LAGRANGIAN SUBMANIFOLDS FROM TROPICAL HYPERSURFACES

DIEGO MATESSI

Abstract. We prove that a smooth tropical hypersurface in $\mathbb{R}^3$ can be lifted to a smooth embedded Lagrangian submanifold in $(\mathbb{C}^*)^3$. This completes the proof of the result announced in the article "Lagrangian pairs pants" [11]. The idea of the proof is to use Lagrangian pairs of pants as the main building blocks.

1. Introduction

1.1. **Main result.** In [11] we introduced a new Lagrangian submanifold of $(\mathbb{C}^*)^n$, which we called a Lagrangian pair of pants. It is a fundamental object in the proof of the following result, announced in the same article

Theorem 1.1. Given a smooth tropical hypersurface Ξ in $\mathbb{R}^2$ or $\mathbb{R}^3$, there is a one parameter family of smooth Lagrangian submanifolds $\mathcal{L}_t$ of respectively $(\mathbb{C}^*)^2$ or $(\mathbb{C}^*)^3$ such that $\mathcal{L}_t$ is homeomorphic to the PL lift $\hat{\Xi}$ of Ξ and converges to it in the Hausdorff topology as $t \to 0$.

In the present article we complete the proof of this theorem by proving the case of hypersurfaces in $\mathbb{R}^3$. The case of curves in $\mathbb{R}^2$ is contained in op.cit. Given the map $\mathrm{Log} : (z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|)$ defined on $(\mathbb{C}^*)^n$, the piecewise linear (PL) Lagrangian lift $\hat{\Xi}$ of a tropical hypersurface Ξ in $\mathbb{R}^n$ is a topological, closed n-dimensional submanifold of $(\mathbb{C}^*)^n$, Lagrangian on the smooth points, such that Log maps $\hat{\Xi}$ onto Ξ. This is similar to other piecewise linear objects associated to tropical subvarieties in the context of complex geometry, e.g. the complexified non-archimedean amoeba in [12], also called phase tropical hypersurfaces (see for instance [9] or [14]).

In [11] we gave many new constructions of Lagrangian surfaces in two dimensional toric varieties, including some monotone Lagrangian tori. After our article appeared on the arXiv, we learned that a few more authors were simultaneously working on similar ideas, finding various other applications. First of all, Mikhalkin released [13] with a different proof of the result for the case of tropical curves in $\mathbb{R}^n$. He also gave many other interesting examples, including a proof of Givental's result on the existence of embedded Lagrangian non-orientable surfaces diffeomorphic to the sum of $2k+1$ Klein bottles, with $k \geq 1$. In addition, in the case of lifts of tropical curves in three dimensional toric varieties, he gives an interpretation of the order of the first homology group of the lift in terms of the multiplicity of the tropical curve. This should have interesting applications in the counting of special Lagrangian submanifolds and homological mirror symmetry. Mak and Ruddat [10] give a construction of Lagrangian submanifolds in the mirror quintic, lifting tropical curves in the boundary of the moment polytope of the ambient toric variety. They also give similar applications to the counting problem of special Lagrangian submanifolds. J. Hicks [7] in a very interesting recent paper proves an application of these Lagrangians to mirror symmetry, showing that they arise as Lagrangian cobordisms between Lagrangian sections. Sheridan and Smith [16] use Lagrangian submanifolds over tropical curves to study the Lagrangian cobordism group on fibres of a Lagrangian fibration. It also turns out that our Lagrangians are are similar to Lagrangian submanifolds in the cotangent bundle of a surface constructed in [15], see also [17] for applications to mirror symmetry.

In the case of tropical curves in $\mathbb{R}^2$ an alternative method of proof of Theorem 1.1 is to use the hyperkähler trick in $(\mathbb{C}^*)^2$, turning complex submanifolds to Lagrangian, so that one can appeal to "tropical to complex" correspondence results. This method was used for instance by Mikhalkin in [13]. The same idea does not apply in the case of tropical hypersurfaces in $\mathbb{R}^3$. This is where our idea of introducing Lagrangian pairs of pants as main building blocks becomes essential.

In Sections 2-4 we recall the main ideas of [11], such as the definition of Lagrangian pair of pants, of the PL lift $\hat{\Xi}$ and we summarize the most useful properties. Section 5 contains some technical results in preparation for the proof of Theorem 1.1 given in Section 6.

1.2. **Examples in toric varieties and Calabi-Yau manifolds?** In the last section we discuss some expected generalizations and examples, extending those in [11] and [13] in the case of tropical curves. In particular we discuss how the same tropical hypersurface can be lifted in different ways, by twisting with local sections. Moreover we give some examples of lifts of non smooth tropical hypersurfaces. Finally we discuss the problem of constructing examples in three dimensional toric varieties. Unfortunately the step from $(\mathbb{C}^*)^3$ to toric varieties is not as straight forward as in the case of tropical curves. A complete construction requires a more detailed analysis of the interaction of the Lagrangian lifts with the toric boundary, which we postpone to a separate paper. Nevertheless we give some interesting candidate examples: a candidate Lagrangian monotone embedding of $S^1 \times S^2$ in $\mathbb{P}^3$ (see Example 7.4), which generalizes the examples in [11] and a candidate non-orientable Lagrangian in $\mathbb{C}^3$ which generalizes Mikhalkin's construction of non-orientable Lagrangian surfaces in $\mathbb{C}^2$. In §7.4 we sketch how one could use the ideas in this article to construct interesting Lagrangian submanifolds in the symplectic Calabi-Yau manifolds with singular Lagrangian torus fibrations which come from our work with Castaño-Bernard [2] and the work of Gross [3], [4]. In particular in Example 7.6 we give a candidate construction of 105 Lagrangian submanifolds (spheres?) in a symplectic manifold homeomorphic (and conjecturally symplectomorphic) to the quintic threefold in $\mathbb{P}^4$.

1.3. **Notation.** Given a set of vectors $u_1, \ldots, u_k$ in a vectors space V, the cone generated by these vectors is the set

$$\text{Cone}\{u_1, \ldots, u_k\} = \left\{ \sum_{j=1}^{k} t_j u_j \mid t_j \in \mathbb{R}_{\geq 0} \right\}.$$

Given a subset A of an affine space, we will denote the convex hull of A by

$$\text{Conv}\, A.$$

Given a subset W of an affine space, the notation

$$\text{Int}\, W$$

stands for the relative interior of W. Namely, we consider the smallest affine subspace containing W, then $\text{Int}\, W$ will be the topological interior relative to this affine subspace. This for examples applies to faces of polyhedra or cones.

Acknowledgments. I wish to thank Ricardo Castaño-Bernard, Mark Gross and Grigory Mikhalkin for useful discussions on this topic. For this project I was partially supported by the grant FIRB 2012 "Moduli spaces and their applications" and by the national research project "Geometria delle varietà proiettive" PRIN 2010-11. I am a member of the INdAM group GNSAGA.

2. Lagrangian PL lifts of tropical hypersurfaces

This section reports notions and results from [11].

2.1. **The set-up.** Let $M \cong \mathbb{Z}^{n+1}$ be a lattice of rank $n+1$ and let $N = \mathrm{Hom}(M, \mathbb{Z})$ be its dual lattice. We define $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and similarly $N_{\mathbb{R}}$. Since $M_{\mathbb{R}}$ is the dual of $N_{\mathbb{R}}$, the space $M_{\mathbb{R}} \oplus N_{\mathbb{R}}$ has a natural symplectic form. We will consider the $n+1$-dimensional torus

$$T = N_{\mathbb{R}}/N$$

whose cotangent bundle is $T^*T = M_{\mathbb{R}} \times N_{\mathbb{R}}/N$. Then ω is the standard symplectic form on T^*T and the projection

$$f : T^*T \to M_{\mathbb{R}}$$

is a Lagrangian torus fibration.

We will often identify $M_{\mathbb{R}}$ with $\mathbb{R}^{n+1}$ by choosing a basis $\{u_1, \ldots, u_{n+1}\}$ of M and denote the corresponding coordinates in $M_{\mathbb{R}}$ by $x = (x_1, \ldots, x_{n+1})$. Similarly we identify $N_{\mathbb{R}}$ with $\mathbb{R}^{n+1}$ by choosing a basis $\{u_1^*, \ldots, u_{n+1}^*\}$ of $N_{\mathbb{R}}$ such that

$$\langle u_j^*, u_k \rangle = \frac{1}{\pi}\delta_{jk} \tag{1}$$

and denote the corresponding coordinates by $y = (y_1, \ldots, y_{n+1})$. In particular N is identified with $\pi\mathbb{Z}^{n+1}$ and thus

$$T = \mathbb{R}^{n+1}/\pi\mathbb{Z}^{n+1}. \tag{2}$$

We denote by $[y]$ the element of T represented by y. The symplectic form ω becomes

$$\omega = \frac{1}{\pi}\sum_{i=1}^{n+1} dx_i \wedge dy_i. \tag{3}$$

We also have that T^*T is symplectomorphic to $(\mathbb{C}^*)^n$ with the symplectic form

$$\omega = \frac{i}{4\pi}\sum_{k=1}^{n+1} \frac{dz_k \wedge d\bar{z}_k}{|z_k|^2}.$$

2.2. **Tropical hypersurfaces.** Let $P \subset N_{\mathbb{R}}$ be a convex lattice polytope. A subdivision of P in smaller lattice polytopes $P_1, \ldots, P_k$ is called *regular* if there exists a convex piecewise affine function $\nu : P \to \mathbb{R}$, such that ν is integral, i.e. $\nu(P \cap N) \subset \mathbb{Z}$, and the P_i's coincide with the domains of affiness of ν. The pair (P, ν) will also denote the set of simplices in the decomposition, i.e. all the P_k's and all of their faces, so that we write $e \in (P, \nu)$ to indicate that e is a simplex in the decomposition. Inclusion of faces will be denoted by

$$f \preceq e.$$

We say that the subdivision is *unimodal* if all the P_i's are elementary simplices.

The discrete Legendre transform of ν is the function $\check{\nu} : M_{\mathbb{R}} \to \mathbb{R}$:

$$\check{\nu}(m) = \min\{\langle v, m\rangle + \nu(v),\ v \in P \cap N\}. \tag{4}$$

Also $\check{\nu}$ gives a decomposition of $M_{\mathbb{R}}$ in the convex polyhedra given by its domains of affiness. As above, the pair $(M_{\mathbb{R}}, \check{\nu})$ will also denote the set of all polyhedra in the subdivision and their faces.

Definition 2.1. The *tropical hypersurface* associated to the pair (P, ν) is the subset $\Xi \subset M_{\mathbb{R}}$ given by the points where $\check{\nu}$ fails to be smooth. We say that Ξ is *smooth* if the subdivision of P induced by ν is unimodal.

The subdivision $(M_{\mathbb{R}}, \check{\nu})$ is dual to the subdivision (P, ν). In particular there is an inclusion reversing bijection between faces of (P, ν) and faces of $(M_{\mathbb{R}}, \check{\nu})$, which we denote by

$$e \mapsto \check{e}.$$

We have that $\dim \check{e} = n + 1 - \dim e$.

2.3. **The tropical hyperplane.** Let $\{u_1, \ldots, u_{n+1}\}$ be a basis of M inducing coordinates $x = (x_1, \ldots, x_{n+1})$ on $M_{\mathbb{R}}$. The standard tropical hyperplane $\Gamma \subset M_{\mathbb{R}}$ is the tropical variety associated to the function $\check{\nu} = \min\{0, x_1, \ldots, x_{n+1}\}$. It can be described as the union of the following cones. Let

$$u_0 = -\sum_{j=1}^{n+1} u_j. \tag{5}$$

Given a proper subset $J \subsetneq \{0, \ldots, n+1\}$, let $|J|$ be its cardinality and let

$$\Gamma_J = \text{Cone}\{u_j,\ j \in J\}.$$

For convenience let us also define

$$\Gamma_{\emptyset} = \{0\},$$

which is the vertex of Γ. We have that

$$\Gamma = \bigcup_{0 \le |J| \le n} \Gamma_J.$$

2.4. **Lagrangian coamoebas.** Let $\{u_1^*, \dots, u_{n+1}^*\}$ be the basis of $N_{\mathbb{R}}$ satisfying (1). Thus the torus T is as in (2). Consider the points

$$p_0 = 0 \quad \text{and} \quad p_k = \frac{\pi}{2} u_k^*, \quad (k = 1, \dots, n+1).$$

Denote by C^+ the set of points $[y] \in T$ which are represented either by a vertex or by an interior point of the simplex with vertices the points $p_0, \dots, p_{n+1}$. Let C^- be the image of C^+ with respect to the involution $[y] \mapsto [-y]$. The *(standard)* $(n+1)$*-dimensional Lagrangian coamoeba* is the set $C = C^+ \cup C^-$ (see Figure 1). The points $[p_0], \dots, [p_{n+1}]$ are called the *vertices* of C.

For any subset $J \subsetneq \{0, \dots, n+1\}$, denote by E_J^+ the set of points $[y] \in T$ which are represented either by a vertex or by a point in the relative interior of the $(n+1-|J|)$-dimensional simplex with vertices the points $\{p_k\}_{k \notin J}$. We let E_J^- be the image of E_J^+ via the involution $[y] \mapsto [-y]$. We define the *J-th face* of C to be the set $E_J = E_J^+ \cup E_J^-$. If $J = \{j\}$ then we denote E_J by E_j and we call it the *j-th facet* of C. We will also denote by T_J the $(n+1-|J|)$-dimensional subtorus of T containing E_J and $\bar{E}_J$ will denote the closure of E_J in T_J. Notice that the facet E_j is contained in a torus T_j which is orthogonal to the vector u_j. For convenience we also define

$$E_\emptyset = C.$$

Faces of dimension 1 are called edges. If we denote by J_k the complement of k in $\{0, \dots, n+1\}$, then

$$p_k = E_{J_k}.$$

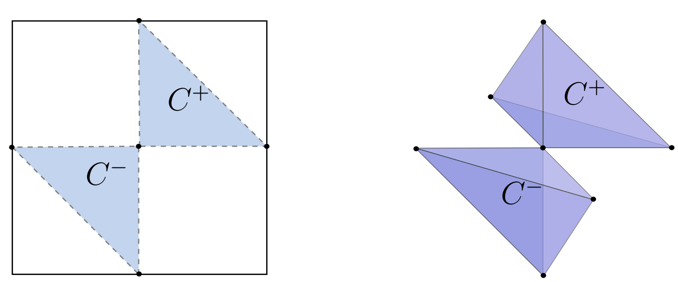

FIGURE 1. The 2 and 3 dimensional standard coamoebas. They contain their vertices but not their higher dimensional faces.

2.5. **The Lagrangian PL-lift of Γ.** For every $J \subset \{0, \dots, n+1\}$ with $0 \leq |J| \leq n$, consider the following $n+1$ dimensional subsets of $M_{\mathbb{R}} \times T$:

$$\hat{\Gamma}_J = \Gamma_J \times E_J. \tag{6}$$

The piecewise linear lift (or PL-lift) of Γ is defined to be

$$\hat{\Gamma} = \bigcup_{0 \leq |J| \leq n} \hat{\Gamma}_J. \tag{7}$$

We have that $\hat{\Gamma}$ is a topological manifold and its smooth part is Lagrangian.

2.6. **Symmetries of Γ and C.** In [11] we also discussed the symmetries of Γ and C, which we summarize here. For every $k = 1, \dots, n+1$ let R_k be the unique affine automorphism of T which maps C^+ to itself, exchanges p_0 and p_k and fixes all other vertices. Define G to be the group generated by the maps R_k. We have that G acts on the coamoeba C. The elements R_k permute the faces of C according to the rule

$$R_k E_J = E_{R_k J},$$

where on the right we use the action induced by exchanging 0 and k. Dually we have the group acting on Γ. If $u_0, \dots, u_{n+1}$ are the vectors in $M_{\mathbb{R}}$ as in §2.3, let R_k^* be the unique linear map which exchanges u_0 and u_k and fixes all other u_j's. We have that R_k^* permutes the cones Γ_J as follows

$$R_k^* \Gamma_J = \Gamma_{R_k J}. \tag{8}$$

Denote by G^* the group generated by the transformations R_k^*. Combining the actions of G and G^*, we get an action on the PL-pair of pants $\hat{\Gamma}$ via the following affine symplectic automorphisms of T^*T:

$$\mathcal{R}_k(x, y) = (R_k^* x, R_k y). \tag{9}$$

Let $\mathcal{G}$ be the group generated by the $\mathcal{R}_k$'s. Then $\mathcal{G}$ acts on $\hat{\Gamma}$.

2.7. **Lagrangian piecewise linear lifts of tropical hypersurfaces.** Let Ξ be a smooth tropical hypersurface in $M_{\mathbb{R}}$ given by a pair (P, ν). Given a k-dimensional face $e \in (P, \nu)$, with $k = 1, \dots, n+1$, let $\check{e}$ be the dual $(n+1) - k$ dimensional face of Ξ. We will use the involution ι of $M_{\mathbb{R}} \times N_{\mathbb{R}}/N$ given by $\iota : (x, [y]) \mapsto (x, [-y])$. Define the following subsets of $N_{\mathbb{R}}/N$:

$$\begin{aligned}\bar{C}_e^+ &= \{[y] \in N_{\mathbb{R}}/N \,|\, 2(y-k) \in e \text{ for some } k \in N\}, \\ \bar{C}_e^- &= \iota\left(\bar{C}_e^+\right), \\ \bar{C}_e &= \bar{C}_e^+ \cup \bar{C}_e^-.\end{aligned}$$

A point $[y] \in N_{\mathbb{R}}/N$ is a *vertex* of $\bar{C}_e$ if $2(y-k)$ is a vertex of e for some $k \in N$. We define C_e^+ (resp. C_e^- and C_e) to be the set of points $[y]$ which are either vertices or relative interior points of $\bar{C}_e^+$ (resp. $\bar{C}_e^-$ and $\bar{C}_e$). Clearly if $f \preceq e$ is a face of e, then C_f is a face of C_e. When we view C_f as a face of C_e we denote it $C_{e,f}$.

Now define the Lagrangian lift of $\check{e}$ to be

$$\hat{e} = \check{e} \times C_e.$$

We define the Lagrangian PL-lift of Ξ to be

$$\hat{\Xi} = \bigcup_e \hat{e}$$

where the union is over all faces in (P,ν) of dimensions $k = 1, \ldots, n+1$. It can be shown that $\hat{\Xi}$ is an $(n+1)$-dimensional topological submanifold of $M_{\mathbb{R}} \times N_{\mathbb{R}}/N$ which is Lagrangian at smooth points.

Given a k-dimensional polyhedron $\check{e}$ of Ξ, define the star-neighborhood of $\check{e}$ to be the union of the polyhedra of Ξ which contain $\check{e}$, i.e.

$$\Xi_{\check{e}} = \bigcup_{f \preceq e,\ \dim f \geq 1} \check{f}. \tag{10}$$

Similarly define its lift

$$\hat{\Xi}_{\check{e}} = \bigcup_{f \preceq e \ \ \dim f \geq 1} \hat{f}.$$

3. Lagrangian pairs of pants

We recall the definition and main properties of a Lagrangian pair of pants from [11].

3.1. **The definition.**

Definition 3.1. We denote by $\tilde{C}$ the *real blow up of the coamoeba* C at all its vertices and by $\pi : \tilde{C} \to C$ the natural projection. Also, for any face E_J of C we denote by $\tilde{E}_J$ its real blow up at its vertices. Let G be the group acting on C defined in §2.6, then this action lifts to an action on $\tilde{C}$.

Define the following function F on C:

$$F(y) = \begin{cases} \left(\cos\left(\sum_{j=1}^{n+1} y_j\right) \prod_{j=1}^{n+1} \sin y_j\right)^{\frac{1}{n+1}} & \text{on } C^+, \\ (-1)^n \left(\cos\left(\sum_{j=1}^{n+1} y_j\right) \prod_{j=1}^{n+1} \sin y_j\right)^{\frac{1}{n+1}} & \text{on } C^-. \end{cases} \tag{11}$$

We have that F is well defined on C and vanishes on the boundary of C. Moreover if $\iota : [y] \mapsto [-y]$ is the involution of the torus then we have that F is G invariant and satisfies $F(\iota(y)) = -F(y)$. The graph of dF over $C - \{p_0, \ldots, p_{n+1}\}$ inside T^*T, i.e. the graph of the map

$$\mathbf{h} = (F_{y_1}, \ldots, F_{y_{n+1}}),$$

where F_{y_j} denotes the partial derivative of F with respect to y_j, is a Lagrangian submanifold. We have the following

Lemma 3.2. Let $F : C \to \mathbb{R}$ be as in (11). Then F and the map $\mathbf{h} : C - \{p_0, \ldots, p_{n+1}\} \to M_{\mathbb{R}}$ extend smoothly to $\tilde{C}$ and the map $\Phi : \tilde{C} \to T^*T$ given by

$$\Phi(q) = (\mathbf{h}(q), \pi(q)). \tag{12}$$

is a Lagrangian embedding of $\tilde{C}$.

Whenever F is a function on C satisfying the above lemma, we say that the map Φ is the *graph of an exact one form over* $\tilde{C}$.

Definition 3.3. We call the submanifold $L = \Phi(\tilde{C})$ the *standard Lagrangian pair of pants* in T^*T. Given $\lambda > 0$, let Φ_λ be the embedding constructed from $\mathbf{h}_\lambda = (\lambda F_{y_1}, \ldots, \lambda F_{y_{n+1}})$ via (12). Then, if $\lambda \neq 1$, we call $\Phi_\lambda(\tilde{C})$ a *λ-rescaled Lagrangian pair of pants*

We have that L has the following symmetries

Lemma 3.4. Given a transformation R_k as in §2.6 and the involution ι of the torus, the map $\mathbf{h} : \tilde{C} \to M_{\mathbb{R}}$ defined using F satisfies

$$\mathbf{h}(R_k(y)) = R_k^* \mathbf{h}(y) \quad \text{and} \quad \mathbf{h}(\iota(y)) = \mathbf{h}(y)$$

In particular the group $\mathcal{G}$ and the involution act on a Lagrangian pair of pants.

3.2. **Some properties.** For every $k \in \{0, \ldots, n+1\}$ define

$$\mathcal{H}_k = \left\{ \sum_{l \neq k} t_l u_l \mid (t_0, \ldots, t_{k-1}, t_{k+1}, \ldots, t_{n+1}) \in (\mathbb{R}_{\geq 0})^{n+1} \text{ and } \prod_{l \neq k} t_l \leq \frac{1}{(n+1)^{n+1}} \right\}.$$

Recall that we defined J_k to be the complement of k in $\{0,\dots,n+1\}$ (see §2.4). Then

$$\mathcal{H}_k \subset \Gamma_{J_k} \quad \text{and} \quad \mathcal{H}_k = R_k^*\mathcal{H}_0.$$

Let

$$\mathcal{H} = \bigcup_{l=0}^{n+1} \mathcal{H}_l, \tag{13}$$

see Figure 2. Let $\mathcal{S}_0$ be the hypersurface

$$\mathcal{S}_0 : \quad (n+1)^{n+1} x_1 \dots x_{n+1} = 1 \text{ and } x_j > 0, \forall j. \tag{14}$$

and let

$$\mathcal{S}_k = R_k^*\mathcal{S}_0. \tag{15}$$

Then the boundary of $\mathcal{H}$ is

$$\partial\mathcal{H} = \bigcup_{l=0}^{n+1} \mathcal{S}_l.$$

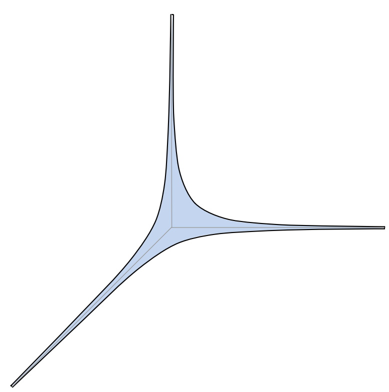

FIGURE 2. The set $\mathcal{H}$ in the case $n = 1$

Proposition 3.5. Assume $n = 1$ or 2. The image of $\mathbf{h} : \tilde{C} \to M_{\mathbb{R}}$ is $\mathcal{H}$ and the hypersurface $\mathcal{S}_k$ is the image of the set $\pi^{-1}(p_k)$. Moreover $\mathbf{h}$ defines a diffeomorphism between $\operatorname{Int} C^+$ and $\operatorname{Int}\mathcal{H}$.

This statement must be true for all values of n, but unfortunately we have been able to write a complete proof only in these dimensions.

Corollary 3.6. Assuming $n = 1$ or 2. Let F be as in (11), then the Hessian of F, restricted to $\operatorname{Int} C^+$, is negative definite.

We expect also this to be true for all n. Let us give a more detailed description of the map $\mathbf{h}$.

Definition 3.7. For every pair of vertices p_k and p_j of C^+, let δ_{jk} be the hyperplane that contains all vertices different from p_k and p_j and passes through the middle point of the edge from p_k to p_j. This hyperplane cuts C^+ in two halves. We denote by Δ_{jk} the half which contains p_k.

Clearly, the set of hyperplanes δ_{jk} cuts C^+ into the first barycentric subdivision of C^+. We have the following inequalities defining Δ_{jk}

$$\Delta_{j0} = \left\{ y \in C^+ \,|\, 2y_j + \sum_{k \neq j} y_k \leq \frac{\pi}{2} \right\} \tag{16}$$

and when $j, k \neq 0$

$$\Delta_{jk} = \left\{ y \in C^+ \,|\, y_k - y_j \geq 0 \right\}. \tag{17}$$

For every face E_J^+ of C^+ let $\mathcal{W}_J^+$ denote its star neighborhood, i.e. the union of simplices of the barycentric subdivision whose closures contain the barycenter of E_J^+. We have that

$$\mathcal{W}_J^+ = \bigcap_{k \notin J, j \in J} \Delta_{jk}. \tag{18}$$

We denote by $\mathcal{W}_J^-$ the image of $\mathcal{W}_J^+$ with respect to ι and

$$\mathcal{W}_J = \mathcal{W}_J^- \cup \mathcal{W}_J^+ \quad \text{and} \quad \tilde{\mathcal{W}}_J = \pi^{-1}(\mathcal{W}_J).$$

We have a dual structure for $\mathcal{H}$.

Definition 3.8. For every $j, k = 0, \ldots, n+1$ with $j \neq k$ let

$$d_{jk} = \operatorname{span}_{\mathbb{R}}\{u_l \,|\, l \neq j, k\}$$

It is a codimension 1 vector subspace which divides $M_{\mathbb{R}}$ in two halves. Denote by D_{jk} the half containing u_j.

If we set $x_0 = 0$, we have the following inequalities defining D_{jk} for all $j, k = 0, \ldots, n+1$ and $j \neq k$

$$D_{jk} = \{x \in M_{\mathbb{R}} \,|\, x_j - x_k \geq 0\}.$$

Let

$$\mathcal{V}_J = \bigcap_{j \in J, k \notin J} D_{jk} \tag{19}$$

When $1 \leq |J| \leq n$, $\mathcal{V}_J$ contains the face Γ_J of Γ and can be regarded as a neighborhood of it, analogous to the star neighborhood $\mathcal{W}_J$ of the face E_J. Moreover

$$\mathcal{V}_{J_k} \cap \mathcal{H} = \mathcal{H}_k.$$

We have the following useful facts:

Lemma 3.9. We have that $R_l^*(\mathcal{V}_J) = \mathcal{V}_{R_l J}$ and $R_l(\mathcal{W}_J) = \mathcal{W}_{R_l J}$.

Lemma 3.10.

$$\mathbf{h}(\tilde{\mathcal{W}}_J) = \mathcal{V}_J \cap \mathcal{H}$$

The following lemma and corollary describes the behavior of $\mathbf{h}$ near the boundary of C^+. Below we denote by h_j the components of $\mathbf{h}$.

Lemma 3.11. Let E_J be a face of C of codimension $1 \leq |J| \leq n$ which has p_0 as a vertex. Let $\{q_\ell = (q_{\ell,1}, \ldots, q_{\ell,n+1})\}_{\ell \in \mathbb{N}}$ be a sequence of points of C which converges to a point in $\operatorname{Int} E_J$ then

$$\lim_{\ell \to \infty} h_k(q_\ell) = 0 \quad \forall k \notin J \cup \{0\}.$$

Moreover if for some $j \in J$ the ratios $q_{\ell,j}/q_{\ell,i}$ are bounded for all $i \in J$, then

$$\lim_{\ell \to \infty} h_j(q_\ell) = +\infty.$$

Corollary 3.12. If $\{q_\ell\}$ is a sequence of points of C which converges to a point in the interior of a face E_J of codimension $1 \leq |J| \leq n$ then $\lim_{\ell \to +\infty} ||\mathbf{h}(q_\ell)|| = +\infty$. If $\{q_\ell\}$ converges to a vertex of C, then any convergent subsequence of $\{\mathbf{h}(q_\ell)\}$ must converge to a point on the boundary of $\mathcal{H}$.

We also have

Proposition 3.13. The Lagrangian pair of pants $\Phi(\tilde{C})$ is homeomorphic to the PL-lift $\hat{\Gamma}$ of Γ.

4. Projections to faces and Legendre transform

Projections onto faces were introduced in [11], where the most important result, Proposition 4.4, was proved. The only new input is the definition of compatible system of projections.

4.1. The projections.

Definition 4.1. Given a face E_J of C of codimension $1 \leq |J| \leq n$, let $L \subseteq N_{\mathbb{R}}$ be a vector subspace of dimension $|J|$ which is transversal to E_J. Let $U_{J,L}$ be the set of points $y \in \operatorname{Int} C$ such that there exists a $y' \in \operatorname{Int} E_J$ such that $y - y' \in L$ and the line segment from y to

y' is entirely contained in $\operatorname{Int} C$. If such a y' exists, it is unique by transversality. Thus we can define the projection

$$\begin{aligned}\mathbf{y}_{J,L} : U_{J,L} &\to \operatorname{Int} E_J \\ y &\mapsto y'.\end{aligned}$$

Recall that $\{p_k\}_{k \notin J}$ is the set of vertices of E_J. Define

$$\tilde{U}_{J,L} = \pi^{-1}(U_{J,L} \cup \{p_k\}_{k \notin J}) \subseteq \tilde{C}$$

Then $\mathbf{y}_{J,L}$ extends uniquely to a map $\mathbf{y}_{J,L} : \tilde{U}_{J,L} \to \tilde{E}_J$.

Dually we give the following definition.

Definition 4.2. Let Γ_J be a face of Γ. Denote by V_J the smallest subspace containing Γ_J. Let L be as in Definition 4.1. Define

$$L^{\perp} = \{x \in M_{\mathbb{R}} \mid \langle x, y \rangle = 0 \ \forall y \in L\}$$

Then $L^{\perp}$ has dimension $n + 1 - |J|$ and it is transversal to V_J. It thus defines the projection $\mathbf{x}_{J,L} : M_{\mathbb{R}} \to V_J$, dual to $\mathbf{y}_{J,L}$, whose fibres are parallel to $L^{\perp}$.

Given a face E_J of C, we denoted by T_J the smallest subtorus of T which contains E_J. By construction $V_J \times T_J$ is a Lagrangian submanifold of $M_{\mathbb{R}} \times T$. Given L and $L^{\perp}$ as in Definitions 4.1 and 4.2, the space $(V_J \times T_J) \times (L^{\perp} \times L)$ is naturally a covering of $M_{\mathbb{R}} \times T$ and thus induces from the latter a symplectic form. On the other hand $(V_J \times T_J) \times (L^{\perp} \times L)$ can also be naturally identified with the cotangent bundle of $V_J \times T_J$. The following Lemma states that the two symplectic forms coincide

Lemma 4.3. The choice of a vector subspace L as in Definition 4.1 induces a natural (linear) symplectomorphism between the cotangent bundle of $V_J \times T_J$ and $(V_J \times T_J) \times (L^{\perp} \times L)$.

Given L, $U_{J,L}$ and $\tilde{U}_{J,L}$ as above, define $\mathbf{h}_{J,L} : \tilde{U}_{J,L} \to V_J$ to be the map

$$\mathbf{h}_{J,L} = \mathbf{x}_{J,L} \circ \mathbf{h}$$

and $\mathbf{g}_{J,L} : \tilde{U}_{J,L} \to V_J \times \tilde{E}_J$ to be

$$\mathbf{g}_{J,L} = (\mathbf{h}_{J,L}, \mathbf{y}_{J,L}). \tag{20}$$

4.2. **Projections and Legendre transform.**

Proposition 4.4. Assume $n = 1$ or 2. The map $\mathbf{g}_{J,L} : \tilde{U}_{J,L} \to V_J \times \tilde{E}_J$ is a diffeomorphism onto its image $Z_{J,L} = \mathbf{g}_{J,L}(\tilde{U}_{J,L}) \subseteq V_J \times \tilde{E}_J$. Moreover, via the identification of the cotangent bundle of $V_J \times T_J$ with (a covering of) $M_{\mathbb{R}} \times T$ given in Lemma 4.3, $\Phi(\tilde{U}_{J,L})$ is the graph of an exact one form over $Z_{J,L}$ obtained as the differential of a Legendre transform of F.

Corollary 4.5. The map $\mathbf{h}_{J,L} : \tilde{U}_{J,L} \to V_J$ is a submersion. The fibres of $\mathbf{h}_{J,L}$ can be identified with open subsets of $\tilde{E}_J$ via the map $\mathbf{y}_{J,L}$.

4.3. **Compatible systems of projections.**

Definition 4.6. A *compatible system of projections* over C is given by a choice of transversal subspaces L_J as in Definition 4.1 for every face E_J of C with the property that if $E_{J_2} \subset E_{J_1}$ then $L_{J_1} \subset L_{J_2}$. In particular this implies that

$$\mathbf{y}_{J_2,L_{J_2}} \circ \mathbf{y}_{J_1,L_{J_1}} = \mathbf{y}_{J_2,L_{J_2}} \quad \text{and} \quad \mathbf{x}_{J_1,L_{J_1}} \circ \mathbf{x}_{J_2,L_{J_2}} = \mathbf{x}_{J_1,L_{J_1}}. \tag{21}$$

For simplicity of notations, once a compatible system of projections is fixed, we will write $\mathbf{x}_J$ instead of $\mathbf{x}_{J,L_J}$ and similarly in all other occurrences of this suffix.

Example 4.7. Given an inner product on $N_{\mathbb{R}}$ let L_J be the orthogonal complement of T_J, then clearly this choice for every E_J forms a compatible system of projections.

Clearly the fact that we have a compatible system of projections implies that whenever $E_{J_2} \subset E_{J_1}$ then the following diagram commutes

$$\begin{array}{ccc} \tilde{U}_{J_1} \cap \tilde{U}_{J_2} & \xrightarrow{\mathbf{g}_{J_2}} & V_{J_2} \times \tilde{E}_{J_2} \\ & \searrow^{\mathbf{h}_{J_1}} & \downarrow \\ & & V_{J_1} \end{array} \tag{22}$$

where the vertical arrow is the projection to V_{J_2} followed by the composition with $\mathbf{x}_{J_1}$. In other words, $\mathbf{h}_{J_2}$ restricted to a fibre of $\mathbf{h}_{J_1}$ is a submersion over the fibre of $\mathbf{x}_{J_1}$ intersected with V_{J_2}.

5. Trimming Lagrangian pairs of pants

Our goal is to use Lagrangian pairs of pants as local models for the smoothing of the PL-lifts of tropical hypersurfaces. For this purpose we need to trim off some parts at infinity. We will discuss the cases $n = 1$ and 2. In this section we consider the λ-rescaled Lagrangian pair of pants $\Phi_\lambda(\tilde{C})$ as in Definition 3.3. Since we are fixing the rescaling factor, in order to avoid cumbersome notation, we will drop the suffix λ from our notations, i.e. we will denote the maps by Φ and $\mathbf{h}$. We will also continue to denote by $\mathcal{H}$ the image of $\mathbf{h}$ and by $\mathcal{S}_k$ the surfaces forming the boundary of $\mathcal{H}$. So, for instance, the surface $\mathcal{S}_0$ is now defined by the equation

$$\mathcal{S}_0 : \quad (n+1)^{n+1} x_1 \dots x_{n+1} = \lambda^{n+1} \text{ and } x_j > 0, \forall j. \tag{23}$$

We also fix a compatible system of projections $\{\mathbf{y}_J\}$ and $\{\mathbf{x}_J\}$.

Let Γ_J be a cone of Γ and r_J a point in the interior of Γ_J. We have

$$r_J = \sum_{j \in J} s_j u_j$$

for some positive coordinates s_j. We will consider the following subsets of Γ_J

$$Q_{r_J} = \left\{ u = \sum_{j \in J} t_j u_j \in \Gamma_J \,|\, t_j \geq s_j \quad \forall j \in J \right\}. \tag{24}$$

Define also the following numbers

$$r_J^+ = \max\{s_j\}_{j \in J} \quad \text{and} \quad r_J^- = \min\{s_j\}_{j \in J}.$$

When Γ_J is one dimensional, there is a canonical identification of Γ_J with $\mathbb{R}_{\geq 0}$ given by $u = tu_j \mapsto t$, therefore we will identify points of Γ_J with their coordinate. In particular when Γ_J is one dimensional $r_J = r_J^+ = r_J^-$.

5.1. **Trimming the ends over n-dimensional cones.** First consider the case when Γ_J has dimension n. We want to understand the preimage of Q_{r_J} by $\mathbf{h}_J$. We will estimate its location inside C and prove that for r_J^- large enough (or equivalently, the scaling factor λ small enough), $\mathbf{h}_J$ defines a circle bundle (with fibre $\tilde{E}_J$) over Q_{r_J}. These will be the "ends at infinity" which we will trim off.

The idea is the following. Let $\mathcal{H}_k$ and $\mathcal{S}_k$ be the sets defined in §3.2 and rescaled by λ. The cone Γ_J is contained in two $n+1$ dimensional cones Γ_{J^+} and Γ_{J^-} corresponding to the two elements k^+ and k^- of $\{0, \dots, n+1\}$ which are not in J. For every point $x \in Q_{r_J}$ consider the line $\mathbf{x}_J^{-1}(x)$. When r_J is large enough the connected component of $\mathbf{x}_J^{-1}(x) \cap \mathcal{H}$ containing x is a closed segment whose endpoints are intersection points of $\mathbf{x}_J^{-1}(x)$ with the hypersurfaces $\mathcal{S}_{k^+}$ and $\mathcal{S}_{k^-}$, contained respectively in the cones Γ_{J^-} and Γ_{J^+}. The union of all these segments, as x moves in Q_{r_J}, together with the map $\mathbf{x}_J$, forms a fibre bundle over Q_{r_J} with fibre the segments. Notice that the preimage of each segment via $\mathbf{h}$ is a circle. This circle is precisely a fibre of $\mathbf{h}_J$. This situation is evident in the case $n = 1$. For instance Figure 3 depicts what happens in the case $J = \{1\}$: for all $r_J > \bar{R}$ and all $x \in Q_{r_J}$ the line $\mathbf{x}_J^{-1}(x)$

intersects both $\mathcal{S}_0$ and $\mathcal{S}_2$. The connected component of $\mathbf{x}_J^{-1}(x) \cap \mathcal{H}$ containing x is marked with a thicker continuous line.

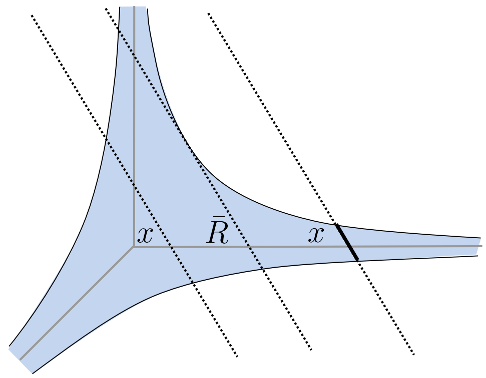

FIGURE 3. The sets $\mathbf{x}_J^{-1}(x) \cap \mathcal{H}$. For the rightmost x, $\mathbf{x}_J^{-1}(x)$ intersects both $\mathcal{S}_0$ and $\mathcal{S}_2$. This does not happen for the leftmost x. The behavior changes after $\bar{R}$.

Since this picture is intuitively clear, we will now only state without proof a few technical lemmas which quantify more precisely the sentence "for r_J^- large enough" and give some estimates on the size of the segments described above.

Lemma 5.1. Let $n = 1$ and $J = \{1\}$. There exist positive constants $\bar{R}_J$ and K_J, depending only on the projection $\mathbf{x}_J$, such that if $r_J > \bar{R}_J\lambda$, then for every $x = (x_1, 0) \in Q_{r_J}$, $\mathbf{x}_J^{-1}(x)$ intersects $\mathcal{S}_0$ transversely in either one or two points. If $x' \in \mathbf{x}_J^{-1}(x) \cap \mathcal{S}_0$ is the point closest to x, then $x' = (x'_1, x'_2)$ with

$$0 < x'_2 < \frac{K_J}{r_J}\lambda^2 \quad \text{and} \quad x'_1 > r_J - \frac{K_J}{r_J}\lambda^2.$$

In the case $n = 2$ we have the following:

Lemma 5.2. Let $J = \{1, 2\}$. There exist positive constants $\bar{R}_J$ and K_J, depending only on the projection $\mathbf{x}_J$ (and not on λ), such that if $r_J^- > \bar{R}_J\lambda$, then for every $x = (x_1, x_2, 0) \in Q_{r_J}$, $\mathbf{x}_J^{-1}(x)$ intersects $\mathcal{S}_0$ transversely in either one or two points. If $x^+ \in \mathbf{x}_J^{-1}(x) \cap \mathcal{S}_0$ is the point closest to x, then $x^+ = (x_1^+, x_2^+, x_3^+)$ with

$$0 < x_3^+ < \frac{K_J}{(r_J^-)^2}\lambda^3 \quad \text{and} \quad x_j^+ > r_J^- - \frac{K_J}{(r_J^-)^2}\lambda^3 \text{ for } j = 1, 2. \tag{25}$$

Let Γ_{J^+} and Γ_{J^-} be the two $n+1$-dimensional cones containing Γ_J. More precisely, if $k^\pm$ are the two elements of $\{0, \ldots, n+1\}$ which are not in J, then $J^{\pm} = J \cup \{k^{\pm}\}$. The hypersurfaces $\mathcal{S}_{k^-}$ and $\mathcal{S}_{k^+}$ are contained in Γ_{J^+} and Γ_{J^-} respectively. We have the following

Corollary 5.3. Let $n = 1$ or 2. For any Γ_J of dimension n, there exists a positive constant $\bar{R}_J$, depending only on the projection $\mathbf{x}_J$, such that for all $r_J \in \Gamma_J$ with $r_J^- > \bar{R}_J \lambda$ and all $x \in Q_{r_J}$, the line $\mathbf{x}_J^{-1}(x)$ intersects $\mathcal{S}_{k^-}$ and $\mathcal{S}_{k^+}$ transversely in either one or two points.

Assume that $\bar{R}_J$ is as in the last corollary and that r_J is such that $r_J^- > \bar{R}_J \lambda$. For every $x \in Q_{r_J}$ let $x^{\pm} \in \mathbf{x}_J^{-1}(x) \cap \mathcal{S}_{k^{\pm}}$ be the intersection point which is closest to x. Clearly the segment joining x^+ and x^- is entirely contained in $\mathcal{H}$ and contains x. Denote such a segment by $I_{J,x}$ and define

$$\mathcal{H}_{r_J} = \bigcup_{x \in Q_{r_J}} I_{J,x}. \tag{26}$$

The following Corollary is quite obvious and follows from the estimates of the last two Lemmas

Corollary 5.4. Let $n = 1$ or 2 and let Γ_J be a cone of dimension n. There exists a constant $\bar{R}_J$, depending only on the projection $\mathbf{x}_J$ and satisfying Corollary 5.3, such that if $r_J \in \Gamma_J$ satisfies $r_J^- > \bar{R}_J \lambda$, then

$$\mathcal{H}_{r_J} \subset \operatorname{Int} \mathcal{V}_J,$$

where the set on the righthand side is defined in (19).

By construction $\mathbf{x}_J : \mathcal{H}_{r_J} \to Q_{r_J}$ is a fibre bundle with fibre $I_{J,x}$. We also have that $\mathbf{h}^{-1}(I_{J,x})$ is a circle, therefore $\mathbf{h}_J : \mathbf{h}^{-1}(\mathcal{H}_{r_J}) \to Q_{r_J}$ is a circle bundle. We would like to prove that $\mathbf{g}_J : \mathbf{h}^{-1}(\mathcal{H}_{r_J}) \to \tilde{E}_J \times Q_{r_J}$ is diffeomorphism, thus providing a trivialization of the fibre bundle, but we first need to show that $\mathbf{g}_J$ is well defined on $\mathbf{h}^{-1}(\mathcal{H}_{r_J})$, i.e. that the latter is contained in $\tilde{U}_J$, the domain of the projection $\mathbf{y}_J$. For this purpose we define certain neighborhoods of an edge E_J of C. Let b_J be the barycenter of E_J^+. Given $\epsilon \in (0, 1)$ and a vertex p_j not in E_J, i.e. $j \in J$, define the point

$$q_{j,\epsilon} = \epsilon p_j + (1 - \epsilon) b_J. \tag{27}$$

Define

$$\mathcal{W}_{J,\epsilon}^+ = \bigcap_{j \in J} \operatorname{Conv}(E_J^+ \cup \{q_{j,\epsilon}, p_j\}) \cap C^+. \tag{28}$$

Notice that if $\epsilon = 1/2$, $\mathcal{W}_{J,\epsilon}^+$ coincides with $\mathcal{W}_J^+$ defined (18), thus $\mathcal{W}_{J,\epsilon}^+$ is a deformation of $\mathcal{W}_J^+$ which comes closer to E_J^+ as ϵ becomes small. Define $\mathcal{W}_{J,\epsilon}$ as usual using the involution and $\tilde{\mathcal{W}}_{J,\epsilon}$ by blowing up.

Lemma 5.5. Given an edge E_J, for every $\epsilon \in (0, 1/2)$ there exists $\bar{R}_J > 0$, depending only on $\mathbf{x}_J$ and ϵ and satisfying Corollaries 5.3 and 5.4, such that for all r_J with $r_J^- > \bar{R}_J \lambda$

$$\mathbf{h}^{-1}(\mathcal{H}_{r_J}) \subseteq \tilde{\mathcal{W}}_{J,\epsilon}.$$

Proof. We prove it for $n = 2$, the case $n = 1$ is similar. We can assume $J = \{1, 2\}$. We do the case $\lambda = 1$, the general case follows easily. We have that $\mathcal{H}_{r_J} \subset \mathcal{V}_J$. Therefore $\mathbf{h}^{-1}(\mathcal{H}_{r_J}) \subset \mathcal{W}_J$. Let $x' \in \mathcal{H}_{r_J}$, i.e. $x' \in I_{J,x}$ for some $x \in Q_{r_J}$. By symmetry we can assume $x' \in \mathcal{H}_0$. By continuity we can also assume that x' is not one of the end points of $I_{J,x}$, so that there is a unique $y \in C^+$ such that $\mathbf{h}(y) = x'$. We have $y \in \mathcal{W}_J \cap \mathcal{W}_{J_0}$. Inequalities (25) imply

$$h_j(y) > r_J^- - \frac{K}{(r_J^-)^2} \text{ for } j = 1, 2. \tag{29}$$

Since $y \in \mathcal{W}_J \cap \mathcal{W}_{J_0}$, we have

$$\begin{aligned} &0 < 2y_j + \sum_{k \neq j} y_k < \pi/2 \quad \forall j = 1, 2, 3 \\ &y_3 > y_j \quad \text{for } j = 1, 2. \end{aligned} \tag{30}$$

Using the symmetries we can also assume that $y_1 \geq y_2$. Then we have that for some constant C

$$(h_1)^3 = \frac{(\cos(2y_1 + y_2 + y_3))^3 \sin y_2 \sin y_3}{[\cos(y_1 + y_2 + y_3) \sin y_1]^2} \leq C \frac{\sin y_3}{\sin y_1} \leq 2C \frac{y_3}{y_1}$$

where we have bounded the factors involving the cosine using the fact that we are on $\mathcal{W}_J \cap \mathcal{W}_{J_0}$. Then (29) implies

$$y_1 \leq \frac{2C}{R'} y_3$$

where

$$R' = \left(r_J^- - \frac{K}{(r_J^-)^2} \right)^3.$$

This implies that $y \in \mathcal{W}_{J,\epsilon}$ with $\epsilon = \frac{C}{R'+C}$. □

Corollary 5.6. Let E_J be an edge. There exists a constant $\bar{R}_J > 0$, depending only on $\mathbf{x}_J$ and satisfying Corollaries 5.3 and 5.4, such that for all r_J with $r_J^- > \bar{R}_J \lambda$, we have

$$\mathbf{h}^{-1}(\mathcal{H}_{r_J}) \subset \tilde{U}_J.$$

Moreover $\mathbf{g}_J : \mathbf{h}^{-1}(\mathcal{H}_{r_J}) \to Q_{r_J} \times \tilde{E}_J$ is a diffeomorphism and $\Phi(\mathbf{h}^{-1}(\mathcal{H}_{r_J}))$ is the graph of dG over $Q_{r_J} \times \tilde{E}_J$, where G is the Legendre transform of F defined in Proposition 4.4.

Proof. It can be seen that, for ϵ small enough, $\tilde{\mathcal{W}}_{J,\epsilon} \subset \tilde{U}_J$. Thus the first part of the Corollary follows from Lemma 5.5. To prove that $\mathbf{g}_J$ is a diffeomorphism we need to prove surjectivity, but this follows from the fact that $\mathbf{y}_J$ restricted to a fibre $\mathbf{h}^{-1}(I_{J,x})$ is a one to one map between a pair of circles, thus it must be surjective. The last claim follows from Proposition 4.4. □

Remark 5.7. In case $n = 1$ also the inverse of Lemma 5.5 holds, namely for any r_J such that $\mathbf{h}^{-1}(\mathcal{H}_{r_J}) \subset \tilde{U}_J$, there exists an ϵ such that $\tilde{\mathcal{W}}_{J,\epsilon} \subset \mathbf{h}^{-1}(\mathcal{H}_{r_J})$. Indeed consider the boundary $\partial\tilde{\mathcal{W}}_{J,\epsilon}$ of $\tilde{\mathcal{W}}_{J,\epsilon}$, i.e. the union of the two segments joining $q_{j,\epsilon}$ to the vertices of E_J. It is a compact set. Then $\mathbf{h}_J$ restricted to $\partial\tilde{\mathcal{W}}_{J,\epsilon}$ tends uniformly to ∞ as $\epsilon \to 0$, therefore for small enough ϵ, $\mathbf{h}_J(\partial\tilde{\mathcal{W}}_{J,\epsilon}) \subset Q_{r_J}$. Since $\mathbf{h}_J$ has no critical points on the fibres of $\mathbf{y}_J$ we must also have that $\mathbf{h}_J(\tilde{\mathcal{W}}_{J,\epsilon}) \subset Q_{r_J}$. The situation is depicted in Figure 4.

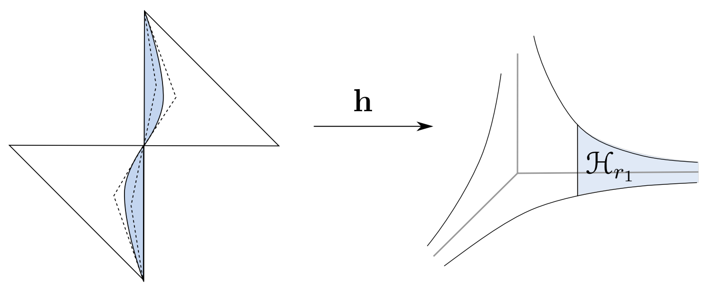

FIGURE 4. The shaded area in the co-amoeba is the preimage of $\mathcal{H}_{r_1}$. The dashed lines represent two neighborhoods of type $\tilde{\mathcal{W}}_{J,\epsilon}$.

We are now ready to do the first trimming of the Lagrangian pair of pants. For every J, with $|J| = 2$, let $\bar{R}_J$ satisfy Corollary 5.6 and choose some $r_J \in \Gamma_J$ such that $r_J^- > \bar{R}_J\lambda$. Then the sets $\mathcal{H}_{r_J}$ are pairwise disjoint (by Corollary 5.4). Define the set

$$\mathcal{H}^{[1]} = \mathcal{H} - \bigcup_{|J|=2} \mathcal{H}_{r_J}. \tag{31}$$

5.2. **Trimming the ends over 1-dimensional cones.** We consider a three dimensional λ-rescaled Lagrangian pair of pants $\Phi(\tilde{C})$ whose set $\mathcal{H}$ has been trimmed over 2-dimensional cones, as in the previous subsection, to form the set $\mathcal{H}^{[1]}$. Given a two dimensional face E_J of C and the restriction of $\mathbf{h}_J$ to $\mathbf{h}^{-1}(\mathcal{H}^{[1]})$. The goal is to study the fibres of this map, i.e. given a point $x \in V_J$ we want to understand $\mathbf{h}_J^{-1}(x) \cap$

$\mathbf{h}^{-1}(\mathcal{H}^{[1]})$. We are particularly interested in the case when $x \in Q_{r_J}$ for r_J large enough. In this case the connected component of $\mathbf{x}_J^{-1}(x) \cap \mathcal{H}^{[1]}$ containing x is homeomorphic to a two dimensional hyperplane amoeba (i.e. to the two dimensional version of $\mathcal{H}$) as in Figure 5. Therefore the preimage of this set with respect to $\mathbf{h}$ is homeomorphic to a two dimensional pair of pants. We will consider the union of all such connected components of $\mathbf{x}_J^{-1}(x) \cap \mathcal{H}^{[1]}$ as x varies in Q_{r_J} and show that its preimage with respect to $\mathbf{h}$ is contained in $\tilde{U}_J$ and thus $\mathbf{h}_J$ restricted to this set is a fibrebundle with fibre a two dimensional pair of pants. We will also study the image of these fibres with respect to $\mathbf{y}_J$ inside $\tilde{E}_J$.

Given a one dimensional cone Γ_J and a point $x \in \Gamma_J$ define $I_{J,x}$ to be the connected component of $\mathbf{x}_J^{-1}(x) \cap \mathcal{H}^{[1]}$ which contains x.

It is convenient to define

$$R_J^+ = \max\{r_{J'}^+\}_{J \subset J'} \quad \text{and} \quad R_J^- = \min\{r_{J'}^-\}_{J \subset J'}.$$

We also assume that the points $r_{J'}$ have been chosen so that

$$R_J^+ \geq \frac{\lambda^3}{(R_J^-)^2}. \tag{32}$$

Lemma 5.8. Let Γ_J be a one dimensional cone. There exists a constant K_J depending only on x_J, such that if $r_J \in \Gamma_J$ satisfies

$$r_J > K_J R_J^+ \tag{33}$$

then for all $x \in Q_{r_J}$, $I_{J,x}$ is homeomorphic to the two dimensional version of $\mathcal{H}$ (defined in (13), with $n = 1$, see Figure 5).

We skip the proof since the result is quite intuitive. It is also obvious that the distance we must move along the cone, before we obtain the required shape, depends on how we trim the neighboring cones. The lemma makes this dependency explicit.

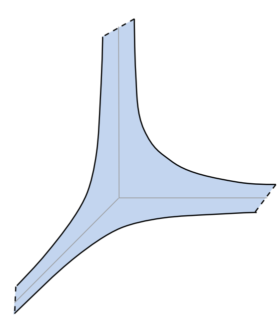

FIGURE 5. The set $I_{J,x}$ when Γ_J is one dimensional

Given a one dimensional cone Γ_J and r_J as in the previous Lemma, let us define $\mathcal{H}_{r_J}$ as in (26). We want to estimate the location of $\mathbf{h}^{-1}(\mathcal{H}_{r_J})$ inside $\tilde{C}$. For this purpose, let us define special neighborhoods of a two dimensional face $\tilde{E}_J$ of $\tilde{C}$. Clearly $|J| = 1$, i.e. $J = \{j\}$. Let b_J be the barycenter of E_J^+ and consider the unique vertex p_j which is not in E_J. Define the point $q_{j,\epsilon}$ on the segment between b_J and p_j as in (27). Let

$$\mathcal{W}^+_{J,\epsilon} = \mathrm{Conv}(E_J^+ \cup q_{j,\epsilon}) \cap C^+$$

and define $\mathcal{W}_{J,\epsilon}$ and $\tilde{\mathcal{W}}_{J,\epsilon}$ by symmetry and blow up as usual. Clearly we have that $\tilde{\mathcal{W}}_J = \tilde{\mathcal{W}}_{J,1/2}$.

Lemma 5.9. Let $J = \{1\}$ and assume that r_J satisfies (33) so that $I_{J,x}$ is homeomorphic to the two dimensional version of $\mathcal{H}$. Then there exists a positive constant C_J, depending only on the projections, such that every $x' = (x'_1, x'_2, x'_3) \in I_{J,x}$ satisfies

$$\begin{aligned} x'_1 &\geq r_J - C_J R_J^+ - \frac{C_J \lambda^3}{(R_J^-)^2} \\ |x'_j| &\leq R_J^+ + \frac{C_J\lambda^3}{(R_J^-)^2} \quad \text{for } j = 2, 3. \end{aligned} \tag{34}$$

We skip the proof, which is just an application of Lemma 5.2.

Corollary 5.10. Let E_J be a two dimensional face. Then there exists a positive constant K'_J, larger than the constant K_J of Lemma 5.8 and depending only on the projections, such that if $r_J > K'_J R_J^+$ then

$$\mathcal{H}_{r_J} \subset \mathrm{Int}\mathcal{V}_J,$$

where the set on the righthand side is defined in (19).

The proof follows easily from inequalities (34) and condition (32).

Lemma 5.11. Let E_J be a two dimensional face and let $\epsilon \in (0, 1/2)$. Then there exists a positive constant C'_J, depending only on the projections, such that if

$$r_J > \frac{C'_J}{\epsilon} R_J^+, \tag{35}$$

then r_J satisfies Lemma 5.8 and the following holds

$$\mathbf{h}^{-1}(\mathcal{H}_{r_J}) \subset \tilde{\mathcal{W}}_{J,\epsilon}.$$

Proof. We can assume $J = 1$ and let $x = (x_1, 0, 0) \in Q_{r_J}$. Let $x' = (x'_1, x'_2, x'_3) \in I_{J,x}$. By imposing that at least $C'_J > K'_J/2$, we can assume that Corollary 5.10 holds. Therefore, using symmetry, we can assume that $x' \in \mathcal{V}_J \cap \mathcal{H}_0$. Then x' satisfies the inequalities (34). By continuity we can assume that $x' \in \operatorname{Int} \mathcal{H}$. Let y be the unique $y \in \operatorname{Int} C^+$ such that $\mathbf{h}(y) = x'$. Since $x' \in \mathcal{V}_J \cap \mathcal{H}_0$ we have $y \in \mathcal{W}^+_{J_0} \cap \mathcal{W}^+_J$ (see Lemma 3.10). In particular for all $j = 1, 2, 3$

$$0 < y_1 \leq y_j \quad \text{and} \quad 0 < 2y_j + \sum_{k \neq j} y_k \leq \pi/2 \tag{36}$$

Moreover we can also assume

$$y_3 \geq y_2. \tag{37}$$

For simplicity denote

$$R = R^+_J + \frac{C_J \lambda^3}{(R^-_J)^2} \quad \text{and} \quad M = r_J - C_J R^+_J - \frac{C_J \lambda^3}{(R^-_J)^2}.$$

Then inequalities (34) imply

$$\mathbf{h}_1(y) = \frac{\lambda \cos(2y_1 + y_2 + y_3) \sin y_2 \sin y_3}{\left(\cos\left(\sum_j y_j\right) \sin y_1 \sin y_2 \sin y_3\right)^{2/3}} \geq M \tag{38}$$

and

$$\frac{\mathbf{h}_1(y)}{\mathbf{h}_2(y)} = \frac{\cos(2y_1 + y_2 + y_3) \sin y_2}{\cos(2y_2 + y_1 + y_3) \sin y_1} \geq \frac{M}{R}. \tag{39}$$

Therefore, using (36) and (37), (38) implies

$$y_1 \leq \frac{\pi}{2} \sin y_1 \leq c \left(\frac{\lambda}{M}\right)^{3/2} \tag{40}$$

for some constant c. On the other hand (39) implies

$$y_1 \leq \frac{\pi}{2} \sin y_1 \leq \frac{\pi R \sin y_2}{2M \cos(2y_2 + y_1 + y_3)} \leq \frac{\pi R\, y_2}{2M \cos(2y_2 + y_1 + y_3)}. \tag{41}$$

When

$$0 \leq 2y_2 + y_1 + y_3 \leq \frac{\pi}{3}$$

(41) implies

$$y_1 \leq \frac{\pi R}{M} y_2. \tag{42}$$

On the other hand when

$$\frac{\pi}{3} \leq 2y_2 + y_1 + y_3 \leq \frac{\pi}{2}$$

we have that (36) and (37) imply

$$y_2 \geq \frac{\pi}{24}$$

therefore, by choosing r_J so that

$$\frac{c\lambda^{3/2}}{\sqrt{M}} \leq \frac{\pi^2 R}{24} \tag{43}$$

we have that (40) implies

$$y_1 \leq c\left(\frac{\lambda}{M}\right)^{3/2} \leq \frac{\pi R}{M} y_2.$$

This implies that $y \in \mathcal{W}^+_{J,\epsilon}$ if for some constant c'

$$\frac{M}{R} \geq \frac{c'}{\epsilon}.$$

It can be easily seen that, if (32) holds, we can suitably choose C'_J so that if r_J satisfies (35), then both (43) and the latter inequality hold. Thus $y \in \mathcal{W}^+_{J,\epsilon}$. □

We then have

Corollary 5.12. Let E_J be a two dimensional face. There exists a constant K''_J, depending only on the projections and larger than the constant K'_J of Corollary 5.10, such that if

$$r_J > K''_J R^+_J \tag{44}$$

then

$$\mathbf{h}^{-1}(\mathcal{H}_{r_J}) \subset \tilde{U}_J.$$

In particular $\mathbf{g}_J : \mathbf{h}^{-1}(\mathcal{H}_{r_J}) \to Q_{r_J} \times \tilde{E}_J$ is a diffeomorphism onto its image and $\Phi(\mathbf{h}^{-1}(\mathcal{H}_{r_J}))$ is the graph of dG, where G is the Legendre transform of F defined in Proposition 4.4.

Proof. Choose ϵ such that $\tilde{\mathcal{W}}_{J,\epsilon} \subset \tilde{U}_J$ and apply Lemma 5.11. □

Corollary 5.13. If r_J is as in Corollary 5.12, then $\mathbf{h}_J : \mathbf{h}^{-1}(\mathcal{H}_{r_J}) \to Q_{r_J}$ is a fibre bundle whose fibre $\mathbf{h}^{-1}(I_{J,x})$ is homeomorphic to a two dimensional pair of pants.

This follows directly from the previous results.

Let us now discuss the compatibility between the fibre bundle structures given in Corollaries 5.6 and 5.13. First of all let us see how the total spaces of these fibre bundles may overlap. Suppose then that Γ_{J_1} and Γ_{J_2} are respectively one and two dimensional cones of Γ such that $\Gamma_{J_1} \subset \Gamma_{J_2}$. Let $r_{J_2} \in \Gamma_{J_2}$ be the point chosen to define $\mathcal{H}^{[1]}$ and let r_{J_1} satisfy Corollary 5.12. Now choose a second point $r'_{J_2} \in \Gamma_{J_2}$ such that

$$Q_{r_{J_2}} \subset Q_{r'_{J_2}}$$

and $r'^{-}_{J_2} > \bar{R}_{J_2}\lambda$, where $\bar{R}_{J_2}$ is as in Corollary 5.6. We have that $\mathcal{H}_{r_{J_1}}$ and $\mathcal{H}_{r'_{J_2}}$ have non-empty intersection. Moreover we have that by construction

$$\mathbf{x}_{J_2}\left(\mathcal{H}_{r_{J_1}} \cap \mathcal{H}_{r'_{J_2}}\right) = (Q_{r'_{J_2}} - Q_{r_{J_2}}) \cap \mathbf{x}_{J_1}^{-1}(Q_{r_{J_1}})$$

Now, the restriction of the commuting diagram (22) gives the following

Corollary 5.14. The following diagram commutes

$$\begin{array}{ccc} \mathbf{h}^{-1}\left(\mathcal{H}_{r_{J_1}} \cap \mathcal{H}_{r'_{J_2}}\right) & \xrightarrow{\mathbf{g}_{J_2}} & \left((Q_{r'_{J_2}} - Q_{r_{J_2}}) \cap \mathbf{x}_{J_1}^{-1}(Q_{r_{J_1}})\right) \times \tilde{E}_{J_2} \\ & \searrow^{\mathbf{h}_{J_1}} & \downarrow \\ & & Q_{r_{J_1}} \end{array}$$

where the horizontal arrow is a diffeomorphism and the vertical one is projection to $(Q_{r'_{J_2}} - Q_{r_{J_2}}) \cap \mathbf{x}_{J_1}^{-1}(Q_{r_{J_1}})$ composed with $\mathbf{x}_{J_1}$.

In particular the above implies that, if we consider a fibre $\mathbf{h}^{-1}(I_{J_1,x})$ of $\mathbf{h}_{J_1}$, then the end of the leg of this fibre corresponding to Γ_{J_2}, i.e. the set $\mathbf{h}^{-1}(I_{J_1,x} \cap \mathcal{H}_{r'_{J_2}})$, has a fibre bundle structure over a segment, with fibre a circle, induced by $\mathbf{h}_{J_2}$.

Definition 5.15. Given a λ-rescaled Lagrangian pair of pants $\Phi(\tilde{C})$, we say that a collection of points $\{r_J \in \operatorname{Int}\Gamma_J\}_{1 \leq |J| \leq n}$ is a **good set of trimming parameters** if for all J with $|J| = 2$ we have $r^{-}_J > \bar{R}_J\lambda$, where $\bar{R}_J$ is as in Corollary 5.6 and for all J with $|J| = 1$ we have that r_J satisfies (44) so that

$$\mathbf{h}^{-1}(\mathcal{H}_{r_J}) \subset \tilde{U}_J.$$

Given a good set of trimming parameters we define $\mathcal{H}^{[1]}$ as in (31). Then, Corollary 5.10 implies that for all J with $|J| = 1$, the sets $\mathcal{H}_{r_J}$ are pairwise disjoint. We can thus define

$$\mathcal{H}^{[2]} = \mathcal{H}^{[1]} - \bigcup_{|J|=1} \mathcal{H}_{r_J}.$$

Notice that $\Phi^{-1}(\mathcal{H}^{[2]})$ is diffeomorphic to $\tilde{C}$.

We have the following useful lemma:

Lemma 5.16. Let $\epsilon_1, \epsilon_2 \in (0, 1/2)$ be such that for every J with $|J| = j$, $\tilde{\mathcal{W}}_{J,\epsilon_j} \subset \tilde{U}_J$. Let $\{r_J \in \operatorname{Int}\Gamma_J\}_{1 \leq |J| \leq 2}$ be a collection of points such that for all J with $|J| = 1$, r_J satisfies (35) for $\epsilon = \epsilon_1$. Then there exists a $\lambda > 0$ such that this collection is a good set of trimming parameters for the λ-rescaled Lagrangian pair of pants. Moreover for all J with $|J| = j$

$$\mathbf{h}_\lambda^{-1}(\mathcal{H}_{r_J}) \subset \tilde{\mathcal{W}}_{J,\epsilon_j}.$$

Proof. Let $\bar{R}_J$ be the constants satisfying Lemma 5.5 for $\epsilon = \epsilon_2$. Then there exists λ such that for all J with $|J| = 2$, $r_J^- > \bar{R}_J \lambda$. Then Lemma 5.11 and the hypothesis guarantee that the given collection is a good set of trimming parameters for the Lagrangian pair of pants. □

5.3. **Estimating the fibres over the ends of 1-dimensional cones.** We consider a three dimensional λ-rescaled Lagrangian pair of pants $\Phi(\tilde{C})$. Given a two dimensional face E_J, we establish a result which allows some control on the image of the map $\mathbf{g}_J : \tilde{U}_J \to \tilde{E}_J \times V_J$. For this purpose we introduce some special subsets of $\tilde{E}_J$. Consider E_J as

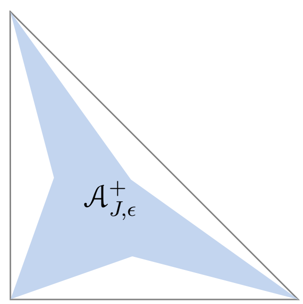

FIGURE 6.

a 2-dimensional Lagrangian coamoeba. Given some $\epsilon \in (0, 1/2)$, for every one dimensional face $E_{J'}$ of E_J define the subset $\mathcal{W}^+_{J,J',\epsilon} \subset E_J^+$ exactly as we defined the sets $\mathcal{W}^+_{J,\epsilon}$ in (28), but where everything is done inside E_J^+ instead of C^+. Then let $\mathcal{W}_{J,J',\epsilon}$ and $\tilde{\mathcal{W}}_{J,J',\epsilon}$ be as usual. Now define

$$\mathcal{A}^+_{J,\epsilon} = E_J^+ - \bigcup_{J \subset J', |J'|=2} \mathrm{Int}(\mathcal{W}^+_{J,J',\epsilon}),$$

see Figure 6. Notice that vertices are included in $\mathcal{A}^+_{J,\epsilon}$. Let $\mathcal{A}_{J,\epsilon}$ and $\tilde{\mathcal{A}}_{J,\epsilon}$ be as usual. We have that $\tilde{\mathcal{A}}_{J,\epsilon}$ is a compact subset of $\tilde{E}_J$ and its interior is homeomorphic to $\tilde{E}_J$. Let $\epsilon_1 \in (0, 1/2)$ be such that

$$\tilde{\mathcal{W}}_{J,\epsilon_1} \subset \tilde{U}_J.$$

Then we have the following

Lemma 5.17. Let $\epsilon, \epsilon_1 \in (0, 1/2)$ and $\tilde{\mathcal{A}}_{J,\epsilon}$ be as above and let K be a neighborhood of the origin in $L_J^{\perp}$. Then there exists a $\bar{R} \in \Gamma_J$ such that for all $r_J > \bar{R}\lambda$ we have

$$Q_{r_J} \times \tilde{\mathcal{A}}_{J,\epsilon} \subseteq \mathbf{g}_J(\tilde{\mathcal{W}}_{J,\epsilon_1}).$$

Moreover, if we identify $V_J \times L_J^{\perp}$ with $M_{\mathbb{R}}$ via $(r, v) \mapsto r + v$, we have for all $r \in Q_{r_J}$

$$\mathbf{h}(\mathbf{g}_J^{-1}(\{r\} \times \tilde{\mathcal{A}}_{J,\epsilon})) \subseteq \{r\} \times K.$$

Proof. It is enough to prove the case $\lambda = 1$. Consider the boundary $\partial\tilde{\mathcal{W}}_{J,\epsilon_1}$ of $\tilde{\mathcal{W}}_{J,\epsilon_1}$, then it is easy to see that

$$\mathcal{A}'_\epsilon = \mathbf{y}_J^{-1}(\tilde{\mathcal{A}}_{J,\epsilon}) \cap \partial\tilde{\mathcal{W}}_{J,\epsilon_1}$$

is a compact subset of $\tilde{C}$. Let

$$\bar{R} > \max_{\mathcal{A}'_\epsilon} \mathbf{h}_J.$$

It is now easy to see that for any $r > \bar{R}$ and any $y' \in \tilde{\mathcal{A}}_{J,\epsilon}$, there exists $y \in \tilde{\mathcal{W}}_{J,\epsilon_1}$ such that $\mathbf{g}_J(y) = (r, y')$. Indeed let

$$y_0 = \mathbf{y}_J^{-1}(y') \cap \partial\tilde{\mathcal{W}}_{J,\epsilon_1}.$$

Then, since $\mathbf{h}_J(y_0) < \bar{R} < r$ and $\lim_{y \to y'} \mathbf{h}_J(y) = +\infty$, there exists a $y \in \mathbf{y}_J^{-1}(y')$ such that $\mathbf{h}_J(y) = r$ (recall that $\mathbf{y}_J^{-1}(y')$ is one dimensional). Thus $\mathbf{g}_J(y) = (r, y')$. This proves that $\{r\} \times \tilde{\mathcal{A}}_{J,\epsilon}$ is in the image of $\mathbf{g}_J$ and hence the first part of the statement if we take $r_J > \bar{R}$.

To prove the last inclusion, we can assume $J = \{1\}$. As $r \to \infty$ we have that $\mathbf{g}_J^{-1}(\{r\} \times \tilde{\mathcal{A}}_{J,\epsilon})$ approaches the face $\tilde{E}_J$. Thus, by Lemma 3.11, the components h_2 and h_3 of $\mathbf{h}$ restricted to $\mathbf{g}_J^{-1}(\{r\} \times \tilde{\mathcal{A}}_{J,\epsilon})$ converge to 0 as $r \to +\infty$. By compactness of $\tilde{\mathcal{A}}_{J,\epsilon}$, this convergence is uniform. Thus by taking a larger $\bar{R}$ also the last inclusion of the lemma holds. ☐

6. Lagrangian lifts of smooth tropical hypersurfaces

In this section we finally prove Theorem 1.1 for the case of tropical hypersurfaces in $M_{\mathbb{R}} \cong \mathbb{R}^3$. We will use the following notation: given two point $q, q' \in M_{\mathbb{R}}$ we will denote

$$[q, q'] = \mathrm{Conv}\{q, q'\}.$$

6.1. **Compatible systems of projections.** For every k-dimensional face $e \in (P,\nu)$, with $k \geq 1$, define the following subspaces

- $N^e_{\mathbb{R}}$ is the k-dimensional vector subspace of $N_{\mathbb{R}}$ parallel to e;
- $T_e \subset T$ is the smallest affine subtorus of T which contains C_e;
- $\Lambda_{\check{e}} \subseteq M_{\mathbb{R}}$ is the $(3-k)$-dimensional vector subspace parallel to $\check{e}$;
- $V_{\check{e}}$ is the smallest affine subspace of $M_{\mathbb{R}}$ which contains $\check{e}$.

Obviously $N^e_{\mathbb{R}}$ is of the form $N^e_{\mathbb{R}} = N^e \otimes \mathbb{R}$, where $N^e = N^e_{\mathbb{R}} \cap N$. When e is 3-dimensional $N^e = N$, $T_e = T$ and $V_{\check{e}} = \check{e}$.

Choose a $(3-k)$-dimensional vector subspace $L_e \subset N_{\mathbb{R}}$ which is transverse to $N^e_{\mathbb{R}}$. This defines a unique projection $\mathbf{y}_e : N_{\mathbb{R}} \to N^e_{\mathbb{R}}$ such that $\ker \mathbf{y}_e = L_e$. We say that the collection of these choices forms a *compatible system of projections* for (P,ν) if, whenever $f \preceq e$, then $L_e \subset L_f$. This implies that $\mathbf{y}_f \circ \mathbf{y}_e = \mathbf{y}_f$. We will use the same notation to denote the projection onto T_e, which is well defined on suitable open neighborhoods of T_e, as $\mathbf{y}_e([y']) = [y]$ where $[y] \in T_e$ is such that $y - y' \in L_e$. When e is 3-dimensional, $L_e = \{0\}$.

Dually the k-dimensional vector subspace $L_e^{\perp}$ is transverse to $V_{\check{e}}$ and it defines the projection $\mathbf{x}_e : M_{\mathbb{R}} \to V_{\check{e}}$. Compatibility of projections implies that if $f \preceq e$ then $L_f^{\perp} \subset L_e^{\perp}$ and $\mathbf{x}_e \circ \mathbf{x}_f = \mathbf{x}_e$. It is easy to construct a compatible system of projections, for instance one can introduce an inner product on $N_{\mathbb{R}}$ and define L_e to be the orthogonal complement of $N^e_{\mathbb{R}}$. When e is 3-dimensional we have that $L_e^{\perp} = M_{\mathbb{R}}$.

As in Lemma 4.3, the choice of L_e induces a natural linear symplectomorphism between the cotangent bundle of $V_{\check{e}} \times T_e$ and $(V_{\check{e}} \times T_e) \times (L_e^{\perp} \times L_e)$. Moreover the latter is naturally a covering of $M_{\mathbb{R}} \times N_{\mathbb{R}}/N$ via

$$\begin{aligned} (V_{\check{e}} \times T_e) \times (L_e^{\perp} \times L_e) &\longrightarrow M_{\mathbb{R}} \times N_{\mathbb{R}}/N \\ ((q,[y]),(v,w)) &\mapsto (q+v,[y+w]) \end{aligned} \tag{45}$$

which is a local symplectomorphism. When e is 3-dimensional this map is just translation by $\check{e}$ on $M_{\mathbb{R}}$.

Remark 6.1. Notice that $L_e^{\perp}$ and L_e can be naturally identified with the cotangent fibres of T_e and $V_{\check{e}}$ respectively, thus $(V_{\check{e}} \times T_e) \times (L_e^{\perp} \times L_e)$ can also be viewed as $T^*V_{\check{e}} \times T^*T_e$. Indeed the symplectic form induced on $(V_{\check{e}} \times T_e) \times (L_e^{\perp} \times L_e)$ as a covering of $M_{\mathbb{R}} \times N_{\mathbb{R}}/N$ coincides with the symplectic form $(-\omega') \oplus \omega''$ where ω' and ω'' are the canonical symplectic forms on $T^*V_{\check{e}}$ and T^*T_e respectively (see Lemma 4.3).

The choice of a point on T_e uniquely identifies T_e with $N^e_{\mathbb{R}}/N^e$. On the other hand, since $L_e^{\perp}$ is naturally a cotangent fibre of T_e, it inherits from

T_e an integral structure, thus it can be written as $L_e^{\perp} = M^e \otimes \mathbb{R} = M^e_{\mathbb{R}}$ where M^e is the dual lattice of N^e. Thus we have an identification

$$L_e^{\perp} \times T_e \cong M^e_{\mathbb{R}} \times N^e_{\mathbb{R}}/N^e. \tag{46}$$

6.2. **Tangent tropical hyperplanes, coamoebas and projections.** Let $e \in (P, \nu)$ be of dimension $k = 2$ or 3. In the following we use the natural identification of $V_{\check{e}} \times L_e^{\perp}$ with $M_{\mathbb{R}}$ given by $(v, w) \mapsto v + w$. Recall definition (10) of the star-neighborhood $\Xi_{\check{e}}$. After fixing a point $q \in \mathrm{Int}(\check{e})$ (or $q = \check{e}$, when $\dim e = 3$), define the tangent tropical hyperplane $\Gamma_e \subseteq L_e^{\perp}$ to be the set

$$\Gamma_e = \{v \in L_e^{\perp} \,|\, \exists t \in \mathbb{R}_{\geq 0} \text{ such that } q + tv \in \Xi_{\check{e}}\}. \tag{47}$$

Obviously Γ_e is independent of q.

There is a one to one correspondence between ℓ-dimensional cones of Γ_e and $(3-k+\ell)$-dimensional polyhedra $\check{f}$ containing $\check{e}$. Let us denote this correspondence by

$$\check{f} \mapsto \Gamma_{e,f}.$$

The cone $\Gamma_{e,f}$ is dual to the face $C_{e,f}$ of C_e. Notice that the smallest subspace containing $\Gamma_{e,f}$ is $L_e^{\perp} \cap \Lambda_{\check{f}}$, where $\Lambda_{\check{f}}$ is as in §6.1.

We have a compatible system of projections $\{\mathbf{y}_{e,f}\}_{f \preceq e}$ from the coamoeba C_e to its faces $C_{e,f}$. When $\dim e = 3$, $\mathbf{y}_{e,f}$ is just the same as $\mathbf{y}_f$. When $\dim e = 2$, then $N^f_{\mathbb{R}} \subset N^e_{\mathbb{R}}$ and $\mathbf{y}_{e,f} : N^e_{\mathbb{R}} \to N^f_{\mathbb{R}}$ is the restriction of $\mathbf{y}_f$ to $N^e_{\mathbb{R}}$, whose kernel is $L_{e,f} = N^e_{\mathbb{R}} \cap L_f$. Using L_f, define $\tilde{U}_{e,f}$ to be the open subset of $\tilde{C}_e$ where $\mathbf{y}_{e,f}$ is defined onto $\tilde{C}_{e,f}$ as in Definition 4.1. Dually we have the projections $\{\mathbf{x}_{e,f}\}_{f \preceq e}$ onto the cones $\Gamma_{e,f}$ of Γ_e, induced by the projections $\mathbf{x}_f$. Indeed, by compatibility of projections, the restriction of $\mathbf{x}_f$ to $L_e^{\perp}$ gives a projection $\mathbf{x}_{e,f} : L_e^{\perp} \to L_e^{\perp} \cap \Lambda_{\check{f}}$ whose kernel is $L_f^{\perp}$. The collections $\{\mathbf{y}_{e,f}\}_{f \preceq e}$ and $\{\mathbf{x}_{e,f}\}_{f \preceq e}$ give a compatible system of projections onto the edges of $\tilde{C}_e$ and cones of Γ_e.

6.3. **Local coordinates.** We can choose a basis $\{u_1, \ldots, u_k\}$ of M^e (see after Remark 6.1 for the definition of M^e) such that each u_j is an integral primitive generator of a one dimensional cone of Γ_e. This choice defines coordinates $x = (x_1, \ldots, x_k)$ on $M^e_{\mathbb{R}}$ which identify Γ_e with the standard tropical hyperplane Γ. Dually, let $\{u_1^*, \ldots, u_k^*\}$ be a basis of $N^e_{\mathbb{R}}$ satisfying (1). Then this basis and the choice of a suitable vertex of the coamoeba C_e as the origin of T_e defines coordinates $y = (y_1, \ldots, y_k)$ such that C_e is identified with the standard Lagrangian coamoeba C. It is clear that such a choice of coordinates is unique up to a transformation in the group G^* and in its dual G. For every

$f \preceq e$ there is a unique face $E_{J_{e,f}}$ of C which, in these coordinates, corresponds to $C_{e,f}$. Moreover $\Gamma_{J_{e,f}}$ corresponds $\Gamma_{e,f}$.

In the previous sections we defined some useful subsets of Γ and $\tilde{C}$ related to their cones and faces, such as the subsets $\tilde{\mathcal{W}}_{J,\epsilon}$ or $\tilde{\mathcal{A}}_{J,\epsilon}$ of $\tilde{C}$. Via the above coordinates, all of these correspond to subsets of T_e or $L_e^{\perp}$. In order to simplify notation, when $f \preceq e$, we will do the following relabeling

$$\tilde{\mathcal{W}}_{\epsilon}^{e,f} := \tilde{\mathcal{W}}_{J_{e,f},\epsilon}$$

and similarly for the other subsets.

6.4. **Inner polyhedrons.** Let $f \in (P, \nu)$ be of dimension $k \geq 1$. Choose and fix a point $b_{\check{f}}$ in the relative interior of the dual polyhedron $\check{f}$ (when $\dim f = 3$, then $b_{\check{f}} = \check{e}$).

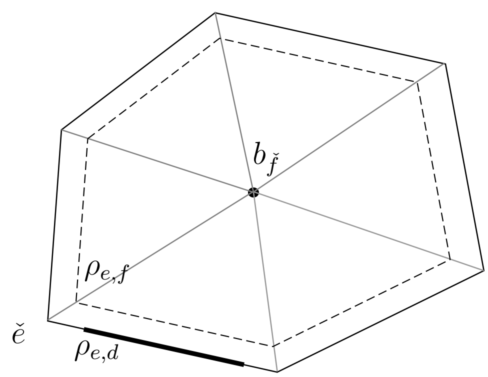

FIGURE 7. Enclosed by dashed lines is the inner polyhedron ρ_f of $\check{f}$. The thicker black line represents the inner polyhedron ρ_d of an edge $\check{d}$ of $\check{f}$.

Now consider, inside $\check{f}$, a polyhedron which is a rescaling of $\check{f}$ with center $b_{\check{f}}$. We call it an *inner polyhedron* of $\check{f}$ and denote it by ρ_f. In Figure 7, ρ_f is drawn in dashed lines when $\check{f}$ is two dimensional, while the inner polyhedron of an edge $\check{d}$ of $\check{f}$ is drawn as a thick black line. Given a face $\check{e}$ of $\check{f}$, let $\rho_{e,f}$ be the face of ρ_f corresponding to $\check{e}$. When $\dim e = 3$, i.e. $\check{e}$ is a vertex of $\check{f}$, then $\rho_{e,f}$ is a point. Define reference points

$$r_{e,f} = \rho_{e,f} - \check{e} \in \Gamma_{e,f}. \tag{48}$$

We will need three collections of inner polyhedrons $\{\rho_f\}, \{\rho'_f\}$ and $\{\rho''_f\}$ satisfying the following strict inclusions

$$\rho'_f \subset \rho''_f \subset \rho_f, \tag{49}$$

when $\dim f = 1$ or 2. For convenience, when $\dim e = 3$, we assume $\rho'_f = \rho''_f = \rho_f = \check{e}$. We will denote by $\rho_{e,f}$, $\rho'_{e,f}$ and $\rho''_{e,f}$ the corresponding faces. We choose inner polyhedrons so that they satisfy the following property

(1) For any two dimensional $d \in (P, \nu)$, any edge $f \preceq d$ and any $q \in \rho_d$, the affine plane $q + L_d^\perp$ intersects the interior of the edges $\rho_{d,f}$, $\rho'_{d,f}$ and $\rho''_{d,f}$ in a point which we can write respectively as $q + r_{d,f}$, $q + r'_{d,f}$ and $q + r''_{d,f}$ for points $r_{d,f}$, $r'_{d,f}$ and $r''_{d,f}$ in $L_d^\perp$ which are independent of q and lie in the interior of the cone $\Gamma_{d,f}$ of Γ_d.

When $\check{f}$ is two dimensional, we can use this data to subdivide it as in Figure 8. The elements of this subdivision are: the inner polyhedron ρ_f, a parallelogram $Y_{d,f}$ for each edge $\check{d}$ of $\check{f}$ and a polyhedral (non-convex) shape $Y_{e,f}$ for each vertex $\check{e}$ of $\check{f}$. For instance $Y_{d,f}$ is constructed as follows: one of its edges is the inner polyhedron ρ_d, the opposite edge is obtained by translating ρ_d by the vector $r_{d,f}$ defined above. By property (1) above, the latter edge is contained in $\rho_{d,f}$. When $\check{e}$ is a vertex the definition of $Y_{e,f}$ follows similarly.

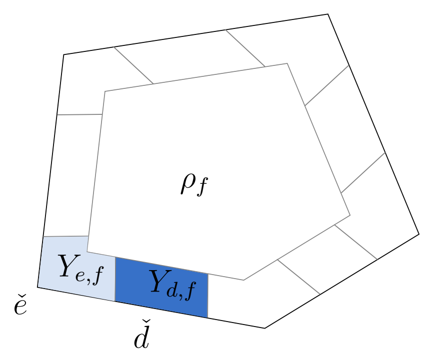

FIGURE 8.

For every $e \in (P, \nu)$ of dimension 3 or 2 we define

$$Y_e = \bigcup_{f \preceq e, \dim f = 1} Y_{e,f}. \tag{50}$$

We will denote by $Y'_{e,f}$ and $Y''_{e,f}$ the elements of the subdivision induced by the collections $\{\rho'_f\}$ and $\{\rho''_f\}$ respectively and by Y'_e and Y''_e their corresponding union as in (50).

For every e of dimension 2 or 3 and every edge f of e, the reference point $r'_{e,f}$ is on the cone $\Gamma_{e,f}$ of Γ_e therefore we can use it to define the sets $Q_{r'_{e,f}}$ as in (24). Denote

$$\Gamma_e^{[1]} = \Gamma_e - \bigcup_{f \preceq e, \dim f = 1} Q_{r'_{e,f}}. \tag{51}$$

6.5. **Neighborhoods.** For every $e \in (P, \nu)$ with $k = \dim e \geq 1$, let $B_e \subset L_e^{\perp}$ be a small convex open neighborhood satisfying

$$Y'_e \subseteq \rho'_e + B_e,$$

when $k = 2$ or 3. Notice that when $f \preceq e$, then $B_f \subset L_e^{\perp}$, then we can define

$$B_{e,f} = Q_{r_{e,f}} + B_f \subset L_e^{\perp}.$$

which is a convex neighborhood of the set $Q_{r_{e,f}} \subset \Gamma_{e,f}$ (see Figure 9). We require that the inner polyhedrons and these neighborhoods satisfy the following properties

(1) $(\rho_e + B_e) \cap (\rho_f + B_f) \neq \emptyset$ if and only if $f \preceq e$ or $e \preceq f$.
(2) for all (e, f) with $f \preceq e$ a codimension 1 face of e

$$[r_{e,f}, r'_{e,f}] + B_f \subset B_e;$$

(3) for all (e, d) with $\dim e = 3$ and d a two dimensional face of e

$$B_{e,d} \cap \Gamma_e^{[1]} = Q_{r_{e,d}} + \Gamma_d^{[1]}, \tag{52}$$

(4) for all (e, f) with $\dim e = 3$ and f and edge of e

$$B_{e,f} \subset \mathcal{V}_{e,f}.$$

It is easy to see that the inner polyhedrons and the neighborhoods can be chosen so that conditions (1) – (4) hold. Condition (4) also implies that,

$$B_{e,f} \cap \Gamma_e = Q_{r_{e,f}}. \tag{53}$$

Moreover it also implies that for all (d, f) with $\dim d = 2$ and f an edge of d

$$B_{d,f} \subset \mathcal{V}_{d,f} \tag{54}$$

and

$$B_{d,f} \cap \Gamma_d = Q_{r_{d,f}}. \tag{55}$$

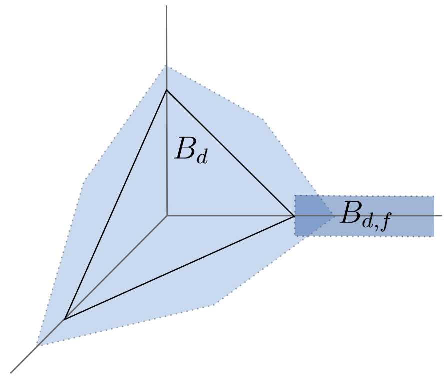

FIGURE 9. The neighborhoods B_d and $B_{d,f}$. The triangle is the convex hull of the points $r_{d,f}$.

6.6. **Fixing the inner polyhedrons.** Consider the pairs (e, d) where $\dim e = 3$ and $d \preceq e$ is a two dimensional face. Choose $\epsilon_1 \in (0, 1/2)$ so that for all such pairs

$$\tilde{W}^{e,d}_{\epsilon_1} \subset \tilde{U}_{e,d}. \tag{56}$$

In the tropical hyperplane Γ_e, for every $f \preceq e$ with $\dim f = 1$ consider the points $r'_{e,f} \in \Gamma_{e,f}$ and for every $d \preceq e$ with $\dim d = 2$ consider the points $r_{e,d} \in \Gamma_{e,d}$. We choose the size of the inner polyhedrons so that these collections of points satisfy (35) with $r_J = r_{e,d}$, $r_{J'} = r'_{e,f}$ and $\epsilon = \epsilon_1$. This can be easily achieved by taking ρ'_f sufficiently close to $\check{f}$ and leaving ρ_d fixed. Notice that conditions (1)-(4) of the previous section still hold, perhaps after taking the segment B_f smaller when f is an edge.

6.7. **Preparing the local models.** Let $e \in (P, \nu)$ be of dimension 2 or 3. We consider a λ-rescaled Lagrangian pair of pants $\overline{\Phi}_e : \tilde{C}_e \to L_e^{\perp} \times T_e$, where λ will be determined later. More precisely, via local coordinates, we can consider the function F_λ (given in (11) and rescaled) as being defined on $\tilde{C}_e$. Then $\overline{\Phi}_e$ is defined as the graph of dF_λ (in the sense of (12)):

$$\begin{aligned} \overline{\Phi}_e : \tilde{C}_e &\to L_e^{\perp} \times T_e \\ y &\mapsto ((dF_\lambda)_y, y). \end{aligned} \tag{57}$$

where $L_e^{\perp}$ is identified with the cotangent fibre of T_e. Let the associated map $\overline{\mathbf{h}}_e$ be given by composition of $\overline{\Phi}_e$ with the projection on $L_e^{\perp}$ and denote its image by $\mathcal{H}_e$. When $f \preceq e$, we define

$$\overline{\mathbf{h}}_{e,f} = \mathbf{x}_{e,f} \circ \overline{\mathbf{h}}_e,$$

corresponding to $\mathbf{h}_{J_{e,f}}$ in local coordinates. Similarly we name by $\overline{\mathbf{g}}_{e,f}$ the map corresponding to $\mathbf{g}_{J_{e,f}}$ (see (20)).

More generally, consider F_λ as a function on $\rho_e \times \tilde{C}_e$ by composing it with the projection $\rho_e \times \tilde{C}_e \to \tilde{C}_e$ and define the local model as the composition of the following maps:

$$\text{(58)} \quad \Phi_e : \quad \rho_e \times \tilde{C}_e \quad \to \quad (V_{\check{e}} \times T_e) \times (L_e^{\perp} \times L_e) \quad \to \quad M_{\mathbb{R}} \times N_{\mathbb{R}}/N.$$

The first map is just the graph of the differential of F, having identified the middle space as the cotangent bundle of $V_{\check{e}} \times T_e$, while the second map is (45). Recall that if $\dim e = 3$, then $\rho_e = \check{e}$. We denote by $\mathbf{h}_e$ the left composition of Φ_e with projection onto $M_{\mathbb{R}}$. Clearly its image is just $\rho_e + \mathcal{H}_e$.

6.8. **Rescaling along the edges.** Here we determine the rescaling factor λ for the local model along and edge $\check{d}$. Consider the pairs (d, f) where $\dim d = 2$ and f is an edge of d. Choose an $\epsilon_3 > 0$ such that for all such pairs, inside $\tilde{C}_d$ we have

$$\tilde{\mathcal{W}}^{d,f}_{\epsilon_3} \subset \tilde{U}_{d,f}$$

(see §6.3 and §6.2 for notation). Given a three dimensional $e \in (P, \nu)$ containing d and viewing $\tilde{C}_d$ as a face of $\tilde{C}_e$, we assume that ϵ_3 is small enough so that the following property holds

$$\text{(59)} \qquad \mathbf{y}^{-1}_{e,d}(\tilde{\mathcal{W}}^{d,f}_{\epsilon_3}) \cap \tilde{\mathcal{W}}^{e,d}_{\epsilon_1} \subset \tilde{\mathcal{W}}^{e,f},$$

where the latter set corresponds to $\tilde{\mathcal{W}}_{J_{e,f}}$ as defined in §3.2.

If $\bar{R}_{J_{d,f}}$ is the constant given in Lemma 5.5 for $\epsilon = \epsilon_3$, we require that λ satisfies $r_{d,f} > \bar{R}_{J_{d,f}}\lambda$ for all such pairs (d, f). Then Lemma 5.5 holds for $\epsilon = \epsilon_3$ and $r_{J_{d,f}} = r_{d,f}$. In particular we can define the subsets $\mathcal{H}_{r_{d,f}} \subset \mathcal{H}_d$, which fibre over $Q_{r_{d,f}}$ with fibres the segments $I^{d,f}_q$. Moreover

$$\text{(60)} \qquad \overline{\mathbf{h}}_d^{-1}(\mathcal{H}_{r'_{d,f}}) \subset \overline{\mathbf{h}}_d^{-1}(\mathcal{H}_{r_{d,f}}) \subset \tilde{\mathcal{W}}^{d,f}_{\epsilon_3} \subset \tilde{U}_{d,f}.$$

Thus also Corollary 5.6 holds. Define the following subsets of $L_d^{\perp}$

$$\text{(61)} \qquad \begin{aligned} H_d &= \mathcal{H}_d - \bigcup_{f \preceq d, \dim f = 1} \mathcal{H}_{r_{d,f}}, \\ H'_d &= \mathcal{H}_d - \bigcup_{f \preceq d, \dim f = 1} \mathcal{H}_{r'_{d,f}}, \\ H''_d &= \mathcal{H}_d - \bigcup_{f \preceq d, \dim f = 1} \mathcal{H}_{r''_{d,f}}. \end{aligned}$$

Obviously we have

$$H_d \subset H''_d \subset H'_d.$$

Recall the neighborhoods B_d and $B_{d,f}$ defined §6.5. After eventually rescaling with a smaller λ, we can also assume

$$\mathcal{H}_d \cap B_{d,f} = \mathcal{H}_{r_{d,f}} \tag{62}$$

and therefore by property (2) of §6.5 and the convexity of B_d

$$H'_d \subset B_d. \tag{63}$$

Following Remark 5.7 we can choose an ϵ'_2, independent of (d,f), such that

$$\tilde{\mathcal{W}}^{d,f}_{\epsilon'_2} \subset \overline{\mathbf{h}}_d^{-1}(\mathcal{H}_{r'_{d,f}}). \tag{64}$$

We will also need the following definition.

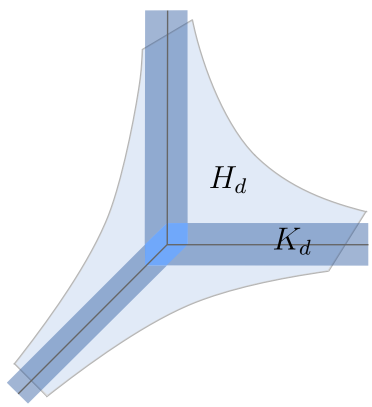

FIGURE 10. The sets H_d and K_d. The small hexagon in the center is K'_d

Definition 6.2. Given the tangent tropical line Γ_d, the vectors $\{u_0, u_1, u_2\}$ generating its one dimensional cones and $t \in \mathbb{R}_{>0}$, define the hexagon

$$K'_d = \text{Conv}\{tu_0, -tu_0, tu_1, -tu_1, tu_2, -tu_2\}.$$

For every edge f of d let

$$K_{d,f} = \Gamma_{d,f} + K'_d$$

and define

$$K_d = \bigcup_{f \preceq d, \dim f = 1} K_{d,f},$$

see Figure 10. For sufficiently small t, these sets have the following properties

a) $K'_d \subset H_d$;

b) the boundary points of $I^{d,f}_{r'_{d,f}}$ are outside K_d;
c) for every $p \in K'_d$ and $q \in H_d$ the segment from p to q lies inside H_d;
d) if a point q lies on the segment between points $p_1 \in K_{d,f} \cap H_d$ and $p_2 \in \mathcal{H}_{r_{d,f}}$ and satisfies $\mathbf{x}_f(q) < r_{d,f}$ then $q \in H_d$.

Properties (c) and (d) hold also if we replace H_d with H'_d or H''_d and $r_{d,f}$ with $r'_{d,f}$ or $r''_{d,f}$ respectively. The fact that these properties hold for some t are easy consequences of the definitions.

6.9. **Rescaling over the vertices.** Here we determine the rescaling parameter λ for the local models over the vertices defined in §6.7. Let $e \in (P, \nu)$ be three dimensional. Given ϵ'_2 as in §6.8, satisfying (64), choose an $\epsilon_2 < \epsilon'_2$ such that for all edges f of e we have

$$\tilde{\mathcal{W}}^{e,f}_{\epsilon_2} \subset \tilde{U}_{e,f} \tag{65}$$

and for all two dimensional faces d of e the following holds

$$\forall \text{ edges } f \preceq d, \quad \mathbf{y}_{e,d}(\tilde{\mathcal{W}}^{e,f}_{\epsilon_2} \cap \tilde{\mathcal{W}}^{e,d}_{\epsilon_1}) \subset \tilde{\mathcal{W}}^{d,f}_{\epsilon'_2}. \tag{66}$$

where ϵ_1 was chosen in §6.6.

We have that by (56), (65) and the criterion in §6.6, the numbers ϵ_1 and ϵ_2 and the collection of points $\{r_{e,f} \in \Gamma_{e,f}\}_{f \preceq e}$ satisfy the hypothesis of Lemma 5.16. Therefore there exists a λ such that the collection $\{r_{e,f} \in \Gamma_{e,f}\}_{f \preceq e}$ is a good set of trimming parameters for a λ-rescaled Lagrangian pair of pants. Notice that, by the criterion in §6.6, also the collection

$$\{r'_{e,f} \in \Gamma_{e,f}, \dim f = 1\} \cup \{r_{e,d} \in \operatorname{Int} \Gamma_{e,d}, \dim d = 2\} \tag{67}$$

forms a good set of trimming parameters.

For every edge f of e the points $r_{e,f} \in \Gamma_{e,f}$ satisfy Corollary 5.3 and for every $x \in Q_{r_{e,f}}$ we can define the segments $I^{e,f}_x$ (see §5.1) and the subsets $\mathcal{H}_{r_{e,f}} \subset \mathcal{H}_e$ as in (26). Moreover, by construction, we have

$$\bar{\mathbf{h}}_e^{-1}(\mathcal{H}_{r'_{e,f}}) \subset \bar{\mathbf{h}}_e^{-1}(\mathcal{H}_{r_{e,f}}) \subset \tilde{\mathcal{W}}^{e,f}_{\epsilon_2} \subset \tilde{U}_{e,f}, \tag{68}$$

so that Corollary 5.6 also holds for the points $r_{e,f}$ and $r'_{e,f}$.

By eventually rescaling by a smaller λ we can also assume the following. For any edge f of e, if $B_{e,f}$ is the set defined in §6.5, then

$$B_{e,f} \cap \mathcal{H}_e = \mathcal{H}_{r_{e,f}}. \tag{69}$$

For any two dimensional face d of e, given the set K'_d as in Definition 6.2, the statement of Lemma 5.17 holds for ϵ_1 chosen as in §6.6, $\epsilon = \epsilon'_2$, $K = K'_d$ and $r_J = r_{e,d}$. In particular we have that

$$Q_{r_{e,d}} \times \tilde{\mathcal{A}}^{e,d}_{\epsilon'_2} \subseteq \bar{\mathbf{g}}_{e,d}(\tilde{W}^{e,d}_{\epsilon_1}). \tag{70}$$

Moreover, for any $r \in Q_{r_{e,d}}$

$$\bar{\mathbf{h}}_e(\bar{\mathbf{g}}_{e,d}^{-1}(\{r\} \times \tilde{\mathcal{A}}^{e,d}_{\epsilon'_2})) \subseteq r + K'_d. \tag{71}$$

We can define the first trimming of $\mathcal{H}_e$ by

$$\mathcal{H}_e^{[1]} = \mathcal{H}_e - \bigcup_{f \preceq e,\, \dim f = 1} \mathcal{H}_{r'_{e,f}}. \tag{72}$$

Moreover, for all two dimensional faces d of e, we have that $r_{e,d}$ satisfies Corollary 5.12. In particular for all $r \in Q_{r_{e,d}}$ we have the fibres $I_r^{e,d} \subset \mathcal{H}_e^{[1]}$, whose preimages under $\bar{\mathbf{h}}_e$ are two dimensional pairs of pants. We also have the subsets $\mathcal{H}_{r_{e,d}}$ which satisfy

$$\bar{\mathbf{h}}_e^{-1}(\mathcal{H}_{r'_{e,d}}) \subset \bar{\mathbf{h}}_e^{-1}(\mathcal{H}_{r_{e,d}}) \subset \tilde{\mathcal{W}}^{e,d}_{\epsilon_1} \subset \tilde{U}_{e,d}. \tag{73}$$

We then define the second trimming

$$\mathcal{H}_e^{[2]} = \mathcal{H}_e^{[1]} - \bigcup_{d \preceq e,\, \dim d = 2} \mathcal{H}_{r'_{e,d}}. \tag{74}$$

By eventually rescaling with a smaller λ, we can assume the following. First of all that

$$\mathcal{H}_e^{[2]} \subset B_e \tag{75}$$

and that for every two dimensional face d of e, the set $B_{e,d}$ defined in §6.5 satisfies

$$B_{e,d} \cap \mathcal{H}_e^{[1]} = \mathcal{H}_{r_{e,d}}. \tag{76}$$

Morover, if $K_d \subset L_d^{\perp}$ is as in Definition 6.2 and H'_d as in (61), then for every $r \in Q_{r_{e,d}}$

$$I_r^{e,d} \subset r + (K_d \cap H'_d). \tag{77}$$

Notice that for all $r \in Q_{r_{e,d}}$ we have

$$\bar{\mathbf{h}}_e(\bar{\mathbf{g}}_{e,d}^{-1}(\{r\} \times \tilde{\mathcal{A}}^{e,d}_{\epsilon'_2})) \subset I_r^{e,d}. \tag{78}$$

Indeed (70), (66), (68) ensure that if $y \in \bar{\mathbf{g}}_{e,d}^{-1}(\{r\} \times \tilde{\mathcal{A}}^{e,d}_{\epsilon'_2})$ then $\bar{\mathbf{h}}_e(y) \in \mathcal{H}_e^{[1]}$. Therefore the inclusion follows from (71), (63), part (a) of Definition 6.2 and (76).

Lemma 6.3. Let $\overline{\mathbf{h}}_d : \tilde{C}_d \to L_d^{\perp}$ be the map from §6.7 and let H'_d be as in (61). Then, for all $r \in Q_{r_{e,d}}$, we have

$$\overline{\mathbf{h}}_d^{-1}(H'_d) \subset \tilde{\mathcal{A}}^{e,d}_{\epsilon'_2} \subset \mathbf{y}_{e,d}(\bar{\mathbf{h}}_e^{-1}(I_r^{e,d})).$$

Proof. These inclusions follow from (64) and (78). □

We can now give a provisional definition of the trimmed local model.

Definition 6.4 (Provisional)**.** The local model at a vertex $\check{e}$ is given by (58), but now Φ_e is rescaled as explained in this subsection and its domain is restricted to the subset

$$\mathcal{Z}_e = \bar{\mathbf{h}}_e^{-1}\left(\mathcal{H}_e^{[2]}\right) \subset \check{e} \times \tilde{C}_e.$$

We denote this local model by $(\Phi_e, \mathcal{Z}_e)$.

6.10. **Gluing the local models.** We now glue the local model over a vertex $\check{e}$ of Ξ to the local model over an adjacent edge. So let d be a two dimensional face of e. We have that the end of $\mathcal{H}_e^{[2]}$ over the cone $\Gamma_{e,d}$ is given by the subset $\mathcal{H}_{r_{e,d}} - \mathcal{H}_{r'_{e,d}}$. Let us denote

$$\mathcal{H}_{[r_{e,d}, r'_{e,d})} := \mathcal{H}_{r_{e,d}} - \mathcal{H}_{r'_{e,d}}.$$

Recall that we have a fibre bundle

$$\bar{\mathbf{h}}_{e,d} : \bar{\mathbf{h}}_e^{-1}(\mathcal{H}_{[r_{e,d}, r'_{e,d})}) \to [r_{e,d}, r'_{e,d})$$

with fibre homeomorphic to a two dimensional pair of pants. Then we have

$$\bar{\mathbf{g}}_{e,d} : \bar{\mathbf{h}}_e^{-1}(\mathcal{H}_{[r_{e,d}, r'_{e,d})}) \to [r_{e,d}, r'_{e,d}) \times \tilde{C}_d$$

which is a diffeomorphism onto its image. Let us denote this image by

$$\mathcal{Z}_{e,d}^0 := \bar{\mathbf{g}}_{e,d}(\bar{\mathbf{h}}_e^{-1}(\mathcal{H}_{[r_{e,d}, r'_{e,d})})).$$

Then $\bar{\Phi}_e(\bar{\mathbf{h}}_e^{-1}(\mathcal{H}_{[r_{e,d}, r'_{e,d})}))$ is the graph of $dG_{e,d}$ for some function $G_{e,d}$ defined over $\mathcal{Z}_{e,d}^0$.

Recall (see (48)) that $[r_{e,d}, r'_{e,d}) \times \tilde{C}_d$ can be identified with $[\rho_{e,d}, \rho'_{e,d}) \times \tilde{C}_d$, a subset of $\rho_d \times \tilde{C}_d$, via translation by $\check{e}$ on the first factor. Then it is convenient to think of $\mathcal{Z}_{e,d}^0$ as a subset of $\rho_d \times \tilde{C}_d$. With this identification, it is $\Phi_e(\bar{\mathbf{h}}_e^{-1}(\mathcal{H}_{[r_{e,d}, r'_{e,d})}))$ which is the graph of $dG_{e,d}$ over $\mathcal{Z}_{e,d}^0$ (recall the difference (58) between $\bar{\Phi}_e$ and Φ_e).

Let us now look at the local model over the edge. Recall that Φ_d is defined in (58) as the graph of dF, for suitable F. The idea is to interpolate the two graphs via a partition of unity.

Recall also that we defined a third inner polyhedron ρ''_d which is nested between ρ_d and ρ'_d, and defines a point $\rho''_{e,d} \in [\rho_{e,d}, \rho'_{e,d}]$. Choose some $\bar{\rho}_{e,d} \in [\rho_{e,d}, \rho'_{e,d}]$ so that

$$\rho_{e,d} < \rho''_{e,d} < \bar{\rho}_{e,d} < \rho'_{e,d}.$$

Define

$$\mathcal{Z}_{e,d}^{\infty} := (\bar{\rho}_{e,d}, \rho'_{e,d}) \times \tilde{C}_d$$

and consider the following open subset of $[\rho_{e,d}, \rho'_{e,d}) \times \tilde{C}_d$

$$\mathcal{Z}_{e,d} = \mathcal{Z}^0_{e,d} \cup \mathcal{Z}^\infty_{e,d}.$$

Let $\eta : [\rho_{e,d}, \rho'_{e,d}) \to \mathbb{R}$ be some smooth, non-increasing function such that

$$\eta(t) = \begin{cases} 1 & t \in [\rho_{e,d}, \rho''_{e,d}], \\ 0 & t \in [\bar{\rho}_{e,d}, \rho'_{e,d}). \end{cases}$$

On the open subset $\mathcal{Z}_{e,d}$ of $\rho_d \times \tilde{C}_d$ define the following function

$$F_{e,d}(t,y) = \begin{cases} \eta(t) G_{e,d}(t,y) + (1-\eta(t))F(y) & \text{on } \mathcal{Z}^0_{e,d}, \\ F(y) & \text{on } \mathcal{Z}^\infty_{e,d}, \end{cases}$$

Definition 6.5. Let

$$\mathcal{Z}_d = (\rho'_d \times \tilde{C}_d) \cup \left(\bigcup_{d \preceq e} \mathcal{Z}_{e,d} \right)$$

and let $F_d : \mathcal{Z}_d \to \mathbb{R}$ be the function which coincides with F on $\rho'_d \times \tilde{C}_d$ and with $F_{e,d}$ on $\mathcal{Z}_{e,d}$. Clearly F_d is smooth. We redefine the local model along the edge as the composition

$$\Phi_d : \quad \mathcal{Z}_d \quad \to \quad \mathcal{Z}_d \times (L_d \times L_d^\perp) \quad \to \quad M_\mathbb{R} \times N_\mathbb{R}/N,$$

where the first map is the graph of the differential of F_d and the second map is (45). As usual we let $\mathbf{h}_d$ be the composition of Φ_d with the projection onto $M_\mathbb{R}$. We denote this local model by $(\Phi_d, \mathcal{Z}_d)$.

The point of this definition is that we have the equality

$$\Phi_e(\bar{\mathbf{h}}_e^{-1}(\mathcal{H}_{[r_{e,d}, r''_{e,d}]})) = \Phi_d(\mathcal{Z}_d \cap ([\rho_{e,d}, \rho''_{e,d}] \times \tilde{C}_d)) \tag{79}$$

and thus the local models over $\check{e}$ and over $\check{d}$ may be glued along this set.

6.11. **Trimming the new local models over the edges.** We consider a pair (d, f) where $d \in (P, \nu)$ is two dimensional and $f \preceq d$ is an edge of d. Let $(\Phi_d, \mathcal{Z}_d)$ be the local model over $\check{d}$ as defined above. Define

$$\mathbf{h}_{d,f} = \mathbf{x}_f \circ \mathbf{h}_d.$$

and let $\mathbf{y}_{d,f}$ be the projection onto the edge $\tilde{C}_f$ of $\tilde{C}_d$.

Lemma 6.6. The following map is a diffeomorphism onto its image

$$\begin{aligned} \mathbf{g}_{d,f} : \qquad & \mathcal{Z}_d \cap (\rho_d \times \tilde{U}_{d,f}) \to V_{\check{f}} \times \tilde{C}_f \\ & (t,y) \mapsto (\mathbf{h}_{d,f}(t,y), \mathbf{y}_{d,f}(y)). \end{aligned}$$

This implies that $\Phi_d(\mathcal{Z}_d \cap (\rho_d \times \tilde{U}_{d,f})$ is the graph of the differential of a Legendre transform of F_d.

Proof. This is analogous to Proposition 4.4. Clearly the Lemma holds when $\mathbf{g}_{d,f}$ is restricted to $\rho_d' \times \tilde{C}_d$, where the local model coincides with the one in §6.7. So we restrict to $\mathcal{Z}_{e,d}$. Let us describe $\mathbf{h}_{d,f}$ in more detail. For every $t \in \rho_d$, define the slice

$$Z_t = \mathcal{Z}_d \cap (\{t\} \times \tilde{C}_d). \tag{80}$$

and let $F_{d,t} : Z_t \to \mathbb{R}$ be the restriction of F_d. Now let

$$\mathbf{h}_{d,t} : Z_t \to L_d^\perp$$

be the differential of $F_{d,t}$ and let

$$\mathbf{h}_{d,f,t} = \mathbf{x}_{d,f} \circ \mathbf{h}_{d,t}.$$

It is easy to show that

$$\begin{aligned} \mathbf{h}_d(t,y) &= t + \mathbf{h}_{d,t}(y) \\ \mathbf{h}_{d,f}(t,y) &= t + \mathbf{h}_{d,f,t}(y) \end{aligned} \tag{81}$$

and therefore that

$$\mathbf{g}_{d,f}(t,y) = (t + \mathbf{h}_{d,f,t}(y), \mathbf{y}_{d,f}(y)). \tag{82}$$

Define

$$\begin{aligned} \mathbf{g}_{d,f,t} : \quad Z_t \cap (\{t\} \times \tilde{U}_{d,f}) &\to \Gamma_{d,f} \times \tilde{C}_f \\ y \quad &\mapsto (\mathbf{h}_{d,f,t}(y), \mathbf{y}_{d,f}(y)). \end{aligned} \tag{83}$$

In particular $\mathbf{g}_{d,f}$ is a diffeomorphism if and only if $\mathbf{g}_{d,f,t}$ is a diffeomorphism for all t. Clearly there is nothing to prove when $\eta(t) = 0$ or 1, since in this case F_d coincides with $G_{e,d}$ or F and the result follows from Proposition 4.4. For other values of η, $F_{d,t}$ is a linear interpolation between $G_{e,d}$ and F. Given local coordinates $y = (y_1, y_2)$ on Z_t, we know that the hessian (in the y coordinates) of both functions is negative definite by construction, therefore also the hessian of $F_{d,t}$ must be negative definite. In particular also the hessian of $F_{d,t}$ restricted to a fibre of $\mathbf{y}_{d,f}$ is negative definite. It follows that $\mathbf{g}_{d,f,t}$ is a local diffeomorphism and that $\mathbf{h}_{d,f,t}$ restricted to a fibre of $\mathbf{y}_{d,f}$ is injective (compare also with Proposition 4.4). Hence $\mathbf{g}_{d,f}$ is a diffeomorphism. The proof of the last statement follows as in Proposition 4.4. □

Recall the definition of the reference points $r_{d,f}$ and $r'_{d,f}$ in $\Gamma_{d,f}$ given in §6.4. We have the following

Lemma 6.7. The set $(\rho_d + [r_{d,f}, r'_{d,f})) \times \tilde{C}_f$ is in the image of $\mathbf{g}_{d,f}$.

Proof. Consider the description (82) of $\mathbf{g}_{d,f}$ and the map $\mathbf{g}_{d,f,t}$ in (83). Let $Z_t \subset \tilde{C}_d$ be as in (80) and let

$$x_t = t + q \in \rho_d + [r_{d,f}, r'_{d,f}).$$

We have to show that $\{x_t\} \times \tilde{C}_f$ is in the image of $\mathbf{g}_{d,f,t}$. When $t \in \rho'_d$, then $Z_t = \{t\} \times \tilde{C}_d$ and $\mathbf{g}_{d,f,t}$ coincides with the map $\overline{\mathbf{g}}_{d,f}$ from §6.8. Therefore the claim follows from (60) and Corollary 5.6.

Otherwise assume $t \in [\rho_{e,d}, \rho'_{e,d})$. In this case F_d interpolates $G_{e,d}$ and F. Let $G_{e,d,t}$ be the restriction of $G_{e,d}$ to Z_t and denote by $\mathbf{h}^+_{d,t}$ and $\mathbf{h}^-_{d,f}$ the differentials (with respect to the y coordinates) respectively of $G_{e,d,t}$ and F and let $\mathbf{h}^{\pm}_{d,f,t} = \mathbf{x}_{d,f} \circ \mathbf{h}^{\pm}_{d,t}$. Then we have that

$$\begin{aligned} \mathbf{h}_{d,t} &= \eta \mathbf{h}^+_{d,t} + (1-\eta)\mathbf{h}^-_{d,t} \\ \mathbf{h}_{d,f,t} &= \eta \mathbf{h}^+_{d,f,t} + (1-\eta)\mathbf{h}^-_{d,f,t} \end{aligned} \tag{84}$$

Observe that $\mathbf{h}^-_{d,t}$ coincides with the map $\overline{\mathbf{h}}_d$ from §6.8. From Lemma 6.3 we have

$$(\mathbf{h}^-_{d,t})^{-1}(H'_d) \subseteq \tilde{\mathcal{A}}^{e,d}_{\epsilon'_2} \subseteq \mathbf{y}_{e,d}(\bar{\mathbf{h}}_e^{-1}(I^{e,d}_{t-\check{e}})) \subseteq Z_t. \tag{85}$$

Given the segment $I^{d,f}_q \subseteq H'_d$ (see §6.8) and the segment $I^{e,f}_{x_t-\check{e}} \subset I^{e,d}_{t-\check{e}}$ define the following curves in Z_t

$$\gamma^-_{d,f,t} := (\mathbf{h}^-_{d,t})^{-1}(I^{d,f}_q) \quad \text{and} \quad \gamma^+_{d,f,t} := \mathbf{y}_{e,d}(\bar{\mathbf{h}}_e^{-1}(I^{e,f}_{x_t-\check{e}})). \tag{86}$$

We have

$$\gamma^-_{d,f,t} \subset \tilde{\mathcal{A}}^{e,d}_{\epsilon'_2}. \tag{87}$$

Moreover, since $I^{e,f}_{x_t-\check{e}} \subset \mathcal{H}_{r_{e,f}} \cap \mathcal{H}_{r_{e,d}}$, by (68), (73) and (66) we have

$$\gamma^+_{d,f,t} \subset \tilde{\mathcal{W}}^{d,f}_{\epsilon'_2}. \tag{88}$$

Notice that since $\bar{\mathbf{g}}_{e,f}$ maps the curve $\bar{\mathbf{h}}_e^{-1}(I^{e,f}_{x_t-\check{e}})$ one to one onto $(x_t - \check{e}) \times \tilde{C}_f$, we have that $\mathbf{y}_{d,f}$ maps the curve $\gamma^+_{d,f,t}$ one to one onto $\tilde{C}_f$. Similarly $\mathbf{y}_{d,f}$ maps $\gamma^-_{d,f,t}$ one to one onto $\tilde{C}_f$. Moreover, by construction,

$$\mathbf{h}^+_{d,f,t}(\gamma^+_{d,f,t}) = \mathbf{h}^-_{d,f,t}(\gamma^-_{d,f,t}) = q \in [r_{d,f}, r'_{d,f}). \tag{89}$$

Now fix a fibre $\mathbf{y}^{-1}_{d,f}(y')$ of $\mathbf{y}_{d,f}$. Let y^+ and y^- be the unique points where this fibre intersects $\gamma^+_{d,f,t}$ and $\gamma^-_{d,f,t}$ respectively. Now recall that the hessians of $F_{d,t}$, $G_{e,d,t}$ and F restricted to a fibre of $\mathbf{y}_{d,f}$ are all negative definite, in particular $\mathbf{h}_{d,f,t}$, $\mathbf{h}^+_{d,f,t}$ and $\mathbf{h}^-_{d,f,t}$ are all injective. It is then easy to see that (84) and (89) together with inclusions (87) and (88) imply that there is a point y on this fibre of $\mathbf{y}_{d,f}$, between y^+ and y^-, such that

$$\mathbf{h}_{d,f,t}(y) = q.$$

Then $\mathbf{g}_{d,f,t}(y) = (q, y')$. This concludes the proof. □

Definition 6.8 (Provisional)**.** Given a two dimensional $d \in (P, \nu)$, let Φ_d be the map in Definition 6.5. Redefine the trimmed domain $\mathcal{Z}_d$ of Φ_d to be the open set of points (t, y) such that $\Phi_d(t, y)$ is defined and for all edges f of d satisfying $y \in \tilde{U}_{d,f}$, we have

$$\mathbf{h}_{d,f,t}(t, y) < r'_{d,f}.$$

We denote this local model by $(\Phi_d, \mathcal{Z}_d)$. Also denote

$$\mathcal{Z}_{d,f} = \{(t, y) \in \mathcal{Z}_d \,|\, y \in \tilde{U}_{d,f} \text{ and } \mathbf{h}_{d,f}(y, t) \in \rho_d + [r_{d,f}, r'_{d,f})\}. \tag{90}$$

It is clear from the construction that $\mathcal{Z}_d$ is homeomorphic to $\rho_d \times \tilde{C}_d$. It is also clear from Lemma 6.7 that $\mathbf{g}_{d,f}$ gives a diffeomorphism from $\mathcal{Z}_{d,f}$ to $(\rho_d + [r_{d,f}, r'_{d,f})) \times \tilde{C}_f$. Moreover for every $t \in \rho_d$ the set

$$Z_{d,t} = \mathcal{Z}_d \cap (\{t\} \times \tilde{C}_d)$$

is homeomorphic to $\tilde{C}_d$. Also define

$$Z_{d,f,t} = \mathcal{Z}_{d,f} \cap (\{t\} \times \tilde{C}_d).$$

The following lemma controls the size of the image of $\mathbf{h}_d$.

Lemma 6.9. If H'_d is the set defined in (61), we have

$$\mathbf{h}_d(\mathcal{Z}_d) \subseteq \rho_d + H'_d.$$

Moreover, given the set $B_{d,f} \subset L_d^{\perp}$ defined in §6.5, we have that, for all $t \in \rho_d$, $y \in Z_{d,t}$ satisfies

$$\mathbf{h}_{d,t}(y) \in B_{d,f}$$

if and only if $y \in Z_{d,f,t}$.

Proof. For the first inclusion we have to show that for all $t \in \rho_d$

$$\mathbf{h}_{d,t}(Z_{d,t}) \subseteq H'_d.$$

By construction, when $t \in \rho'_d$, then $\mathbf{h}_{d,t}$ coincides with the map $\overline{\mathbf{h}}_d$ from §6.8. Therefore Definition 6.8 implies that $\mathbf{h}_{d,t}(Z_{d,t})$ coincides with H'_d.

Now let $t \in [\rho_{e,d}, \rho'_{e,d})$. We use the description (84) of $\mathbf{h}_{d,t}$. Inclusion (77) implies

$$\mathbf{h}^+_{d,t}(Z_{d,t}) \subset K_d \cap H'_d.$$

Moreover (71) implies that

$$\mathbf{h}^+_{d,t}(\mathcal{A}^{e,d}_{\epsilon'_2}) \subseteq K'_d.$$

Given $y \in Z_{d,t}$, assume

$$\mathbf{h}^-_{d,t}(y) \in H'_d.$$

Then (85) implies $y \in \mathcal{A}^{e,d}_{\epsilon'_2}$. Therefore $\mathbf{h}_{d,t}(y)$ is on the segment between $\mathbf{h}^+_{d,t}(y) \in K'_d$ and $\mathbf{h}^-_{d,t}(y) \in H'_d$. Property (c) of Definition 6.2 implies that $\mathbf{h}_{d,t}(y) \in H'_d$.

On the other hand suppose $y \notin H'_d$, then for some edge f of d

$$\mathbf{h}^-_{d,t}(y) \in \mathcal{H}_{r'_{d,f}}. \tag{91}$$

In particular (60) implies $y \in Z_{d,t} \cap \mathcal{W}^{d,f}_{\epsilon_3}$ and (59) ensures that

$$\mathbf{h}^+_{d,t}(y) \in \mathcal{V}_{d,f}, \tag{92}$$

where the latter set is as in (19) for $J = J_{d,f}$. To prove this, recall that

$$(t - \check{e}) + \mathbf{h}^+_{d,t}(y) = \bar{\mathbf{h}}_e(\bar{\mathbf{g}}^{-1}_{e,d}(t - \check{e}, y)).$$

Let $y' = \bar{\mathbf{g}}^{-1}_{e,d}(t - \check{e}, y)$. Since $y \in \mathcal{W}^{d,f}_{\epsilon_3}$ and by (73), $y' \in \mathbf{y}^{-1}_{e,d}(\mathcal{W}^{d,f}_{\epsilon_3}) \cap \mathcal{W}^{e,d}_{\epsilon_1}$. Therefore $y' \in \mathcal{W}^{e,f}$ and $\bar{\mathbf{h}}_e(y') \in \mathcal{V}_{e,f}$ by Lemma 3.10. In particular this implies (92). Now, (92) together with (77) implies

$$\mathbf{h}^+_{d,t}(y) \in K_{d,f} \cap H'_d.$$

The latter, together with (91) and property (d) of Definition 6.2 implies $\mathbf{h}_{d,t}(y) \in H'_d$. This concludes the proof of the first inclusion.

Now suppose $y \in Z_{d,f,t}$. This implies $\mathbf{h}^-_{d,t}(y) \in \mathcal{H}_{r_{d,f}}$ and, by the above arguments, $\mathbf{h}^+_{d,t}(y) \in K_{d,f} \cap H'_d$. Using properties $(b) - (d)$ of Definition 6.2 and the fact that $\mathbf{h}_{d,f,t}(y) \in [r_{d,f}, r'_{d,f})$, we must have

$$\mathbf{h}_{d,t}(y) \in \mathcal{H}_{r_{d,f}} - \mathcal{H}_{r'_{d,f}} \subseteq B_{d,f}.$$

On the other hand suppose $\mathbf{h}_{d,t}(y) \in B_{d,f}$. In particular

$$\mathbf{h}_{d,f,t}(y) > r_{d,f}.$$

It is then enough to prove that $y \in \tilde{U}_{d,f}$. By the first part of the Lemma and by (62), we must have $\mathbf{h}_{d,t}(y) \in \mathcal{H}_{r_{d,f}} - \mathcal{H}_{r'_{d,f}}$. Then we cannot have $y \in (\mathbf{h}^-_{d,t})^{-1}(H_d)$, since if this were true, the same arguments as above would imply $\mathbf{h}_{d,t}(y) \in H_d$, which contradicts $\mathbf{h}_{d,t} \in \mathcal{H}_{r_{d,f}}$. On the other hand we cannot have $y \in \mathcal{W}^{d,f'}_{\epsilon_3}$ for some $f' \neq f$, since this would imply $\mathbf{h}_{d,t}(y) \in \mathcal{V}_{d,f'}$, while (54) implies $\mathbf{h}_{d,t} \in \mathcal{V}_{d,f}$. Therefore we must have $y \in \mathcal{W}^{d,f}_{\epsilon_3}$. In particular $y \in \tilde{U}_{d,f}$. □

6.12. **The local models over faces.** Given an edge $f \in (P, \nu)$, the goal of this subsection is to define a Lagrangian embedding $\Phi_f : \rho_f \times \tilde{C}_f \to M_{\mathbb{R}} \times N_{\mathbb{R}}/N$ which matches with the previous local models on overlaps.

We need to trim further the local models over vertices by replacing (74) with

$$\mathcal{H}_e^{[2]} = \mathcal{H}_e^{[1]} - \bigcup_{d \preceq e,\, \dim d = 2} \mathcal{H}_{r''_{e,d}}. \tag{93}$$

Then $\mathcal{Z}_e$ is as in Definition 6.4. Define, for a three dimensional e and an edge f of e the sets

$$\bar{Y}_e = \mathcal{H}_e^{[2]} \cap \Gamma_e \quad \text{and} \quad \bar{Y}_{e,f} = \bar{Y}_e \cap Q_{r_{e,f}}.$$

Let us now collect some data on $\rho_f \times \tilde{C}_f$ induced by local models over edges and vertices contained in $\check{f}$. For every three dimensional e containing f, define the following subset of $\mathcal{Z}_e$:

$$\mathcal{Z}_{e,f} = \{ y \in \mathcal{Z}_e \cap \tilde{U}_{e,f} \,|\, \bar{\mathbf{h}}_{e,f}(y) \in \bar{Y}_{e,f} \}.$$

We have that by construction and by Corollary 5.6,

$$\bar{\mathbf{g}}_{e,f} : \mathcal{Z}_{e,f} \to \bar{Y}_{e,f} \times \tilde{C}_f$$

is a diffeomorphism and $\bar{\Phi}_e(\mathcal{Z}_{e,f})$ is the graph of the differential of a function G defined on $\bar{Y}_{e,f} \times \tilde{C}_f$. Let us rename this function by $G_{e,f}$. Notice that by translating the first factor of $\bar{Y}_{e,f} \times \tilde{C}_f$ by $\check{e}$, we can assume that $\bar{Y}_{e,f} \times \tilde{C}_f$ is a subset of $\rho_f \times \tilde{C}_f$.

Similarly, for every two dimensional d containing f, in (90) we defined the subset $\mathcal{Z}_{d,f}$ of $\mathcal{Z}_d$. Then by Lemmas 6.6 and 6.7,

$$\mathbf{g}_{d,f} : \mathcal{Z}_{d,f} \to (\rho_d + (r_{d,f}, r'_{d,f})) \times \tilde{C}_f$$

is a diffeomorphism and $\Phi_d(\mathcal{Z}_{d,f})$ is the graph of the differential of a function G defined on $(\rho_d + (r_{d,f}, r'_{d,f})) \times \tilde{C}_f$. Rename this function by $G_{d,f}$.

Notice that when d is a face of e, the domains of definition of the two functions $G_{e,f}$ and $G_{d,f}$ overlap, but we have the following

Lemma 6.10. When d is a face of e the two functions $G_{e,f}$ and $G_{d,f}$ coincide on the overlap $(\bar{Y}_{e,f} \cap (\rho_d + (r_{d,f}, r'_{d,f}))) \times \tilde{C}_f$.

Proof. This is just a consequence of the fact that the local models over $\check{e}$ and $\check{d}$ coincide on the overlaps (as in (79)) and the two functions are defined via a Legendre transform. $\square$

As a consequence, if we consider all the functions $G_{e,f}$, where e varies among two and three dimensional faces containing f, then these patch together to give a unique smooth function

$$G_f : (\rho_f - \rho'_f) \times \tilde{C}_f \to \mathbb{R}.$$

We now extend G_f to the whole of $\rho_f \times \tilde{C}_f$ by interpolating it with the zero function. Let ρ''_f be the third inner polyhedron satisfying (49) and consider a smooth function $\eta : \rho_f \to \mathbb{R}$ such that $0 \leq \rho \leq 1$ and

$$\eta(x) = \begin{cases} 1 & \text{if } x \in \rho_f - \rho''_f \\ 0 & \text{on a neighborhood of } \rho'_f \end{cases}$$

Define $F_f : \rho_f \times \tilde{C}_f \to \mathbb{R}$ by

$$F_f(x, y) = \begin{cases} \eta(x) G_f(x, y) & \text{if } x \in \rho_f - \rho'_f \\ 0 \quad \text{if } x \in \rho'_f. \end{cases}$$

Definition 6.11. Given an edge $f \in (P, \nu)$, let

$$\mathcal{Z}_f = \rho_f \times \tilde{C}_f.$$

The local model over $\check{f}$ is the composition

$$\Phi_f : \quad \mathcal{Z}_f \quad \to \quad \mathcal{Z}_f \times (L_f^{\perp} \times L_f) \quad \to \quad M_{\mathbb{R}} \times N_{\mathbb{R}}/N,$$

where the first map is the graph of the differential of F_f and the second map is (45). We also denote by $\mathbf{h}_f$ the composition of Φ_f with projection to $M_{\mathbb{R}}$.

Lemma 6.12. We have

$$\mathbf{h}_f(\mathcal{Z}_f) \subseteq \rho_f + B_f$$

Proof. We have that the differential of F_f decomposes as the sum $dF_f = d_x F_f + d_y F_f$, i.e. as the sum of the differentials with respect to the x and y coordinates respectively. By the identification of L_f and $L_f^{\perp}$ with the cotangent fibres of ρ_f and T_f respectively, we have that $d_x F_f \in L_f$ and $d_y F_f \in L_f^{\perp}$. Then

$$\mathbf{h}_f(x, y) = x + d_y F_f.$$

When $(x, y) \in \rho'_f \times \tilde{C}_f$, then $F_f = 0$, therefore $\mathbf{h}_f(x, y) \in \rho'_f \subseteq \rho_f + B_f$. Otherwise, when $x \in \rho_f - \rho'_f$, then

$$\mathbf{h}_f(x, y) = x + \eta d_y G_f.$$

Let

$$\mathbf{h}_f^{+}(x, y) = x + d_y G_f.$$

If $x \in \bar{Y}_{e,f}$ for some three dimensional e containing f, then $G_f = G_{e,f}$ and by construction

$$\mathbf{h}_f^+(x,y) - \check{e} = \bar{\mathbf{h}}_e(\bar{\mathbf{g}}_{e,f}^{-1}(x - \check{e}, y))$$

i.e. $\mathbf{h}_f^+(x,y) - \check{e} \in \mathcal{H}_{r_{e,f}}$. Therefore, by (69), $\mathbf{h}_f^+(x,y) \in \rho_f + B_f$. In particular, since $\eta(x) \in [0,1]$, also $\mathbf{h}_f(x,y) \in \rho_f + B_f$.

Similarly, if $(x,y) \in (\rho_d + (r_{d,f}, r'_{d,f})) \times \tilde{C}_f$ for some two dimensional d containing f, then $G_f = G_{d,f}$ and

$$\mathbf{h}_f^+(x,y) = \mathbf{h}_d((\mathbf{g}_{d,f})^{-1}(x,y)).$$

Therefore $\mathbf{h}_f^+(x,y) \in \rho_f + B_f$ by Lemma 6.9. In particular also $\mathbf{h}_f(x,y) \in \rho_f + B_f$. □

6.13. **The last step.** We now glue all the pieces together to form the smooth Lagrangian submanifold $\mathcal{L}$ lifting Ξ.

Definition 6.13 (Final)**.** Given a three dimensional $e \in (P,\nu)$, consider the local model at $\check{e}$ given by (58) (rescaled as in §6.9). Redefine the sets $\mathcal{H}_e^{[1]}$ and $\mathcal{H}_e^{[2]}$ as

$$\begin{aligned} \mathcal{H}_e^{[1]} &= \mathcal{H}_e - \bigcup_{f \preceq e,\, \dim f = 1} \mathcal{H}_{r''_{e,f}}, \\ \mathcal{H}_e^{[2]} &= \mathcal{H}_e^{[1]} - \bigcup_{d \preceq e,\, \dim d = 2} \mathcal{H}_{r''_{e,d}}. \end{aligned} \tag{94}$$

Then restrict the domain of Φ_e to $\mathcal{Z}_e = \bar{\mathbf{h}}_e^{-1}(\mathcal{H}_e^{[2]})$. Notice that we have

$$\mathcal{H}_e^{[2]} \cap \Gamma_e = Y''_e - \check{e},$$

where the latter set was defined in §6.4. For every edge of e we also redefine the subsets $\mathcal{Z}_{e,f}$ of $\mathcal{Z}_e$ as

$$\mathcal{Z}_{e,f} = \{y \in \mathcal{Z}_e \cap \tilde{U}_{e,f} \,|\, \bar{\mathbf{h}}_{e,f}(y) \in (Y''_e - \check{e}) \cap Q_{r_{e,f}}\}.$$

Notice that

$$\mathcal{Z}_{e,f} = \bar{\mathbf{h}}_e^{-1}(\mathcal{H}_e^{[2]} \cap \mathcal{H}_{r_{e,f}}) \tag{95}$$

Similarly we trim the local models along the edges.

Definition 6.14 (Final)**.** Given a two dimensional $d \in (P,\nu)$, let Φ_d be the map in Definition 6.5. We redefine the trimmed domain $\mathcal{Z}_d$ of Φ_d to be the open set of points (t,y) such that $\Phi_d(t,y)$ is defined and for all edges f of d satisfying $y \in \tilde{U}_{d,f}$, we have

$$\mathbf{h}_{d,f,t}(t,y) < r''_{d,f}.$$

We denote this local model by $(\Phi_d, \mathcal{Z}_d)$. For all edges f of d also denote

$$\mathcal{Z}_{d,f} = \{(t,y) \in \mathcal{Z}_d \,|\, y \in \tilde{U}_{d,f} \text{ and } \mathbf{h}_{d,f}(t,y) \in \rho_d + [r_{d,f}, r''_{d,f})\}. \tag{96}$$

Notice that by construction we have the following overlaps. Given a three dimensional e, for every two dimensional face d of e we have

$$\Phi_e(\bar{\mathbf{h}}_e^{-1}(\mathcal{H}_{[r_{e,d}, r''_{e,d})})) = \Phi_d(\mathcal{Z}_d \cap ([\rho_{e,d}, \rho''_{e,d}) \times \tilde{C}_d)), \tag{97}$$

while for every edge f of e

$$\Phi_e(\mathcal{Z}_{e,f}) = \Phi_f((Y''_e \cap \rho_f) \times \tilde{C}_f). \tag{98}$$

Given a two dimensional d and an edge f of d we have

$$\Phi_d(\mathcal{Z}_{d,f}) = \Phi_f((\rho_d + [r_{d,f}, r''_{d,f})) \times \tilde{C}_f). \tag{99}$$

Let us now glue all the pieces together.

Definition 6.15. A Lagrangian smooth lift of Ξ is defined to be the following subset of $M_{\mathbb{R}} \times N_{\mathbb{R}}/N$

$$\mathcal{L} = \bigcup_{1 \leq \dim e \leq 3} \Phi_e(\mathcal{Z}_e) \tag{100}$$

Finally we can prove the following.

Theorem 6.16. $\mathcal{L}$ is a closed Lagrangian submanifold of $M_{\mathbb{R}} \times N_{\mathbb{R}}/N$ homeomorphic to $\hat{\Xi}$.

Proof. Since local models are graphs, the subsets $\Phi_e(\mathcal{Z}_e)$ are Lagrangian submanifolds, for all e. It is enough to prove that (97)–(99) are the only possible intersections between local models.

Inclusions (75) and (63), Lemmas 6.9 and 6.12 and conditions (1) of §6.5 imply that given two simplices e and f of (P, ν) then

$$\Phi_e(\mathcal{Z}_e) \cap \Phi_f(\mathcal{Z}_f) \neq \emptyset,$$

if and only if one is a face of the other.

Suppose now that $\check{e}$ is a vertex of an edge $\check{d}$, then Lemma 6.9, inclusion (63) and (76) imply that

$$\mathbf{h}_d(\mathcal{Z}_d) \cap \mathbf{h}_e(\mathcal{Z}_e) \subseteq \mathcal{H}_{[r_{e,d}, r''_{e,d})}.$$

Thus (97) implies that the latter inclusion is an equality and that

$$\Phi_d(\mathcal{Z}_d) \cap \Phi_e(\mathcal{Z}_e) = \Phi_e(\bar{\mathbf{h}}_e^{-1}(\mathcal{H}_{[r_{e,d}, r''_{e,d})})) = \Phi_d(\mathcal{Z}_d \cap ([\rho_{e,d}, \rho''_{e,d}) \times \tilde{C}_d)).$$

Similar arguments, using Lemmas 6.12 and 6.9, show in the remaining cases that, whenever $\check{e} \preceq \check{d}$, then $\Phi(\mathcal{Z}_e)$ and $\Phi(\mathcal{Z}_d)$ intersect as expected. This concludes the proof of the fact that $\mathcal{L}$ is a submanifold. The closure of $\mathcal{L}$ is a consequence of the construction.

Let us prove that $\mathcal{L}$ is homeomorphic to $\hat{\Xi}$. Let us first describe a decomposition of $\hat{\Xi}$. Given an $e \in (P, \nu)$, of dimension 2 or 3, consider the subset $Y_e'' \subset \Xi$ as defined in §6.4 and let $\hat{Y}_e'' \subset \hat{\Xi}$ be its PL-lift. Then

$$\hat{\Xi} = \left(\bigcup_{\dim e = 2,3} \hat{Y}_e'' \right) \cup \left(\bigcup_{\dim f = 1} (\rho_f'' \times \tilde{C}_f) \right).$$

On the other hand we also have the following decomposition of $\mathcal{L}$

$$\mathcal{L} = \left(\bigcup_{\dim e = 3} \Phi_e \left(\overline{\mathcal{Z}}_e \right) \right) \cup \left(\bigcup_{\dim d = 1,2} \Phi_d \left(\overline{\mathcal{Z}}_d \cap (\rho_d'' \times \tilde{C}_d) \right) \right),$$

where $\overline{\mathcal{Z}}_e$ and $\overline{\mathcal{Z}}_d$ denote the closures of those sets inside $\tilde{C}_e$ and $\rho_d \times \tilde{C}_d$ respectively. By construction and by Proposition 3.13 we have the homeomorphism

$$\Phi_e \left(\overline{\mathcal{Z}}_e \right) \cong \hat{Y}_e''.$$

Similarily

$$\Phi_d' \left(\overline{\mathcal{Z}}_d' \cap (\rho_d'' \times \tilde{C}_d) \right) \cong \hat{Y}_d''$$

when d has dimension 1 or 2. It is also clear that one can arrange these homeomorphisms to match on the intersections.

To construct a family $\mathcal{L}_t$ which converges to $\hat{\Xi}$ in the Hausdorff topology one can uniformly scale the local models by some parameter t and then glue everything together as above. □

7. On more general examples and applications

In [11] we gave various generalizations and examples in the case of Lagrangian lifts of tropical curves. We expect that similar generalizations and examples extend to the case of tropical surfaces, although with some additional subtleties. We briefly comment here these ideas, referring the reader either to [11] when the details are a straight forward generalization or to future work in the more delicate cases. We will use the same notations as in Section 6.

7.1. **Different lifts of the same tropical hypersurface.** As we did for curves in §5.1 of [11], we can twist the Lagrangian lift of a tropical hypersurface by local sections. Let $\check{f}$ be a polyhedron of Ξ of dimension $k = 1, \ldots, n$ and let $C_f \subset N^f_{\mathbb{R}}/N^f$ be the standard coamoeba associated to f. Given a smooth section

$$\sigma_f : \check{f} \to \check{f} \times T \tag{101}$$

Define

$$\hat{f}_{\sigma_f} = C_f \cdot \sigma_f, \tag{102}$$

where the righthand side means that for every $x \in \check{f}$, we consider the set $C_f \cdot \sigma_f(x)$ as a subset of the orbit of $\sigma_f(x)$ under the action of $N^f_{\mathbb{R}}/N^f$ on T. Given the quotient

$$\alpha : \check{f} \times T \to \check{f} \times \frac{T}{N^f_{\mathbb{R}}} \tag{103}$$

then the righthand side is naturally a symplectic manifold. We have that $\hat{f}_{\sigma_f}$ is Lagrangian (at its smooth points) if and only if $\alpha \circ \sigma_f$ is a Lagrangian section of the quotient. So we must impose this condition. Now define the twisted PL-lift to be

$$\hat{\Xi}_\sigma = \left(\bigcup_{\dim e = n+1} \check{e} \times C_e \right) \cup \left(\bigcup_{1 \leq \dim f \leq n} \hat{f}_{\sigma_f} \right).$$

In order for this to be a topological manifold we must impose suitable boundary conditions on the sections σ_f, so that everything matches nicely. The smoothing $\mathcal{L}_\sigma$ of Ξ_σ can be done by suitably adapting the proof of Section 6.

Remark 7.1. We expect that such lifts should be classified by a sheaf of multivalued piecewise linear integral functions, in the spirit of the Gross-Siebert program [6]. Some examples of Lagrangian spheres constructed from piecewise linear integral functions were given in [5], where the underlying tropical surface was just a disk. Moreover, we also expect that the difference $\mathcal{L} - \mathcal{L}_\sigma$ should be, in some sense, related to Lagrangian lifts of lower dimensional tropical varieties. For instance, suppose the lift $\mathcal{L}_\sigma$ is constructed from a piecewise linear integral functions σ, then the difference should be related to the tropical subvariety given by the non-smooth locus of σ. For the relevance of the different lifts of the same tropical variety in homological mirror symmetry see Section 6.3 of [1] and [5].

7.2. **Non smooth tropical hypersurfaces.** We expect to be able to lift also non-smooth tropical hypersurfaces, namely those given by not necessarily unimodal subdivisions of P. An easy case is when $P \subset N_{\mathbb{R}}$ is an integral $n+1$-dimensional simplex (not elementary), with no subdivision. Indeed let $N' \subset N$ be the smallest sublattice in which P is an elementary integral simplex and let $M' \subset M_{\mathbb{R}}$ be its dual. Then the associated tropical subvariety $\Xi \subset M_{\mathbb{R}}$ is a standard tropical hyperplane as a tropical subvariety of $M'_{\mathbb{R}}$. Denote the torus

$$T' = \frac{N_{\mathbb{R}}}{N'}.$$

Inside T' we have the standard Lagrangian coamoeba C' associated to P and Ξ. The action of N' on T defines a covering map

$$\beta : T \to T'.$$

Then we can define

$$C = \beta^{-1}(C')$$

Given the function $F' : C' \to \mathbb{R}$ defined in (11), we let

$$F = F' \circ \beta$$

on C_e. We define the Lagrangian lift of Ξ to be the graph of the differential of F extended to the real blow up of C_e at its vertices.

Example 7.2. An interesting case is when $N = \mathbb{Z}^{n+1}$, $\{u_1, \dots, u_{n+1}\}$ is the standard basis, u_0 is defined as in (5) and

$$P = \mathrm{Conv}\{u_0, \dots, u_{n+1}\}.$$

Then $\beta : T \to T'$ is a covering of degree $n+2$. The associated tropical hypersurface Ξ is the fan whose rays are generated by the vectors

$$\xi_0 = u_0, \quad \text{and} \quad \xi_j = u_0 + (n+2)u_j$$

and the maximal cones are those spanned by all collections of n rays.

7.3. **Lagrangian submanifolds in toric varieties.** We wish to generalize to higher dimensions the examples given in Section 6 of [11] of Lagrangian submanifolds inside a toric variety which lift tropical curves in the moment polytope. We have not yet worked out all the details, since the construction is not as straight forward as in the case of curves, therefore we will only sketch some examples and point out where the difficulties are. Let $\dim M_{\mathbb{R}} = 3$ and let $\Delta \subset M_{\mathbb{R}}$ be a Delzant polyhedron. Denote by $\partial\Delta$ its boundary and by Δ° its interior. Let X_Δ be the associated toric variety, recall that $\Delta^\circ \times T \subset X_\Delta$.

Given a tropical hypersurface $\Xi^\infty \subset M_{\mathbb{R}}$ and $\mathcal{L}^\infty$ a Lagrangian lift of Ξ^∞. Define

$$\Xi = \Delta \cap \Xi^\infty.$$

Then the lift $\mathcal{L}$ of Ξ inside X_Δ is formed by taking the closure of $\mathcal{L}^\infty \cap (\Delta^\circ \times T)$ inside X_Δ. The question is: how nice is $\mathcal{L}$? When is it a smooth submanifold, with or without boundary? In the case of curves and given certain conditions on how Ξ^∞ intersects $\partial\Delta$, it turns out that $\mathcal{L}$ is automatically a smooth manifold with boundary or, in some nicer cases, a smooth manifold without boundary. Some times $\mathcal{L}$ is a non-orientable surface (see [13] or §6.2 of [11]).

In the case of tropical surfaces, it is not hard to find conditions such that $\mathcal{L}$ is a smooth manifold with boundary and corners, but it is not obvious how to obtain smooth manifolds without boundary. The problem is understanding the interaction of $\mathcal{L}$ with the toric boundary of X_Δ.

Example 7.3. This example generalizes Examples 6.2 and 6.3 of [11] and Mikhalkin's tropical wave fronts (Example 3.3 of [13]). The polyhedron Δ is given by an intersection of half spaces

$$\Delta = \bigcap_{\delta} \{ \langle d_\delta, x \rangle \geq t_\delta \}$$

where the boundary of each half space contains a two dimensional face δ of Δ such that d_δ is its inward integral primitive normal direction. Consider the smaller polyhedron inside Δ given by

$$\Delta_\epsilon = \bigcap_{\delta} \{ \langle d_\delta, x \rangle \geq t_\delta + \epsilon \}$$

for some small ϵ. For each edge τ of Δ, let τ_ϵ be the corresponding edge of Δ_ϵ. Consider the two dimensional polyhedron

$$\ell_\tau = \operatorname{Conv}(\tau \cup \tau_\epsilon).$$

Define the tropical surface

$$\Xi = \partial \Delta_\epsilon \cup \left(\bigcup_{\tau} \ell_\tau \right),$$

where the union runs over all edges of Δ. See Figure 11 for a picture of Ξ in the case Δ is a standard simplex. It can be easily seen that since Δ is Delzant, Ξ is smooth and its boundary coincides with the union of the edges of Δ. Each ℓ_τ has the following property. Given its tangent space $M_{\mathbb{R}}^{\ell_\tau}$, choose a basis $\{v_1, v_2\}$ of the lattice M^{ℓ_τ} such that v_1 is tangent to τ. Then we have that for each two dimensional face δ containing τ

$$\langle d_\delta, v_2 \rangle = 1.$$

This is analogous to what we called property (P) in §6.1 of [11] or in Mikhalkin's terminology ℓ_τ is bisectrice (see Definition 1.12 of [13]). In particular each vertex of Δ is the endpoint of an edge of Ξ, all of whose adjacent two dimensional polyhedra are bisectrices. We ask whether one can construct a smooth Lagrangian lift $\mathcal{L}$. We believe this is true but we do not have a complete proof yet. The bisectrice property of the polyhedra ℓ_τ make it possible to construct a lift which is smooth over interior points of the edges τ. The difficulty lies in proving that the lift can be smoothed also over the vertices of Δ. As suggested by Mikhalkin, it would be interesting to follow the dynamics of Δ_ϵ beyond small values of ϵ, such as described by Kalinin and Shkolnikov in [8]. Can this dynamic be translated in a smooth family of Lagrangians?

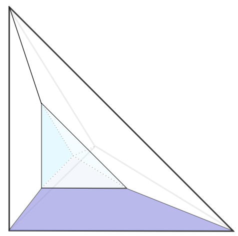

FIGURE 11. The tropical hypersurface Ξ. For clarity, only one of the polyhedra ℓ_τ is colored.

Example 7.4. As a limit case of the above example, let Δ be the polytope of $\mathbb{P}^3$, i.e. the standard simplex in $\mathbb{R}^3$, and let $q \in \Delta$ be its barycenter. For every edge τ of Δ let

$$\ell_\tau = \mathrm{Conv}(\tau \cup q)$$

and define

$$\Xi = \bigcup_\tau \ell_\tau.$$

Then Ξ is a tropical hypersurface which, in a neighborhood of the vertex q in Example 7.2. Therefore we can use the lift constructed there to find the lift $\mathcal{L}$ of Ξ inside $\mathbb{P}^3$. As in the previous example we have not yet proven that one can smooth the lift over the vertices of Δ. We expect $\mathcal{L}$ to be homeomorphic to $S^1 \times S^2$. This example generalizes the monotone Example 6.5 of [11], so it should also be monotone.

Example 7.5. This example in $\mathbb{C}^3$ generalizes Mikhalkin's examples [13] of tropical curves representing non-orientable Lagrangian surfaces in $\mathbb{C}^2$. Let

$$\Delta = (\mathbb{R}_{\geq 0})^3,$$

then $X_\Delta = \mathbb{C}^3$. Consider the points

$$Q_0 = (0,0,0), \quad Q_1 = (3,3,3),$$
$$P_{1,1} = (4,3,3), \quad P_{2,1} = (7,2,2), \quad P_{3,1} = (12,0,0).$$

Let $P_{j,k}$ be the point obtained from $P_{j,1}$ by exchanging the first and the k-th coordinate. Define three dimensional polytopes

$$\Sigma_1 = \mathrm{Conv}\{Q_1, P_{1,1}, P_{1,2}, P_{1,3}\},$$
$$\Sigma_2 = \mathrm{Conv}\{P_{1,1}, P_{1,2}, P_{1,3}, P_{2,1}, P_{2,2}, P_{2,3}.\}$$

Clearly Σ_1 is a standard simplex and Σ_2 is a truncated simplex. Define the two dimensional polytopes

$$\beta_1 = \mathrm{Conv}\{Q_0,\ Q_1,\ P_{1,1},\ P_{2,1},\ P_{3,1}\}$$
$$\gamma_1 = \mathrm{Conv}\{P_{2,2},\ P_{3,2},\ P_{2,3},\ P_{3,3}\}.$$

Let β_k and γ_k be obtained from β_1 and γ_1 by the symmetry exchanging the first and k-th coordinate. Now let

$$\Xi = \partial\Sigma_1 \cup \partial\Sigma_2 \cup \left(\bigcup_k \beta_k \right) \cup \left(\bigcup_k \gamma_k \right).$$

It can be checked that this is a smooth tropical hypersurface in Δ, see Figure 12.

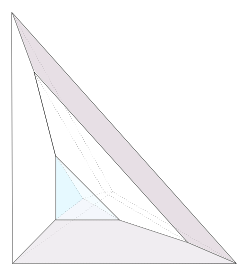

FIGURE 12. We have only colored $\partial\Sigma_1$, β_1 and γ_2

The two dimensional polyhedra which hit the boundary of Δ are the β_k's and γ_k's. We have that β_k has one edge lying on a coordinate axis of Δ and it is a bisectrice (see previous example). The γ_k's have an edge lying on a coordinate plane δ of Δ. They have the property that if $\{v_1, v_2\}$ is a basis of the lattice M^{γ_k} such that v_1 is tangent to δ, then

$$\langle d_\delta, v_2 \rangle = 2, \tag{104}$$

where d_δ is the inward, primitive integral normal direction of δ. This is analogous to the condition satisfied by the edges of tropical curves representing non-orientable surfaces (see §3.4 of [13]).

Therefore, it seems reasonable to expect that such a tropical hypersurface admits a smooth (non-orientable) Lagrangian lift $\mathcal{L}$ in $\mathbb{C}^3$. Indeed the above properties guarantee that $\mathcal{L}$ can be constructed so that it is smooth everywhere except over the points Q_0, $P_{3,1}$, $P_{3,2}$, $P_{3,3}$

which are the points where an edge of Ξ hits the boundary of Δ. While the point Q_0 is of the type already present in Example 7.3 (i.e. the vertices of Δ), the points $P_{3,k}$ have a different nature. They are the end points of an edge of Ξ which is adjacent to a bisectrice (i.e. β_k) and two polyhedra satisfying (104) (i.e. two of the γ_j's).

7.4. **Lagrangian submanifolds of Calabi-Yau manifolds.** An interesting generalization of the above constructions would be to find Lagrangian submanifolds inside the symplectic Calabi-Yau manifolds with a Lagrangian torus fibration constructed in [2], based on Gross's topological torus fibrations [3]. Indeed, given a symplectic manifold (X, ω) with a Lagrangian torus fibration $f : X \to B$, let B_0 be the locus in B of smooth fibres and let $D = B - B_0$ be the discriminant locus. Action coordinates on B define an integral affine structure on B_0, i.e. an atlas with change of coordinate maps inside $\mathrm{GL}_n(\mathbb{Z}) \ltimes \mathbb{R}^n$. Therefore B_0 is a natural ambient space where tropical subvarieties can be defined. If we also have a Lagrangian section $\sigma : B \to X$, then the Arnold-Liouville theorem tells us that $X_0 = f^{-1}(B_0)$ is symplectomorphic to T^*B_0/Λ, where Λ is a lattice of maximal rank in T^*B_0. Therefore, locally X_0 is like $M_{\mathbb{R}} \times N_{\mathbb{R}}/N$. Hence we can define the Lagrangian PL lift $\hat{\Xi}$ of a tropical hypersurface Ξ in B_0. If $\dim_{\mathbb{R}} B_0 = 3$ then we can also find a smoothing $\mathcal{L}_0 \subset X_0$ of $\hat{\Xi}$. Suppose now that Ξ is a tropical hypersurface which has boundary on the discriminant locus D. What is the closure $\mathcal{L}$ of $\mathcal{L}_0$? When is it a smooth manifold, without boundary? The Lagrangian 3-torus fibrations constructed in [2] have prescribed singular fibres modeled on those described [3]. Indeed D is a (thickening of a) 3-valent graph, with two types of singular fibres over the vertices: positive and negative. We believe that it should not be hard to understand when the closure $\mathcal{L}$ of $\mathcal{L}_0$ is smooth. Indeed the examples in [5] of Lagrangian spheres were constructed using this idea. The following examples are inside a symplectic Calabi-Yau homeomorphic to the quintic threefold in $\mathbb{P}^4$.

Example 7.6. In [3] and [4], Gross describes a 3-valent graph D inside a 3-sphere B and an integral affine structure on $B_0 = B - D$ such that one can compactify $X_0 = T^*B_0/\Lambda$ to a topological manifold X by adding canonical singular fibres over D. Gross proves that X is homeomorphic to a smooth quintic threefold in $\mathbb{P}^4$. In [2] it is shown that one can find a symplectic form on X (extending the natural one on X_0) so that the fibration is Lagrangian. The 3-sphere B is identified with the boundary $\partial\Delta$ of the standard simplex in $\mathbb{R}^3$. Let $\Delta^{[2]}$ be the two skeleton of $\partial\Delta$, i.e. the union of two dimensional faces. Then $D \subset \Delta^{[2]}$ and D divides $\Delta^{[2]}$ in 105 connected components. Each of these components is a smooth tropical hypersurface with boundary on D. The components are divided into three different types which are pictured in Figure 13. Type (a) are contained in the interior of

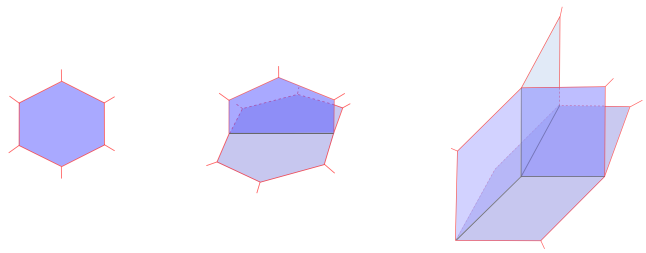

FIGURE 13.

2-faces of Δ and there are 60 of these (6 in each face). Type (b) are defined along edges of Δ and there are 40 of these (4 along each edge). Type (c) are defined around vertices of Δ and there are 5 of these. In Example 4.10 of [5] it is shown how to construct Lagrangian spheres over type (a) components. It should be possible, combining the methods of this article with a detailed analysis of the interaction of $\mathcal{L}_0$ with the singular fibres, to construct smooth Lagrangian submanifolds (spheres?) over components of type (b) and (c). Similarly we should be able to construct Lagrangian submanifolds over tropical curves with boundary on D using the constructions in [13] and [11], together with a similar analysis of interactions with the singular fibres. For an explicit construction of Lagrangian lifts of tropical curves in the mirror of the quintic, using toric degenerations, see also [10].

REFERENCES

[1] Paul S. Aspinwall, Tom Bridgeland, Alastair Craw, Michael R. Douglas, Mark Gross, Anton Kapustin, Gregory W. Moore, Graeme Segal, Balázs Szendroi, and P. M. H. Wilson. *Dirichlet branes and mirror symmetry*, volume 4 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI, 2009.

[2] R. Castano-Bernard and D. Matessi. Lagrangian 3-torus fibrations. *J. Differential Geom.*, 81(3):483–573, 2009. arXiv:math/0611139.

[3] M. Gross. Topological Mirror Symmetry. *Invent. Math.*, 144:75–137, 2001.

[4] M. Gross, D. Huybrechts, and D. Joyce. *"Calabi-Yau manifolds and related geometries" Lecture notes at a summer school in Nordfjordeid, Norway, June 2001*. Springer Verlag, 2003.

[5] M. Gross and D. Matessi. On homological mirror symmetry of toric Calabi-Yau three-folds. *J. Symplectic Geom.*, 16(5):1249–1349, 2018. arXiv:1503.03816.

[6] M. Gross and B. Siebert. Mirror Symmetry via Logarithmic degeneration data I. *J. Differential Geom.*, 72(2):169–338, 2006. math.AG/0309070.
[7] Jeff Hicks. Tropical Lagrangians and Homological Mirror Symmetry. arXiv:1904.06005.
[8] Nikita Kalinin and Mikhail Shkolnikov. Introduction to tropical series and wave dynamic on them. arXiv:1706.03062.
[9] Gabriel Kerr and Ilia Zharkov. Phase tropical hypersurfaces. *Geom. Topol.*, 22(6):3287–3320, 2018. arXiv:1610.05290.
[10] Cheuk Yu Mak and Helge Ruddat. Tropically constructed Lagrangians in mirror quintic threefolds. arXiv:1904.11780.
[11] Diego Matessi. Lagrangian pairs of pants. To appear in International Mathematics Research Notices. doi:10.1093/imrn/rnz126. arXiv:1802.02993.
[12] G. Mikhalkin. Decomposition into pairs-of-pants for complex algebraic hypersurfaces. *Topology*, 43(5):1035–1065, 2004.
[13] G. Mikhalkin. Examples of tropical-to-Lagrangian correspondences. *European Journal of Mathematics*, 5(3):1033–1066, 2019. arXiv:1802.06473.
[14] Mounir Nisse and Frank Sottile. Non-Archimedean coamoebae. In *Tropical and non-Archimedean geometry*, volume 605 of *Contemp. Math.*, pages 73–91. Amer. Math. Soc., Providence, RI, 2013. arXiv:1110.1033.
[15] Vivek Shende, David Treumann, Harold Williams, and Eric Zaslow. Cluster varieties from Legendrian knots. To appear in Duke Mathematical Journal. arXiv:1512.08942.
[16] Nick Sheridan and Ivan Smith. Lagrangian cobordism and tropical curves. arXiv:1805.07924.
[17] David Treumann, Harold Williams, and Eric Zaslow. Kasteleyn operators from mirror symmetry. arXiv:1810.05985.

Diego MATESSI
Dipartimento di Matematica
Università degli Studi di Milano
Via Saldini 50
I-20133 Milano, Italy
E-mail address: diego.matessi@unimi.it