ON THE EXISTENCE OF CANONICAL MULTI–PHASE BRAKKE FLOWS

SALVATORE STUVARD AND YOSHIHIRO TONEGAWA

Abstract. This paper establishes the global-in-time existence of a multi-phase mean curvature flow, evolving from an arbitrary closed rectifiable initial datum, which is a Brakke flow and a BV solution at the same time. In particular, we prove the validity of an explicit identity concerning the change of volume of the evolving grains, showing that their boundaries move according to the generalized mean curvature vector of the Brakke flow. Under suitable assumptions on the initial datum, such additional property resolves the non-uniqueness issue of Brakke flows.

Contents

Date: October 1, 2021.

1. Introduction

Arising as the gradient flow of the area functional, the *mean curvature flow* (henceforth abbreviated as MCF) is arguably the most fundamental geometric flow involving extrinsic curvatures. The unknown of MCF is a one-parameter family $\{\Gamma(t)\}_{t\geq 0}$ of surfaces in Euclidean space (or in an ambient Riemannian manifold) such that the normal velocity of the flow equals the mean curvature vector at each point for every time t. The initial value problem for MCF starting with a smooth closed surface $\Gamma(0) = \Gamma_0$ is locally well-posed in time, until the appearance of singularities such as shrinking or neck pinching. Numerous frameworks of generalized solutions past singularities have been proposed: we mention, among others, the Brakke flow [5], level set flows [7, 11], BV solutions [12, 22] and L^2 flows [26, 4]. Existence of these possibly different generalized solutions to the MCF as well as their relations have been studied intensively in the past 40 years or so.

The aim of the present paper is to establish the global-in-time existence of a *"canonical multi-phase" Brakke flow* evolving from an arbitrary rectifiable initial datum. The attribute *"multi-phase"* here refers to the fact that the evolving surfaces are, in fact, *boundaries* of finitely many, but at least two, labelled open subsets of $\mathbb{R}^{n+1}$ (henceforth referred to also as *"grains"* or *"phases"*). The MCF evolution of such objects is strongly motivated by material sciences, as it describes the motion and growth of crystallites in polycrystalline materials; see e.g. [27]. While the literature concerning the two-phase MCF is rich, fewer works have been dedicated to the more general case of multi-phase MCF with at least three grains, despite its relevance in the modeling of physical processes governed by surface tension type energies.

In the present paper, we work with an arbitrary (but finite) number of grains. The solution we construct consists of two objects: the flow of the evolving grains and a Brakke flow, intertwined as follows. The Brakke flow – a measure-theoretic generalization of MCF, particularly suited to describe the evolution of surfaces through singularities (see Definition 2.1) – is essentially supported on the topological boundary of the grains, and it keeps track of multiplicities. Additionally, the mean curvature of the Brakke flow determines the distributional velocity at which the reduced boundary of each grain moves. As a result of the latter property, the change of volume of each grain between two instants of time can be recovered by integrating the mean curvature over the reduced boundary, a property certainly expected for a smooth MCF but quite non-trivial in a context where singularities and multiplicities occur. The attribute *"canonical"* refers to this very precise interplay between the Brakke flow and the evolution of the grains. Note that Brakke flows are non-unique in general due to the nature of the formulation, but the existence of the grains prevents redundant non-uniqueness such as sudden vanishing, for example. In the absence of higher multiplicities of the Brakke flow, we show that the collection of the grains (or, more precisely, the collection of their indicator functions) constitutes a *BV solution* of the MCF (see Definition 2.4). In this case, the grain boundaries are, in fact, a smooth MCF almost everywhere in space and time by Brakke's partial regularity theory [5, 17, 35].

In certain instances, the additional BV characterization may lead to the uniqueness of the canonical Brakke flow. For instance, in ambient dimension $n + 1 = 2$, the recent work of Fischer et al. [12] shows that, as soon as a *strong solution* of the network flow exists (see [12, Definition 11] for the definition of strong solution) then any BV solution must coincide with it at least for all times until the first topology changes occur. One conclusion derived from [12] and the present paper is then that, if $n = 1$ and if a regular network flow starting from the given initial datum exists, then the canonical Brakke flow constructed in the present paper from the same initial datum necessarily coincides with that regular flow until the first topology changes; see [12, Theorem 19] for the precise statement (also see [14] for a similar uniqueness result in $n = 2$). In fact, the work [12] inspired the study carried out in the present paper.

In a more precise and technical terminology, the highlight of the main results of the present paper may be stated as follows (see the complete statements in Section 2.3).

Theorem A. Let $E_{0,1},\ldots,E_{0,N} \subset \mathbb{R}^{n+1}$ be mutually disjoint non-empty open sets with $N \geq 2$ such that $\Gamma_0 := \mathbb{R}^{n+1} \setminus \cup_{i=1}^N E_{0,i}$ is countably n-rectifiable. Assume that the n-dimensional Hausdorff measure of Γ_0 is finite or grows at most exponentially fast at infinity. Then there exist a Brakke flow $\{V_t\}_{t\geq 0}$ as well as one-parameter families $\{E_i(t)\}_{t\geq 0}$ ($i \in \{1,\ldots,N\}$) of open sets, with $\{E_1(t),\ldots,E_N(t)\}$ mutually disjoint for each $t \geq 0$, such that

$$\|V_0\| = \mathcal{H}^n \llcorner_{\Gamma_0}\,, \qquad E_i(0) = E_{0,i} \quad \text{for every } i = 1,\ldots,N\,,$$

and satisfying the following properties. Writing $\Gamma(t) := \mathbb{R}^{n+1} \setminus \cup_{i=1}^N E_i(t)$:

(a) $\mathcal{H}^{n-1+\delta}(\Gamma(t) \triangle \operatorname{spt}\|V_t\|) = 0$ for any $\delta > 0$ and for a.e. $t \geq 0$.

(b) For each $i = 1,\ldots,N$ and for any arbitrary test function $\phi = \phi(x,t)$, we have, in the sense of distributions on $[0,\infty)$, that

$$\frac{d}{dt}\int_{E_i(t)} \phi\,dx = \int_{\partial^* E_i(t)} \phi\, h\cdot\nu_i\,d\mathcal{H}^n + \int_{E_i(t)} \frac{\partial\phi}{\partial t}\,dx\,. \tag{1.1}$$

(c) If the Brakke flow is locally a unit density flow, then, locally, we have

$$\mathcal{H}^n(\Gamma(t) \triangle \cup_{i=1}^N \partial^* E_i(t)) = 0 \text{ and } \|V_t\| = \mathcal{H}^n \llcorner_{\cup_{i=1}^N \partial^* E_i(t)} \tag{1.2}$$

for a.e. t.

In the above statement, $A\triangle B$ denotes the symmetric difference of two sets A and B, and $\operatorname{spt}\|V_t\|$ is the support of the weight measure $\|V_t\|$ of the varifold V_t. The symbol h denotes the generalized mean curvature vector of V_t, and ν_i is the outer unit normal vector field to the reduced boundary $\partial^* E_i(t)$ of $E_i(t)$. Since $\Gamma(t) = \cup_{i=1}^N \partial E_i(t)$ for all $t > 0$ (see Theorem 2.11(3)), the claim (a) shows that the support of the Brakke flow coincides – up to a lower dimensional set – with the union of the topological boundaries of the grains for a.e. $t > 0$. The claim (b) states that each reduced boundary $\partial^* E_i(t)$ is a solution to the MCF in the integral sense specified in (1.1): that is, the generalized velocity of $\partial^* E_i(t)$ – defined as the distributional time derivative of the indicator function of $E_i(t)$ – is precisely $h\cdot\nu_i\,\mathcal{H}^n\llcorner_{\partial^* E_i(t)}$.

When integrated, formula (1.1) provides, as a byproduct, the change of the $(n+1)$-dimensional volume of each grain $E_i(t)$ in any bounded open set U:

$$\mathcal{L}^{n+1}(U\cap E_i(t_2)) - \mathcal{L}^{n+1}(U\cap E_i(t_1)) = \int_{t_1}^{t_2}\int_{U\cap\partial^* E_i(t)} h\cdot\nu_i\,d\mathcal{H}^n dt\,. \tag{1.3}$$

We emphasize the following point in particular: the formulae (1.1) and (1.3) hold true even in case there is a “higher multiplicity portion” of $\|V_t\|$ on $\partial^* E_i(t)$, or some “interior boundary” $\partial E_i(t) \setminus \partial^* E_i(t)$. As far as the authors are aware of, for generalized MCF, (1.3) had never been established prior to the present paper, even for the two-phase MCF. Even though we have $\|V_0\| = \mathcal{H}^n\llcorner_{\Gamma_0}$, and thus the unit density condition is satisfied at the initial time of the Brakke flow, it is unfortunately not possible to exclude, in the very general framework under consideration, the occurrence of higher multiplicities at a later time. Despite all those possible singular behaviors, (1.1) and (1.3) are guaranteed time-globally. The claim (c) states that, if the higher multiplicity of $\|V_t\|$ does not occur for a.e. t locally in space-time, then the phase boundary measure and the Brakke flow may be identified with one another in that region and we may say that the N-tuple $\chi = (\chi_{E_1}, \dots, \chi_{E_N})$ is a BV solution to MCF (see Definition 2.4). We can guarantee the existence of a unit density flow for some initial time interval $[0, T_0)$ if we additionally assume a suitable density ratio upper bound on Γ_0 (see Theorem 2.13). Such assumption would still allow for an initial datum Γ_0 which consists of a union of Lipschitz curves joined by triple junctions in $n = 1$ and Lipschitz bubble clusters with tetrahederal singularities in $n = 2$, for example.

For general Brakke flows, there is no clear pathway leading from the characterization of Brakke flow, which consists of a variational inequality dictating an upper bound on the rate of change of the mass of the evolving surfaces (see Section 2), to formula (1.1), even under the unit density assumption. In fact, as mentioned already, by the partial regularity theorem for unit density Brakke flows [5, 17, 34], in this case it is known that $\Gamma(t)$ is a C^∞ MCF in a space-time neighborhood of (x_0, t_0) for a.e. t_0 and for $\mathcal{H}^n$-a.e. $x_0 \in \Gamma(t_0)$: nonetheless, this alone is not sufficient to guarantee that $\chi = (\chi_{E_1}, \dots, \chi_{E_N})$ is a BV solution. The same remark goes for the opposite implication, i.e., from BV solution to Brakke flow. Since there is no known partial regularity theory for general BV solutions, these two notions appear far from being equivalent in any case. In the present paper, instead, the Brakke flow arises as the limit of a suitable time-discrete approximation scheme, analogous to that introduced by Kim and the second-named author in [18], and we prove (1.1) by showing that an analogous identity holds approximately true for the approximating flows, with vanishing errors in the limit. In order to gain enough control on the change of volumes in the approximation scheme and consequently obtain good estimates on the error terms, we will need to implement an appropriate modification to the construction of the time-discrete approximate flows devised in [18]: the details of such modification will be explained thoroughly later; see Section 3.1 and Appendix A.

We next discuss closely related works, particularly on the aspect of existence of generalized MCF. For two-phase MCF, the level-set method [7, 11] provides a general existence and uniqueness result even past the time after singularities appear. On the other hand, the level-set may develop a non-trivial interior, a phenomenon called "fattening", due to the singular behavior of the MCF. Also the uniqueness of the level-set solutions depends essentially on the maximum principle and it cannot handle general multi-phase MCF of more than two phases.

For the general multi-phase problem, it is natural to consider an initial datum Γ_0 with singularities to start with. For example, in dimension $n = 1$, a typical Γ_0 in a three-phase problem is a union of curves meeting at triple junctions. In the parametric setting, Bronsard and Reitich [6] first showed the short-time existence of a unique solution for $C^{2,\alpha}$ initial datum. Since then, there have been numerous studies (mostly for $n = 1$ but also for higher dimensions [13, 8]), and we refer the reader to the survey [24] for the references on the parametric approaches. Due to the nature of the solutions and the need to heavily employ PDE techniques, these existence results do not extend beyond the time of topological changes. With a non-parametric approach and for the existence of MCF with regular triple junctions, one can adopt the elliptic regularization [16] for the class of flat chains with coefficients in a finite group, see [29].

Luckhaus and Sturzenhecker [22] introduced the formulation of BV solution of the two-phase MCF, which can be extended naturally to the multi-phase MCF. Their existence result is conditional, in the sense that a BV solution is shown to exist under the assumption that the time-discrete approximate solutions converge to their limit without loss of surface energy. Laux and Otto [20] proved that a sequence arising from the thresholding scheme of Merriman, Bence, and Osher converges conditionally to a BV solution using the interpretation in terms of minimizing movement due to Esedoḡlu and Otto [9], again under an assumption similar to [22] (see also [21] for a similar convergence result of the parabolic Allen-Cahn system). The BV solution of [22] was partly motivated by the minimizing movements scheme of Almgren-Taylor-Wang [1], and the multi-phase version has been studied recently by Bellettini and Kholmatov [3].

On the side of Brakke flows, Ilmanen [15] proved the existence of a rectifiable Brakke flow arising as a limit of solutions to a parabolic Allen-Cahn equation, and the second-named author proved the integrality of the Brakke flow [33]. These results are for two-phase MCF, but the relation to the BV solution as formulated in [22] remained obscure. In a different but related problem – the Allen-Cahn action functional –, Mugnai and Röger [26] introduced a notion of L^2 flow to describe a weak formulation of MCF with additional L^2 forcing term for $n = 1, 2$. The existence of the L^2 flow depends on the result of Röger and Schätzle [28] which solved one of De Giorgi's conjectures. The work [26] essentially contains the result that the limit phase boundary of the parabolic Allen-Cahn equation satisfies an analogous equation to (1.1) (see [26, Proposition 4.5]). The solution constructed in the present paper is, in fact, also an L^2 flow in the sense of Mugnai-Röger, and, even though the approach leading to (1.1) is different, some properties of generalized velocities of L^2 flows established in [26] are used in the present paper as well.

Finally, we mention again that a global-in-time existence result of a multi-phase Brakke flow which is equipped with moving grain boundaries was given by Kim and the second-named author in [18], reworking the pioneering paper by Brakke [5] within a different formulation. The grains in [18] move continuously with respect to the Lebesgue measure, but the problem concerning the validity of an exact identity involving the volume change was not addressed in there. Previous works by the authors of the present paper (see [32, 31]), in which certain Brakke flows are constructed with an approximation scheme analogous to that introduced in [18], could be reworked so that the additional conclusions concerning the interplay between the flow of the grains and the Brakke flow can be drawn in those contexts as well: in particular, it is possible to have the Brakke flow with prescribed boundary constructed in [32] satisfy the formulae (1.1) and (1.3) (see Section 7.2).

Acknowledgments. S.S. is partially supported by the *GNAMPA - Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* of INdAM; Y.T. is partially supported by JSPS 18H03670, 19H00639, 17H01092.

2. Notation and main results

2.1. **Basic notation.** We shall use the same notation adopted in [18, Section 2]. In particular, the ambient space we will be working in is Euclidean space $\mathbb{R}^{n+1}$, and $\mathbb{R}^+$ will denote the interval $[0, \infty)$. Coordinates (x, t) are set in the product space $\mathbb{R}^{n+1} \times \mathbb{R}^+$, and t will be thought of and referred to as "time". For a subset A of Euclidean space, $\operatorname{clos} A$, $\operatorname{int} A$, ∂A, and $\operatorname{conv} A$ will denote the closure, interior, boundary, and convex hull of A, respectively. If $A \subset \mathbb{R}^{n+1}$ is (Borel) measurable, $\mathcal{L}^{n+1}(A)$ or $|A|$ will denote the Lebesgue measure of A, whereas $\mathcal{H}^k(A)$ denotes the k-dimensional Hausdorff measure of A. When $x \in \mathbb{R}^{n+1}$ and $r > 0$, $U_r(x)$ and $B_r(x)$ denote the open ball and the closed ball centered at x with radius r, respectively. More generally, if k is an integer then $U^k_r(x)$ and $B^k_r(x)$ will denote open and closed balls in $\mathbb{R}^k$, and $\omega_k := \mathcal{L}^k(U^k_1(0))$.

A positive Radon measure μ on $\mathbb{R}^{n+1}$ (or in "space-time" $\mathbb{R}^{n+1}\times\mathbb{R}^+$) is always also regarded as a positive linear functional on the space $C_c(\mathbb{R}^{n+1})$ of continuous and compactly supported functions on $\mathbb{R}^{n+1}$, with the pairing denoted $\mu(\phi)$ for $\phi \in C_c(\mathbb{R}^{n+1})$. The restriction of μ to a Borel set A is denoted $\mu\llcorner_A$, so that $(\mu\llcorner_A)(E) := \mu(A\cap E)$ for any $E \subset \mathbb{R}^{n+1}$. The support of μ is denoted $\operatorname{spt}\mu$, and it is the closed set defined by

$$\operatorname{spt}\mu := \left\{ x \in \mathbb{R}^{n+1} \, : \, \mu(B_r(x)) > 0 \text{ for every } r > 0 \right\} .$$

The upper and lower k-dimensional densities of a Radon measure μ at $x \in \mathbb{R}^{n+1}$ are

$$\Theta^{*k}(\mu, x) := \limsup_{r\to 0^+} \frac{\mu(B_r(x))}{\omega_k\, r^k} , \qquad \Theta^k_*(\mu, x) := \liminf_{r\to 0^+} \frac{\mu(B_r(x))}{\omega_k\, r^k} ,$$

respectively. If $\Theta^{*k}(\mu,x) = \Theta^k_*(\mu,x)$ then the common value is denoted $\Theta^k(\mu,x)$, and is called the k-dimensional density of μ at x. For $1 \le p \le \infty$, the space of p-integrable (resp. locally p-integrable) functions with respect to μ is denoted $L^p(\mu)$ (resp. $L^p_{\rm loc}(\mu)$). If $U \subset \mathbb{R}^{n+1}$ is an open set, $L^p(\mathcal{L}^{n+1}\llcorner_U)$ and $L^p_{\rm loc}(\mathcal{L}^{n+1}\llcorner_U)$ are simply written $L^p(U)$ and $L^p_{\rm loc}(U)$. For a signed or vector-valued measure μ, $|\mu|$ denotes its total variation.

Given an open set $U \subset \mathbb{R}^{n+1}$, we say that a function $f \in L^1(U)$ has bounded variation in U, written $f \in \mathrm{BV}(U)$, if

$$\sup\left\{ \int_U f \operatorname{div} g \, dx \, : \, g \in C^1_c(U;\mathbb{R}^{n+1}) \text{ with } \|g\|_{C^0} \le 1 \right\} < \infty .$$

If $f \in \mathrm{BV}(U)$ then there exists an $\mathbb{R}^{n+1}$-valued Radon measure on U, which we will call the measure derivative of f and denote ∇f, such that

$$\int_U f \operatorname{div} g\, dx = -\int g \cdot d\nabla f \qquad \text{for all } g \in C^1_c(U;\mathbb{R}^{n+1}) .$$

We say that $f \in \mathrm{BV}_{\rm loc}(U)$ if $f \in \mathrm{BV}(U')$ for all $U' \Subset U$.

For a set $E \subset \mathbb{R}^{n+1}$, χ_E is the characteristic (or indicator) function of E, defined by $\chi_E(x) = 1$ if $x \in E$, and $\chi_E(x) = 0$ otherwise. We say that E has locally finite perimeter in $\mathbb{R}^{n+1}$ if $\chi_E \in \mathrm{BV}_{\rm loc}(\mathbb{R}^{n+1})$. When E is a set of locally finite perimeter, then the measure derivative $\nabla\chi_E$ is the associated Gauss-Green measure, and its total variation $|\nabla\chi_E|$ is the perimeter measure; by De Giorgi's structure theorem, $|\nabla\chi_E| = \mathcal{H}^n\llcorner_{\partial^* E}$, where $\partial^* E$ is the reduced boundary of E, and $\nabla\chi_E = -\nu_E\,|\nabla\chi_E| = -\nu_E\,\mathcal{H}^n\llcorner_{\partial^*E}$, where ν_E is the outer pointing unit normal vector field to $\partial^* E$.

A subset $\Gamma \subset \mathbb{R}^{n+1}$ is countably k-rectifiable if it is $\mathcal{H}^k$-measurable and it admits a covering

$$\Gamma \subset Z \cup \bigcup_{h\in\mathbb{N}} f_h(\mathbb{R}^k) ,$$

where $\mathcal{H}^k(Z) = 0$ and $f_h \colon \mathbb{R}^k \to \mathbb{R}^{n+1}$ is Lipschitz. A countably k-rectifiable set Γ is (locally) $\mathcal{H}^k$-rectifiable if, moreover, $\mathcal{H}^k(\Gamma)$ is (locally) finite. Countably k-rectifiable sets Γ are characterized by the existence of approximate tangent planes $\mathcal{H}^k$-almost everywhere ([30, Theorem 11.6]). In other words, Γ is countably k-rectifiable if and only if there exists a *positive* function

$g \in L^1_{\rm loc}(\mathcal{H}^k \llcorner_\Gamma)$ such that the following holds: for $\mathcal{H}^k$-a.e. $x \in \Gamma$ there exists a k-dimensional linear subspace of $\mathbb{R}^{n+1}$, denoted $T_x\Gamma$ and referred to as the (approximate) tangent plane to Γ at x such that

$$g(x + r\,\cdot)\,\mathcal{H}^k \llcorner_{\frac{\Gamma - \{x\}}{r}} \overset{*}{\rightharpoonup} g(x)\,\mathcal{H}^k \llcorner_{T_x\Gamma} \qquad \text{in the sense of measures, as } r \to 0^+\,.$$

A (positive) measure μ on $\mathbb{R}^{n+1}$ is said to be k-rectifiable if there are a countably k-rectifiable set Γ and a positive function $g \in L^1_{\rm loc}(\mathcal{H}^k \llcorner_\Gamma)$ such that $\mu = g\,\mathcal{H}^k \llcorner_\Gamma$. If μ is k-rectifiable, $\mu = g\,\mathcal{H}^k \llcorner_\Gamma$, then for μ-a.e. x the measures $\mu_{x,r}$ defined, for $r > 0$, by $\mu_{x,r}(A) := r^{-k}\,\mu(x + r\,A)$ satisfy

$$\mu_{x,r} \overset{*}{\rightharpoonup} g(x)\,\mathcal{H}^k \llcorner_{T_x\Gamma} \qquad \text{as } r \to 0^+\,,$$

that is $g(x)\,\mathcal{H}^k \llcorner_{T_x\Gamma}$ is the tangent measure of μ at x. In this case the notation $T_x\mu$ may be used interchangeably with $T_x\Gamma$ to denote the approximate tangent plane to the k-rectifiable measure μ at x for μ-a.e. x. More generally, given a Radon measure μ on $\mathbb{R}^{n+1}$ and $x \in \mathbb{R}^{n+1}$, we say that μ has an approximate tangent k-plane at x if there exist $g(x) \in (0, \infty)$ and a k-dimensional linear subspace $\pi \subset \mathbb{R}^{n+1}$ such that

$$\mu_{x,r} \overset{*}{\rightharpoonup} g(x)\,\mathcal{H}^k \llcorner_\pi \qquad \text{as } r \to 0^+\,.$$

When this happens, the plane π is unique, and it is denoted $T_x\mu$.

A k-dimensional varifold in $\mathbb{R}^{n+1}$ is a positive Radon measure V on the space $\mathbb{R}^{n+1} \times \mathbf{G}(n+1,k)$, where $\mathbf{G}(n+1,k)$ is the Grassmannian of (unoriented) k-dimensional linear subspaces of $\mathbb{R}^{n+1}$. If V is a k-varifold in $\mathbb{R}^{n+1}$, we write $V \in \mathbf{V}_k(\mathbb{R}^{n+1})$, and we let $\|V\|$ and δV denote its weight and first variation, respectively. When δV is locally bounded and absolutely continuous with respect to $\|V\|$, we let $h(\cdot, V)$ denote the generalized mean curvature vector of V, so that $\delta V = -h(\cdot, V)\,\|V\|$ in the sense of $\mathbb{R}^{n+1}$-valued measures on $\mathbb{R}^{n+1}$. If Γ is countably k-rectifiable, and $\theta \in L^1_{\rm loc}(\mathcal{H}^k \llcorner_\Gamma)$ is positive and integer-valued, we let $\mathbf{var}(\Gamma, \theta)$ denote the varifold $\mathbf{var}(\Gamma, \theta) = \theta\,\mathcal{H}^k \llcorner_\Gamma \otimes \delta_{T.\Gamma}$. When V admits a representation $V = \mathbf{var}(\Gamma, \theta)$ as above, we say that V is an integral k-varifold, and we write $V \in \mathbf{IV}_k(\mathbb{R}^{n+1})$. All above notions concerning measures (and varifolds) in Euclidean space $\mathbb{R}^{n+1}$ can be immediately localized to open sets $U \subset \mathbb{R}^{n+1}$.

2.2. **Three weak notions of MCF.** As anticipated in the introduction, in the last few decades several alternative notions of weak solution to the MCF have been proposed. In this section we briefly define and comment upon the three of interest in the present paper: Brakke flows, L^2 flows, and BV flows. We begin with the notion of Brakke flow, introduced by Brakke in [5].

Definition 2.1 (Brakke flow)**.** Let $0 < T \leq \infty$, and let $U \subset \mathbb{R}^{n+1}$ be an open set. A *k-dimensional (integral) Brakke flow* in U is a one-parameter family of varifolds $\{V_t\}_{t \in [0,T)}$ in U such that all of the following hold:

(a) For a.e. $t \in [0, T)$, $V_t \in \mathbf{IV}_k(U)$;
(b) For a.e. $t \in [0, T)$, δV_t is locally bounded and absolutely continuous with respect to $\|V_t\|$;
(c) The generalized mean curvature $h(\cdot, V_t)$ (which exists for a.e. t by (b)) satisfies $h(\cdot, V_t) \in L^2_{\rm loc}(\|V_t\|; \mathbb{R}^{n+1})$, and for every compact set $K \subset U$ and for every $t < T$ it holds $\sup_{s \in [0,t]} \|V_s\|(K) < \infty$;

(d) For all $0 \le t_1 < t_2 < T$ and $\phi \in C^1_c(U \times [0,T)\,;\mathbb{R}^+)$, it holds

$$\begin{aligned}&\|V_{t_2}\|(\phi(\cdot,t_2)) - \|V_{t_1}\|(\phi(\cdot,t_1)) \\ &\quad \le \int_{t_1}^{t_2}\int_U \left\{-\phi(x,t)\,|h(x,V_t)|^2 + \nabla\phi(x,t)\cdot h(x,V_t) + \frac{\partial\phi}{\partial t}(x,t)\right\} d\|V_t\|(x)\,dt\,.\end{aligned} \tag{2.1}$$

The inequality in (2.1) is typically referred to as *Brakke's inequality*. It is not difficult to show that if $\{\Gamma(t)\}_{t\in[0,T)}$ is a flow of smooth submanifolds with mean curvature $h(x,t) = h(x,\Gamma(t))$ and normal velocity $v(x,t)$ then one has, for any $0 \le t_1 < t_2 < T$ and for every $\phi \in C^1_c(U\times[t_1,t_2])$, the identity

$$\int_{\Gamma(t)} \phi(x,t)\,d\mathcal{H}^k\Big|_{t=t_1}^{t_2} = \int_{t_1}^{t_2}\int_{\Gamma(t)} \left\{-\phi\, v\cdot h + \nabla\phi\cdot v + \frac{\partial\phi}{\partial t}\right\} d\mathcal{H}^k\,dt\,. \tag{2.2}$$

In particular, if $\{\Gamma(t)\}_t$ is a smooth MCF then setting $V_t := \mathbf{var}(\Gamma(t),1)$ defines a Brakke flow for which (2.1) in satisfied with an equality. Conversely, if $V_t = \mathbf{var}(\Gamma(t),1)$ with $\Gamma(t)$ smooth, then the inequality (2.1) is sufficient to guarantee that $\{\Gamma(t)\}$ is a classical solution to the MCF. For further details on the definition, the reader can consult the original work by Brakke in [5] or the more recent monograph [35]. Next, we define the notion of unit density flow.

Definition 2.2. A Brakke flow is said to be a *unit density flow* in $U\times[t_1,t_2]$ if $\Theta^k(\|V_t\|,x) = 1$ for $\|V_t\|$-a.e. $x\in U$ and $\mathcal{L}^1$-a.e. $t\in[t_1,t_2]$. If $U = \mathbb{R}^{n+1}$, we may simply say that $\{V_t\}_{t\in[t_1,t_2]}$ is a unit density Brakke flow.

The following definition of an L^2 flow has been given by Mugnai and Röger in [26].

Definition 2.3 (L^2 flow)**.** Let $0 < T < \infty$, and let $U\subset\mathbb{R}^{n+1}$ be an open and bounded set. A one-parameter family $\{V_t\}_{t\in[0,T)}$ of varifolds in U is a *k-dimensional* L^2 *flow* if it satisfies (a)-(b) in Definition 2.1 as well as the following:

(c') The generalized mean curvature $h(\cdot,V_t)$ (which exists for a.e. $t\in[0,T)$ by (b)) satisfies $h(\cdot,V_t)\in L^2(\|V_t\|;\mathbb{R}^{n+1})$, and $d\mu := d\|V_t\|dt$ is a Radon measure on $U\times(0,T)$;

(d') There exist a vector field $v\in L^2(\mu;\mathbb{R}^{n+1})$ and a positive constant C such that

(d'1) $v(x,t)\perp T_x\|V_t\|$ for μ-a.e. $(x,t)\in U\times(0,T)$;

(d'2) For every $\phi\in C^1_c(U\times(0,T))$ it holds

$$\left|\int_0^T\int_U \frac{\partial\phi}{\partial t}(x,t) + \nabla\phi(x,t)\cdot v(x,t)\,d\|V_t\|(x)\,dt\right| \le C\,\|\phi\|_{C^0}\,. \tag{2.3}$$

Any function $v\in L^2(\mu;\mathbb{R}^{n+1})$ satisfying (d') above is called a *generalized velocity vector* for the flow.

The definition can be easily motivated by considering (2.2) once again: the latter, indeed, implies (2.3) if both h and v are in $L^2(\mu;\mathbb{R}^{n+1})$. L^2 flows can then be described as flows of generalized surfaces with generalized mean curvature and velocity vectors in L^2.

Finally, the following definition of BV flow is related to a weak motion of *hypersurfaces* which are the boundaries of a finite family of sets of locally finite perimeter; see [22, 20]. Given $N\ge 2$, we say that $\{E_1,\ldots,E_N\}$ is an $\mathcal{L}^{n+1}$-partition of $\mathbb{R}^{n+1}$ if $E_i\subset\mathbb{R}^{n+1}$ for every i, they are pairwise disjoint, and $\mathcal{L}^{n+1}(\mathbb{R}^{n+1}\setminus\bigcup_{i=1}^N E_i) = 0$.

Definition 2.4 (BV flow)**.** Let $N \geq 2$ be an integer, and let $0 < T < \infty$. N one-parameter families $\{E_i(t)\}_{t\in[0,T)}$ $(i = 1, \ldots, N)$ identify a BV solution for multi-phase MCF in $\mathbb{R}^{n+1}$ if all of the following hold:

(a") For a.e. $t \in [0,T)$, $\{E_1(t), \ldots, E_N(t)\}$ is an $\mathcal{L}^{n+1}$-partition of $\mathbb{R}^{n+1}$, $E_i(t)$ is a set of locally finite perimeter, and, setting $I_{i,j}(t) := \partial^* E_i(t) \cap \partial^* E_j(t)$ for $i \neq j$,

$$\operatorname*{ess\,sup}_{t\in[0,T)} \sum_{i,j=1,\, i\neq j}^{N} \mathcal{H}^n(I_{i,j}(t)) < \infty\,; \tag{2.4}$$

(b") There exist scalar functions $v_1, \ldots, v_N$ such that

$$\int_0^T \int_{\partial^* E_i(t)} |v_i(x,t)|^2 \, d\mathcal{H}^n(x)\, dt < \infty \qquad \text{for every } i\,, \tag{2.5}$$

and with the property that

$$\begin{aligned} \int_{E_i(t)} \phi(x,t)\, dx \bigg|_{t=t_1}^{t_2} &= \int_{t_1}^{t_2} \int_{E_i(t)} \frac{\partial \phi}{\partial t}(x,t)\, dx\, dt \\ &\quad + \int_{t_1}^{t_2} \int_{\partial^* E_i(t)} \phi(x,t)\, v_i(x,t)\, d\mathcal{H}^n(x)\, dt \end{aligned} \tag{2.6}$$

for a.e. $0 \leq t_1 < t_2 < T$ and for all $\phi \in C^1_c(\mathbb{R}^{n+1} \times [0,T))$;

(c") Setting $\nu_i(x,t) = \nu_{E_i(t)}(x)$ for the outer unit normal to the reduced boundary of $E_i(t)$ at x, it holds

$$v_i(\cdot,t)\, \nu_i(\cdot,t) = v_j(\cdot,t)\, \nu_j(\cdot,t) \qquad \mathcal{H}^n\text{-a.e. on } I_{i,j}(t), \text{ for a.e. } 0 < t \leq T\,; \tag{2.7}$$

(d") The functions v_i further satisfy

$$\sum_{i\neq j} \int_0^T \int_{I_{i,j}(t)} \operatorname{div} g - (\nu_i \otimes \nu_i) \cdot \nabla g \, d\mathcal{H}^n\, dt = -\sum_{i\neq j} \int_0^T \int_{I_{i,j}(t)} v_i\, \nu_i \cdot g\, d\mathcal{H}^n\, dt \tag{2.8}$$

for all vector fields $g \in C^1_c(\mathbb{R}^{n+1} \times [0,T]\,; \mathbb{R}^{n+1})$;

(e) The following inequality holds for a.e. $0 < t \leq T$:

$$\sum_{i,j=1,\, i\neq j}^{N} \mathcal{H}^n(I_{i,j}(t)) + \sum_{i,j=1,\, i\neq j}^{N} \int_0^t \int_{I_{i,j}(s)} |v_i(x,s)|^2\, d\mathcal{H}^n(x)\, ds \leq \sum_{i,j=1,\, i\neq j}^{N} \mathcal{H}^n(I_{i,j}(0))\,. \tag{2.9}$$

The equality (2.6) characterizes $v_i(\cdot,t)$ as the velocity of $\partial^* E_i(t)$, as anticipated in Section 1. By [23, Proposition 29.4], for each $i = 1, \ldots, N$, we have

$$\mathcal{H}^n\Big(\partial^* E_i(t) \setminus \cup_{j=1, j\neq i}^N I_{i,j}(t)\Big) = 0 \;\text{ and }\; \mathcal{H}^n(I_{i,j}(t) \cap I_{i,j'}(t)) = 0 \;\text{ for } j \neq j', \tag{2.10}$$

thus (2.8) formally characterizes $v_i \nu_i$ as the mean curvature vector of $\cup_{i=1}^N \partial^* E_i(t)$ on $I_{i,j}(t)$. Since

$$\mathcal{H}^n(\cup_{i=1}^N \partial^* E_i(t)) = \frac{1}{2} \sum_{i,j=1, i\neq j}^{N} \mathcal{H}^n(I_{i,j}(t)), \tag{2.11}$$

(e) corresponds precisely to the energy dissipation inequality for the hypersurface measures of the MCF.

2.3. **Main results.**

Definition 2.5. We will denote by Ω a function in $C^2(\mathbb{R}^{n+1})$ such that

$$0 < \Omega(x) \le 1\,, \quad |\nabla\Omega(x)| \le c_1\Omega(x)\,, \quad \|\nabla^2\Omega(x)\| \le c_1\Omega(x) \tag{2.12}$$

for all $x \in \mathbb{R}^{n+1}$, where $c_1 \ge 0$ is a constant and $\|\nabla^2\Omega(x)\|$ is the Hilbert-Schmidt norm of the Hessian matrix $\nabla^2\Omega(x)$.

The function Ω is introduced as a weight function to treat unbounded MCF, which may have infinite n-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$. A typical choice of Ω in this case can be $\Omega(x) = \exp(-\sqrt{1+|x|^2})$ with a suitable choice of c_1. If one is interested in MCF with finite measure, one can choose $\Omega \equiv 1$, with $c_1 = 0$ in this case. With a function Ω as specified above, we consider an initial datum Γ_0 satisfying the following:

Assumption 2.6. Suppose that $\Gamma_0 \subset \mathbb{R}^{n+1}$ is a closed, countably n-rectifiable set such that

$$\mathcal{H}^n \llcorner_\Omega (\Gamma_0) := \int_{\Gamma_0} \Omega(x)\, d\mathcal{H}^n(x) < \infty\,. \tag{2.13}$$

Moreover, assume that there are $N \ge 2$ mutually disjoint non-empty open sets $E_{0,1}, \dots, E_{0,N}$ such that $\Gamma_0 = \mathbb{R}^{n+1} \setminus \cup_{i=1}^N E_{0,i}$. In particular, $\{E_{0,1}, \dots, E_{0,N}\}$ is an $\mathcal{L}^{n+1}$-partition of $\mathbb{R}^{n+1}$.

By (2.13) and $\Omega > 0$, Γ_0 has locally finite $\mathcal{H}^n$-measure, and thus Γ_0 has no interior point. In particular, we have

$$\Gamma_0 = \cup_{i=1}^N \partial E_{0,i}.$$

For a given Γ_0 as in Assumption 2.6, the assignment of the partition $\mathbb{R}^{n+1} \setminus \Gamma_0 = \bigcup_{i=1}^N E_{0,i}$ is certainly non-unique. Each $E_{0,i}$ may not be connected, for example, and the different choice may result in different MCF. In general, $E_{0,i}$ has locally finite perimeter, and $\partial^* E_{0,i} \subset \operatorname{spt}|\nabla\chi_{E_{0,i}}| \subset \partial E_{0,i}$ for each $i = 1, \dots, N$.

The following Theorems 2.7-2.12 all hold under Assumption 2.6. First, we claim the existence of a Brakke flow starting with Γ_0 which is also an L^2 flow in any bounded domain of $\mathbb{R}^{n+1}$, and whose generalized velocity vector coincides with the generalized mean curvature vector of the evolving varifolds.

Theorem 2.7. *There exists an n-dimensional Brakke flow $\{V_t\}_{t\ge 0}$ in $\mathbb{R}^{n+1}$ such that*

1. $\|V_0\| = \mathcal{H}^n\llcorner_{\Gamma_0}$,
2. *if $\mathcal{H}^n(\cup_{i=1}^N(\partial E_{0,i} \setminus \partial^* E_{0,i})) = 0$, then $\lim_{t\to 0+}\|V_t\| = \mathcal{H}^n\llcorner_{\Gamma_0}$,*
3. $\|V_t\|(\Omega) \le \mathcal{H}^n \llcorner_\Omega (\Gamma_0)\exp(c_1^2 t/2)$ *and* $\int_0^t \int_{\mathbb{R}^{n+1}} |h(\cdot, V_s)|^2\, \Omega\, d\|V_s\| ds < \infty$ *for all* $t > 0$.
4. *If $\mathcal{H}^n(\Gamma_0) < \infty$, and thus if one can choose $\Omega \equiv 1$ in Assumption 2.6, then $\|V_t\|(\mathbb{R}^{n+1}) + \int_0^t \int_{\mathbb{R}^{n+1}} |h(\cdot, V_s)|^2\, d\|V_s\| ds \le \mathcal{H}^n(\Gamma_0)$ for all $t > 0$.*
5. *For any $0 < T < \infty$ and for any open and bounded set $U \subset \mathbb{R}^{n+1}$, the one-parameter family $\{V_t \llcorner_U\}_{t\in[0,T)}$ (where we regard $V_t \llcorner_U$ as a varifold in U) is an n-dimensional L^2 flow with generalized velocity vector $v(x,t) = h(x, V_t)$ on $U \times (0,T)$.*

Remark 2.8. If $n = 1$, the above Brakke flow satisfies the additional regularity properties obtained in [19]: for $\mathcal{L}^1$-a.e. $t > 0$, $\operatorname{spt}\|V_t\|$ is locally the union of a finite number of $W^{2,2}$ curves meeting at junctions with angles of either 0, 60, or 120 degrees for $N \ge 3$, and only 0 degree (no transverse crossing) for $N = 2$. See [19, Theorem 2.2, 2.3] for the further details.

Definition 2.9. With $\{V_t\}_{t\geq 0}$ as in Theorem 2.7, let μ be the Radon measure on $\mathbb{R}^{n+1}\times\mathbb{R}^+$ given by $d\mu = d\|V_t\|dt$, and for $t\in\mathbb{R}^+$, define the closed set

$$(\operatorname{spt}\mu)_t := \{x\in\mathbb{R}^{n+1} \,:\, (x,t)\in\operatorname{spt}\mu\}\,.$$

The following Theorem 2.10 is satisfied in general for Brakke flows, and thus it holds, in particular, for the flow produced in Theorem 2.7.

Theorem 2.10. *The Radon measure μ and the Brakke flow $\{V_t\}_{t\geq 0}$ are related as follows.*

(1) $\operatorname{spt}\|V_t\|\subset(\operatorname{spt}\mu)_t$ *and* $\mathcal{H}^n(B_r\cap(\operatorname{spt}\mu)_t)<\infty$ *for all* $t>0$ *and* $r>0$,
(2) $\mathcal{H}^{n-1+\delta}((\operatorname{spt}\mu)_t\setminus\operatorname{spt}\|V_t\|)=0$ *for every* $\delta>0$ *and for* $\mathcal{L}^1$*-a.e.* $t\in\mathbb{R}^+$,
(3) $\operatorname{spt}\|V_t\|$ *is countably* n*-rectifiable for* $\mathcal{L}^1$*-a.e.* $t\in\mathbb{R}^+$,
(4) $V_t=\mathbf{var}(\operatorname{spt}\|V_t\|,\theta_t)$ *with* $\theta_t(x)=\Theta^n(\|V_t\|,x)$ *(*$\|V_t\|$*-a.e.* x*) for* $\mathcal{L}^1$*-a.e.* $t\in\mathbb{R}^+$.

The following theorem shows that, in addition to the Brakke flow of Theorem 2.7, there are evolving domains $\{E_i(t)\}_{t\geq 0}$ starting from $E_{0,i}$ defining a (generalized) BV solution for multi-phase MCF in $\mathbb{R}^{n+1}$.

Theorem 2.11. *For each* $i=1,\dots,N$*, there exists a family of open sets* $\{E_i(t)\}_{t\geq 0}$ *such that, setting* $\Gamma(t):=\mathbb{R}^{n+1}\setminus\cup_{i=1}^N E_i(t)$*:*

(1) $E_i(0)=E_{0,i}$ *for* $i=1,\dots,N$,
(2) $E_1(t),\dots,E_N(t)$ *are pairwise disjoint for* $t\in\mathbb{R}^+$,
(3) $(\operatorname{spt}\mu)_t=\Gamma(t)=\cup_{i=1}^N\partial E_i(t)$ *for all* $t>0$,
(4) *for all* $t\in\mathbb{R}^+$, $\|V_t\|\geq|\nabla\chi_{E_i(t)}|$ *for every* $i=1,\dots,N$*, and* $2\|V_t\|\geq\sum_{i=1}^N|\nabla\chi_{E_i(t)}|$,
(5) $S(i):=\{(x,t)\,:\,x\in E_i(t),\ t\in\mathbb{R}^+\}$ *is open in* $\mathbb{R}^{n+1}\times\mathbb{R}^+$ *for* $i=1,\dots,N$,
(6) *for every* $0<T<\infty$*, the families* $\{E_i(t)\}_{t\in[0,T)}$ *define a generalized* BV *solution for multi-phase mean curvature flow as in Definition 2.4, in the following sense: (a") holds when* $\mathcal{H}^n\llcorner_\Omega$ *replaces* $\mathcal{H}^n$ *in* (2.4)*; (b") holds when* $\mathcal{H}^n\llcorner_\Omega$ *replaces* $\mathcal{H}^n$ *in* (2.5)*; (c") holds. In fact, the scalar velocity fields* v_i *are precisely* $v_i(x,t)=h(x,V_t)\cdot\nu_i(x,t)$*, so that* (2.6) *reads as follows: for every* $i\in\{1,\dots,N\}$
$$\int_{E_i(t)}\phi(x,t)\,dx\Big|_{t=t_1}^{t_2}=\int_{t_1}^{t_2}\Big(\int_{\partial^*E_i(t)}\phi\,h\cdot\nu_i\,d\mathcal{H}^n+\int_{E_i(t)}\frac{\partial\phi}{\partial t}\,dx\Big)\,dt \tag{2.14}$$
for any $0\leq t_1<t_2<\infty$ *and* $\phi\in C^1_c(\mathbb{R}^{n+1}\times\mathbb{R}^+)$,
(7) *if* $N\geq 3$*, then, for* $\mathcal{L}^1$*-a.e.* $t\in\mathbb{R}^+$ *and* $\mathcal{H}^n$*-a.e.* $x\in\Gamma(t)$, $\theta_t(x)=1$ *implies* $x\in\cup_{i=1}^N\partial^*E_i(t)$,
(8) *if* $N=2$*, then, for* $\mathcal{L}^1$*-a.e.* $t\in\mathbb{R}^+$ *and* $\mathcal{H}^n$*-a.e.* $x\in\Gamma(t)$,
$$\theta_t(x)=\begin{cases}\text{odd integer} & \text{for } x\in\partial^*E_1(t)(=\partial^*E_2(t)),\\ \text{even integer} & \text{for } x\in\Gamma(t)\setminus\partial^*E_1(t).\end{cases} \tag{2.15}$$

Notice that in (6) we speak about *generalized* BV flow, because we cannot guarantee, in general, the validity of (2.8). We will comment further on this point in Section 7.1. We anticipate, nonetheless, that further information on the flow, including the validity of (2.8), can be inferred when the Brakke flow is *unit density.* Though the result follows immediately from Theorem 2.11(7), we state it separately.

Theorem 2.12. *Suppose that, for* $0<t_1<t_2<\infty$ *and for an open set* U*, the Brakke flow in Theorem 2.7 is a unit density flow in* $U\times(t_1,t_2)$*. Then, for* $\mathcal{L}^1$*-a.e.* $t\in(t_1,t_2)$,
$$\|V_t\|\llcorner_U=\mathcal{H}^n\llcorner_{U\cap\cup_{i=1}^N\partial^*E_i(t)}\,. \tag{2.16}$$

Furthermore, (2.8) *holds true with* $v_i = h \cdot \nu_i$ *for all vector fields* $g \in C_c^1(U \times [t_1, t_2]\,; \mathbb{R}^{n+1})$.

If we assume the following additional conditions on Γ_0, we can guarantee that the resulting Brakke flow $\{V_t\}_{t\geq 0}$ as above is unit density for short time.

Theorem 2.13. *Suppose that* $\mathcal{H}^n(\Gamma_0) < \infty$ *and there exist* $r_0 > 0$ *and* $\delta_0 > 0$ *such that*

$$\sup_{x\in\mathbb{R}^{n+1},\,0<r<r_0} \frac{\mathcal{H}^n(\Gamma_0 \cap B_r(x))}{\omega_n r^n} < 2 - \delta_0. \tag{2.17}$$

Then there exists $T_0 = T_0(n, r_0, \delta_0, \mathcal{H}^n(\Gamma_0)) \in (0,\infty)$ *such that* $\{V_t\}_{t\in[0,T_0)}$ *in Theorem 2.7 is a unit density Brakke flow and* $\{(\chi_{E_1(t)}, \ldots, \chi_{E_N(t)})\}_{t\in[0,T_0)}$ *in Theorem 2.11 is a BV solution of MCF.*

If we set "the maximal existence time" of unit density Brakke flow as $\hat{T} := \sup\{t \geq 0 : \int_0^t \|V_s\|(\{\theta_s(x) \geq 2\})\,ds = 0 \text{ and } \|V_t\| \neq 0\}$, under the assumption of Theorem 2.13, either $\|V_t\| = 0$ before T_0, or we have $\hat{T} \geq T_0$ and V_t is a non-trivial unit density flow on $[0, \hat{T})$ and $\{(\chi_{E_1(t)}, \ldots, \chi_{E_N(t)})\}_{t\in[0,\hat{T})}$ is also a BV solution of MCF. There is a simple argument to obtain a lower bound on the "extinction time" for V_t, see Section 7.3.

2.4. **Outline of the proofs and plan of the paper.** In what follows, we assume to have fixed

- a function Ω as in Definition 2.5,
- a set Γ_0 and domains $\{E_{0,i}\}_{i=1}^N$ as in Assumption 2.6.

The strategy towards the proof of the existence of the Brakke flow $\{V_t\}_{t\geq 0}$ and of the one-parameter families $\{E_i(t)\}_{t\geq 0}$ of open sets for $i \in \{1, \ldots, N\}$ from Theorems 2.7 and 2.11 respectively is analogous to that employed by Kim and the second-named author in [18], with an important technical modification which is crucial to gain enough control on the change of volume of the grains and conclude the identity (2.14). Let us explain this point in further detail. The scheme introduced in [18] can be roughly summarized as follows. One constructs, starting from $\{E_{0,i}\}_{i=1}^N$, a sequence (indexed by $j \in \mathbb{N}$) of piecewise constant-in-time flows of *open partitions* of $\mathbb{R}^{n+1}$ of N elements (see Section 2.5): more precisely, for each j the corresponding flow $\{\mathcal{E}_j(t)\}_{t\in[0,j]}$ consists of constant open partitions $\mathcal{E}_j(t) = \mathcal{E}_{j,k} = \{E_{j,k,i}\}_{i=1}^N$ for t in intervals (*epochs*) $((k-1)\,\Delta t_j, k\,\Delta t_j]$ of length $\Delta t_j \to 0^+$. The goal is then to show that if the partition $\mathcal{E}_{j,k+1}$ is constructed from the partition $\mathcal{E}_{j,k}$ appropriately then, along a suitable subsequence $j_\ell \to \infty$, the (varifolds associated to) the boundaries $\partial\mathcal{E}_{j_\ell}(t)$ converge to the desired Brakke flow V_t.

The open partition $\mathcal{E}_{j,k+1}$ at a given epoch is constructed from the open partition $\mathcal{E}_{j,k}$ at the previous epoch by applying two operations, which we call *steps*. The first step is a small *Lipschitz deformation* of partitions with the effect of regularizing singularities by locally minimizing the area of the boundary of partitions at a suitably small scale; the second step consists of flowing the boundary of partitions by an appropriately defined *approximate mean curvature vector*, obtained by smoothing the surfaces' first variation via convolution with a localized heat kernel.

The only difference between the scheme employed in [18] and the one devised in the present paper is in the choice of the Lipschitz deformations in the first step. While in [18] one only requires that the change of volumes of the grains due to Lipschitz deformation is small (for a certain smallness scale), in the present paper we shall require that the change of volume is small *compared to the reduction in surface measure*. We define such new class of "*volume-controlled*" Lipschitz deformations in Section 3.1 and then we claim that the construction of [18] can be carried over using volume-controlled deformations, as outlined in Section 3.2: by the results of [18], this completes the proof of Theorem 2.7(1)-(4) (with some additional remarks), Theorem 2.10(1), and Theorem 2.11(1)-(5). There are technical details for verifying that the modifications to the volume controlled deformations do not pose any difficulties, and we defer it to Appendix A.

In Section 4, the main result is Theorem 2.11(6), namely the identity (2.14). For that, we first prove that any Brakke flow is L^2 flow in Section 4.1, which is Theorem 2.7(5). This fact seems to be first noted in [4] but we include the proof for completeness. Next Section 4.2 is the main part of the proof, showing the existence of measure-theoretic velocities. The proof consists of calculating explicitly the rate of change of integrals of test functions across different epochs in the approximating flows. We will prove the validity of an approximate identity for the sets $E_{j_\ell,i}(t)$, and the use of volume-controlled Lipschitz deformations will be crucial to conclude that the errors vanish in the limit as $\ell \to \infty$. The last Section 4.3 shows that the existence of tangent space of μ on the reduced boundary of grains (seen as a set in space-time) and concludes the proof of Theorem 2.11(6).

In Section 5 we prove Theorem 2.10(2)-(4), which, in particular, implies that the support $\mathrm{spt}\|V_t\|$ of the evolving varifolds is $\mathcal{H}^n$-equivalent to the boundary of partition $\Gamma(t) = \bigcup_{i=1}^N \partial E_i(t)$. This result is used in Section 6. The argument we follow was suggested by Ilmanen in [15, Section 7.1], and we include it for the sake of completeness. It is based on the so-called *Clearing Out Lemma* (see Lemma 5.1), a very robust result, which in turn follows from Huisken's monotonicity formula, stating roughly that if the localized density of $\|V_t\|$ at some point y is too small at a scale $r > 0$ then necessarily the point $(y, t+r^2)$ does not belong to the support of the space-time measure μ.

Section 6 contains the proof of Theorem 2.11(7)(8). They are improved version of integrality theorem of [18, Theorem 8.6] in that the behaviors of approximating flows and their grains are tracked more in detail. A characterization in [19, Section 4] of limiting behavior within a length scale of $o(1/j^2)$, where measure minimizing property dominates, is essentially used. At the end, we describe the proofs of Theorem 2.12 and 2.13.

2.5. **Further notation.** We collect here some further notation and results which will be extensively used throughout the paper. We begin with the notion of open partitions of $\mathbb{R}^{n+1}$ of N elements and corresponding admissible maps.

Definition 2.14. Let $N \geq 2$ be an integer, and let Ω be as in Definition 2.5. A collection $\mathcal{E} = \{E_1, \ldots, E_N\}$ of subsets $E_i \subset \mathbb{R}^{n+1}$ is called an *Ω-finite open partition of N elements* if

(a) $E_1, \ldots, E_N$ are open and mutually disjoint,
(b) $\int_{\mathbb{R}^{n+1}\setminus\cup_{i=1}^N E_i} \Omega(x)\, d\mathcal{H}^n(x) < \infty$,
(c) the set $\cup_{i=1}^N \partial E_i$ is countably n-rectifiable.

The set of all Ω-finite open partitions of N elements is denoted by $\mathcal{OP}^N_\Omega$.

Note that it is allowed for some E_i to be the empty set $\emptyset$. Since $\Omega > 0$ everywhere, the property (b) implies that the closed set $\mathbb{R}^{n+1} \setminus \cup_{i=1}^N E_i$ has locally finite $\mathcal{H}^n$-measure, and thus no interior point. In particular, it holds $\mathbb{R}^{n+1} \setminus \bigcup_{i=1}^N = \cup_{i=1}^N \partial E_i$. We define

$$\partial\mathcal{E} := \bigcup_{i=1}^N \partial E_i \, ,$$

and with a slight abuse of notation we use the same symbol to denote the varifold

$$\mathbf{var}\left(\bigcup_{i=1}^{N} \partial E_i, 1\right) \in \mathbf{IV}_n(\mathbb{R}^{n+1}),$$

namely the unit density varifold induced from the countably n-rectifiable set $\partial\mathcal{E}$.

Definition 2.15. Given $\mathcal{E} = \{E_i\}_{i=1}^N \in \mathcal{OP}^N_\Omega$, a function $f\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is $\mathcal{E}$-*admissible* if it is Lipschitz and if it satisfies the following. Set, for every i, $\tilde{E}_i := \mathrm{int}(f(E_i))$. Then:

(a) $\tilde{E}_1, \ldots, \tilde{E}_N$ are pairwise disjoint;
(b) $\mathbb{R}^{n+1} \setminus \bigcup_{i=1}^N \tilde{E}_i \subset f\left(\bigcup_{i=1}^N \partial E_i\right)$;
(c) $\sup_{x\in\mathbb{R}^{n+1}} |f(x) - x| < \infty$.

The following is [18, Lemma 4.4].

Lemma 2.16. *Let $\mathcal{E} = \{E_i\}_{i=1}^N \in \mathcal{OP}^N_\Omega$, and let f be $\mathcal{E}$-admissible. Set $\tilde{\mathcal{E}} := \{\tilde{E}_i\}_{i=1}^N$, where $\tilde{E}_i = \mathrm{int}(f(E_i))$. Then, $\tilde{\mathcal{E}} \in \mathcal{OP}^N_\Omega$. We will call $\tilde{\mathcal{E}}$ the* push-forward *of $\mathcal{E}$ through f, denoted $\tilde{\mathcal{E}} =: f_\star\mathcal{E}$.*

Next, we define a class of test functions with good properties. For $j \in \mathbb{N}$, we set

$$\mathcal{A}_j := \Big\{\phi \in C^2(\mathbb{R}^{n+1};\mathbb{R}^+) \quad : \quad \phi(x) \le \Omega(x)\,, \, |\nabla\phi(x)| \le j\,\phi(x)\,, \\ \|\nabla^2\phi(x)\| \le j\,\phi(x) \quad \text{for every } x \in \mathbb{R}^{n+1}\Big\}. \tag{2.18}$$

Note that $\Omega \in \mathcal{A}_j$ for all $j \ge c_1$.

Finally, we give the notion of smoothed mean curvature vector of a varifold. Let $\psi \in C^\infty(\mathbb{R}^{n+1})$ be a radially symmetric cut-off function such that

$$\psi(x) = 1 \text{ for } |x| \le 1/2\,, \qquad \psi(x) = 0 \text{ for } |x| \ge 1\,,$$
$$0 \le \psi(x) \le 1\,, \qquad |\nabla\psi(x)| \le 3\,, \qquad \|\nabla^2\psi(x)\| \le 9 \text{ for all } x \in \mathbb{R}^{n+1}\,.$$

Then, for every $\varepsilon \in (0,1)$ we define

$$\hat{\Phi}_\varepsilon(x) := \frac{1}{(2\pi\varepsilon^2)^{\frac{n+1}{2}}} \exp\left(-\frac{|x|^2}{2\varepsilon^2}\right)\,, \qquad \Phi_\varepsilon(x) := c(\varepsilon)\,\psi(x)\,\hat{\Phi}_\varepsilon(x)\,, \tag{2.19}$$

where $1 < c(\varepsilon) \le c(n)$ is a normalization constant chosen in such a way that

$$\int_{\mathbb{R}^{n+1}} \Phi_\varepsilon(x)\,dx = 1\,.$$

We shall call Φ_ε the smoothing kernel at scale ε. For a varifold $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$, we then define the ε-smoothed mean curvature vector of V to be the vector field $h_\varepsilon(\cdot, V) \in C^\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ defined by

$$h_\varepsilon(\cdot, V) := -\Phi_\varepsilon * \left(\frac{\Phi_\varepsilon * \delta V}{\Phi_\varepsilon * \|V\| + \varepsilon\,\Omega^{-1}}\right)\,. \tag{2.20}$$

In the above formula, $\Phi_\varepsilon * \|V\|$ is the measure on $\mathbb{R}^{n+1}$ defined by

$$(\Phi_\varepsilon * \|V\|)(\phi) := \|V\|(\Phi_\varepsilon * \phi) = \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} \Phi_\varepsilon(x-y)\,\phi(y)\,dy\,d\|V\|(x)$$

for all $\phi \in C_c(\mathbb{R}^{n+1})$, identified with the smooth function

$$(\Phi_\varepsilon * \|V\|)(x) := \int_{\mathbb{R}^{n+1}} \Phi_\varepsilon(y-x)\, d\|V\|(y)$$

by means of the identity

$$(\Phi_\varepsilon * \|V\|)(\phi) = \langle \Phi * \|V\| \,,\, \phi \rangle_{L^2(\mathbb{R}^{n+1})} \,.$$

Analogously, $\Phi_\varepsilon * \delta V$ is the $\mathbb{R}^{n+1}$-valued measure on $\mathbb{R}^{n+1}$ defined by

$$(\Phi_\varepsilon * \delta V)(g) := \delta V(\Phi_\varepsilon * g) = \int_{\mathbb{R}^{n+1}} g(y) \cdot \int_{\mathbb{R}^{n+1}\times \mathbf{G}(n+1,n)} S\left(\nabla \Phi_\varepsilon(x-y)\right)\, dV(x,S)\, dy$$

for all $g \in C_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$, identified with the smooth vector field

$$(\Phi_\varepsilon * \delta V)(x) := \int_{\mathbb{R}^{n+1}\times \mathbf{G}(n+1,n)} S(\nabla \Phi_\varepsilon(y-x))\, dV(y,S)\,.$$

We state the following lemma concerning the smoothed mean curvature vector to be used in the subsequent sections. For the proof, the reader can consult [18, Lemma 5.1].

Lemma 2.17. *For every $M > 0$, there exists a constant $\varepsilon_1 \in (0,1)$, depending only on n, c_1, and M such that the following holds. Let $V \in \mathbf{V}_n(\mathbb{R}^{n+1})$ be an n-dimensional varifold in $\mathbb{R}^{n+1}$ such that $\|V\|(\Omega) \le M$, and, for every $\varepsilon \in (0,\varepsilon_1)$, let $h_\varepsilon(\cdot,V)$ be its smoothed mean curvature vector. Then:*

$$|h_\varepsilon(x,V)| \le 2\,\varepsilon^{-2}\,, \tag{2.21}$$

$$\|\nabla h_\varepsilon(x,V)\| \le 2\,\varepsilon^{-4}\,. \tag{2.22}$$

3. Existence of multi-phase Brakke flow

3.1. **Volume-controlled Lipschitz deformations.** In this subsection we introduce the modified class of Lipschitz deformations used in the present paper to gain improved control on the volume change of partitions. For $\mathcal{E} \in \mathcal{OP}^N_\Omega$ and $j \in \mathbb{N}$, the class is denoted by $\mathbf{E}^{vc}(\mathcal{E},j)$ (upperscript vc indicating "volume controlled"), and it replaces the class $\mathbf{E}(\mathcal{E},j)$ defined in [18, Definition 4.8].

Definition 3.1. For $\mathcal{E} = \{E_i\}_{i=1}^N \in \mathcal{OP}^N_\Omega$ and $c_1 \le j \in \mathbb{N}$, define $\mathbf{E}^{vc}(\mathcal{E},j)$ to be the set of all $\mathcal{E}$-admissible functions $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that, writing $\{\tilde{E}_i\}_{i=1}^N := f_\star \mathcal{E}$, we have

(a) $|f(x) - x| \le 1/j^2$ for all $x \in \mathbb{R}^{n+1}$,
(b) $\mathcal{L}^{n+1}(\tilde{E}_i \triangle E_i) \le \{\|\partial\mathcal{E}\|(\Omega) - \|\partial f_\star\mathcal{E}\|(\Omega)\}/j$ for all $i = 1,\dots,N$,
(c) $\|\partial f_\star \mathcal{E}\|(\phi) \le \|\partial\mathcal{E}\|(\phi)$ for all $\phi \in \mathcal{A}_j$.

The difference between the above class $\mathbf{E}^{vc}(\mathcal{E},j)$ and the class $\mathbf{E}(\mathcal{E},j)$ in [18, Definition 4.8] lies in the condition (b), which in [18] was simply $\mathcal{L}^{n+1}(\tilde{E}_i \triangle E_i) \le 1/j$. Since $\Omega \in \mathcal{A}_j$ for all $j \ge c_1$, (c) implies $\|\partial\mathcal{E}\|(\Omega) \ge \|\partial f_\star\mathcal{E}\|(\Omega)$, and thus the right-hand side of (b) is non-negative. In particular, the identity map $f(x) = x$ belongs to $\mathbf{E}^{vc}(\mathcal{E},j)$ for $j \ge c_1$. We similarly modify the definitions of $\Delta_j\|\partial\mathcal{E}\|(\Omega)$, $\mathbf{E}(\mathcal{E},C,j)$, and $\Delta_j\|\partial\mathcal{E}\|(C)$ in [18, (4.11)-(4.13)] by introducing their volume controlled counterparts, as follows.

Definition 3.2. For $\mathcal{E} \in \mathcal{OP}^N_\Omega$ and $c_1 \le j \in \mathbb{N}$, define

$$\Delta^{vc}_j\|\partial\mathcal{E}\|(\Omega) := \inf_{f \in \mathbf{E}^{vc}(\mathcal{E},j)} \{\|\partial f_\star\mathcal{E}\|(\Omega) - \|\partial\mathcal{E}\|(\Omega)\} \le 0\,, \tag{3.1}$$

and for a compact set $C \subset \mathbb{R}^{n+1}$,

$$\mathbf{E}^{vc}(\mathcal{E}, C, j) := \{f \in \mathbf{E}^{vc}(\mathcal{E}, j) : \{x : f(x) \neq x\} \cup \{f(x) : f(x) \neq x\} \subset C\}, \tag{3.2}$$

$$\Delta_j^{vc}\|\partial\mathcal{E}\|(C) := \inf_{f \in \mathbf{E}^{vc}(\mathcal{E}, C, j)} \{\|\partial f_\star \mathcal{E}\|(C) - \|\partial\mathcal{E}\|(C)\}. \tag{3.3}$$

Even with this modification, claims in [18, Section 4] remain essentially the same:

Lemma 3.3. *For compact sets $C \subset \tilde{C}$, we have*

$$\Delta_j^{vc}\|\partial\mathcal{E}\|(\tilde{C}) \leq \Delta_j^{vc}\|\partial\mathcal{E}\|(C) \tag{3.4}$$

and

$$\Delta_j^{vc}\|\partial\mathcal{E}\|(\Omega) \leq (\max_C \Omega)\{\Delta_j^{vc}\|\partial\mathcal{E}\|(C) + (1 - \exp(-c_1 \mathrm{diam}\, C))\|\partial\mathcal{E}\|(C)\}. \tag{3.5}$$

Lemma 3.4. *Suppose that $\{C_l\}_{l=1}^\infty$ is a sequence of compact sets which are mutually disjoint and suppose that C is a compact set with $\cup_{l=1}^\infty C_l \subset C$. Then*

$$\Delta_j^{vc}\|\partial\mathcal{E}\|(C) \leq \sum_{l=1}^\infty \Delta_j^{vc}\|\partial\mathcal{E}\|(C_l). \tag{3.6}$$

We remark that the corresponding statement in [18, Lemma 4.11] contains the additional assumption $\mathcal{L}^{n+1}(C) < 1/j$, which we do not need to assume in the present setting. For completeness, we provide the proof.

Proof. Of course, if $\Delta_j^{vc}\|\partial\mathcal{E}\|(C) = -\infty$ then there is nothing to prove. Observe that, by Lemma 3.3, if $\Delta_j^{vc}\|\partial\mathcal{E}\|(C) > -\infty$, then also $\Delta_j^{vc}\|\partial\mathcal{E}\|(C_l) > -\infty$ for all l. Let $m \in \mathbb{N}$ and $\varepsilon \in (0,1)$ be arbitrary. For all $l \leq m$, choose $f_l \in \mathbf{E}^{vc}(\mathcal{E}, C_l, j)$ such that $\Delta_j^{vc}\|\partial\mathcal{E}\|(C_l) + \varepsilon \geq \|\partial(f_l)_\star\mathcal{E}\|(C_l) - \|\partial\mathcal{E}\|(C_l)$. We define a map $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by setting $f \llcorner_{C_l}(x) = (f_l)\llcorner_{C_l}(x)$ for $l = 1, \ldots, m$ and $f \llcorner_{\mathbb{R}^{n+1} \setminus \cup_{i=l}^m C_l}(x) = x$. The $\mathcal{E}$-admissibility of f follows from that of f_l and from the fact that $\{C_l\}$ are mutually disjoint. To prove $f \in \mathbf{E}^{vc}(\mathcal{E}, j)$, we need to check Definition 3.1(a)-(c) and (a) follows immediately. Writing $\{\tilde{E}_i\}_{i=1}^N := f_\star\mathcal{E}$, we have $\tilde{E}_i \triangle E_i = \cup_{l=1}^m C_l \cap (\tilde{E}_i \triangle E_i)$, and

$$\begin{aligned}
\mathcal{L}^{n+1}(\tilde{E}_i \triangle E_i) &= \sum_{l=1}^m \mathcal{L}^{n+1}(C_l \cap (\tilde{E}_i \triangle E_i)) \leq \sum_{l=1}^m \{\|\partial\mathcal{E}\|(\Omega) - \|\partial(f_l)_\star\mathcal{E}\|(\Omega)\}/j \\
&= \sum_{l=1}^m \{\|\partial\mathcal{E}\| \llcorner_{C_l}(\Omega) - \|\partial(f_l)_\star\mathcal{E}\| \llcorner_{C_l}(\Omega)\}/j = \{\|\partial\mathcal{E}\|(\Omega) - \|\partial f_\star\mathcal{E}\|(\Omega)\}/j,
\end{aligned}$$

where we used (b) for $f_l \in \mathbf{E}^{vc}(\mathcal{E}, C_l, j)$ to conclude (b) for f. The condition (c) can be checked similarly. These show that $f \in \mathbf{E}^{vc}(\mathcal{E}, j)$, and in fact $f \in \mathbf{E}^{vc}(\mathcal{E}, C, j)$ by construction. By the definition of $\Delta_j^{vc}\|\partial\mathcal{E}\|(C)$, we have

$$\begin{aligned}
\Delta_j^{vc}\|\partial\mathcal{E}\|(C) &\leq \|\partial f_\star\mathcal{E}\|(C) - \|\partial\mathcal{E}\|(C) = \sum_{l=1}^m (\|\partial(f_l)_\star\mathcal{E}\|(C_l) - \|\partial\mathcal{E}\|(C_l)) \\
&\leq m\varepsilon + \sum_{l=1}^m \Delta_j^{vc}\|\partial\mathcal{E}\|(C_l).
\end{aligned}$$

By letting $\varepsilon \to 0^+$ first and then $m \to \infty$, we obtain (3.6). □

The following is similar to [18, Lemma 4.12] with a minor change. The proof is identical.

Lemma 3.5. *Suppose that* $\mathcal{E} = \{E_i\}_{i=1}^N \in \mathcal{OP}^N_\Omega$*,* $j \in \mathbb{N}$*,* C *is a compact subset of* $\mathbb{R}^{n+1}$*,* $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ *is a* $\mathcal{E}$*-admissible function such that, writing* $\{\tilde{E}_i\}_{i=1}^N := f_\star\mathcal{E}$*,*

(a) $\{x : f(x) \neq x\} \cup \{f(x) : f(x) \neq x\} \subset C$*,*
(b) $|f(x) - x| \leq 1/j^2$ *for all* $x \in \mathbb{R}^{n+1}$*,*
(c) $\mathcal{L}^{n+1}(\tilde{E}_i \triangle E_i) \leq \{\|\partial\mathcal{E}\|(\Omega) - \|\partial f_\star\mathcal{E}\|(\Omega)\}/j$ *for all* $i = 1, \ldots, N$*,*
(d) $\|\partial f_\star \mathcal{E}\|(C) \leq \exp(-j \operatorname{diam} C)\|\partial\mathcal{E}\|(C)$*.*

Then we have $f \in \mathbf{E}^{vc}(\mathcal{E}, C, j)$*.*

3.2. **The constructive scheme.** In this subsection we provide the detailed construction of the sequence of piecewise constant-in-time approximating flows of open partitions leading to the existence result of a multi-phase Brakke flow. We let the weight function Ω be as in Definition 2.5, and we consider an initial rectifiable set Γ_0 with a corresponding Ω-finite open partition of N elements $\mathcal{E}_0$ as in Assumption 2.6. For every natural number $j \geq c_1$, and for times $t \in [0, j]$, we define open partitions $\mathcal{E}_j(t) = \{E_{j,1}(t), \ldots, E_{j,N}(t)\}$ according to the following rule:

$$\mathcal{E}_j(0) = \mathcal{E}_0\,, \tag{3.7}$$

$$\mathcal{E}_j(t) = \mathcal{E}_{j,k} \qquad \text{for all } t \in ((k-1)\Delta t_j, k\Delta t_j]\,. \tag{3.8}$$

In (3.8), the epoch length is $\Delta t_j = 2^{-p_j}$ for some $p_j \in \mathbb{N}$, and $k \in \{1, \ldots, j\,2^{p_j}\}$. For each k, the open partition $\mathcal{E}_{j,k}$ is obtained from the open partition $\mathcal{E}_{j,k-1}$ (with the convention $\mathcal{E}_{j,0} = \mathcal{E}_0$) through successive modifications, encoded in the following two-step algorithm:

(1) First, one chooses $f_1 \in \mathbf{E}^{vc}(\mathcal{E}_{j,k-1}, j)$ with the property that

$$\|\partial(f_1)_\star\mathcal{E}_{j,k-1}\|(\Omega) - \|\partial\mathcal{E}_{j,k-1}\|(\Omega) \leq (1 - j^{-5})\,\Delta_j^{vc}\|\partial\mathcal{E}_{j,k-1}\|(\Omega)\,, \tag{3.9}$$

and sets

$$\mathcal{E}^*_{j,k} := (f_1)_\star(\mathcal{E}_{j,k-1}); \tag{3.10}$$

thus, in particular,

$$E^*_{j,k,i} := \operatorname{int}(f_1(E_{j,k-1,i})) \quad \text{for every } i \in \{1, \ldots, N\}\,. \tag{3.11}$$

(2) Next, one defines the map

$$f_2(x) := x + \Delta t_j\, h_{\varepsilon_j}(x, \partial\mathcal{E}^*_{j,k})\,, \tag{3.12}$$

where $\varepsilon_j \in (0,1)$, and $h_{\varepsilon_j}(\cdot, \partial\mathcal{E}^*_{j,k})$ is the ε_j-smoothed mean curvature vector of the multiplicity one varifold on $\partial\mathcal{E}^*_{j,k}$. Notice that f_2 is a diffeomorphism of $\mathbb{R}^{n+1}$ due to Lemma 2.17 as soon as $\Delta t_j \ll \varepsilon_j^4$. We set

$$\mathcal{E}_{j,k} := (f_2)_\star\mathcal{E}^*_{j.k}\,, \tag{3.13}$$

and thus

$$E_{j,k,i} = f_2(E^*_{j,k,i}) \quad \text{for every } i \in \{1, \ldots, N\}\,. \tag{3.14}$$

Notice that the scheme just defined differs from that adopted in [18] only in step (1) of the algorithm, where volume-controlled Lipschitz deformations are used. In spite of such modification, we claim that the proof from [18] can be essentially carried over to this framework, thus leading to the following theorem.

Theorem 3.6. *There is a constant* $c_2 = c_2(n) \gg 1$ *with the following property. Let* Ω*,* Γ_0*, and* $\mathcal{E}_0 \in \mathcal{OP}^N_\Omega$ *be as in Definition 2.5 and Assumption 2.6. Then there exist*

- *a subsequence* j_ℓ *of* $\mathbb{N}$*,*

- *reals* $\varepsilon_{j_\ell} \in \left(0, j_\ell^{-6}\right)$ *with* $\lim_{\ell\to\infty} \varepsilon_{j_\ell} = 0$,
- *integers* $p_{j_\ell} \in \mathbb{N}$ *with* $\Delta t_{j_\ell} = 2^{-p_j} \in \left(2^{-1}\varepsilon_{j_\ell}^{c_2}, \varepsilon_{j_\ell}^{c_2}\right]$,
- *a family* $\{\mu_t\}_{t\in\mathbb{R}^+}$ *of Radon measures on* $\mathbb{R}^{n+1}$,
- *and a family* $\mathcal{E}(t) = \{E_1(t), \ldots, E_N(t)\}_{t\geq 0}$ *of open sets*

such that the approximating flow of open partitions $\mathcal{E}_{j_\ell}(t)$ *defined by* (3.7)-(3.8) *satisfies for all* $T < \infty$*:*

$$\limsup_{\ell\to\infty} \sup_{t\in[0,T]} \|\partial\mathcal{E}_{j_\ell}(t)\|(\Omega) \leq \|\partial\mathcal{E}_0\|(\Omega)\, \exp(c_1^2 T/2)\,, \tag{3.15}$$

$$\limsup_{\ell\to\infty} \int_0^T \left(\int_{\mathbb{R}^{n+1}} \frac{|\Phi_{\varepsilon_{j_\ell}} * \delta(\partial\mathcal{E}_{j_\ell}(t))|^2\, \Omega}{\Phi_{\varepsilon_{j_\ell}} * \|\partial\mathcal{E}_{j_\ell}(t)\| + \varepsilon_{j_\ell}\, \Omega^{-1}}\, dx - \frac{1}{\Delta t_{j_\ell}}\, \Delta^{vc}_{j_\ell} \|\partial\mathcal{E}_{j_\ell}(t)\|(\Omega) \right) dt < \infty\,, \tag{3.16}$$

$$\lim_{\ell\to\infty} j_\ell^{2(n+1)}\, \Delta^{vc}_{j_\ell} \|\partial\mathcal{E}_{j_\ell}(t)\|(\Omega) = 0 \qquad \text{for a.e. } t\in\mathbb{R}^+\,, \tag{3.17}$$

$$\lim_{\ell\to\infty} \|\partial\mathcal{E}_{j_\ell}(t)\|(\phi) = \mu_t(\phi) \qquad \text{for all } \phi\in C_c(\mathbb{R}^{n+1}) \text{ and any } t\in\mathbb{R}^+\,, \tag{3.18}$$

$$\chi_{E_{j_\ell,i}(t)} \to \chi_{E_i(t)} \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}) \text{ as } \ell\to\infty \text{ for every } i\in\{1,\ldots,N\} \text{ and for every } t\geq 0\,. \tag{3.19}$$

Furthermore, the following holds:

(a) *There exists a subset* $Z \subset \mathbb{R}^+$ *with* $\mathcal{L}^1(Z) = 0$ *such that, for every* $t \in \mathbb{R}^+ \setminus Z$, μ_t *is integral: that is, there exists* $V_t \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ *such that* $\|V_t\| = \mu_t$*;*

(b) *If* V_t *is defined to be an arbitrary varifold in* $\mathbf{V}_n(\mathbb{R}^{n+1})$ *with* $\|V_t\| = \mu_t$ *also for* $t \in Z$, *then the family* $\{V_t\}_{t\in\mathbb{R}^+}$ *is an* n*-dimensional Brakke flow in* $\mathbb{R}^{n+1}$ *satisfying the conclusions of Theorem 2.7(1)-(4) and Theorem 2.10(1);*

(c) *The flow of grains* $E_i(t)$ *satisfies the conclusions of Theorem 2.11(1)-(5).*

The proofs of the claims contained in the statement of Theorem 3.6 are analogous to the corresponding ones outlined in [18], modulo few technical modifications in consequence of the use of volume controlled Lipschitz deformations. More precisely:

- the conclusions (3.15) to (3.18) are contained in [18, Proposition 6.1, Proposition 6.4];
- the existence of the flow of grains $E_i(t)$, the convergence in (3.19), and the conclusion (c) is [18, Theorem 3.5];
- the conclusion (a) is [18, Lemma 9.1];
- the conclusion (b) is [18, Theorem 3.2, Proposition 3.4], except that Theorem 2.7(2) follows from the argument in [32, Proposition 6.10] and Theorem 2.7(4) follows from Brakke's inequality and Theorem 2.7(3) with $\Omega = 1$: Fix $\phi \in C_c^\infty(\mathbb{R}^{n+1};[0,1])$ with $\phi(x) = 1$ on B_1 and $\phi(x) = 0$ on $\mathbb{R}^{n+1}\setminus B_2$, and for $k\in\mathbb{N}$, set $\phi_k(x) := \phi(x/k)$. Use ϕ_k in (2.1) with $t_1 = 0$ and arbitrary $t = t_2 > 0$ and let $k\to\infty$. In doing so,
$$\int_0^t \int |\nabla\phi_k||h|\, d\|V_s\| ds \leq \Big(\int_0^t \int |\nabla\phi_k|^2\, d\|V_s\| ds\Big)^{\frac12} \Big(\int_0^t \int |h|^2\, d\|V_s\| ds\Big)^{\frac12} \to 0$$
as $k\to\infty$ due to $|\nabla\phi_k| = |\nabla\phi|/k$ and uniform bounds from Theorem 2.7(3). Then one can see that Theorem 2.7(4) holds true.

We will not repeat the proofs of the above results, but we will detail the aforementioned changes due to volume controlled Lipschitz deformations in Appendix A. For the time being, we will assume the validity of Theorem 3.6, and we will proceed with the derivation of all remaining results claimed in Section 2.

4. BV flow: proof of Theorem 2.11(6)

The main result of this section is the proof of Theorem 2.11(6), which we isolate as the following

Theorem 4.1. *The one-parameter family $\{E(t)\}_{t\in\mathbb{R}^+}$ of open partitions defined in Theorem 3.6 is a generalized* BV *solution to multi-phase mean curvature flow with scalar velocity fields $v_i = h\cdot\nu_i$. More precisely, for every $i\in\{1,\dots,N\}$ it holds*

$$\begin{aligned}\int_{E_i(t)}\phi(x,t)\,dx\Big|_{t=t_1}^{t_2} &= \int_{t_1}^{t_2}\int_{E_i(t)}\frac{\partial\phi}{\partial t}(x,t)\,dx\,dt \\ &\quad+\int_{t_1}^{t_2}\int_{\partial^* E_i(t)}\phi(x,t)\,h(x,V_t)\cdot\nu_{E_i(t)}(x)\,d\mathcal{H}^n(x)\,dt\end{aligned} \tag{4.1}$$

for every $0\le t_1<t_2<\infty$ and all test functions $\phi\in C^1_c(\mathbb{R}^{n+1}\times\mathbb{R}^+)$.

Remark 4.2. Introducing the notation χ_i for the indicator function $\chi_i(x,t) := \chi_{E_i(t)}(x)$, and recalling that $\nabla\chi_{E_i(t)} = -\nu_{E_i(t)}\,\mathcal{H}^n\llcorner_{\partial^* E_i(t)}$ as $\mathbb{R}^{n+1}$-valued Radon measures on $\mathbb{R}^{n+1}$, we see that the identity in (4.1) can be rephrased as

$$\begin{aligned}\int_{\mathbb{R}^{n+1}}\phi(x,t)\,\chi_i(x,t)\,dx\Big|_{t=t_1}^{t_2} &= \int_{t_1}^{t_2}\int_{\mathbb{R}^{n+1}}\frac{\partial\phi}{\partial t}(x,t)\,\chi_i(x,t)\,dx\,dt \\ &\quad-\int_{t_1}^{t_2}\int_{\mathbb{R}^{n+1}}\phi(x,t)\,h(x,V_t)\cdot d\nabla\chi_{E_i(t)}(x)\,dt\,.\end{aligned} \tag{4.2}$$

This implies, in particular, that $\chi_i\in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^{n+1}\times\mathbb{R}^+)$ with derivative

$$\begin{aligned}\nabla'\chi_i(x,t) &= \left(\nabla\chi_{E_i(t)}(x), -h(x,V_t)\cdot\nabla\chi_{E_i(t)}(x)\right)\,dt \\ &= \left(-\nu_i(x,t), h(x,V_t)\cdot\nu_i(x,t)\right)\,d\mathcal{H}^n\llcorner_{\partial^* E_i(t)}\,dt\,,\end{aligned} \tag{4.3}$$

where $\nabla' = (\nabla,\partial_t)$ and the identity holds in the sense of $(\mathbb{R}^{n+1}\times\mathbb{R})$-valued Radon measures on $\mathbb{R}^{n+1}\times\mathbb{R}^+$.

The proof of Theorem 4.1 will be obtained in three main steps, which are the content of Subsection 4.1 to Subsection 4.3, respectively. First, we check that Brakke flow implies L^2 flow and recall the important observation Corollary 4.4 due to Mugnai-Röger [26]. Second, we will prove that, for each $i\in\{1,\dots,N\}$, $\chi_i\in\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^{n+1}\times\mathbb{R}^+)$, see Proposition 4.5; then, we will characterize the measure-theoretic time-derivative of χ_i to show that (4.3) holds.

4.1. Characterization as L^2 flow.

In this subsection, we first prove Theorem 2.7(4), which we isolate as the following

Theorem 4.3. *Let $\{V_t\}_{t\in\mathbb{R}^+}$ be the Brakke flow defined in Theorem 3.6. Then, for every $0<T<\infty$ and for every open and bounded $U\subset\mathbb{R}^{n+1}$, the varifolds $V_t\llcorner_{(U\times\mathbf{G}(n+1,n))}$ ($t\in[0,T)$) are an n-dimensional L^2 flow with generalized velocity vector $v(\cdot,t)=h(\cdot,V_t)$.*

Proof. We verify the requirements of Definition 2.3. Conditions (a) and (b) are satisfied by V_t in $\mathbb{R}^{n+1}$, and thus they are trivially satisfied when the varifolds are restricted to the open set U. Concerning (c'), we have immediately that

$$\int_0^T\int_U |h(\cdot,V_t)|^2\,d\|V_t\|\,dt \le \left(\max_{\mathrm{clos}\,U}\Omega^{-1}\right)\int_0^T\int_{\mathbb{R}^{n+1}}|h(\cdot,V_t)|^2\,\Omega\,d\|V_t\|\,dt<\infty\,,$$

so that $h(\cdot, V_t) \in L^2(\|V_t\| \llcorner_U; \mathbb{R}^{n+1})$ for a.e. $t \in [0,T)$. Analogously, $d\|V_t\|\,dt$ is a Radon measure on $U \times (0,T)$, and in fact we have

$$\left| \int_0^T \int_U \phi(x,t)\, d\|V_t\|(x)\, dt \right| \leq \|\phi\|_{C^0} \left(\max_{\operatorname{clos} U} \Omega^{-1} \right) \mathcal{H}^n \llcorner_\Omega (\Gamma_0) \int_0^T \exp(c_1^2 t/2)\, dt < \infty$$

for every $\phi \in C_c(U \times (0,T))$. To complete the proof, we are then only left with checking that (2.3) holds with $v(\cdot,t) = h(\cdot, V_t)$: with this choice, indeed, the condition in (d'1) is automatically satisfied due to the perpendicularity of mean curvature. Consider first a test function $\phi \in C^1_c(\mathbb{R}^{n+1} \times (0,T]\,; \mathbb{R}^+)$. Brakke's inequality (2.1) then implies that

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^{n+1}} &\frac{\partial \phi}{\partial t}(x,t) + h(x,V_t) \cdot \nabla \phi(x,t)\, d\|V_t\|(x)\, dt \\ &\geq \int_0^T \int_{\mathbb{R}^{n+1}} \phi(x,t)\, |h(x,V_t)|^2\, d\|V_t\|(x)\, dt \geq 0\,. \end{aligned} \tag{4.4}$$

In particular, the assignment

$$\phi \mapsto L\phi := \int_0^T \int_{\mathbb{R}^{n+1}} \frac{\partial \phi}{\partial t}(x,t) + h(x,V_t) \cdot \nabla \phi(x,t)\, d\|V_t\|(x)\, dt$$

defines a positive linear functional on $C^1_c(\mathbb{R}^{n+1} \times (0,T])$: hence, L is monotone, that is $L\phi_1 \leq L\phi_2$ whenever $\phi_1 \leq \phi_2$ everywhere. For every $\varepsilon \in (0,1)$, let $\psi_{U,\varepsilon} \in C^1_c(\mathbb{R}^{n+1} \times (0,T])$ be the cut-off function $\psi_{U,\varepsilon}(x,t) = \psi_U(x)\psi_\varepsilon(t)$ defined according to the following prescriptions:

(i) $0 \leq \psi_U \leq 1$ and $0 \leq \psi_\varepsilon \leq 1$,
(ii) $\psi_U \equiv 1$ on $\operatorname{clos} U$ and $\psi_U \equiv 0$ on $\operatorname{dist}(x, \operatorname{clos} U) \geq 1$,
(iii) $\psi_\varepsilon \equiv 1$ on $[\varepsilon, T]$ and $\psi_\varepsilon \equiv 0$ on $(0, \varepsilon/2]$,
(iv) $\|\nabla \psi_U\|_{C^0} \leq C$ and $\|\psi'_\varepsilon\|_{C^0} \leq C/\varepsilon$ for a geometric constant C.

Let now $\phi \in C^1_c(U \times (0,T))$ be a non-zero function such that $\operatorname{spt}(\phi) \subset U \times [\varepsilon, T]$. For such a function, by the definition of $\psi_{U,\varepsilon}$, it holds

$$-\psi_{U,\varepsilon} \leq \frac{\phi}{\|\phi\|_{C^0}} \leq \psi_{U,\varepsilon}\,,$$

so that the linearity and monotonicity of L yield

$$|L\phi| \leq L\psi_{U,\varepsilon}\, \|\phi\|_{C^0}\,.$$

Notice that for such a ϕ the space integration can be restricted to U (as the – continuous – derivatives of ϕ are necessarily zero on U^c for every t), and thus the above argument shows that whenever $\phi \in C^1_c(U \times (0,T))$ is supported on $U \times [\varepsilon, T]$ it holds

$$\left| \int_0^T \int_U \frac{\partial \phi}{\partial t}(x,t) + h(x,V_t) \cdot \nabla \phi(x,t)\, d\|V_t\|(x)\, dt \right| \leq L\psi_{U,\varepsilon}\, \|\phi\|_{C^0}\,. \tag{4.5}$$

We next proceed to estimate $L\psi_{U,\varepsilon}$. We have, setting $U_1 := \{\operatorname{dist}(x, \operatorname{clos} U) < 1\}$,

$$\begin{aligned} L\psi_{U,\varepsilon} &\leq \frac{C}{\varepsilon} \int_{\varepsilon/2}^{\varepsilon} \|V_t\|(U_1)\, dt + \int_0^T \int_{\mathbb{R}^{n+1}} |h(\cdot, V_t)|\, |\nabla \psi_U|\, d\|V_t\|\, dt \\ &\leq C \left(\max_{\operatorname{clos} U_1} \Omega^{-1} \right) \left(\mathcal{H}^n \llcorner_\Omega (\Gamma_0) + (\mathcal{H}^n \llcorner_\Omega (\Gamma_0))^{1/2} \left(\int_0^T \int_{\mathbb{R}^{n+1}} |h(\cdot, V_t)|^2\, \Omega\, d\|V_t\|\, dt \right)^{1/2} \right)\,. \end{aligned}$$

We see then that $\sup_{\varepsilon>0} L\psi_{U,\varepsilon} < \infty$: in particular, thanks to (4.5), we conclude that (2.3) holds with $v(\cdot,t) = h(\cdot, V_t)$ for every $\phi \in C^1_c(U \times (0,T))$, thus completing the proof. ☐

The following is a simple corollary of Theorem 4.3 and [26, Proposition 3.3].

Corollary 4.4. *Let $\{V_t\}_{t\in\mathbb{R}^+}$ be the Brakke flow defined in Theorem 3.6, and let μ be the space-time measure $d\mu = d\|V_t\|\,dt$. Then,*

$$\begin{pmatrix} h(x_0, V_{t_0}) \\ 1 \end{pmatrix} \in T_{(x_0,t_0)}\mu \tag{4.6}$$

at μ-a.e. (x_0,t_0) such that the tangent space $T_{(x_0,t_0)}\mu$ exists.

4.2. **Existence of measure-theoretic velocities.** We have the following

Proposition 4.5. *Let $\{\mathcal{E}(t)\}_{t\in\mathbb{R}^+}$ and $\{V_t\}_{t\in\mathbb{R}^+}$ be as in Theorem 3.6, and let $\mu\llcorner_\Omega$ denote the Radon measure $\mu\llcorner_\Omega := \Omega\, d\|V_t\|\,dt$. Also, for every $i \in \{1,\ldots,N\}$, set*

$$S(i) := \left\{(x,t) \in \mathbb{R}^{n+1}\times\mathbb{R}^+ \,:\, x \in E_i(t)\right\} .$$

Then, for every i there exists a function $u_i = u_i(x,t) \in L^2(\mu\llcorner_\Omega)$ with $\mathrm{spt}(u_i) \subset \partial S(i)$ and such that

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} \phi(x,t)\,\chi_i(x,t)\,dx\Big|_{t=t_1}^{t_2} &= \int_{t_1}^{t_2}\int_{\mathbb{R}^{n+1}} \frac{\partial\phi}{\partial t}(x,t)\,\chi_i(x,t)\,dx\,dt \\ &\quad + \int_{t_1}^{t_2}\int_{\mathbb{R}^{n+1}} \phi(x,t)\,u_i(x,t)\,d\|V_t\|\,dt\end{aligned} \tag{4.7}$$

for every $0 \le t_1 < t_2 < \infty$ and all test functions $\phi \in C^1_c(\mathbb{R}^{n+1}\times[0,\infty))$.

Proposition 4.5 readily implies the following

Corollary 4.6. *For every $i \in \{1,\ldots,N\}$, $\chi_i \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^{n+1}\times\mathbb{R}^+)$, and*

$$\nabla'\chi_i(x,t) = \left(\nabla\chi_{E_i(t)}\,dt, u_i(x,t)\,d\|V_t\|\,dt\right) . \tag{4.8}$$

In particular, since χ_i is the indicator function of $S(i)$, the latter is a set of locally finite perimeter in $\mathbb{R}^{n+1}\times\mathbb{R}^+$.

Proof of Proposition 4.5. We shall divide the proof into several steps. At first, we will drop the dependence of the test function on time, which will be introduced again at a later step. The idea of the proof is to obtain an evolution identity for the approximating flow $\partial\mathcal{E}_{j_\ell}(t)$ defined in Subsection 3.2 (where j_ℓ is the sequence of Theorem 3.6), and then to prove that, as $\ell\to\infty$, such identity converges to (4.7). Notice that we can assume without loss of generality that $t_1 = 0$, since then the seemingly more general identity can be retrieved by simply taking differences. We will then also rename $t_2 = T \in (0,\infty)$.

Step one: preliminary reductions. By density, it is evidently sufficient to show that the identity (4.7) holds when $\phi = \phi(x)$ is a function in $C^2_c(\mathbb{R}^{n+1})$. Furthermore, since (4.7) is linear in ϕ, we can assume without loss of generality that $|\phi| \le \Omega$ everywhere on $\mathbb{R}^{n+1}$. Consider the approximating flow $\mathcal{E}_{j_\ell}(t)$, and fix ℓ so large that the flow is defined on the interval $[0,T]$. We will deduce the validity of an approximate identity for the approximating flow, with vanishing errors in the limit as $\ell\to\infty$. We fix the index $i \in \{1,\ldots,N\}$, and, for the sake of simplicity in the notation, we drop the corresponding subscripts, so to write $\mathcal{E}(t)$ in place of $\mathcal{E}_{j_\ell}(t)$ and $E(t)$ in place of $E_{j_\ell,i}(t)$. Recalling the construction of $\mathcal{E}(t)$, we let k_T

be the integer in $\{1,\ldots,j_\ell\,2^{p_{j_\ell}}\}$ such that $T\in((k_T-1)\Delta t,k_T\Delta t]$ (with $\Delta t=\Delta t_{j_\ell}$), so that $E(T)=E(k_T\,\Delta t)=E_{k_T}:=E_{j_\ell,k_T,i}$. We then have the discretization

$$\int_{E(T)}\phi(x)\,dx-\int_{E_0}\phi(x)\,dx=\sum_{k=0}^{k_T-1}\left\{\int_{E_{k+1}}\phi(x)\,dx-\int_{E_k}\phi(x)\,dx\right\}\,, \tag{4.9}$$

and we can further decompose each summand on the right-hand side of (4.9) as

$$\int_{E_{k+1}}\phi(x)\,dx-\int_{E_k}\phi(x)\,dx=D_1+D_2\,, \tag{4.10}$$

where

$$D_1 := \int_{E^*_{k+1}}\phi(x)\,dx-\int_{E_k}\phi(x)\,dx\,, \tag{4.11}$$

$$D_2 := \int_{E_{k+1}}\phi(x)\,dx-\int_{E^*_{k+1}}\phi(x)\,dx\,. \tag{4.12}$$

Here, $E^*_{k+1}=\mathrm{int}(f_1(E_k))$ for some Lipschitz function $f_1\in\mathbf{E}^{vc}(\mathcal{E}_k,j_\ell)$ satisfying the almost-optimality condition

$$\|\partial(f_1)_\star\mathcal{E}_k\|(\Omega)-\|\partial\mathcal{E}_k\|(\Omega)\le(1-j_\ell^{-5})\,\Delta^{vc}\|\partial\mathcal{E}_k\|(\Omega)\,, \tag{4.13}$$

where $\Delta^{vc}\|\partial\mathcal{E}_k\|(\Omega):=\Delta^{vc}_{j_\ell}\|\partial\mathcal{E}_k\|(\Omega)$, and $E_{k+1}=f_2(E^*_{k+1})$, with $f_2(x)=x+\Delta t\,h_\varepsilon(x,\partial\mathcal{E}^*_{k+1})$ ($\varepsilon=\varepsilon_{j_\ell}$). We will now proceed evaluating D_1 and D_2 separately.

Step two: evaluation of D_1 and D_2. For D_1, we simply observe that

$$D_1=\int_{E^*_{k+1}\setminus E_k}\phi(x)\,dx-\int_{E_k\setminus E^*_{k+1}}\phi(x)\,dx\,,$$

so that, using Definition 3.1(b) and (3.1), it holds

$$|D_1|\le 2\,\|\phi\|_\infty\,\mathcal{L}^{n+1}(E_k\triangle E^*_{k+1})\le -2\,\|\phi\|_\infty\,\frac{\Delta^{vc}\|\partial\mathcal{E}_k\|(\Omega)}{\Delta t}\,\frac{\Delta t}{j_\ell}\,. \tag{4.14}$$

We then proceed with D_2, and in order to further ease the notation we set $f(x):=f_2(x)$, $F(x)=f(x)-x$, as well as $g:=f^{-1}$ and $G(x):=g(x)-x$. We also simply write

$$E:=E_{k+1}\,,\qquad E^*:=E^*_{k+1}=g(E)\,.$$

Using that g is a diffeomorphism and changing variable in the second integral in D_2, we have

$$\begin{aligned}D_2&=\int_E\phi(x)\,dx-\int_E\phi(g(x))\,\mathbf{J}g(x)dx\\&=\int_E\{\phi(x)-\phi(g(x))\}\,dx+\int_E\phi(x)\,\{1-\mathbf{J}g(x)\}\,dx+\int_E(\phi(x)-\phi(g(x)))\,(\mathbf{J}g(x)-1)\,dx\,,\end{aligned} \tag{4.15}$$

where $\mathbf{J}g$ denotes the Jacobian determinant of g. Using that $G(x)=-F(g(x))$ and that $F(y)=\Delta t\,h_\varepsilon(y)$ ($h_\varepsilon(\cdot)=h_\varepsilon(\cdot,\partial\mathcal{E}^*_{k+1})$) together with (2.21), we have that if $(\mathrm{spt}\,\phi)_1:=\{\mathrm{dist}(x,\mathrm{spt}\,\phi)\le1\}$ then

$$|\phi(g(x))-\phi(x)|\le\|\nabla\phi\|_\infty\,|G(x)|\,\chi_{(\mathrm{spt}\,\phi)_1}(x)\le\Delta t\,\varepsilon^{-3}\,\chi_{(\mathrm{spt}\,\phi)_1}(x) \tag{4.16}$$

$$|\phi(g(x))-\phi(x)-\nabla\phi(x)\cdot G(x)|\le\|\nabla^2\phi\|_\infty\,|G(x)|^2\,\chi_{(\mathrm{spt}\,\phi)_1}(x)\le\Delta t\,\varepsilon^{c_2-6}\,\chi_{(\mathrm{spt}\,\phi)_1}(x)\,. \tag{4.17}$$

Notice that, in order to apply the estimate (2.21), we are using that $\|\partial\mathcal{E}^*_{k+1}\|(\Omega) \le \|\partial\mathcal{E}_k\|(\Omega)$ and we are assuming that $\varepsilon = \varepsilon_{j_\ell}$ is smaller that the ε_1 given by Lemma 2.17 corresponding to the choice $M = \sup_{t\in[0,T]} \|\partial\mathcal{E}_{j_\ell}(t)\|(\Omega)$, which is bounded by a quantity depending only on T (and the initial datum) due to (3.15). In deducing (4.16)-(4.17), we have also used that, by the definition of g and (2.21), and for all ℓ large enough, if $x \notin (\operatorname{spt}\phi)_1$ then $g(x) \notin \operatorname{spt}\phi$, and that ε can be assumed sufficiently small, so that $\varepsilon\,\|\nabla\phi\|_\infty$ and $\varepsilon\,\|\nabla^2\phi\|_\infty$ are both bounded by 1. Next, we also estimate

$$|\mathbf{J}g(x) - 1| \le c(n)\,\|\nabla G(x)\| \le c(n)\,\Delta t\,\varepsilon^{-4}\,, \tag{4.18}$$

$$|\mathbf{J}g(x) - 1 - \operatorname{div}(G(x))| \le c(n)\,\|\nabla G(x)\|^2 \le \Delta t\,\varepsilon^{c_2-9}\,, \tag{4.19}$$

where we have used that $\nabla G = -[(\nabla F)\circ g]\,\nabla g = -[(\nabla F)\circ g]\,[(\nabla f)^{-1}\circ g]$, that $\nabla F = \Delta t\,\nabla h_\varepsilon$, and the estimate (2.22). We can then conclude from (4.17) and (4.19) that the sum of the first two integrals in (4.15) is

$$= -\int_E \{\nabla\phi(x)\cdot G(x) + \phi(x)\operatorname{div}(G(x))\}\,dx + \mathrm{Err}_1\,,$$

where

$$|\mathrm{Err}_1| \le \Delta t\,\varepsilon^{c_2-10} \tag{4.20}$$

as soon as ε is small enough to absorb the constants depending on $\mathcal{L}^{n+1}((\operatorname{spt}\phi)_1)$. On the other hand, using that $\nabla\phi(x)\cdot G(x) + \phi(x)\operatorname{div}(G(x)) = \operatorname{div}(\phi(x)G(x))$ and the divergence theorem we have

$$\begin{aligned} -\int_E \{\nabla\phi(x)\cdot G(x) + \phi(x)\operatorname{div}(G(x))\}\,dx &= -\int_{\partial^* E} \phi(x)\,G(x)\cdot\nu_E(x)\,d\mathcal{H}^n(x) \\ &= \Delta t\int_{\partial^* E}\phi(x)\,h_\varepsilon(g(x))\cdot\nu_E(x)\,d\mathcal{H}^n(x)\,. \end{aligned} \tag{4.21}$$

Recall that, in the right-hand side of (4.21), $h_\varepsilon(\cdot) = h_\varepsilon(\cdot,\partial\mathcal{E}^*_{k+1})$. We then continue the chain of identities in (4.21) as

$$= \Delta t\int_{\partial^* E}\phi(x)\,h_\varepsilon(x,\partial\mathcal{E}_{k+1})\cdot\nu_E(x)\,d\mathcal{H}^n(x) + \mathrm{Err}_2\,,$$

where

$$\mathrm{Err}_2 := \Delta t\int_{\partial^* E}\phi(x)\,\nu_E(x)\cdot\big(h_\varepsilon(g(x),\partial\mathcal{E}^*_{k+1}) - h_\varepsilon(x,\partial\mathcal{E}_{k+1})\big)\,d\mathcal{H}^n(x)\,.$$

Since $\mathcal{E}_{k+1} = f_\star\mathcal{E}^*_{k+1}$, and f is a diffeomorphism, we have that $\partial\mathcal{E}_{k+1} = f_\sharp\partial\mathcal{E}^*_{k+1}$, where $f_\sharp$ denotes the push-forward operator on integral varifolds through f. Calling, for simplicity, $V = \partial\mathcal{E}^*_{k+1}$ and $\hat V = \partial\mathcal{E}_{k+1} = f_\sharp V$, we can write

$$h_\varepsilon(x,\hat V) - h_\varepsilon(g(x),V) = [h_\varepsilon(x,\hat V) - h_\varepsilon(g(x),\hat V)] + [h_\varepsilon(g(x),\hat V) - h_\varepsilon(g(x),V)]\,. \tag{4.22}$$

Recall, now, that the ε-smoothed mean curvature vector $h_\varepsilon(\cdot,V)$ is defined by

$$h_\varepsilon(\cdot,V) = -\Phi_\varepsilon * \frac{\Phi_\varepsilon * \delta V}{\Phi_\varepsilon * \|V\| + \varepsilon\,\Omega^{-1}}\,.$$

In particular, using [18, Equation (5.74) and Lemma 4.17], it is easy to estimate the second summand on the right-hand side of (4.22) by

$$|h_\varepsilon(g(x),\hat V) - h_\varepsilon(g(x),V)| \le \varepsilon^{c_2-2n-14}(\|V\|(\Omega) + \|V\|(\Omega)^2)\,. \tag{4.23}$$

For the first summand on the right-hand side of (4.22), instead, we write explicitly

$$h_\varepsilon(x,\hat V) - h_\varepsilon(g(x),\hat V) = \int_{\mathbb{R}^{n+1}} \{\Phi_\varepsilon(g(x)-y) - \Phi_\varepsilon(x-y)\}\, \frac{\Phi_\varepsilon * \delta\hat V}{\Phi_\varepsilon * \|\hat V\| + \varepsilon\,\Omega^{-1}}(y)\,dy\,,$$

and standard calculations involving the definition of the kernel Φ_ε (see e.g. the estimate in [18, Eq. (5.68)]) give

$$\begin{aligned} |h_\varepsilon(x,\hat V) - h_\varepsilon(g(x),\hat V)| &\le c(n)\,\varepsilon^{c_2-n-6} \left(\int_{\mathbb{R}^{n+1}} \frac{|\Phi_\varepsilon * \delta\hat V|^2\,\Omega}{\Phi_\varepsilon * \|\hat V\| + \varepsilon\,\Omega^{-1}}\,dy\right)^{\frac12} \\ &\le \varepsilon^{c_2-n-7} \left(\int_{\mathbb{R}^{n+1}} \frac{|\Phi_\varepsilon * \delta\hat V|^2\,\Omega}{\Phi_\varepsilon * \|\hat V\| + \varepsilon\,\Omega^{-1}}\,dy\right)^{\frac12}\,. \end{aligned} \tag{4.24}$$

Thus, we can finally estimate

$$\begin{aligned} |\mathrm{Err}_2| &\le \Delta t\,\varepsilon^{c_2-2n-14}\,\|\hat V\|(\Omega) \left(\|V\|(\Omega) + \|V\|(\Omega)^2 + \left(\int_{\mathbb{R}^{n+1}} \frac{|\Phi_\varepsilon * \delta\hat V|^2\,\Omega}{\Phi_\varepsilon * \|\hat V\| + \varepsilon\,\Omega^{-1}}\,dy\right)^{\frac12}\right) \\ &\le \Delta t\,\varepsilon^{c_2-2n-14}\,\|\hat V\|(\Omega) \left(\|\hat V'\|(\Omega) + \|\hat V'\|(\Omega)^2 + \left(\int_{\mathbb{R}^{n+1}} \frac{|\Phi_\varepsilon * \delta\hat V|^2\,\Omega}{\Phi_\varepsilon * \|\hat V\| + \varepsilon\,\Omega^{-1}}\,dy\right)^{\frac12}\right)\,, \end{aligned} \tag{4.25}$$

where $\hat V' = \partial\mathcal{E}_k$ and we used $\|V\|(\Omega) = \|\partial(f_1)_\star\mathcal{E}_k\|(\Omega) \le \|\partial\mathcal{E}_k\|(\Omega)$. Concerning the third integral in (4.15), instead, we use (4.16) and (4.18) to estimate

$$\mathrm{Err}_3 := \int_E (\phi(x) - \phi(g(x)))\,(\mathbf{J}g(x) - 1)\,dx$$

by

$$|\mathrm{Err}_3| \le \Delta t\,\varepsilon^{c_2-8} \tag{4.26}$$

for all ε sufficiently small. We have thus proved that

$$\int_{E_{k+1}} \phi(x)\,dx - \int_{E_k} \phi(x)\,dx = \Delta t\,\left(\int_{\partial^* E_{k+1}} \phi(x)\,\nu_{E_{k+1}}\cdot h_\varepsilon(x,\partial\mathcal{E}_{k+1})\,d\mathcal{H}^n(x) + \mathrm{Err}_{k+1}\right)\,, \tag{4.27}$$

where

$$\mathrm{Err}_{k+1} := (\Delta t)^{-1}\,(D_1 + \mathrm{Err}_1 + \mathrm{Err}_2 + \mathrm{Err}_3)$$

satisfy the estimates (4.14), (4.20), (4.25), and (4.26).

Step three: sum of contributions and limit $\ell \to \infty$. Now we sum the identities (4.27) for $k = 0, \ldots, k_T - 1$. Introducing back the subscripts j_ℓ and i, and using that (see (3.8))

$$\partial\mathcal{E}_{j_\ell}(t) = \partial\mathcal{E}_{j_\ell,k+1} \qquad \text{for all } t \in (k\Delta t_{j_\ell}, (k+1)\Delta t_{j_\ell}]\,,$$

we obtain from (4.9) that

$$\int_{E_{j_\ell,i}(T)} \phi(x)\,dx - \int_{E_{0,i}} \phi(x)\,dx = \int_0^{k_T\,\Delta t_{j_\ell}} \int_{\mathbb{R}^{n+1}} \phi(x)\,h_{\varepsilon_{j_\ell}}(x,\partial\mathcal{E}_{j_\ell}(t))\cdot d\mu_{E_{j_\ell,i}(t)}(x)\,dt + \mathrm{Err}_{j_\ell}\,, \tag{4.28}$$

with $\mu_{E_{j_\ell,i}(t)} = \nabla\chi_{E_{j_\ell,i}(t)} = \nu_{E_{j_\ell,i}(t)}\,\mathcal{H}^n_{\llcorner\partial^* E_{j_\ell,i}(t)}$ and, since $k_T\,\Delta t_{j_\ell} \le T+1$

$$\begin{aligned}|\mathrm{Err}_{j_\ell}| \le \frac{1}{j_\ell}\int_0^{T+1} -\frac{\Delta^{vc}_{j_\ell}\|\partial\mathcal{E}_{j_\ell}(t)\|(\Omega)}{\Delta t_{j_\ell}}\,dt - \frac{\Delta^{vc}_{j_\ell}\|\partial\mathcal{E}_0\|(\Omega)}{j_\ell}\\ + \varepsilon_{j_\ell}^{c_2-11}(T+1) + \varepsilon_{j_\ell}^{c_2-2n-14}\,(T+1)\,(M_{j_\ell}^2+M_{j_\ell}^3)\\ + \varepsilon_{j_\ell}^{c_2-2n-14}\sqrt{T+1}\,M_{j_\ell}\left(\int_0^{T+1}\int_{\mathbb{R}^{n+1}}\frac{|\Phi_{\varepsilon_{j_\ell}}*\delta(\partial\mathcal{E}_{j_\ell}(t))|^2\,\Omega}{\Phi_{\varepsilon_{j_\ell}}*\|\partial\mathcal{E}_{j_\ell}(t)\|+\varepsilon_{j_\ell}\Omega^{-1}}\,dx\,dt\right)^{\frac12}\,,\end{aligned}\tag{4.29}$$

where $M_{j_\ell} := \sup_{t\in[0,T+1]}\|\partial\mathcal{E}_{j_\ell}(t)\|(\Omega)$. Next, Lemma 2.17 implies that

$$\left|\int_T^{k_T\Delta t_{j_\ell}}\int_{\mathbb{R}^{n+1}}\phi\,h_{\varepsilon_{j_\ell}}(\cdot,\partial\mathcal{E}_{j_\ell}(t))\cdot d\mu_{E_{j_\ell,i}(t)}\right| \le M_{j_\ell}\,\varepsilon_{j_\ell}^{c_2-3}\,,$$

so that the end-point $k_T\,\Delta t_{j_\ell}$ in the time integral of (4.28) can be replaced by T. Letting $\ell\to\infty$ in the identity (4.28), we have that the left-hand side converges to

$$\int_{E_i(T)}\phi(x)\,dx - \int_{E_{i,0}}\phi(x)\,dx \tag{4.30}$$

due to (3.19), whereas $\mathrm{Err}_{j_\ell}\to 0$ thanks to (4.29), (3.15), and (3.16). We are then left with studying the limit

$$\lim_{\ell\to\infty}\int_0^T\int_{\mathbb{R}^{n+1}}\phi(x)\,h_{\varepsilon_{j_\ell}}(x,\partial\mathcal{E}_{j_\ell}(t))\cdot d\mu_{E_{j_\ell,i}(t)}(x)\,dt\,. \tag{4.31}$$

To this aim, we first observe that, since

$$h_\varepsilon(x,V) = -\Phi_\varepsilon*\left(\frac{\Phi_\varepsilon*\delta V}{\Phi_\varepsilon*\|V\|+\varepsilon\,\Omega^{-1}}\right)(x)\,,$$

and using that, by definition, $\Omega(x)\le\Omega(y)\exp(c_1\,|x-y|)$ for all $x,y\in\mathbb{R}^{n+1}$ together with the properties of the kernel Φ_ε, we can then estimate

$$\begin{aligned}&\int_{\partial^* E_{j_\ell,i}(t)}\Omega\,|h_{\varepsilon_{j_\ell}}(\cdot,\partial\mathcal{E}_{j_\ell}(t))|^2\,d\mathcal{H}^n\\ &\quad\le C\int_{\mathbb{R}^{n+1}}\frac{|\Phi_{\varepsilon_{j_\ell}}*\delta(\partial\mathcal{E}_{j_\ell}(t))|^2\,\Omega(y)}{\Phi_{\varepsilon_{j_\ell}}*\|\partial\mathcal{E}_{j_\ell}(t)\|+\varepsilon_{j_\ell}\Omega^{-1}}\,\frac{\Phi_{\varepsilon_{j_\ell}}*(\mathcal{H}^n\llcorner_{\partial^* E_{j_\ell,i}(t)})}{\Phi_{\varepsilon_{j_\ell}}*\|\partial\mathcal{E}_{j_\ell}(t)\|+\varepsilon_{j_\ell}\Omega^{-1}}\,dy\\ &\quad\le C\int_{\mathbb{R}^{n+1}}\frac{|\Phi_{\varepsilon_{j_\ell}}*\delta(\partial\mathcal{E}_{j_\ell}(t))|^2\,\Omega(y)}{\Phi_{\varepsilon_{j_\ell}}*\|\partial\mathcal{E}_{j_\ell}(t)\|+\varepsilon_{j_\ell}\Omega^{-1}}\,dy\,.\end{aligned}\tag{4.32}$$

Therefore, for every $T>0$ and for every $\phi\in C_c(\mathbb{R}^{n+1}\times[0,T])$ it holds

$$\begin{aligned}&\left|\int_0^T\int_{\mathbb{R}^{n+1}}\phi(x,t)\,\Omega(x)\,|h_{\varepsilon_{j_\ell}}(x,\partial\mathcal{E}_{j_\ell}(t))\cdot d\mu_{E_{j_\ell,i}(t)}(x)|\,dt\right|\\ &\quad\le\left(\int_0^T\int_{\partial^* E_{j_\ell,i}(t)}\Omega\,|h_{\varepsilon_{j_\ell}}(x,\partial\mathcal{E}_{j_\ell}(t))|^2\,d\mathcal{H}^n\,dt\right)^{1/2}\left(\int_0^T\int_{\partial^* E_{j_\ell,i}(t)}\Omega\,|\phi(x,t)|^2\,d\mathcal{H}^n\,dt\right)^{1/2}\\ &\quad\overset{(4.32)}{\le} C\left(\int_0^T\int_{\mathbb{R}^{n+1}}\frac{|\Phi_{\varepsilon_{j_\ell}}*\delta(\partial\mathcal{E}_{j_\ell}(t))|^2\,\Omega(y)}{\Phi_{\varepsilon_{j_\ell}}*\|\partial\mathcal{E}_{j_\ell}(t)\|+\varepsilon_{j_\ell}\Omega^{-1}}\,dy\right)^{1/2}\left(\int_0^T\int_{\partial^* E_{j_\ell,i}(t)}\Omega\,|\phi(x,t)|^2\,d\mathcal{H}^n\,dt\right)^{1/2}.\end{aligned}\tag{4.33}$$

In particular, using (3.15) and (3.16), for any fixed $T > 0$,

$$\Omega\, h_{\varepsilon_{j_\ell}}(\cdot, \partial\mathcal{E}_{j_\ell}(t)) \cdot d\mu_{E_{j_\ell,i}(t)}\, dt$$

are (signed) Radon measures in $\mathbb{R}^{n+1} \times [0,T]$ with uniformly bounded total variation, and we can let σ_i denote a subsequential limit as $\ell \to \infty$. Since $\mathcal{H}^n \llcorner_{\partial^* E_{j_\ell,i}(t)} \le \|\partial\mathcal{E}_{j_\ell}(t)\|$, and the latter measures converge, as $\ell \to \infty$, to $\|V_t\|$ for every $t \in \mathbb{R}^+$, it is clear from (4.33) that σ_i is absolutely continuous with respect to the measure $\mu\llcorner_\Omega = \Omega\, d\|V_t\|\, dt$. We will let u_i denote the Radon-Nikodým derivative of σ_i with respect to $\mu\llcorner_\Omega$, so that

$$\sigma_i = u_i\, \Omega\, d\|V_t\|\, dt \tag{4.34}$$

in the sense of measures. The estimate (4.33) also implies that $u_i \in L^2(\mu\llcorner_\Omega)$, with

$$\|u_i\|_{L^2(\mu\llcorner_\Omega)} \le C \limsup_{\ell\to\infty} \left(\int_0^T \int_{\mathbb{R}^{n+1}} \frac{|\Phi_{\varepsilon_{j_\ell}} * \delta(\partial\mathcal{E}_{j_\ell}(t))|^2\, \Omega(y)}{\Phi_{\varepsilon_{j_\ell}} * \|\partial\mathcal{E}_{j_\ell}(t)\| + \varepsilon_{j_\ell}\Omega^{-1}}\, dy \right)^{1/2} . \tag{4.35}$$

Hence, we can now calculate the limit in (4.31) (along the aforementioned subsequence, not relabeled), which gives

$$\lim_{\ell\to\infty} \int_0^T \int_{\mathbb{R}^{n+1}} (\phi\, \Omega^{-1})\, \Omega\, h_{\varepsilon_{j_\ell}}(\cdot, \partial\mathcal{E}_{j_\ell}(t)) \cdot d\mu_{E_{j_\ell,i}(t)}\, dt = \int_0^T \int_{\mathbb{R}^{n+1}} \phi\, u_i\, d\|V_t\|\, dt\, .$$

We have then concluded the existence of $u_i \in L^2(\mu\llcorner_\Omega)$ such that

$$\int_{E_i(T)} \phi(x)\, dx - \int_{E_{i,0}} \phi(x)\, dx = \int_0^T \int_{\mathbb{R}^{n+1}} \phi(x)\, u_i(x,t)\, d\|V_t\|(x)\, dt \tag{4.36}$$

for all $\phi \in C_c(\mathbb{R}^{n+1})$, that is we have obtained (4.7) when ϕ does not depend on t.

Step four: the case of time-dependent ϕ. We now consider the general case of a test function $\phi \in C^1_c(\mathbb{R}^{n+1} \times [0,T])$. The proof is analogous, with a few minor modifications which take the dependence on t into account. First, formula (4.9) becomes

$$\int_{E(T)} \phi(x,T)\, dx - \int_{E_0} \phi(x,0)\, dx = \sum_{k=0}^{k_T-1} \left\{ \int_{E_{k+1}} \phi(x,(k+1)\Delta t)\, dx - \int_{E_k} \phi(x,k\Delta t)\, dx \right\} , \tag{4.37}$$

whereas formula (4.10) becomes

$$\int_{E_{k+1}} \phi(x,(k+1)\Delta t)\, dx - \int_{E_k} \phi(x,k\Delta t)\, dx = D_1 + D_2 + D_3\, , \tag{4.38}$$

where

$$D_1 := \int_{E^*_{k+1}} \phi(x,k\Delta t)\, dx - \int_{E_k} \phi(x,k\Delta t)\, dx\, , \tag{4.39}$$

$$D_2 := \int_{E_{k+1}} \phi(x,k\Delta t)\, dx - \int_{E^*_{k+1}} \phi(x,k\Delta t)\, dx\, , \tag{4.40}$$

$$D_3 := \int_{E_{k+1}} \phi(x,(k+1)\Delta t)\, dx - \int_{E_{k+1}} \phi(x,k\Delta t)\, dx\, . \tag{4.41}$$

By the Fundamental theorem of Calculus and Fubini's theorem we can immediately calculate

$$D_3 = \int_{k\Delta t}^{(k+1)\Delta t} \int_{E_{k+1}} \frac{\partial \phi}{\partial t}(x,t)\, dxdt\,. \tag{4.42}$$

The summands D_1 and D_2 are, instead, treated as in the time-independent case, with ϕ replaced by $\phi(\cdot, k\Delta t)$. The identity (4.27) then becomes

$$\int_{E_{k+1}} \phi(x,(k+1)\Delta t)\, dx - \int_{E_k} \phi(x, k\Delta t)\, dx = \int_{k\Delta t}^{(k+1)\Delta t} \int_{E_{k+1}} \frac{\partial \phi}{\partial t}(x,t)\, dxdt$$
$$+ \Delta t \left(\int_{\partial^* E_{k+1}} \phi(x, k\Delta t)\, \nu_{E_{k+1}} \cdot h_\varepsilon(x, \partial\mathcal{E}_{k+1})\, d\mathcal{H}^n(x) + \mathrm{Err}_{k+1} \right) . \tag{4.43}$$

Introducing back the subscripts $_{j_\ell}$ and $_i$, we can then write

$$\Delta t \int_{\partial^* E_{k+1}} \phi(x, k\Delta t)\, \nu_{E_{k+1}} \cdot h_\varepsilon(x, \partial\mathcal{E}_{k+1})\, d\mathcal{H}^n(x)$$
$$= \int_{k\Delta t_{j_\ell}}^{(k+1)\Delta t_{j_\ell}} \int_{\partial^* E_{j_\ell,i}(t)} \phi(x,t)\, \nu_{E_{j_\ell,i}(t)} \cdot h_{\varepsilon_{j_\ell}}(x, \partial\mathcal{E}_{j_\ell}(t))\, d\mathcal{H}^n(x)\, dt + \widetilde{\mathrm{Err}}_{k+1}\,, \tag{4.44}$$

where

$$|\widetilde{\mathrm{Err}}_{k+1}| \le \Delta t_{j_\ell}\, \|\partial_t \phi\|_{C^0} \left(\max_{\operatorname{spt}\phi} \Omega^{-1} \right) \int_{k\Delta t_{j_\ell}}^{(k+1)\Delta t_{j_\ell}} \int_{\partial^* E_{j_\ell,i}(t)} \Omega\, |h_{\varepsilon_{j_\ell}}(x, \partial\mathcal{E}_{j_\ell}(t))|\, d\mathcal{H}^n(x)\,. \tag{4.45}$$

Summing over $k = 0, \ldots, k_T - 1$, we can then replace (4.28) with

$$\int_{E_{j_\ell,i}(T)} \phi(x,T)\, dx - \int_{E_{0,i}} \phi(x,0)\, dx = \int_0^{k_T \Delta t_{j_\ell}} \int_{E_{j_\ell,i}(t)} \frac{\partial \phi}{\partial t}(x,t)\, dx\, dt$$
$$+ \int_0^{k_T \Delta t_{j_\ell}} \int_{\mathbb{R}^{n+1}} \phi(x,t)\, h_{\varepsilon_{j_\ell}}(x, \partial\mathcal{E}_{j_\ell}(t)) \cdot d\mu_{E_{j_\ell,i}(t)}(x)\, dt + \widetilde{\mathrm{Err}}_{j_\ell}\,, \tag{4.46}$$

where $\widetilde{\mathrm{Err}}_{j_\ell}$ contains, with respect to Err_{j_ℓ}, an additional error term which can be estimated by

$$|\widetilde{\mathrm{Err}}_{j_\ell} - \mathrm{Err}_{j_\ell}| \le C\, \varepsilon_{j_\ell}^{c_2}\, \|\partial_t \phi\|_{C^0} \left(\max_{\operatorname{spt}\phi} \Omega^{-1} \right) M_{j_\ell}^{1/2} \sqrt{T+1}\, \times$$
$$\times \left(\int_0^{T+1} \int_{\mathbb{R}^{n+1}} \frac{|\Phi_{\varepsilon_{j_\ell}} * \delta(\partial\mathcal{E}_{j_\ell}(t))|^2\, \Omega(y)}{\Phi_{\varepsilon_{j_\ell}} * \|\partial\mathcal{E}_{j_\ell}(t)\| + \varepsilon_{j_\ell}\Omega^{-1}}\, dy \right)^{1/2} , \tag{4.47}$$

where we have used (4.45) in combination with the Cauchy-Schwarz inequality and (4.32). In particular, when $\ell \to \infty$ also $\widetilde{\mathrm{Err}}_{j_\ell} \to 0$. Next, also in this case we can replace $k_T \Delta t_{j_\ell}$ with T in the integrals on the right-hand side of (4.46), paying an additional error term which we estimate by

$$\left| \int_T^{k_T \Delta t_{j_\ell}} \int_{E_{j_\ell,i}(t)} \frac{\partial \phi}{\partial t}\, dx\, dt \right| \le C\, \varepsilon_{j_\ell}^{c_2}\, \|\partial_t \phi\|_{C^0}\, \mathcal{L}^{n+1}((\operatorname{spt}\phi)_1)\,,$$

which vanishes in the limit as $\ell \to \infty$. We can then finally conclude that

$$\int_{E_i(T)} \phi(x,T)\,dx - \int_{E_{i,0}} \phi(x,0)\,dx = \int_0^T \int_{E_i(t)} \frac{\partial \phi}{\partial t}(x,t)\,dx\,dt + \int_0^T \int_{\mathbb{R}^{n+1}} \phi(x,t)\,u_i(x,t)\,d\|V_t\|(x)\,dt\,, \tag{4.48}$$

where $u_i \in L^2(\mu \llcorner_\Omega)$ is defined by (4.34).

Step five: support of u_i. To conclude the proof of Proposition 4.5, it only remains to show that $\mathrm{spt}(u_i) \subset \partial S(i)$, where

$$S(i) = \left\{ (x,t) \in \mathbb{R}^{n+1} \times \mathbb{R}^+ \,:\, x \in E_i(t) \right\}\,.$$

Notice that $S(i)$ is open (relatively to $\mathbb{R}^{n+1} \times \mathbb{R}^+$) by [18, Theorem 3.5 (5)]. Suppose now that $(\hat x, \hat t) \notin \partial S(i)$. Then, either $(\hat x, \hat t) \in S(i)$ or $(\hat x, \hat t) \notin \mathrm{clos}(S(i))$. In the first case, there is a neighborhood $(\hat x, \hat t) \in U \subset \mathbb{R}^{n+1} \times \mathbb{R}^+$ such that $x \in E_i(t)$ for all $(x,t) \in U$. In particular, $(\hat x, \hat t) \notin \mathrm{spt}\mu \cup (\mathrm{spt}\|V_0\| \times \{0\})$, and since σ_i is absolutely continuous with respect to μ, also $(\hat x, \hat t) \notin \mathrm{spt}(\sigma_i) = \mathrm{spt}(u_i)$.
In the second case, we have that, necessarily, $\hat x \notin \mathrm{clos}(E_i(\hat t))$. Of course, if $\hat x \in E_{i'}(\hat t)$ for some $i' \neq i$ then $(\hat x, \hat t) \in S(i')$, and the above proof implies again that $(\hat x, \hat t) \notin \mathrm{spt}\mu \cup (\mathrm{spt}\|V_0\| \times \{0\})$, which then gives $(\hat x, \hat t) \notin \mathrm{spt}(u_i)$. Thus, we can assume that $\hat x \in \Gamma(\hat t) \setminus \mathrm{clos}(E_i(\hat t))$. Since $(\hat x, \hat t) \notin \mathrm{clos}(S(i))$, there is an open neighborhood $(\hat x, \hat t) \in U \subset \mathbb{R}^{n+1} \times \mathbb{R}^+$ such that $U \cap \mathrm{clos}(S(i)) = \emptyset$, and we can choose U of the form $U = U_r(\hat x) \times [0, r^2)$ if $\hat t = 0$ or $U = U_r(\hat x) \times \left(\hat t - r^2, \hat t + r^2\right)$ if $\hat t > 0$. In both cases, if $(x,t) \in U$ then $x \notin E_i(t)$. For any function $\phi \in C^1_c(U)$, the identity (4.48) then gives

$$0 = \iint_U \phi(x,t)\,u_i(x,t)\,d\|V_t\|(x)\,dt = \iint_U \phi(x,t)\,\Omega^{-1}(x)\,d\sigma_i(x,t)\,. \tag{4.49}$$

Since ϕ is arbitrary, we deduce then that $|\sigma_i|(U) = 0$, and thus that $(\hat x, \hat t) \notin \mathrm{spt}(\sigma_i) = \mathrm{spt}(u_i)$. The proof of the theorem is now complete. □

4.3. **Boundaries move by their mean curvature.** In this subsection we complete the proof of Theorem 4.1, by achieving the representation for the measure-theoretic time derivative $\partial_t \chi_i$ as specified in Remark 4.2. The crucial step is to prove the following lemma, for which the L^2 flow property of $\{V_t\}$ plays a pivotal role. We shall denote $\mathbf{p}$ and $\mathbf{q}$ the projections of $\mathbb{R}^{n+1} \times \mathbb{R}$ onto its factors, so that $\mathbf{p}(x,t) = x$ and $\mathbf{q}(x,t) = t$.

Lemma 4.7. *For every $i \in \{1, \ldots, N\}$, there is a set $G_i \subset \partial^* S(i)$ with $\mathcal{H}^{n+1}(\partial^* S(i) \setminus G_i) = 0$ such that the following holds. For every $(x,t) \in G_i$:*

1. *the tangent $T_{(x,t)}\mu$ exists, and $T_{(x,t)}\mu = T_{(x,t)}(\partial^* S(i))$;*
2. *(4.6) holds;*
3. *$x \in \partial^* E_i(t)$, and $T_x\|V_t\| = T_x(\partial^* E_i(t))$;*
4. *$\mathbf{p}(\nu_{S(i)}(x,t)) \neq 0$, and $\nu_{E_i(t)}(x) = |\mathbf{p}(\nu_{S(i)}(x,t))|^{-1}\, \mathbf{p}(\nu_{S(i)}(x,t))$;*
5. *$T_x(\partial^* E_i(t)) \times \{0\}$ is a linear subspace of $T_{(x,t)}\mu$.*

The proof of Lemma 4.7 can be deduced with relatively little effort from arguments already contained in [26], but we include it for the reader's convenience. Before proceeding, we will need some consequences of the celebrated monotonicity formula by Huisken for mean curvature flow. Since said consequences will be needed also later on in the paper, we record them here. First, let us set some notation. For $(y,s) \in \mathbb{R}^{n+1}\times\mathbb{R}^+$, we define the n-dimensional backwards heat kernel $\rho_{(y,s)}$ by

$$\rho_{(y,s)}(x,t) := \frac{1}{(4\pi(s-t))^{n/2}}\,\exp\left(-\frac{|x-y|^2}{4(s-t)}\right) \qquad \text{for all } t<s \text{ and } x\in\mathbb{R}^{n+1}\,, \tag{4.50}$$

as well as the truncated kernel

$$\hat\rho^r_{(y,s)}(x,t) := \eta\left(\frac{x-y}{r}\right)\,\rho_{(y,s)}(x,t)\,, \tag{4.51}$$

where $r>0$ and $\eta\in C^\infty_c(U_2;\mathbb{R}^+)$ is a radially symmetric function such that $\eta\equiv 1$ on B_1, $0\le\eta\le 1$, $|\nabla\eta|\le 2$ and $\|\nabla^2\eta\|\le 4$. The following is [18, Lemma 10.3], and it is a variant of Huisken's monotonicity formula for MCF.

Lemma 4.8. *There exists $c(n)>0$ with the following property. For every $0\le t_1<t_2<s<\infty$, $y\in\mathbb{R}^{n+1}$, and $r>0$ it holds*

$$\|V_t\|(\hat\rho^r_{(y,s)}(\cdot,t))\Big|_{t=t_1}^{t_2} \le c(n)\,r^{-2}\,(t_2-t_1)\sup_{t\in[t_1,t_2]} r^{-n}\,\|V_t\|(B_{2r}(y))\,. \tag{4.52}$$

As a consequence, one has the following [18, Lemma 10.4], which provides a uniform upper bound on mass density ratios.

Lemma 4.9. *For every $L>1$ there exists $\Lambda=\Lambda(n,L,\Omega,\|\partial\mathcal{E}_0\|(\Omega))\in(1,\infty)$ such that*

$$\sup\left\{r^{-n}\,\|V_t\|(B_r(x))\,:\, r\in(0,1]\,, x\in B_L(0)\,, t\in[L^{-1},L]\right\}\le\Lambda\,. \tag{4.53}$$

Proof of Lemma 4.7. Fix $i\in\{1,\dots,N\}$, and observe that, since $\chi_i\in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^{n+1}\times\mathbb{R}^+)$ is the characteristic function of $S(i)$, it holds $|\nabla'\chi_i| = \mathcal{H}^{n+1}\llcorner_{\partial^*S(i)}$. We then let g_i be the Radon-Nikodým derivative of μ with respect to $|\nabla'\chi_i|$, namely

$$g_i(x,t) = \frac{d\mu}{d|\nabla'\chi_i|}(x,t) := \lim_{r\to 0^+}\frac{\mu(B^{n+2}_r(x,t))}{|\nabla'\chi_i|(B^{n+2}_r(x,t))}\,, \tag{4.54}$$

where $B^{n+2}_r(x,t)$ is the closed ball with radius r and center (x,t) in $\mathbb{R}^{n+1}\times\mathbb{R}$. By the Lebesgue-Radon-Nikodým theorem, it holds $g_i(x,t)<\infty$ for $\mathcal{H}^{n+1}$-a.e. $(x,t)\in\partial^*S(i)$, and

$$\mu = g_i\,\mathcal{H}^{n+1}\llcorner_{\partial^*S(i)} + \mu\llcorner_{\Sigma_i}\,, \qquad \text{for a set } \Sigma_i \text{ with } \mathcal{H}^{n+1}(\partial^*S(i)\cap\Sigma_i)=0\,. \tag{4.55}$$

On the other hand, it is not difficult to see that

$$\mu \ll \mathcal{H}^{n+1}\,. \tag{4.56}$$

Indeed, first notice that, as an immediate corollary of Lemma 4.9, one has

$$\Theta^{*n+1}(\mu,(x,t)) := \limsup_{r\to 0^+}\frac{\mu(B^{n+1}_r(x,t))}{\omega_{n+1}r^{n+1}} < \infty \qquad \text{for every } (x,t)\in\mathbb{R}^{n+1}\times(0,\infty)\,. \tag{4.57}$$

Let then A be a set with $\mathcal{H}^{n+1}(A)=0$, and set, for $Q\in\mathbb{N}$:

$$D_Q := \left\{(x,t)\in\mathbb{R}^{n+1}\times(0,\infty)\,:\,\Theta^{*n+1}(\mu,(x,t))\le Q\right\}\,.$$

Then, by [30, Theorem 3.2], one has

$$\mu(D_Q\cap A)\le 2^{n+1}\,Q\,\mathcal{H}^{n+1}(D_Q\cap A) = 0\,,$$

so that the conclusion $\mu(A) = 0$ follows because

$$\mu\left(\mathbb{R}^{n+1} \times \mathbb{R}^+ \setminus \bigcup_{Q\in\mathbb{N}} D_Q\right) = 0$$

due to (4.57).
Combining (4.55) with (4.56), we immediately have that

$$\mu\llcorner_{\partial^* S(i)} = g_i\, \mathcal{H}^{n+1} \llcorner_{\partial^* S(i)} \,. \tag{4.58}$$

Similarly, by taking the Radon-Nikodým derivative of $|\nabla'\chi_i|$ with respect to μ we see that

$$\frac{d|\nabla'\chi_i|}{d\mu}(x,t) \quad \text{is finite for } \mu\text{-a.e. } (x,t)\,. \tag{4.59}$$

Since $|\nabla'\chi_i| \ll \mu$ by Theorem 2.11(4) and Proposition 4.5, this is also finite for $|\nabla'\chi_i|$-a.e. and (4.54) shows that this is equal to $1/g(x,t)$, so that, in particular, $g(x,t) > 0$ for $\mathcal{H}^{n+1}$-a.e. $(x,t) \in \partial^* S(i)$. Together with (4.58), the latter information implies that $\mu\llcorner_{\partial^* S(i)}$ is an $(n+1)$-rectifiable measure, so that $T_{(x,t)}(\mu\llcorner_{\partial^* S(i)})$ exists and is equal to $T_{(x,t)}(\partial^* S(i))$ for $\mathcal{H}^{n+1}$-a.e. $(x,t) \in \partial^* S(i)$. On the other hand, by [30, Theorem 3.5], for $\mathcal{H}^{n+1}$-a.e. $(x,t) \in \partial^* S(i)$ it holds

$$\lim_{r\to 0^+} \frac{\mu(B_r^{n+2}(x,t) \setminus \partial^* S(i))}{r^{n+1}} = 0\,. \tag{4.60}$$

If $(x_0,t_0) \in \partial^* S(i)$ is such that both (4.60) holds and $\phi \in C_c(B_1^{n+2})$ is arbitrary then

$$\begin{aligned} \limsup_{r\to 0^+} &\left| \int_{(\mathbb{R}^{n+1}\times\mathbb{R}^+)\setminus\partial^* S(i)} \phi(r^{-1}(x-x_0,t-t_0))\, r^{-(n+1)}\, d\mu(x,t) \right| \\ &\le \|\phi\|_0 \lim_{r\to 0^+} \frac{\mu(B_r^{n+2}(x_0,t_0) \setminus \partial^* S(i))}{r^{n+1}} = 0\,. \end{aligned} \tag{4.61}$$

This shows that

$$\begin{aligned} \lim_{r\to 0^+} \int \phi(r^{-1}(x-x_0,t-t_0))\, r^{-(n+1)}\, d\mu(x,t) \\ &= \lim_{r\to 0^+} \int_{\partial^* S(i)} \phi(r^{-1}(x-x_0,t-t_0))\, r^{-(n+1)}\, d\mu(x,t) \\ &= g_i(x_0,t_0) \int_{T_{(x_0,t_0)}(\partial^* S(i))} \phi(x,t)\, d\mathcal{H}^{n+1}(x,t) \end{aligned}$$

for every $\phi \in C_c(\mathbb{R}^{n+1}\times\mathbb{R})$, and for $\mathcal{H}^{n+1}$-a.e. $(x_0,t_0) \in \partial^* S(i)$. This completes the proof of (1), whereas (2) is an immediate consequence of (1) and Corollary 4.4, since, on $\partial^* S(i)$, μ and $\mathcal{H}^{n+1}$ have the same null sets by (4.58).

Next, to prove (3) and (4), we are going to use the coarea formula for sets of finite perimeter, slicing $\partial^* S(i)$ according to hyperplanes $\{\mathbf{q} = t\}$. Set, for $t \in \mathbb{R}^+$, $(\partial^* S(i))_t := \partial^* S(i) \cap \{\mathbf{q} = t\} = \{x : (x,t) \in \partial^* S(i)\}$. Then, using e.g. [23, Theorem 18.11] together with the fact that $S(i) \cap \{\mathbf{q} = t\} = E_i(t)$ we have that

$$\mathcal{H}^n((\partial^* S(i))_t \,\Delta\, \partial^* E_i(t)) = 0 \tag{4.62}$$

for a.e. $t \in \mathbb{R}^+$; moreover, for such t

$$\mathbf{p}(\nu_{S(i)}(x,t)) \neq 0\,, \tag{4.63}$$

$$\nu_{E_i(t)}(x) = \frac{\mathbf{p}(\nu_{S(i)}(x,t))}{|\mathbf{p}(\nu_{S(i)}(x,t))|}\,, \tag{4.64}$$

for $\mathcal{H}^n$-a.e. $x \in (\partial^* S(i))_t$. Let

$$Z := \left\{ t \in \mathbb{R}^+ \,:\, (4.62)\,\text{fails} \right\}\,,$$

and for every $t \in \mathbb{R}^+$ set

$$Z_t := \{ x \in (\partial^* S(i))_t \,:\, x \notin \partial^* E_i(t)\,\text{or}\ (4.63)\text{-}(4.64)\ \text{fail} \}\,,$$

so that $\mathcal{L}^1(Z) = 0$ and, for every $t \notin Z$, $\mathcal{H}^n(Z_t) = 0$. Consider then the Borel function $\kappa(x,t) := \chi_{Z_t}(x)$ on $\mathbb{R}^{n+1} \times \mathbb{R}^+$. We have:

$$\begin{aligned}
0 &= \int_0^\infty \chi_{\mathbb{R}^+ \setminus Z}(t)\, \mathcal{H}^n(Z_t)\, dt = \int_0^\infty \mathcal{H}^n(Z_t)\, dt \\
&= \int_0^\infty \int_{\partial^* S(i) \cap \{\mathbf{q} = t\}} \kappa(x,t)\, d\mathcal{H}^n(x)\, dt \\
&= \int_{\partial^* S(i)} \kappa(x,t)\, \left| \nabla^{\partial^* S(i)} \mathbf{q}(x,t) \right| d\mathcal{H}^{n+1}(x,t)\,,
\end{aligned}$$

where in the last identity we have used the coarea formula [2, Eq. (2.72)], and where $\nabla^{\partial^* S(i)} \mathbf{q}(x,t)$ is the tangential gradient, along $\partial^* S(i)$, of $\mathbf{q}$ at (x,t). Combining now (1) and Corollary (4.4), we see that

$$\begin{pmatrix} h(x,V_t) \\ 1 \end{pmatrix} \in T_{(x,t)}(\partial^* S(i)) \qquad \text{at } \mathcal{H}^{n+1}\text{-a.e. } (x,t) \in \partial^* S(i)\,, \tag{4.65}$$

which readily implies that $\left| \nabla^{\partial^* S(i)} \mathbf{q}(x,t) \right| > 0$ for $\mathcal{H}^{n+1}$-a.e. $(x,t) \in \partial^* S(i)$. Hence, it must be $\kappa(x,t) = 0$ for $\mathcal{H}^{n+1}$-a.e. $(x,t) \in \partial^* S(i)$, thus proving the first part of (3) and (4). At such points, the identity $T_x\|V_t\| = T_x(\partial^* E_i(t))$ is then obtained by repeating the argument in (1) at fixed t.

Finally, (5) is always true at points satisfying (1)-(4): indeed, if $\xi \in T_x(\partial^* E_i(t))$ then

$$\begin{pmatrix} \xi \\ 0 \end{pmatrix} \cdot \nu_{S(i)}(x,t) = \xi \cdot \mathbf{p}(\nu_{S(i)}(x,t)) = |\mathbf{p}(\nu_{S(i)}(x,t))|\, \xi \cdot \nu_{E_i(t)}(x) = 0\,.$$

This completes the proof. □

Proof of Theorem 4.1. By virtue of Proposition 4.5, we only need to show the validity of (4.3). Fix $i \in \{1, \ldots, N\}$. Using that $S(i)$ is a set of locally finite perimeter in $\mathbb{R}^{n+1} \times \mathbb{R}^+$, for any $0 \le t_1 < t_2 < \infty$ and any $\phi \in C^1_c(\mathbb{R}^{n+1} \times (t_1,t_2))$ we have that

$$\begin{aligned}
\int_{\mathbb{R}^{n+1} \times \mathbb{R}^+} \frac{\partial \phi}{\partial t}(x,t)\, \chi_i(x,t)\, dxdt &= \int_{S(i)} \frac{\partial \phi}{\partial t}(x,t)\, dxdt \\
&= \int_{\partial^* S(i)} \phi(x,t)\, \mathbf{q}(\nu_{S(i)}(x,t))\, d\mathcal{H}^{n+1}(x,t)\,.
\end{aligned} \tag{4.66}$$

Let G_i be the set of Lemma 4.7. For every $(x,t) \in G_i$, we have

$$T_{(x,t)}\mu = (T_x(\partial^* E_i(t)) \times \{0\}) \oplus \operatorname{span} \begin{pmatrix} h(x,V_t) \\ 1 \end{pmatrix} , \tag{4.67}$$

and thus

$$\nu_{S(i)}(x,t) = \frac{1}{\sqrt{1+|h(x,V_t)|^2}} \begin{pmatrix} \nu_{E_i(t)}(x) \\ -h(x,V_t)\cdot \nu_{E_i(t)}(x) \end{pmatrix} , \tag{4.68}$$

where we have used that $h(x,V_t) \perp T_x\|V_t\| = T_x(\partial^* E_i(t))$. In particular, since $\mathcal{H}^{n+1}(\partial^* S(i) \setminus G_i) = 0$, (4.66) yields

$$\begin{aligned} &\int_{\mathbb{R}^{n+1}\times\mathbb{R}^+} \frac{\partial \phi}{\partial t}(x,t)\, \chi_i(x,t)\, dxdt \\ &\qquad = -\int_{G_i} \phi(x,t)\, h(x,V_t)\cdot \nu_{E_i(t)}(x)\, \frac{1}{\sqrt{1+|h(x,V_t)|^2}}\, d\mathcal{H}^{n+1}(x,t)\,. \end{aligned} \tag{4.69}$$

Due to (4.68), at points $(x,t) \in G_i$ the coarea factor with respect to the slicing via $\mathbf{q}$ is precisely

$$|\nabla^{\partial^* S(i)} \mathbf{q}(x,t)| = \frac{1}{\sqrt{1+|h(x,V_t)|^2}}\,,$$

so that the chain of identities in (4.69) can be continued as

$$\begin{aligned} &= -\int_0^\infty \int_{G_i \cap \{\mathbf{q}=t\}} \phi(x,t)\, h(x,V_t)\cdot \nu_{E_i(t)}(x)\, d\mathcal{H}^n(x)\, dt \\ &= -\int_0^\infty \int_{\partial^* E_i(t)} \phi(x,t)\, h(x,V_t)\cdot \nu_{E_i(t)}(x)\, d\mathcal{H}^n(x)\, dt\,, \end{aligned} \tag{4.70}$$

again by the coarea formula. This completes the proof. □

5. Ilmanen's density lower bound

This section contains a proof of the following fact (Theorem 5.3 below), from which we may conclude the proofs of Theorem 2.10(2)-(4): if $\{V_t\}_{t\in\mathbb{R}^+}$ is as in Proposition 3.6, then there exists a threshold $\theta_0 > 0$ such that, at "most" points (in a sense to be suitably specified) on the support of the evolving varifolds the mass density is not smaller than θ_0. Such density lower bound is already stated in the work of Ilmanen [15, Section 7.1], and we include here its proof only for the sake of completeness.

Using Lemma 4.8 and Lemma 4.9, we prove next the following lemma, which is a variant of Brakke's clearing out lemma (cf. [5, Section 6.3], [15, Section 6.1], and [18, Lemma 10.6]). Recalling the notation set in (4.50)-(4.51), we define, for $\delta > 0$,

$$\hat\rho^{r,\delta}_y(x) := \hat\rho^r_{(y,t+\delta\, r^2)}(x,t) = \eta\left(\frac{x-y}{r}\right)\, \frac{1}{(4\pi\delta r^2)^{n/2}}\, \exp\left(-\frac{|x-y|^2}{4\delta r^2}\right)\,. \tag{5.1}$$

Lemma 5.1. *For any $L > 1$ there exists $\delta_0 = \delta_0(n, L, \Omega, \|\partial\mathcal{E}_0\|(\Omega)) \in (0,1)$ with the following property. If $(y,t) \in B_L(0)\times[L^{-1}, L]$ and $r \in \left(0, \frac{1}{2}\right)$ are such that*

$$\|V_t\|(\hat\rho^{r,\delta_0}_y) < \frac{1}{2}\,, \tag{5.2}$$

then $(y, t+\delta_0 r^2) \notin \operatorname{spt}\mu$.

Proof. For $\delta_0 \in (0,1)$ to be specified later, suppose towards a contradiction that $(y, t+\delta_0 r^2) \in \mathrm{spt}\mu$. Then, by [18, Lemma 10.1(2)] there is a sequence (y_i, t_i) converging to $(y, t+\delta_0 r^2)$ such that $V_{t_i} \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ and $y_i \in \mathrm{spt}\|V_{t_i}\|$. Since V_{t_i} is an integral varifold $\mathbf{var}(M_{t_i}, \theta_{t_i})$, in any arbitrarily small neighborhood of a point $x \in \mathrm{spt}\|V_{t_i}\|$ there is a point $\tilde{x} \in M_{t_1}$ where M_{t_i} has an approximate tangent plane $\mathrm{Tan}(M_{t_i}, \tilde{x})$ and the density $\Theta_{V_{t_i}}(\tilde{x}) = \theta_{t_i}(\tilde{x})$ is an integer. Thus, we can assume without loss of generality that $\mathrm{Tan}(M_{t_i}, y_i)$ exists and $\Theta_{V_{t_i}}(y_i) \geq 1$. Assume also without loss of generality that $t_i > t$ for all i, and fix $\varepsilon > 0$. Then, apply Lemma 4.8 with $t_1 = t$, $t_2 = t_i$, $s = t_i + \varepsilon$, and $y = y_i$ to get

$$\|V_s\|(\hat{\rho}^r_{(y_i, t_i+\varepsilon)}(\cdot, s))\Big|_{s=t}^{t_i} \leq c(n)\, r^{-2}\, (t_i - t) \sup_{s\in[t,t_i]} r^{-n}\|V_s\|(B_{2r}(y_i))\,. \tag{5.3}$$

Consider first the term

$$\begin{aligned} \|V_{t_i}\|(\hat{\rho}^r_{(y_i,t_i+\varepsilon)}(\cdot, t_i)) &= \int_{M_{t_i}} \eta\left(\frac{x-y_i}{r}\right) \rho_{(y_i,\varepsilon)}(x,0)\, \theta_{t_i}(x)\, d\mathcal{H}^n(x) \\ &\geq \int_{M_{t_i}\cap B_r(y_i)} \rho_{(y_i,\varepsilon)}(x,0)\, \theta_{t_i}(x)\, d\mathcal{H}^n(x)\,. \end{aligned}$$

By changing variable $z = \frac{x-y_i}{\sqrt{\varepsilon}}$ in the integral on the right-hand side, it is not difficult to see that, in the limit as $\varepsilon \to 0^+$, the latter converges to

$$\theta_{t_i}(y_i) \int_{\mathrm{Tan}(M_{t_i}, y_i)} \rho_{(0,1)}(z,0)\, d\mathcal{H}^n(z) = \theta_{t_i}(y_i) \geq 1\,.$$

We can then conclude

$$1 \leq \|V_t\|(\hat{\rho}^r_{(y_i,t_i)}(\cdot, t)) + c(n)\, r^{-2}\, (t_i - t) \sup_{s\in[t,t_i]} r^{-n}\|V_s\|(B_{2r}(y_i))\,. \tag{5.4}$$

We let then $i \to \infty$: using that $(y_i, t_i) \to (y, t+\delta_0 r^2)$, and recalling (5.1) we get from (4.53)

$$\begin{aligned} 1 &\leq \|V_t\|(\hat{\rho}^{r,\delta_0}_y) + c(n)\, \delta_0 \sup_{s\in[t,t+\delta_0 r^2]} r^{-n}\|V_s\|(B_{2r}(y)) \\ &\leq \|V_t\|(\hat{\rho}^{r,\delta_0}_y) + c(n)\, \delta_0\, \Lambda(n, L+1, \Omega, \|\partial\mathcal{E}_0\|(\Omega))\,. \end{aligned} \tag{5.5}$$

Choosing δ_0 such that the second summand in the right-hand side of (5.5) is $\leq \frac{1}{2}$ leads to a contradiction with (5.2). □

Remark 5.2. Observe that, since

$$\|V_t\|(\hat{\rho}^{r,\delta_0}_y) \leq (4\pi\delta_0)^{-n/2}\, r^{-n}\, \|V_t\|(U_{2r}(y))\,,$$

Lemma 5.1 immediately implies the following: if $(y,t) \in B_L(0) \times [L^{-1}, L]$ is such that, for some $r \in (0,1)$,

$$r^{-n}\, \|V_t\|(U_r(y)) < \theta_0 := \frac{(4\pi\delta_0)^{n/2}}{2^{n+1}}\,, \tag{5.6}$$

then $(y, t + 4^{-1}\delta_0 r^2) \notin \mathrm{spt}\mu$.

Theorem 5.3. *For $L > 1$, let $\theta_0 = \theta_0(n, L+1, \Omega, \|\partial\mathcal{E}_0\|(\Omega))$ be the number defined in* (5.6), *and define, for $t \geq 0$, the sets*

$$\begin{aligned} \mathcal{Z}^0 &:= \left\{ (y,t) \in \mathrm{spt}\mu \cap (B_L(0) \times [L^{-1}, L]) :\ \limsup_{r\to 0^+} r^{-n}\, \|V_t\|(U_r(y)) < \theta_0 \right\}, \\ \mathcal{Z}^0_t &:= \mathcal{Z}^0 \cap (\mathbb{R}^{n+1} \times \{t\})\,. \end{aligned}$$

Then, there exists $G \subset \mathbb{R}^+$ with $\mathcal{L}^1(G) = 0$ such that $\mathcal{H}^{n-1+\alpha}(\mathcal{Z}^0_t) = 0$ for every $\alpha > 0$ for every $t \in \mathbb{R}^+ \setminus G$.

Proof. For every $\theta < \theta_0$, and for every $\sigma \in (0,1)$, define

$$\mathcal{Z}^0_{\theta,\sigma} := \Big\{ (y,t) \in \mathrm{spt}\mu \cap (B_L(0) \times [L^{-1}, L]) :\, r^{-n}\, \|V_t\|(U_r(y)) < \theta \quad \text{for every } r \in (0,\sigma) \Big\}\,,$$

so that

$$\mathcal{Z}^0 = \bigcup_{\theta<\theta_0\,,\sigma\in(0,1)} \mathcal{Z}^0_{\theta,\sigma}\,. \tag{5.7}$$

Let $(y,t) \in \mathcal{Z}^0_{\theta,\sigma}$. For any $t' \in \big(t, t+4^{-2}\delta_0\sigma^2\big]$, let $r>0$ be such that $r^2 = 4\,\delta_0^{-1}\,(t'-t)$ (so that necessarily $r \in (0,\sigma/2])$, and let $y' \in B_{\gamma r}(y)$ (with $\gamma \in (0,1)$ to be specified) be arbitrary. It holds then $(y',t) \in B_{L+1}(0) \times [(L+1)^{-1}, L+1]$. Furthermore, since $U_r(y') \subset U_{(1+\gamma)r}(y)$, and since $(1+\gamma)r < 2r < \sigma$, it holds

$$r^{-n}\,\|V_t\|(U_r(y')) < (1+\gamma)^n\,\theta\,. \tag{5.8}$$

Hence, for γ small enough (depending on the ratio θ_0/θ) it holds $r^{-n}\,\|V_t\|(U_r(y')) < \theta_0$, so that Remark 5.2 implies that $(y', t+4^{-1}\delta_0 r^2) = (y',t') \notin \mathrm{spt}\mu$. In particular, $(y',t') \notin \mathcal{Z}^0_{\theta,\sigma}$. Analogously, if $t' \in \big[t-4^{-2}\delta_0\sigma^2, t\big)$ and $y' \in B_{\gamma r}(y)$ for $r^2 = 4\delta_0^{-1}(t-t')$ and γ small enough, we have that necessarily $(y',t') \notin \mathcal{Z}^0_{\theta,\sigma}$. For otherwise, it would be $((1+\gamma)r)^{-n}\,\|V_{t'}\|(U_{(1+\gamma)r}(y')) < \theta$, and thus

$$r^{-n}\,\|V_{t'}\|(U_r(y)) < (1+\gamma)^n\,\theta < \theta_0\,, \tag{5.9}$$

which would then imply $(y, t'+4^{-1}\delta_0 r^2) = (y,t) \notin \mathrm{spt}\mu$, a contradiction.

We have then concluded the following dichotomy:

$$\begin{aligned} &\text{either } (y,t) \notin \mathcal{Z}^0_{\theta,\sigma}\,,\\ &\text{or } (y',t') \notin \mathcal{Z}^0_{\theta,\sigma} \text{ whenever } 0 < |t-t'| \le 4^{-2}\delta_0\sigma^2 \text{ and } |y'-y|^2 \le 4\,\gamma^2\,\delta_0^{-1}\,|t-t'|\,. \end{aligned} \tag{5.10}$$

Therefore, if $(y,t) \in \mathcal{Z}^0_{\theta,\sigma}$ then the truncated double paraboloid

$$\mathcal{P}(y,t) := \{ |y'-y|^2 \le 4\,\gamma^2\,\delta_0^{-1}\,|t-t'| \le 4^{-1}\,\gamma^2\,\sigma^2 \} \tag{5.11}$$

intersects $\mathcal{Z}^0_{\theta,\sigma}$ only in (y,t). Next, setting $2\,\tau := 4^{-1}\,\gamma^2\,\sigma^2$, we consider sets

$$\mathcal{Z}^0_{\theta,\sigma,y_0,t_0} := \mathcal{Z}^0_{\theta,\sigma} \cap (B_1(y_0) \times [t_0-\tau, t_0+\tau])\,, \qquad (y_0 \in B_L(0)\,, t_0 \in [L^{-1}, L])\,,$$

so that a countable union of such sets covers $\mathcal{Z}^0_{\theta,\sigma}$. Fix any such set, and call it $\mathcal{Z}'$ for the sake of simplicity: it will suffice to show that, setting $\mathcal{Z}'_t := \mathcal{Z}' \cap (\mathbb{R}^{n+1} \times \{t\})$, it holds $\mathcal{H}^{n-1+\alpha}(\mathcal{Z}'_t) = 0$ for every $\alpha > 0$ and a.e. $t \ge 0$. Notice that if $(y,t) \in \mathcal{Z}'$ then, by the definition of τ, the set $\mathcal{Z}' \cap (\{y\} \times \mathbb{R})$ is contained in $\mathcal{P}(y,t)$: in particular, for $y \in B_1(y_0)$ the fiber $\{y\} \times \mathbb{R}$ intersects $\mathcal{Z}'$ in at most one point. Let $\mathbf{p}$ be the coordinate projection $\mathbf{p}(x,t) = x$, fix $\delta > 0$, and cover the set $\mathbf{p}(\mathcal{Z}') \subset B_1(y_0)$ by countably many open balls $U_{r_i}(y_i)$ so that

$$r_i \le \delta\,, \qquad \sum_i \omega_{n+1}\, r_i^{n+1} \le 2\,\mathcal{L}^{n+1}(B_1(y_0))\,. \tag{5.12}$$

For every center y_i of the balls in the covering, let t_i be the only point such that $(y_i,t_i) \in \mathcal{Z}'$, and notice that, as a consequence of the first part of the proof, if $(y,t) \in \mathcal{Z}'$ with $y \in U_{r_i}(y_i)$

then necessarily $|t-t_i| < 4^{-1}\gamma^{-2}\delta_0 r_i^2$. In other words, for δ suitably small the cylinders $U_{r_i}(y_i)\times\big(t_i-4^{-1}\gamma^{-2}\delta_0 r_i^2, t_i+4^{-1}\gamma^{-2}\delta_0 r_i^2\big)$ are a covering of $\mathcal{Z}'$. We can then estimate

$$\begin{aligned}\int_{t_0-\tau}^{t_0+\tau}\mathcal{H}^{n-1+\alpha}_\delta(\mathcal{Z}'_t)\,dt &\le \int_{t_0-\tau}^{t_0+\tau}\sum_{i\,:\,|t-t_i|<4^{-1}\gamma^{-2}\delta_0 r_i^2}\omega_{n-1+\alpha}\, r_i^{n-1+\alpha}\,dt\\ &\le \sum_i\int_{t_i-4^{-1}\gamma^{-2}\delta_0 r_i^2}^{t_i+4^{-1}\gamma^{-2}\delta_0 r_i^2}\omega_{n-1+\alpha}\, r_i^{n-1+\alpha}\,dt\\ &\le C(n,\alpha)\,\gamma^{-2}\,\delta_0\,\delta^\alpha\,,\end{aligned}$$

where we have used (5.12). Letting $\delta\to 0^+$, we find then

$$\int_{t_0-\tau}^{t_0+\tau}\mathcal{H}^{n-1+\alpha}(\mathcal{Z}'_t)\,dt = 0 \tag{5.13}$$

by monotone convergence, and hence, by taking countable unions,

$$\int_0^\infty\mathcal{H}^{n-1+\alpha}((\mathcal{Z}^0_{\theta,\sigma})_t)\,dt = 0\,, \tag{5.14}$$

with the obvious meaning of the symbols, and taking into account that $\mathcal{Z}'_t$ is empty when $t<L^{-1}$ or $t>L$. The conclusion follows from (5.7). □

The following is the immediate corollary of Theorem 5.3, which proves Theorem 2.10(2).

Corollary 5.4. *There exists $G\subset\mathbb{R}^+$ with $\mathcal{L}^1(G)=0$ such that for every $t\in\mathbb{R}^+\setminus G$:*

$$\mathcal{H}^{n-1+\alpha}\left(\left\{y\in\mathbb{R}^{n+1}\,:\,(y,t)\in\mathrm{spt}\mu\quad\textit{and}\quad\lim_{r\to0^+}r^{-n}\,\|V_t\|(U_r(y))=0\right\}\right)=0\qquad\forall\,\alpha>0\,. \tag{5.15}$$

In particular, recalling from Theorem 2.10(1) and Theorem 2.11(3) that $\mathrm{spt}\|V_t\|\subset\{x\,:\,(x,t)\in\mathrm{spt}\,\mu\}=\Gamma(t)$*, the set* $\Gamma(t)=\mathbb{R}^{n+1}\setminus\bigcup_{i=1}^N E_i(t)=\bigcup_{i=1}^N\partial E_i(t)$ *is* $\mathcal{H}^n$*-equivalent to* $\mathrm{spt}\|V_t\|$ *for a.e.* $t\ge 0$*, and in fact*

$$\dim_{\mathcal{H}}(\Gamma(t)\setminus\mathrm{spt}\|V_t\|)\le n-1\qquad\textit{for a.e. } t\ge0\,. \tag{5.16}$$

Since $V_t\in\mathbf{IV}_n(\mathbb{R}^{n+1})$ for a.e. $t\ge0$, there exists a countably n-rectifiable set $\tilde\Gamma(t)$ such that $V_t=\mathbf{var}(\tilde\Gamma(t),\theta_t)$ and $\theta_t(x)=\Theta^n(\|V_t\|,x)$. By definition, we have $\mathcal{H}^n(\tilde\Gamma(t)\setminus\mathrm{spt}\|V_t\|)=0$ and by [30, Theorem 3.5], $\Theta^n(\|V_t\|,x)=0$ for $\mathcal{H}^n$-a.e. $x\in\mathrm{spt}\|V_t\|\setminus\tilde\Gamma(t)$. The latter claim shows $\mathrm{spt}\|V_t\|\setminus\tilde\Gamma(t)$ is included in the set appearing in (5.15), and we can conclude that $\tilde\Gamma(t)$ is $\mathcal{H}^n$-equivalent to $\mathrm{spt}\|V_t\|$. This proves Theorem 2.10(3)(4).

6. Two-sidedness at unit density point

In this section, we prove Theorem 2.11(7)(8), which are restated in Prposition 6.3 and 6.4, respectively. To do so, we need to analyze the behavior of approximating flows. The first Lemma 6.1 shows that, for a.e. t, only the reduced boundaries of approximating grains contribute to the limit measure and that the measures coming from “interior boundaries” $\partial E_{j_\ell,k}(t)\setminus\partial^* E_{j_\ell,k}(t)$ vanish in the limit. Roughly speaking, this is due to the measure minimizing property in the length scale of $o(1/j^2)$ which has the effect of eliminating the interior boundaries.

Lemma 6.1. *For a.e. $t \in [0,\infty)$ and $\mathcal{E}_{j_\ell}(t) = \{E_{j_\ell,k}(t)\}_{k=1}^N$, we have*

$$\lim_{\ell\to\infty} \sum_{k=1}^N \|\partial^* E_{j_\ell,k}(t)\| = 2\mu_t. \tag{6.1}$$

Proof. We fix t such that

$$\lim_{\ell\to\infty} \frac{\Delta^{vc}_{j_\ell}\|\partial\mathcal{E}_{j_\ell}(t)\|(\Omega)}{j_\ell \Delta t_{j_\ell}} = 0, \tag{6.2}$$

which holds for a.e. $t \in [0,\infty)$ and drop t for simplicity in the following. It is sufficient to prove that

$$\lim_{\ell\to\infty} \mathcal{H}^n(U_R \cap \mathrm{spt}\|\partial\mathcal{E}_{j_\ell}\| \setminus \cup_{k=1}^N \partial^* E_{j_\ell,k})) = 0$$

for arbitrary $R \geq 1$ since $\mathrm{spt}\|\partial\mathcal{E}_{j_\ell}\| \subset \cup_{k=1}^N \partial E_{j_\ell,k}$, $\mathcal{H}^n(\cup_{k=1}^N \partial E_{j_\ell,k} \setminus \mathrm{spt}\|\partial\mathcal{E}_{j_\ell}\|) = 0$ and

$$2\|\partial\mathcal{E}_{j_\ell}\| = \sum_{k=1}^N \|\partial^* E_{j_\ell,k}\| + 2\mathcal{H}^n \llcorner_{\mathrm{spt}\|\partial\mathcal{E}_{j_\ell}\|\setminus\cup_{k=1}^N \partial^* E_{j_\ell,k}}.$$

As in [19, Section 4], let $r_\ell := 1/(j_\ell)^{2.5}$. Define

$$Z_\ell := \{x \in U_R \cap \mathrm{spt}\|\partial\mathcal{E}_{j_\ell}\| \,:\, \|\partial\mathcal{E}_{j_\ell}\|(B_{r_\ell}(x)) \leq c_3 r_\ell^n\} \text{ and } Z_\ell^c := U_R \cap \mathrm{spt}\|\partial\mathcal{E}_{j_\ell}\| \setminus Z_\ell,$$

where c_3 is the same constant appearing in [18, Proposition 7.2] and apply it to each ball $B_{r_\ell}(\hat{x})$ with $\hat{x} \in Z_\ell$ to estimate $\|\partial\mathcal{E}_{j_\ell}\|(B_{r_\ell/2}(\hat{x}))$: there exists $c_4 = c_4(n)$ as in the claim and a $\mathcal{E}_{j_\ell}$-admissible function f and $r \in [r_\ell/2, r_\ell]$ such that

(1) $f(x) = x$ for $x \in \mathbb{R}^{n+1} \setminus U_r(\hat{x})$,
(2) $f(x) \in B_r(\hat{x})$ for $x \in B_r(\hat{x})$,
(3) $\|\partial f_\star \mathcal{E}_{j_\ell}\|(B_r(\hat{x})) \leq \frac{1}{2}\|\partial\mathcal{E}_{j_\ell}\|(B_r(\hat{x}))$,
(4) $\mathcal{L}^{n+1}(E_{j_\ell,k} \triangle \tilde{E}_{j_\ell,k}) \leq c_4(\|\partial\mathcal{E}_{j_\ell}\|(B_r(\hat{x})))^{\frac{n+1}{n}}$ for all k, where $\{\tilde{E}_{j_\ell,k}\}_{k=1}^N = f_\star \mathcal{E}_{j_\ell}$.

We may use Lemma 3.5 with $C = B_r(\hat{x})$ and above (1)-(4) to check that $f \in \mathbf{E}^{vc}(\mathcal{E}_{j_\ell}, j_\ell)$ as long as j_ℓ is large enough (actually if $\exp(-j_\ell/2r_\ell) = \exp(-j_\ell^{3/2}/2) < 1/2$). Then, by the definition of $\Delta^{vc}_{j_\ell}\|\partial\mathcal{E}_{j_\ell}\|(\Omega)$, (1) and (3) as well as $\max_{B_r(\hat{x})} \Omega \leq \exp(2c_1 r_\ell) \min_{B_r(\hat{x})} \Omega$, we have

$$\begin{aligned}
\Delta^{vc}_{j_\ell}\|\partial\mathcal{E}_{j_\ell}\|(\Omega) &\leq \|\partial f_\star \mathcal{E}_{j_\ell}\|(\Omega) - \|\partial\mathcal{E}_{j_\ell}\|(\Omega) \\
&\leq (\min_{B_r(\hat{x})} \Omega)\big\{\exp(2c_1 r_\ell)\|\partial f_\star\mathcal{E}_{j_\ell}\|(B_r(\hat{x})) - \|\partial\mathcal{E}_{j_\ell}\|(B_r(\hat{x}))\big\} \\
&\leq (\min_{B_r(\hat{x})} \Omega)(\frac{1}{2}\exp(2c_1 r_\ell) - 1)\|\partial\mathcal{E}_{j_\ell}\|(B_r(\hat{x})) \\
&\leq -\frac{\min_{B_{2R}} \Omega}{4}\|\partial\mathcal{E}_{j_\ell}\|(B_r(\hat{x})) \leq -\frac{\min_{B_{2R}} \Omega}{4}\|\partial\mathcal{E}_{j_\ell}\|(B_{r_\ell/2}(\hat{x}))
\end{aligned} \tag{6.3}$$

for all sufficiently large j_ℓ. By the Besicovitch covering theorem, we have a mutually disjoint set of closed balls $\{B_{r_\ell/2}(\hat{x}_i)\}_{\hat{x}_i \in Z_\ell}$ (whose number is at most $c(n) r_\ell^{-n-1}$) such that

$$\|\partial\mathcal{E}_{j_\ell}\|(Z_\ell) \leq \mathbf{B}(n) \sum \|\partial\mathcal{E}_{j_\ell}\|(B_{r_\ell/2}(\hat{x}_i)) \tag{6.4}$$

and (6.3) and (6.4) show that

$$\|\partial\mathcal{E}_{j_\ell}\|(Z_\ell) \leq -\frac{c(n)}{r_\ell^{n+1} \min_{B_{2R}} \Omega} \Delta^{vc}_{j_\ell}\|\partial\mathcal{E}_{j_\ell}\|(\Omega). \tag{6.5}$$

Since $-\Delta_{j_\ell}^{vc}\|\partial\mathcal{E}_{j_\ell}\|(\Omega) \ll r_\ell^{n+1}$ due to (6.2), the right-hand side of (6.5) converges to 0 as $\ell\to\infty$. By (6.5), we need to prove

$$\lim_{\ell\to\infty}\mathcal{H}^n(Z_\ell^c\setminus\cup_{k=1}^N\partial^*E_{j_\ell,k})=0. \tag{6.6}$$

By the Besicovitch covering theorem, we have a set of mutually disjoint balls $\{B_{r_\ell}(x_i)\}$ with $x_i\in Z_\ell^c\setminus\cup_{k=1}^N\partial^*E_{j_\ell,k}$ such that

$$\mathcal{H}^n(Z_\ell^c\setminus\cup_{k=1}^N\partial^*E_{j_\ell,k})\le\mathbf{B}(n)\sum_i\mathcal{H}^n(B_{r_\ell}(x_i)\cap Z_\ell^c\setminus\cup_{k=1}^N\partial^*E_{j_\ell,k}). \tag{6.7}$$

Because of the lower bound of measure $\|\partial\mathcal{E}_{j_\ell}\|(B_{r_\ell}(x_i))\ge c_3r_\ell^n$ for $x_i\in Z_\ell^c$, the number of disjoint balls is bounded by $c_3^{-1}r_\ell^{-n}\|\partial\mathcal{E}_{j_\ell}\|(B_{2R})$. If we prove that

$$\lim_{\ell\to\infty}\sup_i r_\ell^{-n}\mathcal{H}^n(B_{r_\ell}(x_i)\cap Z_\ell^c\setminus\cup_{k=1}^N\partial^*E_{j_\ell,k})=0, \tag{6.8}$$

combined with (6.7), we would prove (6.6), ending the proof. Assume for a contradiction that (6.8) were not true and we had a subsequence (denoted by the same index) $x_\ell\in Z_\ell^c$ such that

$$0<\alpha\le r_\ell^{-n}\mathcal{H}^n(B_{r_\ell}(x_\ell)\cap Z_\ell^c\setminus\cup_{k=1}^N\partial^*E_{j_\ell,k}). \tag{6.9}$$

Consider a rescaling $F_\ell:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ defined by $F_\ell(x)=(x-x_\ell)/r_\ell$ and consider a sequence $V_\ell:=(F_\ell)_\sharp(\partial\mathcal{E}_{j_\ell})$ and $F_\ell(E_{j_\ell,k})$ $(k=1,\ldots,N)$. This rescaling was discussed in [19, Section 4], and there exists a subsequence (denoted by the same index) and a limit $V\ne0$ with the properties stated in [19, Theorem 4.1], which shows that V is a unit density varifold and $\mathrm{spt}\|V\|$ is a real-analytic minimal hypersurface away from a closed lower-dimensional singularity. Moreover, $\mathrm{spt}\|V\|$ is two-sided in the following sense. Let $E_1,\ldots,E_N\subset\mathbb{R}^{n+1}$ be limits of $F_\ell(E_{j_\ell,1}),\ldots,F_\ell(E_{j_\ell,N})$ (see [19, p.517]). By the lower-semicontinuity of BV function, we have $\|\partial^*E_k\|\le\|V\|$ and $\{E_1,\ldots,E_N\}$ can be defined so that they are mutually disjoint open sets such that $\mathbb{R}^{n+1}\setminus\mathrm{spt}\|V\|=\cup_{k=1}^NE_k$. The two-sidedness means that, at each regular point x of $\mathrm{spt}\|V\|$, there are two distinct indices $k_1,k_2\in\{1,\ldots,N\}$ such that $x\in\overline{E_{k_1}}\cap\overline{E_{k_2}}$ ([19, Lemma 4.8]). In particular, this implies that

$$\|V\|=\frac{1}{2}\sum_{k=1}^N\|\partial^*E_k\|. \tag{6.10}$$

On the other hand, (6.9) implies that

$$\begin{aligned}\|V_\ell\|(U_2)=&\|V_\ell\|(U_2\setminus\cup_{k=1}^N\partial^*(F_\ell(E_{j_\ell,k})))+\|V_\ell\|(U_2\cap\cup_{k=1}^N\partial^*(F_\ell(E_{j_\ell,k})))\\ &\ge r_\ell^{-n}\mathcal{H}^n(U_{2r_\ell}(x_\ell)\cap Z_\ell^c\setminus\cup_{k=1}^N\partial^*E_{j_\ell,k})+\|V_\ell\|(U_2\cap\cup_{k=1}^N\partial^*(F_\ell(E_{j_\ell,k})))\\ &\ge\alpha+\frac{1}{2}\sum_{k=1}^N\|\partial^*(F_\ell(E_{j_\ell,k}))\|(U_2).\end{aligned} \tag{6.11}$$

Since

$$\|\partial^*E_k\|(U_2)\le\liminf_{\ell\to\infty}\|\partial^*(F_\ell(E_{j_\ell,k}))\|(U_2) \tag{6.12}$$

for each k, (6.10)-(6.12) show

$$\|V\|(U_2)\le\liminf_{\ell\to\infty}\|V_\ell\|(U_2)-\alpha\le\|V\|(B_2)-\alpha.$$

Since $\|V\|(\partial U_2)=0$, this is a contradiction. The argument up to this point shows (6.8), which in turn shows the claim (6.1). $\square$

The next Lemma 6.2 shows that $\partial\mathcal{E}_{j_\ell}(t)$ is locally and subsequencially close to a "θ-layered sheets" after appropriate blow-ups, for almost all time and place.

Lemma 6.2. *For $\mathcal{L}^1$-a.e. $t \in \mathbb{R}^+$ and $\mathcal{H}^n$-a.e. $x \in \operatorname{spt}\|V_t\|$, there exist $\theta := \Theta^n(\|V_t\|, x) \in \mathbb{N}$, $T := \operatorname{Tan}^n(\|V\|, x) \in \mathbf{G}(n+1, n)$, $r_\ell \to 0+$, a subsequence $\{j'_\ell\}_{\ell=1}^\infty \subset \{j_\ell\}_{\ell=1}^\infty$ and $\mathcal{H}^n$-measurable sets $W_\ell \subset T \cap B_{r_\ell}$ with the following property (after a change of variable, we may assume that $x = 0$ in the following).*
Define $f_{(r_\ell)}(y) := y/r_\ell$ for $y \in \mathbb{R}^{n+1}$ and $\{E_{j'_\ell,1}, \ldots, E_{j'_\ell,N}\} := \mathcal{E}_{j'_\ell}(t)$. Then we have

$$\lim_{\ell\to\infty} (f_{(r_\ell)})_\sharp \partial\mathcal{E}_{j'_\ell}(t) = \theta|T| \tag{6.13}$$

as varifolds,

$$\mathcal{H}^0(B_{r_\ell} \cap T^{-1}(a) \cap \cup_{i=1}^N \partial E_{j'_\ell,i}) = \theta \tag{6.14}$$

for all $a \in W_\ell$,

$$\lim_{\ell\to\infty} \sup_{a\in W_\ell} \{|T^\perp(x/r_\ell)| \,:\, x \in B_{r_\ell} \cap T^{-1}(a) \cap \cup_{i=1}^N \partial E_{j'_\ell,i}\} = 0, \tag{6.15}$$

$$\lim_{\ell\to\infty} \frac{\mathcal{H}^n(W_\ell)}{\omega_n r_\ell^n} = 1. \tag{6.16}$$

The claims (6.14) *and* (6.15) *also hold with $\partial^* E_{j'_\ell,i}$ in place of $\partial E_{j'_\ell,i}$.*

Proof. Without loss of generality, assume that $t \in \mathbb{R}^+$ satisfies

$$\liminf_{\ell\to\infty} \Big(\int_{\mathbb{R}^{n+1}} \frac{|\Phi_{\varepsilon_{j_\ell}} * \delta(\partial\mathcal{E}_{j_\ell}(t))|^2\Omega}{\Phi_{\varepsilon_{j_\ell}} * \|\partial\mathcal{E}_{j_\ell}(t)\| + \varepsilon_{j_\ell}\Omega^{-1}}\, dx - \frac{1}{\Delta t_{j_\ell}} \Delta_{j_\ell}^{vc} \|\partial\mathcal{E}_{j_\ell}(t)\|(\Omega) \Big) < \infty, \tag{6.17}$$

which holds for a.e. $t \in \mathbb{R}^+$ by (3.16) and Fatou's Lemma. By the compactness theorem of [18, Theorem 8.6], we may conclude that there exists a converging subsequence in the sense of varifold $\{\partial\mathcal{E}_{j_\ell}(t)\}_{\ell=1}^\infty$ (denoted by the same index) and the limit $V_t \in \mathbf{IV}_n(\mathbb{R}^{n+1})$ with $\mu_t = \|V_t\|$. By (3.19), each $\{E_{j_\ell,i}(t)\}_{\ell=1}^\infty$ also converges to $E_i(t)$ in $L^1_{loc}(\mathbb{R}^{n+1})$ for $i = 1, \ldots, N$.

By Corollary 5.4, for a.e. $t \in \mathbb{R}^+$, we have

$$\theta_t\, \mathcal{H}^n \llcorner_{\cup_{i=1}^N \partial E_i(t)} = \theta_t\, \mathcal{H}^n \llcorner_{\operatorname{spt}\|V_t\|} = \|V_t\|, \tag{6.18}$$

where $\theta_t(x) := \Theta^n(\|V_t\|, x)$ is $\mathcal{H}^n$-a.e. integer-valued. Note in particular that $\operatorname{spt}\|V_t\|$ as well as $\cup_{i=1}^N \partial E_i(t)$ are countably n-rectifiable. We fix such a generic t and subsequently drop the dependence of t.

In the following, we use the same argument in the proof of [18, Theorem 8.6]. Let $\{\mathcal{E}_{j_\ell}\}_{\ell=1}^\infty$ be a subsequence (denoted by the same index) such that the quantity in (6.17) is uniformly bounded. Let $U \subset \mathbb{R}^{n+1}$ be a bounded open set. It is sufficient to prove the claim on U. As in [18, p.112-113], for each $j, q \in \mathbb{N}$, let $A_{j,q}$ be a set consisting of all $x \in \operatorname{clos} U$ such that

$$\|\delta(\Phi_{\varepsilon_j} * \partial\mathcal{E}_j)\|(B_r(x)) \le q \|\Phi_{\varepsilon_j} * \partial\mathcal{E}_j\|(B_r(x)) \tag{6.19}$$

for all $r \in (j^{-2}, 1)$ and additionally define

$$A_q := \{x \in \operatorname{clos} U \,:\, \text{there exist } x \in A_{j_\ell,q} \text{ for infinitely many } \ell \text{ with } x_\ell \to x\}. \tag{6.20}$$

Following the argument in [18, p.113], one can prove that

$$\|V\|(U \setminus \cup_{q=1}^\infty A_q) = 0. \tag{6.21}$$

Thus, for $\mathcal{H}^n$-a.e. $x \in \operatorname{spt}\|V\|$, we have some $q \in \mathbb{N}$ such that $x \in A_q$, and additionally, the approximate tangent space exists with multiplicity $\theta \in \mathbb{N}$. Without loss of generality, we may assume that $x = 0$ and write $T := \mathrm{Tan}^n(\|V\|, x)$. Since $0 \in A_q$, there exists a further subsequence of $\{j_\ell\}_{\ell=1}^\infty$ (denoted by the same index) such that $x_{j_\ell} \in A_{j_\ell,q}$ with $\lim_{\ell\to\infty} x_{j_\ell} = 0$. Set $r_\ell = 1/\ell$ and define $f_{(r_\ell)}(x) := x/r_\ell$ and

$$V_{j_\ell} := (f_{(r_\ell)})_\sharp \partial\mathcal{E}_{j_\ell},\ \tilde V_{j_\ell} := (f_{(r_\ell)})_\sharp(\Phi_{\varepsilon_{j_\ell}} * \partial\mathcal{E}_{j_\ell}),\ V^*_{j_\ell} := (f_{(r_\ell)})_\sharp \partial^*\mathcal{E}_{j_\ell}, \tag{6.22}$$

where $\partial^*\mathcal{E}_{j_\ell}$ denotes the unit density varifold defined from $\cup_{i=1}^N \partial^* E_{j_\ell,i}$. We may choose a further subsequence with the following properties:

$$\lim_{\ell\to\infty} V_{j_\ell} = \lim_{\ell\to\infty} \tilde V_{j_\ell} = \theta|T|, \tag{6.23}$$

$$\lim_{\ell\to\infty} \frac{x_{j_\ell}}{r_\ell} = 0, \qquad \lim_{\ell\to\infty} \frac{j_\ell^{-1}}{r_\ell} = 0, \tag{6.24}$$

$$\lim_{\ell\to\infty} r_\ell^{-n}\mathcal{H}^n(B_{r_\ell} \cap \partial\mathcal{E}_{j_\ell} \setminus \partial^*\mathcal{E}_{j_\ell}) = 0. \tag{6.25}$$

Note that the choice with (6.25) is possible due to (6.1). We then proceed verbatim as in [18, p.114-120] with $\nu = \theta + 1$ and $d = \theta$. In particular, we fix $\lambda \in (1, 2)$ such that $\lambda^{n+1}\theta < \theta + 1$ (as in [18, (8.111)]) and use [18, Lemma 8.5]. In summary, with the same notation as in [18], we obtain a sequence of sets $G_\ell^{**} \subset \partial\mathcal{E}_{j_\ell} \cap B_{(\lambda-1)r_\ell}$ with the following properties for all large enough ℓ;

$$\lim_{\ell\to\infty} r_\ell^{-n}\|\partial\mathcal{E}_{j_\ell}\|(B_{(\lambda-1)r_\ell} \setminus G_\ell^{**}) = 0, \tag{6.26}$$

$$\lim_{\ell\to\infty} \sup\{|T^\perp(x/r_\ell)| \,:\, x \in G_\ell^{**}\} = 0, \tag{6.27}$$

$$\sup_{a\in T\cap B_{(\lambda-1)r_\ell}} \mathcal{H}^0(\{x \in G_\ell^{**} \,:\, T(x) = a\}) \le \theta. \tag{6.28}$$

These claims are, respectively, (8.154), (8.156) (stated differently) and (8.159) of [18]. The measurability of G_ℓ^{**} is not stated in [18]. However, since each step to estimate $\|\partial\mathcal{E}_{j_\ell}\|(B_{(\lambda-1)r_\ell} \setminus G_\ell^{**})$ uses covering arguments, if necessary, we may simply take the complement of these coverings with no change of estimates and obtain a possibly smaller G_ℓ^{**} which is a Borel set. We next prove that, writing $\tilde g_\ell(a) := \mathcal{H}^0(\{x \in G_\ell^{**} \,:\, T(x) = a\})$ for $a \in T$, we have

$$\lim_{\ell\to\infty} \frac{\mathcal{H}^n(\{a \in T \cap B_{(\lambda-1)r_\ell} \,:\, \tilde g_\ell(a) = \theta\})}{\omega_n(\lambda-1)^n r_\ell^n} = 1. \tag{6.29}$$

Note that $\tilde g_\ell(a)$ as above on T is $\mathcal{H}^n$-measurable (see [10, Lemma 5.8]). To see (6.29), first, by (6.23) and (6.26), we have

$$\lim_{\ell\to\infty} (f_{(r_\ell)})_\sharp(\partial\mathcal{E}_{j_\ell} \llcorner_{G_\ell^{**}\times\mathbf{G}(n+1,n)}) = \theta|T \cap B_{\lambda-1}| \tag{6.30}$$

as varifolds. We also have

$$\|T_\sharp \circ (f_{(r_\ell)})_\sharp(\partial\mathcal{E}_{j_\ell} \llcorner_{G_\ell^{**}\times\mathbf{G}(n+1,n)})\|(B_{\lambda-1}) = (r_\ell)^{-n}\int_T \tilde g_\ell(a)\, d\mathcal{H}^n(a). \tag{6.31}$$

Since $T_\sharp$ commutes with $\lim_{\ell\to\infty}$ and $T_\sharp|T \cap B_{\lambda-1}| = |T \cap B_{\lambda-1}|$, (6.30) and (6.31) show that

$$\theta\omega_n(\lambda-1)^n = \lim_{\ell\to\infty} (r_\ell)^{-n}\int_T \tilde g_\ell(a)\, d\mathcal{H}^n(a). \tag{6.32}$$

Since $\tilde g_\ell \le \theta$ due to (6.28) (note also $G_\ell^{**} \subset B_{(\lambda-1)r_\ell}$), (6.32) shows

$$0 = \lim_{\ell\to\infty} (r_\ell)^{-n}\int_{T\cap B_{(\lambda-1)r_\ell}} (\theta - \tilde g_\ell)\, d\mathcal{H}^n \ge \lim_{\ell\to\infty} (r_\ell)^{-n}\mathcal{H}^n(T \cap B_{(\lambda-1)r_\ell} \cap \{\tilde g_\ell \le \theta - 1\}). \tag{6.33}$$

Then (6.28) and (6.33) show (6.29). Finally, we define

$$W_\ell := \{a \in T \,:\, \tilde g(a) = \theta\} \setminus \big(T(\partial\mathcal{E}_{j_\ell} \cap B_{(\lambda-1)r_\ell} \setminus G_\ell^{**}) \cup T(\partial\mathcal{E}_{j_\ell} \cap B_{(\lambda-1)r_\ell} \setminus \partial^*\mathcal{E}_{j_\ell})\big). \quad (6.34)$$

Since $\tilde g$ is $\mathcal{H}^n$-measurable, so is W_ℓ. Due to (6.26), (6.25) and (6.29), we can deduce (6.16). For any $a \in W_\ell$, $T^{-1}(a) \cap \partial\mathcal{E}_{j_\ell} \cap B_{(\lambda-1)r_\ell}$ consists of θ points belonging to $G_\ell^{**} \cap \partial^*\mathcal{E}_{j_\ell}$ by (6.34). This proves (6.14) (both with $\partial\mathcal{E}_{j_\ell}$ and $\partial^*\mathcal{E}_{j_\ell}$). This combined with (6.27) also proves (6.15), and we may conclude the proof after renaming $(\lambda-1)r_\ell$ as r_ℓ. □

Proposition 6.3. *For V_t and $\{E_i(t)\}_{i=1}^N$ in Proposition 3.6, for a.e. $t \in \mathbb{R}^+$, we have*

$$\mathcal{H}^n(\{x \,:\, \theta_t(x) = 1\} \setminus \cup_{i=1}^N \partial^* E_i(t)) = 0. \quad (6.35)$$

The function $\theta_t(x)$ is the n-dimensional density of $\|V_t\|$ at x.

Proof. We prove (6.35) for a.e. t such that the conclusion of Lemma 6.2 holds. With such t fixed, we drop t and suppose that (6.35) were not true for a contradiction. We may apply the result of Lemma 6.2 and find a point $x \in \operatorname{spt}\|V\| \setminus \cup_{i=1}^N \partial^* E_i$ with $\Theta^n(\|V\|, x) = 1$ and $T := \operatorname{Tan}^n(\|V\|, x)$. Moreover, by the well-known property of the set of finite perimeter, there exists some $k \in \{1, \ldots, N\}$ such that $\lim_{r\to 0} \frac{\mathcal{L}^{n+1}(E_k \cap B_r(x))}{\omega_{n+1} r^{n+1}} = 1$. Without loss of generality, we may assume that $x = 0$, $T = \{x_{n+1} = 0\}$ and $k = 1$. Since $\chi_{E_{j_\ell,1}} \to \chi_{E_1}$ in L^1_{loc} as $\ell \to \infty$, in choosing the subsequence (denoted by the same index) in Lemma 6.2, we may additionally arrange the choice so that we have

$$\lim_{\ell\to\infty} \frac{\mathcal{L}^{n+1}(E_{j_\ell,1} \cap B_{r_\ell})}{\omega_{n+1} r_\ell^{n+1}} = 1. \quad (6.36)$$

For simplicity, write the sum of the measure theoretic boundaries as $\partial_*\mathcal{E}_{j_\ell} := \cup_{i=1}^N \partial_* E_{j_\ell,i}$. Since $\partial\mathcal{E}_{j_\ell}$ is closed, we remind the reader that

$$\partial^*\mathcal{E}_{j_\ell} \subset \partial_*\mathcal{E}_{j_\ell} \subset \partial\mathcal{E}_{j_\ell}. \quad (6.37)$$

We also define the measure theoretic interior and exterior of $E_{j_\ell,1}$ ([10, Definition 5.13]) as:

$$I_\ell := \Big\{x \in \mathbb{R}^{n+1} \,:\, \lim_{r\to 0} \frac{\mathcal{L}^{n+1}(B_r(x) \setminus E_{j_\ell,1})}{r^n} = 0\Big\} \quad (6.38)$$

and

$$O_\ell := \Big\{x \in \mathbb{R}^{n+1} \,:\, \lim_{r\to 0} \frac{\mathcal{L}^{n+1}(B_r(x) \cap E_{j_\ell,1})}{r^n} = 0\Big\}. \quad (6.39)$$

We note from the definition that

$$I_\ell \cap \partial_*\mathcal{E}_{j_\ell} = \emptyset. \quad (6.40)$$

We next use some results in the proof of [10, Theorem 5.23] on the property of measure theoretic interior and exterior. For each $m, k \in \mathbb{N}$, define

$$\begin{aligned} G_\ell(k) &:= \Big\{x \in \mathbb{R}^{n+1} \,:\, \mathcal{L}^{n+1}(B_r(x) \cap O_\ell) \le \frac{\omega_n r^{n+1}}{3^{n+2}} \text{ for } 0 < r < \frac{3}{k}\Big\}, \\ H_\ell(k) &:= \Big\{x \in \mathbb{R}^{n+1} \,:\, \mathcal{L}^{n+1}(B_r(x) \cap I_\ell) \le \frac{\omega_n r^{n+1}}{3^{n+2}} \text{ for } 0 < r < \frac{3}{k}\Big\}, \end{aligned} \quad (6.41)$$

and

$$\begin{aligned} G_\ell^{\pm}(k,m) &:= G_\ell(k) \cap \Big\{x \,:\, x \pm s e_{n+1} \in O_\ell \text{ for } 0 < s < \frac{3}{m}\Big\}, \\ H_\ell^{\pm}(k,m) &:= H_\ell(k) \cap \Big\{x \,:\, x \pm s e_{n+1} \in I_\ell \text{ for } 0 < s < \frac{3}{m}\Big\}. \end{aligned} \quad (6.42)$$

Here e_{n+1} is the unit vector pointing towards the positive direction of x_{n+1}-axis. These sets have the property (see Step 3 of [10, Theorem 5.23]) that

$$\mathcal{L}^n(T(G_\ell^{\pm}(k,m))) = \mathcal{L}^n(T(H_\ell^{\pm}(k,m))) = 0 \tag{6.43}$$

for all $k, m \in \mathbb{N}$. Moreover, for all

$$a \in T \setminus \cup_{k,m=1}^{\infty} T\big(G_\ell^{*}(k,m) \cup G_\ell^{-}(k,m) \cup H_\ell^{+}(k,m) \cup H_\ell^{-}(k,m)\big) \tag{6.44}$$

and with $\mathcal{H}^0(T^{-1}(a) \cap \partial_* E_{j_\ell,1}) < \infty$, if $x_1, x_2 \in T^{-1}(a)$ with $T^{\perp}(x_1) < T^{\perp}(x_2)$ and $x_1 \in I_\ell$ and $x_2 \in O_\ell$, then there exists $x_3 \in T^{-1}(a) \cap \partial_* E_{j_\ell,1}$ such that $T^{\perp}(x_1) < T^{\perp}(x_3) < T^{\perp}(x_2)$ (see Step 5 of [10, Theorem 5.23]). Here, x_1 is in interior and x_2 is in exterior of $E_{j_\ell,1}$ over a, and the claim is that there must be a "boundary point" x_3 between these two points. The same claim holds if $x_1 \in O_\ell$ and $x_2 \in I_\ell$ instead.

Let W_ℓ be the set obtained in Lemma 6.2. By [10, Lemma 5.9], we have $\mathcal{L}^{n+1}(I_\ell \triangle E_{j_\ell,1}) = 0$, so with (6.36),

$$\lim_{\ell \to \infty} \frac{\mathcal{L}^{n+1}(I_\ell \cap B_{r_\ell})}{\omega_{n+1} r_\ell^{n+1}} = 1. \tag{6.45}$$

Then, by the Fubini Theorem and (6.45), we may choose a sequence $\{b_\ell\}_{\ell=1}^{\infty} \subset \mathbb{R}^+$ such that $b_\ell \in [r_\ell/3, r_\ell/2]$ and so that, writing

$$A_\ell^+ := B_{r_\ell} \cap \{x_{n+1} = b_\ell\},$$
$$A_\ell^- := B_{r_\ell} \cap \{x_{n+1} = -b_\ell\},$$

we have

$$\lim_{\ell \to \infty} \frac{\mathcal{H}^n(I_\ell \cap A_\ell^+)}{\mathcal{H}^n(A_\ell^+)} = \lim_{\ell \to \infty} \frac{\mathcal{H}^n(I_\ell \cap A_\ell^-)}{\mathcal{H}^n(A_\ell^-)} = 1. \tag{6.46}$$

On cylinders $T(A_\ell^+) \times [-b_\ell, b_\ell] \subset \mathbb{R}^{n+1}$, by (6.41), (6.42) and the stated property thereafter, for $\mathcal{H}^n$-a.e. $a \in W_\ell$, we have the following property:

$$\begin{gathered} \text{if } (a, s_1) \in I_\ell \text{ and } (a, s_2) \in O_\ell \text{ with } -b_\ell \le s_1 < s_2 \le b_\ell, \\ \text{then there exists } \hat{s} \in (s_1, s_2) \text{ such that } (a, \hat{s}) \in \partial_* E_{j_\ell,1}, \end{gathered} \tag{6.47}$$

and similarly if $(a, s_1) \in O_\ell$ and $(a, s_2) \in I_\ell$. On the other hand, we know that $T^{-1}(a) \cap B_{r_\ell} \cap \partial_* \mathcal{E}_{j_\ell}$ is a singleton located close to T due to (6.14) and (6.15). Here, we used the fact that (6.14) is satisfied for both $\partial \mathcal{E}_{j_\ell}$ and $\partial^* \mathcal{E}_{j_\ell}$ as well as (6.37). We use this fact to

$$a \in \tilde{W}_\ell \cap T(I_\ell \cap A_\ell^+) \cap T(I_\ell \cap A_\ell^-) =: W_\ell^*.$$

Note that (a, r_ℓ) and $(a, -r_\ell)$ are both in I_ℓ and $(\{a\} \times [-r_\ell, r_\ell]) \cap \partial_* \mathcal{E}_{j_\ell}$ is a singleton due to the way $\tilde{W}_\ell$ is defined. If $(\{a\} \times [-r_\ell, r_\ell]) \cap O_\ell \neq \emptyset$, (6.47) implies that there must be at least two points of $\partial_* E_{j_\ell,1}$ in $\{a\} \times [-r_\ell, r_\ell]$, since both crossing from I_ℓ to O_ℓ and the other way around have to happen. Since $\partial_* \mathcal{E}_{j_\ell} = \cup_{i=1}^{N} \partial_* E_{j_\ell,i}$, this is a contradiction. Combined with (6.40), we conclude that for $\mathcal{H}^n$-a.e. $a \in W_\ell^*$, $\{a\} \times [-r_\ell, r_\ell]$ is a disjoint union of one point of $\partial_* \mathcal{E}_{j_\ell}$ and two line segments included in I_ℓ, with no point of O_ℓ. Because of (6.16) and (6.46), one also sees

$$\lim_{\ell \to \infty} \frac{\mathcal{H}^n(W_\ell^*)}{\mathcal{H}^n(A_\ell^+)} = 1. \tag{6.48}$$

In particular, W_ℓ^* has a positive $\mathcal{H}^n$ measure in $T \subset \mathbb{R}^n \times \{0\}$ and there must be a Lebesgue point a of W_ℓ^* such that $(\{a\} \times [-r_\ell, r_\ell]) \cap \partial_* \mathcal{E}_{j_\ell}$ is a singleton, say, $\{(a,s)\}$. Then, by the Fubini theorem and the property of W_ℓ^*,

$$\begin{aligned} r^{-(n+1)}\mathcal{L}^{n+1}(O_\ell \cap B_r((a,s))) &\le r^{-(n+1)}\mathcal{L}^{n+1}((B_r^n(a) \setminus W_\ell^*) \times [s-r, s+r]) \\ &\le 2r^{-n}\mathcal{H}^n(B_r^n(a) \setminus W_\ell^*) \end{aligned}$$

which converges to 0 as $r \to 0$ since a is a Lebesgue point of W_ℓ^* in T. Since $\mathcal{L}^{n+1}(O_\ell \cap B_r(x)) = \mathcal{L}^{n+1}(B_r(x) \setminus E_{j_\ell,1})$, this implies that $(a,s) \in I_\ell$. On the other hand, $(a,s) \in \partial_* \mathcal{E}_{j_\ell}$, a contradiction to (6.40). This concludes the proof. □

Proposition 6.4. *Assume $N = 2$. For V_t and $\{E_i(t)\}_{i=1}^2$ in Proposition 3.6, for a.e. $t \in \mathbb{R}^+$, we have*

$$\theta_t(x) = \begin{cases} odd & \mathcal{H}^n \text{ a.e. } x \in \partial^* E_1(t) (= \partial^* E_2(t)), \\ even & \mathcal{H}^n \text{ a.e. } x \in \operatorname{spt}\|V_t\| \setminus \partial^* E_1(t). \end{cases} \tag{6.49}$$

Proof. The proof proceeds similarly as the proof of Proposition 6.3, except that we need to localize the argument to each layers. Fix a bounded open set $U \subset \mathbb{R}^{n+1}$. We may choose a generic $t \in \mathbb{R}^+$ as before and drop the subscript t. Since $\partial^* E_1 = \partial^* E_2$ by the definition of the reduced boundary, for $\mathcal{H}^n$-a.e. $x \in \operatorname{spt}\|V\| \setminus \partial^* E_1$, we have

$$\lim_{r\to 0} \frac{\mathcal{L}^{n+1}(E_i \cap B_r(x))}{\omega_{n+1} r^{n+1}} = 1 \text{ for either } i = 1 \text{ or } 2. \tag{6.50}$$

On the other hand, for $\mathcal{H}^n$-a.e. $x \in \partial^* E_1$, there exists a unit outer normal ν to $\partial^* E_1$ such that, letting $B_r^{+(-)}(x) := \{y \in B_r(x) \,:\, (y-x)\cdot\nu \ge (\le) 0\}$, we have

$$\lim_{r\to 0} \frac{\mathcal{L}^{n+1}(E_1 \cap B_r^+(x))}{\omega_{n+1} r^{n+1}} = 0 \;\text{ and }\; \lim_{r\to 0} \frac{\mathcal{L}^{n+1}(E_1 \cap B_r^-(x))}{\omega_{n+1} r^{n+1}} = \frac{1}{2}. \tag{6.51}$$

We use Lemma 6.2, and in the proof of Lemma 6.2 we may additionally assume that the chosen subsequence satisfies

$$\lim_{\ell\to\infty} \frac{\mathcal{L}^{n+1}(E_{j_\ell,i} \cap B_{r_\ell})}{\omega_{n+1} r_\ell^{n+1}} = 1 \text{ for either } i = 1 \text{ or } 2 \tag{6.52}$$

if $0 \in \operatorname{spt}\|V\| \setminus \partial^* E_1$ and

$$\lim_{\ell\to\infty} \frac{\mathcal{L}^{n+1}(E_{j_\ell,1} \cap B_{r_\ell}^+)}{\omega_{n+1} r_\ell^{n+1}} = 0 \;\text{ and }\; \lim_{\ell\to\infty} \frac{\mathcal{L}^{n+1}(E_{j_\ell,1} \cap B_{r_\ell}^-)}{\omega_{n+1} r_\ell^{n+1}} = \frac{1}{2} \tag{6.53}$$

if $0 \in \partial^* E_1$. Without loss of generality, we may assume $i = 1$ in (6.52), $T = \{x_{n+1} = 0\}$ and $B_{r_\ell}^+ = B_{r_\ell} \cap \{x_{n+1} \ge 0\}$ in (6.53). By (6.13), we have

$$\lim_{\ell\to\infty} (r_\ell)^{-n} \int_{B_{2r_\ell}} \|S - T\| \, d(\partial\mathcal{E}_{j_\ell})(x,S) = 0. \tag{6.54}$$

Set

$$C_\ell := \Big\{x \in \partial\mathcal{E}_{j_\ell} \cap B_{r_\ell} \,:\, \|\operatorname{Tan}^n(\|\partial\mathcal{E}_{j_\ell}\|, x) - T\| \le 1/10\Big\}. \tag{6.55}$$

Given any $0 < \delta < r_\ell$ and $x \in \partial\mathcal{E}_{j_\ell}$, by the rectifiability of $\partial\mathcal{E}_{j_\ell}$, there exists $0 < r < \delta$ such that

$$\frac{1}{2} \le \frac{1}{\omega_n r^n}\|\partial\mathcal{E}_{j_\ell}\|(B_r(x)) \text{ and } \frac{1}{\omega_n r^n}\int_{B_r(x)} \|\operatorname{Tan}^n(\|\partial\mathcal{E}_{j_\ell}\|, x) - S\| \, d(\partial\mathcal{E}_{j_\ell}) \le \frac{1}{40}. \tag{6.56}$$

Then, for $x \in \partial\mathcal{E}_{j_\ell} \cap B_{r_\ell} \setminus C_\ell$, (6.56) and (6.55) show

$$\begin{aligned}\int_{B_r(x)} \|T - S\| \, d(\partial\mathcal{E}_{j_\ell}) &\geq \|\mathrm{Tan}^n(\|\partial\mathcal{E}_{j_\ell}\|, x) - T\| \, \|\partial\mathcal{E}_{j_\ell}\|(B_r(x)) - \omega_n r^n/40 \\ &\geq \omega_n r^n(1/20 - 1/40) = \omega_n r^n/40.\end{aligned} \tag{6.57}$$

We cover $\partial\mathcal{E}_{j_\ell} \cap B_{r_\ell} \setminus C_\ell$ by such balls and use the Besicovitch covering theorem to show that

$$\mathcal{H}^n(\partial\mathcal{E}_{j_\ell} \cap B_{r_\ell} \setminus C_\ell) \leq 40\mathbf{B}_{n+1} \int_{B_{2r_\ell}} \|T - S\| \, d(\partial\mathcal{E}_{j_\ell}), \tag{6.58}$$

where $\mathbf{B}_{n+1}$ is the Besicovitch constant. Then (6.54) and (6.58) show

$$\lim_{\ell\to\infty} (r_\ell)^{-n}\mathcal{H}^n(\partial\mathcal{E}_{j_\ell} \cap B_{r_\ell} \setminus C_\ell) = \lim_{\ell\to\infty} (r_\ell)^{-n}\mathcal{H}^n(T(\partial\mathcal{E}_{j_\ell} \cap B_{r_\ell} \setminus C_\ell)) = 0, \tag{6.59}$$

so that $T^{-1}(a) \cap \partial\mathcal{E}_{j_\ell} \cap B_{r_\ell} \subset C_\ell$ for a in a large portion of $T \cap B_{r_\ell}$ for large enough ℓ. Now recall the property of W_ℓ in Lemma 6.2. We may redefine W_ℓ by $W_\ell \setminus T(\partial\mathcal{E}_{j_\ell} \cap B_{r_\ell} \setminus C_\ell)$ and keep the properties (6.14)-(6.16). By this, we additionally have that

$$\|\mathrm{Tan}^n(\|\partial\mathcal{E}_{j_\ell}\|, x) - T\| \leq 1/10 \text{ for all } x \in T^{-1}(W_\ell) \cap B_{r_\ell} \cap \partial^*\mathcal{E}_{j_\ell}. \tag{6.60}$$

We now proceed similarly to the previous Proposition 6.3, and choose b_ℓ satisfying (6.46) in the case of (6.52). The case of (6.53) can be handled similarly so we discuss the former case. For all large ℓ, we may choose a Lebesgue point a of W_ℓ in T such that $(a, -b_\ell)$ and (a, b_ℓ) are also Lebesgue points of $I_\ell \cap A_\ell^-$ and $I_\ell \cap A_\ell^+$, respectively. By (6.14) and (6.15), there are $-b_\ell := u_0 < u_1 < \ldots < u_\theta < u_{\theta+1} := b_\ell$ such that $\cup_{k=1}^{\theta}\{(a, u_k)\} = T^{-1}(a) \cap B_{r_\ell} \cap \partial^*\mathcal{E}_{j_\ell}$. At each point (a, u_k), since it is in $\partial^* E_{j_\ell,1}$, the blow-up of $E_{j_\ell,1}$ converges to a half-space, with the approximate tangent space having a small slope relative to T due to (6.55). Then, for sufficiently small $0 < \delta < \min_{0\leq k\leq\theta}\{|u_{k+1} - u_k|\}$ and $k = 1, \ldots, \theta$, we may choose $b_{\ell,k} \in [\delta/3, \delta/2]$ so that

$$\frac{\mathcal{H}^n(T^{-1}(B^n_\delta(a)) \cap \{x_{n+1} = u_k + b_{\ell,k}\} \cap O_\ell)}{\omega_n\delta^n} \geq 1 - \frac{1}{6\theta} \tag{6.61}$$

and

$$\frac{\mathcal{H}^n(T^{-1}(B^n_\delta(a)) \cap \{x_{n+1} = u_k - b_{\ell,k}\} \cap I_\ell)}{\omega_n\delta^n} \geq 1 - \frac{1}{6\theta}, \tag{6.62}$$

or the inequalities replacing the role of O_ℓ and I_ℓ. We may also assume that

$$\frac{\mathcal{H}^n(B^n_\delta(a) \cap W_\ell)}{\omega_n\delta^n} \geq \frac{8}{9} \tag{6.63}$$

since a is a Lebesgue point of W_ℓ, and similarly

$$\frac{\mathcal{H}^n(T^{-1}(B^n_\delta(a)) \cap I_\ell \cap A_\ell^\pm)}{\omega_n\delta^n} \geq \frac{8}{9}. \tag{6.64}$$

With these properties, we can make sure that, with respect to $\mathcal{H}^n$, 1/3 of $W_\ell \cap B^n_\delta(a)$ has the property that, if $\tilde{a}$ is in this set,

$$(\tilde{a}, \pm b_\ell) \in I_\ell,\ (\tilde{a}, u_k + b_{\ell,k}) \in O_\ell,\ (\tilde{a}, u_k - b_{\ell,k}) \in I_\ell \text{ or vice-versa for } k = 1, \ldots, \theta. \tag{6.65}$$

Using (6.47), for $\mathcal{H}^n$-a.e. $\tilde{a}$ as above, there exists some $s_k \in (u_k - b_{\ell,k}, u_k + b_{\ell,k})$ with $(\tilde{a}, s_k) \in \partial_* E_{j_\ell,1}$ for each $k = 1, \ldots, \theta$, and there are no other point of $\partial_* E_{j_\ell}$ along the line segment connecting $(\tilde{a}, -b_\ell)$ and $(\tilde{a}, b_\ell)$. Looking at this line segment and the intersection of O_ℓ and I_ℓ, since these two endpoints are in I_ℓ and each $(\tilde{a}, s_k)$ is sided by O_ℓ and I_ℓ, θ has to be necessarily even. This finishes the proof in the case of (6.52). For (6.53), the similar argument results in the situation that $(\tilde{a}, -b_\ell) \in I_\ell$ and $(\tilde{a}, b_\ell) \in O_\ell$, which necessitates that θ is odd. This concludes the proof. □

Finally, we comment on the proofs of Theorem 2.12 and 2.13. If V_t is a unit density flow in $U \times (t_1, t_2)$, then $\theta_t(x) = 1$ for $\|V_t\|$-a.e. $x \in U$ and a.e. $t \in (t_1, t_2)$. By Theorem 2.10(2)(4), we may assume that $V_t = \mathbf{var}(\Gamma(t), 1)$, and by Theorem 2.11(7), $\Gamma(t)$ may be replaced by $\cup_{i=1}^N \partial^* E_i(t)$. Thus (2.16) follows immediately. To check that (2.8) holds, since $\sum_{i\neq j} \mathcal{H}^n \llcorner_{I_{i,j}(t)} = 2\mathcal{H}^n \llcorner_{\Gamma(t)} = 2\|V_t\|$ (see (2.10) and (2.11)), the left-hand side of (2.8) is equal to $2\int_0^T \delta V_t(g)\, dt$. Since $v_i \nu_i = (h \cdot \nu_i)\nu_i = h$ for $\mathcal{H}^n$-a.e. $x \in I_{i,j}(t)$ due to the perpendicularity of the mean curvature vector, the right-hand side of (2.8) is $-2\int_0^T \int h \cdot g\, d\|V_t\| dt$. Since the generalized mean curvature vector exists for a.e. $t > 0$, they are equal indeed. This proves the claim of Theorem 2.12. For Theorem 2.13, under the assumption, one can show that there exists $T_0 = T_0(n, \mathcal{H}^n(\Gamma_0), r_0, \delta_0) > 0$ such that $\int_{\Gamma_0} \rho_{(y,s)}(x, 0)\, d\mathcal{H}^n(x) < 2 - \delta_0/2$ (recall (4.50)) for all $y \in \mathbb{R}^{n+1}$ and $0 < s \leq T_0$. Then, Huisken's monotonicity formula shows that $\int_{\mathbb{R}^{n+1}} \rho_{(y,s)}(x,t)\, d\|V_t\|(x)$ is non-increasing on $t \in [0, s)$ and thus $< 2 - \delta_0/2$. For a contradiction, if V_t is not unit density on $[0, T_0]$, there would exist some $t \in (0, T_0)$ with $V_t \in \mathbf{IV}_n(\mathbb{R}^{n+1})$, and $y \in \mathrm{spt}\|V_t\|$ such that $\Theta^n(\|V_t\|, y) \geq 2$ and where $T_y\|V_t\|$ exists. Then, one can prove that $\lim_{\epsilon \to 0+} \int_{\mathbb{R}^{n+1}} \rho_{(y,t+\epsilon)}(x,t)\, d\|V_t\|(x) = \Theta^n(\|V_t\|, y) \geq 2$. Since $t + \epsilon < T_0$ for all small $\epsilon > 0$, this would be a contradiction. Thus, V_t is a unit density Brakke flow on $[0, T_0]$. Once this is proved, by Theorem 2.12, the claim of BV solution also follows. This is the outline of proof of Theorem 2.13.

7. Final remarks

7.1. **On generalized BV solutions.** As explained in Definition 2.4, a BV flow is classically defined as consisting of two objects: N families of sets of finite perimeter $E_i(t)$, and velocities v_i. From them, naturally, one can define a unit density varifold $V_t = \mathbf{var}(\cup_{i=1}^N \partial^* E_i(t), 1)$ and check that (2.8) implies that the generalized mean curvature $h(\cdot, V_t)$ is equal to $v_i \nu_i$ on $\partial^* E_i(t)$ for $i = 1, \ldots, N$. The generalized BV flow of Theorem 2.11(6) involves the accompanying Brakke flow V_t in addition to the families of sets of finite perimeters, and one may wonder if the definition makes sense even without the reference to the Brakke flow. In fact, it is interesting to observe that each $\partial^* E_i(t)$ for a.e. $t > 0$ is C^2-rectifiable due to Menne's C^2-rectifiability theorem [25, Theorem 4.8] and one can define a unique second fundamental form for $\partial^* E_i(t)$ as well as mean curvature vector by the C^2-approximability property, independent of V_t. The mean curvature vector defined in this sense coincides with $h(\cdot, V_t)$ $\|V_t\|$-a.e. on $\partial^* E_i(t)$. Thus, $h \cdot \nu_i$ on $\partial^* E_i(t)$ is uniquely defined from the C^2-rectifiability without reference to V_t. On the other hand, the summation $\tilde{h} := \frac{1}{2}\sum_{i=1}^N (h \cdot \nu_i)\nu_i$ may not correspond, in general, to the generalized mean curvature vector of $\mathbf{var}(\cup_{i=1}^N \partial^* E_i(t), 1)$ if there is some non-trivial higher multiplicity portion of V_t. For example on $\mathbb{R}^2$, define $E_+ := \{(x,y) : y > 0 \text{ if } x \leq 0, y > x^2 \text{ if } x > 0\}$ and $E_- := \{(x,y) : y < 0 \text{ if } x \leq 0, y < -x^2 \text{ if } x > 0\}$. Then $V := \mathbf{var}(\partial E_+, 1) + \mathbf{var}(\partial E_-, 1)$ has a bounded generalized mean curvature while $\mathbf{var}(\partial^* E_+ \cup \partial^* E_-, 1)$ has a singular first variation at the origin. Note that V has multiplicity $= 2$ on the negative x-axis. In this sense, the formula (2.8) does not hold in general. Note that (2.9) is relevant only when $\mathcal{H}^n(\Gamma_0)$ is finite, and it follows from Theorem 2.7(4) with $v_i = h \cdot \nu_i$. Over all, for generalized BV solution, it makes sense to consider the pair of sets of finite perimeters and Brakke flow together, unlike the original BV solutions discussed in Definition 2.4.

7.2. **MCF with fixed boundary conditions.** In [32], given a strictly convex bounded domain $U \subset \mathbb{R}^{n+1}$ with C^2 boundary ∂U, a countably n-rectifiable set $\Gamma_0 \subset U$ with $\mathcal{H}^n(\Gamma_0) < \infty$ and an open partition $E_{0,1}, \ldots, E_{0,N}$ of U such that $\Gamma_0 = U \setminus \cup_{i=1}^N E_{0,i}$, existence of a Brakke flow and a family of open partitions with fixed boundary condition is established for the given initial datum. The construction method is along the lines of [18], and we may also carry it out using the volume-controlled Lipschitz maps. If one compares the construction in [32] with that in [18], one sees that differences occur only near the boundary ∂U: more precisely, the approximate smoothed mean curvature vector is damped near the portion of Γ_0 close to ∂U, and there is another step in each epoch – a Lipschitz retraction step (see [32, Section 2.6]). Hence, the proof of the present paper works with no essential change away from ∂U, and (2.14) holds for $\phi \in C_c^1(U \times \mathbb{R}^+)$; since the formula does not involve $\nabla\phi$, by approximation, the same formula holds even for $\phi \in C^1(\operatorname{clos} U \times [0,T])$ for arbitrary $T > 0$. Since the existence results in [31] are based on [32], the same applies to the solutions discussed in [31].

7.3. **A lower bound estimate for extinction time.** The formula (2.14) gives a lower bound on the time of extinction of the flow.

Lemma 7.1. *Suppose that $\mathcal{H}^n(\Gamma_0) < \infty$ and $|E_{0,i}| := \mathcal{L}^{n+1}(E_{0,i}) < \infty$. Then*

$$0 < |E_i(t)| < \infty \text{ for } t \in [0, (|E_{0,i}|/\mathcal{H}^n(\Gamma_0))^2).$$

The proof is simple: note that we have (1.3) with $U = \mathbb{R}^{n+1}$ from (2.14). If $E_i(T) = \emptyset$, then by the Hölder inequality, Theorem 2.11(4) and Theorem 2.7(4),

$$|E_{0,i}| \leq \Big(\int_0^T \int_{\partial^* E_i(t)} |h|^2 \, d\mathcal{H}^n dt\Big)^{\frac{1}{2}} \Big(\int_0^T \mathcal{H}^n(\partial^* E_i(t)) \, dt\Big)^{\frac{1}{2}} \leq \mathcal{H}^n(\Gamma_0)\sqrt{T}.$$

This gives the lower bound for T. The same inequality shows $|E_i(t)| \leq |E_{0,i}| + \mathcal{H}^n(\Gamma_0)\sqrt{t} < \infty$ in general, so the claim follows. Argument utilizing the relative isoperimetric inequality shows that there is a unique $i_0 \in \{1, \ldots, N\}$ such that $|E_{0,i_0}| = \infty$ (let this i_0 be N without loss of generality). Suppose that $E_{0,i'}$ has the maximum volume among $i = 1, \ldots, N-1$. Since $0 < \|\nabla\chi_{E_{i'}(t)}\|(\mathbb{R}^{n+1}) \leq \|V_t\|(\mathbb{R}^{n+1})$ for $t \in [0, (|E_{0,i'}|/\mathcal{H}^n(\Gamma_0))^2)$, we can guarantee that $V_t \neq 0$ during this time interval. On the other hand, if Γ_0 is bounded, by avoidance lemma of Brakke flows (see for example [16, 10.6 and 10.7]) and a comparison with a shrinking sphere, V_t must vanish in finite time.

Appendix A. The existence theorem of [18] revisited

Here we point out places which require a change to $\mathbf{E}^{vc}(\mathcal{E}, j)$ from $\mathbf{E}(\mathcal{E}, j)$ in [18, 19]. It turned out that the proofs require no essential change and the only point to be checked is that the same Lipschitz maps used in the proofs satisfy the condition of Definition 3.1(b).

A.1. **Construction of approximate flows.** For the construction of discrete approximate sequence in [18, Section 6], we simply replace $\mathbf{E}(\mathcal{E}_{j,l}, j)$ by $\mathbf{E}^{vc}(\mathcal{E}_{j,l}, j)$ and $\Delta_j\|\partial\mathcal{E}_{j,l}\|(\Omega)$ by $\Delta_j^{vc}\|\partial\mathcal{E}_{j,l}\|(\Omega)$ when f_1 is chosen in [18, (6.9)]. As in [18, (6.10)], if we define $\{E_{j,l,i}\}_{i=1}^N := \mathcal{E}_{j,l}$, $\mathcal{E}^*_{j,l+1} := (f_1)_\star\mathcal{E}_{j,l}$ and $\{E^*_{j,l+1,i}\}_{i=1}^N = \mathcal{E}^*_{j,l+1}$, by Definition 3.1(b) and (3.1), we have for each $i = 1, \ldots, N$

$$\mathcal{L}^{n+1}(E^*_{j,l+1,i} \triangle E_{j,l,i}) \leq \{\|\partial\mathcal{E}_{j,l}\|(\Omega) - \|\partial\mathcal{E}^*_{j,l+1}\|(\Omega)\}/j \leq -(\Delta_j^{vc}\|\partial\mathcal{E}_{j,l}\|(\Omega))/j.$$

The change in [18, (6.9)] is also reflected in the estimate [18, (6.4)] and we have

$$\frac{\|\partial\mathcal{E}_{j,l}\|(\Omega)-\|\partial\mathcal{E}_{j,l-1}\|(\Omega)}{\Delta t_j}+\frac{1}{4}\int_{\mathbb{R}^{n+1}}\frac{|\Phi_{\varepsilon_j}\ast\delta(\partial\mathcal{E}_{j,l})|^2\Omega}{\Phi_{\varepsilon_j}\ast\|\partial\mathcal{E}_{j,l}\|+\varepsilon_j\Omega^{-1}}\,dx$$
$$-\frac{(1-j^{-5})}{\Delta t_j}\Delta_j^{vc}\|\partial\mathcal{E}_{j,l-1}\|(\Omega)\le\varepsilon_j^{\frac{1}{8}}+\frac{c_1^2}{2}\|\partial\mathcal{E}_{j,l-1}\|(\Omega).$$

This leads to the estimates (3.16) and (3.17).

A.2. **Proofs of rectifiability and integrality.** For the proofs of rectifiability and integrality of μ_t for $\mathcal{L}^1$-a.e. $t\in\mathbb{R}^+$, the smallness of $\Delta_j\|\partial\mathcal{E}(t)\|$ is essential in [18, Section 7&8]. The general idea is that, whenever $\Delta_j\|\partial\mathcal{E}(t)\|$ is used in [18], the proofs use contradiction arguments and some appropriate Lipschitz deformations with drastic measure reduction are constructed. Here, “drastic” means that the measure is typically reduced by some factor of the measure itself, so that the reduction is typically much larger than the volume change caused by the deformation. Thus, even if we impose the additional condition, it is satisfied by the same Lipschitz deformations in [18] and we only need to check that it is indeed the case. We point out the following three separate places.

(1) In [18, Proposition 7.2], it is proved that there exists a $\mathcal{E}$-admissible function f which reduces the measure $\|\partial\mathcal{E}\|(B_r)$ by the factor of $1/2$ (see (3)) for some $r\in[R/2,R]$ when the measure in B_R is sufficiently small. Note that (4) gives the desired estimate on the change of volume of each grain in terms of $\|\partial\mathcal{E}\|(B_r)$. Since the radii r of balls used later are typically $O(1/j^2)$, thus the volume change is $O(r^{n+1})=O(r^n/j^2)$, and $\|\partial\mathcal{E}\|(B_r)=O(r^n)$, we can deduce that f belongs to $\mathbf{E}^{vc}(\mathcal{E},j)$. Since the claim of [18, Proposition 7.2] is about the existence of $\mathcal{E}$-admissible function, no change is required in the proof.

(2) In [18, Theorem 7.3], the assumption (4) should be replaced by the volume-controlled counterpart. In the proof, on p.94-95, a Lipschitz map f is defined with the properties stated in the bottom of p.94 using [18, Proposition 7.2]. Using [18, (7.40)] which gives

$$(1-2^{-1/2})\|\partial\mathcal{E}_{j_l}\|\llcorner_\Omega(B_{r_x}(x))\le\|\partial\mathcal{E}_{j_l}\|\llcorner_\Omega(B_{r_x}(x))-\|\partial(f_x)_\star\mathcal{E}_{j_l}\|\llcorner_\Omega(B_{r_x}(x)),$$

one can proceed in [18, (7.43)] as

$$\begin{aligned}\mathcal{L}^{n+1}(E_i\triangle\tilde{E}_i)&\le c_4\sum_{k=1}^{\Lambda}(\|\partial\mathcal{E}_{j_l}\|(B(k))^{\frac{n+1}{n}}\le\frac{c_4c_3^{1/n}}{2j_l^2}\sum_{k=1}^{\Lambda}\|\partial\mathcal{E}_{j_l}\|(B(k))\\&\le\frac{c_4c_3^{1/n}(\min_{B_3(x_0)}\Omega)^{-1}}{2(1-2^{-1/2})j_l^2}\sum_{k=1}^{\Lambda}(\|\partial\mathcal{E}_{j_l}\|\llcorner_\Omega(B(k))-\|\partial f_\star\mathcal{E}_{j_l}\|\llcorner_\Omega(B(k)))\\&=\frac{c_4c_3^{1/n}(\min_{B_3(x_0)}\Omega)^{-1}}{2(1-2^{-1/2})j_l^2}(\|\partial\mathcal{E}_{j_l}\|(\Omega)-\|\partial f_\star\mathcal{E}_{j_l}\|(\Omega)).\end{aligned}$$

Thus, for all sufficiently large l, Definition 3.1(b) is satisfied. The use of [18, Lemma 4.12] is also justified, and we have $f\in\mathbf{E}^{vc}(\mathcal{E}_{j_l},j_l)$.

(3) Throughout Section 8 of [18], the only crucial point that needs to be checked is in [18, Lemma 8.1] which involves the actual construction of a measure-reducing Lipschitz deformation. It proves roughly that when $\partial\mathcal{E}$ is flat and close to being measure-minimizing within a cylinder of size $O(1/j^2)$, then the measure has to be an integer multiple of discs. The argument proceeds by assuming the contrary. The intuitive picture is that, if $\partial\mathcal{E}$ does not have a measure close to a multiple of discs, then one can locate a hole which can be expanded

horizontally. This would cause a drastic reduction of measure and lead to a contradiction to the almost measure-minimizing property. More precisely, besides the change of $\Delta_j\|\partial\mathcal{E}\|$ to $\Delta_j^{vc}\|\partial\mathcal{E}\|$ throughout, near the end of the proof of [18, Lemma 8.1], p.106, one needs to check the "expansion map" f_a is in $\mathbf{E}^{vc}(\mathcal{E}, E(r_1,\rho_1), j)$. We recall that $Y \subset T^\perp$ is a set of ν points with

$$\begin{aligned} \operatorname{diam} Y < j^{-2} \ \text{([18, Lemma 8.1(3)])} \\ r_1 < R < j^{-2}/2 \ \text{(4 lines above [18, (8.7)] and [18, Lemma 8.1(1)])} \\ \rho_1 = (1+R^{-1}r_1)\rho < 2\rho < j^{-2} \ \text{([18, (8.7)] and [18, Lemma 8.1(1)])} \end{aligned}$$

and, with $T \in \mathbf{G}(n+1,n)$ fixed,

$$E(r_1,\rho_1) = \{x \in \mathbb{R}^{n+1} \,:\, |T(x)| \le r_1,\, \operatorname{dist}(T^\perp(x), Y) \le \rho_1\} \quad \text{([18, (8.1)])}.$$

The map f_a is defined to be the identity map outside of $E(r_1,\rho_1)$, so the change of volume of grains caused by f_a is at most

$$\mathcal{L}^{n+1}(E(r_1,\rho_1)) \le 2\omega_n \nu r_1^n \rho_1 < 2\omega_n\nu r_1^n/j^2. \tag{A.1}$$

As one can see in [18, (8.67)], the reduction of measure is

$$\|\partial(f_a)_\star\mathcal{E}\|(E(r_1,\rho_1)) - \|\partial\mathcal{E}\|(E(r_1,\rho_1)) < -\frac{1}{2}(1-\zeta)\omega_n r_1^n \tag{A.2}$$

and we also have from [18, (8.8)] that

$$\|\partial\mathcal{E}\|(E(r_1,\rho_1)) = (\nu-\zeta)\omega_n r_1^n. \tag{A.3}$$

We need to see the difference with the weight Ω, and since $\operatorname{diam} E(r_1,\rho_1) < 4/j^2$,

$$\begin{aligned} &\|\partial(f_a)_\star\mathcal{E}\|(\Omega) - \|\partial\mathcal{E}\|(\Omega) \\ &= \|\partial(f_a)_\star\mathcal{E}\|\llcorner_\Omega\,(E(r_1,\rho_1)) - \|\partial\mathcal{E}\|\llcorner_\Omega\,(E(r_1,\rho_1)) \\ &\le (\max_{E(r_1,\rho_1)}\Omega)\|\partial(f_a)_\star\mathcal{E}\|(E(r_1,\rho_1)) - (\min_{E(r_1,\rho_1)}\Omega)\|\partial\mathcal{E}\|(E(r_1,\rho_1)) \\ &\le (\min_{E(r_1,\rho_1)}\Omega)(e^{4c_1/j^2}\|\partial(f_a)_\star\mathcal{E}\|(E(r_1,\rho_1)) - \|\partial\mathcal{E}\|(E(r_1,\rho_1))) \\ &\le (\min_{E(r_1,\rho_1)}\Omega)\{e^{4c_1/j^2}(\|\partial(f_a)_\star\mathcal{E}\|(E(r_1,\rho_1)) - \|\partial\mathcal{E}\|(E(r_1,\rho_1))) \\ &\quad + (e^{4c_1/j^2}-1)\|\partial\mathcal{E}\|(E(r_1,\rho_1)\} \\ &\le -\frac{1}{2}(1-\zeta)\omega_n(\min_{E(r_1,\rho_1)}\Omega)e^{4c_1/j^2}r_1^n + \frac{4c_1}{j^2}e^{4c_1/j^2}(\nu-\zeta)\omega_n r_1^n. \end{aligned} \tag{A.4}$$

In the last line, we used (A.2) and (A.3). Note that the first term of the last line is a negative term of order $O(r_1^n)$, while the change of volume expressed in (A.1) is $O(r_1^n/j^2)$. Thus (A.1) and (A.4) give the desired inequality

$$\mathcal{L}^{n+1}(E_i \triangle \tilde{E}_i) \le \mathcal{L}^{n+1}(E(r_1,\rho_1)) \le (\|\partial\mathcal{E}\|(\Omega) - \|\partial(f_a)_\star\mathcal{E}\|(\Omega))/j$$

for all sufficiently large j, and we have $f_a \in \mathbf{E}^{vc}(\mathcal{E}, E(r_1,\rho_1), j)$. The rest of the proof is not affected by the change of volume controlled deformation.

A.3. **Volume change of grains.** The motivation of having $\mathcal{L}^{n+1}(E_i \triangle \tilde{E}_i) < 1/j$ in [18] is the use in the proof of [18, Lemma 10.10], and in fact, it is the only place that this inequality is essentially used to derive any conclusion. In the proof, see the second line from the bottom of [18, p.134], it is used to make sure that the volume change of grains is small for each discrete time step and the continuity of the labelling of each grain is derived in the end. The similar smallness of volume change is available with the volume-controlled counterparts since $\|\partial \mathcal{E}_{j_\ell}(t)\|(\Omega)$ is uniformly bounded for a fixed time interval $[0,T]$ by (3.15). Thus the proof can be carried out similarly.

A.4. **Changes in** [19]**.** The results from [19] are used in the present paper and the modifications are needed there as well. On the other hand, similarly, one can check that the change to the volume-controlled counterpart does not cause any difficulties. The part which is relevant to the change is Section 4 of [19]. In the proof of Lemma 4.2, a Lipschitz retraction map $\hat{F}$ is used, with the reduction of measure inside of $B_{r_\ell R}(z^{(\ell)})$ being $\beta(r_\ell R)^n$, while the volume of the ball is $O((r_\ell R)^{n+1})$ (see (4.7) and (4.9)). Since $r_\ell = 1/j_\ell^{2.5}$ (see just after (4.3)), one can check that $\hat{F} \in \mathbf{E}^{vc}(\mathcal{E}_{j_\ell}, j_\ell)$ for all large ℓ. The similar argument can be applied to the proof of Lemma 4.3 and Lemma 4.5. The proof of Lemma 4.6 uses [18, Lemma 8.1], with the modifications discussed above in A.2(3). In the proofs of Lemma 4.7–4.9, the Lipschitz maps reduce the measure in similar manners, and they belong to the volume-controlled counterparts. In particular, all of the results in [19] hold true even with the modifications.

References

[1] Fred Almgren, Jean E. Taylor, and Lihe Wang. Curvature-driven flows: a variational approach. *SIAM J. Control Optim.*, 31(2):387–438, (1993).

[2] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[3] Giovanni Bellettini and Shokhrukh Yu. Kholmatov. Minimizing movements for mean curvature flow of partitions. *SIAM J. Math. Anal.*, 50(4):4117–4148, (2018).

[4] Lorenzo Bertini, Paolo Buttà, and Adriano Pisante. Stochastic Allen-Cahn approximation of the mean curvature flow: large deviations upper bound. *Arch. Ration. Mech. Anal.*, 224(2):659–707, (2017).

[5] Kenneth A. Brakke. *The motion of a surface by its mean curvature*, volume 20 of *Mathematical Notes.* Princeton University Press, Princeton, N.J., 1978.

[6] Lia Bronsard and Fernando Reitich. On three-phase boundary motion and the singular limit of a vector-valued Ginzburg-Landau equation. *Arch. Rational Mech. Anal.*, 124(4):355–379, (1993).

[7] Yun Gang Chen, Yoshikazu Giga, and Shun'ichi Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.*, 33(3):749–786, (1991).

[8] Daniel Depner, Harald Garcke, and Yoshihito Kohsaka. Mean curvature flow with triple junctions in higher space dimensions. *Arch. Ration. Mech. Anal.*, 211(1):301–334, (2014).

[9] Selim Esedoḡlu and Felix Otto. Threshold dynamics for networks with arbitrary surface tensions. *Comm. Pure Appl. Math.*, 68(5):808–864, (2015).

[10] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions.* Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.

[11] Lawrence C. Evans and Joel Spruck. Motion of level sets by mean curvature. I. *J. Differential Geom.*, 33(3):635–681, (1991).

[12] Julian Fischer, Sebastian Hensel, Tim Laux, and Theresa Simon. The local structure of the energy landscape in multiphase mean curvature flow: Weak-strong uniqueness and stability of evolutions. *arXiv:2003.05478*, (2020).

[13] Alexandre Freire. Mean curvature motion of triple junctions of graphs in two dimensions. *Comm. Partial Differential Equations*, 35(2):302–327, (2010).

[14] Sebastian Hensel and Tim Laux. Weak-strong uniqueness for the mean curvature flow of double bubbles. (2021). Preprint arXiv:2108.01733.
[15] Tom Ilmanen. Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature. *J. Differential Geom.*, 38(2):417–461, (1993).
[16] Tom Ilmanen. Elliptic regularization and partial regularity for motion by mean curvature. *Mem. Amer. Math. Soc.*, 108(520):x+90, (1994).
[17] Kota Kasai and Yoshihiro Tonegawa. A general regularity theory for weak mean curvature flow. *Calc. Var. Partial Differential Equations*, 50(1-2):1–68, (2014).
[18] Lami Kim and Yoshihiro Tonegawa. On the mean curvature flow of grain boundaries. *Ann. Inst. Fourier (Grenoble)*, 67(1):43–142, (2017).
[19] Lami Kim and Yoshihiro Tonegawa. Existence and regularity theorems of one-dimensional Brakke flows. *Interfaces Free Bound.*, 22(4):505–550, (2020).
[20] Tim Laux and Felix Otto. Convergence of the thresholding scheme for multi-phase mean-curvature flow. *Calc. Var. Partial Differential Equations*, 55(5):Art. 129, 74, (2016).
[21] Tim Laux and Theresa Simon. Convergence of the Allen-Cahn equation to multiphase mean curvature flow. *Comm. Pure Appl. Math.*, 71(8):1597–1647, (2018).
[22] Stephan Luckhaus and Thomas Sturzenhecker. Implicit time discretization for the mean curvature flow equation. *Calc. Var. Partial Differential Equations*, 3(2):253–271, (1995).
[23] Francesco Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012.
[24] Carlo Mantegazza, Matteo Novaga, Alessandra Pluda, and Felix Schulze. Evolution of networks with multiple junctions. (2018). Preprint arXiv:1611.08254.
[25] Ulrich Menne. Second order rectifiability of integral varifolds of locally bounded first variation. *J. Geom. Anal.*, 23(2):709–763, (2013).
[26] Luca Mugnai and Matthias Röger. The Allen-Cahn action functional in higher dimensions. *Interfaces Free Bound.*, 10(1):45–78, (2008).
[27] W. W. Mullins. Two-dimensional motion of idealized grain boundaries. *J. Appl. Phys.*, 27:900–904, (1956).
[28] Matthias Röger and Reiner Schätzle. On a modified conjecture of De Giorgi. *Math. Z.*, 254(4):675–714, (2006).
[29] Felix Schulze and Brian White. A local regularity theorem for mean curvature flow with triple edges. *J. Reine Angew. Math.*, 758:281–305, (2020).
[30] Leon Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University*. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.
[31] Salvatore Stuvard and Yoshihiro Tonegawa. Dynamical instability of minimal surfaces at flat singular points. 2020. Preprint arXiv:2008.13728.
[32] Salvatore Stuvard and Yoshihiro Tonegawa. An existence theorem for Brakke flow with fixed boundary conditions. *Calc. Var. Partial Differential Equations*, 60(1):Paper No. 43, 53, (2021).
[33] Yoshihiro Tonegawa. Integrality of varifolds in the singular limit of reaction-diffusion equations. *Hiroshima Math. J.*, 33(3):323–341, (2003).
[34] Yoshihiro Tonegawa. A second derivative Hölder estimate for weak mean curvature flow. *Adv. Calc. Var.*, 7(1):91–138, (2014).
[35] Yoshihiro Tonegawa. *Brakke's mean curvature flow: An introduction*. SpringerBriefs in Mathematics. Springer, Singapore, 2019.

Dipartimento di Matematica, Università degli Studi di Milano, Via Saldini 50, I-20133 Milano (MI), Italy
Email address: salvatore.stuvard@unimi.it

Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551, Japan
Email address: tonegawa@math.titech.ac.jp