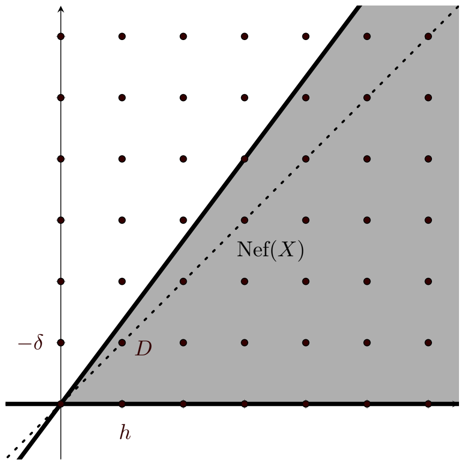


Figure 4.3.1: The nef cone of $S_4^{[2]}$

- **Case 2, t=5.** Both $P_t(-1)$ and $P_{4t}(5)$ are solvable, and this is the only case where the divisor class D with $q_X(D) = 2$ is nef and big but not ample, we will explain why later. Let $X := S_{10}^{[2]}$: by Theorem 4.3.1 and Theorem 4.3.2 we have $\mathrm{Aut}(X) = \{\mathrm{id}\}$ and $\mathrm{Bir}(X) \cong \langle \iota \rangle$, where ι is an anti-symplectic involution whose indeterminacy locus is a subvariety $P \subset X$ isomorphic to $\mathbb{P}^2$. Moreover, the class of $D = h - 2\delta$ is fixed by ι, i.e., $\iota^* D \cong D$. We will see details in Section 4.4.2. By Theorem 4.1.1, the nef cone is strictly contained in the movable cone. We represent the two cones in Figure 4.3.2. The extremal rays of $\overline{\mathrm{Mov}(X)}$, call them R_1 and R_2, are the orthogonal to the (-2)-classes δ and $\iota^*\delta$. The other extremal ray of $\mathrm{Nef}(X)$, which we call R_D, passes through D and it is fixed by ι. Since $P_{4t}(5)$ is solvable, there is a (-10)-class $\rho = 2h - 5\delta$ which is, in this case, fixed by ι, i.e., $\iota^*\rho = \rho$. The extremal ray R_D is the orthogonal to ρ. We conclude that $\overline{\mathrm{Mov}(X)}$ has two chambers, $\mathrm{Nef}(X)$ and $\mathrm{Nef}(X')$, where X' is an IHS manifold birational to X with $\mathrm{Aut}(X') = \{\mathrm{id}\}$ and $\mathrm{Bir}(X') \cong \langle \iota \rangle$. By Theorem 4.3.2 and Remark 4.3.3, the action of ι on $\mathrm{NS}(X)$ preserves the closure of the movable cone, exchanges the two chambers of $\overline{\mathrm{Mov}(X)}$, and $\iota^* R_1 = R_2$. Let $g : X \dashrightarrow X'$ be a birational map and consider the pullback map $g^* : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$. Then g^* is a parallel transport operator by Theorem 2.2.39 and it is an isomorphism of Hodge structures by Theorem 2.2.39. Similarly $\iota^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is a parallel transport operator which is an isomorphism of Hodge structures, so the same holds for $(g \circ \iota)^*$. Moreover, $(g \circ \iota)^*$ maps ample classes of X' to ample classes of X. By Theorem 2.2.38 we conclude that X and X' are isomorphic.

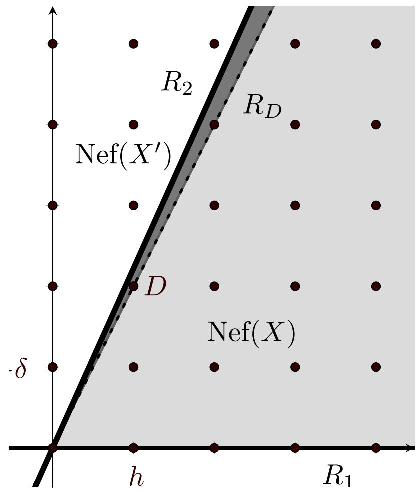


Figure 4.3.2: Case 2, t=5

- **Case 3.** Suppose that both $P_t(-1)$ and $P_{4t}(5)$ are solvable, and $t \neq 5$. Let $X := S_{2t}^{[2]}$ and $D = bh - a\delta$, where (a, b) is the minimal solution of $P_t(-1)$, so $q_X(D) = 2$. In Figure 4.3.3 we represent $\overline{\mathrm{Mov}(X)}$ and $\mathrm{Nef}(X)$. The class D is outside $\mathrm{Nef}(X)$, otherwise by Theorem 4.3.1 the Pell-type equation $P_t(-1)$ would be solvable. Moreover it is in the interior of $\overline{\mathrm{Mov}(X)}$, otherwise by Remark 4.3.3 the action of ι on $\mathrm{NS}(X)$ would exchange it with h, but we know that $\iota^* D \cong D$ by Theorem 4.3.2. Hence D is ample for another smooth birational model of X, call it X'. As for the case $t = 5$, we have $\mathrm{Aut}(X) = \{\mathrm{id}\}$, $\mathrm{Bir}(X) = \langle \iota \rangle$ where ι is a non-regular anti-symplectic involution. The class D is fixed by ι, i.e., $\iota^* D \cong D$, hence if R_D is the ray passing through D we have $\iota^* R_D = R_D$. Then by Theorem 4.3.2 and Remark 4.3.3 the action of ι on $\mathrm{NS}(X)$ preserves the closure of the movable cone $\overline{\mathrm{Mov}(X)}$, exchanging the two extremal rays, call them R_1 and R_2. Moreover, R_1 and R_2 are the orthogonal to the classes δ and $\iota^*\delta$. Since $P_{4t}(5)$ is solvable, there is a (-10)-class ρ, and $\iota^*\rho$ is another (-10)-class. The other extremal ray of $\mathrm{Nef}(X)$, call it R_3, is the orthogonal to ρ, and if we consider the ray $R_4 := \iota^* R_3$, we have that R_4 is the orthogonal to the class $\iota^*\rho$. We conclude that $\mathrm{Mov}(X)$ is divided into three chambers: $\mathrm{Nef}(X)$, $\mathrm{cone}(R_3, R_4)$ and $\mathrm{cone}(R_4, R_2)$, where $\mathrm{cone}(R_i, R_j)$ is the closed (semi)cone of $\mathbb{R}^2 \cong \mathrm{NS}(X)_\mathbb{R}$ generated by the rays R_i and R_j. The second chamber is the nef cone $\mathrm{Nef}(X')$ of the birational model of X which contains the class D in its ample cone, while the third chamber is the nef cone $\mathrm{Nef}(X'')$ of an IHS manifold X'' birational to X. If $g : X \dashrightarrow X'$ is a birational morphism and $\iota' := g \circ \iota \circ g^{-1}$, we have $(\iota')^*(\mathrm{Nef}(X')) = \mathrm{Nef}(X')$, which implies by Theorem 2.2.38 that ι' extends to a biregular involution on the variety X', hence $\mathrm{Aut}(X') = \langle \iota' \rangle$. Moreover, $\iota^*(\mathrm{Nef}(X)) = \mathrm{Nef}(X'')$: proceeding as seen for the case $t = 5$, we conclude that X and X'' are isomorphic. The IHS manifold X' is not isomorphic to the Hilbert square of a K3 surface by [BC20, Corollary 6.5].

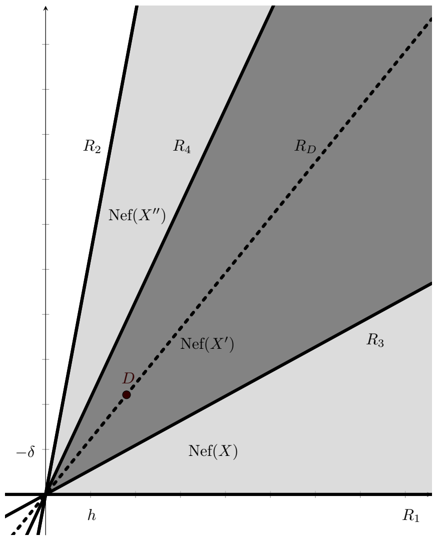


Figure 4.3.3: Case 3

Given a generic K3 surface S_{2t} of degree $2t$, the value $t \geq 1$ is said *n-irregular* if the group $\mathrm{Bir}(S_{2t}^{[n]})$ contains an involution which is not biregular on any IHS birational model of $S_{2t}^{[n]}$, see [BC20, §1]. By [BC20, Table 1], the only 2-irregular positive integer t is $t = 5$. This shows that $t = 5$ is the only value such that $S_{2t}^{[2]}$ admits a non-ample big and nef divisor D with $q_X(D) = 2$, otherwise, proceeding as in Case 3, the class D would be in the interior of the nef cone of a smooth birational model X' of X, and $\mathrm{Aut}(X') = \langle \iota' \rangle$, with ι' not trivial.

Let now X be a variety of the type discussed in Case 1: so X is the Hilbert square of a generic K3 surface S_{2t} of degree $2t$, with $t > 1$, such that $P_t(-1)$ is solvable and $P_{4t}(5)$ is not solvable. Let $D \in \mathrm{Div}(X)$ be the ample divisor with $q_X(D) = 2$. Since $D \in \mathrm{Div}(X)$ is ample, by Theorem 1.1.18 we have $H^i(X, \mathcal{O}_X(D)) = 0$ for $i > 0$, recalling that the canonical bundle of an IHS manifold is trivial. Moreover, since $q_X(D) = 2$, by Theorem 3.1.9 we obtain

$$\dim(H^0(X, \mathcal{O}_X(D))) = 6.$$

This shows that the rational map associated to the complete linear system $|D|$ has $\mathbb{P}^5$ as codomain, i.e.,

$$\varphi_{|D|} : X \dashrightarrow \mathbb{P}^5.$$

We state the main problem of the second part of this thesis.

Problem 4.3.4. *Let $X = S_{2t}^{[2]}$ be the Hilbert square of a generic K3 surface of degree $2t$, and suppose that X admits an ample divisor $D \in \mathrm{Div}(X)$ such that $q_X(D) = 2$. Determine the base locus of the complete linear system $|D|$ and describe the rational map*

$$\varphi_{|D|} : X \dashrightarrow \mathbb{P}^5.$$

The case $t = 5$ will be studied separately: this is the only one with $t \neq 1$ where there exists a big and nef divisor $D \in \mathrm{Div}(X)$ with $q_X(D) = 2$ which is not ample. The same problem can be studied for varieties of Case 3 denoted by X', i.e., smooth birational models of generic K3 surfaces of degree $2t$, where $t > 1$ is such that $P_t(-1)$ and $P_{4t}(5)$ are solvable, admitting an ample divisor $D \in \mathrm{Div}(X')$ with $q_{X'}(D) = 2$. All the results that we will see, except for Theorem 4.6.5 which is still an open problem, hold also for these varieties.

Remark 4.3.5. As observed in [BCNWS16, §1], the first values of t which gives varieties of Case 1 are $t = 2, 10, 13, \text{etc.}$ One verifies that the first value of t which gives a variety of Case 3 is $t = 29$. If the Pell-type equation $P_t(-1)$ is solvable and (a, b) is the minimal solution, then for $t \geq 10$ we have $a \geq 3$. This will be useful in the proof of Theorem 4.6.5, see also Appendix B.

4.4 Base locus of $|D|$ for $t = 2, 5$

In this section we study Problem 4.3.4 for $t = 2$, i.e., when $X = S_4^{[2]}$ is the Hilbert square of a generic smooth quartic surface of $\mathbb{P}^3$, following [Bea83a] and [BCNWS16]. We also analyse the special case $t = 5$, following [O'G05], [O'G08a], where the divisor D is big and nef, but not ample.

4.4.1 Case $t = 2$

Let $t = 2$: then $X = S_4^{[2]}$ is the Hilbert square of a smooth quartic surface $S_4 \subset \mathbb{P}^3$ with $\mathrm{Pic}(S_4) = \mathbb{Z}H$, $H^2 = 4$. Let $h \in \mathrm{Pic}(X)$ be the line bundle induced by H. The movable cone $\mathrm{Mov}(X)$, which is closed by Theorem 4.2.3, coincides with the nef cone $\mathrm{Nef}(X)$ by Theorem 4.1.1, and its extremal rays are generated by h and $3h - 4\delta$, see Figure 4.3.1.

Consider the class $D := h - \delta$. Figure 4.3.1 shows that D is in the ample cone. Moreover, $q_X(D) = 2$, so by Theorem 1.1.18 and Theorem 3.1.9 we get a rational map

$$\varphi_{|D|} : S_4^{[2]} \dashrightarrow \mathbb{P}^5 .$$

Beauville in [Bea83a] gave a finite morphism

$$f : S_4^{[2]} \to \mathbb{G}(1, \mathbb{P}^3),$$

where $\mathbb{G}(1, \mathbb{P}^3)$ is the Grassmannian of lines in $\mathbb{P}^3$, which can be embedded as a smooth quadric hypersurface in $\mathbb{P}^5$ through the Plücker embedding. Using the notation of Section 2.2.2, a point $x \in S_4^{[2]}$ is either of the form $p + q$, with $p, q \in S_4$ distinct points, if x is a reduced subscheme, or of the form (p, t), where $t \in \mathbb{P}^1$ represents a tangent direction through the point $p \in S_4$ if x is a non-reduced subscheme. In both cases, $x \in S_4^{[2]}$ gives a unique line $l_x \subset \mathbb{P}^3$: the one passing through p and q if $x = p + q$, otherwise the one tangent to S in p with direction $t \in \mathbb{P}^1$ if $x = (p, t)$. We then set

$$f(x) := l_x \in \mathbb{G}(1, \mathbb{P}^3).$$

Since $S_4 \subset \mathbb{P}^3$ is a generic K3 surface, S_4 does not contain any line. Then f is a finite morphism of degree 6, since a generic line of $\mathbb{P}^3$ intersects S_4 in four distinct points. Moreover, $S_4^{[2]}$ admits an involution called *Beauville involution*: given $x \in S_4^{[2]}$, the line $l_x \subset \mathbb{P}^3$ intersects S_4 in

$$l_x \cap S_4 = x \cup x',$$

for some $x' \in S_4^{[2]}$. For instance, in the very general case, if $x = p + q$, then $l_x \cap S_4 = \{p, q, r, s\}$, with $p, q, r, s \in S_4$ pairwise distinct, and then $x' = r + s$. The Beauville involution is defined as follows:

$$\iota : S_4^{[2]} \to S_4^{[2]}, \qquad x \mapsto x'.$$

Note that ι is everywhere well-defined since $S_4 \subset \mathbb{P}^3$ does not contain any line. Moreover, by Theorem 4.3.1 we have $\mathrm{Aut}(S_4^{[2]}) = \langle \iota \rangle$, and ι is an anti-symplectic involution. By the geometrical constructions of f and ι we see that $f \circ \iota = f$, so we obtain the following commutative diagram:

$$\begin{tikzcd}
S_4^{[2]} \arrow[r, "f"] \arrow[d, "\iota"] & \mathbb{G}(1, \mathbb{P}^3) \\
S_4^{[2]}. \arrow[ur, "f"] &
\end{tikzcd}$$

Let $F := \mathrm{Fix}(\iota)$ be the locus of points in $S_4^{[2]}$ which are fixed by ι. Then F is given by the following points $x \in S_4^{[2]}$:

- $x = p + q$, with $p, q \in S_4$ distinct points, such that the line $l_x \subset \mathbb{P}^3$ is bitangent to the surface S_4, i.e., $S_4 \cap l_x = \{p, q\}$ and l_x intersects S_4 with multiplicity 2 in both the points.
- $x = (p, t)$, with $t \in \mathbb{P}^1$ tangent direction to $p \in S_4$, and $S_4 \cap l_x = \{p\}$, so the line l_x intersects S_4 in p with multiplicity 4.

As we will see in Lemma 4.5.2 the fixed locus $F \subset S_4^{[2]}$ is a Lagrangian surface, we will give details in Section 4.5. We denote by $\phi : S_4^{[2]} \to \mathbb{P}^5$ the composition

$$\phi : S_4^{[2]} \xrightarrow{f} \mathbb{G}(1, \mathbb{P}^3) \xhookrightarrow{\mathrm{Pl}} \mathbb{P}^5,$$

where $\mathrm{Pl} : \mathbb{G}(1, \mathbb{P}^3) \hookrightarrow \mathbb{P}^5$ is the Plücker embedding. We now show that the rational map $\varphi_{|D|}$ coincides with ϕ.

Theorem 4.4.1. *Keep notation as above. The complete linear system $|D|$ is basepoint free, and the morphism $\varphi_{|D|}$ induced by the complete linear system $|D|$ coincides with*

$$\phi : X \to \mathbb{G}(1, \mathbb{P}^3) \hookrightarrow \mathbb{P}^5.$$

described above.

Proof. Let $\mathcal{O}_{\mathbb{P}^5}(1) \in \mathrm{Pic}(\mathbb{P}^5)$ be the hyperplane bundle of $\mathbb{P}^5$, and denote by $\mathcal{O}_{\mathbb{G}(1,\mathbb{P}^3)}(1) := \mathcal{O}_{\mathbb{P}^5}(1)|_{\mathbb{G}(1,\mathbb{P}^3)}$ its restriction to the Grassmannian $\mathbb{G}(1,\mathbb{P}^3)$ embedded in $\mathbb{P}^5$ through the Plücker embedding. By Proposition 1.1.15 the line bundle $\mathcal{O}_{\mathbb{G}(1,\mathbb{P}^3)}(1)$ is ample, and since $f : S_4^{[2]} \to \mathbb{G}(1,\mathbb{P}^3)$ is a finite morphism, the pullback $f^*\mathcal{O}_{\mathbb{G}(1,\mathbb{P}^3)}(1)$ is ample on $S_4^{[2]}$. Consider

$$\int_{S_4^{[2]}} c_1\left(f^*\mathcal{O}_{\mathbb{G}(1,\mathbb{P}^3)}(1)\right)^4.$$

By Proposition 1.2.2 this is equal to

$$\deg(f)\cdot\int_{\mathbb{G}(1,\mathbb{P}^3)} c_1\left(\mathcal{O}_{\mathbb{G}(1,\mathbb{P}^3)}(1)\right)^4.$$

We have seen that $\deg(f) = 6$, and since $\mathbb{G}(1,\mathbb{P}^3)$ is embedded in $\mathbb{P}^5$ as a quadric hypersurface, we have

$$\int_{\mathbb{G}(1,\mathbb{P}^3)} c_1\left(\mathcal{O}_{\mathbb{G}(1,\mathbb{P}^3)}(1)\right)^4 = 2.$$

Thus we obtain

$$\int_{S_4^{[2]}} c_1\left(f^*\mathcal{O}_{\mathbb{G}(1,\mathbb{P}^3)}(1)\right)^4 = 12, \tag{4.4.1}$$

which is equal to

$$3\cdot q_X(f^*\mathcal{O}_{\mathbb{G}(1,\mathbb{P}^3)}(1))^2 \tag{4.4.2}$$

by Theorem 2.2.4 and Proposition 2.2.8. Since $f^*\mathcal{O}_{\mathbb{G}(1,\mathbb{P}^3)}(1)$ is ample as seen above, by Corollary 2.2.14 we have $q_X(f^*\mathcal{O}_{\mathbb{G}(1,\mathbb{P}^3)}(1)) > 0$, so by (4.4.1) and (4.4.2) we obtain

$$q_X(f^*\mathcal{O}_{\mathbb{G}(1,\mathbb{P}^3)}(1)) = 2.$$

By Theorem 4.3.1 there is a unique ample divisor $D \in \mathrm{Div}(S_4^{[2]})$ with $q_X(D) = 2$. The class of D is $h - \delta$. This implies that $\mathcal{O}_X(D) = f^*\mathcal{O}_{\mathbb{G}(1,\mathbb{P}^3)}(1) = \phi^*\mathcal{O}_{\mathbb{P}^5}(1)$. Then ϕ is induced by a linear system of $|D|$. As remarked in [BCNWS16, §6.1], if ϕ was induced by a proper linear system of $|D|$, the span of the image of ϕ would be contained in a hyperplane of $\mathbb{P}^5$, but the image of the morphism ϕ is a quadric hypersurface of $\mathbb{P}^5$. Thus ϕ is the map induced by the complete linear system $|D|$, i.e., $\varphi_{|D|} = \phi$. In particular, since ϕ is a morphism, the base locus of $|D|$ is empty. □

We now describe the cohomology ring of $\mathbb{G}(1,\mathbb{P}^3)$, see [GH78, §1.5, §6.2], [EH16] and [KNT17, §6] for more details. Fix a complete flag

$$v_0 \in L_0 \subset H_0 \subset \mathbb{P}^3,$$

i.e., let v_0 be a point in $\mathbb{P}^3$, L_0 be a line in $\mathbb{P}^3$ and H_0 be a plane in $\mathbb{P}^3$ such that $v_0 \in L_0 \subset H_0$. We define the following *Schubert varieties* in $\mathbb{G}(1,\mathbb{P}^3)$, see [GH78, §1.5] to see explicitly that these are projective subvarieties of $\mathbb{G}(1,\mathbb{P}^3)$:

$$\begin{aligned}
&\Sigma_0 := \mathbb{G}(1,\mathbb{P}^3),\\
&\Sigma_1 := \{L \in \mathbb{G}(1,\mathbb{P}^3) \mid L \cap L_0 \neq \emptyset\},\\
&\Sigma_{1,1} := \{L \in \mathbb{G}(1,\mathbb{P}^3) \mid L \subset H_0\}, \qquad \Sigma_2 := \{L \in \mathbb{G}(1,\mathbb{P}^3) \mid v_0 \in L\},\\
&\Sigma_{2,1} := \{L \in \mathbb{G}(1,\mathbb{P}^3) \mid v_0 \in L \subset H_0\},\\
&\Sigma_{2,2} := \{L_0\}.
\end{aligned} \tag{4.4.3}$$

We denote the fundamental cohomological classes of $\Sigma_{i,j}$ in the cohomology ring $H^*(\mathbb{G}(1,\mathbb{P}^3))$ by $\sigma_{i,j}$. Then these classes generates the cohomology ring:

$$\begin{aligned}
&H^0(\mathbb{G}(1,\mathbb{P}^3),\mathbb{Z}) \cong \mathbb{Z}\sigma_0,\\
&H^2(\mathbb{G}(1,\mathbb{P}^3),\mathbb{Z}) \cong \mathbb{Z}\sigma_1,\\
&H^4(\mathbb{G}(1,\mathbb{P}^3),\mathbb{Z}) \cong \mathbb{Z}\sigma_{1,1} \oplus \mathbb{Z}\sigma_2,\\
&H^6(\mathbb{G}(1,\mathbb{P}^3),\mathbb{Z}) \cong \mathbb{Z}\sigma_{2,1},\\
&H^8(\mathbb{G}(1,\mathbb{P}^3),\mathbb{Z}) \cong \mathbb{Z}\sigma_{2,2}.
\end{aligned}$$

The cup products between the Schubert cycles are given by:

$$\begin{aligned}
&\sigma_1^2 = \sigma_{1,1} + \sigma_2,\\
&\sigma_{1,1}^2 = \sigma_2^2 = \sigma_1 \cdot \sigma_{2,1} = \sigma_{2,2},\\
&\sigma_{1,1} \cdot \sigma_2 = 0,\\
&\sigma_1 \cdot \sigma_2 = \sigma_1 \cdot \sigma_{1,1} = \sigma_{2,1}.
\end{aligned} \tag{4.4.4}$$

As shown in [GH78], the cohomology ring of $\mathbb{G}(1,\mathbb{P}^3)$ is isomorphic to its Chow ring, see [Ful13] for details on the theory of Chow groups. We have seen in Section 1.1.3 that for smooth varieties the class group and the Picard group are isomorphic. Moreover, the class group coincide with the Chow group of cycles of codimension 1, see [Ful13, §1]. Since the Chow ring and the cohomology ring of $\mathbb{G}(1,\mathbb{P}^3)$ coincide, we obtain

$$\operatorname{Pic}(\mathbb{G}(1,\mathbb{P}^3)) \cong H^2(\mathbb{G}(1,\mathbb{P}^3)) \cong \mathbb{Z}\sigma_1,$$

so we can identify σ_1 with $c_1(\mathcal{O}_{\mathbb{G}(1,\mathbb{P}^3)}(1))$. Thus by Theorem 4.4.1 we have $c_1(\mathcal{O}_X(D)) = f^*\sigma_1$, and $c_1(\mathcal{O}_X(D))^2 = f^*\sigma_1^2 = f^*(\sigma_2 + \sigma_{1,1})$. From (4.4.3) the hyperplane section of $\mathbb{G}(1,\mathbb{P}^3)$ can be associated to the set of lines in $\mathbb{P}^3$ incident to the fixed line $L_0 \subset \mathbb{P}^3$. Hence the intersection of two hyperplane sections, which we denote by H and H', is represented by the set of lines in $\mathbb{P}^3$ incident to two fixed lines $L_0, L_0' \subset \mathbb{P}^3$. If these two lines are disjoint, then $H \cap H'$ is irreducible. If they intersect in a point v_0, then $H \cap H'$ has two irreducible components: the set of lines contained in the plane generated by L_0 and L_0', and the set of lines passing through v_0. The classes of these two components are $\sigma_{1,1}$ and σ_2. This explains the equality $\sigma_1^2 = \sigma_{1,1} + \sigma_2$.

Let $\operatorname{Bit}(S_4) \subset \mathbb{G}(1,\mathbb{P}^3)$ be the surface of bitangents line to $S_4 \subset \mathbb{P}^3$.

Remark 4.4.2. Consider the restriction of the map $f : S_4^{[2]} \to \mathbb{G}(1,\mathbb{P}^3)$ to the fixed locus $F = \operatorname{Fix}(\iota)$ of the Beauville involution. By the geometrical description of F seen above, we obtain the following isomorphism:

$$f|_F : F \xrightarrow{\sim} \operatorname{Bit}(S_4), \qquad x \mapsto l_x.$$

Moreover, if $X := S_4^{[2]}$, then $[F] = 5D^2 - \frac{2}{5}q_X^\vee \in H^{2,2}(S_4^{[2]},\mathbb{Q})$: we will show that an equality of this form holds in a more general context, cf. Theorem 4.5.11, so we do not prove it now.

Remark 4.4.3. Let $S_4^{[2]} \to S_4^{(2)}$ be the Hilbert–Chow morphism and consider the exceptional divisor $E \in \operatorname{Div}(S_4^{[2]})$. Recall that

$$\operatorname{Pic}(S_4^{[2]}) = \mathbb{Z}h \oplus \mathbb{Z}\delta,$$

where $h \in \mathrm{Pic}(S_4^{[2]})$ is the line bundle induced by the ample generator of $\mathrm{Pic}(S_4)$, and $\delta \in \mathrm{Pic}(S_4^{[2]})$ is such that $2\delta = [E]$. Consider the supports $\mathrm{Supp}(E)$ and $\mathrm{Supp}(\iota^* E)$. The former is the set of points in $S_4^{[2]}$ of the form (p,t), where $t \in \mathbb{P}^1$ is a tangent direction through $p \in S_4$, the latter is the set of points $x \in S_4^{[2]}$ such that $\iota(x) \in \mathrm{Supp}(E)$. Hence

$$B := \mathrm{Supp}(E) \cap \mathrm{Supp}\,(\iota^* E) = \{x \in S_4^{[2]} \mid x = (p,t) \text{ and } \iota((p,t)) = (q,t)\},$$

i.e., B is the set of points of the form (p,t) such that the line of $\mathbb{P}^3$ passing through p with tangent direction t is tangent to S_4 also in another point $q \in S_4$. Note that we can have $p = q$. In other words, if $x \in B$, then $S_4 \cap l_x = x \cup x'$, and either $l_x \subset \mathbb{P}^3$ is bitangent to S_4 in two distinct points $p, q \in S_4$ with $x = (p,t)$ and $x' = (q,t)$, or $l_x \subset \mathbb{P}^3$ intersects S_4 in a unique point p with multiplicity 4, and $x = x' = (p,t)$. We put on B the induced closed subscheme structure. Thus the restriction of the map f to B has degree two:

$$f|_B : B \xrightarrow{2:1} \mathrm{Bit}(S_4), \qquad x \mapsto l_x.$$

We write $\mathrm{Bit}(S_4)$ also for the fundamental cohomological class of the surface $\mathrm{Bit}(S_4)$ in $H^4(\mathbb{G}(1,\mathbb{P}^3),\mathbb{Z})$. We now compute the classes in $H^4(S_4^{[2]},\mathbb{Z})$ of the pullbacks $f^*\mathrm{Bit}(S_4)$, $f^*\sigma_{1,1}$ and $f^*\sigma_2$.

Proposition 4.4.4. *Let $S_4 \subset \mathbb{P}^3$ be a generic smooth quartic surface, i.e., $\mathrm{Pic}(S_4) \cong \mathbb{Z}H$, $H^2 = 4$. Let $X := S_4^{[2]}$ be the Hilbert square of S_4, denote by $h \in \mathrm{Pic}(X)$ the line bundle induced by H and by $\delta \in \mathrm{Pic}(X)$ the line bundle such that 2δ is the class of the exceptional divisor of the Hilbert–Chow morphism. Let $f : X \to \mathbb{G}(1,\mathbb{P}^3)$ be the map defined above. Then the following equalities hold in $H^{2,2}(X,\mathbb{Z})$:*

$$f^*\mathrm{Bit}(S_4) = 20h^2 + 8\delta^2 - 32h\delta - \tfrac{8}{5}q_X^\vee,$$
$$f^*\sigma_{1,1} = \tfrac{1}{2}h^2 - \tfrac{1}{4}\delta^2 - \tfrac{1}{2}h\delta - \tfrac{1}{10}q_X^\vee,$$
$$f^*\sigma_2 = \tfrac{1}{2}h^2 + \tfrac{5}{4}\delta^2 - \tfrac{3}{2}h\delta + \tfrac{1}{10}q_X^\vee.$$

Moreover, $f^*\mathrm{Bit}(S)$, $f^*\sigma_{1,1}$ and $f^*\sigma_2$ are pseudoeffective classes of $H^{2,2}(X,\mathbb{Z})$.

Proof. We have

$$\mathrm{Bit}(S_4) = 12\sigma_2 + 28\sigma_{1,1} \in H^{2,2}(\mathbb{G}(1,\mathbb{P}^3),\mathbb{Z}). \tag{4.4.5}$$

see for instance [Wel81] and [ABT01, Proposition 3.3]. Let

$$f^*\mathrm{Bit}(S_4) = xh^2 + y\delta^2 + zh\delta + w \cdot \frac{2}{5}q_X^\vee \in H^{2,2}(X,\mathbb{Z})$$

for some $x, y, z, w \in \mathbb{Q}$. We want to determine the coefficients $x, y, z, w \in \mathbb{Q}$. We denote by $D := h - \delta \in \mathrm{Pic}(S_4^{[2]})$ the class of the ample divisor on $S_4^{[2]}$ with $q_X(D) = 2$.

- By (4.4.4) and (4.4.5) we have $\int_{\mathbb{G}(1,\mathbb{P}^3)} \mathrm{Bit}(S_4)\cdot(\sigma_2+\sigma_{1,1}) = 40$. Recall that rational equivalence and the homological equivalence coincide in $\mathbb{G}(1,\mathbb{P}^3)$. Then, since $\deg(f) = 6$, by (1.2.1) we have $f_*f^*\mathrm{Bit}(S_4) = 6 \cdot \mathrm{Bit}(S_4)$. By the projection formula we have

$$f_*f^*\mathrm{Bit}(S_4)\cdot(\sigma_2+\sigma_{1,1}) = f_*\left(f^*\mathrm{Bit}(S_4)\cdot f^*(\sigma_2+\sigma_{1,1})\right).$$

By Theorem 4.4.1 we have $f^*\sigma_1 = c_1(D)$, where $D = h - \delta \in \mathrm{Pic}(X)$, hence $f^*(\sigma_{1,1} + \sigma_2) = c_1(D)^2$ and

$$\langle f^*\mathrm{Bit}(S_4), c_1(D)^2\rangle = 240. \tag{4.4.6}$$

Equation (4.4.6) gives

$$40x + 4y + 16z + 20w = 240. \tag{4.4.7}$$

- Similarly,

$$\int_{\mathbb{G}(1,\mathbb{P}^3)} \mathrm{Bit}(S_4)^2 = \int_{\mathbb{G}(1,\mathbb{P}^3)} (12\sigma_2 + 28\sigma_{1,1})^2 = 928,$$

hence

$$\langle f^*\mathrm{Bit}(S_4), f^*\mathrm{Bit}(S_4)\rangle = 5568,$$

which gives

$$48x^2 - 16xy + 80xw + 12y^2 - 40yw - 8z^2 + 92w^2 = 5568. \tag{4.4.8}$$

- By Remark 4.4.2 we have $\deg(f|_F) = 1$, hence $f_*[F] = \mathrm{Bit}(S_4)$, where $[F] \in H^4(S_4^{[2]}, \mathbb{Z})$ is the fundamental cohomological class of the surface F. Recall that by Remark 4.4.2 we have $[F] = 5D^2 - \frac{2}{5}q_X^\vee$, where $X = S_4^{[2]}$, cf. Theorem 4.5.11. By the projection formula we have

$$f_*\left(f^*\mathrm{Bit}(S_4)\cdot F\right) = \mathrm{Bit}(S_4)\cdot f_*F = \mathrm{Bit}(S_4)\cdot \mathrm{Bit}(S_4) = 928,$$

so we obtain

$$\langle f^*\mathrm{Bit}(S_4), F\rangle = 928,$$

which gives

$$160x + 40y + 80z + 8w = 928. \tag{4.4.9}$$

- By Remark 4.4.3 we have $\deg(f|_B) = 2$, hence $f_*B = 2\mathrm{Bit}(S_4)$. By the projection formula we have

$$\begin{aligned}
\int_{\mathbb{G}(1,\mathbb{P}^3)} f_*\left(f^*\mathrm{Bit}(S_4)\cdot B\right) &= \int_{\mathbb{G}(1,\mathbb{P}^3)} \mathrm{Bit}(S_4)\cdot f_*B \\
&= \int_{\mathbb{G}(1,\mathbb{P}^3)} \mathrm{Bit}(S_4)\cdot 2\mathrm{Bit}(S_4) \\
&= 1856.
\end{aligned}$$

Since $B = \mathrm{Supp}(E) \cap \mathrm{Supp}(\iota^*E)$, where $E \in \mathrm{Div}(S_4^{[2]})$ is the exceptional divisor of the Hilbert–Chow morphism, and $[E] = 2\delta$ in $\mathrm{Pic}(S_4^{[2]})$, we have

$$\langle f^*\mathrm{Bit}(S_4), 2\delta(4h - 6\delta)\rangle = 1856,$$

which gives

$$96x - 144y - 64z + 240w = 1856. \tag{4.4.10}$$

The system given by (4.4.7), (4.4.8), (4.4.9) and (4.4.10) has the following solution with multiplicity 2:

$$\begin{cases} x = 20, \\ y = 8, \\ z = -32, \\ w = -4. \end{cases}$$

Hence we obtain

$$f^*\mathrm{Bit}(S_4) = 20h^2 + 8\delta^2 - 32h\delta - \frac{8}{5}q_X^\vee \in H^{2,2}(X, \mathbb{Z}). \tag{4.4.11}$$

We now compute $f^*\sigma_{1,1}$ and $f^*\sigma_2$. As already remarked, $f^*\sigma_1 = D$ implies $f^*(\sigma_2 + \sigma_{1,1}) = D^2 = h^2 + \delta^2 - 2h\delta$. Hence

$$\begin{aligned} f^*\mathrm{Bit}(S_4) &= f^*(12\sigma_2 + 28\sigma_{1,1}) \\ &= f^*(12\sigma_2 + 12\sigma_{1,1} + 16\sigma_{1,1}) \\ &= 12D^2 + f^*(16\sigma_{1,1}) \\ &= 12h^2 + 12\delta^2 - 24h\delta + 16f^*\sigma_{1,1}. \end{aligned} \tag{4.4.12}$$

From (4.4.11) and (4.4.12) we obtain

$$\begin{aligned} f^*\sigma_{1,1} &= \tfrac{1}{2}h^2 - \tfrac{1}{4}\delta^2 - \tfrac{1}{2}h\delta - \tfrac{1}{10}q_X^\vee, \\ f^*\sigma_2 &= \tfrac{1}{2}h^2 + \tfrac{5}{4}\delta^2 - \tfrac{3}{2}h\delta + \tfrac{1}{10}q_X^\vee, \end{aligned}$$

and the first part of the lemma is proven. We pass to the pseudo-effectiveness of $f^*\mathrm{Bit}(S_4)$, $f^*\sigma_{1,1}$ and $f^*\sigma_2$. As observed in [Voi10, proof Lemma 2.8], the extremal rays of the pseudoeffective cone of 2-cycles in $\mathbb{G}(1, \mathbb{P}^3)$ are generated by the Schubert cycles $\sigma_{1,1}$ and σ_2, which are in particular pseudoeffective. Moreover, recall that by Theorem 2.2.45 the numerical equivalence and the homological equivalence coincide in $S_4^{[2]}$. Hence by Proposition 2.2.44 the pullbacks $f^*\mathrm{Bit}(S_4)$, $f^*\sigma_{1,1}$ and $f^*\sigma_2$ are pseudoeffective. $\square$

Keep notation as above. Proposition 4.4.4 says that, if $D_1, D_2 \in |D|$ are two distinct divisors, then the surface $D_1 \cap D_2$ can have two irreducible components, whose fundamental cohomological classes in $H^{2,2}(X, \mathbb{Z})$ are $f^*\sigma_{1,1}$ and $f^*\sigma_2$. Theorem 4.6.5 will show that if $D_1 \cap D_2$ is a reducible surface, then necessarily $[D_1 \cap D_2] = f^*\sigma_{1,1} + f^*\sigma_2$, where $[D_1 \cap D_2]$ is the fundamental cohomological class of the surface $D_1 \cap D_2$ in $H^4(S_4^{[2]}, \mathbb{Z})$.

The equality $\mathrm{Bit}(S_4) = 12\sigma_2 + 28\sigma_{1,1}$ can be seen geometrically in the following way. As shown in [HV14, §8.1], the number of bitangents passing through a fixed point $p \in S_4$ is six, and each bitangent, if seen as (p, t) with $t \in \mathbb{P}^1$ tangent direction through p, must be counted twice, since $(\iota(p), t)$ represents the same line. Moreover, by a classical result of Plücker a plane quartic curve has 28 bitangents.

We conclude this section by showing that every big and nef line bundle on $X = S_4^{[2]}$ is basepoint free.

Proposition 4.4.5. *Let $S_4 \subset \mathbb{P}^3$ be a generic K3 surface of degree* 4. *Then every big and nef line bundle on $X = S_4^{[2]}$ is basepoint free.*

Proof. By Theorem 4.1.1 the nef cone of $S_4^{[2]}$ is generated by the classes h and $3h-4\delta$ in $\mathrm{NS}(S_4^{[2]})$. The Hilbert basis of $\mathrm{Nef}(S_4^{[2]})$ is given by $\{h, h-\delta, 3h-4\delta\}$, i.e., every class in $\mathrm{Nef}(S_4^{[2]})$ is a linear combination with positive coefficients of the classes h, $h-\delta$, $3h-4\delta$. Hence it suffices to show that every element in the Hilbert basis of $\mathrm{Nef}(S_4^{[2]})$ is basepoint free. By Lemma 2.3.2 the class h is basepoint free, the class $h-\delta$ is basepoint free by Theorem 4.4.1, and $3h-4\delta$ is basepoint free since $\iota^* h = 3h-4\delta$, see Theorem 4.3.1 and Remark 4.3.3, where we denote by ι the Beauville involution. $\square$

4.4.2 Case $t = 5$

Let $t = 5$, and consider $X := S_{10}^{[2]}$ the Hilbert square of the generic K3 surface of degree 10. As already remarked, $t = 5$ is the only 2-irregular value, i.e., it is the only value of t such that $\mathrm{Bir}(S_{2t}^{[2]})$ contains an involution which is not biregular on any IHS birational model of $S_{2t}^{[2]}$. The Pell-type equations $P_t(-1)$ and $P_{4t}(5)$ are both solvable, so there exists a unique divisor $D \in \mathrm{Div}(X)$ with $q_X(D) = 2$ whose class is in the moving cone of X and the nef cone is strictly contained in the moving cone, see Figure 4.3.2. The class of the divisor D is $h - 2\delta \in \mathrm{Pic}(X)$, and this is on the boundary of the nef cone, hence it is big and nef, but not ample. We now recall the geometric description, given by O'Grady, of the map induced by the complete linear system $|D|$. The main references are [O'G05], [O'G08a], [O'G13] and [IM15].

First of all, we describe the generic K3 surface S_{10}. Mukai showed that S_{10} is given by the following intersection:

$$S_{10} = F \cap Q,$$

where F is the Fano 3-fold of index 2, i.e., $\omega_F = \mathcal{O}_F(-2)$, and degree 5, and Q is a quadric hypersurface of $\mathbb{P}^9$. More precisely, the Fano 3-fold F is obtained as

$$F = Gr(2,5) \cap \Sigma \subset \mathbb{P}^9,$$

where $Gr(2,5)$ is the Grassmannian parametrizing 2-dimensional linear subspaces of a vector space of dimension 5, equivalently this is the Grassmannian $\mathbb{G}(1,\mathbb{P}^4)$ parametrizing lines in $\mathbb{P}^4$, and $\Sigma \cong \mathbb{P}^6$ is a linear subspace of $\mathbb{P}^9$ of dimension 6. Here the Grassmannian $Gr(2,5)$ is embedded in $\mathbb{P}^9$ as a smooth subvariety of dimension 6 given by the intersection of five quadrics $Q_1, \dots, Q_5 \subset \mathbb{P}^9$. Hence the variety S_{10} is the intersection of 6 quadrics in $\mathbb{P}^9$ and of a linear 6-dimensional subspace $\Sigma \subseteq \mathbb{P}^9$. See [Muk88] for details. Since S_{10} is a generic K3 surface, it does not contain lines and conics.

We now define a birational involution on $S_{10}^{[2]}$. First, we introduce some notation from [O'G05, §4.3].

- Let $R(F)$ be the Hilbert scheme parametrizing lines contained in F.
- Let $W(F)$ be the Hilbert scheme parametrizing conics contained in F.
- Let $B_F := \{[Z] \in F^{[2]} \,|\, \mathrm{span}(Z) \subset F\}$, i.e., B_F is the closed subset of $F^{[2]}$ of 0-dimensional subschemes of length 2 contained in lines contained in F. We put on B_F the reduced induced scheme structure.

- For every $[Z] \in F^{[2]}$, we set $W_Z := \{[C] \in W(F) \mid Z \subset C\}$, i.e., W_Z is the closed subset of $F^{[2]}$ of conics of F containing Z. We put on W_Z the reduced induced scheme structure.

We can now state the following lemma, see [Isk78, Corollary 6.6] for item (i) and [O'G05, Lemma 4.20] for items (ii), (iii) and (iv).

Lemma 4.4.6 (Iskovskih–O'Grady)**.** *Keep notation as above.*

(i) $R(F) \cong \mathbb{P}^2$.

(ii) $W(F) \cong \mathbb{P}^4$.

(iii) *Let* $[Z] \in F^{[2]} \setminus B_F$. *Then* W_Z *consists of a single conic* C_Z.

(iv) *Let* $[Z] \in B_F$. *Then* W_Z *parametrizes the reducible conics in* F *containing* $\operatorname{span}(Z)$.

We now define the *O'Grady involution*

$$\iota : S_{10}^{[2]} \dashrightarrow S_{10}^{[2]}.$$

Since $S_{10}^{[2]} \subset F^{[2]}$, we can set

$$P := B_F \cap S_{10}^{[2]}, \qquad U := S_{10}^{[2]} \setminus P,$$

hence U is the open subset of points $x \in S_{10}^{[2]}$ such that the line $\operatorname{span}(x)$ is not contained in F. By Lemma 4.4.6, (iii), for every $x \in U$ there exists a unique conic C_x contained in F which contains x: using the notation of Section 2.2.2, if $x = p + q$ with $p, q \in S_{10}$ distinct points, there exists a unique conic C_x in F passing through p and q, and if $x = (p, t)$, with $t \in \mathbb{P}^1$ tangent direction through $p \in S_{10}$, there exists a unique conic C_x in F passing through p with tangent direction t. Since $S_{10} = F \cap Q$, the conic C_x intersects Q in two other points, which can coincide, in such a case C_x is tangent to Q in this other point. Hence there exists $x' \in S_{10}^{[2]}$ such that

$$C_x \cap S_{10} = \operatorname{Supp}(x) \cup \operatorname{Supp}(x').$$

We then set $\iota(x) := x'$. Note that the restriction of ι to U is a regular map

$$\iota|_U : U \to S_{10}^{[2]}$$

whose image is U, i.e., $\iota|_U : U \xrightarrow{\sim} U$ is a biregular involution. Moreover, $\iota : S_{10}^{[2]} \dashrightarrow S_{10}^{[2]}$ is a birational involution whose indeterminacy locus is P. Indeed, if $x \in P$, then $\operatorname{span}(x)$ is contained in F, and conics in F containing x are reducible conics given by the union of $\operatorname{span}(x)$ and another line contained in F. Note that by Lemma 4.4.6, (i), we have $R(F) \cong \mathbb{P}^2$, so we see that we cannot define the involution ι on P. Moreover, $P \cong \mathbb{P}^2$.

We now define a morphism

$$f : S_{10}^{[2]} \to |\mathcal{I}_{S_{10}}(2)|^\vee.$$

First of all, we remark that $|\mathcal{I}_{S_{10}}(2)|^\vee \cong \mathbb{P}^5$. Indeed, consider the exact sequence of the ideal sheaf of S_{10} given by the embedding $\varphi_{|H|} : S_{10} \hookrightarrow \mathbb{P}^6$, where $H \in \mathrm{Div}(S_{10})$ is the divisor whose class generates $\mathrm{Pic}(S_{10})$:

$$0 \to \mathcal{I}_{S_{10}} \to \mathcal{O}_{\mathbb{P}^6} \to \mathcal{O}_{S_{10}} \to 0.$$

Tensorising by $\mathcal{O}_{\mathbb{P}^6}(2)$ we get

$$0 \to \mathcal{I}_{S_{10}}(2) \to \mathcal{O}_{\mathbb{P}^6}(2) \to \mathcal{O}_{S_{10}}(2H) \to 0,$$

which gives a long exact sequence in cohomology

$$\begin{array}{ccccccc} 0 & \to & H^0(\mathbb{P}^6, \mathcal{I}_{S_{10}}(2)) & \to & H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2)) & \to & H^0(S_{10}, \mathcal{O}_{S_{10}}(2H)) \\ & \to & H^1(\mathbb{P}^6, \mathcal{I}_{S_{10}}(2)) & \to & \ldots . & & \end{array}$$

We recall the following result, see [Huy16, Corollary 2.2.5].

Lemma 4.4.7. *Let C be an irreducible, smooth, non-hyperelliptic curve of genus $g > 2$ on a K3 surface S. Then the linear system $L = \mathcal{O}_S(C)$ is projectively normal, i.e., the pullback under φ_L defines for all $k \geq 0$ a surjective map*

$$H^0(\mathbb{P}^g, \mathcal{O}_{\mathbb{P}^g}(k)) \twoheadrightarrow H^0(S, L^{\otimes k}).$$

Lemma 4.4.7 implies the surjectivity of $H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2)) \to H^0(S_{10}, \mathcal{O}_{S_{10}}(2H))$. Moreover, we have

$$\dim(H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))) = 28, \qquad \dim(H^0(S_{10}, \mathcal{O}_{S_{10}}(2H))) = 22,$$

where the second equality is given by Theorem 2.1.6. Thus we obtain

$$\dim(H^0(\mathbb{P}^6, \mathcal{I}_{S_{10}}(2))) = 6,$$

so $|\mathcal{I}_{S_{10}}(2)|^\vee \cong \mathbb{P}^5$. Since $S_{10} = Q_1 \cap \cdots \cap Q_5 \cap Q \cap \Sigma \subset \mathbb{P}^9$, a basis of $|\mathcal{I}_{S_{10}}(2)|$ is given by $\{Q_1, \ldots, Q_5, Q\}$. We set

$$f : S_{10}^{[2]} \to |\mathcal{I}_{S_{10}}(2)|^\vee, \qquad [Z] \mapsto \{\tilde{Q} \in |\mathcal{I}_{S_{10}}(2)| \,|\, \mathrm{span}(Z) \subset \tilde{Q}\}.$$

For instance, let $x \in S_{10}^{[2]}$ such that $\mathrm{span}(x) \subset F$. Then $\mathrm{span}(x)$ is contained in $Q_1, \ldots, Q_5$, and since S_{10} does not contain any line, $\mathrm{span}(x)$ is not contained in Q. This shows that f maps x to the point of $|\mathcal{I}_{S_{10}}(2)|^\vee$ which corresponds to Q, hence P is contracted to a point. Moreover, we show that $\deg(f|_U) = 2$ and $f|_U$ is induced by $\iota|_U : U \xrightarrow{\sim} U$. If $x \in U$, there exists a unique conic $C_x \subset F$ containing x. Consider $\iota(x)$. Then $\mathrm{span}(x)$ and $\mathrm{span}(\iota(x))$ intersect the same conic C_x, so they are in the same plane, and they intersect in a non-empty set. If a quadric in $|\mathcal{I}_{S_{10}}(2)|$ contains $\mathrm{span}(x)$, then it contains $\iota(x)$ and $\mathrm{span}(x) \cap \mathrm{span}(\iota(x))$, so it contains $\mathrm{span}(\iota(x))$. This shows that $f(x) = f(\iota(x))$ and $f(x) \neq f(y)$ for $y \neq x, \iota(x)$.

Denote by $Y \subset \mathbb{P}^5$ the image of f. Then O'Grady has shown in [O'G13, §4] that Y is an EPW sextic. The pullback $f^*\mathcal{O}_{\mathbb{P}^5}(1)|_Y$ is a big and nef divisor which is not ample by Proposition 1.1.15, since f contacts P, which is isomorphic to $\mathbb{P}^2$. Moreover, since $|\mathcal{I}_{S_{10}}(2)|^\vee \cong \mathbb{P}^5$, we have $q_X(f^*\mathcal{O}_{\mathbb{P}^5}(1)) = 2$. The only big and nef divisor $D \in \mathrm{Div}(S_{10}^{[2]})$ with $q_X(D) = 2$ is the one with class $h - 2\delta \in \mathrm{Pic}(S_{10}^{[2]})$, so we have

$$f^*\mathcal{O}_{\mathbb{P}^5}(1) = D = h - 2\delta \in \mathrm{Pic}(S_{10}^{[2]}), \tag{4.4.13}$$

which shows that f is a morphism induced by a linear system of $|D|$. If f was induced by a proper linear system of $|D|$, its image would be contained in a hyperplane of $\mathbb{P}^5$, similarly to the proof of Theorem 4.4.1. Now, $Y \subset \mathbb{P}^5$ is an EPW sextic, so its span is $\mathbb{P}^5$, thus $f = \varphi_{|D|}$ is the morphism induced by the complete linear system $|D|$. See [O'G05, §4.3, §5.2.4] and [O'G13, §4] for more details. In particular we obtain that the base locus of $|D|$ is empty, and the morphism $\varphi_{|D|}$ is the double cover of an EPW sextic ramified in its singular locus. Moreover, by Theorem 4.3.2, the action induced by ι on $H^2(S_{10}^{[2]}, \mathbb{Z})$ is the reflection in the span of the class D, and $\mathrm{Aut}(S_{10}^{[2]}) = \{\mathrm{id}\}$, $\mathrm{Bir}(S_{10}^{[2]}) = \langle \iota \rangle$.

We conclude this section by showing that every big and nef line bundle on the Hilbert square $S_{10}^{[2]}$ is basepoint free. We recall, without going into details, the following result, see [Cat20, Lemma 1.3].

Lemma 4.4.8. *Let S_{2t} be a generic K3 surface of degree $2t$, with $t \geq 2$, and denote by $h \in \mathrm{Pic}(S_{2t}^{[2]})$ the line bundle induced by the ample generator of $\mathrm{Pic}(S_{2t})$. Then $h - \delta \in \mathrm{Pic}(S_{2t}^{[2]})$ is basepoint free.*

We can now state the following.

Proposition 4.4.9. *Let S_{10} be the generic K3 surface of degree 10. Then every big and nef line bundle on $S_{10}^{[2]}$ is basepoint free.*

Proof. By Theorem 4.1.1 the nef cone of $S_{10}^{[2]}$ is generated by the classes h and $h - 2\delta$ in $\mathrm{NS}(S_{10}^{[2]})$, and every nef class is big, see the proof of Theorem 4.2.3. The Hilbert basis of $\mathrm{Nef}(S_{10}^{[2]})$ is given by $\{h, h-\delta, h-2\delta\}$, i.e., every class in $\mathrm{Nef}(S_{10}^{[2]})$ is a linear combination with positive coefficients of the classes h, $h-\delta$ and $h-2\delta$. If every element of the Hilbert basis of $\mathrm{Nef}(S_{10}^{[2]})$ is basepoint free, we are done. Now, h is basepoint free by Lemma 2.3.2, the class $h - \delta$ is basepoint free by Lemma 4.4.8, and $h - 2\delta$ is basepoint free by the discussion above, see in particular (4.4.13). □

We have seen in Section 4.3, Case $t = 5$, that the movable cone $\mathrm{Mov}(S_{10}^{[2]})$ is divided into two chambers: the first chamber is the nef cone of $S_{10}^{[2]}$, the other chamber is the nef cone of an IHS manifold isomorphic to $S_{10}^{[2]}$, that we have called X'.

Corollary 4.4.10. *Let X' be the IHS manifold described above. Then every big and nef line bundle on X' is basepoint free.*

Proof. We have seen in Section 4.3, Case $t = 5$, that X' is isomorphic to $S_{10}^{[2]}$, so we conclude by Proposition 4.4.9. □

4.5 Fixed locus of anti-symplectic involutions on IHS manifolds

Let X be the Hilbert square of a generic K3 surface S_{2t} of degree $2t$, and assume that X admits an ample divisor $D \in \mathrm{Div}(X)$ with $q_X(D) = 2$. We have seen that $\mathrm{Aut}(X) \cong \langle \iota \rangle$, where ι is an anti-symplectic involution. In this section we study some properties of the fixed locus $F = \mathrm{Fix}(\iota)$, which is a Lagrangian surface, cf. Lemma 4.5.2. We will compute the fundamental cohomological class of F in $H^{2,2}(X, \mathbb{Z})$ and we will show that the map induced by the complete linear system $|D|$, denoted by $\varphi_{|D|} : X \dashrightarrow \mathbb{P}^5$, factors through the quotient $\pi : X \to X/\langle\iota\rangle$. We will remark that these results hold also for a smooth birational model X of the Hilbert square of a generic K3 surface such that X admits an ample divisor D with $q_X(D) = 2$.

4.5.1 Connectedness of Fix(ι)

We first recall the following general definition.

Definition 4.5.1. Let X be a complex manifold which admits a symplectic form $\omega \in H^{2,0}(X)$, i.e., a nowhere vanishing holomorphic closed 2-form. A submanifold $Y \subseteq X$ is an *isotropic submanifold* if the symplectic form restricts to zero on Y, and it is a *Lagrangian submanifold* if it is an isotropic submanifold of maximal dimension $\dim(Y) = \frac{1}{2}\dim(X)$.

Note that, as seen in the definition of IHS manifold in Section 2.2, the dimension of a complex manifold which admits a symplectic form is always even. The following result by Beauville gives a description of the fixed locus of an anti-symplectic involution on an IHS manifold.

Lemma 4.5.2 (Lemma 1 in [Bea11])**.** *Let X be an IHS manifold which admits an anti-symplectic involution ι. Then the fixed locus $F = \mathrm{Fix}(\iota)$ of ι is a smooth Lagrangian submanifold of X.*

The fixed locus of an anti-symplectic involution on an IHS manifold is not necessarily connected. For instance, the following result holds for K3 surfaces, see [Nik83, Theorem 4.2.], [Kon00, Theorem 6.1], [AST11, Theorem 4.1] and [AS15, Theorem 1.1].

Theorem 4.5.3. *Let ι be an anti-symplectic involution on a K3 surface. The fixed locus of ι is either empty, the disjoint union of two elliptic curves or the disjoint union of a smooth curve of genus $g \geq 0$ and j smooth rational curves.*

We show that in the case that we are considering the fixed locus $F = \mathrm{Fix}(\iota)$ of the anti-symplectic involution ι is connected. This property will hold also for varieties of Case 3 of Section 4.3, so we assume that X is a smooth birational model of the Hilbert square of a generic K3 surface of degree $2t$ which admits an ample divisor D with $q_X(D) = 2$. We need a result from [BCMS19]. Let $\mathcal{M}^{\rho}_{\langle 2\rangle}$ be the moduli space which parametrizes triples (X, ι_X, i_X), where X is an IHS manifold of $K3^{[n]}$-type, $\iota_X \in \mathrm{Aut}(X)$ is an anti-symplectic involution whose action on $H^2(X, \mathbb{Z})$ is the reflection in the class of an ample divisor D with $q_X(D) = 2$, and $i_X : \langle 2\rangle \hookrightarrow \mathrm{NS}(X)$ is a primitive embedding such that $i(\langle 2\rangle)$ contains the class of D. Such an X is said to be $\langle 2\rangle$-*polarised*. Then we have the following result.

Theorem 4.5.4 (Corollary 4.1 and Theorem 5.2 in [BCMS19])**.** *Any two points $(X, \iota_X, i_X), (Y, \iota_Y, i_Y) \in \mathcal{M}^{\rho}_{\langle 2\rangle}$, where $\mathcal{M}^{\rho}_{\langle 2\rangle}$ is the moduli space described above, are deformation equivalent.*

We can now prove the following corollary of Theorem 4.5.4.

Corollary 4.5.5. *Let X be a smooth birational model of a Hilbert square of a generic K3 surface S_{2t} such that X admits an ample divisor D with $q_X(D) = 2$. Let $\iota : X \to X$ be the anti-symplectic involution which generates* $\mathrm{Aut}(X)$*. Then the locus $F = \mathrm{Fix}(\iota)$ of the points fixed by ι is connected.*

Proof. By Theorem 4.5.4, the triples $(X, \iota_X, i_X) \in \mathcal{M}^{\rho}_{\langle\iota\rangle}$, where X is as in the statement of the corollary, are deformation equivalent. Since deformation equivalence preserves topological properties of $\mathrm{Fix}(\iota)$, it suffices to find a triple $(X, \iota_X, i_X) \in \mathcal{M}^{\rho}_{\langle\iota\rangle}$ such that $\mathrm{Fix}(\iota_X)$ is connected, where X is as in the statement of the corollary. Let $X = S_4^{[2]}$ be the Hilbert square of the generic smooth quartic surface of $\mathbb{P}^3$, and let $\iota_X \in \mathrm{Aut}(X)$ be the Beauville involution of Section 4.4.1. Then by [Wel81, Corollary 3.4.4], we have that $\mathrm{Fix}(\iota_X)$ is connected. This proves the corollary. $\square$

Another way to obtain the connectedness of the fixed locus $F = \mathrm{Fix}(\iota) \subset X$ is using the following theorem in [FMOS20].

Theorem 4.5.6 (Main Theorem in [FMOS20])**.** *Let (X, λ) be a polarized IHS manifold of $K3^{[n]}$-type such that $q_X(\lambda) = 2$, and let $\iota \in \mathrm{Aut}(X)$ be the involution associated to λ. Then the number of connected components of* $\mathrm{Fix}(\iota)$ *is equal to the divisibility* $\mathrm{div}(\lambda)$ *in the lattice* $(H^2(X, \mathbb{Z}), q_X)$*.*

If X is a smooth birational model of the Hilbert square of a generic K3 surface S_{2t} which admits an ample divisor $D \in \mathrm{Div}(X)$ with $q_X(D) = 2$, then $D = bh - a\delta$, where (a, b) is the minimal solution of the Pell-type equation $P_t(-1)$. Note that the integers a and b are coprime, and b is odd by Proposition 1.5.8. Moreover $H^2(S_{2t}, \mathbb{Z})$ is unimodular, so there exists $x \in H^2(X, \mathbb{Z})$ such that $(h, x) = 1$. Note that if the variety X is of the form X' in Case 3 of Section 4.3, we are using the isomorphism $H^2(X, \mathbb{Z}) \cong H^2(S_{2t}^{[2]}, \mathbb{Z})$ of Theorem 2.2.39, which is compatible with the BBF forms. Thus there exist $\alpha, \beta \in \mathbb{Z}$ such that

$$(D, \alpha\delta + \beta x) = b\beta + 2a\alpha = 1.$$

This shows that $\mathrm{div}(D) = 1$. We conclude by Theorem 4.5.6 that $F = \mathrm{Fix}(\iota)$ has one connected component.

4.5.2 Action induced by ι on $|D|$

Let X the Hilbert square of a generic K3 surface S_{2t} such that X admits an ample divisor D with $q_X(D) = 2$, and let ι be the anti-symplectic involution which generates $\mathrm{Aut}(X)$. By [CGM19, §3], the quotient variety $X/\langle\iota\rangle$ is singular along $\pi'(F)$, where $\pi' : X \to X/\langle\iota\rangle$ is the quotient map, more precisely the singularities of $X/\langle\iota\rangle$ are canonical and not terminal. Hence the desingularization of $X/\langle\iota\rangle$, which we call W, is the blow-up of $X/\langle\iota\rangle$ on its singular locus. Equivalently, consider the blow-up $\mathrm{Bl}_F(X)$ of X in the fixed locus F. The involution ι gives rise to an involution $\tilde{\iota}$ on $\mathrm{Bl}_F(X)$ which fixes the exceptional divisor $E \subset \mathrm{Bl}_F(X)$. One can show that the quotient $\mathrm{Bl}_F(X)/\langle\tilde{\iota}\rangle$ is isomorphic to W, obtaining the following commutative diagram, see [CGM19, Theorem 3.6] for more details:

$$\begin{tikzcd} \mathrm{Bl}_F(X) \arrow[r, "\pi"] \arrow[d, "\beta"] & W \arrow[d, "\beta'"] \\ X \arrow[r, "\pi'"] & X/\langle\iota\rangle \end{tikzcd} \quad . \tag{4.5.1}$$

Let $B \in \mathrm{Div}(W)$ be the branch divisor of $\pi : \mathrm{Bl}_F(X) \to W$. Then there exists a divisor $N \in \mathrm{Div}(W)$ such that $\mathcal{O}_W(2N) \cong \mathcal{O}_W(B)$ in $\mathrm{Pic}(W)$, see Section 1.6.

Let D be the ample divisor on X with $q_X(D) = 2$, and $\varphi_{|D|} : X \dashrightarrow \mathbb{P}^5$ be the map induced by the complete linear system $|D|$. We show the following result, similar to [vGS07, Proposition 2.7,(2)] obtained in the case of K3 surfaces admitting a symplectic involution.

Proposition 4.5.7. *Keep notation as above. Consider the diagram in* (4.5.1). *Let $\mathcal{D} := \mathcal{O}_X(D)$ and $\mathcal{N} := \mathcal{O}_W(N)$. There exists a line bundle $\mathcal{D}_W \in \mathrm{Pic}(W)$ such that $\pi^*\mathcal{D}_W = \beta^*\mathcal{D}$. Moreover, the vector space $H^0(X, \mathcal{D})$ decomposes as*

$$H^0(X, \mathcal{D}) \cong H^0(W, \mathcal{D}_W) \oplus H^0(W, \mathcal{D}_W - \mathcal{N}), \tag{4.5.2}$$

which is the decomposition of $H^0(X, \mathcal{D})$ into ι^*-eigenspaces.

In order to prove Proposition 4.5.7, we need the following technical lemma, see [DHH+15, Lemma 4.3] for a more general statement.

Lemma 4.5.8. *Let $\pi : X \to Y$ be a double cover of a smooth projective variety such that the branch locus is a smooth prime divisor $B \in \mathrm{Div}(Y)$, and denote by $\iota \in \mathrm{Aut}(X)$ the involution associated to the double cover. If $\mathrm{Pic}(X)^\iota$ is the subgroup of ι-invariant line bundles on X, then $\pi^*\mathrm{Pic}(Y) \cong \mathrm{Pic}(X)^\iota$.*

Proof Lemma 4.5.8. We show that every divisor $D \in \mathrm{Div}(X)$ whose class is in $\mathrm{Pic}(X)^\iota$ is linearly equivalent to a pullback of a divisor on Y.

Suppose that $\iota_*(D) = D$ as divisors, where $\iota_* : \mathrm{Div}(X) \to \mathrm{Div}(X)$ is the pushforward defined in [Ful13, §1.4]: then D is of the form $a_1E_1 + \cdots + a_sE_s$, where $a_i \in \mathbb{Z}$ and the E_i's are reduced invariant divisors, i.e., every E_i is the sum of all the prime divisors contained in a $\langle\iota\rangle$-orbit. Thus it suffices to consider the case $D = E_i$. If D is the ramification divisor, i.e., $\pi_*(D) = B$, then we apply Lemma 1.6.3, (i). If $\pi_*(D) \neq B$, since by assumption $\iota_*(D) = D$, we have $\mathrm{Supp}(D) = \pi^{-1}(\mathrm{Supp}(\pi_*(D)))$ and π is not ramified in $X \setminus \pi^{-1}(\mathrm{Supp}(B))$, so $D = \pi^*\pi_*(D)$.

Suppose now that $\iota_*(D) \neq D$. By definition ι acts trivially on $\mathrm{Pic}(X)^\iota$, so $D - \iota_*(D) = \mathrm{div}(g)$ for some $g \in \mathcal{M}_X^*(X)$. Consider the following map

$$\tau : H^0(X, \mathcal{O}_X(D)) \to H^0(X, \mathcal{O}_X(D)), \qquad f \mapsto \frac{\iota^* f}{g}.$$

Note that $\frac{\iota^* f}{g} \in H^0(X, \mathcal{O}_X(D))$: indeed,

$$\begin{aligned} \mathrm{div}\left(\tfrac{\iota^* f}{g}\right) + D &= \mathrm{div}(\iota^* f) - \mathrm{div}(g) + D \\ &= \iota_*(\mathrm{div}(f)) - \mathrm{div}(g) + D \\ &= \iota_*(\mathrm{div}(f)) + \iota_*(D) - D + D \\ &= \iota_*(\mathrm{div}(f)) + \iota_*(D), \end{aligned}$$

and, since ι is a regular involution, $\iota_*(\mathrm{div}(f)) + \iota_*(D)$ is effective if and only if $\mathrm{div}(f) + D \geq 0$, which is true since $f \in H^0(X, \mathcal{O}_X(D))$.

We show that τ is a linear isomorphism and, after rescaling g, the order of τ divides 2. Since $D - \iota_*(D) = \mathrm{div}(g)$ and $\iota_*(D) - D = \mathrm{div}(\iota^* g)$, we have $\mathrm{div}(g) + \mathrm{div}(\iota^* g) = 0$. Then $g\iota^* g = 1$ after rescaling g. Hence

$$f \overset{\tau}{\mapsto} \frac{\iota^* f}{g} \overset{\tau}{\mapsto} \iota^* \left(\frac{\iota^* f}{g}\right) \cdot \frac{1}{g} = \frac{f}{g\iota^* g} = f,$$

which shows that the order of τ divides 2. Let now $h \in H^0(X, \mathcal{O}_X(D))$ be an eigenvector of τ. We set $D' := \mathrm{div}(h) + D$. Then D' is ι-invariant, i.e., $\iota_*(D') = D'$ as divisors, since

$$\begin{aligned}
\iota_*(D') &= \mathrm{div}(\iota^* h) + \iota_*(D) \\
&= \mathrm{div}(g \cdot \tau(h)) + D - \mathrm{div}(g) \\
&= \mathrm{div}(g) + \mathrm{div}(\pm h) + D - \mathrm{div}(g) \\
&= \mathrm{div}(h) + D \\
&= D'.
\end{aligned}$$

By the first part of the proof D' is a pullback, then D is linearly equivalent to a pullback of a divisor. □

Proof of Proposition 4.5.7. Consider $\pi : \mathrm{Bl}_F(X) \to W$ appearing in diagram (4.5.1): by Lemma 4.5.8 we obtain a divisor $D_W \in \mathrm{Div}(W)$ whose class $\mathcal{D}_W$ in $\mathrm{Pic}(W)$ is such that $\pi^*\mathcal{D}_W = \beta^*\mathcal{D}$ in $\mathrm{Pic}(\mathrm{Bl}_F(X))$. Lemma 4.5.8 can be applied: $F \subset X$ is smooth by Lemma 4.5.2 and connected by Corollary 4.5.5, so the exceptional divisor of β and the branch divisor of π are smooth prime divisors. Since β is a birational morphism, we have $H^0(X, \mathcal{D}) \cong H^0(\mathrm{Bl}_F(X), \beta^*\mathcal{D})$, which is isomorphic to $H^0(\mathrm{Bl}_F(X), \pi^*\mathcal{D}_W)$, being $\beta^*\mathcal{D} = \pi^*\mathcal{D}_W$. Moreover,

$$\begin{aligned}
\pi_*(\pi^*\mathcal{D}_W) &\cong \pi_*(\pi^*\mathcal{D}_W \otimes \mathcal{O}_{\mathrm{Bl}_F(X)}) \\
&\cong \mathcal{D}_W \otimes \pi_*\mathcal{O}_{\mathrm{Bl}_F(X)}
\end{aligned}$$

where the second isomorphism is obtained by applying the Projection formula. By Lemma 1.6.4 we have

$$\pi_*\mathcal{O}_{\mathrm{Bl}_F(X)} \cong \mathcal{O}_W \oplus \mathcal{O}_W(-N),$$

where $B \in \mathrm{Div}(W)$ is the branch divisor and $N \in \mathrm{Div}(W)$ is the divisor such that $\mathcal{O}_W(2N) \cong \mathcal{O}_W(B)$ in $\mathrm{Pic}(W)$. Thus we obtain the isomorphism

$$\pi_*(\pi^*\mathcal{D}_W) \cong \mathcal{D}_W \otimes (\mathcal{O}_W \oplus \mathcal{O}_W(-N)) \cong \mathcal{D}_W \oplus (\mathcal{D}_W - \mathcal{N}).$$

Hence decomposition (4.5.2) holds. Moreover, this is the decomposition of $H^0(X, \mathcal{D})$ in ι^*-eigenspaces. Indeed, two global sections $s, t \in H^0(X, \mathcal{D})$ are in the same eigenspace if and only if the rational function $f = s/t$ is ι-invariant, and this is true when the sections belong both to $H^0(W, \mathcal{D}_W)$ or $H^0(W, \mathcal{D}_W - \mathcal{N})$ in the decomposition, hence each of these two spaces is contained in an eigenspace of $H^0(X, \mathcal{D})$. We conclude that $H^0(W, \mathcal{D}_W)$ and $H^0(W, \mathcal{D}_W - \mathcal{N})$ are isomorphic to the two eigenspaces by decomposition (4.5.2). □

Let $\mathcal{D}_W \in \mathrm{Pic}(W)$ be the line bundle such that $\pi^*\mathcal{D}_W \cong \beta^*\mathcal{D}$ given by Proposition 4.5.7. We recall the following relation between the dimension of the vector space $H^0(W, \mathcal{D}_W)$ and $(D|_F)^2$, see [CGM19, Proposition 7.3] for a more general statement.

Proposition 4.5.9 (Camere–Garbagnati–Mongardi)**.** *Keep notation as above. Then*

$$\dim(H^0(W, \mathcal{D}_W)) = \frac{7}{2} + \frac{1}{16}(D|_F)^2.$$

Proof. By [CGM19, Proposition 7.3] we have

$$\chi(\mathcal{D}_W) = \frac{1}{2}\chi(\mathcal{D}) + \frac{1}{16}(D|_F)^2 - \frac{1}{2}\chi(\mathcal{O}_X) + \chi(\mathcal{O}_W).$$

Since W is a Calabi–Yau variety and X is an IHS manifold of $K3^{[2]}$-type, we have the following equalities:

- $\chi(\mathcal{D}) = \frac{1}{8}(q_X(D)+4)(q_X(D)+6)$ by Theorem 3.1.9.
- $\chi(\mathcal{O}_W) = 2$, since by definition of Calabi–Yau variety $\dim(H^{i,0}(W)) = 0$ for $i = 1, \dots, \dim(W) - 1$.
- $\chi(\mathcal{O}_X) = 3$ because $\dim(H^{i,0}(X)) = 0$ for i odd and $\dim(H^{2,0}(X)) = 1$.
- By Theorem 1.1.29 we have

  $$\chi(\mathcal{D}_W) = \dim(H^0(W, \mathcal{D}_W)) \qquad \text{and} \qquad \chi(\mathcal{D}) = \dim(H^0(X, \mathcal{D})),$$

  since D is ample, D_W is big and nef by [CGM19, Lemma 7.1] and the canonical bundles of X and W are trivial.
- $\dim(H^0(X, \mathcal{D})) = 6$ by Theorem 3.1.9, since $q_X(D) = 2$ by assumption.

We conclude that

$$\dim(H^0(W, \mathcal{D}_W)) = 3 + \frac{1}{16}(D|_F)^2 - \frac{3}{2} + 2 = \frac{7}{2} + \frac{1}{16}(D|_F)^2,$$

as we wanted. $\square$

Another useful result is the following computation of the Euler characteristic of the fixed locus of the anti-symplectic involution ι by Beauville. We give the result in our setting, see [Bea11, Theorem 2] for a more general statement.

Lemma 4.5.10. *Let X be the Hilbert square of a generic K3 surface S_{2t} such that X admits an ample divisor D with $q_X(D) = 2$ Let $\iota : X \to X$ be the anti-symplectic involution which generates* $\mathrm{Aut}(X)$. *Let $F = \mathrm{Fix}(\iota)$ be the fixed locus of ι. Then the Euler characteristic of F is*

$$\chi(F) = 192.$$

Proof. By [Bea11, Theorem 2], we have $\chi(F) = \frac{1}{2}(t^2 + 23)$, where t is the trace of ι^* acting on $H^{1,1}(X)$. If a and b are the dimensions respectively of the $(+1)$ and (-1)-eigenspaces of ι^* on $H^2(X)$, then $a + b = 23$ and $a - b = t - 2$. In our case, the first eigenspace is generated by the class of D, hence $a = 1$ and $b = 22$, hence $t = -19$ and $\chi(F) = 192$. $\square$

Similarly to the case studied in [vGS07, §2.6], since $\iota^* D \cong D$, there is an induced involution on $\mathbb{P}(H^0(X, \mathcal{D})^\vee)$, which has two fixed spaces $\mathbb{P}^a$ and $\mathbb{P}^b$, where $a + 1 + b + 1 = 6$, and $a = -1$ if the corresponding eigenspace of ι^* on $H^0(X, \mathcal{D})$ is zero, similarly for b. By Proposition 4.5.7 the direct sum (4.5.2) is the direct sum of the ι^*-eigenspaces of $H^0(X, \mathcal{D})$, so $\mathbb{P}^a$ and $\mathbb{P}^b$ are isomorphic respectively to $\mathbb{P}(H^0(W, \mathcal{D}_W)^\vee)$ and to $\mathbb{P}(H^0(W, \mathcal{D}_W - \mathcal{N})^\vee)$. If $\{s_0, \dots, s_5\}$ is a basis of $H^0(X,\mathcal{D})$, the induced involution $\bar\iota$ on $\mathbb{P}(H^0(X,\mathcal{D})^\vee)$ is described by the following:

$$\begin{tikzcd} X \arrow[r, dashed, "\varphi_{|D|}"] \arrow[d, "\iota"] & \mathbb{P}(H^0(X,\mathcal{D})^\vee) \arrow[d, "\bar\iota"] \\ X \arrow[r, dashed, "\varphi_{|D|}"] & \mathbb{P}(H^0(X,\mathcal{D})^\vee), \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & (s_0(x):\cdots:s_5(x)) \arrow[d, mapsto] \\ \iota(x) \arrow[r, mapsto] & (s_0(\iota(x)):\cdots:s_5(\iota(x))). \end{tikzcd}$$

Then $\varphi_{|D|}$ factors through the quotient $X\to X/\langle\iota\rangle$ if and only if the action of $\bar\iota$ on $\mathbb{P}(H^0(X,\mathcal{D})^\vee)$ is trivial, which happens if and only if either $H^0(W,\mathcal{D}_W)$ or $H^0(W,\mathcal{D}_W-\mathcal{N})$ is zero. We can now state the main theorem of this section.

Theorem 4.5.11. *Let X be the Hilbert square of a generic K3 surface S_{2t} such that X admits an ample divisor D with $q_X(D)=2$. Let $\iota: X\to X$ be the anti-symplectic involution which generates $\mathrm{Aut}(X)$. Let $F=\mathrm{Fix}(\iota)$ be the fixed locus of ι. Then*

$$[F]=5D^2-\frac{2}{5}q_X^\vee\in H^{2,2}(X,\mathbb{Z}), \tag{4.5.3}$$

where $[F]$ denotes the fundamental cohomological class of F in $H^{2,2}(X,\mathbb{Z})$. Moreover, let $\varphi_{|D|}: X\dashrightarrow\mathbb{P}^5$ be the map induced by the complete linear system $|D|$. Then the following diagram is commutative:

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi_{|D|}"] \arrow[dr, "\pi'"'] & & \mathbb{P}^5 \\ & X/\langle\iota\rangle, \arrow[ur, dashed] & \end{tikzcd} \tag{4.5.4}$$

where $\pi': X\to X/\langle\iota\rangle$ is the quotient map.

Proof. Let $\mathcal{D}_W\in\mathrm{Pic}(W)$ be the line bundle such that $\pi^*\mathcal{D}_W\cong\beta^*\mathcal{D}$. By Proposition 4.5.9 we have

$$\dim(H^0(W,\mathcal{D}_W))=\frac{7}{2}+\frac{1}{16}(D|_F)^2.$$

By decomposition (4.5.2) and $\dim(H^0(X,\mathcal{D}))=6$ we get

$$\dim(H^0(W,\mathcal{D}_W))\in\{0,1,\dots,6\},$$

so we obtain the following possible values for $(D|_F)^2$:

$$\begin{aligned} \dim(H^0(W,\mathcal{D}_W))=0 &\quad\Leftrightarrow\quad (D|_F)^2=-56.\\ \dim(H^0(W,\mathcal{D}_W))=1 &\quad\Leftrightarrow\quad (D|_F)^2=-40.\\ \dim(H^0(W,\mathcal{D}_W))=2 &\quad\Leftrightarrow\quad (D|_F)^2=-24.\\ \dim(H^0(W,\mathcal{D}_W))=3 &\quad\Leftrightarrow\quad (D|_F)^2=-8.\\ \dim(H^0(W,\mathcal{D}_W))=4 &\quad\Leftrightarrow\quad (D|_F)^2=8.\\ \dim(H^0(W,\mathcal{D}_W))=5 &\quad\Leftrightarrow\quad (D|_F)^2=24.\\ \dim(H^0(W,\mathcal{D}_W))=6 &\quad\Leftrightarrow\quad (D|_F)^2=40. \end{aligned} \tag{4.5.5}$$

Since D is ample, $D|_F$ is ample on F by Proposition 1.1.15, hence $(D|_F)^2 > 0$ by Theorem 1.1.12. This implies that

$$\dim(H^0(W, \mathcal{D}_W)) \in \{4, 5, 6\}.$$

We show that $\dim(H^0(W, \mathcal{D}_W)) = 6$ by computing $[F] \in H^{2,2}(X, \mathbb{Z})$, where we denote by $[F]$ the fundamental cohomological class of the fixed locus $F = \mathrm{Fix}(\iota)$ of the involution ι. Let $h \in \mathrm{Pic}(X)$ be the line bundle induced by the ample generator of $\mathrm{Pic}(S_{2t})$. By Proposition 3.4.1 we can write

$$[F] = xh^2 + yh\delta + z\delta^2 + w \cdot \frac{2}{5} q_X^\vee \in H^{2,2}(X, \mathbb{Z}), \tag{4.5.6}$$

with $x, y, z, w \in \mathbb{Q}$ to determine. Recall that $D = bh - a\delta$, with (a, b) minimal solution of the Pell-type equation $P_t(-1)$. We denote by $\langle \cdot\,, \cdot \rangle$ the bilinear form of $H^4(X, \mathbb{Z})$ of Proposition 3.1.5. We have the following four conditions.

1. $\langle [F], (\sigma + \bar{\sigma})^2 \rangle = 0$, where σ is the symplectic form $\sigma \in H^0(X, \Omega_X^2)$, since F is Lagrangian by Lemma 4.5.2. If $\eta := (\sigma + \bar{\sigma}, \sigma + \bar{\sigma})$, we have:

$$\langle h^2, (\sigma + \bar{\sigma})^2 \rangle = q_X(h)\eta + 2(h, \sigma + \bar{\sigma}) = 2t\eta, \tag{4.5.7}$$

and similarly

$$\begin{aligned} &\langle \delta^2, (\sigma + \bar{\sigma})^2 \rangle = -2\eta \\ &\langle h\delta, (\sigma + \bar{\sigma})^2 \rangle = 0 \\ &\langle \tfrac{2}{5} q_X^\vee, (\sigma + \bar{\sigma})^2 \rangle = 10\eta. \end{aligned} \tag{4.5.8}$$

Note that $\eta \neq 0$, see Definition 2.2.3 and Theorem 2.2.4, so we obtain from $\langle [F], (\sigma + \bar{\sigma})^2 \rangle = 0$ and (4.5.6) the following condition:

$$tx - z + 5w = 0. \tag{4.5.9}$$

2. From Lemma 4.5.10 we have $c_2(F) = 192$, i.e., $\langle [F], [F] \rangle = 192$. This gives, together with (4.5.6), the following condition:

$$3t^2x^2 - ty^2 + 3z^2 + 23w^2 - 2txz + 10xwt - 10zw = 48. \tag{4.5.10}$$

3. Consider the action induced by ι on $H^{2,2}(X, \mathbb{Q})$, described in the basis $\{h^2, h\delta, \delta^2, q_X^\vee\}$ by

$$\begin{aligned} &\iota^*(h^2) = \iota^* h \cdot \iota^* h, \\ &\iota^*(h\delta) = \iota^* h \cdot \iota^* \delta, \\ &\iota^*(\delta^2) = \iota^* \delta \cdot \iota^* \delta, \\ &\iota^*(q_X^\vee) = q_X^\vee, \end{aligned} \tag{4.5.11}$$

where $\cdot$ denotes the cup product. Since F is the locus of points fixed by ι, we have $\iota^*([F]) = [F]$, i.e., if

$$\iota^*([F]) = \tilde{x}h^2 + \tilde{y}h\delta + \tilde{z}\delta^2 + \frac{2}{5}\tilde{w} q_X^\vee,$$

with $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w} \in \mathbb{Q}$, then $x = \tilde{x}$, $y = \tilde{y}$, $z = \tilde{z}$, $w = \tilde{w}$. The last equation comes from the equality $c_2(X) = \frac{6}{5} q_X^\vee$ of Proposition 3.1.8 and the fact that ι is an automorphism. Alternatively, from Theorem 2.2.39 we have

$q_X(\alpha) = q_X(\iota(\alpha))$ for every $\alpha \in H^2(X, \mathbb{Z})$. Imposing $x = \tilde{x}$ we obtain the following condition:

$$x - c^2x - cdy - d^2z = 0. \tag{4.5.12}$$

One remarks a posteriori that $y = \tilde{y}$, $z = \tilde{z}$ and $w = \tilde{w}$ give the same condition (4.5.12).

4. By $(D|_F)^2 \in \{8, 24, 40\}$ we have $\langle [F], D^2 \rangle \in \{8, 24, 40\}$, since the bilinear form $\langle \cdot , \cdot \rangle$ represents the intersection form by Proposition 3.1.5. This gives, together with (4.5.6), the following:

$$\begin{aligned} (t + 2t^2b^2)x + 2abty + (2a^2 - 1)z + 5w = 2 \quad &\Longleftrightarrow \quad \langle [F], D^2 \rangle = 8, \\ (t + 2t^2b^2)x + 2abty + (2a^2 - 1)z + 5w = 6 \quad &\Longleftrightarrow \quad \langle [F], D^2 \rangle = 24, \\ (t + 2t^2b^2)x + 2abty + (2a^2 - 1)z + 5w = 10 \quad &\Longleftrightarrow \quad \langle [F], D^2 \rangle = 40. \end{aligned} \tag{4.5.13}$$

If $\langle [F], D^2 \rangle = 8$, which is equivalent to $\dim(H^0(W, \mathcal{D}_W)) = 4$ by (4.5.5), the system given by (4.5.9), (4.5.10), (4.5.12) and the first condition in (4.5.13) has solutions with $x, y, z, w \notin \mathbb{Q}$, which is impossible. We obtain the same contradiction if $\langle [F], D^2 \rangle = 24$, i.e., if $\dim(H^0(W, \mathcal{D}_W)) = 5$. We conclude that $\langle [F], D^2 \rangle = 40$, which is equivalent to $\dim(H^0(W, \mathcal{D}_W)) = 6$. With the help of a computer, the system given by (4.5.9), (4.5.10), (4.5.12) and the third condition in (4.5.13) implies $w \in \{-1, -\frac{13}{12}\}$. By Corollary 3.4.11 we cannot have $w = -\frac{13}{12}$, hence $w = -1$. Imposing $w = -1$, we necessarily obtain only one admissible solution, which is the following:

$$\begin{cases} x = 5b^2, \\ y = -10ab, \\ z = 5a^2, \\ w = -1. \end{cases}$$

We conclude that

$$[F] = 5D^2 - \frac{2}{5}q_X^\vee \in H^{2,2}(X, \mathbb{Z}).$$

Moreover, we have obtained

$$H^0(X, \mathcal{D}) \cong H^0(W, \mathcal{D}_W),$$

so by (4.5.2) we have $H^0(W, \mathcal{D}_W - \mathcal{N}) = \{0\}$, which shows that the action $\bar{\iota}$ on $\mathbb{P}(H^0(X, \mathcal{D})^\vee)$ is trivial, i.e., $\varphi_{|D|}$ factors through the quotient $X \to X/\langle\iota\rangle$. Thus we obtain the commutative diagram (4.5.4). □

We conclude this section with a useful corollary of Theorem 4.5.11. We first need to introduce some notation. We follow [O'G08b, §4]. Let X and D be as in Theorem 4.5.11, and consider

$$\varphi_{|D|} : X \dashrightarrow \mathbb{P}^5.$$

More precisely, we choose an isomorphism $|D|^\vee \xrightarrow{\sim} \mathbb{P}^5$ and we denote by

$$f : X \dashrightarrow \mathbb{P}^5$$

the composition $X \dashrightarrow |D|^\vee \xrightarrow{\sim} \mathbb{P}^5$. Let B be the base locus of $|D|$, and $\beta_B : \tilde{X} \to X$ be the blow-up of X in B. We denote by $\tilde{f} : \tilde{X} \to \mathbb{P}^5$ the regular map which resolves the indeterminacies of f. Let $Y := \mathrm{Im}(\tilde{f})$, which is a closed subset of $\mathbb{P}^5$. We then obtain a dominant map, which we call f by abuse of notation

$$f : X \dashrightarrow Y.$$

Let $\deg(f)$ be the degree of f.

Corollary 4.5.12. *Let X and D be as in Theorem* 4.5.11. *Let $f : X \dashrightarrow Y \subset \mathbb{P}^5$ be the map induced by the complete linear system $|D|$. Then* $\deg(f)$ *is even.*

Proof. By Theorem 4.5.11 the map f factors through $\pi' : X \to X/\langle\iota\rangle$, which has degree two, so $\deg(f)$ is even. □

Remark 4.5.13. Ferretti in [Fer12, Lemma 4.1] obtained the same relation of (4.5.3) in the Chow ring of a smooth double EPW sextic X: in his case F was the branch locus of the double cover $f : X \to Y \subseteq \mathbb{P}^5$, where Y is an EPW sextic.

4.6 Case $t \neq 2, 5$, first part

Let $X = S_{2t}^{[2]}$ be the Hilbert square of a generic K3 surface of degree $2t$ such that there exists an ample divisor $D \in \mathrm{Div}(X)$ with $q_X(D) = 2$. In the first part of this section we recall a result by Rieß in [Rie18] which describes the divisorial base component of the complete linear system $|D|$, in particular we will see that $|D|$ is movable, i.e., its fixed part is zero, see Definition 1.1.22. In the second part of this section we study the reducibility of the surface $D_1 \cap D_2$ for the case $t \neq 2$, where $D_1, D_2 \in |D|$. In the final part we show which of the results obtained hold for a smooth birational model X of the Hilbert square of a generic K3 surface of degree $2t$ such that X admits an ample divisor D with $q_X(D) = 2$ and with both $P_t(-1)$ and $P_{4t}(5)$ solvable, see Case 3 of Section 4.3.

4.6.1 Divisorial base component

Let X be an IHS manifold of $K3^{[n]}$-type, and consider a big and nef divisor $D \in \mathrm{Div}(X)$. The fixed part of the complete linear system $|D|$ is described by the following result obtained by Rieß in [Rie18, Proposition 8.2], which can be seen as a partial generalization of Theorem 2.1.7. See [Rie18, Theorem 4.7] for a more general statement.

Theorem 4.6.1 (Rieß)**.** *Let X be an IHS manifold of $K3^{[n]}$-type and consider a big and nef divisor $D \in \mathrm{Div}(X)$. Then $|D|$ has a fixed part if and only if $D = mL + F$, where $m \geq 2$, the class of L is movable with $q_X(L) = 0$ and F is a reduced and irreducible divisor with $q_X(F) < 0$ and $(L, F) = 1$. In this case F is the fixed part of $|D|$.*

We obtain the following corollary.

Corollary 4.6.2. *Let X be the Hilbert square of a generic K3 surface S_{2t} of degree $2t$. Consider a big and nef divisor $D \in \mathrm{Div}(X)$. Then $|D|$ has no fixed part.*

Proof. Let $h \in \operatorname{Pic}(S_{2t}^{[2]})$ be the line bundle induced by the ample generator of $\operatorname{Pic}(S_{2t})$. Then $\operatorname{Pic}(X) = \mathbb{Z}h \oplus \mathbb{Z}\delta$, and $q_X(h) = 2t$, $q_X(\delta) = -2$, $(h,\delta) = 0$. Hence (D_1, D_2) is an even integer for every $D_1, D_2 \in \operatorname{Div}(X)$, in particular we have $(D_1, D_2) \neq 1$. By Theorem 4.6.1 the complete linear system $|D|$ has fixed part if and only if $D = mL + F$, with $m \geq 2$, where L is movable with $q_X(L) = 0$ and F is a reduced and irreducible divisor of negative square with $(L, F) = 1$. The Gram matrix for $\operatorname{Pic}(X)$ in the basis $\{h, -\delta\}$ is

$$\begin{pmatrix} 2t & 0 \\ 0 & -2 \end{pmatrix},$$

hence there are no $L, F \in \operatorname{Pic}(X)$ such that $(L, F) = 1$. We conclude that the complete linear system $|D|$ has no fixed part. $\square$

While Theorem 2.1.7 completely characterises the base locus of a complete linear system associated to a big and nef divisor on a K3 surface, for higher dimensional IHS manifolds of $K3^{[n]}$-type Theorem 4.6.1 does not give any information on the non-divisorial component of the base locus of the complete linear system $|D|$. There exist examples of complete linear systems on Hilbert squares of a generic K3 surface S_{2t} of degree $2t$ whose base locus is non empty and of codimension greater than 1: if $X = S_2^{[2]}$ and $\operatorname{Pic}(X) = \mathbb{Z}h \oplus \mathbb{Z}\delta$, where as usual h is the line bundle induced by the ample generator of $\operatorname{Pic}(S_2)$, Rieß has shown in [Rie20] that the base locus of the complete linear system $|D|$, where D is the ample divisor with class $2h - \delta$, is isomorphic to $\mathbb{P}^2$.

4.6.2 Reducibility of the surface $D_1 \cap D_2$

Let $X = S_{2t}^{[2]}$ be the Hilbert square of a generic K3 surface such that X admits an ample divisor D with $q_X(D) = 2$. In this section we study the reducibility of the surface $D_1 \cap D_2$, where $D_1, D_2 \in |D|$ are distinct divisors. The case $t = 2$ will be studied separately: it is the only case in which $D_1 \cap D_2$ can be reducible. If $t \neq 2$, we will show that $D_1 \cap D_2$ is reduced and irreducible.

First of all, we show that every divisor in the complete linear system $|D|$ is reduced and irreducible.

Proposition 4.6.3. *Let X and D be as in Theorem* 4.5.11*. Then every divisor $D' \in |D|$ is reduced and irreducible.*

Proof. By abuse of notation, we write D for an effective divisor D' which belongs to the complete linear system $|D|$. Since $q_X(D) = 2$, the divisor D is reduced, i.e., it is not of the form $D = \alpha E$ with $\alpha \in \mathbb{Z}$, $\alpha \neq \pm 1$, and $E \in \operatorname{Div}(X)$. Suppose by contradiction that

$$D = D_1 + D_2 \tag{4.6.1}$$

where $D_1 = \sum_i n_i D_{1,i}$ and $D_2 = \sum_j m_j D_{2,j}$ are effective divisors with $D_{1,i}, D_{2,j}$ prime divisors which are pairwise distinct, and $n_i, m_j \in \mathbb{Z}_{>0}$. Without loss of generality we can assume that D_1 has only one component, i.e., it is irreducible. We have

$$q_X(D) = 2 = q_X(D_1) + q_X(D_2) + 2(D_1, D_2). \tag{4.6.2}$$

Since D_1 and D_2 are effective divisors with no common components, we can apply Proposition 2.3.4, obtaining $(D_1, D_2) \geq 0$. Note that we cannot have neither $q_X(D_1) = 0$ nor $q_X(D_2) = 0$. Indeed, the equation $q_X(xh - y\delta) = 0$ has a non zero solution $(x, y) \in \mathbb{Z}^2$ only when t is a perfect square. If t was a perfect square, the Pell-type equation $P_t(-1)$ would be solvable only for $t = 1$, which is a case that we are not considering. Hence t is not a perfect square and we have $q_X(D_1) \neq 0$, $q_X(D_2) \neq 0$.

Since $H^2(X, \mathbb{Z})$ is an even lattice, we have only two possibilities: either one between D_1 and D_2 is zero, or (at least) one of the two has negative square with respect to the BBF form. We show that $q_X(D_i) > 0$ for $i = 1, 2$. Assume by contradiction that at least one between D_1 and D_2 has negative square. This can happen only if there exists a component $D_{1,i}$ or $D_{2,j}$ whose square with respect to the BBF form is negative. Without loss of generality, we can suppose that $q_X(D_{1,1}) < 0$. Recall that $H^2(X, \mathbb{Z}) \cong H^2(S_{2t}, \mathbb{Z}) \oplus \langle -2 \rangle$, where S_{2t} is a K3 surface, and $H^2(S_{2t}, \mathbb{Z})$ is a unimodular lattice. Then the divisibility in $H^2(X, \mathbb{Z})$ of a primitive class can be only 1 or 2. Hence by Lemma 2.3.5 we have either $q_X(D_{1,1}) = -2$ or $q_X(D_{1,1}) = -4$. We show that $q_X(D_{1,1}) = -4$ is not possible. Indeed, if $t = 2$, the class of $D_{1,1}$ in $\mathrm{Pic}(X)$ is necessarily $h - 2\delta$, which is outside the pseudoeffective cone, whose extremal rays are generated by δ and $2h - 3\delta$ by Theorem 4.1.1, obtaining a contradiction. If $t \neq 2$, there exists a (-4)-class if and only if the Pell-type equation $P_t(2)$ is solvable: since by assumption $P_t(-1)$ has solutions, by Proposition 1.5.7 we have that $P_t(2)$ has no solution, hence there are no (-4)-classes. We conclude that $D_{1,1}$ must be a (-2)-class, hence it is either δ or $\iota^*\delta$, where ι is the anti-symplectic involution on X which generates $\mathrm{Aut}(X)$, see Theorem 4.3.1. Since $\iota^* D = D$, it is enough to show that $D_{1,1} = \delta$ is not possible. If $D_{1,1} = \delta$, from (4.6.1) we get

$$D = n_1\delta + D_2,$$

where $n_1 \geq 1$ since D_1 is effective by assumption, hence

$$D_2 = D - n_1\delta = bh - (a + n_1)\delta,$$

where as usual (a, b) is the minimal solution of the negative Pell equation $P_t(-1)$. We show that D_2 is outside the pseudoeffective cone. By Theorem 4.1.1 the extremal rays of the pseudoeffective cone are generated by the classes δ and $\iota^*\delta = dh - c\delta$, where (c, d) is the minimal solution of the Pell equation $P_t(1)$. Using (1.5.1), we obtain $\iota^*\delta = 2abh - (a^2 + tb^2)\delta$, hence we need to check that $a + n_1 > \frac{a^2+tb^2}{2a}$ in order to show that D_2 is outside the pseudoeffective cone. This is true since $a^2 - tb^2 = -1$. We obtain a contradiction, so $q_X(D_i) \geq 0$ for $i = 1, 2$. Moreover, we have already remarked that $q_X(D_i) \neq 0$ for $i = 1, 2$, thus $q_X(D_i) > 0$. We get a contradiction with (4.6.2), so one between D_1 and D_2 is zero. If $D_2 = 0$, then $n_1 = 1$ since $q_X(D) = n_1^2 q_X(D_{1,1}) = 2$, so D is reduced and irreducible. If $D_1 = 0$, we repeat the argument for $D = D_2$ until we obtain a D which is reduced and irreducible. □

Suppose now that $D_1, D_2 \in |D|$ are two distinct divisors. We want to study the surface $D_1 \cap D_2$ and see if it is reduced and irreducible. The fundamental tool to do this is Corollary 3.4.11. First of all, we need the following technical lemma.

Lemma 4.6.4. *Let X and D be as in Theorem 4.5.11 and let $D_1, D_2 \in |D|$ be two distinct divisors. Denote by ι the anti-symplectic involution which generates* *the automorphism group* $\mathrm{Aut}(X)$*. Suppose that there is no decomposition of the form*

$$[D_1 \cap D_2] = A + B \in H^{2,2}(X, \mathbb{Z}),$$

where $[D_1 \cap D_2]$ *is the fundamental cohomological class of the surface* $D_1 \cap D_2$*, and* $A, B \in H^{2,2}(X, \mathbb{Z})$ *are effective classes such that* $\iota^*(A) = A$ *and* $\iota^*(B) = B$*. Then the surface* $D_1 \cap D_2$ *is reduced and irreducible.*

Proof. Suppose that

$$[D_1 \cap D_2] = A_1 + A_2 + \cdots + A_n \in H^{2,2}(X, \mathbb{Z}),$$

where $A_i \in H^{2,2}(X, \mathbb{Z})$ are effective classes, not necessarily pairwise distinct. Recall that by Theorem 4.3.1 we have $\iota^* D = D$, hence

$$\iota^*(A_1 + \cdots + A_n) = A_1 + \cdots + A_n.$$

If $n > 1$ is odd, then there exists i such that $\iota^* A_i = A_i$ since ι is an involution, hence we take $A := A_i$ and $B := A_1 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_n$. Thus $\iota^* A = A$ and $\iota^* B = B$. We obtain a contradiction with the assumption that there is no decomposition of this form. Suppose now that $n = 2$ and $\iota^* A_1 = A_2$. We show that this is not possible. Recall that $D = bh - a\delta$, with (a, b) minimal solution of the Pell-type equation $P_t(-1)$. By Corollary 3.4.11 a basis of the lattice $H^{2,2}(X, \mathbb{Z})$ is given by $\{h^2, \frac{h^2 - h\delta}{2}, \frac{1}{8}\left(\delta^2 + \frac{2}{5} q_X^\vee\right), \frac{2}{5} q_X^\vee\}$ (note that we have substituted δ^2 in the basis given by Corollary 3.4.11 with $\frac{2}{5} q_X^\vee$: this slightly simplifies the computations). Hence the classes A_1 and A_2 in $H^{2,2}(X, \mathbb{Z})$ are of the form

$$\begin{aligned}
A_1 &= \left(x + \tfrac{y}{2}\right) h^2 + \tfrac{z}{8} \delta^2 - \tfrac{y}{2} h\delta + \left(\tfrac{1}{8} z + w\right) \tfrac{2}{5} q_X^\vee, \\
A_2 &= \left(b^2 - x - \tfrac{y}{2}\right) h^2 + \left(a^2 - \tfrac{z}{8}\right) \delta^2 + \left(-2ab + \tfrac{y}{2}\right) h\delta - \left(\tfrac{1}{8} z + w\right) \tfrac{2}{5} q_X^\vee,
\end{aligned}$$

for some $x, y, z, w \in \mathbb{Z}$. By Theorem 4.3.1 and (4.5.11), we obtain

$$\begin{aligned}
\iota^* A_1 \quad = \quad & \left(c^2 \left(x + \tfrac{y}{2}\right) + cd \left(-\tfrac{y}{2}\right) + \tfrac{z}{8} d^2\right) h^2 \\
& + \left(t^2 d^2 \left(x + \tfrac{y}{2}\right) - cdt \tfrac{y}{2} + c^2 \tfrac{z}{8}\right) \delta^2 \\
& + \left(-2cdt \left(x + \tfrac{y}{2}\right) + c^2 \tfrac{y}{2} + td^2 \tfrac{y}{2} - 2cd \tfrac{z}{8}\right) h\delta \\
& + \left(\tfrac{1}{8} z + w\right) \tfrac{2}{5} q_X^\vee,
\end{aligned}$$

and similarly

$$\begin{aligned}
\iota^* A_2 \quad = \quad & \left(c^2 \left(b^2 - x - \tfrac{y}{2}\right) + cd \left(\tfrac{y}{2} - 2ab\right) + d^2 \left(a^2 - \tfrac{z}{8}\right)\right) h^2 \\
& + \left(t^2 d^2 \left(b^2 - x - \tfrac{y}{2}\right) + \left(\tfrac{y}{2} - 2ab\right) cdt + c^2 \left(a^2 - \tfrac{z}{8}\right)\right) \delta^2 \\
& + \left(-2cdt \left(b^2 - x - \tfrac{y}{2}\right) + (c^2 + td^2) \left(2ab - \tfrac{y}{2}\right) - 2cd \left(a^2 - \tfrac{z}{8}\right)\right) h\delta \\
& - \left(\tfrac{1}{8} z + w\right) \tfrac{2}{5} q_X^\vee.
\end{aligned}$$

Imposing $\iota^* A_1 = A_2$, we obtain a system whose solution is

$$\begin{cases}
x = \frac{1}{2} b^2 - ab, \\
y = 2ab, \\
z = 4a^2, \\
w = -\frac{a^2}{2}.
\end{cases}$$

Recall that b is odd by Proposition 1.5.8, so $x \notin \mathbb{Z}$, which is not possible. Since by assumption we cannot have $\iota^* A_1 = A_1$, one between A_1 and A_2 is zero, so $D_1 \cap D_2$ is reduced and irreducible. If $n > 1$ is even, if there exists an i such that $\iota^* A_i = A_i$, we proceed as in the case of n odd, otherwise without loss of generality we can assume that $\iota^* A_1 = A_2$. Then, taking $A := A_1 + A_2$ and $B := A_3 + \cdots + A_n$, we have $\iota^* A = A$ and $\iota^* B = B$, which contradicts the assumption. We conclude that $D_1 \cap D_2$ is reduced and irreducible. $\square$

We can now state the main theorem of this section.

Theorem 4.6.5. *Let X be the Hilbert square of a generic K3 surface S_{2t} of degree $2t$ such that X admits an ample divisor $D \in \mathrm{Div}(X)$ with $q_X(D) = 2$. Let $D_1, D_2 \in |D|$ be two distinct divisors.*

(i) If $t = 2$, then the surface $D_1 \cap D_2$ can be reducible. If $[D_1 \cap D_2] = A + B$, where $[D_1 \cap D_2]$ is the fundamental cohomological class of $D_1 \cap D_2$ in $H^{2,2}(X, \mathbb{Z})$, then the effective classes A and B are:

$$A = \tfrac{1}{2}h^2 - \tfrac{1}{4}\delta^2 - \tfrac{1}{2}h\delta - \tfrac{1}{10}q_X^\vee \in H^{2,2}(X, \mathbb{Z}),$$
$$B = \tfrac{1}{2}h^2 + \tfrac{5}{4}\delta^2 - \tfrac{3}{2}h\delta + \tfrac{1}{10}q_X^\vee \in H^{2,2}(X, \mathbb{Z}).$$

Moreover, if $f : X = S_4^{[2]} \to \mathbb{G}(1, \mathbb{P}^3)$ is the map seen in Section 4.4.1 and $\sigma_{1,1}$, σ_2 are the generators of $H^4(\mathbb{G}(1, \mathbb{P}^3), \mathbb{Z})$, then $A = f^*\sigma_{1,1}$ and $B = f^*\sigma_2$.

(ii) If $t \neq 2$, then $D_1 \cap D_2$ is a reduced and irreducible surface.

Proof. We begin with $t = 2$, so $X = S_4^{[2]}$ is the Hilbert square of a generic smooth quartic surface of $\mathbb{P}^3$ and the class of D is $h - \delta \in \mathrm{Pic}(X)$, where h is the class induced by the ample generator of $\mathrm{Pic}(S_4)$. By Lemma 4.6.4, the surface $D_1 \cap D_2$ can be reducible only if there exist effective classes $A, B \in H^{2,2}(X, \mathbb{Z})$ such that $[D_1 \cap D_2] = A + B$ and $\iota^*(A) = A$, $\iota^*(B) = B$, where ι is the Beauville involution. By Corollary 3.4.11 a basis of the lattice $H^{2,2}(X, \mathbb{Z})$ is given by $\{h^2, \frac{h^2 - h\delta}{2}, \frac{1}{8}\left(\delta^2 + \frac{2}{5}q_X^\vee\right), \frac{2}{5}q_X^\vee\}$ (we have again substituted δ^2 in the basis given by Corollary 3.4.11 with $\frac{2}{5}q_X^\vee$ to simplify the computations). Hence we have

$$\begin{aligned} A &= \left(x + \tfrac{y}{2}\right) h^2 + \tfrac{z}{8}\delta^2 - \tfrac{y}{2}h\delta + \left(\tfrac{1}{8}z + w\right) \tfrac{2}{5}q_X^\vee \in H^{2,2}(X, \mathbb{Z}), \\ B &= \left(1 - x - \tfrac{y}{2}\right) h^2 + \left(1 - \tfrac{z}{8}\right) \delta^2 + \left(\tfrac{y}{2} - 2\right) h\delta - \left(\tfrac{1}{8}z + w\right) \tfrac{2}{5}q_X^\vee \in H^{2,2}(X, \mathbb{Z}), \end{aligned}$$

for some $x, y, z, w \in \mathbb{Z}$.

- By assumption A and B are effective. Moreover, $h \in \mathrm{Pic}(X)$ is nef by Theorem 4.1.1. Then by Theorem 1.1.14 we have

  $$\langle A, h^2 \rangle \geq 0, \qquad \langle B, h^2 \rangle \geq 0,$$

  where $\langle \cdot, \cdot \rangle$ is the bilinear form of $H^4(X, \mathbb{Z})$ given in Proposition 3.1.5, which coincides with the intersection pairing. We obtain the following condition:
  $$0 \leq 12x + 6y + z + 10w \leq 10. \tag{4.6.3}$$

- Let $\sigma \in H^0(X, \Omega_X^2)$ be the symplectic form. Then, since A, and B are effective classes in $H^{2,2}(X, \mathbb{Z})$ by assumption, we have

$$\int_A (\sigma + \bar{\sigma})^2 = 2\int_A \sigma \wedge \bar{\sigma} \geq 0, \qquad \int_B (\sigma + \bar{\sigma})^2 = 2\int_B \sigma \wedge \bar{\sigma} \geq 0,$$

since $\sigma \wedge \bar{\sigma} \in H^{2,2}(X)$ is a volume form on A and on B. Note that $\sigma \wedge \bar{\sigma}$ can be zero on A or B, for instance when A or B are Lagrangian. Hence

$$\langle A, (\sigma + \bar{\sigma})^2 \rangle \geq 0, \qquad \langle B, (\sigma + \bar{\sigma})^2 \rangle \geq 0.$$

Using (4.5.7) and (4.5.8) we obtain the following condition:

$$0 \leq 4x + 2y + z + 10w \leq 2. \tag{4.6.4}$$

- The class D is ample, and by assumption A and B are effective. Hence by Theorem 1.1.12 we have

$$\langle A, D^2 \rangle > 0, \qquad \langle B, D^2 \rangle > 0,$$

and we obtain the following condition:

$$0 < 40x + 12y + 3z + 20w < 12. \tag{4.6.5}$$

- By Theorem 4.3.1 and (4.5.11) we have

$$\begin{aligned} \iota^* A \;\; = \;\; & \left(9x + \tfrac{3}{2}y + \tfrac{1}{2}z\right) h^2 \\ & + \left(16x + 2y + \tfrac{9}{8}\right) \delta^2 \\ & - \left(24x + \tfrac{7}{2}y + \tfrac{3}{2}z\right) h\delta \\ & + \left(\tfrac{1}{8}z + w\right) \tfrac{2}{5} q_X^\vee. \end{aligned}$$

Since $\iota^* A = A$, we obtain the system

$$\begin{cases} 9x + \frac{3}{2}y + \frac{1}{2}z = x + \frac{y}{2}, \\ 16x + 2y + \frac{9}{8}z = \frac{z}{8}, \\ 24x + \frac{7}{2}y + \frac{3}{2}z = \frac{y}{2}, \end{cases}$$

which gives the following condition

$$16x + 2y + z = 0. \tag{4.6.6}$$

We look for $x, y, z, w \in \mathbb{Z}$ which satisfy (4.6.3), (4.6.4), (4.6.5) and (4.6.6). Note that (4.6.3) and (4.6.4) imply

$$-2 \leq 8x + 4y \leq 10,$$

and since $x, y \in \mathbb{Z}$ we have

$$2x + y \in \{0, 1, 2\}. \tag{4.6.7}$$

- Suppose that $2x + y = 0$. By (4.6.6) we have $z = -12x$, and (4.6.5) becomes

$$0 < w - x < \frac{3}{5}.$$

Since $w - x \in \mathbb{Z}$, this condition is never satisfied.

- Suppose that $2x + y = 1$. By (4.6.6) we have $z = -12x - 2$, and (4.6.5) becomes
$$-\frac{3}{10} < w - x < \frac{3}{10}.$$
Since $w - x \in \mathbb{Z}$, we get $x = w$, and (4.6.4) gives $x \in \{-1, 0\}$. We obtain the following two solutions:
$$\begin{cases} x = 0, \\ y = 1, \\ z = -2, \\ w = 0, \end{cases} \quad \text{and} \quad \begin{cases} x = -1, \\ y = 3, \\ z = 10, \\ w = -1, \end{cases}$$
which coincide by Proposition 4.4.4 with the effective classes $f^*\sigma_{1,1}$ and $f^*\sigma_2$ respectively. Moreover, with the same technique it is possible to show that $f^*\sigma_{1,1}$ and $f^*\sigma_2$ are reduced and irreducible.

- The case $2x + y = 2$ is symmetric to the case $2x + y = 0$, i.e., if A is a class obtained in this case, then A coincides with a class B obtained in the case $2x + y = 0$. Since there are no classes in the case $2x + y = 0$, there are no classes in the cases $2x + y = 2$.

We conclude that if $D_1 \cap D_2$ is a reducible surface, then it is reduced with two irreducible components whose fundamental cohomological classes in $H^{2,2}(X, \mathbb{Z})$ are $f^*\sigma_{1,1}$ and $f^*\sigma_2$.

Suppose now that $t \neq 2$. We want to show that the surface $D_1 \cap D_2$ is reduced and irreducible. By Lemma 4.6.4 it is enough to show that we cannot have $[D_1 \cap D_2] = A + B$ for effective $A, B \in H^{2,2}(X, \mathbb{Z})$ such that $\iota^*A = A$ and $\iota^*B = B$. The technique is the same seen before. Recall that $D = bh - a\delta$, where (a, b) is the minimal solution of the Pell-type equation $P_t(-1)$. Since $t \neq 2$, we have $t \geq 10$ and $a \geq 3$, see Remark 4.3.5. By Corollary 3.4.11 we can write
$$\begin{aligned} A &= \left(x + \tfrac{y}{2}\right) h^2 + \tfrac{z}{8}\delta^2 - \tfrac{y}{2} h\delta + \left(\tfrac{1}{8}z + w\right) \tfrac{2}{5} q_X^\vee, \\ B &= \left(b^2 - x - \tfrac{y}{2}\right) h^2 + \left(a^2 - \tfrac{z}{8}\right) \delta^2 + \left(-2ab + \tfrac{y}{2}\right) h\delta - \left(\tfrac{1}{8}z + w\right) \tfrac{2}{5} q_X^\vee, \end{aligned}$$
for some $x, y, z, w \in \mathbb{Z}$.

- By assumption A and B are effective. Moreover, the class h is nef by Theorem 4.1.1. Thus by Theorem 1.1.14 we have
$$\langle A, h^2 \rangle \geq 0, \qquad \langle B, h^2 \rangle \geq 0.$$
We obtain the following condition:
$$0 \leq 6tx + 3ty + z + 10w \leq 6tb^2 - 2a^2. \tag{4.6.8}$$

- Let $\sigma \in H^0(X, \Omega_X^2)$ be the symplectic form. Since A and B are effective, similarly to the case $t = 2$ we have
$$\langle A, (\sigma + \bar{\sigma})^2 \rangle \geq 0, \qquad \langle B, (\sigma + \bar{\sigma})^2 \rangle \geq 0.$$
Using (4.5.7) and (4.5.8), and recalling that $a^2 - tb^2 = -1$, we obtain the following condition:
$$0 \leq 2tx + ty + z + 10w \leq 2. \tag{4.6.9}$$

- The class D is ample, and by assumption A and B are effective. Hence by Theorem 1.1.12 we have

$$\langle A, D^2\rangle > 0, \qquad \langle B, D^2\rangle > 0,$$

and we obtain, after some computations, the following condition:

$$0 < (4t+8t^2b^2)x + (2t+4t^2b^2-4abt)y + (1+tb^2)z + 20w < 12. \quad (4.6.10)$$

- By Theorem 4.3.1 and (4.5.11) we have

$$\begin{aligned}\iota^*A \;=\; & \left(c^2\left(x+\tfrac{y}{2}\right)+cd(-\tfrac{y}{2})+\tfrac{z}{8}d^2\right)h^2\\ & +\left(t^2d^2\left(x+\tfrac{y}{2}\right)-cdt\tfrac{y}{2}+c^2\tfrac{z}{8}\right)\delta^2\\ & +\left(-2cdt\left(x+\tfrac{y}{2}\right)+c^2\tfrac{y}{2}+td^2\tfrac{y}{2}-2cd\tfrac{z}{8}\right)h\delta\\ & +\left(\tfrac{1}{8}z+w\right)\tfrac{2}{5}q_X^\vee.\end{aligned}$$

Since $\iota^*A = A$ we obtain the following condition:

$$8tdx + 4(td-c)y + dz = 0. \quad (4.6.11)$$

Note that (4.6.8) and (4.6.9) implies

$$-\frac{1}{t} \le 2x+y \le 2b^2 + \frac{1}{t}.$$

Since $2x+y\in\mathbb{Z}$ and $t\ge 10$ we have

$$2x+y\in\{0,1,\dots,2b^2\}.$$

Note that, similarly to the case $t=2$ seen above, a class A obtained by imposing $2x+y\in\{b^2+1,\dots,2b^2\}$ coincide with a class B obtained for $2x+y\in\{0,1,\dots,b^2\}$. Hence it suffices to study $2x+y\in\{0,1,\dots,b^2\}$. Suppose that $2x+y=k$, where $k\in\{0,1,\dots,,b^2\}$. By (4.6.9) we have

$$-tk \le z+10w \le 2-tk,$$

and since $z+10w\in\mathbb{Z}$ we have

$$z+10w\in\{-tk,-tk+1,-tk+2\}.$$

Suppose that $z+10w=-tk$. Then (4.6.10) gives, after some computations,

$$-4t^2b^2k < -4abty + a^2z < -4t^2b^2k + 12.$$

Since $-4abty+a^2z\in\mathbb{Z}$, we have

$$-4abty + a^2z = -4t^2b^2k + h, \qquad h\in\{1,2,\dots,,11\}.$$

With the help of a computer we obtain

$$w = \frac{-5a^2tk + 2ha^2 - 4tk + h}{10a^2}.$$

Then w is an integer only if

$$4tk - h \equiv 0 \pmod{a^2}.$$

If $k = 0$, then $-h \equiv_{a^2} 0$ only if $h = 9$ and $a = 3$, being $a \geq 3$. This happens only when $t = 10$, and we get $w = \frac{19}{10}$, which is not an integer. Suppose now that $k \neq 0$. Since $k \leq b^2$, we have $4tk - h \leq 4a^2 + 4 - h$, hence in order to get $4tk - h \equiv_{a^2} 0$ we must have

$$4tk - h \in \{a^2, 2a^2, 3a^2, 4a^2\}.$$

If $4tk - h = a^2$, then

$$k = \frac{tb^2 + h - 1}{4t}.$$

If $h \neq 1, 11$, then $h - 1$ is not divisible by $t \geq 10$, and k is not an integer. If $h = 1$, then $k = \frac{b^2}{4}$, which is not an integer by Proposition 1.5.8.
If $h = 11$, then $h - 1 = 10$ is divisible by $t \geq 10$ if and only if $t = 10$, which implies $b = 1$: thus $k = \frac{1}{2}$, which is not an integer.

In a similar way it is possible to show that all the other remaining cases are not possible: we give details of these computations in Appendix B.

We conclude that there are no effective classes $A, B \in H^{2,2}(X, \mathbb{Z})$ such that $[D_1 \cap D_2] = A + B$, hence $D_1 \cap D_2$ is a reduced and irreducible surface.

□

We conclude this chapter with two corollaries in the case $t \neq 2$, which are the analogue of [O'G08b, Corollary 4.2]. Keep the notation given at the end of Section 4.5.2. Let $X_0 := X \setminus B$, which is open and dense in X, and let $Y_0 := f(X_0)$, which contains an open dense subset of Y. We then obtain a regular map

$$f_0 : X_0 \to Y_0$$

by restricting f to X_0.

Corollary 4.6.6. *Let X and D be as in Theorem 4.6.5 with $t \neq 2$, and keep notation as above. If $L \subset \mathbb{P}^5$ is a linear subspace of codimension at most 2, then $L \cap Y_0$ is reduced and irreducible and, if non empty, it has pure codimension equal to* $\operatorname{cod}(L, \mathbb{P}^5)$.

Proof. The proof is the same as [O'G08b, Corollary 4.2, (i)]. If $L \cong \mathbb{P}^5$, there is nothing to prove. Let $\operatorname{cod}(L, \mathbb{P}^5) = 1$. Denote by $D_1 \in |D|$ the divisor which corresponds to L in the isomorphism $|D|^\vee \cong \mathbb{P}^5$. Then $[D_1 \cap X_0] = f_0^* L$, where $[D_1 \cap X_0]$ is the fundamental cohomological class of $D_1 \cap X_0$ and f_0^* is the pullback in cohomology. Since X_0 is open and dense in X and f_0 is surjective, Proposition 4.6.3 implies the result.

Let $\operatorname{cod}(L, \mathbb{P}^5) = 2$, so $L = L_1 \cap L_2$ with $L_1, L_2 \subset \mathbb{P}^5$ hyperplanes. As before, let $D_1, D_2 \in |D|$ the divisors which correspond to L_1, L_2 in the isomorphism $|D|^\vee \cong \mathbb{P}^5$. We have $[D_1 \cap D_2 \cap X_0] = f_0^* L$, moreover X_0 is an open and dense subset of X and f_0 is surjective, so the thesis follows from Theorem 4.6.5, (ii). □

Corollary 4.6.7. *Let X and D be as in Theorem 4.6.5 with $t \neq 2$. Then the base locus $B = \operatorname{Bs}|D|$ has dimension at most 1. Let B_{red} be the reduced scheme associated to B and B^1_{red} be the union of the irreducible components of B_{red} of* *dimension* 1*. If* $D_1, D_2, D_3 \in |D|$ *are linearly independent, then* $D_1 \cap D_2 \cap D_3$ *has pure dimension* 1 *and there exists a unique decomposition*

$$[D_1 \cap D_2 \cap D_3] = \Gamma + \Sigma, \tag{4.6.12}$$

where Γ, Σ *are effective* 1*-cycles such that*

- $\mathrm{Supp}(\Gamma) \cap B_{\mathrm{red}}$ *is either* 0*-dimensional or empty,*
- $\mathrm{Supp}(\Sigma) = B^1_{\mathrm{red}}$.

Note that $[D_1 \cap D_2 \cap D_3]$ in the statement of Corollary 4.6.7 denotes the fundamental cycle of $D_1 \cap D_2 \cap D_3$, see [Ful13, §1.5].

Proof. Since by Theorem 4.6.5, (ii), the surface $D_1 \cap D_2$ is reduced and irreducible, and D_1, D_2, D_3 are linearly independent, the intersection $D_1 \cap D_2 \cap D_3$ has pure dimension 1 and the base locus has dimension at most 1.

We pass to the unicity of the decomposition (4.6.12). Let $\Gamma_0 \in Z_1(X_0)$ be the fundamental cycle of $(D_1 \cap D_2 \cap D_3) \setminus B$ and $\Gamma \in Z_1(X)$ be its closure. Since $D_1 \cap D_2 \cap D_3 \supset B$ and $\dim(B) \leq 1$, we obtain the desired decomposition with this choice of Γ. Vice versa, if we have such a decomposition, then Γ is necessarily the closure of Γ_0 and so the decomposition is unique. □

Remark 4.6.8. In Chapter 5 we will see the equality in (4.6.12), by abuse of notation, as an equality between the cohomological class induced in $H^{3,3}(X, \mathbb{Z})$ by $[D_1 \cap D_2 \cap D_3]$ and $\Gamma + \Sigma$.

4.6.3 Other birational models

Let S_{2t} be a generic K3 surface of degree $2t$ and let X be a smooth birational model of $S_{2t}^{[2]}$. Then there is an isomorphism $H^2(S_{2t}^{[2]}, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$ which is compatible with the BBF forms by Theorem 2.2.39 and an isomorphism of graded rings $H^*(S_{2t}^{[2]}, \mathbb{Z}) \cong H^*(X, \mathbb{Z})$. Note that the image of the dual of the BBF form of $S_{2t}^{[2]}$ under this isomorphism is the dual of the BBF form on X, being the isomorphism compatible with the BBF forms. Moreover, the Hodge structures of the cohomology groups of $S_{2t}^{[2]}$ and X are isomorphic. Let now $t \neq 1, 5$ such that the Pell-type equations $P_t(-1)$ and $P_{4t}(5)$ are solvable. Suppose that X admits an ample divisor D with $q_X(D) = 2$, so X is a variety denoted by X' in Case 3 of Section 4.3. Recall that there exists an anti-symplectic involution ι such that $\mathrm{Aut}(X) \cong \langle \iota \rangle$ and $\iota^*(D) = D$ in $\mathrm{Pic}(X)$. We now see which of the results obtained in Section 4.5.2, Section 4.6.1 and Section 4.6.2 hold also for these varieties. We begin by stating the results which correspond respectively to Theorem 4.5.11, Corollary 4.6.2 and Proposition 4.6.3.

Theorem 4.6.9. *Let* S_{2t} *be a generic K3 surface of degree* $2t$ *with* $t \neq 1, 5$ *such that both the Pell-type equations* $P_t(-1)$ *and* $P_{4t}(5)$ *are solvable. Let* X *be a smooth birational model of* $S_{2t}^{[2]}$ *which admits an ample divisor* D *with* $q_X(D) = 2$*. Denote by* $\iota : X \to X$ *the anti-symplectic involution which generates* $\mathrm{Aut}(X)$ *and by* F *the fixed locus of* ι*. Then*

$$[F] = 5D^2 - \frac{2}{5} q_X^\vee \in H^{2,2}(X, \mathbb{Z}),$$

where $[F]$ denotes the fundamental cohomological class of F in $H^{2,2}(X, \mathbb{Z})$. Moreover, let $\varphi_{|D|} : X \dashrightarrow \mathbb{P}^5$ be the map induced by the complete linear system $|D|$. Then the following diagram is commutative:

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi_{|D|}"] \arrow[dr, "\pi"'] & & \mathbb{P}^5 \\ & X/\langle\iota\rangle, \arrow[ur, dashed] & \end{tikzcd}$$

where $\pi : X \to X/\langle\iota\rangle$ is the quotient map.

Proof. All the results of Section 4.5.2 stated before Theorem 4.5.11 hold also for X. Recall that the Hodge structures of $S_{2t}^{[2]}$ and X are isomorphic. Then the proof is the same of Theorem 4.5.11, since we have only used properties of intersection numbers in $H^{2,2}(S_{2t}^{[2]}, \mathbb{Z})$, which are preserved by the isomorphism of graded rings $H^*(S_{2t}^{[2]}, \mathbb{Z}) \cong H^*(X, \mathbb{Z})$, and we have remarked that in this isomorphism $q_X^\vee$ corresponds to $q_{S_{2t}^{[2]}}^\vee$. □

Proposition 4.6.10. *Let S_{2t} be a generic K3 surface of degree $2t$ and X be a smooth birational model of $S_{2t}^{[2]}$. Consider a big and nef divisor $D \in \mathrm{Div}(X)$. Then $|D|$ has no fixed part.*

Proof. Since the isomorphism $H^2(S_{2t}^{[2]}, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$ is compatible with the BBF forms, the proof is the same of Corollary 4.6.2. □

Proposition 4.6.11. *Let X and D be as in Theorem 4.6.9. Then every divisor $D' \in |D|$ is reduced and irreducible.*

Proof. Since the isomorphism $H^2(S_{2t}^{[2]}, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$ is compatible with the BBF forms, the proof is the same of Proposition 4.6.3. Moreover, pseudoeffective classes are preserved by this isomorphism by Proposition 4.1.2, since clearly the closure of the movable cone of $S_{2t}^{[2]}$ coincides with the closure of the movable cone of X. □

Let X and D be as in Theorem 4.6.9. Consider $D_1, D_2 \in |D|$ two distinct divisors. It is still an open problem to determine if the surface $D_1 \cap D_2$ is reduced and irreducible. Indeed, if one tries to use the same procedure of the proof of Theorem 4.6.5, if $h \in \mathrm{Pic}(S_{2t}^{[2]})$ is the line bundle induced by the ample generator of $\mathrm{Pic}(S_{2t})$, then h is not nef as a class of X, thus it is not clear if the products $\langle A, h^2\rangle$ and $\langle B, h^2\rangle$ are still both non-negative.

Chapter 5

Geometric description of $\langle 2\rangle$-polarised Hilbert squares of generic K3 surfaces

In this chapter we prove the following theorem, which solves Problem 4.3.4 introduced in Section 4.3.

Theorem 5.0.1. *Let X be the Hilbert square of a generic $K3$ surface S_{2t} of degree $2t$ such that X admits an ample divisor D with $q_X(D) = 2$. Suppose that $t \neq 2$, and denote by ι the anti-symplectic involution which generates* $\mathrm{Aut}(X)$*. Then the complete linear system $|D|$ is basepoint free, and the morphism*

$$\varphi_{|D|} : X \to Y \subset \mathbb{P}^5$$

is a double cover whose ramification locus is the surface F of points fixed by ι. Moreover, $Y \cong X/\langle\iota\rangle$ and X is a double EPW sextic.

Such an X, as seen in Section 4.5.1, is also known as a $\langle 2\rangle$-polarised Hilbert square of a generic K3 surface. Double EPW sextics, introduced in Section 2.2.10, are IHS manifolds of $K3^{[2]}$-type. From now on we denote by $f : X \dashrightarrow Y \subset \mathbb{P}^5$ the map induced by the complete linear system $|D|$, recall the notation given in Section 4.5.2 and in Section 4.6.2. We will prove Theorem 5.0.1 in several steps, which we now summarize.

(i) $\dim(Y) = 4$.

(ii) $\deg(f) \cdot \deg(Y) \leq 12$ and the equality holds if and only if $\mathrm{Bs}|D| = \emptyset$.

(iii) Either $\deg(f) = 2$ and $\deg(Y) = 6$ or $\deg(f) = 4$ and $\deg(Y) = 3$. In particular, by Item (ii), $\mathrm{Bs}|D| = \emptyset$ and f is a morphism.

(iv) The case $\deg(f) = 4$ and $\deg(Y) = 3$ never holds.

(v) The variety X is a double EPW sextic.

Each step will be studied in a dedicated section. The structure of the proof is similar to the case studied by O'Grady in [O'G08b], which we briefly recall. Two

IHS manifolds M_1 and M_2 of dimension $2n$ are *numerically equivalent* if there is an isomorphism of abelian groups

$$\psi : H^2(M_1, \mathbb{Z}) \xrightarrow{\sim} H^2(M_2, \mathbb{Z})$$

such that

$$\int_{M_1} \alpha^{2n} = \int_{M_2} \psi(\alpha)^{2n}$$

for all $\alpha \in H^2(M_1, \mathbb{Z})$. An IHS manifold M of dimension 4 is a *numerical* $K3^{[2]}$ if it is numerically equivalent to $S^{[2]}$, where S is a K3 surface. Then O'Grady showed the following, see [O'G08b, Theorem 1.1].

Theorem 5.0.2 (O'Grady). *A numerical $K3^{[2]}$ is deformation equivalent to one of the following.*

(i) An IHS manifold X of dimension 4 carrying an anti-symplectic involution $\iota : X \to X$ such that the quotient $X/\langle\iota\rangle$ is isomorphic to an EPW sextic $Y \subset \mathbb{P}^5$, hence X is a double EPW sextic.

(ii) An IHS manifold X of dimension 4 admitting a rational map $f : X \dashrightarrow \mathbb{P}^5$ which is birational onto its image Y, with $6 \leq \deg(Y) \leq 12$.

The first step of the proof of Theorem 5.0.2 is the following: using the surjectivity of the period map and the projectivity criterion of Theorem 2.2.12, a numerical $K3^{[2]}$ is deformation equivalent to a 4-dimensional IHS manifold X such that:

(1) X admits an ample divisor H with $q_X(H) = 2$.

(2) $H^{1,1}(X, \mathbb{Z}) = \mathbb{Z}h$, where $h = c_1(\mathcal{O}_X(H))$.

(3) If $\Sigma \in Z_1(X)$ is an integral algebraic 1-cycle on X and $[\Sigma] \in H^{3,3}(X, \mathbb{Q})$ is its fundamental cohomological class, then $[\Sigma] = mh^3/6$ for some $m \in \mathbb{Z}$.

(4) If $V \subset H^4(X)$ is a rational sub Hodge structure, then $V_{\mathbb{C}} = V_1 \oplus V_2 \oplus V_3$, where $V_1 \subset \mathbb{C}h^2 \oplus \mathbb{C}q_X^\vee$, V_2 is either 0 or equal to $\mathbb{C}h \otimes h^\perp$, where the orthogonality is with respect to the BBF form, and V_3 is either 0 or equal to

$$W(h) := (q_X^\vee)^\perp \cap \mathrm{Sym}^2(h^\perp),$$

where $(q_X^\vee)^\perp$ is the orthogonal with respect to the BBF form.

(5) If $H^4(X, \mathbb{Z})_f$ is the torsion free quotient group of $H^4(X, \mathbb{Z})$, then the image of h^2 in $H^4(X, \mathbb{Z})_f$ is indivisible.

(6) $H^{2,2}(X, \mathbb{Z})_f \subset \mathbb{Z}\frac{h^2}{2} \oplus \mathbb{Z}\frac{q_X^\vee}{5}$.

Items (5) and (6) imply the so-called *irreducibility property* of $|H|$: the surface $D_1 \cap D_2$, where $D_1, D_2 \in |H|$ are distinct divisors, is reduced and irreducible, see [O'G08b, Proposition 4.1]. The proof of this property is quite easy. This corresponds to Theorem 4.6.5 in our case, which is much more complicated to obtain: we have seen that the key point was the explicit description of the lattice $H^{2,2}(S_{2t}^{[2]}, \mathbb{Z})$ given by Corollary 3.4.11, where S_{2t} is a generic K3 surface of degree $2t$, whose proof was basically the aim of the whole Chapter 3. Moreover, note that in our case Item (1) is true by assumption, Item (3) corresponds to Theorem 3.5.6 and Item (5) can be shown to follow from Corollary 3.4.11. Since we will not use this last property, we omit the proof. Using the ampleness of H and the irreducibility property of $|H|$, O'Grady showed that one of the following holds.

- Item (i) or Item (ii) of Theorem 5.0.2 holds.
- If $f : X \dashrightarrow \mathbb{P}^5$ is the map induced by $|H|$, then $Y = \mathrm{Im}(f)$ is one of the following.
  - (a) A 3-fold of degree at most 6.
  - (b) A 4-fold of degree at most 4.

Items (a) and (b) never hold, since they contradict either the irreducibility of the surface $D_1 \cap D_2$ or Item (4). This is not at all trivial, see [O'G08b] for details. O'Grady could not show that Item (ii) of Theorem 5.0.2 never holds, since under his assumptions the variety X does not admit necessarily an anti-symplectic involution. In our case this holds by Theorem 4.3.1. Once having obtained Corollary 3.4.11 and Theorem 4.6.5, the proof of Theorem 5.0.1 becomes much easier than the one of Theorem 5.0.2, thanks to the existence of the anti-symplectic involution ι such that $\iota^* D = D$, where D is the ample divisor with $q_X(D) = 2$. This will also show that a case similar to Item (ii) of Theorem 5.0.2 never holds in our situation. Moreover, a variety X satisfying the hypothesis of Theorem 5.0.1 is itself a double EPW sextic, and not only deformation equivalent to a double EPW sextic, giving a strong geometric characterization.

5.1 Step 1: $\dim(Y) = 4$

We begin by showing that $\dim(Y) = 4$. Recall the following classical result.

Proposition 5.1.1 (Proposition 0 in [EH87]). *If $X \subset \mathbb{P}^r$ is a non-degenerate variety, then $\deg(X) \geq 1 + \mathrm{cod}(X, \mathbb{P}^r)$.*

First of all, we prove an analogue of [O'G08b, Corollary 4.3]. Keep the notation given at the end of Section 4.5.2 and before Corollary 4.6.6.

Proposition 5.1.2. *Let X and D be as in Theorem 4.6.5 with $t \neq 2$. Consider*

$$f : X \dashrightarrow Y \subset \mathbb{P}^5,$$

the map induced by the complete linear system $|D|$. Then $\dim(Y) \in \{3, 4\}$.

Proof. Clearly we have $1 \leq \dim(Y) \leq 4$. Suppose that $\dim(Y) = 1$: then Y is a non-degenerate irreducible curve in $\mathbb{P}^5$, hence by Proposition 5.1.1 we have $\deg(Y) \geq 5$. Let $L \subset \mathbb{P}^5$ be a generic hyperplane. Then $Y_0 \cap L$ is given by $\deg(Y)$ points, where Y_0 was introduced before Corollary 4.6.6, since Y_0 contains an open dense subset of Y. By Corollary 4.6.6 this is irreducible and reduced, hence we have a contradiction.

Suppose now that $\dim(Y) = 2$, so Y is a non-degenerate irreducible surface in $\mathbb{P}^5$. By Proposition 5.1.1 we have $\deg(Y) \geq 4$. Let $L \subset \mathbb{P}^5$ be a generic linear subspace of codimension 2. Then $Y_0 \cap L$ is given by $\deg(Y)$ points, since by construction Y_0 contains an open dense subset of Y. But this is irreducible and reduced by Corollary 4.6.6, and we obtain a contradiction. We conclude that $\dim(Y)$ is either 3 or 4. □

We now focus on the case $\dim(Y) = 3$. We want to show that this never holds. First of all, similarly to [O'G08b, Proposition 4.5], we give boundaries to the value $\deg(Y)$.

Proposition 5.1.3. *Keep notation as above and suppose that* $\dim(Y) = 3$. *Then* $3 \leq \deg(Y) \leq 6$. *Moreover, if* $\deg(Y) = 6$, *then the base locus* $\mathrm{Bs}|D|$ *is* 0*-dimensional.*

Proof. Let $d_Y := \deg(Y)$. Since $Y \subset \mathbb{P}^5$ is a non-degenerate subvariety, from Proposition 5.1.1 we have $d_Y \geq 3$. Let now $L_1, L_2, L_3 \subset \mathbb{P}^5$ be three generic hyperplanes. Then the intersection $Y \cap L_1 \cap L_2 \cap L_3$ is transverse and it is given by d_Y points, which we call $p_1, \ldots, p_{d_Y}$. Let $\Gamma_{0,i} := f_0^{-1}(p_i)$ for $i = 1, \ldots, d_Y$, and let Γ_i be the closure of $\Gamma_{0,i}$ in X. Let $D_1, D_2, D_3 \in |D|$ be the divisors which correspond to L_1, L_2, L_3 in the isomorphism $|D|^\vee \cong \mathbb{P}^5$. Since L_1, L_2, L_3 are generic, D_1, D_2, D_3 are linearly independent, and by Corollary 4.6.7 we have

$$[D_1 \cap D_2 \cap D_3] = \sum_{i=1}^{d_Y} \Gamma_i + \Sigma, \tag{5.1.1}$$

where the equality is in $H^{3,3}(X, \mathbb{Z})$, see Remark 4.6.8, and we write by abuse of notation Γ_i and Σ for their classes in $H^{3,3}(X, \mathbb{Z})$. Recall that $D = bh - a\delta$ with (a, b) minimal solution of the Pell-type equation $P_t(-1)$, and b is odd by Proposition 1.5.8. Similarly to Example 3.5.3 we get $D^\vee = bh^\vee - 2a\delta^\vee$. Moreover, as remarked in Section 4.5.1 the divisibility of D in $H^2(X, \mathbb{Z})$ is $\mathrm{div}(D) = 1$, hence by Proposition 3.5.1 we have

$$D \cdot D^\vee = \left(bh - a\delta, bh - 2\frac{a}{2}\delta\right) = 2.$$

Since $D_1, D_2, D_3 \in |D|$ and $\iota^* D \cong D$, we have

$$\iota^*[D_1 \cap D_2 \cap D_3] = [D_1 \cap D_2 \cap D_3],$$

where $[D_1 \cap D_2 \cap D_3]$ is the fundamental cohomological class of $D_1 \cap D_2 \cap D_3$ in $H^{3,3}(X, \mathbb{Z})$, hence

$$\sum_{i=1}^{d_Y} \iota^*\Gamma_i + \iota^*\Sigma = \sum_{i=1}^{d_Y} \Gamma_i + \Sigma.$$

We know from Theorem 4.3.1 that the only class in $\mathrm{Pic}(X)$ which is fixed by the anti-symplectic involution ι is D. Since from Theorem 3.5.6 we have $H^{3,3}(X, \mathbb{Z}) \cong \mathbb{Z}h_6^\vee \oplus \mathbb{Z}\delta_6^\vee$, the only class fixed by the action induced by ι on $H^{3,3}(X, \mathbb{Z}) \cong H_{1,1}(X, \mathbb{Z})$ is $D^\vee$. Moreover, from Theorem 4.5.11 we know that f factors through the quotient by ι, so $\iota^*\Gamma_i \cong \Gamma_i$, since $\Gamma_i = \overline{f_0^{-1}(p_i)}$. Then the class of Γ_i is either some positive multiple of $D^\vee$, or it is of the form $\Gamma_i = \Gamma'_i + \iota^*\Gamma'_i$, where Γ'_i is an effective class. Since $q_X(D) = 2$, which gives $\int_X c_1(\mathcal{O}_X(D))^4 = 12$ by Theorem 2.2.4 and Proposition 2.2.8, we have

$$12 = \langle D, \sum_{i=1}^{d_Y} \Gamma_i + \Sigma \rangle.$$

Note that $\langle D, \Gamma_i \rangle$ is either $\langle D, \alpha D^\vee \rangle = 2\alpha$, where $\alpha \in \mathbb{Z}_{\geq 1}$, or $\langle D, \Gamma_i' + \iota^* \Gamma_i' \rangle \geq 2$, being D ample. Hence $12 \geq \langle D, \Sigma \rangle + 2d_Y$, i.e.,

$$d_Y \leq 6.$$

Moreover, if $d_Y = 6$, then $\Sigma = 0$, otherwise $\langle D, \Sigma \rangle > 0$ by Theorem 1.1.12, thus the base locus $\mathrm{Bs}|D|$ is 0-dimensional, since the 1-dimensional component of $\mathrm{Bs}|D|$ is contained in Σ by Corollary 4.6.7. □

Note that in the proof of Proposition 5.1.3 the anti-symplectic involution ι is strongly used, together with the fact that f factors through the quotient $\pi : X \to X/\langle \iota \rangle$. This gives some differences with the analogue result by O'Grady in [O'G08b, Proposition 4.5]. We now recall the definition of *linearly normal variety.*

Definition 5.1.4. Let $Y \subseteq \mathbb{P}^n$ be a closed irreducible subvariety of $\mathbb{P}^n$. Then $Y \subseteq \mathbb{P}^n$ is *linearly normal* for the given embedding if the natural map

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(Y, \mathcal{O}_Y(1))$$

is surjective.

Geometrically an irreducible variety $Y \subseteq \mathbb{P}^n$ is linearly normal if it cannot be obtained by a linear projection from a projective space of higher dimension, except the trivial case of being contained in a proper linear subspace. Note that if $f : X \dashrightarrow \mathbb{P}^n$ is the map induced by a complete linear system $|D|$ on X and $Y := \mathrm{Im}(\tilde{f})$, where we denote by $\tilde{f} : \tilde{X} \to \mathbb{P}^n$ the morphism which resolves the indeterminacies of the map f, then $Y \subseteq \mathbb{P}^n$ is linearly normal by definition of map induced by a complete linear system, see also Theorem 1.1.8 applied to $\tilde{f}$.

We now come back to our problem. We have shown that if $\dim(Y) = 3$, then $Y \subset \mathbb{P}^5$ is a non-degenerate irreducible subvariety of degree at most 6. Moreover, since it is obtained from the map induced by a complete linear system, it is linearly normal. We prove that this is in contradiction with Corollary 4.6.6. We follow [O'G08b, §5]. The key point is the following result obtained by O'Grady, see [O'G08b, Proposition 5.1].

Proposition 5.1.5 (O'Grady)**.** *Let $Y \subset \mathbb{P}^5$ be a 3-dimensional non-degenerate linearly normal subvariety of degree at most 6.*

(i) If $\deg(Y) \leq 5$, then given an arbitrary non-empty subset $U \subset Y$ there exists a 3-dimensional linear subspace $L \subset \mathbb{P}^5$ such that $L \cap U$ is reducible.

(ii) If $\deg(Y) = 6$ then there exists a 3-dimensional linear subspace $L \subset \mathbb{P}^5$ such that $L \cap Y$ is not reduced or reducible.

We omit the proof of Proposition 5.1.5. We can now show the main result of this section, which is the analogue of [O'G10, §5].

Proposition 5.1.6. *Let X and D be as in Theorem 4.6.5 with $t \neq 2$. Consider*

$$f : X \dashrightarrow Y \subset \mathbb{P}^5$$

the map induced by the complete linear system $|D|$. Then $\dim(Y) = 4$.

Proof. By Proposition 5.1.2 we know that $\dim(Y)$ is either 3 or 4. If $\dim(Y) = 3$, by Proposition 5.1.3 we have $3 \leq \deg(Y) \leq 6$. Suppose that $\deg(Y) \leq 5$. We denote by $U \subset Y$ the interior of Y_0. Then $U \neq \emptyset$, and by Proposition 5.1.5 there exists a 3-dimensional linear subspace $L \subset \mathbb{P}^5$ such that $L \cap U$ is reducible. Moreover, U is an open subset of Y_0, so $L \cap Y_0$ is reducible, which contradicts Corollary 4.6.6.

Suppose now that $\deg(Y) = 6$. First of all, we show that $Y = Y_0$. By Proposition 5.1.3 we have $\dim(\mathrm{Bs}|D|) = 0$. Let n be a positive integer such that nD is very ample, and let $D_1 \in |nD|$ be a generic divisor. Since the base locus is given by a finite number of points, we can choose D_1 such that $\mathrm{Supp}(D_1) \subset X \setminus B = X_0$. We show that $f_0(D_1) = Y$. Since $\dim(Y_0) = 3$, by the fiber dimension theorem, see [Har13, Exercise II.3.22], the generic fiber of f_0 has dimension 1. Hence D_1 intersects a generic fiber in a finite number of points. In particular $f_0(D_1)$ has dimension 3. Since $f_0(D_1)$ is a closed subset of Y and the variety Y is irreducible of dimension 3, we have $f_0(D_1) = Y$, hence $Y = Y_0$. Now, since Y is an irreducible, non-degenerate, linearly normal variety and $\deg(Y) = 6$, by Proposition 5.1.5 there exists a 3-dimensional linear subspace $L \subset \mathbb{P}^5$ such that $L \cap Y_0$ is not reduced or not irreducible. This contradicts Corollary 4.6.6. We conclude that $\dim(Y) = 3$ does not hold, so $\dim(Y) = 4$. □

5.2 Step 2: $\deg(f) \cdot \deg(Y) \leq 12$

Keep notation as above. We have seen that, if $f : X \dashrightarrow Y \subset \mathbb{P}^5$ is the map induced by the complete linear system $|D|$, then Y has dimension 4. We now state the analogues of [O'G08b, Proposition 4.6] and [O'G08b, Corollary 4.7].

Proposition 5.2.1. *Keep notation as above. Let $D_1, D_2, D_3, D_4 \in |D|$ be generic divisors, in particular we assume that D_1, D_2 and D_3 are linearly independent. Let Γ and Σ be the effective 1-cycle of Corollary 4.6.7. Then*

$$\deg(Y) \cdot \deg(f) + \sum_{p \in B_{\mathrm{red}}} \mathrm{mult}_p\,(\Gamma \cdot D_4) + \int_\Sigma c_1(\mathcal{O}_X(D)) = 12. \tag{5.2.1}$$

Proof. The sum which appears in (5.2.1) is finite since $\mathrm{Supp}(\Gamma) \cap B_{\mathrm{red}}$ is either 0-dimensional or empty by Corollary 4.6.7. Let $L_1, L_2, L_3, L_4 \subset \mathbb{P}^5$ be the hyperplanes which correspond to D_1, D_2, D_3, D_4 in the isomorphism $|D|^\vee \cong \mathbb{P}^5$. Since D_1, D_2, D_3, D_4 are generic, also L_1, L_2, L_3, L_4 are generic. Let $\tilde{f} : \tilde{X} \to \mathbb{P}^5$ be the resolution of $f : X \dashrightarrow \mathbb{P}^5$ and let $Z \subset Y$ be the subset of points p such that $\dim(\tilde{f}^{-1}(p)) > 0$. If $E \subset \tilde{X}$ is the exceptional divisor of the blow-up $\beta : \tilde{X} \to X$ in the base locus $\mathrm{Bs}|D|$, then $Z = \tilde{f}(E)$ because D is ample, thus no curve in $X \setminus B$ is contracted. This implies that $\dim(Z) \leq 2$. Moreover, $\tilde{f}$ is a projective morphism, hence it is a closed map, so Z is closed and we get the inequality $\dim(\tilde{f}(\mathrm{supp}(E))) \leq \dim(\mathrm{supp}(E)) = 3$. Since L_1, L_2, L_3, L_4 are generic, we have

$$\emptyset = L_1 \cap L_2 \cap L_3 \cap L_4 \cap Z = L_1 \cap L_2 \cap L_3 \cap L_4 \cap \tilde{f}(E) \tag{5.2.2}$$

and

$$|L_1 \cap L_2 \cap L_3 \cap L_4 \cap Y| < \infty. \tag{5.2.3}$$

From (5.2.3) the effective divisors $\tilde{f}^*L_1, \dots \tilde{f}^*L_4$ intersect properly on $\tilde{X}$, and (5.2.2) shows that this intersection is contained in the open subset X_0, seen as a subset of $\tilde{X}$. We obtain

$$\deg(Y)\cdot\deg(f) = \tilde{f}^*L_1\cdot\tilde{f}^*L_2\cdot\tilde{f}^*L_3\cdot\tilde{f}^*L_4 = \sum_{p\in X_0} \operatorname{mult}_p(D_1\cdot D_2\cdot D_3\cdot D_4). \quad (5.2.4)$$

By the decomposition of Corollary 4.6.7 and $q_X(D) = 2$, which implies as already seen $\int_X c_1(\mathcal{O}_X(D))^4 = 12$, we have

$$12 = \int_X c_1(\mathcal{O}_X(D))^4 = (\Gamma + \Sigma)\cdot D = \Gamma\cdot D_4 + \int_\Sigma c_1(\mathcal{O}_X(D)). \quad (5.2.5)$$

Since $\operatorname{supp}(\Sigma) = B^1_{\mathrm{red}}$, the restrictions of Γ and $[D_1 \cap D_2 \cap D_3]$ to X_0 coincide, so we obtain

$$\Gamma\cdot D_4 = \sum_{p\in X_0} \operatorname{mult}_p(D_1\cdot D_2\cdot D_3\cdot D_4) + \sum_{p\in B_{\mathrm{red}}} \operatorname{mult}_p(\Gamma\cdot D_4). \quad (5.2.6)$$

From (5.2.4), (5.2.5) and (5.2.6) we get

$$\deg(Y)\cdot\deg(f) + \sum_{p\in B_{\mathrm{red}}} \operatorname{mult}_p(\Gamma\cdot D_4) + \int_\Sigma c_1(\mathcal{O}_X(D)) = 12,$$

which is exactly (5.2.1). □

Corollary 5.2.2. *Keep notation as above. Then*

$$\deg(Y)\cdot\deg(f) \le 12$$

with equality if and only if $\mathrm{Bs}|D| = \emptyset$*, i.e., if and only if* f *is a morphism.*

Proof. Since D is ample and Σ is either effective or empty, by Theorem 1.1.12 we have $\int_\Sigma c_1(\mathcal{O}_X(D)) \ge 0$, and the equality holds if and only if Σ is the empty set. Clearly $\deg(Y)\cdot\deg(f) \ge 0$ and

$$\sum_{p\in B_{\mathrm{red}}} \operatorname{mult}_p(\Gamma\cdot D_4) \ge 0,$$

thus Proposition 5.2.1 implies $\deg(Y)\cdot\deg(f) \le 12$.
Suppose that $\mathrm{Bs}|D| = \emptyset$. We have

$$\sum_{p\in B_{\mathrm{red}}} \operatorname{mult}_p(\Gamma\cdot D_4) = 0, \qquad \Sigma = \emptyset,$$

hence Proposition 5.2.1 implies $\deg(Y)\cdot\deg(f) = 12$.
Suppose now that $\deg(Y)\cdot\deg(f) = 12$. If $\Sigma \neq \emptyset$, by Theorem 1.1.12 we have $\int_\Sigma D > 0$, which contradicts Proposition 5.2.1. Hence we get $\Sigma = \emptyset$ and $[D_1 \cap D_2 \cap D_3] = \Gamma$, and $\operatorname{supp}(B) \subseteq \operatorname{supp}(\Gamma)$. By Definition 1.1.21 we also have $\operatorname{supp}(B) \subseteq \operatorname{supp}(D)$. Then every point $p \in \operatorname{supp}(B)$ is in $\operatorname{supp}(D) \cap \operatorname{supp}(\Gamma)$. Since by assumption $\deg(Y)\cdot\deg(f) = 12$, we have that Proposition 5.2.1 implies

$$\sum_{p\in \operatorname{supp} B} \operatorname{mult}_p(D\cdot\Gamma) = 0$$

We conclude that $\mathrm{Bs}|D| = \emptyset$. □

5.3 Step 3: the divisor D is basepoint free

We now study the possible values that $\deg(Y)$ and $\deg(f)$ can take.

Proposition 5.3.1. *Let X and D be as in Theorem 4.6.5 with $t \neq 2$. Then $|D|$ is basepoint free, i.e., $\mathrm{Bs}|D| = \emptyset$. Moreover, if $f : X \to Y \subset \mathbb{P}^5$ is the morphism induced by $|D|$, one of the following holds.*

(i) $\deg(f) = 2$ *and* $\deg(Y) = 6$.

(ii) $\deg(f) = 4$ *and* $\deg(Y) = 3$.

Proof. Recall that the following diagram is commutative by Theorem 4.5.11:

$$\begin{tikzcd} X \arrow[rr, dashed, "f"] \arrow[dr, "\pi"'] & & Y \subset \mathbb{P}^5 \\ & X/\langle \iota \rangle, \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

where ι is the anti-symplectic involution which generates $\mathrm{Aut}(X)$. In particular by Corollary 4.5.12 we know that $\deg(f)$ is even. Suppose that $\deg(f) = 2$. In this case $\deg(\bar{f}) = 1$, so $\bar{f}$ is birational. As observed by O'Grady in the proof of [O'G08b, Proposition 4.9], if σ is the symplectic form on X, since $\iota^*(\sigma) = -\sigma$ we have $\iota^*(\sigma \wedge \sigma) = \sigma \wedge \sigma$, so if W is any desingularization of $X/\langle \iota \rangle$, we have $H^0(W, \omega_W) \neq 0$ and so $H^0(\tilde{Y}, \omega_{\tilde{Y}}) \neq 0$, where $\tilde{Y}$ is any desingularization of the variety Y. Alternatively, in our case from [CGM19, §3] we know that a desingularization of $X/\langle \iota \rangle$ is a Calabi–Yau variety W, see Section 4.5.2, so $H^0(W, \omega_W) \cong \mathbb{C}$, hence $H^0(\tilde{Y}, \omega_{\tilde{Y}}) \cong \mathbb{C}$ for a desingularization of Y. By [O'G08b, (4.0.30)] we have

$$H^0(\omega_{\tilde{Y}}) = H^0(\mathcal{I}_Z(\deg(Y) - 6)),$$

where $Z \subset \mathbb{P}^5$ is a subscheme supported on $\mathrm{sing}(Y)$. Since $H^0(\omega_{\tilde{Y}}) \cong \mathbb{C}$, we have $\deg(Y) \geq 6$. By Corollary 5.2.2 we conclude that $\deg(Y) = 6$ and $\deg(f) = 2$, hence $\mathrm{Bs}|D| = \emptyset$.

The remaining cases to analyse are the following:

(i) $\deg(Y) = 2$.

(ii) $\deg(Y) = 3$, $\deg(f) = 3$, $\mathrm{Bs}|D| \neq \emptyset$.

(iii) $\deg(Y) = 3$, $\deg(f) = 4$, $\mathrm{Bs}|D| = \emptyset$.

(iv) $\deg(Y) = 4$, $\deg(f) = 3$, $\mathrm{Bs}|D| = \emptyset$.

(v) $\deg(Y) = 6$, $\deg(f) = 2$, $\mathrm{Bs}|D| = \emptyset$.

By Corollary 4.5.12, cases (ii) and (iv) are not possible. Consider case (i), so we suppose that $\deg(Y) = 2$, i.e., Y is a quadric hypersurface. We show that there exists a 3-dimensional linear subspace $L \subset \mathbb{P}^5$ such that $L \cap Y$, and so $L \cap Y_0$, is reducible. We show it for Y of rank 6, the other cases are similar. We can suppose that in the homogeneous coordinates $(x_0 : \cdots : x_5)$ of $\mathbb{P}^5$ the quadric Y is represented by the zero locus of the homogeneous polynomial $x_0^2 + x_1^2 + \cdots + x_5^2$. If L is obtained as the intersection of the hyperplanes H_1 and H_2 represented by the polynomials $x_0 + ix_1$ and $x_2 + ix_3$ respectively, then $Y \cap H_1 \cap H_2$ can be represented by $(x_4 + ix_5)(x_4 - ix_5)$, i.e., $Y \cap H_1 \cap H_2$ is reducible. This contradicts Corollary 4.6.6, hence we conclude that (i) is not possible. The only remaining possible cases are (iii) and (v), in particular we have $\mathrm{Bs}|D| = \emptyset$. □

Note that we have again strongly used the anti-symplectic involution ι and the fact that f factors through the quotient $\pi : X \to X/\langle\iota\rangle$. This simplifies the situation, compared to the one in [O'G08b], where much longer discussions are needed to show that Items (ii) and (iv) never hold.

5.4 Step 4: $\deg(Y) = 6$ and $\deg(f) = 2$

Keep notation as above. The aim of this section is to prove that $\deg(Y) = 6$ and $\deg(f) = 2$. Before doing that, we remark that we cannot proceed following [O'G08b, §5.5]. In that case, the fact that the variety considered is a *deformation* of the Hilbert square of a K3 surface plays a central role in the proof. If M is an IHS manifold of $K3^{[2]}$-type and $h \in H^{1,1}(M, \mathbb{Q})$ is an element such that $q_M(h) \neq 0$, then O'Grady shows that $H^4(M, \mathbb{C})$ can be decomposed as

$$H^4(M, \mathbb{C}) = \left(\mathbb{C}h^2 \oplus \mathbb{C}q_M^\vee\right) \oplus \left(\mathbb{C}h \otimes h^\perp\right) \oplus W(h),$$

where the orthogonality is with respect to the BBF form and

$$W(h) := (q_M^\vee)^\perp \cap \mathrm{Sym}^2(h^\perp),$$

where $(q_M^\vee)^\perp$ is the orthogonal with respect to the intersection product. As already seen, O'Grady takes an IHS variety X deformation equivalent to M such that properties (1)-(6) seen in the introduction to this Chapter hold. In particular property (4), which we now recall, is strongly used in the proof. See [O'G08b, Proposition 3.2, Lemma 3.3] for more details.

- If $V \subset H^4(X)$ is a rational sub Hodge structure, then $V_\mathbb{C} = V_1 \oplus V_2 \oplus V_3$, where $V_1 \subset \left(\mathbb{C}h^2 \oplus \mathbb{C}q_X^\vee\right)$, V_2 is either 0 or equal to $\mathbb{C}h \otimes h^\perp$ and V_3 is either 0 or equal to $W(h)$.

In our case, there is no reason why a similar result holds, without deforming the variety $X = S_{2t}^{[2]}$ that we are studying. In our setting we exploit once again the anti-symplectic involution ι and Theorem 4.5.11. Recall that we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y \subset \mathbb{P}^5 \\ & X/\langle\iota\rangle. \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Note that f is a morphism by Proposition 5.3.1.

Proposition 5.4.1. *Let X and D be as in Theorem 4.6.5 with $t \neq 2$. Then the morphism $f : X \to Y \subset \mathbb{P}^5$ induced by the complete linear system $|D|$ is such that $\deg(f) = 2$ and $\deg(Y) = 6$.*

Proof. By Proposition 5.3.1 it suffices to show that $\deg(f) = 4$ and $\deg(Y) = 3$ never holds. Suppose by contradiction that $\deg(f) = 4$ and $\deg(Y) = 3$. First we show that $Y \subset \mathbb{P}^5$ is a normal variety. Since $Y \subset \mathbb{P}^5$ is a hypersurface, then by [Har13, Proposition II.8.23] we have that Y is normal if and only if $\operatorname{cod}_Y(\operatorname{Sing}(Y)) \geq 2$. Suppose that $\dim(\operatorname{Sing}(Y)) = 3$. Note that Y does not contain planes, otherwise we would get a contradiction with Corollary 4.6.6. As observed by O'Grady in [O'G08b, Claim 5.10], since in $\mathbb{P}^5$ hyperplanes and quadrics contain planes, a cubic hypersurface of $\mathbb{P}^5$ which is either non reduced or reducible contains planes. Hence in our case Y is reduced and irreducible. Then, as in [O'G08b, Lemma 5.17], the intersection between the variety Y and a generic plane of $\mathbb{P}^5$ is a singular cubic curve which is reduced and irreducible. In particular this has only one singular point, so $\operatorname{Sing}(Y)$ has exactly one irreducible component $\Sigma \cong \mathbb{P}^3$ of degree 1. Thus $Y \supset \Sigma$, and it contains planes. This contradicts Corollary 4.6.6. We conclude that $\operatorname{cod}_Y(\operatorname{Sing}(Y)) \geq 2$, hence Y is normal.

Since $\deg(f) = 4$ by assumption, by the commutativity of the diagram above we have $\deg(\tilde{f}) = 2$. Let $\tilde{Y} := X/\langle\iota\rangle$. Since X is smooth and $\operatorname{Aut}(X) \cong \langle\iota\rangle$ is a finite group, the quotient $\tilde{Y}$ is a normal variety. Both $\tilde{Y}$ and Y are normal varieties, hence by Remark 1.6.2 there is a non trivial involution $\tau : \tilde{Y} \to \tilde{Y}$ such that $Y \cong \tilde{Y}/\tau$. We show that τ lifts to an automorphism on X. We use a technique from [O'G13, p.179] and [DK18, Proposition B.8]. Let $F = \operatorname{Fix}(\iota)$ be the fixed locus of the anti-symplectic involution ι. Since $\operatorname{Sing}(\tilde{Y}) = \pi(F)$ and τ is an automorphism on $\tilde{Y}$, we have $\tau(\operatorname{Sing}(\tilde{Y})) = \operatorname{Sing}(\tilde{Y})$. Let $Y' := \tilde{Y} \setminus \operatorname{Sing}(\tilde{Y})$. Then the restriction of τ to Y' gives an involution $\tau|_{Y'} : Y' \to Y'$ of Y'. We set $\tau' := \tau|_{Y'}$. Since $\operatorname{cod}_X(F) = 2$ and X is simply connected, we have that $X' := X \setminus F$ is simply connected. Thus if we restrict π to X' we obtain the following universal cover:

$$\pi' := \pi|_{X'} : X' \to Y'.$$

Then by the lifting criterion we have that τ' lifts to an automorphism on X'. Thus we obtain a birational map $\tilde{\tau} : X \dashrightarrow X$ which is not defined a priori on the fixed locus F. By Theorem 4.3.2 we have $\operatorname{Bir}(X) \cong \operatorname{Aut}(X) \cong \langle\iota\rangle$, so $\tilde{\tau}$ is either the identity or ι. This implies that the involution $\tau : \tilde{Y} \to \tilde{Y}$ is the identity, which is a contradiction. We conclude that $\deg(\tilde{f})$ cannot be 2, hence we get $\deg(f) = 2$ and $\deg(Y) = 6$. □

5.5 Step 5: the variety $S_{2t}^{[2]}$ is a double EPW sextic

Keep notation as above. In order to complete the proof of Theorem 5.0.1, we show that $Y \subset \mathbb{P}^5$ is an EPW sextic and that f can be identified with the natural double cover of an EPW sextic. We need a result in [O'G06]. We first recall the definition of numerical $(K3)^{[2]}$ from [O'G08b].

Definition 5.5.1. An irreducible symplectic 4-fold X is a *numerical* $(K3)^{[2]}$ if there exists a K3 surface S and an isomorphism of abelian groups

$$\psi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} H^2(S^{[2]}, \mathbb{Z})$$

such that

$$\int_M \alpha^4 = \int_{S^{[2]}} \psi(\alpha)^4$$

for all $\alpha \in H^2(X,\mathbb{Z})$.

Recall the definition of double EPW sextics and the notation introduced in Section 2.2.10. We have the following result, see [O'G06, Theorem 1.1].

Theorem 5.5.2 (O'Grady)**.** *Let X be a numerical $(K3)^{[2]}$ and H be an ample divisor on X such that $q_X(H) = 2$. Suppose that there exists an anti-symplectic involution $\iota : X \to X$ and denote by $Y := X/\langle\iota\rangle$ the quotient. Assume that $\mathrm{Bs}|D| = \emptyset$ and that the morphism induced by the complete linear system $|H|$ is the composition of the quotient map $f : X \to Y$ and an embedding $Y \hookrightarrow |H|^\vee$. Then there exists $A \in \mathbb{LG}(\bigwedge^3 V_6)$ such that $f : X \to Y$ is identified with the natural double cover $X_A \to Y_A$.*

We can finally prove Theorem 5.0.1.

Proof of Theorem 5.0.1. By Proposition 5.4.1 the morphism

$$f : X \to Y \subset \mathbb{P}^5$$

induced by the complete linear system $|D|$ has degree 2 and $\deg(Y) = 6$. Moreover, we have obtained the following commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y \subset \mathbb{P}^5 \\ & X/\langle\iota\rangle. \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

where $\pi : X \to X/\langle\iota\rangle$ is the quotient with respect to ι, the anti-symplectic involution which generates $\mathrm{Aut}(X)$. Thus we apply Theorem 5.5.2, and we obtain that X is a double EPW sextic. $\square$

Chapter 6

Future perspectives

In this chapter we present some open problems concerning the topics studied in this thesis.

6.1 Problem 4.3.4 for birational models of $S_{2t}^{[2]}$

Theorem 5.0.1 gives a geometric description of Hilbert squares $X = S_{2t}^{[2]}$ of generic K3 surfaces of degree $2t$ admitting an ample divisor D with $q_X(D) = 2$: if $t \neq 2$, the complete linear system $|D|$ induces a morphism of degree 2 which is the double cover of an EPW sextic, so X is a double EPW sextic. These are the varieties appearing in Case 1 in Section 4.3. However, Problem 4.3.4 is still open for varieties of Case 3 in Section 4.3, i.e., when X is a smooth birational model of the Hilbert square $S_{2t}^{[2]}$, where t is such that both $P_t(-1)$ and $P_{4t}(5)$ are solvable, and X admits an ample divisor D with $q_X(D) = 2$. The problem is that we do not know if Theorem 4.6.5 holds in this case: in the proof given for varieties of Case 1 we use the nefness of $h \in \mathrm{Pic}(S_{2t}^{[2]})$, the class induced by the ample generator of $\mathrm{Pic}(S_{2t})$, but this class is not nef when seen as a class for varieties appearing in Case 3, see Figure 4.3.3. By Theorem 2.2.40 we know that X is a elementary Mukai flop of $S_{2t}^{[2]}$. Using the notation introduced in the proof of Theorem 4.6.5, if one shows that the products $\langle A, h^2 \rangle$ and $\langle B, h^2 \rangle$ are still non negative, then the result holds, together with all the statements of Chapter 5, so D is basepoint free and the variety X is a double EPW sextic. Otherwise, if one shows that the surface $D_1 \cap D_2$ is not reduced or not irreducible for some distinct divisors $D_1, D_2 \in |D|$, then the procedure of Chapter 5 cannot be used. In this case, we expect that the variety X has a different and special description, as seen for the case $t = 2$ in Section 4.4.1. Moreover, in such a case we conjecture that the surface $D_1 \cap D_2$ contains as a component the subvariety $P^* \subset X$, where we denote by P^* the dual of the subvariety $P \subset S_{2t}^{[2]}$ where the Mukai flop is performed, and P^* can be the base locus of the complete linear system $|D|$, if this is not empty.

Problem. *Solve Problem* 4.3.4 *for varieties appearing in Case* 3 *in Section* 4.3.

6.2 Pseudoeffective cone of 2-cycles

Consider Theorem 4.6.5. The technique used in the proof exploits necessary conditions for the psedoeffectiveness of a class in $H^{2,2}(S_{2t}^{[2]},\mathbb{Z})$, i.e., if the procedure gives a decomposition

$$[D_1 \cap D_2] = A + B \in H^{2,2}(S_{2t}^{[2]},\mathbb{Z})$$

where $D_1, D_2 \in |D|$ are distinct divisors and $A, B \in H^{2,2}(S_{2t}^{[2]},\mathbb{Z})$, then it can happen that A and B are not effective. This is why in the case $t = 2$ once we have obtained such a decomposition we have remarked that $A = f^*\sigma_{1,1}$ and $B = f^*\sigma_2$, where $f : S_4^{[2]} \to \mathbb{G}(1,\mathbb{P}^3)$ is the map described in Section 4.4.1 and $\sigma_{1,1}$ and σ_2 are the generators of $H^4(\mathbb{G}(1,\mathbb{P}^3),\mathbb{Z})$: since $\sigma_{1,1}$ and σ_2 are effective, A and B are effective. Let now $t = 5$, the special case studied in Section 4.4.2: the big and nef divisor D with class $h - 2\delta$ is not ample and $q_X(D) = 2$. Repeating the procedure of the proof of Theorem 4.6.5, we obtain the following possible decomposition:

$$\begin{aligned}
&[D_1 \cap D_2] = 2A + B \in H^{2,2}(S_{10}^{[2]},\mathbb{Z}),\\
&A = \tfrac{1}{2}h^2 + \tfrac{25}{8}\delta^2 - \tfrac{5}{2}h\delta + \tfrac{1}{20}q_X^\vee \in H^{2,2}(S_{10}^{[2]},\mathbb{Z}),\\
&B = -\tfrac{9}{4}\delta^2 + h\delta - \tfrac{1}{10}q_X^\vee \in H^{2,2}(S_{10}^{[2]},\mathbb{Z}).
\end{aligned}$$

Note that A is the class of a 2-dimensional subvariety $P \subset S_{10}^{[2]}$ with $P \cong \mathbb{P}^2$, see (3.4.4) and [Bak15, p.17], hence A is effective. We do not know if B is effective or not. This shows that in general it could be useful to have more information on the pseudoeffective cone of 2-cycles $\overline{\mathrm{Eff}}_2(S_{2t}^{[2]})$. There are some results on effective classes of IHS manifolds: if S is a K3 surface, then the classes of Lagrangian surfaces contained in $S^{[2]}$ lie in the boundary of $\overline{\mathrm{Eff}}_2(S^{[2]})$, see [Ott15, Proposition 2.3], and if S_{2t} is a generic K3 surface, then the Chern classes $c_k(S_{2t}^{[2]})$ are not big for $1 \le k \le n$, see [Ott15, Lemma 2.5]. In the case of Hilbert squares of generic K3 surfaces, it could be interesting to understand if Theorem 4.2.3 can be useful. Note that describing explicitly $\overline{\mathrm{Eff}}_2(S_{2t}^{[2]})$ could be very hard: this is a cone in the 4-dimensional vector space $H^{2,2}(S_{2t}^{[2]},\mathbb{Q})$.

Problem. *Find information on $\overline{\mathrm{Eff}}_2(S_{2t}^{[2]})$ for a generic K3 surface S_{2t}.*

6.3 Generalize Problem 4.3.4 to $K3^{[n]}$

It is natural to study Problem 4.3.4 for Hilbert schemes of n points on generic K3 surfaces, where $n \ge 3$. Cattaneo obtained in [Cat19] a complete description of the group of biregular automorphisms $\mathrm{Aut}(S_{2t}^{[n]})$ of the Hilbert scheme of n points on a generic K3 surface S_{2t} of degree $2t$. If the integer $t > 0$ respects some numerical conditions, then $\mathrm{Aut}(S_{2t}^{[n]})$ is generated by a non-symplectic involution which fixes the class of an ample divisor D with square 2 with respect to the BBF form. In [BC20] Beri and Cattaneo described the group of birational automorphisms $\mathrm{Bir}(S_{2t}^{[n]})$. Starting from these results, we can try to study the following problem.

Problem. *Let S_{2t} be a generic K3 surface of degree $2t$ and let X be a smooth birational model $S_{2t}^{[2]}$ which admits an ample divisor D with $q_X(D) = 2$. Determine* $\mathrm{Bs}|D|$ *and describe geometrically the map induced by the complete linear system $|D|$:*

$$\varphi_{|D|} : X \dashrightarrow \mathbb{P}^N.$$

As for $n = 2$, the group of biregular automorphisms on such an X is generated by an anti-symplectic involution which fixes the class of the ample divisor D. We wonder if, similarly to the case $n = 2$ in Theorem 4.5.11, the map $\varphi_{|D|}$ factors through the quotient $X \to X/\langle\iota\rangle$. Moreover, we have seen in Theorem 5.0.1 that, for $n = 2$, the Hilbert square $S_{2t}^{[2]}$ admitting an ample divisor D with $q_X(D) = 2$ with $t \neq 2$ is a double EPW sextic. If $t = 2$, then $\varphi_{|D|}$ is a finite morphism of degree 6 with image the Grassmannian $\mathbb{G}(1, \mathbb{P}^3)$. What makes the case $t = 2$ different from the others is the fact that the surface $D_1 \cap D_2$, for $D_1, D_2 \in |D|$, can be reducible, while for $t \neq 2$ this is always reduced and irreducible, as seen in Theorem 4.6.5. For $n = 3$ we expect that special examples arise whenever a similar property of reducibility holds. In order to study such a problem, we need to explicitly describe the lattice $H^{i,i}(S_{2t}^{[n]}, \mathbb{Z})$ for every $1 \leq i \leq 2n - 1$. In general it could be complicated to obtain a description which does not depend directly on Nakajima operators, as in Corollary 3.4.11.

Problem. *Let S_{2t} be a generic K3 surface of degree $2t$. Describe the lattice $H^{i,i}(S_{2t}^{[n]}, \mathbb{Z})$ for $n \geq 3$ and $1 \leq i \leq 2n - 1$. Study the same problem for any projective K3 surface S.*

6.4 Base loci of other complete linear systems

Let $X := S_{2t}^{[2]}$ be the Hilbert square of a generic K3 surface S_{2t}. We denote by $h \in \mathrm{Pic}(X)$ the class induced by the ample generator of $\mathrm{Pic}(S_{2t})$. If $t = 1$, then Rieß in [Rie20] showed that the only big and nef divisor which is not basepoint free is $2h - \delta$, whose base locus is isomorphic to $\mathbb{P}^2$. Moreover, X admits another smooth birational model X', whose big and nef divisors are all basepoint free. For $t = 2$ and $t = 5$, by Proposition 4.4.5 and Proposition 4.4.9 every big and nef divisor on X is basepoint free. We would like to describe the base loci of big and nef divisors for other values of t. The examples seen above make us conjecture the following.

Conjecture. *Let $X = S_{2t}^{[2]}$ be the Hilbert square of a generic K3 surface. Suppose that D is a big and nef divisor on X. Then the base locus* $\mathrm{Bs}|D|$ *is either empty or isomorphic to a disjoint union of copies of $\mathbb{P}^2$.*

Clearly it is enough to study the problem for divisors whose classes belong to the Hilbert basis of the nef cone.

Appendices

Appendix A

Example 3.4.2: code

We present the Sage program used in Example 3.4.2: I thank Simon Brandhorst for useful explanations and clarifications on Sage and on this code. The Sage packages used are:
sage.modules.free_ quadratic_ module_ integer_ symmetric,
sage.modules.torsion_ quadratic_ module.

```
sage: M=matrix(ZZ, 4, 4, [[12, 6, 2, -4], [6, 2, 1, -2], [2, 1, 1, -1],
      [-4, -2, -1, 12]])
sage: L=IntegralLattice(M)
sage: D=L.discriminant_group()
sage: print(D)
Finite quadratic module over Integer Ring with invariant (2, 42)
Gram matrix of the quadratic form with values in Q/Z:
[1/2  1/2]
[1/2 2/21]
sage: G1=[g.lift() for g in L.gens()] #the generating set of D used
sage: D.normal_form()
Finite quadratic module over Integer Ring with invariants (2, 42)
Gram matrix of the quadratic form with values in Q/Z:
[1/2   0   0   0]
[  0 1/2   0   0]
[  0   0 2/3   0]
[  0   0   0 3/7]
sage: G2=D.normal_form().gens()
((1, 0), (1, 21), (0, 14), (0, 6))
```

We explain some details on the code. Keep notation of Example 3.4.2. We denote by g_1 and g_2 the two generators given by G1 in the code, i.e.,

$$\begin{aligned} g_1 &= \tfrac{1}{2}w_2 - \tfrac{1}{2}w_4, \\ g_2 &= \tfrac{1}{42}w_1 - \tfrac{20}{21}w_3 - \tfrac{19}{21}w_4. \end{aligned} \tag{A.0.1}$$

Then G2 gives us the elements v_1, v_2, v_3, v_4 in the form

$$\begin{aligned} v_1 &= g_1, \\ v_2 &= g_1 + 21g_2, \\ v_3 &= 14g_2, \\ v_4 &= 6g_2. \end{aligned} \tag{A.0.2}$$

Then (A.0.1) and (A.0.2) give the following equivalences modulo L:

$$\begin{aligned}
v_1 &\equiv \tfrac{1}{2}w_2 + \tfrac{1}{2}w_4,\\
v_2 &\equiv \tfrac{1}{2}w_1 + \tfrac{1}{2}w_2 + \tfrac{1}{2}w_4,\\
v_3 &\equiv \tfrac{1}{3}w_1 + \tfrac{2}{3}w_2 + \tfrac{1}{3}w_4,\\
v_4 &\equiv \tfrac{1}{7}w_1 + \tfrac{2}{7}w_3 + \tfrac{4}{7}w_4,
\end{aligned}$$

which are exactly the equivalences in (3.4.7) of Example 3.4.2.

Appendix B

Theorem 4.6.5: details of the proof

We present details of the proof of Theorem 4.6.5. Recall that $X = S_{2t}^{[2]}$ is the Hilbert square of a generic K3 surface S_{2t} of degree $2t$, admitting an ample divisor D with $q_X(D) = 2$. The class of D is $bh - a\delta \in \mathrm{Pic}(X)$, where (a, b) is the minimal solution of $P_t(-1)$. Let $D_1, D_2 \in |D|$ be two distinct divisors, and suppose that

$$[D_1 \cap D_2] = A + B \in H^{2,2}(X, \mathbb{Z}),$$

where $[D_1 \cap D_2]$ is the fundamental cohomological class of $D_1 \cap D_2$ in $H^{2,2}(X, \mathbb{Z})$ and $A, B \in H^{2,2}(X, \mathbb{Z})$ are effective classes such that $\iota^*(A) = A$ and $\iota^*(B) = B$. Here ι is the anti-symplectic involution which generates $\mathrm{Aut}(X)$, as shown by Theorem 4.3.1. We recall conditions (4.6.8), (4.6.9), (4.6.10), (4.6.11) respectively:

$$0 \leq 6tx + 3ty + z + 10w \leq 6tb^2 - 2a^2,$$
$$0 \leq 2tx + ty + z + 10w \leq 2,$$
$$0 < (4t + 8t^2b^2)x + (2t + 4t^2b^2 - 4abt)y + (1 + tb^2)z + 20w < 12,$$
$$8tdx + 4(td - c)y + dz = 0,$$

where (c, d) is the minimal solution of the Pell equation $P_t(1)$. Recall that we are considering $t \geq 10$ and $a \geq 3$. In the proof of Theorem 4.6.5 we have obtained

$$2x + y \in \{0, 1, \dots, 2b^2\},$$

and we have remarked that it suffices to study the cases

$$2x + y \in \{0, 1, \dots, b^2\}.$$

Suppose that $2x + y = k$, where $k \in \{0, 1, \dots, b^2\}$. By (4.6.9) we have

$$-tk \leq z + 10w \leq -tk + 2,$$

and since $z + 10w \in \mathbb{Z}$ we have

$$z + 10w \in \{-tk, -tk + 1, -tk + 2\}.$$

Case 1: suppose that $z + 10w = -tk$. Then (4.6.10) gives, after some computations,

$$-4t^2b^2k < -4abty + a^2z < -4t^2b^2k + 12.$$

Since $-4abty + a^2z \in \mathbb{Z}$, we have

$$-4abty + a^2z = -4t^2b^2k + h, \qquad h \in \{1, 2, \ldots, , 11\}.$$

With the help of a computer we obtain

$$w = \frac{-5a^2tk + 2ha^2 - 4tk + h}{10a^2}.$$

Then w is an integer only if

$$4tk - h \equiv 0 \pmod{a^2}.$$

If $k = 0$, then $-h \equiv_{a^2} 0$ only if $h = 9$ and $a = 3$. This happens only when $t = 10$, and we get $w = \frac{19}{10}$, which is not an integer. Suppose now that $k \neq 0$. Since $k \leq b^2$, we have $4tk - h \leq 4a^2 + 4 - h$, hence in order to get $4tk - h \equiv_{a^2} 0$ we must have

$$4tk - h \in \{a^2, 2a^2, 3a^2, 4a^2\}.$$

- If $4tk - h = a^2$, then
$$k = \frac{tb^2 + h - 1}{4t}.$$
If $h \neq 1, 11$, then $h - 1$ is not divisible by $t \geq 10$, and k is not an integer.
If $h = 1$, then $k = \frac{b^2}{4}$, which is not an integer by Proposition 1.5.8.
If $h = 11$, then $h - 1 = 10$ is divisible by t if and only if $t = 10$, which implies $b = 1$: thus $k = \frac{1}{2}$, which is not an integer.

- If $4tk - h = 2a^2$, then
$$k = \frac{2tb^2 + h - 2}{4t}.$$
If $h \neq 2$, then $h - 2$ is not divisible by $t \geq 10$, and k is not an integer.
If $h = 2$, then $k = \frac{b^2}{2}$, which is not an integer by Proposition 1.5.8.

- If $4tk - h = 3a^2$, then
$$k = \frac{3tb^2 + h - 3}{4t}.$$
If $h \neq 3$, then $h - 3$ is not divisible by $t \geq 10$, and k is not an integer.
If $h = 3$, then $k = \frac{3}{4}b^2$, which is not an integer by Proposition 1.5.8.

- If $4tk - h = 4a^2$, then
$$k = \frac{4tb^2 + h - 4}{4t}.$$
If $h \neq 4$, then $h - 4$ is not divisible by $t \geq 10$, and k is not an integer.
If $h = 4$, then $k = b^2$, and we obtain
$$w = \frac{-5tb^2 + 4}{10},$$
which is not an integer since 4 is not divisible by 5.

Since we always obtain a contradiction with $k, w \in \mathbb{Z}$, we conclude that Case 1 never holds.

Case 2: suppose that $z + 10w = -tk + 1$. Then (4.6.10) gives

$$-4t^2b^2k - 2 < -4abty + a^2z < -4t^2b^2k + 10,$$

and since $-4abty + a^2z \in \mathbb{Z}$ we have

$$-4abty + a^2z = -4t^2b^2k + h, \qquad h \in \{-1, 0, \dots, 9\}.$$

With the help of a computer we obtain

$$w = \frac{-5a^2tk + (2h+1)a^2 - 4tk + h}{10a^2}.$$

Then w is an integer only if

$$4tk - h \equiv 0 \pmod{a^2}.$$

If $k = 0$, then $-h \equiv_{a^2} 0$ only if either $h = 0$ or $h = 9$, being $a \geq 3$. If $h = 0$, then $w = \frac{1}{10}$, which is not an integer. If $h = 9$, then necessarily $a = 3$ and $t = 10$: after some computations we obtain $x = \frac{3}{4}$, which is not an integer. Suppose now that $k \neq 0$. Since $k \leq b^2$, we have $4tk - h \leq 4a^2 + 4 - h$, hence in order to get $4tk - h \equiv_{a^2} 0$ we must have

$$4tk - h \in \{a^2, 2a^2, 3a^2, 4a^2\}.$$

- If $4tk - h = a^2$, then
$$k = \frac{tb^2 + h - 1}{4t}.$$
If $h \neq 1$, then $h - 1$ is not divisible by $t \geq 10$, and k is not an integer. If $h = 1$, then $k = \frac{b^2}{4}$, which is not an integer by Proposition 1.5.8.

- If $4tk - h = 2a^2$, then
$$k = \frac{2tb^2 + h - 2}{4t},$$
If $h \neq 2$, then $h - 2$ is not divisible by $t \geq 10$, and k is not an integer. If $h = 2$, then $k = \frac{b^2}{2}$, which is not an integer by Proposition 1.5.8.

- If $4tk - h = 3a^2$, then
$$k = \frac{3tb^2 + h - 3}{4t}.$$
If $h \neq 3$, then $h - 3$ is not divisible by $t \geq 10$ and k is not an integer. If $h = 3$, then $k = \frac{3}{4}b^2$, which is not an integer by Proposition 1.5.8.

- If $4tk - h = 4a^2$, then
$$k = \frac{4tb^2 + h - 4}{4t}.$$
If $h \neq 4$, then $h - 4$ is not divisible by $t \geq 10$ and k is not an integer. If $h = 4$, then $k = b^2$, and with the help of a computer we get the following equation for x:
$$x^2 - xb^2 - tb^4 + \frac{b^4}{4} + b^2 = 0. \tag{B.0.1}$$
Then x is not an integer, otherwise (B.0.1) is not true, since $\frac{b^4}{4}$ is not an integer by Proposition 1.5.8.

We always obtain a contradiction with $x, k \in \mathbb{Z}$, then Case 2 never holds.

Case 3: suppose that $z + 10w = -tk + 2$. Then (4.6.10) gives

$$-4t^2b^2k - 4 < -4abty + a^2z < -4t^2b^2k + 8.$$

Since $-4abty + a^2z \in \mathbb{Z}$ we have

$$-4abty + a^2z = -4t^2b^2k + h, \qquad h \in \{-3, -2, \dots, 7\}.$$

With the help of a computer we obtain

$$w = \frac{-5a^2tk + (2h+2)a^2 - 4tk + h}{10a^2}.$$

Then w is an integer only if

$$4tk - h \equiv 0 \pmod{a^2}.$$

If $k = 0$, then $-h \equiv_{a^2} 0$ only if $h = 0$, being $a \geq 3$. If $h = 0$, we obtain $w = \frac{1}{5}$, which is not an integer. Suppose now that $k \neq 0$. Since $k \leq b^2$ we have $4tk - h \leq 4a^2 + 4 - h$, hence in order to get $4tk - h \equiv_{a^2} 0$ we must have

$$4tk - h \in \{a^2, 2a^2, 3a^2, 4a^2\}.$$

- If $4tk - h = a^2$, then
$$k = \frac{tb^2 + h - 1}{4t}.$$
If $h \neq 1$, then $h - 1$ is not divisible by $t \geq 10$ and k is not an integer.
If $h = 1$, then $k = \frac{b^2}{4}$, which is not an integer by Proposition 1.5.8.

- If $4tk - h = 2a^2$, then
$$k = \frac{2tb^2 + h - 2}{4t}.$$
If $h \neq 2$, then $h - 2$ is not divisible by $t \geq 10$ and k is not an integer.
If $h = 2$, then $k = \frac{b^2}{2}$, which is not an integer by Proposition 1.5.8.

- If $4tk - h = 3a^2$, then
$$k = \frac{3tb^2 + h - 3}{4t}.$$
If $h \neq 3$, then $h - 3$ is not divisible by $t \geq 10$, and k is not an integer.
If $h = 3$, then $k = \frac{3}{4}b^2$, which is not an integer by Proposition 1.5.8.

- If $4tk - h = 4a^2$, then
$$k = \frac{4tb^2 + h - 4}{4t}.$$
If $h \neq 4$, then $h - 4$ is not divisible by $t \geq 10$, and k is not an integer.
If $h = 4$, then $k = b^2$ and
$$w = \frac{-5tb^2 + 6}{10a^2},$$
which is not an integer since 6 is not divisible by 5.

Since we always obtain a contradiction with $k, w \in \mathbb{Z}$, Case 3 never holds. We conclude that there are no effective 2-cycles $A, B \in H^{2,2}(X, \mathbb{Z})$ such that $[D_1 \cap D_2] = A + B$, hence $D_1 \cap D_2$ is a reduced and irreducible surface.

Bibliography

[ABT01] E. Arrondo, M. Bertolini, and C. Turrini. A focus on focal surfaces. *Asian Journal of Mathematics*, 5(3):535–560, 2001.

[AH59] Michael F Atiyah and Friedrich Hirzebruch. Riemann–Roch theorems for differentiable manifolds. *Bulletin of the American Mathematical Society*, 65(4):276–281, 1959.

[And04] Yves André. Une introduction aux motifs. *Panoramas et syntheses*, 17, 2004.

[Ara06] Donu Arapura. Motivation for Hodge cycles. *Advances in Mathematics*, 207(2):762–781, 2006.

[AS15] Michela Artebani and Alessandra Sarti. Symmetries of order four on K3 surfaces. *Journal of the Mathematical Society of Japan*, 67(2):503–533, 2015.

[AST11] Michela Artebani, Alessandra Sarti, and Shingo Taki. K3 surfaces with non-symplectic automorphisms of prime order. *Mathematische Zeitschrift*, 268(1-2):507–533, 2011.

[Bak15] Benjamin Bakker. A classification of Lagrangian planes in holomorphic symplectic varieties. *Journal of the Institute of Mathematics of Jussieu*, 16(4):859–877, 2015.

[BC20] Pietro Beri and Alberto Cattaneo. On birational transformations of Hilbert schemes of points on K3 surfaces. *arXiv preprint arXiv:2005.05187*, 2020.

[BCMS19] Samuel Boissière, Andrea Cattaneo, Dmitrii Genrikhovich Markushevich, and Alessandra Sarti. On the non-symplectic involutions of the Hilbert square of a K3 surface. *Izvestiya: Mathematics*, 83(4):731, 2019.

[BCNWS16] Samuel Boissiere, Andrea Cattaneo, Marc Nieper-Wißkirchen, and Alessandra Sarti. The automorphism group of the Hilbert scheme of two points on a generic projective K3 surface. In *K3 surfaces and their moduli*, pages 1–15. Springer, 2016.

[Bea83a] Arnaud Beauville. Some remarks on Kahler manifolds with $c_1 = 0$. *Classification of algebraic and analytic manifolds, Katata, 1982*, 1983.

[Bea83b] Arnaud Beauville. Variétés Kähleriennes dont la premiere classe de Chern est nulle. *Journal of Differential Geometry*, 18(4):755–782, 1983.

[Bea96] Arnaud Beauville. *Complex algebraic surfaces.* Number 34. Cambridge University Press, 1996.

[Bea11] Arnaud Beauville. Antisymplectic involutions of holomorphic symplectic manifolds. *Journal of Topology*, 4(2):300–304, 2011.

[Ber20] Pietro Beri. *On birational transformations and automorphisms of some hyperkähler manifolds.* PhD thesis, Université de Poitiers, 2020.

[BHPVdV15] Wolf Barth, Klaus Hulek, Chris Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4. Springer, 2015.

[BM14] Arend Bayer and Emanuele Macrì. MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations. *Inventiones mathematicae*, 198(3):505–590, 2014.

[BNW07] Samuel Boissière and Marc A Nieper-Wißkirchen. Generating series in the cohomology of Hilbert schemes of points on surfaces. *LMS Journal of Computation and Mathematics*, 10:254–270, 2007.

[BNWS13] Samuel Boissière, Marc Nieper-Wißkirchen, and Alessandra Sarti. Smith theory and irreducible holomorphic symplectic manifolds. *Journal of Topology*, 6(2):361–390, 2013.

[Bog78] Fedor Alekseevich Bogomolov. Hamiltonian Kahler manifolds. In *Doklady Akademii Nauk*, volume 243, pages 1101–1104. Russian Academy of Sciences, 1978.

[Boi05] Samuel Boissière. Chern classes of the tangent bundle on the Hilbert scheme of points on the affine plane. *Journal of Algebraic Geometry*, 14(4):761–787, 2005.

[Bou04] Sébastien Boucksom. Divisorial Zariski decompositions on compact complex manifolds. In *Annales scientifiques de l'Ecole normale supérieure*, volume 37, pages 45–76, 2004.

[BR75] Dan Burns and Michael Rapoport. On the Torelli problem for kählerian K3 surfaces. In *Annales scientifiques de l'École Normale Supérieure*, volume 8, pages 235–273, 1975.

[Bre13] Glen E Bredon. *Topology and geometry*, volume 139. Springer Science & Business Media, 2013.

[Bri77] Joël Briançon. Description de $\mathrm{Hilb}^n\mathbb{C}\{x, y\}$. *Inventiones mathematicae*, 41(1):45–89, 1977.

[Bus19] Nikolay Buskin. Every rational Hodge isometry between two K3 surfaces is algebraic. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2019(755):127–150, 2019.

[Cat19] Alberto Cattaneo. Automorphisms of Hilbert schemes of points on a generic projective K3 surface. *Mathematische Nachrichten*, 292(10):2137–2152, 2019.

[Cat20] Andrea Cattaneo. The degree of the tangent and secant variety to a projective surface. *Advances in Geometry*, 20(2):233–248, 2020.

[CDE+02] Manuel Clavel, Francisco Durán, Steven Eker, Patrick Lincoln, Narciso Martı-Oliet, José Meseguer, and José F Quesada. Maude: Specification and programming in rewriting logic. *Theoretical Computer Science*, 285(2):187–243, 2002.

[CGM19] Chiara Camere, Alice Garbagnati, and Giovanni Mongardi. Calabi–Yau quotients of hyperkähler four-folds. *Canadian Journal of Mathematics*, 71(1):45–92, 2019.

[Che55] Claude Chevalley. Invariants of finite groups generated by reflections. *American Journal of Mathematics*, 77(4):778–782, 1955.

[CM13] François Charles and Eyal Markman. The standard conjectures for holomorphic symplectic varieties deformation equivalent to Hilbert schemes of K3 surfaces. *Compositio Mathematica*, 149(3):481–494, 2013.

[CS13] John Horton Conway and Neil James Alexander Sloane. *Sphere packings, lattices and groups*, volume 290. Springer Science & Business Media, 2013.

[Deb13] Olivier Debarre. *Higher-dimensional algebraic geometry*. Springer Science & Business Media, 2013.

[DHH+15] Ulrich Derenthal, Jürgen Hausen, Armand Heim, Simon Keicher, and Antonio Laface. Cox rings of cubic surfaces and Fano threefolds. *Journal of Algebra*, 436:228–276, 2015.

[DK18] Olivier Debarre and Alexander Kuznetsov. Gushel-Mukai varieties: Classification and birationalities. *Algebraic Geometry*, pages 15–76, 2018.

[DK19] Olivier Debarre and Alexander Kuznetsov. Double covers of quadratic degeneracy and Lagrangian intersection loci. *Mathematische Annalen*, pages 1–35, 2019.

[DM19] Olivier Debarre and Emanuele Macrì. On the period map for polarized hyperkähler fourfolds. *International Mathematics Research Notices*, 2019(22):6887–6923, 2019.

[EGL01] Geir Ellingsrud, Lothar Göttsche, and Manfred Lehn. On the cobordism class of the hilbert scheme of a surface. *J. Algebraic Geom.*, 10:81–100, 2001.

[EH87] David Eisenbud and Joe Harris. On varieties of minimal degree. In *Proc. Sympos. Pure Math*, volume 46, pages 3–13, 1987.

[EH16] David Eisenbud and Joe Harris. *3264 and all that: A second course in algebraic geometry*. Cambridge University Press, 2016.

[Eis13] David Eisenbud. *Commutative algebra: with a view toward algebraic geometry*, volume 150. Springer Science & Business Media, 2013.

[EPW01] David Eisenbud, Sorin Popescu, and Charles Walter. Lagrangian subbundles and codimension 3 subcanonical subschemes. *Duke Mathematical Journal*, 107(3), 2001.

[Fer12] Andrea Ferretti. The Chow ring of double EPW sextics. *Algebra & Number Theory*, 6(3):539–560, 2012.

[FL17] Mihai Fulger and Brian Lehmann. Positive cones of dual cycle classes. *Algebraic Geometry*, 4(1):1–28, jan 2017.

[FMOS20] Laure Flapan, Emanuele Macrì, Kieran O'Grady, and Giulia Saccà. Antisymplectic involutions on hyperkähler manifolds. In *2020 Fall Eastern Virtual Sectional Meeting*. AMS, 2020.

[FN71] Akira Fujiki and Shigeo Nakano. Supplement to "On the inverse of monoidal transformation". *Publications of the Research Institute for Mathematical Sciences*, 7(3):637–644, 1971.

[Fog68] John Fogarty. Algebraic families on an algebraic surface. *American Journal of Mathematics*, 90(2):511–521, 1968.

[Fuj83] Akira Fujiki. On primitive symplectic compact Kähler V-manifolds of dimension four, in "Classification of Algebraic and Analytic Manifolds". *Progress in Mathematics, Birkhauser*, 39:71–125, 1983.

[Fuj87] Akira Fujiki. On the de Rham cohomology group of a compact Kähler symplectic manifold. In *Algebraic Geometry, Sendai, 1985*, pages 105–165. Mathematical Society of Japan, 1987.

[Ful13] William Fulton. *Intersection theory*, volume 2. Springer Science & Business Media, 2013.

[GD66] A. Grothendieck and J. Dieudonné. Éléments de géométrie algébrique. *Publications mathématiques de l'IHÉS*, 28(1):5–248, 1966.

[GH78] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*, volume 19. Wiley Online Library, 1978.

[GHJ12] Mark Gross, Daniel Huybrechts, and Dominic Joyce. *Calabi–Yau manifolds and related geometries: lectures at a summer school in Nordfjordeid, Norway, June 2001*. Springer Science & Business Media, 2012.

[Göt90] Lothar Göttsche. The Betti numbers of the Hilbert scheme of points on a smooth projective surface. *Mathematische Annalen*, 286(1):193–207, 1990.

[Gri79] Phillip A Griffiths. A theorem concerning the differential equations satisfied by normal functions associated to algebraic cycles. *American Journal of Mathematics*, 101(1):94–131, 1979.

[Gro61] Alexandre Grothendieck. Techniques de construction et theoremes d'existence en geometrie algebrique, IV: Les schemas de Hilbert. *Séminaire Bourbaki*, 221, 1961.

[Gua01] Daniel Guan. On the Betti numbers of irreducible compact hyperkähler manifolds of complex dimension four. *Mathematical Research Letters*, 8(5):663–669, 2001.

[HA62] Friedrich Hirzebruch and Michael Francis Atiyah. The Riemann–Roch theorem for analytic embeddings. *Topology*, 1:151–166, 1962.

[Har06] Robin Hartshorne. *Ample subvarieties of algebraic varieties*, volume 156. Springer, 2006.

[Har13] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 2013.

[Hat05] Allen Hatcher. *Algebraic topology.* Cambridge, 2005.

[HBS66] Friedrich Hirzebruch, Armand Borel, and RLE Schwarzenberger. *Topological methods in algebraic geometry*, volume 175. Springer Berlin, 1966.

[HHT12] David Harvey, Brendan Hassett, and Yuri Tschinkel. Characterizing projective spaces on deformations of Hilbert schemes of K3 surfaces. *Communications on Pure and Applied Mathematics*, 65(2):264–286, 2012.

[HK00] Yi Hu and Sean Keel. Mori dream spaces and GIT. *Michigan Mathematical Journal*, 48(1), 2000.

[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves.* Cambridge University Press, 2010.

[HT01] Brendan Hassett and Yuri Tschinkel. Rational curves on holomorphic symplectic fourfolds. *Geometric And Functional Analysis*, 11(6):1201–1228, 2001.

[HT09] Brendan Hassett and Yuri Tschinkel. Moving and ample cones of holomorphic symplectic fourfolds. *Geometric and Functional Analysis*, 19(4):1065, 2009.

[Huy99] Daniel Huybrechts. Compact hyperkähler manifolds: basic results. *Inventiones mathematicae*, 135(1):63–113, 1999.

[Huy02] Daniel Huybrechts. Moduli spaces of hyperkähler manifolds and mirror symmetry. *arXiv preprint math/0210219*, 2002.

[Huy03] Daniel Huybrechts. The Kähler cone of a compact hyperkähler manifold. *Mathematische Annalen*, 326(3):499–513, 2003.

[Huy16] Daniel Huybrechts. *Lectures on K3 surfaces*, volume 158. Cambridge University Press, 2016.

[Huy19] Daniel Huybrechts. Motives of isogenous K3 surfaces. *Commentarii Mathematici Helvetici*, 94(3):445–458, 2019.

[HV14] Daniel Huybrechts and Claire Voisin. Curves and cycles on K3 surfaces. *Algebraic Geometry*, pages 69–106, jan 2014.

[IM15] Atanas Iliev and Carlo Madonna. EPW sextics and Hilbert squares of K3 surfaces. *Serdica Mathematical Journal*, 41(4):343–354, 2015.

[Isk78] VA Iskovskih. Fano 3-folds. II. *Mathematics of the USSR-Izvestiya*, 12(3):469, 1978.

[Ive06] Birger Iversen. *Linear determinants with applications to the Picard scheme of a family of algebraic curves*, volume 174. Springer, 2006.

[Kap16a] Simon Kapfer. Computing cup products in integral cohomology of Hilbert schemes of points on K3 surfaces. *LMS Journal of Computation and Mathematics*, 19(1):78–97, 2016.

[Kap16b] Simon Kapfer. *On the cohomology of irreducible holomorphically symplectic varieties*. PhD thesis, Universität Augsburg, 2016.

[Kaw92] Yujiro Kawamata. Unobstructed deformations, a remark on a paper of Z. Ran. *J. Alg. Geom*, 1:183–190, 1992.

[KMM87] Yujiro Kawamata, Katsumi Matsuda, and Kenji Matsuki. Introduction to the minimal model problem. In *Algebraic geometry, Sendai, 1985*, pages 283–360. Mathematical Society of Japan, 1987.

[KNT17] Kathlén Kohn, Bernt Ivar Utstøl Nødland, and Paolo Tripoli. Secants, bitangents, and their congruences. In *Combinatorial Algebraic Geometry*, pages 87–112. Springer, 2017.

[Kod64] Kunihiko Kodaira. On the structure of compact complex analytic surfaces, I. *American Journal of Mathematics*, 86(4):751–798, 1964.

[Kod06] Kunihiko Kodaira. *Complex manifolds and deformation of complex structures*. Springer, 2006.

[Kon00] Shigeyuki Kondo. A complex hyperbolic structure for the moduli space of curves of genus three. *Journal für die reine und angewandte Mathematik*, 2000(525):219–232, 2000.

[Kul77] Viktor S Kulikov. Degenerations of K3 surfaces and Enriques surfaces. *Mathematics of the USSR-Izvestiya*, 11(5):957, 1977.

[Kur62] Masatake Kuranishi. On the locally complete families of complex analytic structures. *Annals of Mathematics*, pages 536–577, 1962.

[Kur65] Masatake Kuranishi. New proof for the existence of locally complete families of complex structures. In *Proceedings of the conference on complex analysis*, pages 142–154. Springer, 1965.

[Laz17] Robert K Lazarsfeld. *Positivity in algebraic geometry I: Classical setting: line bundles and linear series*, volume 48. Springer, 2017.

[Leh99] Manfred Lehn. Chern classes of tautological sheaves on Hilbert schemes of points on surfaces. *Inventiones mathematicae*, 136(1):157–207, 1999.

[LS03] Manfred Lehn and Christoph Sorger. The cup product of Hilbert schemes for K3 surfaces. *Inventiones mathematicae*, 152(2):305–329, 2003.

[Mar07] Eyal Markman. Integral generators for the cohomology ring of moduli spaces of sheaves over Poisson surfaces. *Advances in Mathematics*, 208(2):622–646, 2007.

[Mar11] Eyal Markman. A survey of Torelli and monodromy results for holomorphic-symplectic varieties. In *Complex and differential geometry*, pages 257–322. Springer, 2011.

[Mar13] Eyal Markman. Prime exceptional divisors on holomorphic symplectic varieties and monodromy reflections. *Kyoto Journal of Mathematics*, 53(2):345–403, 2013.

[Mat17] Daisuke Matsushita. On isotropic divisors on irreducible symplectic manifolds. In *Higher Dimensional Algebraic Geometry: In honour of Professor Yujiro Kawamata's sixtieth birthday*, pages 291–312. Mathematical Society of Japan, 2017.

[May72] Alan L Mayer. Families of K-3 surfaces. *Nagoya Mathematical Journal*, 48:1–17, 1972.

[Men19] Paolo Menegatti. *Action du groupe de Klein sur une surface K3*. PhD thesis, Université de Poitiers, 2019.

[Miy87] Yoichi Miyaoka. The Chern classes and Kodaira dimension of a minimal variety. In *Algebraic geometry, Sendai, 1985*, pages 449–476. Mathematical Society of Japan, 1987.

[MS16] John Milnor and James D Stasheff. *Characteristic Classes*, volume 76. Princeton university press, 2016.

[Muk84] Shigeru Mukai. Symplectic structure of the moduli space of sheaves on an abelian or K3 surface. *Inventiones mathematicae*, 77(1):101–116, 1984.

[Muk88] Shigeru Mukai. Moduli of vector bundles on K3 surfaces, and symplectic manifolds. *Sugaku Expositions*, 1(2):138–174, 1988.

[Nag58] Masayoshi Nagata. Remarks on a Paper of Zariski on the Purity of Branch-Loci. *Proceedings of the National Academy of Sciences of the United States of America*, 44(8):796, 1958.

[Nak63] Yoshikazu Nakai. Some fundamental lemmas on projective schemes. *Transactions of the American Mathematical Society*, 109(2):296–302, 1963.

[Nak97] Hiraku Nakajima. Heisenberg algebra and Hilbert schemes of points on projective surfaces. *Annals of mathematics*, 145(2):379–388, 1997.

[Nik80] Viacheslav V Nikulin. Integral symmetric bilinear forms and some of their applications. *Mathematics of the USSR-Izvestiya*, 14(1):103, 1980.

[Nik83] Viacheslav V Nikulin. Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections. Algebrogeometric applications. *Journal of Soviet Mathematics*, 22(4):1401–1475, 1983.

[NW02] Marc A Nieper-Wißkirchen. Characteristic classes and Rozansky–Witten invariants of compact Hyperkähler manifolds. *PhD Thesis, Köln*, 2002.

[NW04] Marc Nieper-Wißkirchen. *Chern Numbers and Rozansky–Witten Invariants of Compact Hyperkähler Manifolds*. World Scientific, 2004.

[O'G99] K. G. O'Grady. Desingularized moduli spaces of sheaves on a K3. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 1999(512):49–117, 1999.

[O'G03] Kieran G. O'Grady. A new six-dimensional irreducible symplectic variety. *Journal of Algebraic Geometry*, 12(3):435–505, 2003.

[O'G05] Kieran G. O'Grady. Involutions and Linear Systems on Holomorphic Symplectic Manifolds. *Geometric & Functional Analysis GAFA*, 15(6):1223–1274, 2005.

[O'G06] Kieran G. O'Grady. Irreducible symplectic 4-folds and Eisenbud–Popescu–Walter sextics. *Duke Mathematical Journal*, 134(1):99–137, 2006.

[O'G08a] Kieran G. O'Grady. Dual Double EPW-sextics and Their Periods. *Pure and Applied Mathematics Quarterly*, 4(2):427–468, 2008.

[O'G08b] Kieran G O'Grady. Irreducible symplectic 4-folds numerically equivalent to $(K3)^{[2]}$. *Communications in Contemporary Mathematics*, 10(04):553–608, 2008.

[O'G10] Kieran G O'Grady. Higher-dimensional analogues of K3 surfaces. *Current developments in algebraic geometry*, 59:257–293, 2010.

[O'G12] Kieran G O'Grady. EPW-sextics: taxonomy. *manuscripta mathematica*, 138(1):221–272, 2012.

[O'G13] Kieran O'Grady. Double covers of EPW-sextics. *Michigan Mathematical Journal*, 62(1), 2013.

[Ott15] John Christian Ottem. Nef cycles on some hyperkahler fourfolds. *arXiv preprint arXiv:1505.01477*, 2015.

[Per13] Oskar Perron. *Die Lehre von den Kettenbrüchen: B and I: Elementare Kettenbrüche*. Springer-Verlag, 2013.

[PS08] Chris AM Peters and Joseph HM Steenbrink. *Mixed hodge structures*, volume 52. Springer Science & Business Media, 2008.

[QW05] Zhenbo Qin and Weiqiang Wang. Integral operators and integral cohomology classes of Hilbert schemes. *Mathematische Annalen*, 331(3):669–692, 2005.

[Ran92] Ziv Ran. Deformations of manifolds with torsion or negative canonical bundles. *J. Algebraic Geom.*, 1:279–291, 1992.

[Rie18] Ulrike Rieß. Base divisors of big and nef line bundles on irreducible symplectic varieties. *arXiv preprint arXiv:1807.05192*, 2018.

[Rie20] Ulrike Rieß. On the non-divisorial base locus of big and nef line bundles on $K3^{[2]}$-type varieties. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 151(1):52–78, 2020.

[Sam58] Pierre Samuel. Relations d'équivalence en géométrie algébrique. In *Proc. Internat. Congress Math*, pages 470–487, 1958.

[Sch] Christian Schnell. The Serre construction in codimension two. Available at `http://www.math.stonybrook.edu/%7ecschnell/pdf/notes/serre.pdf`.

[Sch10] Ulrich Schlickewei. The Hodge conjecture for self-products of certain K3 surfaces. *Journal of Algebra*, 324(3):507–529, 2010.

[SD74] Bernard Saint-Donat. Projective models of K-3 surfaces. *American Journal of Mathematics*, 96(4):602–639, 1974.

[SDWSDJ+20] The Sage Developers, William Stein, David Joyner, David Kohel, John Cremona, and Burçin Eröcal. Sagemath, version 9.0, 2020.

[Siu81] Yum-Tong Siu. A simple proof of the surjectivity of the period map of K3 surfaces. *manuscripta mathematica*, 35(3):311–321, 1981.

[Siu83] Y-T Siu. Every K3 surface is Kähler. *Inventiones mathematicae*, 73(1):139–150, 1983.

[Smi61] Henry John Stephen Smith. On systems of linear indeterminate equations and congruences. *Philosophical transactions of the royal society of london*, XV(151):293–326, 1861.

[ST54] Geoffrey C Shephard and John A Todd. Finite unitary reflection groups. *Canadian Journal of Mathematics*, 6:274–304, 1954.

[Tia87] Gang Tian. Smoothness of the universal deformation space of compact Calabi–Yau manifolds and its Peterson–Weil metric. In *Mathematical aspects of string theory*, pages 629–646. World Scientific, 1987.

[Tod80] Andrei N Todorov. Applications of the Kähler–Einstein–Calabi–Yau metric to moduli of K3 surfaces. *Inventiones mathematicae*, 61(3):251–265, 1980.

[Tod89] Andrey N Todorov. The Weil–Petersson geometry of the moduli space of SU ($n \geq 3$) (calabi–yau) manifolds I. *Communications in mathematical physics*, 126(2):325–346, 1989.

[Ver96] Mikhail Verbitsky. Cohomology of compact hyperkähler manifolds and its applications. *Geometric & Functional Analysis GAFA*, 6(4):601–611, 1996.

[Ver13] Misha Verbitsky. Mapping class group and a global Torelli theorem for hyperkähler manifolds. *Duke Mathematical Journal*, 162(15):2929–2986, 2013.

[vGS07] Bert van Geemen and Alessandra Sarti. Nikulin involutions on K3 surfaces. *Mathematische Zeitschrift*, 255(4):731–753, 2007.

[Voi02] Claire Voisin. *Théorie de Hodge et géométrie algébrique complexe*, volume 10. Société mathématique de France Paris, 2002.

[Voi10] Claire Voisin. Coniveau 2 complete intersections and effective cones. *Geometric and Functional Analysis*, 19(5):1494–1513, 2010.

[Wel81] Gerald E Welters. Abel–Jacobi isogenies for certain types of Fano threefolds. *MC Tracts*, 1981.

[WW03] Jan Wierzba and Jarosław Wiśniewski. Small contractions and symplectic 4-folds. *Duke Mathematical Journal*, 120(1):65–95, 2003.

[Yok94] Hideo Yokoi. Solvability of the diophantine equation $x^2 - dy^2 = \pm 2$ and new invariants for real quadratic fields. *Nagoya Mathematical Journal*, 134:137–149, 1994.

[Zar58] Oscar Zariski. On the purity of the branch locus of algebraic functions. *Proceedings of the National Academy of Sciences of the United States of America*, 44(8):791, 1958.