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A note on moduli spaces of conformal classes for flat tori of higher dimension and on their conformal multiplication

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Abstract

Motivated by the theory of *complex multiplication* of abelian varieties, in this paper we study the conformality classes of flat tori in $\mathbb{R}^n$ and investigate criteria to determine whether a n-dimensional flat torus has non trivial (i.e. bigger than $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$) semigroup of conformal endomorphisms (the analogs of *isogenies* for abelian varieties). We then exhibit several geometric constructions of tori with this property and study the class of conformally equivalent lattices in order to describe the moduli space of the corresponding tori.

Keywords Flat tori · Conformal multiplication

1 Introduction

In this paper we consider n-dimensional flat tori $\mathbb{T}^n_\Gamma$ as quotients of $\mathbb{R}^n$ under the action of a subgroup $\Gamma \cong \mathbb{Z}^n$ induced by a (maximal rank) lattice $\Lambda \subset \mathbb{R}^n$. For this reason we also use the notation $\mathbb{T}^n_\Lambda$ for the same torus.

In order to introduce a torus as a quotient of $\mathbb{R}^n$, one can give a lattice Λ and an equivalence relation in terms of this lattice i.e., the space of orbits. But one can also define flat tori starting from a (parallel) polytope or from a tessellation cell of $\mathbb{R}^n$. Indeed, in correspondence to a *tessellation* of the n-dimensional Euclidean space $\mathbb{R}^n$, we can consider the lattice of the barycenters of the cells of the tessellation, call it Λ_B and then the torus obtained from this lattice, namely $\mathbb{T}_B$.

One can then study the corresponding conformal classes for tori and determine the moduli space of conformal tori. In general, in $\mathbb{R}^n$ two real tori $\mathbb{T}^n_\Lambda$ and $\mathbb{T}^n_{\Lambda'}$ are conformal if and only if

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their lattices can be obtained one from the other via a conformal linear map i.e. $\Lambda' = \mu A(\Lambda)$ with $A \in \mathrm{SO}(n)$, $\mu > 0$.

Definition 1 If $\mathbb{T}^n_\Lambda$ is a flat n-dimensional torus a conformal endomorphism is a group epimorphism $\iota : \mathbb{T}^n_\Lambda \to \mathbb{T}^n_\Lambda$ such that ι lifts to a conformal linear map $\tilde{\iota} : \mathbb{R}^n \to \mathbb{R}^n$. We denote by $End_c(\mathbb{T}^n_\Lambda)$ the set of conformal endomorphisms of $\mathbb{T}^n_\Lambda$.

Then $End_c(\mathbb{T}^n_\Lambda)$ with product given by composition of functions is a unitary semigroup.

Definition 2 The subset of $End_c(\mathbb{T}^n_\Lambda)$ consisting of invertible endomorphims is the group $Aut_c(\mathbb{T}^n_\Lambda)$ of conformal automorphisms of $\mathbb{T}^n_\Lambda$.

Sometimes it is convenient to include the trivial map $x \overset{\iota}{\mapsto} 0$ in the definition of conformal endomorphism; for instance, in the case of elliptic curves with complex multiplication, one can then consider the ring of endomorphisms where addition is the pointwise addition of morphisms of an elliptic curve to itself, and product is given by composition of morphisms.

A conformal endomorphism is the analog for a flat torus of an isogeny for an elliptic curve or an abelian variety (see [10,16,22]). Given an integer $m \in \mathbb{Z}, m \neq 0$, we have the conformal transformation $[m] : \mathbb{T}^n_\Lambda \to \mathbb{T}^n_\Lambda$, $\mathbf{t} \simeq (e^{it_1}, e^{it_2}, \ldots, e^{it_n}) \overset{[m]}{\mapsto} m\mathbf{t} \simeq (e^{imt_1}, e^{imt_2}, \ldots, e^{imt_n})$. Therefore for any lattice Λ, $End_c(\mathbb{T}^n_\Lambda)$ contains the multiplicative semigroup $\mathbb{Z}^*$ of non zero integers under the multiplication operation. The kernel of $[m]$ consists of the m-torsion points of $\mathbb{T}^n_\Lambda$.

Our aim is to describe those tori that have *conformal multiplication*, i.e. those tori whose semigroup of endomorphisms $End_c(\mathbb{T}^n_\Lambda)$ is larger than $\mathbb{Z}^*$.

This problem arises from the natural generalization of the complex case, in which the conformal multiplication is usually called *complex multiplication* (see [2,3,15,21,22]). Let us recall briefly this fact. Since a flat metric on a real 2-torus induces a complex structure, the torus is a complex one dimensional curve of genus one i.e. a complex one-dimensional torus. All conformally equivalent metrics induce the same complex structure. Therefore if one considers $\mathbb{C} \simeq \mathbb{R}^2$, the complex curve is of the form $E_\tau := \mathbb{C}/\Lambda_\tau$, where Λ_τ is the lattice generated by $(\mathbf{1}, \tau)$ with $\mathrm{Im}\tau > 0$. These curves are called *elliptic curves if one fixes a base point in the curve*. Complex tori obtained by considering the quotient of $\mathbb{C}$ under the action induced by the (square) lattice $\Lambda_{\mathcal{C}}$ or by the (hexagonal) lattice $\Lambda_{\mathcal{F}}$, whose sets of generators respectively are $\mathcal{C} = (\mathbf{1}, i)$ and $\mathcal{F} = (\mathbf{1}, \tau)$ with $\mathbf{1} = (1, 0)$ and $\tau = e^{i\pi/3}$, have complex multiplication. Indeed, $\Lambda_{\mathcal{C}}$ and $\Lambda_{\mathcal{F}}$ are the ring of integers of the imaginary quadratic fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, i.e. the Gaussian integers and Eisenstein integers, respectively, so multiplication by elements of this rings provides isogenies of the corresponding elliptic curves (see also [3]).

It is well known that, starting from a lattice, it is possible to define a canonical fundamental region of the lattice, namely the *Dirichlet–Voronoi region*. The two lattices $\Lambda_{\mathcal{C}}$ and $\Lambda_{\mathcal{F}}$ have, for example, as corresponding Voronoi regions, respectively, a square and a regular hexagon.

We will give examples of tori with conformal multiplications in higher dimensions, inspired by the fact that, in general, the corresponding conformal group of automorphisms is closely related to the symmetry group of the Voronoi region associated with the lattice that defines the torus. This is due to the fact that if a finite group of $\mathrm{SO}(n)$ leaves invariant a lattice then the group also preserves the Voronoi region. This happens very clearly in the case of complex dimension 1. Indeed, all the examples of tori with conformal automorphisms are defined by lattices whose Voronoi regions (triangles, hexagons and squares) have many symmetries. The lattices from these examples have particular importance either in Crystallography or in application for Coincidence Site Lattice Theory (see [7]). Lattices with a large group of symmetries play also an important role in the theory of sphere packings and the theory of simple groups (for instance the *monster group*, see [9]; see also [21]). Furthermore, we study the class of conformally equivalent lattices and describe the moduli space of the corresponding tori and use a similar approach (as in [4]) for biregular quaternionic tori to study their moduli space in higher dimension.

2 Flat tori: lattices and tessellations in $\mathbb{R}^n$

In order to define a n-dimensional torus it is sufficient to give a lattice Λ of maximal rank in $\mathbb{R}^n$. Indeed, starting from such a lattice Λ in $\mathbb{R}^n$, we can find $n+1$ points of the lattice Λ, call them $O, P_1, \dots, P_n$ such that the vectors $\mathbf{v}_1 = P_1 - O, \mathbf{v}_2 = P_2 - O, \dots, \mathbf{v}_n = P_n - O$ form a basis F of $\mathbb{R}^n$ which turns out to be a frame for Λ, in the sense that $\Lambda = \{(m_1\mathbf{v}_1, \dots, m_n\mathbf{v}_n) : (m_1, \dots, m_n) \in \mathbb{Z}^n\}$.

Now, one can consider the group Γ_Λ which acts on $\mathbb{R}^n$ as integer translations in the directions of the vectors of the frame F. In symbols, if $g \in \Gamma_\Lambda$ and $\mathbf{w}_F = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ (whose coordinates are taken with respect to the frame F), the action of Γ_Λ on $\mathbb{R}^n$ is given by

$$g \cdot \mathbf{w}_F := (a_1 + m_1, a_2 + m_2, \dots, a_n + m_n)$$

where $(m_1, m_2, \dots, m_n) \in \mathbb{Z}^n$.

Definition 3 Two frames F_1 and F_2 of $\mathbb{R}^n$ are said to be *equivalent* if they define the same lattice Λ.

Notice that different frames can give rise to the same lattice as shown in the next.

Remark 1 Let $\{\mathbf{e_1}, \mathbf{e_2}, \dots, \mathbf{e_n}\}$ be the standard orthonormal basis in $\mathbb{R}^n$. Then the two frames $F_1 = \{\mathbf{e_1}, \mathbf{e_2}, \dots, \mathbf{e_n}\}$ and $F_2 = \{A(\mathbf{e_1}), A(\mathbf{e_2}), \dots, A(\mathbf{e_n})\}$ where $A \in SL(n, \mathbb{Z})$, are equivalent since they generate the same standard cubic lattice Λ_C in $\mathbb{R}^n$.

Let $F = (\mathbf{u}_1, \dots, \mathbf{u}_n)$, $\mathbf{u}_i \in \mathbb{R}^n$ be a positively oriented frame (i.e. a positively oriented basis) of $\mathbb{R}^n$ and let $M(F)$ be the $n \times n$ matrix whose j-th column is $\mathbf{u}_j$, $\forall j = 1, \dots, n$. We will consider now a relation for n-dimensional lattices induced by the following.

Definition 4 The lattice Λ_1 and Λ_2 (of the same rank) are said to be *conformally equivalent* if there exist a frame F_1 of Λ_1, a frame F_2 of Λ_2, a real number $\mu > 0$ and matrices $O \in SO(n)$ and $A \in GL(n, \mathbb{Z})$ such that

$$AM(F_1) = M(F_2)\ (\mu O). \tag{1}$$

As soon as the group Γ_Λ of integer translations is canonically defined from the lattice Λ, one can also consider the associated n-dimensional flat torus $\mathbb{R}^n/\Gamma_\Lambda := \mathbb{T}^n_{\Gamma_\Lambda}$ equipped with the naturally induced real differentiable structure and Riemannian metric; therefore any such torus is a compact oriented real n-manifold. Furthermore, the quotient map $\pi_{\Gamma_\Lambda} : \mathbb{R}^n \to \mathbb{T}^n_{\Gamma_\Lambda}$ can be also considered as the projection map of the universal covering $(\mathbb{R}^n, \pi_{\Gamma_\Lambda})$ of the torus $\mathbb{T}^n_{\Gamma_\Lambda}$. For the sake of shortening the notation, in the sequel we will also use the notation $\mathbb{T}^n_\Lambda$ or $\mathbb{T}^n_\Gamma$ to indicate $\mathbb{T}^n_{\Gamma_\Lambda}$.

Using charts and local coordinates, one can define diffeomorphisms between tori in the standard way. In particular if the diffeomorphism is conformal the tori are said to be *conformally equivalent*. Equivalently, a way to look at diffeomorphism φ between tori $\mathbb{T}^n_{\Gamma_1}$ and $\mathbb{T}^n_{\Gamma_2}$

is to consider those (differentiable) functions $f : \mathbb{R}^n \to \mathbb{R}^n$ that are *equivariant* with respect to the action induced by Γ_1 and Γ_2, i.e.

$$f(g \cdot \mathbf{w}_{F_1}) = g \cdot (f(\mathbf{w})_{F_2}) \quad \text{for any integer translation } g \text{ and } \mathbf{w} \in \mathbb{R}^n.$$

Actually, by lifting to the universal coverings, these functions can be regarded as functions which make the following diagram commute

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\ f\ } & \mathbb{R}^n \\ {\scriptstyle \pi_{\Gamma_1}}\downarrow & & {\scriptstyle \pi_{\Gamma_2}}\downarrow \\ \mathbb{T}^n_{\Gamma_1} & \xrightarrow{\ \varphi\ } & \mathbb{T}^n_{\Gamma_2} \end{array}$$

Remark 2 Since any torus $\mathbb{T}^n_\Gamma$ is an abelian compact Lie group with a natural flat invariant metric, it follows that any equivariant translation in $\mathbb{R}^n$ induces an isometry and thus a conformal self-map of $\mathbb{T}^n_\Gamma$. Furthermore, for each flat torus $\mathbb{T}^n_\Gamma$, the universal covering projection map $\pi_\Gamma : \mathbb{R}^n \to \mathbb{T}^n_\Gamma$ is a local isometry.

Proposition 1 *Conformally equivalent tori have conformally equivalent lattices.*

Proof If $\mathbb{T}^n_{\Lambda_1}$ is conformally equivalent to $\mathbb{T}^n_{\Lambda_2}$ and $\varphi : \mathbb{T}^n_{\Lambda_1} \to \mathbb{T}^n_{\Lambda_2}$ is an orientation-preserving conformal diffeomorphism, then precomposing φ with a translation, we can assume that φ fixes the origin of the lattices. Let $\tilde{\varphi} : \mathbb{R}^n \to \mathbb{R}^n$ be the lifting of φ which fixes 0. Then $\tilde{\varphi}$ is a conformal map of $\mathbb{R}^n$. If $n = 2$ it follows that $\tilde{\varphi}$ is an orientation preserving homothetic transformation of $\mathbb{R}^2$ into itself. If $n > 2$, it follows from Liouville's theorem that $\tilde{\varphi}$ is a Möbius transformation that fixes the origin and therefore must be of the form $\mathbf{v} \mapsto \mu O(\mathbf{v})$ with $O \in SO(n)$ and μ a positive real number. If we choose two positively oriented frames F_1 and F_2 respectively, then there exists $A \in GL^+(n, \mathbb{Z})$ such that

$$AM(F_2) = \mu O M(F_1),$$

for a positive real number μ, i.e., Λ_1 is conformally equivalent to Λ_2. In particular, one can actually choose $A' \in SL(n, \mathbb{Z})$ and $\mu' > 0$ such that

$$A'M(F_2) = \mu' O M(F_1),$$

and so the claim follows.

Another way to study flat tori is to approach these tori as special manifolds in the class of parallel orbifolds as introduced in [26]. If one considers a not necessarily convex polytope P of maximal dimension n in $\mathbb{R}^n$ with an even number of $(n-1)$-subpolytopes (called "faces of P") contained in the boundary of P, such that, given any face S of P, there exists another face S' of P which is the image of S under a translation of $\mathbb{R}^n$; in short if $S' = S + \mathbf{v}$, with $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} \neq 0$, then the identification of such parallel (and isometrical) faces of P defines the quotient space which inherits a structure of topological orbifold, known as *parallel orbifold*.

As a non trivial example of parallel orbifold one can consider the quotient space obtained starting from the polytope P which is the union of two isometric regular dodecahedra in $\mathbb{R}^3$ glued along a face. The orbifold obtained via identification of opposite parallel isometric faces by translations is a natural generalization of the surfaces called *translational surfaces* as introduced in [26].

In some cases, parallel orbifolds may become parallel manifolds i.e. they don't have singularities. Flat tori $\mathbb{R}^n/\Gamma$ are examples of *parallel manifolds* (with Γ a group acting on $\mathbb{R}^n$ via integer translations, where any fundamental region for the action of Γ plays the role of P).

More generally, one can give an interpretation of flat tori looking at uniform tessellations of $\mathbb{R}^n$, with special polytopes as tiling cells, in the following sense

Definition 5 A *tessellation* or *tiling* of $\mathbb{R}^n$ is a cover by a finite number of countable families of closed sets, called tiles, such that the tiles intersect only on their boundaries and two tiles in the same family are congruent.

We define a *uniform* tessellation Θ of $\mathbb{R}^n$ to be any tessellation of $\mathbb{R}^n$ made by regions of $\mathbb{R}^n$ or cells such that any cell can be reached from an other by means of a finite number of translations along prescribed n directions of $\mathbb{R}^n$.

For example any *regular* tessellation (i.e. whose cells are regular polytopes with an even number of faces) turn out to be a uniform evenly-sided polytopic tessellation of $\mathbb{R}^n$; since in $\mathbb{R}^n$ (for any n) there is always the cubic tessellation which is a regular tessellation, one can say that there is an evenly-sided polytopic tessellation Θ in any $\mathbb{R}^n$.

Observe that, starting from any tessellation Θ of $\mathbb{R}^n$, one can consider the barycenters of the tiling cells which form a lattice Λ_{B_Θ} of $\mathbb{R}^n$ and then define the torus $\mathbb{T}^n_{\Lambda_{B_\Theta}}$. It is easily seen that the torus obtained as a parallel manifold, starting from the cubic tessellation is the same as the torus obtained from its barycentric lattice.

Notice that different cells can give rise to the same barycentric lattice. For example, following the classification of hexagons that tile the space, due to Reinhardt [18], we can consider the two hexagons in Fig. 1 with opposite sides parallel and isometric. These define the same barycentric lattice and thus the same torus.

Example 1 If we consider the plane and the tessellation given by equilateral triangles, this tiling doesn't define a parallel manifold, indeed we have that the polytope has not an even number of edges. Consider now the "duplication" of these triangles, gluing every two triangles along an edge, we get a rhombic tiling of the plane, where the rhombi are special (indeed their angles measure $\pi/3$ and $2\pi/3$) and this tiling defines a parallel manifold, which is moreover a torus (this is a torus with complex multiplication).

We point out that the tori corresponding to the three different tiling cells, namely regular hexagons, equilateral triangles and rhombi as above are conformally equivalent. Indeed their barycentric lattices are conformally equivalent. In particular the torus corresponding to the lattice with tiling cells equilateral triangles is a cover of the torus with tiling cells regular hexagons.

Remark 3 If we consider the cubic tessellation of $\mathbb{R}^3$ and the barycenters of the cubes, taking the edges from the vertices of the cube to the barycenter we get six pyramids. Each pyramid obtained in this way is a tiling cell of $\mathbb{R}^3$ which does not define a parallel manifold and the corresponding tessellation of $\mathbb{R}^3$ is not uniform. With the same argument used in the plane, we now consider the octahedron obtained gluing the square basis of two pyramids as a tiling cell. In this case, it defines a parallel manifold, but the corresponding tessellation of $\mathbb{R}^3$ is not uniform. Finally, gluing together six of these octahedra sharing a common vertex, one gets the *rhombic dodecahedron* as tiling cell which gives rise to a parallel manifold and the corresponding tessellation of $\mathbb{R}^3$ is uniform (see Fig. 2). We point out that the manifolds corresponding to the three different tiling cells are conformally equivalent tori. Indeed the barycentric lattices are conformally equivalent.

As seen in the previous considerations, one can associate a lattice starting with a tessellation. But also the converse is true. The idea of Dirichlet–Voronoi cell (also Dirichlet cell or

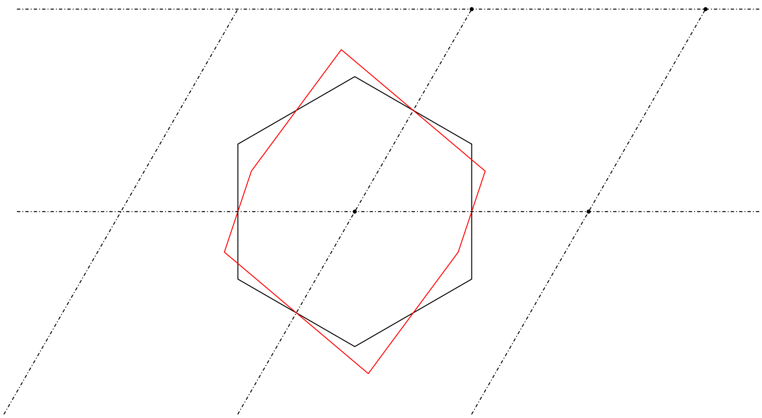

Fig. 1 Different hexagonal tiles (both with parallel pair of sides) associated with the same lattice

Voronoi region, briefly D–V cell[1]) was introduced in two classical papers by Dirichlet and Voronoi, respectively.

Definition 6 Let us give a discrete point set L in the n-dimensional Euclidean space $\mathbb{R}^n$. The Dirichlet–Voronoi tiling of L is a tiling with convex tiles $D(z)$ centered at $z \in L$, where

$$D(z) = \{y \in \mathbb{R}^n \ : \ |y - z| \leq |y - x|\}$$

$\forall x \in L$, where the function $| \cdot |$ is the usual Euclidean norm of $\mathbb{R}^n$.

This means that $D(z)$ consists of those points y of $\mathbb{R}^n$ whose distance from the point z (the origin of D) is not greater than its distance from any other point of the set L. The tile $D(z)$ is called the cell of z; when L is a lattice these cells are translated copies of the cell $D(0)$ of the origin.

Therefore if one considers a Dirichlet–Voronoi tiling (i.e. a tessellation obtained from a lattice and considering the corresponding Voronoi regions as tiles), then

Proposition 2 *There is a one to one correspondence between Voronoi tessellations and tori associated with barycentric lattices.*

Remark 4 The tessellations of $\mathbb{R}^2$ as in Example 1 are not Dirichlet–Voronoi tessellations.

Example 2 A standard cubic lattice Λ_c gives rise to the cubic honeycomb and a hexagonal close-packed lattice gives rise to a tessellation of the space with trapezo-rhombic dodecahedra. The body-centred cubic (BCC) lattice of $\mathbb{R}^3$ (i.e. the cubic lattice together with the barycenters of any cube) has as Dirichlet–Voronoi regions truncated octahedra which define a uniform evenly–sided polytopic tessellation of $\mathbb{R}^3$. Similarly, a face-centred cubic (FCC) lattice of $\mathbb{R}^3$ (i.e. the cubic lattice together with the barycenters of the faces of each cube) has as Dirichlet–Voronoi regions the rhombic dodecahedra which again define a uniform evenly–sided polytopic tessellation of $\mathbb{R}^3$. Certain body centred tetragonal lattices give rise to a tessellation of $\mathbb{R}^3$ with rhombo-hexagonal dodecahedra.[2]

[1] In Crystallography the notion of Dirichlet–Voronoi regions is in relations with *Brillouin zones* delimited by Bragg planes, the planes perpendicular to a connection line from the origin to each lattice point and passing through the midpoint of two vertices.

[2] Some of these notions have several applications in Crystallography, and in particular some of these lattices are known as *Bravais* lattices.

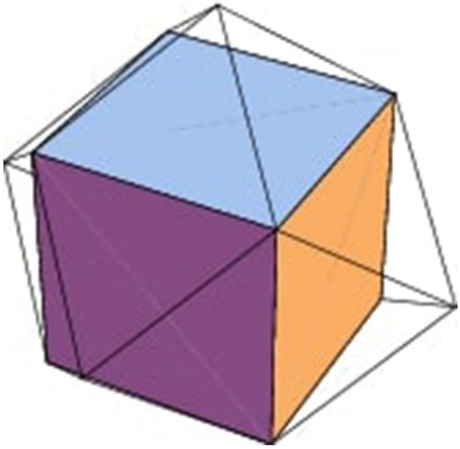

Fig. 2 Construction of the rhombic dodecahedron

Another way to construct uniform polytopic tessellation of $\mathbb{R}^n$ is obtained by starting from a uniform tessellation Θ in $\mathbb{R}^{n-1}$ and a vector $\mathbf{w}$ in $\mathbb{R}^n \setminus \mathbb{R}^{n-1}$; then one can consider the infinite polytopes obtained by taking as sides the straight lines in the directions of $\mathbf{w}$ passing through the vertices of the regular tessellation Θ in $\mathbb{R}^{n-1}$. These are examples of uniform polytopic tessellations of $\mathbb{R}^n$ and will be called *pencils*).

Beside the case of $\mathbb{R}^n$ (which represents the *flat* model), one can consider the problem of uniformly tessellating different spaces and then the quotient spaces of the spheres S^n and the hyperbolic spaces $\mathbb{H}^n$, which are, respectively, the elliptic and hyperbolic models. For example, there is a dodecahedral tessellation of 3-dimensional sphere S^3 and similarly $\mathbb{H}^3$ admits a regular tessellations of hyperbolic dodecahedra.

The tessellations of S^n and $\mathbb{H}^n$ are very complicated and so are the related theory of quotient spaces which certainly deserve further investigations which the authors aim to complete in a future paper.[3,4]

3 Complex multiplication for complex tori and further extensions

Given a uniform polytopic tessellation Θ of $\mathbb{R}^n$ one then considers the quotient space $\mathbb{R}^n/\Lambda_{B_\Theta}$ which will be also denoted by $\mathbb{R}^n/\Theta$ for the sake of shortness. In principle, if a uniform polytopic tessellation Θ of $\mathbb{R}^n$ has many symmetries, then there should be (crystallographic) groups acting on $\mathbb{R}^n$ which preserve Θ and therefore induce endomorphisms of the quotient

[3] In general the major difficulty in the description of uniform polytopic tiling is the lack of results on the solid angles of the polytopes involved. On the other hand, in $\mathbb{R}^n$ there are many interesting tools for this purpose. Among the others, we recall that the *Dehn invariant*.

[4] It is named after Max Dehn, who used it to solve Hilbert's third problem on whether all polyhedra with equal volume could be dissected into each other. of a polyhedron is a value used to determine whether polyhedra can be dissected into each other or whether they can tile space. Two polyhedra have a dissection into polyhedral pieces that can be reassembled into another one, if and only if their volumes and Dehn invariants are equal. A polyhedron can be cut up and reassembled to tile space if and only if its Dehn invariant is zero, so having Dehn invariant zero is a necessary condition for being a space-filling polyhedron. It is also an open problem whether the Dehn invariant of a self-intersection free flexible polyhedron is invariant as it flexes.
The Dehn invariant is zero for the cube but nonzero for the other Platonic solids, implying that the other Platonic solids cannot tile space and that they cannot be dissected into a cube. All of the Archimedean solids have Dehn invariants that are rational combinations of the invariants for the Platonic solids.

space. In particular, for every $\mathbb{R}^n/\Theta$, one can consider the endomorphism induced by the function $z \mapsto -z$, beside the identity. The set of all endomorphisms of $\mathbb{R}^n/\Theta$ (not necessarily conformal) is a ring (where addition is pointwise addition of functions and the standard composition as product) which will be denoted by $End(\mathbb{R}^n/\Theta)$ and which contains the group of automorphisms of $\mathbb{R}^n/\Theta$, i.e. $Aut(\mathbb{R}^n/\Theta)$.

The ring of holomorphic endomorphisms of an elliptic curve $E_\tau := \mathbb{C}/\Lambda_\tau$ defined over the field $\mathbb{C}$ can be of one of two forms: (i) the integers $\mathbb{Z}$, (ii) an order in an imaginary quadratic number field. If the elliptic curve is defined over a field of positive characteristic there is a third possibility, namely (iii) an order in a definite quaternion algebra over $\mathbb{Q}$ (see [19] Corollary 9.4 for precise statement and definitions).

One says that an elliptic curve (or a complex torus) has conformal multiplication if its ring of endomorphisms is not $\mathbb{Z}$.
Complex tori are also particular examples of abelian varieties and, in general, manifolds with complex multiplication turn out to be very interesting, in fact we have the following quotation by David Hilbert:

"*The theory of complex multiplication is not only the most beautiful part of mathematics but also of all science.*"

It is known that complex tori with non trivial group of automorphisms are (up to biholomorphisms) the ones corresponding to $\tau = i$ and to $\tau = e^{\frac{i\pi}{3}}$ (see e.g. [6,24]). Notice that the corresponding lattices give rise to the regular tessellations (square and hexagonal, respectively) of the plane. Complex tori with complex multiplication are those with the following (equivalent) property

$$(m + l\tau)\Lambda_\tau \subseteq \Lambda_\tau$$

with $m, l \in \mathbb{Z}$. This condition says that τ is a solution of a quadratic equation with rational coefficients, i.e. τ satisfies

$$\tau^2 + A\tau + B = 0, \tag{2}$$

with $A, B \in \mathbb{Q}$. Notice that the root of the previous equation, for τ such that $Im\tau > 0$ is $\tau = a + ib$ with $a = -\dfrac{A}{2}$ and $b = \dfrac{\sqrt{4B - A^2}}{2}$. With some additional computations one can prove that the only solutions in the fundamental domain for the modular group (see [8,19]) are $\tau = i$ ($A = 0, B = 1$) and $\tau' = e^{\frac{\pi i}{3}}$ (i.e $A = -1, B = 1$).

In $\mathbb{C}$ two lattices Λ_1 and Λ_2 of rank 2 are said to be *homothetic* if there is a complex number α such that $\alpha\Lambda_1 = \Lambda_2$.

Remark 5 If there exists an $\alpha \in \mathbb{C}\backslash\mathbb{Z}$ such that $\alpha\Lambda \subseteq \Lambda$ then the mapping $\varphi_\alpha : \Lambda \to \Lambda$ such that $\varphi_\alpha(\lambda) = \alpha\lambda$ is a conformal mapping in $\mathbb{C}$ which preserves the lattice Λ and if $\vartheta_\alpha \in [0, 2\pi)$ is the principal argument of α the mapping φ_α can be written as $\lambda \mapsto |\alpha|e^{i\vartheta_\alpha}\lambda$. In this way one can easily recognise that φ_α is a dilation of modulus $|\alpha|$ and a rotation of angle ϑ_α.

For an example of (complex) multiplication on a manifold of dimension greater than 2, we can consider the product of the two elliptic curves $M = E_\tau \times E_{\tau'}$, with $\tau = i$ and $\tau' = e^{\frac{\pi i}{3}}$; in this case, M turns out to be an abelian variety. The complex multiplication in M is realised by the action induced by the product $(m + \tau l, r + \tau' s)$ of non trivial endomorphisms on E_τ and $E_{\tau'}$; thus, when applied on the unitary generators $(\mathbf{1}, \mathbf{1})$, in order to be homothetic we get

$$(m + \tau l)(m + \overline{\tau} l) = (r + s\tau')(r + s\overline{\tau'})$$

where $m, l, r, s \in \mathbb{Z}$. Hence, for τ and τ' as above, one has

$$r^2 + s^2 - rs = m^2 + l^2$$

that is satisfied by any pythagorean triple $(m, l, r = s)$, i.e. any three integers such that $m^2 + l^2 = r^2$. Similar examples can be generalized in any dimension when considering suitable splitting of $\mathbb{R}^n = \mathbb{R}^{m_1} \oplus \mathbb{R}^{m_2}$ (with $m_1 + m_2 = n$) in order to use complex multiplication on the factors; all these examples will be said to be in the class of "decomposable" spaces.

Definition 7 A torus $\mathbb{R}^n/\Theta$ has

- only trivial conformal endomorphisms if $End(\mathbb{R}^n/\Theta) \simeq \mathbb{Z}$;
- nontrivial conformal endomorphisms if $\mathbb{Z} \underset{\neq}{\subset} End(\mathbb{R}^n/\Theta)$.

Those tori $\mathbb{R}^n/\Theta$ with semigroup of endomorphisms bigger than $\mathbb{Z}^*$ are the n-dimensional analogs of *elliptic curves with complex multiplication*.

Remark 6 Since any torus $\mathbb{T}^n_\Gamma$ is an abelian compact Lie group with a natural flat metric (see [15]), it follows that any translation is an isometry and thus a conformal self-map of $\mathbb{T}^n_\Gamma$ without fixed points. Furthermore, for each flat torus $\mathbb{T}^n_\Gamma$, the universal covering projection map $\pi_\Gamma : \mathbb{R}^n \to \mathbb{T}^n_\Gamma$ is a local isometry.

Assume now that $n > 2$ and $\varphi : \mathbb{T}^n_\Gamma \to \mathbb{T}^n_\Gamma$ is a *conformal* transformation with a fixed point $x_0 \in \mathbb{T}^n_\Gamma$; then the corresponding lifting $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a conformal transformation in $\mathbb{R}^n$. From Liouville Theorem on conformal mappings in high dimensions [5] it follows that Φ—with respect to a basis in $\mathbb{R}^n$—can be written as $\lambda \cdot O$, where λ is a positive real number and O is an orthogonal matrix of $\mathbb{R}^n$ such that $\lambda \cdot O(\Lambda) \subseteq \Lambda$ like in the case of the mapping φ_α showed in Remark 5. In other words φ is an endomorphisms of $\mathbb{T}^m_\Lambda$. Therefore

Proposition 3 *If $n > 2$, a torus $\mathbb{T}^n_\Lambda$ has conformal multiplication if there exists a conformal map $f : \mathbb{T}^n_\Lambda \to \mathbb{T}^n_\Lambda$ with a fixed point p such that the derivative at p is not a homothetic transformation, (i.e. $df(p) : T_p\mathbb{T}^n_\Lambda \to T_p\mathbb{T}^n_\Lambda$ is not of the form μI, $\mu > 0$). Then by Liouville Theorem f lifts to an affine map $\tilde{f}$ on $\mathbb{R}^n$ which preserves the lattice Λ. After composing with a translation, we can assume that $\tilde{f} = \mu O$ with $\mu > 0$, $O \in SO(n)$ and $O \neq I$.*

Remark 7 As already observed, the result of Proposition 3 is a generalization of what was mentioned in Remark 5 for complex tori; furthermore it justifies the use of the term *conformal* multiplication to extend the notion of *complex* multiplication as in the complex case.

From the geometric ideas of the previous considerations, we give the following

Definition 8 Given a lattice Λ in $\mathbb{R}^n$, we say that $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is

- a *Λ-orthogonal transformation* if Φ is an orthogonal transformation of $\mathbb{R}^n$ (with respect to the standard Euclidean inner product) which preserves the lattice Λ, i.e. such that $\Phi(\Lambda) \subseteq \Lambda$.
- a *Λ-conformal transformation* if there is a positive real number ϱ and an orthogonal transformation Ψ such that $\Phi = \varrho\Psi$ and $\Phi(\Lambda) \subseteq \Lambda$.

In general, for orthogonal transformations preserving hypercubic lattices there is a very exhaustive result (see [1]) which makes use of Clifford algebra approach as a computational tool to apply the well-known Cartan–Dieudonné Theorem.

Proposition 4 *Given a hypercubic lattice Λ in $\mathbb{R}^n$, an orthogonal transformation Ψ preserves Λ if and only if there exist lattice vectors $c_1, \ldots, c_k \in \Lambda$ such that*

$$\Psi(x) = (-1)^k (c_1 c_2 \cdots c_k) x (c_1 c_2 \cdots c_k)^{-1}$$

where the product of the vectors c_j's is to be interpreted in terms of the product in the Clifford algebra $\mathbb{R}_{n,0}$ which contains $\mathbb{R}^n$ as the space $\mathbb{R}^1_{n,0}$.

We recall that an orthogonal transformation T in $\mathbb{R}^n$ is a conformal isometry with respect to the Euclidean norm of $\mathbb{R}^n$. Now we can state the most general result for Λ-conformal transformations in $\mathbb{R}^n$, with Λ a lattice in $\mathbb{R}^n$ which contains a frame for $\mathbb{R}^n$ consisting of vectors of the same length which are pairwise orthogonal.

Theorem 1 *Let Λ be a lattice in $\mathbb{R}^n$ which contains a frame $\mathcal{F}$ for $\mathbb{R}^n$ consisting of vectors of the same length which are pairwise orthogonal. Let M be the $n \times n$ matrix which represents the change of bases in $\mathbb{R}^n$ from the basis $\mathcal{F}$ to the canonical basis $\mathcal{E}$ of $\mathbb{R}^n$. Then any transformation of the form*

$$\Phi(x) = \varrho M^{-1}[(-1)^k (c_1 c_2 \cdots c_k) M(x) (c_1 c_2 \cdots c_k)^{-1}]$$

(*where ϱ is a positive real number and the product of the vectors c_j in the hypercubic lattice is defined as in Proposition 4*) *is a Λ-conformal transformation.*

4 Examples of tori with conformal multiplication

The next task is to give examples of flat tori $\mathbb{T}^n_\Lambda$ with conformal multiplication in dimension $n > 2$. We begin by considering the examples already exhibited of the FCC (face-centered cubic) and BCC (body-centered cubic) lattices[5] in $\mathbb{R}^3$: for these two lattices we know that the corresponding Voronoi regions (namely the rhombic dodecahedron and the truncated octahedron) have many symmetries so that the associated tori are expected to have conformal multiplications. This intuition is now confirmed by Theorem 1, since both FCC and BCC lattices contain the hypercubic lattice of $\mathbb{R}^3$.

Then we will focus our attention to some lattices in $\mathbb{R}^4$, $\mathbb{R}^8$ and $\mathbb{R}^{24}$, namely the Hurwitz lattice in $\mathbb{R}^4 \simeq \mathbb{H}$, the Γ_8 lattice in $\mathbb{R}^8 \simeq \mathbb{O}$ and the Leech lattice in $\mathbb{R}^{24} \simeq \mathbb{O} \times \mathbb{O} \times \mathbb{O}$. One of the reasons for this choice is that these lattices have interesting symmetries and, furthermore, $\mathbb{R}^4 \simeq \mathbb{H}$ (quaternions) and $\mathbb{R}^8 \simeq \mathbb{O}$ (octonions) are (the only other than complex) division algebras.

4.1 Hurwitz quaternionic tori

Let

$$\mathbb{H} = \{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \ : \ x_n \in \mathbb{R}, \ n = 0, 1, 2, 3, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ij} = -\mathbf{ji} = \mathbf{k}\}.$$

be the non commutative division algebra of quaternions.

Definition 9 A *Lipschitz quaternion* (or Lipschitz integer) is a quaternion whose components are all integers. The ring of all Lipschitz quaternions is the subset of quaternions with integer coefficients:

$$Lip := \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H} \ : \ a, b, c, d \in \mathbb{Z}\}\,.$$

[5] These lattices are also known as Bravais lattices in Crystallography.

This is a subring of the ring of *Hurwitz quaternions*:

$$Hur := \left\{ a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H} \ : \ a, b, c, d \in \mathbb{Z} \ \text{ or } \ a, b, c, d \in \mathbb{Z} + \frac{1}{2} \right\}.$$

Indeed it can be proven (see e.g. [26]) that Hur is closed under quaternion multiplication and addition, which makes it a subring of the ring of all quaternions $\mathbb{H}$.

As a group, Hur is free abelian with generators $\frac{1}{2}(1 + \mathbf{i} + \mathbf{j} + \mathbf{k}), \mathbf{i}, \mathbf{j}, \mathbf{k}$. Therefore Hur defines a lattice in $\mathbb{R}^4$. This lattice is known as the F_4 lattice since it is the root lattice of the semisimple Lie algebra $\mathfrak{f}_4$. The Lipschitz quaternions Lip form an index 2 sublattice of Hur and it is a subring of the ring of quaternions. These two lattices will be also denoted by Λ_{Hur} and Λ_{Lip} respectively.

The lattice Λ_{Lip} is precisely the hypercubic lattice in $\mathbb{R}^4$. We now give the proofs of two lemmas.

Lemma 1 *Given an orthonormal frame $\mathcal{F}$ of $\mathbb{R}^4$ in Λ_{Hur}, then either all vectors of $\mathcal{F}$ are in Λ_{Lip} or are in $\Lambda_{Hur} \setminus \Lambda_{Lip}$.*

Proof Indeed, any unitary Hurwitz number orthogonal to the unitary Lipschitz number, say $(0, 1, 0, 0)$ is of the form $(a, 0, c, d)$ with $a^2 + c^2 + d^2 = 1$ and necessarily $a, c, d \in \mathbb{Z}$, i.e. is a unitary Lipschitz number. Now let us consider the case of unitary Hurwitz numbers orthogonal to a unitary Hurwitz non Lipschitz number, say $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. These are of the form (a, b, c, d) such that $a + b + c + d = 0$, $a^2 + b^2 + c^2 + d^2 = 1$ and therefore a, b, c, d cannot be all integers.

The next lemma can be considered as a generalization of the previous one.

Lemma 2 *Given a frame $\mathcal{F}$ of $\mathbb{R}^4$ in Λ_{Hur} whose vectors have the same norm and are pairwise orthogonal, then either all vectors of $\mathcal{F}$ are in Λ_{Lip} or are in $\Lambda_{Hur} \setminus \Lambda_{Lip}$.*

Proof If $w_1 = (a, b, c, d) \in \Lambda_{Lip}$ and $w_1 \in \mathcal{F}$ then any $(x, y, z, t) \in \Lambda_{Hur}$ belongs to $\mathcal{F}$ if and only if $xa + yb + cz + dt = 0$ and $x^2 + y^2 + z^2 + t^2 = a^2 + b^2 + c^2 + d^2$. In other words,

$$(x - a)^2 + (y - b)^2 + (z - c)^2 + (t - d)^2 = 2(a^2 + b^2 + c^2 + d^2).$$

If $(x, y, z, t) \notin \Lambda_{Lip}$ then $x = m_1/2, \ y = m_2/2, \ z = m_3/2, \ t = m_4/2$ with $m_j = 2k_j + 1$ and $k_j \in \mathbb{Z}$ for $j = 1, 2, 3, 4$. Therefore it follows that

$$\begin{aligned} &\frac{(2(k_1 - a) + 1)^2}{4} + \frac{(2(k_2 - b) + 1)^2}{4} + \frac{(2(k_3 - c) + 1)^2}{4} + \frac{(2(k_4 - d) + 1)^2}{4} \\ &\quad = (k_1 - a)^2 + (k_1 - a) + (k_2 - b)^2 + (k_2 - b) \\ &\qquad + (k_3 - c)^2 + (k_3 - c) + (k_4 - d)^2 + (k_4 - d) + 1 = 2(a^2 + b^2 + c^2 + d^2) \end{aligned}$$

and this is a contradiction since the last number is even but the previous ones are odd, due to the fact that for any integer n it turns out that $n^2 + n = n(n + 1)$ is even.

From the proof of Lemma 2, one can also easily obtain a one-to-one correspondence between $\Lambda_{Hur} \setminus \Lambda_{Lip}$ and Λ_{Lip}; indeed if $(m_1/2, m_2/2, m_3/2, m_4/2)$ with $m_j = 2k_j + 1$ and $k_j \in \mathbb{Z}$ for $j = 1, 2, 3, 4$, then the correspondence is given by

$$\left(\frac{m_1}{2}, \frac{m_2}{2}, \frac{m_3}{2}, \frac{m_4}{2}\right) = \left(\frac{2k_1 + 1}{2}, \frac{2k_2 + 1}{2}, \frac{2k_3 + 1}{2}, \frac{2k_4 + 1}{2}\right) \mapsto (k_1, k_2, k_3, k_4).$$

Consider now a Hurwitz–conformal transformation Φ, that is a conformal transformation which preserves the Hurwitz lattice Λ_{Hur}. Therefore, from Theorem 1, we have

Proposition 5 *If Φ is a conformal transformation which preserves the Hurwitz lattice, then there exist a positive real number $\varrho = \sqrt{m}$ ($m \in \mathbb{N}$) and $c_1, \ldots, c_k \in Lip$ such that*

$$\Phi(x) = \varrho M^{\delta'}[(-1)^k(c_1c_2\cdots c_k)M^{\delta}(x)(c_1c_2\cdots c_k)^{-1}]. \tag{3}$$

where $\delta = -1, 0, 1$ $\delta' = -1, 0, 1$ and up to a permutation of rows/columns,

$$M = \begin{pmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix}$$

with $M^0 = I$ and $M^1 = M$ and $M^{-1} = M^T$.

Proof Due to Lemma 2, we distinguish 4 possible cases, namely

(a) Φ is a Λ_{Lip}-conformal transformation, i.e. $\Phi(\Lambda_{Lip}) \subseteq \Lambda_{Lip}$;
(b) Φ is a conformal transformation such that $\Phi(\Lambda_{Lip}) \subseteq \Lambda_{Hur}\backslash\Lambda_{Lip}$;
(c) Φ is a conformal transformation such that $\Phi(\Lambda_{Hur}\backslash\Lambda_{Lip}) \subseteq \Lambda_{Lip}$;
(d) Φ is a conformal transformation such that $\Phi(\Lambda_{Hur}\backslash\Lambda_{Lip}) \subseteq \Lambda_{Hur}\backslash\Lambda_{Lip}$.

If Φ is a conformal transformation which preserves the Lipschitz lattice, then, since the Lipschitz lattice is the hypercube lattice in $\mathbb{R}^4$, we actually conclude from Proposition 4 that there exist $c_1, \ldots, c_k \in Lip$ and a positive real number ϱ such that

$$\Phi(v) = \varrho(-1)^k(c_1c_2\cdots c_k)v(c_1c_2\cdots c_k)^{-1}.$$

Furthermore, from the definition of conformality and from Lemma 2, we know that a Λ_{Lip}-conformal transformation Φ maps an orthogonal frame of Λ_{Lip} onto an orthogonal frame of Λ_{Lip} which is represented by vectors in Λ_{Lip} all of the same norm. Since the squared norm of a Lipschitz number is a natural number, we conclude that $\varrho = \sqrt{m}$ with $m \in \mathbb{N}$ so that Φ is as in (3) with $\delta = \delta' = 0$.

If Φ is a conformal transformation such that $\Phi(\Lambda_{Lip}) \subseteq \Lambda_{Hur}\backslash\Lambda_{Lip}$, then $S^{-1} \circ \Phi$ is a Λ_{Lip}-conformal transformation and then case a) applies so that Φ is as in (3) with $\delta = 0$, $\delta' = 1$.

If Φ is a conformal transformation such that $\Phi(\Lambda_{Hur}\backslash\Lambda_{Lip}) \subseteq \Lambda_{Lip}$, then $\Phi \circ S$ is a Λ_{Lip}-conformal transformation and then case a) applies so that Φ is as in (3) with $\delta = 1$, $\delta' = 0$.

If Φ is a conformal transformation such that $\Phi(\Lambda_{Hur}\backslash\Lambda_{Lip}) \subseteq \Lambda_{Hur}\backslash\Lambda_{Lip}$, then $S^{-1} \circ \Phi \circ S$ is a Λ_{Lip}-conformal transformation and then case a) applies so that Φ is as in (3) with $\delta = \delta' = 1$.

This result is in line with the following result proved in [12].[6]

Proposition 6 *The orientation preserving self-similarities of Hur onto itself are precisely the maps $M_{(u,v)}[\mathbf{q}] = u\mathbf{q}\bar{v}$ for $u, v \in Hur$ with $|u| = |v| = 1$ and the maps $\frac{1}{2}M_{(u,v)}$ for $u, v \in Hur$ with $|u|^2 = |v|^2 = 2$.*

[6] Actually in [12] all finite groups of automorphism of complex tori of (complex) dimension 2 are completely classified and listed.

4.2 Γ_8 torus

The Γ_8 lattice is a discrete full rank subgroup of $\mathbb{R}^8$. It can be given explicitly by the set of points in $\mathbb{R}^8$ such that all the coordinates are integers or all the coordinates are half-integers (a mixture of integers and half-integers is not allowed) and the sum of the eight coordinates is an even integer.

In symbols,

$$\Gamma_8 = \left\{ (x_l) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \tfrac{1}{2})^8 : \sum_i x_l \equiv 0 \ (\mathrm{mod}\ 2) \right\}.$$

The lattice Γ_8 is also known as the E_8 lattice since it is the root lattice of the Lie algebra $\mathfrak{e}_8$. The lattice Γ_8 can be characterized as the unique lattice in $\mathbb{R}^8$ with the following properties: it is integral, meaning that all scalar products of lattice elements are integer numbers. It is unimodular, meaning that it is integral, and can be generated by the columns of an 8×8 matrix with determinant ± 1 (i.e. the volume of the fundamental parallelotope of the lattice is 1). Equivalently, Γ_8 is self-dual, meaning it is equal to its dual lattice. It is even, meaning that the norm of any lattice vector is even.

The minimal norm of a non trivial vector in Γ_8 is $\sqrt{2}$, actually there are 240 elements in Γ_8 with this minimal distance from the origin. Among them we consider the following 8 vectors

$$\begin{pmatrix}1\\0\\0\\0\\1\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\\0\\0\\1\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\\0\\0\\0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\0\\1\\0\\0\\0\\1\end{pmatrix} \begin{pmatrix}1/2\\1/2\\1/2\\1/2\\-1/2\\-1/2\\-1/2\\-1/2\end{pmatrix} \begin{pmatrix}1/2\\-1/2\\-1/2\\1/2\\-1/2\\1/2\\1/2\\-1/2\end{pmatrix} \begin{pmatrix}1/2\\-1/2\\1/2\\-1/2\\-1/2\\1/2\\-1/2\\1/2\end{pmatrix} \begin{pmatrix}-1/2\\-1/2\\1/2\\1/2\\1/2\\1/2\\-1/2\\-1/2\end{pmatrix}$$

which represent a frame for $\mathbb{R}^8$ in Γ_8 formed by orthogonal vectors of the same (minimal) length. Therefore a result similar to Lemma 1 or to Lemma 2 does not hold in Γ_8.

There are other frames of $\mathbb{R}^8$ formed by vectors of minimal length in Γ_8 which are orthogonal and whose coordinates are all integers or half integers. Thus it is not possible to choose (up to permutation of rows/columns) a preferred matrix M of change of bases as in Proposition 5 for the Hurwitz lattice case.

For instance one can choose

$$\begin{pmatrix}1&0&0&0&1&0&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&1&0\\0&0&0&1&0&0&0&1\\1&0&0&0&-1&0&0&0\\0&1&0&0&0&-1&0&0\\0&0&1&0&0&0&-1&0\\0&0&0&1&0&0&0&-1\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}1/2&1/2&1/2&1/2&1/2&1/2&1/2&1/2\\1/2&-1/2&-1/2&1/2&1/2&-1/2&-1/2&1/2\\1/2&-1/2&1/2&-1/2&1/2&-1/2&1/2&-1/2\\-1/2&-1/2&1/2&1/2&-1/2&-1/2&1/2&1/2\\1/2&1/2&1/2&1/2&-1/2&-1/2&-1/2&-1/2\\1/2&-1/2&-1/2&1/2&-1/2&1/2&1/2&-1/2\\1/2&-1/2&1/2&-1/2&-1/2&1/2&-1/2&1/2\\-1/2&-1/2&1/2&1/2&1/2&1/2&-1/2&-1/2\end{pmatrix}.$$

These matrices induce conformal transformations which map the canonical orthonormal basis of $\mathbb{R}^8$ onto the frame given by the columns of these matrices. Furthermore, these column vectors are in Γ_8, pairwise orthogonal and of minimal length. In any case, if $\mathcal{U}$ is a frame of

$\mathbb{R}^8$ in Γ_8 formed by orthogonal vectors $u_1, \ldots, u_8$ of minimal length, there exists a conformal transformation $M_\mathcal{U}$ such that $M_\mathcal{U}(u_j) = e_j$ $(j = 1, \ldots, 8)$, where e_j is the j-th element of the canonical basis of $\mathbb{R}^8$. Thus, from Proposition 4, we have

Proposition 7 *Let Φ be a conformal transformation which maps an orthogonal frame $\mathfrak{U} = \{mu_1, \ldots, mu_8\}$ in $\mathbb{R}^8$ with $m \in \mathbb{Z}$ (where $\mathcal{U} = \{u_1, \ldots, u_8\}$ is formed by orthogonal vectors of minimal length in Γ_8) onto an orthogonal frame $\mathfrak{V} = \{lv_1, \ldots, lv_8\}$ in $\mathbb{R}^8$ with $l \in \mathbb{Z}$ (where $\mathcal{V} = \{v_1, \ldots, v_8\}$ is formed by orthogonal vectors of minimal length in Γ_8) then there exist $c_1, \ldots, c_k$ in the hypercubic lattice of $\mathbb{R}^8$ such that*

$$\Phi(x) = \frac{l}{m} M_\mathcal{V}{}^{-1}[(-1)^k (c_1 c_2 \cdots c_k) M_\mathcal{U}(x)(c_1 c_2 \cdots c_k)^{-1})] \tag{4}$$

where the product of the vectors c_j's is to be interpreted in terms of the product in the Clifford algebra $\mathbb{R}_{8,0}$ which contains $\mathbb{R}^8$ as the space $\mathbb{R}^1_{8,0}$ and the matrices $M_\mathcal{U}$ and $M_\mathcal{V}$ are such that $M_\mathcal{U}(u_j) = e_j$ $M_\mathcal{V}(v_j) = e_j$ $(j = 1, \ldots, 8)$, where e_j is the j-th element of the canonical basis of $\mathbb{R}^8$.

The automorphism group (or symmetry group) of a lattice in $\mathbb{R}^n$ is defined as the subgroup of the orthogonal group $O(n)$ that preserves the lattice. The symmetry group of the Γ_8 lattice is the Weyl/Coxeter group of type E_8. This is the group generated by reflections in the hyperplanes orthogonal to the 240 roots of the lattice. Its order is given by

$$|W(\mathrm{E}_8)| = 696729600 = 4! \cdot 6! \cdot 8!.$$

Even unimodular lattices can occur only in dimensions divisible by 8. In dimension 16 there are two such lattices: one reducible, namely the direct sum of two copies of Γ_8 and another one Γ_{16} constructed in an analogous fashion to Γ_8. In dimension 24 there are 24 such lattices, called *Niemeier lattices*. The most important of these is the *Leech lattice*.

4.3 Leech torus

First of all we want to find a frame for $\mathbb{R}^{24}$ of vectors in the Leech lattice. A nice description of the Leech lattice, via three copies of the lattice Γ_8, with appropriate relations is suggested in the following (see [24]).

Proposition 8 *Let $r_1, \ldots, r_8$ be a basis of simple roots of Γ_8. There exists a rotation*

$$R : \mathbb{R}^8 \to \mathbb{R}^8$$

such that

- *the transformation $T_1 = \sqrt{2}R^{-1}$ maps each r_i to a vector $w_{i,1} \in \Gamma_8$ whose dot product with r_i is 1;*
- *the transformation $T_2 = \sqrt{2}R$ maps each r_i to a vector $w_{i,2} \in \Gamma_8$ whose dot product with r_i is 1.*

For any rotation R with these properties, define the lattices $L_1 = T_1(\Gamma_8)$ and $L_2 = T_2(\Gamma_8)$. Then the lattice consisting of all triples $(a, b, c) \in \mathbb{R}^{24}$ such that:

1. $a, b, c \in \Gamma_8$
2. $a + b, a + c, b + c \in L_1$
3. $a + b + c \in L_2$

is isometric to a copy of the Leech lattice that has been rescaled by a factor of $\sqrt{2}$.

In [24] the author also shows that given any $u \in \Gamma_8$, then the vectors of $\mathbb{R}^{24}$ of the form

$$(2u, 0, 0),\ (0, 2u, 0),\ (0, 0, 2u)$$

belong to the corresponding Leech lattice. In particular this proves that the Leech lattice does not contain the hypercubic lattice of $\mathbb{R}^{24}$. Therefore given three frames $\mathcal{U} = \{u_1, \ldots, u_8\}$, $\mathcal{U}' = \{u'_1, \ldots, u'_8\}$ and $\mathcal{U}'' = \{u''_1, \ldots, u''_8\}$ of Γ_8 whose vectors are orthogonal and of minimal length, it turns out that the set

$$\mathfrak{U} = \{2u_1, \ldots, 2u_8, 2u'_1, \ldots, 2u'_8, 2u''_1, \ldots, 2u''_8\}$$

is a frame in the corresponding Leech lattice formed by vectors pairwise orthogonal and of minimal length. Following the same strategy of the previous sections, we have this generalization of Propositions 4 and 7.

Proposition 9 *Let* Φ *be a conformal transformation which maps an orthogonal frame* $\mathfrak{U} = \{mu_1, \ldots, mu_8, mu'_1, \ldots, mu'_8, mu''_1, \ldots, mu''_8\}$ *in* $\mathbb{R}^{24}$, *with* $m \in \mathbb{Z}$ (*where each set* $\mathcal{U} = \{u_1, \ldots, u_8\}$, $\mathcal{U}' = \{u'_1, \ldots, u'_8\}$, $\mathcal{U}'' = \{u''_1, \ldots, u''_8\}$ *consists of pairwise orthogonal vectors of minimal length in* Γ_8) *onto an orthogonal frame* $\mathfrak{V}$ *of* $\mathbb{R}^{24}$ *formed by vectors* $lv_1, \ldots, lv_8, lv'_1, \ldots, lv'_8, v''_1, \ldots, lv''_8$ *with* $l \in \mathbb{Z}$ (*where each set* $\mathcal{V} = \{v_1, \ldots, v_8\}$, $\mathcal{V}' = \{v'_1, \ldots, v'_8\}$, $\mathcal{V}'' = \{v''_1, \ldots, v''_8\}$ *consists of pairwise orthogonal vectors of minimal length in* Γ_8) *then there exist* $c_1, \ldots, c_k$ *in the hypercubic lattice of* $\mathbb{R}^{24}$ *such that*

$$\Phi(x) = \frac{l}{m} M_{\mathcal{V},\mathcal{V}',\mathcal{V}''}{}^{-1}[(-1)^k (c_1 c_2 \cdots c_k) M_{\mathcal{U},\mathcal{U}',\mathcal{U}''}(x)(c_1 c_2 \cdots c_k)^{-1})] \tag{5}$$

where the product of the vectors c_j*'s is to be interpreted in terms of the product in the Clifford algebra* $\mathbb{R}_{24,0}$ *which contains* $\mathbb{R}^{24}$ *as the space* $\mathbb{R}^1_{24,0}$ *and the matrices* $M_{\mathcal{U},\mathcal{U}',\mathcal{U}''}$ *and* $M_{\mathcal{V},\mathcal{V}',\mathcal{V}'''}$ *are*

$$M_{\mathcal{U},\mathcal{U}',\mathcal{U}''} = \left(\begin{array}{c|c|c} M_{\mathcal{U}} & 0 & 0 \\ \hline 0 & M_{\mathcal{U}'} & 0 \\ \hline 0 & 0 & M_{\mathcal{U}''} \end{array}\right) \quad M_{\mathcal{V},\mathcal{V}',\mathcal{V}''} = \left(\begin{array}{c|c|c} M_{\mathcal{V}} & 0 & 0 \\ \hline 0 & M_{\mathcal{V}'} & 0 \\ \hline 0 & 0 & M_{\mathcal{V}''} \end{array}\right)$$

with $M_{\mathcal{U}}(u_j) = e_j$, $M_{\mathcal{U}'}(u'_j) = e_j$, $M_{\mathcal{U}''}(u''_j) = e_j$, $M_{\mathcal{V}}(v_j) = e_j$, $M_{\mathcal{V}'}(v'_j) = e_j$, $M_{\mathcal{V}''}(v''_j) = e_j$, $(j = 1, \ldots, 8)$, *where* e_j *is the* j*-th element of the canonical basis of* $\mathbb{R}^8$.

The examples provided for the Γ_8 lattice and the Leech lattice seem to suggest an algorithm to detect whether a lattice which does not contain a copy of the hypercubic lattice might have a rich group of endomorphisms. To be more precise, one starts from looking for generators of the lattice whose length is minimal and which are pairwise orthogonal. Then if possible one has to complete this set and form an orthogonal set of generators whose length is proportional to the minimal one in the lattice. This condition guarantees in fact the existence of a non trivial endomorphism of the corresponding torus.
The examples of manifolds with conformal multiplication given above are all obtained as quotients of $\mathbb{R}^n$ over lattices induced by uniform and regular tessellations of $\mathbb{R}^n$, in particular most of these lattices resemble the cubic or barycentric cubic tessellations in higher dimension.

From our previous general considerations on manifolds obtained as quotients of $\mathbb{R}^n$ over lattices, we can observe that these manifolds have a conformal multiplication if they are associated with lattices whose Voronoi regions have some non trivial symmetries as domains of $\mathbb{R}^n$. In particular this forces the lattices to have (at least) a frame of generators with the same norm. Thus a manifold $\mathbb{R}^n/\Lambda$ has conformal multiplication if there exists an orthonormal transformation of $\mathbb{R}^n$ (different from $\pm Id$) which leaves invariant the Voronoi region of Λ.

We conclude by remarking that for a manifold $\mathbb{R}^n/\Lambda$ the property of having conformal multiplication is invariant for conformal equivalence of $\mathbb{R}^n/\Lambda$. For example the torus obtained from the lattice FCC (whose Voronoi region is the rhombic dodecahedron) has conformal multiplication and, thanks to Remark 3, also the tori corresponding to tessellations obtained by octahedra or pyramids which decompose each rhombic dodecahedron have conformal multiplications.

5 Ellipsoids and moduli space of conformal tori

Starting from a frame F in $\mathbb{R}^n$, as seen before, we can consider the $n \times n$ matrix $M(F)$ with the vectors of the frame as columns. It is then easy to associate to $M(F)$ a symmetric matrix as

$$S_F := M(F)^T \cdot M(F)$$

where A^T denotes the transpose of the matrix A. This symmetric matrix S_F provides a Riemannian metric on the torus $\mathbb{R}^n/\Lambda_F$ since it is positive definite. The matrix S_F admits a diagonal form with all the positive eigenvalues $\lambda_1, \ldots, \lambda_n$ along the diagonal. In other words, given a frame F in $\mathbb{R}^n$ and considered the symmetric matrix $S_F := M(F)^T \cdot M(F)$, there exists an orthogonal matrix $O \in O(n)$ such that

$$OM(F)O^T = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}.$$

Thus any such a matrix represents an ellipsoid in $\mathbb{R}^n$ and its eigenvalues are regarded as the axes or the momenta of the ellipsoid. We will denote by $\mathcal{E}(n)$ the set of all ellipsoids in $\mathbb{R}^n$ or equivalently of all symmetric, positive-definite and real $n \times n$ matrices.

In particular for the frame $F = (\mathbf{1}, \tau)$ the eigenvalues of S_F can be seen as the axes of the ellipse represented by S_F itself. They can be also written in terms of the coefficients of the quadratic polynomial that has τ as a root, namely if τ is the root of $\tau^2 + A\tau + B = 0$ with positive imaginary part as in (2), then the eigenvalues of S_F are $\lambda_{1,2} = \frac{1+B\pm\sqrt{(1-B)^2+A^2}}{2}$.

Observe furthermore that if one considers the frame $F_k = (k, k\tau)$ with $k \in \mathbb{C}$ not zero, then the corresponding symmetric matrices S_F and S_{F_k} are related in the following way

$$S_{F_k} = |k|^2 S_F$$

so that the Riemannian metrics associated with these frames are conformal. A similar result holds in any dimension.

Proposition 10 *Conformally equivalent lattices in $\mathbb{R}^n$ induce conformal Riemannian metrics on the corresponding tori.*

Proof With the usual notation, if F_1 and F_2 are frames in $\mathbb{R}^n$ such that there exist $A \in GL(n, \mathbb{R})$, $O \in SO(n)$ and a positive real number μ so that

$$A\ M(F_1) = M(F_2)\ (\mu O)$$

then $M(F_1) = A^{-1}\ M(F_2)\ (\mu O)$ which implies

$$S_{F_1} = M(F_1)^T M(F_1) = \mu^2 O^{-1} M(F_2)^T M(F_2)\ O = \mu^2 O^{-1} S_{F_2}\ O.$$

We conclude this section by showing the correspondence between the class of conformal flat tori and the double coset space

$$C(n) \stackrel{def}{=} SL(n, \mathbb{Z})\backslash[SL(n, \mathbb{R})/SO(n)].$$

Proposition 11 *The set $C(n)$ is in 1 to 1 correspondence with the moduli space of conformally equivalent n-dimensional tori.*

Proof The proof easily follows from Proposition 1 since two conformally equivalent tori $\mathbb{T}^n_{\Lambda_1}$ and $\mathbb{T}^n_{\Lambda_2}$ correspond to the same point in $C(n)$.

Remark 8 The (symmetric) space $C(n) = SL(n, \mathbb{Z})\backslash[SL(n, \mathbb{R})/SO(n)]$ can be regarded as the *Teichmüller space* of n-dimensional flat tori.

From [15], we recall that $SL(n, \mathbb{R})/SO(n)$ is a symmetric space of non-compact type of rank $n-1$. Furthermore it can be equipped with a Riemannian metric which makes the space of nonpositive curvature. In particular, when $n = 2$ the symmetric space $SL(2, \mathbb{R})/SO(2)$ can be identified with the hyperbolic (right) half-plane $H^2_{\mathbb{R}}$ which is biholomorphic to the Poincaré disk.

The geometric interpretation of the symmetric space $SL(n, \mathbb{R})/SO(n)$ is given in the following

Proposition 12 *The symmetric space of non-compact type $SL(n, \mathbb{R})/SO(n)$ is in a natural way diffeomorphic (isometric with the appropriate metrics) to the space $P_1(n)$ of positive definite $n \times n$ symmetric matrices with determinant one.*

Proof Let $M(n)$ be the real vector space of $n \times n$ matrices. Let

$$P_1(n) = \{A \in M(n) : A = A^t,\ \det(A) = 1,\ \langle Av, v\rangle > 0, \forall v \neq 0\}$$

be the set of symmetric and positive-definite $n \times n$ real matrices with determinant equal to one. Then $SL(n, \mathbb{R})$ acts on $P_1(n)$ in the following way

$$g_G(A) := GAG^t, \quad \text{with } G \in SL(n, \mathbb{R}), \quad A \in P_1(n).$$

By an argument of elementary linear algebra we see that $g_G(A) \in P_1(n)$ and that the action is transitive. Furthermore the $n \times n$ identity matrix I_n belongs to $P_1(n)$ and its isotropy group is precisely $SO(n)$. This completes the proof.

The proof presented here appears also in [20]; notice that the result will be in accordance with the calculus of real dimension of $C(n)$ as in the following

Remark 9 From direct computation and from the definition of $C(n)$ one can calculate the real dimension of (the orbifold) $C(n)$ as

$$\dim_{\mathbb{R}}(C(n)) = n^2 - 1 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2} - 1.$$

Remark 10 The space $P_1(n)$ is the space of non degenerate $n-1$-dimensional ellipsoids of volume 1 in $\mathbb{R}^n$ and also the set of positive definite quadratic forms up to a positive scaling factor. This symmetric space has a compactification obtained by adding symmetric matrices of determinant 0 and has a very complicated boundary.

Siegel has shown the remarkable fact that the volume (with respect to the canonical symmetric metric) of $C(n)$ is a function of $\zeta(2), \ldots, \zeta(n)$, where ζ is the Riemann's zeta function. This is a number-theoretical result which depends on Eisenstein series (see [14]).

In particular, using an idea of Minkowski, Siegel obtained the formula

$$Vol(C(n)) = n2^{n-1}\prod_{l=2}^{n} \frac{\zeta(l)}{Vol(\mathbb{S}^{l-1})}, \tag{6}$$

where

$$Vol(\mathbb{S}^{l-1}) = \frac{2(\sqrt{\pi})^l}{\Gamma(l/2)} \tag{7}$$

is the volume of the $(l-1)$-sphere and Γ is the Euler Gamma function.

6 Quaternionic moduli space

Let $\mathbb{H}$ be the non-commutative field of quaternions. We identify $\mathbb{H}$ with $\mathbb{R}^4$ as usual and $\mathbb{H}^n$ with the real vector space $\mathbb{R}^{4n}$. Let Λ be a lattice of $\mathbb{H}^n$ of maximal rank. Then we consider the quaternionic n-dimensional torus as the quotient $\mathbb{H}^n/\Lambda$. This is also a real torus of real dimension $4n$. All the examples of quaternionic tori turn out to be quaternionic affine manifolds (see [13,23]). Here we extend the definition of biregularity for quaternionic tori as introduced in [4] for the case $n=1$.

Definition 10 We say that the lattices of rank $4n$, Λ_1 and Λ_2, are *biregularly* equivalent if there exist two frames F_1 and F_2 (in Λ_1 and Λ_2 respectively), $Q \in GL^+(n, \mathbb{H}) = GL^+(4n, \mathbb{R})$ and $A \in SL(4n, \mathbb{Z})$, such that

$$A\ M(F_1) = M(F_2)Q$$

where $M(F_1)$ and $M(F_2)$ are the matrices associated with F_1 and F_2 respectively.

Proposition 13 *Two biregularly equivalent lattices Λ_1 and Λ_2 determine two quaternionic tori which are biregularly equivalent. Moreover, the moduli space of biregularly equivalent quaternionic tori is the orbifold*

$$B(n) \stackrel{def}{=} SL(4n, \mathbb{Z})\backslash[GL^+(4n, \mathbb{R})/GL(n, \mathbb{H})].$$

The argument of the proof is exactly the same as in the proof of Proposition 11.

Here we are using the canonical inclusions $SL(4n, \mathbb{Z}) \subset GL^+(4n, \mathbb{R})$. Indeed it can be shown that $GL(n, \mathbb{H})$ can be embedded in $GL(4n, \mathbb{R})$. The real dimension of $B(n)$ is $12n^2$.

Remark 11 Since $GL(1, \mathbb{H}) = \mathbb{H}\backslash\{0\} = \mathbb{H}^*$, our definition coincides with the one given in [4] for the case of the moduli space $B(1)$ of quaternionic one dimensional biregular tori.

The moduli space $B(1)$ and $C(4)$ are related by the following.

Lemma 3 *$B(1)$ is of real dimension 12 and fibers over $C(4)$ (which is of real dimension 9) with fiber $SO(3)$*

$$p : B(1) \stackrel{SO(3)}{\longrightarrow} C(4).$$

Proof The result follows immediately from the fact that $SO(4)$ fibers over $SU(2)$ with fiber $SO(3)$.

As an application we calculate the volume of $B(1)$. From the lemma above we have $Vol(B(1)) = Vol(C(4))Vol(SO(3))$ and therefore, from formula (6) we obtain the following

$$Vol(B(1)) = 16Vol(\mathbb{S}^3)\prod_{l=2}^{4}\frac{\zeta(l)}{Vol(\mathbb{S}^{l-1})} = 32\pi^2\prod_{l=2}^{4}\frac{\zeta(l)}{Vol(\mathbb{S}^{l-1})}$$

since the volume of $SO(3)$ is $\frac{1}{2}Vol(\mathbb{S}^3)$ (in fact $SU(2)$ double covers $\mathbb{S}^3$).
Now, from $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$ and $Vol(\mathbb{S}^1)Vol(\mathbb{S}^2)Vol(\mathbb{S}^3) = 16\pi^4$ we finally have

$$Vol(B(1)) = \frac{1}{270}\pi^4\zeta(3).$$

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