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Anisotropic tubular neighborhoods of sets

Antonin Chambolle[1] · **Luca Lussardi**[2] · **Elena Villa**[3]

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Abstract
Let $E \subset \mathbb{R}^N$ be a compact set and $C \subset \mathbb{R}^N$ be a convex body with $0 \in \operatorname{int} C$. We prove that the topological boundary of the anisotropic enlargement $E + rC$ is contained in a finite union of Lipschitz surfaces. We also investigate the regularity of the volume function $V_E(r) := |E + rC|$ proving a formula for the right and the left derivatives at any $r > 0$ which implies that V_E is of class C^1 up to a countable set completely characterized. Moreover, some properties on the second derivative of V_E are proved.

Keywords Rectifiability · anisotropic outer Minkowski content · viscosity solutions

Mathematics Subject Classification 28A75 · 35D40

1 Introduction

The study of the tubular neighborhood $E_r := \{x \in \mathbb{R}^N : \operatorname{dist}(x, E) \leq r\}$ of a convex set E in $\mathbb{R}^N$ plays a crucial role in convex geometry. Of course, is not without interest to investigate the tubular neighborhood also for non convex sets, and it turns out that the boundary of E_r becomes more regular than the boundary of E, which could be very irregular: more precisely, in 1985 Fu [11] proves that ∂E_r is a Lipschitz manifold whenever E is compact in $\mathbb{R}^N$ and $r > r_0$ for some $r_0 > 0$. The approach of Fu is essentially based on the fact that the sublevels of regular values of a proper and semiconcave function are sets of positive reach: this argument can be applied since the distance function is semiconcave far from E. The semiconcavity of

✉ Luca Lussardi
luca.lussardi@polito.it

Antonin Chambolle
chambolle@ceremade.dauphine.fr

Elena Villa
elena.villa@unimi.it

1 Ceremade, CNRS and Université de Paris-Dauphine PSL, Place de Lattre de Tassigny, 75775 Paris Cedex 16, France

2 Dipartimento di Scienze Matematiche “G.L. Lagrange”, Politecnico di Torino, c.so Duca degli Abruzzi 24, 10129 Turin, Italy

3 Dipartimento di Matematica “F. Enriques”, Università degli Studi di Milano, Via C. Saldini 50, 20133 Milan, Italy

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the distance is strongly related with the smoothness of the ball in $\mathbb{R}^N$: notice indeed that E_r can also be written as $E_r = E + rB$, where B is the unit closed ball centered in the origin. In this paper first of all we investigate the extension of such results to the anisotropic case, that is in the case $E_r = E + rC$ where C is a prescribed convex body, i.e. a compact convex set in $\mathbb{R}^N$ with $0 \in \operatorname{int} C$. In this case the appropriate anisotropic distance to E, which we denote by d_E, could not be semiconcave outside E, since we are not assuming any kind of regularity of the boundary of C, unless locally Lipschitz coming from convexity: notice that we are really interested in enlarging E with a convex body C, since this case recovers also the crystalline anisotropy where C is convex but not necessarily strictly convex nor smooth. We will prove (see Thm. 3.1) that for any $r > 0$ the boundary of E_r is contained in a finite union of Lipschitz surfaces when E is bounded and C is Lipschitz with $0 \in \operatorname{int} C$. Of course since C is not sufficiently smooth we cannot use the Fu's approach, but the key idea of our proof is very easy: we first prove that enlarging C by a very small set, like εK with $\varepsilon > 0$ small and $K \subset B$, we still obtain a Lipschitz domain, and then we use the same idea of Rataj and Winter [17] covering ∂E_r by a finite union of sets with small diameter. The rectifiability of ∂E_r is an independent interesting result, but actually we need to prove the regularity of ∂E_r in order to study the regularity of the volume function $V_E(r) := |E_r|$ (see [21,22] for the isotropic case). We therefore characterize the set, at most countable, where V_E is not differentiable (see Thm. 5.2) and we find explicit formulae for left and right derivatives of V_E. Moreover, V_E is of class C^1 whenever it is differentiable (see Thm. 5.3). We mention that such a result finds application also in different fields, for example, in Stochastic Geometry, where V_E' is strictly related to the notion of covariogram and of contact distribution function associated to a random closed set (e.g., see [23, Sec. 4] for the isotropic case, and the recent paper [15] where dilation by finite sets is considered). Finally, an easy characterization of V_E'' is proved (see Thm. 5.4). Our result is a generalization of the isotropic case [14] and our proof is partially based on the so called *anisotropic outer Minkowski content* (see [6] and [16] for details). We also need to base our argument on the existence of a so called *Cahn-Hoffmann vector field* for C with divergence measure bounded from above (see Prop. 4.1), and this is, in our opinion, an interesting result independent on the rest, since we are not assuming the strict convexity of C, so that such an existence result holds true also in the crystalline case.

2 Notation and preliminaries

2.1 Notation

For any A subset of $\mathbb{R}^N$ we will denote by $|A|$ the Lebesgue measure of A while $\mathcal{H}^k(A)$ stands for the k-dimensional Hausdorff measure of A, where $k \in \{0, \dots, N\}$; of course $\mathcal{H}^N$ is the Lebesgue measure. For any $x \in \mathbb{R}^N$ the euclidean norm of x will be denoted by $|x|$ while $x \cdot y$ stands for the euclidean scalar product in $\mathbb{R}^N$ between x and y. For any $r > 0$ and $x \in \mathbb{R}^N$ the closed ball centered in x with radius r will be denoted by $B_r(x)$; we let $B_r := B_r(0)$ and $S^{N-1} := \partial B_1$. We finally denote by ω_k the volume of the k-dimensional unit ball in $\mathbb{R}^k$.

2.2 Convex analysis

Here we recall some basic notions of convex analysis; for all details we refer to [18]. In this paragraph C will be a *convex body in* $\mathbb{R}^N$, that is a compact convex subset of $\mathbb{R}^N$ with $0 \in$

int C. We denote by $h_C\colon \mathbb{R}^N \to \mathbb{R}$ the *support function of* C, that is $h_C(v) := \max_{x\in C} x\cdot v$. We will use also the *polar of* h_C, denoted by h_C° and defined by $h_C^\circ(v) := \max_{h_C(x)\le 1} x\cdot v$ for each $v\in\mathbb{R}^N$; it turns out that both h_C and h_C° are convex and positively 1-homogeneous. We will need also to consider convex sets for which the support function and its polar are more regular. Let C be of class C^2. We say that C is *elliptic* if the curvature of ∂C is bounded from below by some positive constant. It turns out that if C is C^2 and elliptic then both h_C and h_C° are in $C^2(\mathbb{R}^N\setminus\{0\})$.

A very useful notion related with convexity is given by semiconcavity. Let A be a subset of $\mathbb{R}^N$ and let $f\colon A\to\mathbb{R}$. We say that f is *concave* if the inequality

$$\lambda f(x) + (1-\lambda) f(y) \le f(\lambda x + (1-\lambda) y)$$

holds true whenever $x, y\in A$, $\lambda\in[0,1]$ and $\{\lambda x + (1-\lambda)y : \lambda\in[0,1]\}\subset A$. A function $f\in C^0(A)$ is said to be *semiconcave* if there exists $\alpha>0$ such that for any $x,y\in A$ and for any $\lambda\in[0,1]$ with $\{\lambda x+(1-\lambda)y:\lambda\in[0,1]\}\subset A$ it holds

$$\lambda f(x) + (1-\lambda) f(y) \le f(\lambda x + (1-\lambda) y) + \frac{\alpha}{2}\lambda(1-\lambda)|x-y|^2. \tag{2.1}$$

Notice that if f is semiconcave and smooth enough, for instance of class C^2, then $D^2 f\le \alpha I$, where I is the identity matrix and the inequality holds in the sense of matrices. A useful class of semiconcave functions can be constructed; we have the following well known proposition, see for instance [9].

Proposition 2.1 *Let $A\subset\mathbb{R}^N$ and let $S\subset\mathbb{R}^M$ be compact. Let $F\in C^0(S\times A)$. Then the function $f\colon A\to\mathbb{R}$ defined by $f(x) := \inf_{s\in S} F(s,x)$ is semiconcave provided $F(s,\cdot)$ satisfies (2.1) uniformly with respect to s.*

2.3 Geometric measure theory

In this paragraph we recall some notions of Geometric Measure Theory we will need; for all details we refer the reader to [2], [10] and [20]. Let $N\ge 1$ be integer and let $k\in\mathbb{N}$ with $k\le N$. Let $S\subset\mathbb{R}^N$. We say that S is *k-rectifiable* if there exist a bounded set $B\subset\mathbb{R}^k$ and a Lipschitz function $f\colon B\to\mathbb{R}^N$ such that $S=f(B)$; equivalently, by the Kirszbraun's extension Theorem, we can say that S is k-rectifiable if S is contained in a finite union of Lipschitz surfaces in $\mathbb{R}^N$. We say that $S\subset\mathbb{R}^n$ is *countably $\mathcal{H}^k$-rectifiable* if there exist countably many Lipschitz functions $f_h\colon\mathbb{R}^k\to\mathbb{R}^N$ such that

$$\mathcal{H}^k\left(S\setminus\bigcup_{h=0}^{+\infty} f_h(\mathbb{R}^k)\right)=0.$$

A useful characterization of rectifiability is the Besicovitch–Marstrand–Mattila's Theorem (see, for instance, [2, Thm. 2.63]): a Borel set $S\subset\mathbb{R}^N$ with $\mathcal{H}^k(S)<+\infty$ is countably $\mathcal{H}^k$-rectifiable if and only if for $\mathcal{H}^k$-a.e. $x\in S$ we have

$$\lim_{\rho\to 0^+}\frac{\mathcal{H}^k(S\cap B_\rho(x))}{\omega_k\rho^k}=1. \tag{2.2}$$

It turns out that if S is countably $\mathcal{H}^k$-rectifiable then for $\mathcal{H}^k$-almost any point $x_0\in S$ it is well defined the *approximate tangent space* $\mathrm{Tan}^k(S,x_0)$, that is

$$\lim_{\rho\to 0^+}\frac{1}{\rho^k}\int_S \phi\left(\frac{x-x_0}{\rho}\right)d\mathcal{H}^k(x)=\int_{\mathrm{Tan}^k(S,x_0)}\phi(y)\,d\mathcal{H}^k(y),\quad \forall\phi\in C_c^\infty(\mathbb{R}^N).$$

In particular, if $k = N-1$ then $\mathrm{Tan}^{N-1}(S, x_0)^\perp$ is generated by some unit vector denoted by ν_S.

Let now $E \subset \mathbb{R}^N$ be a measurable set and $\Omega \subset \mathbb{R}^N$ be an open domain; we denote by χ_E the characteristic function of E. We say that E has *finite perimeter in* Ω if $\chi_E \in BV(\Omega)$; the perimeter of E in Ω is defined by $\mathcal{P}(E;\Omega) := |D\chi_E|(\Omega)$, where $|D\chi_E|$ denotes the total variation of $D\chi_E$; we also let $\mathcal{P}(E) := \mathcal{P}(E;\mathbb{R}^N)$. For sufficiently smooth boundaries the perimeter coincides with the $(N-1)$-dimensional Hausdorff measure of the topological boundary. The upper and lower *N-dimensional densities of E at x are* respectively defined by

$$\Theta_N^*(E,x) := \limsup_{\rho\to 0} \frac{|E\cap B_\rho(x)|}{\omega_N \rho^N}, \quad \Theta_{*N}(E,x) := \liminf_{\rho\to 0} \frac{|E\cap B_\rho(x)|}{\omega_N \rho^N}.$$

If $\Theta_N^*(E,x) = \Theta_{*N}(E,x)$ their common value is denoted by $\Theta_N(E,x)$. For every $t \in [0,1]$ we define $E^t := \{x \in \mathbb{R}^N : \Theta_N(E,x) = t\}$. The *essential boundary* of E is defined as $\partial^* E := \mathbb{R}^N \setminus (E^0 \cup E^1)$. It turns out that if E has finite perimeter in Ω, then $\mathcal{H}^{N-1}(\partial^* E \setminus E^{1/2}) = 0$, and $\mathcal{P}(E;\Omega) = \mathcal{H}^{N-1}(\partial^* E \cap \Omega)$. Moreover, one can define a subset of $E^{1/2}$ as the set of points x where there exists a unit vector $\nu_E(x)$ such that

$$\frac{E-x}{\rho} \to \{y \in \mathbb{R}^N : y\cdot\nu_E(x) \le 0\}, \quad \mathrm{in} L^1_{\mathrm{loc}}(\mathbb{R}^N) \mathrm{as} \rho \to 0^+,$$

and which is referred to as the outer normal to E at x. The set where $\nu_E(x)$ exists is called the *reduced boundary* and is denoted by $\mathcal{F}E$. One can show that $\mathcal{H}^{N-1}(\partial^* E \setminus \mathcal{F}E) = 0$, moreover, one has the decomposition $D\chi_E = (-\nu_E)\mathcal{H}^{N-1} \llcorner \mathcal{F}E$. Let us collect some elementary properties of sets with countably $\mathcal{H}^{N-1}$-rectifiable boundary and with finite perimeter in Ω; for any $E \subseteq \mathbb{R}^N$ we let $E^c := \mathbb{R}^N \setminus E$. Assume that E has finite perimeter in Ω and ∂E is countably $\mathcal{H}^{N-1}$-rectifiable. Then the following relations hold true: $\mathcal{H}^{N-1}(\mathcal{F}E) = \mathcal{H}^{N-1}(\mathcal{F}E^c)$ and $\nu_{E^c}(x) = -\nu_E(x)$ for any $x \in \mathcal{F}E$.

We finally recall an anisotropic version of the coarea formula which we will need; for details see [13, Thm. 3]. Let $u \in BV(\Omega)$ and let $\alpha\colon \mathbb{R}^N \to (0,+\infty)$ be a convex and positively one-homogeneous function with $c^{-1}|v| \le \alpha(v) \le c|v|$ for any $v \in \mathbb{R}^N$ and for some constant $c > 0$. Then the following formula holds true:

$$\int \alpha(Du) = \int_{-\infty}^{+\infty} \int_{\Omega\cap\mathcal{F}\{u\le t\}} \alpha(\nu_{\{u\le t\}})\, d\mathcal{H}^{N-1}\, dt. \tag{2.3}$$

2.4 Anisotropic outer Minkowski content

We now briefly recall the notion of outer Minkowski content; for details see [1] and [22] (isotropic case), [6] and [16] (anisotropic case). Let $C \subset \mathbb{R}^N$ be a convex body. For each closed set $E \subset \mathbb{R}^N$ we also define the *anisotropic outer Minkowski content of E* as

$$\mathcal{SM}_C(E) := \lim_{\varepsilon\to 0^+} \frac{|(E+\varepsilon C)\setminus E|}{\varepsilon}$$

whenever such a limit exists.

The following existence and characterization result for $\mathcal{SM}_C$ holds true.

Theorem 2.2 [16, Thm. 4.4] *Let $E \subset \mathbb{R}^N$ be a closed set such that:*

(a) *∂E is a countably $\mathcal{H}^{N-1}$-rectifiable bounded set;*

(b) *there exist* $\gamma > 0$ *and a probability measure* η *in* $\mathbb{R}^N$ *absolutely continuous with respect to* $\mathcal{H}^{N-1}$ *such that* $\eta(B_r(x)) \geq \gamma r^{N-1}$ *for all* $x \in \partial E$ *and for all* $r \in (0, 1)$.

Then

$$\mathcal{SM}_C(E) = \int_{\mathcal{F}E} h_C(\nu_E)\, d\mathcal{H}^{N-1} + 2\int_{\partial E\cap E^0} \phi_C(\nu_E)\, d\mathcal{H}^{N-1} \tag{2.4}$$

where

$$\phi_C(v) := \frac{h_C(v) + h_C(-v)}{2}.$$

Notice that any compact set in $\mathbb{R}^N$ whose boundary is contained in a finite union of Lipschitz surfaces satisfies property (b): see for instance [1, Rem. 1]. We also observe that even if ν_E is not well defined on $\partial E \cap E^0$, the expression $\phi_C(\nu_E)$ turns out to be well defined.

2.5 Anisotropic tubular neighborhoods

In this paragraph we will introduce all the objects we want to investigate. Let $N \geq 1$ be integer. Let $E \subset \mathbb{R}^N$ be compact and $C \subset \mathbb{R}^N$ be a compact Lipschitz set with $0 \in \operatorname{int} C$. For any $r > 0$ denote $E_r := E + rC$. Moreover, let

$$E'_r := \bigcup_{s<r} E_s.$$

It is convenient to introduce the anisotropic distance from E, that is

$$d_E(x) := \inf_{y\in E} h^\circ_C(x-y).$$

Notice that $E_r = \{d_E \leq r\}$ and $E'_r = \{d_E < r\}$. It turns out (for details see [6]) that d_E is Lipschitz continuous and, if C is a convex body,

$$h_C(\nabla d_E) = 1 \quad \text{a.e. on } \{d_E > 0\}. \tag{2.5}$$

Finally, let $V_E\colon [0, +\infty) \to \mathbb{R}$ be given by $V_E(r) := |E_r|$. Note that for $C = B_1$, it is also named *volume function of E* (see also [21,22]). It is easy to see that V_E is continuous.

3 Regularity of the boundaries

In this section we prove that ∂E_r and $\partial E'_r$ are sufficiently smooth, in the sense of geometric measure theory.

Theorem 3.1 *For any* $r > 0$ *the sets* ∂E_r, $\partial E'_r$ *are finite union of Lipschitz surfaces.*

Proof We divide the proof in two steps.

Step 1: Let $K \subset \mathbb{R}^N$ be a bounded set. We claim that for ε positive and sufficiently small the set $C + \varepsilon K$ is a Lipschitz set.

Without loss of generality we can assume $K \subset B_1$. For any $\xi \in \mathbb{R}^N$, $\xi \neq 0$, we let

$$\xi^\perp := \{x \in \mathbb{R}^N : x\cdot\xi = 0\} \quad \text{and} \quad S^\xi_\eta := \{x \in \mathbb{R}^N : |\pi_\xi(x)| < \eta \text{ and } x\cdot\xi > 0\}$$

where π_ξ denotes the orthogonal projection on $\xi^\perp$. Since C is Lipschitz and compact we can write its boundary locally as a graph of a Lipschitz function in a uniform way: precisely, we can find $r > 0$ such that $B_r \subset C$ and such that for any $z \in \partial C$ there exists a Lipschitz function $f_z\colon B_r \cap z^\perp \to \mathbb{R}$ with

$$\{x + f_z(x)\hat{z} : x \in B_r \cap z^\perp\} = \partial C \cap S_r^z, \quad \hat{z} := z/|z|.$$

Let $\varepsilon < r/2$ and fix $x_0 \in \partial(C + \varepsilon K)$. There exists $k_0 \in K$ such that $x_0 \in \partial C + \varepsilon k_0$, thus $x_0 = z_0 + \varepsilon k_0$ for some $z_0 \in \partial C$. For any $x \in B_{r/2} \cap z_0^\perp$ and any $k \in K$ let:

$$g(x) := \sup\{f_{z_0}(\pi_{z_0}(x - \varepsilon k)) + \varepsilon k \cdot \hat{z}_0 : k \in K\}.$$

For $\xi \in \partial(C + \varepsilon K) \cap S_{r/2}^{z_0}$, writing $\xi = \eta + \varepsilon k$, $\eta \in \partial C$, $k \in \overline{K}$, we observe that for $y = \pi_{z_0}(\eta)$ one has $\eta = y + f_{z_0}(y)\hat{z}_0$ with $|y| \leq r$. Thus, one finds that ξ decomposes as $x + t\hat{z}_0$ with $x = \pi_{z_0}(\xi)$ and $t \leq g(x)$. On the other hand, if $t < g(x)$ then there exists $k \in K$ with $t < f_{z_0}(y) + \varepsilon k \cdot \hat{z}_0$, but then one would have $\eta \cdot \hat{z}_0 = t - \varepsilon k \cdot \hat{z}_0 < f_{z_0}(y)$, a contradiction since $\eta \notin \mathring{C}$. Hence $t = g(x)$ and it follows that:

$$\partial(C + \varepsilon K) \cap S_{r/2}^{z_0} \subset \{x + g(x)\hat{z}_0 : x \in B_{r/2} \cap z_0^\perp\}.$$

Conversely, if $\xi = x + g(x)\hat{z}_0$ for $x \in B_{r/2} \cap z_0^\perp$, first it is clear that $\xi \in \partial C + \varepsilon\overline{K}$ by definition of g (as there exist $k_n \in K$, $n \geq 1$, with $\lim_{n\to\infty} f_{z_0}(\pi_{z_0}(x - \varepsilon k_n) + \varepsilon k_n \cdot \hat{z}_0) = g(x)$, and then $\eta_n = \xi - \varepsilon k_n \in \partial C$). On the other hand if one lets now $\xi_n := x + (g(x) + \frac{1}{n})\hat{z}_0$ then for any $k \in K$, $(\xi_n - \varepsilon k) \cdot \hat{z}_0 > g(x) \geq f(\pi_{z_0}(x - \varepsilon k)) = f(\pi_{z_0}(\xi_n - \varepsilon k))$, hence $(\xi_n - \varepsilon K) \cap C = \emptyset$. This shows that ξ is not in the interior of $C + \varepsilon K$, hence $\xi \in \partial(C + \varepsilon K)$. We deduce:

$$\{x + g(x)\hat{z}_0 : x \in B_{r/2} \cap z_0^\perp\} = \partial(C + \varepsilon K) \cap S_{r/2}^{z_0}.$$

We notice eventually that g is Lipschitz continuous with the same Lipschitz constant L of f_{z_0}, which achieves the proof that $\partial(C + \varepsilon K)$ is locally a Lipschitz graph: indeed for any $x, y \in B_{r/2} \cap z_0^\perp$ it holds

$$g(x) - g(y) \leq \sup_{k\in K}\{f_{z_0}(\pi_{z_0}(x - \varepsilon k)) - f_{z_0}(\pi_{z_0}(y - \varepsilon k))\} \leq L|x - y|.$$

Step 2: Now it is relatively easy to conclude the proof for $\partial E_r'$; the rectifiablity of ∂E_r follows since $\partial E_r \subseteq \partial E_r'$. The idea is to use the same argument as in the proof of [17, Prop. 2.3]. If $r > 0$ by step 1 we can say that for any $x \in \mathbb{R}^N$ the set $rC + (B_{r'}(x) \cap E)$ has Lipschitz boundary for $r' < r$ sufficiently small (apply step 1 to rC instead of C). We cover now E, which has compact closure, with balls $B_{r'}(x_1), \ldots, B_{r'}(x_d)$ and we let $E_i := E \cap B_{r'}(x_i)$. Then

$$\partial E_r' \subseteq \bigcup_{i=1}^{d} \partial(E_i)_r$$

that is $\partial E_r'$ is contained in a finite union of Lipschitz surfaces, and this yields the conclusion. $\square$

4 Construction of a Cahn–Hoffmann vector field for C

First of all, we recall some basic results of the theory of viscosity solutions; for details we refer to [8]. Let $\mathrm{Sym}_N(\mathbb{R})$ be the set of all symmetric $N \times N$ matrices with real entries, let

Ω be a subset of $\mathbb{R}^N$ and let $F\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathrm{Sym}_N(\mathbb{R}) \to \mathbb{R}$ be a continuous function such that the following monotonicity condition holds:

$$F(x, r, p, X) \leq F(x, s, p, Y)$$

whenever $r \leq s$ and $Y \leq X$ in the sense of matrices. Let $u\colon \Omega \to \mathbb{R}$ be upper semicontinuous. We say that u is a *viscosity subsolution* of the equation $F(x, u, Du, D^2u) = 0$ on Ω if for any $\phi \in C^2(\Omega)$ and for any $\bar{x} \in \Omega$ local maximum point of $u - \phi$ it holds

$$F(\bar{x}, u(\bar{x}), D\phi(\bar{x}), D^2\phi(\bar{x})) \leq 0.$$

Let now $u\colon \Omega \to \mathbb{R}$ be lower semicontinuous. We say that u is a *viscosity supersolution* of the equation $F(x, u, Du, D^2u) = 0$ on Ω if for any $\phi \in C^2(\Omega)$ and for any $\bar{x} \in \Omega$ local minimum point of $u - \phi$ it holds

$$F(\bar{x}, u(\bar{x}), D\phi(\bar{x}), D^2\phi(\bar{x})) \geq 0.$$

If u is both a viscosity subsolution and supersolution then u is called *viscosity solution* of $F(x, u, Du, D^2u) = 0$ on Ω.

We are ready to start the construction of a Cahn-Hoffmann vector field for C in the smooth case.

Proposition 4.1 *Assume that C is a convex body of class C^2 and elliptic. Let $n := \nabla h_C(\nabla d_E)$. Then $n \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$,*

$$\|n\|_\infty \leq \max_{z\in C} |z| \tag{4.1}$$

and div n *is a Radon measure on $\mathbb{R}^N \backslash \overline{E}$ with*

$$\operatorname{div} n \leq \frac{N-1}{r} \tag{4.2}$$

in the distributional sense out of E_r.

Proof First of all we point out that the assumptions on C guarantee that both h_C and h_C° are in $C^2(\mathbb{R}^N \backslash \{0\})$. From the standard fact that $h_C^2/2$ and $(h_C^\circ)^2/2$ are Legendre-Fenchel convex conjugates, so that their gradients $h_C \nabla h_C$ and $h_C^\circ \nabla h_C^\circ$ are inverse mappings, we deduce that for any $z \in \mathbb{R}^N \backslash \{0\}$

$$\nabla h_C(\nabla h_C^\circ(z)) = \frac{z}{h_C^\circ(z)}. \tag{4.3}$$

For the sake of simplicity we will denote $d := d_E$.

Step 1. The proof of (4.1) is easy: indeed, if we fix $x \in \mathbb{R}^N$ with $d(x) > 0$ and $y \in \overline{E^1}$ is such that $d(x) = h_C^\circ(x - y)$ then formula (4.3) reads as

$$n(x) = \frac{x - y}{h_C^\circ(x - y)}$$

from which we immediately get (4.1) since $C = \{x \in \mathbb{R}^N : h_C^\circ(x) \leq 1\}$.

Step 2. We prove (4.2). First of all, it turns out that d is a viscosity supersolution of

$$-\operatorname{div} \nabla h_C(\nabla d) = -\frac{N-1}{r} \tag{4.4}$$

in $\mathbb{R}^N \setminus E_r$. This is a variant of a classical result, see [3]. The proof is quite straightforward. Indeed, if ϕ is a smooth function which touches the graph of d from below at a point $\bar{x} \notin E_r$ (that is, $\phi \le d$, $\phi(\bar{x}) = d(\bar{x})$) then by definition of d, ϕ also touches the graph of $x \mapsto h_C^\circ(x - \bar{y})$ from below at $\bar{x}$, where $\bar{y} \in E$ is a point of minimal distance to $\bar{x}$. Being both functions smooth at $\bar{x}$, it follows that $\nabla\phi(\bar{x}) = \nabla h_C^\circ(\bar{x} - \bar{y})$ and $D^2\phi(\bar{x}) \le D^2 h_C^\circ(\bar{x} - \bar{y})$. In particular,

$$\begin{aligned} -\mathrm{div}\, \nabla h_C(\nabla\phi)(\bar{x}) &= -D^2 h_C(\nabla\phi(\bar{x})) : D^2\phi(\bar{x}) \\ &\ge -D^2 h_C(\nabla h_C^\circ(\bar{x} - \bar{y})) : D^2 h_C^\circ(\bar{x} - \bar{y}) \\ &= -\mathrm{div}\, \nabla h_C(\nabla h_C^\circ)(\bar{x} - \bar{y}). \end{aligned}$$

Combining (4.3) with the Euler's identity, for any $z \in \mathbb{R}^N \setminus \{0\}$ we obtain, also by direct computation,

$$\mathrm{div}\, \nabla h_C(\nabla h_C^\circ(z)) = \mathrm{div}\, \frac{z}{h_C^\circ(z)} = \frac{N h_C^\circ(z) - z \cdot \nabla h_C^\circ(z)}{|h_C^\circ(z)|^2} = \frac{N-1}{h_C^\circ(z)}$$

and therefore finally

$$-\mathrm{div}\, \nabla h_C(\nabla\phi)(\bar{x}) \ge -\frac{N-1}{h_C^\circ(\bar{x} - \bar{y})}.$$

We find that not only d is a viscosity supersolution of (4.4) out of E_r, but the more precise inequality

$$-\mathrm{div}\, \nabla h_C(\nabla\phi)(\bar{x}) \ge -\frac{N-1}{d(\bar{x})}$$

holds. Since $h_C^\circ \in C^2(\mathbb{R}^N \setminus \{0\})$ by Proposition 2.1 we can say that d is (locally) semiconcave out of E_r, and in particular $D^2 d \le c$ in both the viscosity and distributional sense. It is not obvious however to deduce from these facts that

$$-\mathrm{div}\, \nabla h_C(\nabla d) = D^2 h_C(\nabla d) : D^2 d \le (N-1)/r$$

out of E_r in the sense of distributions, as the left-hand side is the product of a L^∞, yet discontinuous function, and a Radon measure.

We pick now $R > r$, $\lambda > 0$, and we introduce u^λ a solution of the problem

$$\begin{aligned} \min \Bigg\{ \int_{E_R \setminus E_r} h_C(Du) + \left(\frac{N-1}{r} + \lambda \right) \int_{E_R \setminus E_r} u(x)dx \ : \\ u \in BV(E_{2R} \setminus E_{r/2}), u \ge d, u = d \text{ if } d \ge R \text{ or } d \le r \Bigg\}. \end{aligned} \tag{4.5}$$

Notice that we can easily apply on the functional in (4.5) direct method of the Calculus of Variations: we have lower semicontinuity in the strong convergence of L^1 essentially by Reshetnyak's lower semicontinuity and we have strong L^1-compactness of sequences bounded in energy since $h_C(v) \ge c|v|$ for some $c > 0$. Moreover, observe that by truncation arguments we clearly have $r \le u^\lambda \le R$ in $E_R \setminus E_r$. Standard density estimates for the level sets of u^λ show also that u^λ is a.e. equal to a lower and a upper-semicontinuous function. We assume that u^λ is upper-semicontinuous, and is a.e. equal to its lower-semicontinuous envelope. We check then that u^λ is a strict viscosity subsolution of (4.4) in $\{u^\lambda > d\}$, in the following sense: if $\phi \ge u^\lambda$, ϕ smooth, $\phi(\bar{x}) = u^\lambda(\bar{x})$, then if $\nabla\phi(\bar{x}) \ne 0$ one has

$$-\mathrm{div}\, \nabla h_C(\nabla\phi)(\bar{x}) \le -\frac{N-1}{r} - \lambda.$$

The proof is easy and quite standard. Possibly replacing ϕ with $\phi+\eta|\cdot-\bar{x}|^2$, η small, we may assume that $\bar{x}$ is the only contact point. Then, one checks that $\{\phi-\delta<u^\lambda\}$ has nonempty interior and goes to $\{\bar{x}\}$ in the Hausdorff distance as $\delta\to 0$. We denote $H_\lambda=(N-1)/r+\lambda$. For $\delta>0$ small we have

$$\int_{\overline{E_R\setminus E_r}} h_C(Du^\lambda)+H_\lambda\int_{E_R\setminus E_r} u^\lambda dx \leq \int_{\overline{E_R\setminus E_r}} h_C(D(u^\lambda\wedge(\phi-\delta)))$$
$$+H_\lambda\int_{E_R\setminus E_r}(u^\lambda\wedge(\phi-\delta))dx.$$

Moreover, since for any open set A the functional

$$u\mapsto\int_A h_C(Du)$$

satisfies the generalized coarea formula (2.3) and it is convex, we get submodularity (see [5, Prop. 3.2]), which reads as

$$\int_A h_C(D(u^\lambda\wedge(\phi-\delta)))+\int_A h_C(D(u^\lambda\vee(\phi-\delta)))\leq\int_A h_C(Du^\lambda)+\int_A h_C(\nabla\phi)dx.$$

Therefore, we obtain that (letting A a small open set containing $\{\phi-\delta<u^\lambda\}$, for δ small)

$$-H_\lambda\int_{\{\phi-\delta<u^\lambda\}}(\phi-\delta-u^\lambda)\,dx\leq\int_A h_C(\nabla\phi)dx-\int_A h_C(D(u^\lambda\vee(\phi-\delta))).$$

If $\nabla\phi(\bar{x})\neq 0$ then one may assume that $\nabla\phi\neq 0$ in A, so that it follows

$$\begin{aligned}H_\lambda\int_{\{\phi-\delta<u^\lambda\}}(u^\lambda-(\phi-\delta))\,dx&\leq\int_A\nabla h_C(\nabla\phi)\cdot(D(\phi-\delta-(u^\lambda\vee(\phi-\delta))))\\&=\int_A\operatorname{div}\nabla h_C(\nabla\phi)(u^\lambda-(\phi-\delta))^+dx\\&=\int_{\{\phi-\delta<u^\lambda\}}\operatorname{div}\nabla h_C(\nabla\phi)(u^\lambda-(\phi-\delta))dx.\end{aligned}$$

We deduce that $\operatorname{div}\nabla h_C(\nabla\phi)(\bar{x})\geq H_\lambda$, as claimed, otherwise one reaches a contradiction for small δ.

Now, we can deduce that $u^\lambda\leq d$ (so that in particular $u^\lambda=d$), using a standard comparison result for viscosity sub and supersolution (with one possibly discontinuous). We sketch the argument, see [4] and [8] for details. Let $m:=\max\{u^\lambda-d\}$ and assume by contradiction that $m>0$. For $\delta>0$ small, we consider

$$m_\delta:=\max_{x,y}\left\{u^\lambda(y)-d(x)-\frac{|x-y|^2}{2\delta}\right\}\geq m$$

which is reached at (x_δ,y_δ). We have that x_δ is a point of maximum of $u^\lambda_\delta-d$ where

$$u^\lambda_\delta(x):=\max_y\left\{u^\lambda(y)-\frac{|x-y|^2}{2\delta}\right\}\geq u^\lambda(x) \tag{4.6}$$

is a sup-convolution. In particular, if $x\in\{u^\lambda_\delta>d+m/2\}$, a point $\bar{y}$ which reaches the maximum in (4.6) is such that $u^\lambda(\bar{y})>d(\bar{y})$ as soon as $\delta<m/L^2$ (L denoting the Lipschitz constant of d), and in this case u^λ_δ is still a strict subsolution of (4.4) in $\{u^\lambda_\delta>d+m/2\}$:

take ϕ smooth with $\phi \geq u_\delta^\lambda$ on $\{u_\delta^\lambda > d + m/2\}$ and with $\phi(\bar{x}) = u_\delta^\lambda(\bar{x})$ for some $\bar{x} \in \{u_\delta^\lambda > d + m/2\}$, and use

$$\psi(y) := \phi(y + \bar{x} - \bar{y}) + \frac{|\bar{x} - \bar{y}|^2}{2\delta}$$

as a test function in the definition of strict subsolution of (4.4) applied to u^λ. Now, since u_δ^λ is (near x_δ) semiconvex while d is semiconcave, we can invoke Jensen's Lemma (see [8] for details), and find that there are points $x_n \to x_\delta$ which are local maximum points of

$$x \mapsto u_\delta^\lambda(x) - d(x) + p_n \cdot x - \alpha_n \frac{|x - x_\delta|^2}{2},$$

with $p_n \to 0, \alpha_n \to 0, u_\delta^\lambda(x_n) > d(x_n) + m/2$; notice that we have to add the term $\alpha_n \frac{|x - x_\delta|^2}{2}$ since, in order to apply Jensen's Lemma, we need x_δ be a strict local maximum of the function we perturb with the linear term $p_n \cdot x$. By Aleksandrov's Theorem (see again [8] for details) we can also assume that u_δ^λ and d are both twice differentiable at x_n. In particular, for n large

$$\nabla u_\delta^\lambda(x_n) = \nabla d(x_n) - p_n + \alpha_n(x_n - x_\delta) \neq 0$$

and $D^2 u_\delta^\lambda(x_n) \leq D^2 d(x_n) + \alpha_n I$ so that

$$\begin{aligned}
\frac{N-1}{r} + \lambda &\leq D^2 h_C(\nabla u_\delta^\lambda(x_n)) : D^2 u_\delta^\lambda(x_n) \\
&\leq D^2 h_C(\nabla d(x_n) - p_n + \alpha_n(x_n - x_\delta)) : D^2 d(x_n) \\
&\quad + \alpha_n \mathrm{Tr}(D^2 h_C(\nabla d(x_n) - p_n + \alpha_n(x_n - x_\delta))) \leq \frac{N-1}{r} + o(1)
\end{aligned}$$

where $o(1) \to 0$ as $n \to \infty$. Since $\lambda > 0$ this yields a contradiction. Hence $u^\lambda = d$ for any $\lambda > 0$, and it follows that d is the only minimizer of (4.5) for any $\lambda > 0$, and in the limit is also a minimizer for $\lambda = 0$.

Finally, we have shown that the functional in (4.5) is minimized by d, including for $\lambda = 0$. But then, the Euler–Lagrange equation for the problem is easily derived: using perturbations $d + \delta\phi$ with $\delta > 0$ small, ϕ smooth, nonnegative, with compact support in $E_R \setminus E_r$, we readily find

$$\int_{E_R \setminus E_r} \left(\nabla h_C(\nabla d) \cdot \nabla \phi + \frac{N-1}{r} \phi \right) dx \geq 0,$$

that is precisely (4.2) in the distributional sense. $\square$

We are ready to prove essentially the same result stated in Proposition 4.1 for a general convex body C.

Theorem 4.2 *Let C be a convex body. There exists $n \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ such that a.e. on $\mathbb{R}^N$ we have*

$$n \in \partial h_C(\nabla d_E) \tag{4.7}$$

and $\operatorname{div} n$ *is a Radon measure on $\mathbb{R}^N \setminus \overline{E}$ with*

$$\operatorname{div} n \leq \frac{N-1}{r} \tag{4.8}$$

in the distributional sense out of E_r.

Proof We use again the notation $d = d_E$. We prove (4.7) and (4.8) approximating C by smooth, elliptic, uniformly bounded and convex sets C^σ, with $C^\sigma \supseteq C$, and using Proposition 4.1. Let $E_r^\sigma := E + rC^\sigma$ and denote by d^σ the anisotropic distance from C^σ. Then $n^\sigma := \nabla h_{C^\sigma}(\nabla d^\sigma) \in C^\sigma$ is well defined a.e.,and (4.2) reads

$$\operatorname{div} n^\sigma \leq \frac{N-1}{r} \tag{4.9}$$

out of E_r^σ. As $\sigma \to 0^+$ we can assume, up to a subsequence, since $||n^\sigma||_\infty$ remains bounded by (4.1), that $n^\sigma \overset{*}{\rightharpoonup} n$ in $L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ and we have for any nonnegative C^1 function ϕ with compact support in $\mathbb{R}^N \backslash E_r$, for σ small enough (using the Hausdorff convergence of E_r^σ to E_r),

$$-\frac{N-1}{r}\int \phi\, dx \leq \int n^\sigma \cdot \nabla\phi\, dx \to \int n \cdot \nabla\phi\, dx$$

as $\sigma \to 0^+$, showing that in $\mathbb{R}^N \backslash E_r$, $\operatorname{div} n$ is a measure bounded from above by $(N-1)/r$, so that we get (4.8). On the other hand, if $\eta : \mathbb{R}_+ \to \mathbb{R}$ is any smooth nonincreasing function with $\eta(t) = 1$ for $t \leq r$, $\eta(t) = 0$ for t large, one has (since $n^\sigma = \nabla h_{C^\sigma}(\nabla d^\sigma) \in \partial h_{C^\sigma}(-\eta'(d^\sigma)\nabla d^\sigma)$, using that ∇h_{C^σ} is zero-homogeneous and always contained in $\partial h_{C^\sigma}(0)$):

$$\int n^\sigma \cdot (-\nabla(\eta \circ d^\sigma))\, dx = \int h_{C^\sigma}(-\nabla(\eta \circ d^\sigma))\, dx$$

Since $h_{C^\sigma} \geq h_C$, we easily see that, from $\eta \circ d^\sigma \to \eta \circ d$ in any L^p and using standard lower semicontinuity results for integral functionals,

$$\int h_C(-\nabla(\eta \circ d))\, dx \leq \liminf_{\sigma \to 0^+} \int h_C(-\nabla(\eta \circ d^\sigma))\, dx \leq \liminf_{\sigma \to 0^+} \int h_{C^\sigma}(-\nabla(\eta \circ d^\sigma))\, dx,$$

that is

$$\int h_C(-\nabla(\eta \circ d))\, dx \leq \liminf_{\sigma \to 0^+} \int n^\sigma \cdot (-\nabla(\eta \circ d^\sigma))\, dx. \tag{4.10}$$

On the other hand (using (4.9)),

$$\begin{aligned}\int n^\sigma \cdot (-\nabla(\eta \circ d^\sigma))\, dx &= \int n^\sigma \cdot (-\nabla(\eta \circ d))\, dx - \int n^\sigma \cdot \nabla(\eta \circ d^\sigma - \eta \circ d)\, dx \\ &\leq \int n^\sigma \cdot (-\nabla(\eta \circ d))\, dx + \frac{N-1}{r}\int (\eta \circ d^\sigma - \eta \circ d)\, dx\end{aligned}$$

since we have assumed that $C^\sigma \supseteq C$, so that $d^\sigma \leq d$ and $\eta \circ d^\sigma - \eta \circ d \geq 0$. Since $d^\sigma \to d$ uniformly, $n^\sigma \overset{*}{\rightharpoonup} n$ and $\nabla(\eta \circ d) \in L^1(\mathbb{R}^N; \mathbb{R}^N)$, we deduce that

$$\limsup_{\sigma \to 0^+} \int n^\sigma \cdot (-\nabla(\eta \circ d^\sigma))\, dx \leq \int n \cdot (-\nabla(\eta \circ d))\, dx$$

which together with (4.10) yields

$$\int h_C(-\nabla(\eta \circ d))\, dx \leq \int n \cdot (-\nabla(\eta \circ d))\, dx.$$

Since $n \in C$ a.e. we obtain (4.7) and this ends the proof. □

Remark 4.3 It turns out that $\operatorname{div} n$ is absolutely continuous with respect to $\mathcal{H}^{N-1} \llcorner (\mathbb{R}^N \backslash \overline{E})$ (see, for instance, [19, Thm. 3.2-b]).

Remark 4.4 Recently Giga and Pozar [12] provided a construction of an n satisfying (4.7) with minimal $\int |\mathrm{div}\, n|^2$. Also, an alternative way to build a Cahn-Hoffmann field satisfying (4.7) can be deduced from the construction in Chambolle, Morini and Ponsiglione [7], in addition this should also provide a field with minimal curvature.

5 Regularity of the volume function

In this section we investigate the regularity of the volume function V_E. Our result extends [14, Eq. (2.20)], where an expression for V_E' has been given whenever C is strictly convex. In what follows n is given as in Theorem 4.2.

Let

$$J := \{r > 0 : \mathcal{H}^{N-1}(\partial E_r' \cap E_r^1) > 0\}$$

Remark 5.1 We will prove (see (5.7)) that for any $r > 0$, $\mathcal{H}^{N-1}(\partial E_r' \cap E_r^1) \ll |\mathrm{div}\, n|$ so that J is at most countable.

In what follow we denote by $V_E'(r^+)$ and $V_E'(r^-)$ respectively the right and the left derivative of V_E.

Theorem 5.2 *For any $r > 0$ we have*

$$V_E'(r^+) = \int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1} \tag{5.1}$$

and

$$V_E'(r^-) = \int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1} + 2\int_{\partial E_r' \cap E_r^1} \phi_C(\nu_{E_r'})\, d\mathcal{H}^{N-1}. \tag{5.2}$$

In particular, V_E is differentiable at r if and only if $r \notin J$.

Proof Notice that from the fact that ∂C is locally Lipschitz and compact we easily deduce that

$$\theta_C := \inf\{\Theta_N^*(C, x) : x \in \partial C\} > 0.$$

As a consequence, we obtain $\partial E_r \cap E_r^0 = \emptyset$: indeed, if $x \in \partial E_r$ then $x \in y + r\partial C$ for some $y \in \partial E$, hence $\Theta_N^*(E_r, x) \geq \theta_C > 0$. By Theorem 3.1 and Theorem 2.2 it follows that for any $r > 0$

$$\lim_{s \to 0^+} \frac{V_E(r+s) - V_E(r)}{s} = \mathcal{SM}_C(E_r) = \int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1}$$

that is formula (5.1). It remains to compute the left derivative of V_E. We divide the rest of the proof in some steps.

Step 1. Let $C^* := -C$, that is the symmetrical of C with respect to the origin; notice that $h^\circ_{C^*}(-v) = h^\circ_C(v)$ for all $v \in \mathbb{R}^N$. We also introduce the corresponding anisotropic distance to ${E_r'}^c$:

$$d^*(x) := \inf_{z \in {E_r'}^c} h^\circ_{C^*}(x - z)$$

where we have denoted $E_r'^{\,c} := (E_r')^c$. Let $s \in (0, r)$. Notice that

$$E_r \backslash E_{r-s} = \{x : r - s < d_E(x) \le r\}.$$

Let $x \in \mathbb{R}^N$ with $d^*(x) < s$. By definition there exist $\varepsilon > 0$ and $z_\varepsilon \in E_r'^{\,c}$ such that $h^\circ_{C^*}(x - z_\varepsilon) = s - \varepsilon$. Then, for any $y \in E$ we obtain, by the subadditivity of h°_C,

$$\begin{aligned} h^\circ_C(x - y) &\ge h^\circ_C(z_\varepsilon - y) - h^\circ_C(z_\varepsilon - x) \\ &= h^\circ_C(z_\varepsilon - y) - h^\circ_{C^*}(x - z_\varepsilon) \ge r - s + \varepsilon \end{aligned}$$

that is $d_E(x) > r - s$. Thus $\{d^*(x) < s,\ d_E(x) \le r\} \subseteq E_r \backslash E_{r-s}$. Taking into account Lemma 3.1 we can say that $|\{d^* = s\}| = 0$ and $|E_r^c| = |E_r'^{\,c}|$, hence

$$|(E_r'^{\,c} + sC^*) \backslash E_r'^{\,c}| = |\{d^*(x) < s,\ d_E(x) \le r\}| \le |E_r \backslash E_{r-s}|.$$

Passing to the limit as $s \to 0^+$ we deduce that

$$\mathcal{SM}_{C^*}(E_r'^{\,c}) \le \liminf_{s \to 0^+} \frac{|E_r \backslash E_{r-s}|}{s}. \tag{5.3}$$

Using Theorem 2.2 we get

$$\mathcal{SM}_{C^*}(E_r'^{\,c}) = \int_{\mathcal{F}E_r'^{\,c}} h_{C^*}(\nu_{E_r'^{\,c}})\, d\mathcal{H}^{N-1} + 2 \int_{\partial E_r'^{\,c} \cap (E_r'^{\,c})^0} \phi_{C^*}(\nu_{E_r'^{\,c}})\, d\mathcal{H}^{N-1}.$$

From $\mathcal{F}E_r = \mathcal{F}E_r'^c$, $\nu_{E_r'^c} = -\nu_{E_r}$, $\partial E_r'^c = \partial E_r'$ and $(E_r'^c)^0 = E_r'^1 = E_r^1$ it follows

$$\mathcal{SM}_{C^*}(E_r'^c) = \int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1} + 2 \int_{\partial E_r' \cap E_r^1} \phi_C(\nu_{E_r'})\, d\mathcal{H}^{N-1}. \tag{5.4}$$

Notice now that if $r \notin J$ then $\mathcal{H}^{N-1}(\partial E_r' \cap E_r^1) = 0$. We obtain that for any $r \in (0, +\infty) \backslash J$

$$\int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1} \le \liminf_{s \to 0^+} \frac{|E_r \backslash E_{r-s}|}{s}. \tag{5.5}$$

Step 2. We prove now that for any $r > 0$

$$\limsup_{s \to 0^+} \frac{|E_r \backslash E_{r-s}|}{s} \le \int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1} - \operatorname{div} n(E_r \backslash E_r'). \tag{5.6}$$

For any $s \in (0, r)$ we have, using the coarea formula and (5.15),

$$\begin{aligned} |E_r \backslash E_{r-s}| &= \int_{r-s}^{r} \int_{\mathcal{F}E_t} h_C(\nu_{E_t})\, d\mathcal{H}^{N-1}\, dt \\ &= \int_0^s \int_{\mathcal{F}E_{r-s+u}} h_C(\nu_{E_{r-s+u}})\, d\mathcal{H}^{N-1}\, du \\ &\le \int_0^s \int_{\mathcal{F}E_{r-s}} h_C(\nu_{E_{r-s}})\, d\mathcal{H}^{N-1}\, du + \int_0^s \frac{N-1}{r-s} |E_{r-s+u} \backslash E_{r-s}|\, du \\ &= s \int_{\mathcal{F}E_{r-s}} h_C(\nu_{E_{r-s}})\, d\mathcal{H}^{N-1} + o(s). \end{aligned}$$

Therefore, by (5.9) we obtain

$$\begin{aligned} \limsup_{s \to 0^+} \frac{|E_r \backslash E_{r-s}|}{s} &\le \lim_{t \to r^-} \int_{\mathcal{F}E_t} h_C(\nu_{E_t})\, d\mathcal{H}^{N-1} \\ &= \int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1} - \operatorname{div} n(E_r \backslash E_r') \end{aligned}$$

which is (5.6).

Step 3. We now conclude the proof showing that for any $r > 0$ it holds

$$- \operatorname{div} n(E_r \setminus E'_r) = 2 \int_{\partial E'_r \cap E^1_r} \phi_C(\nu_{E'_r})\, d\mathcal{H}^{N-1}. \tag{5.7}$$

The inequality "$\geq$" in (5.7) follows combining (5.3) with (5.6). We prove "$\leq$". We have

$$E_r \setminus E'_r = E_r \cap E'^c_r = \bigcap_{s>0} \left[(E'^c_r + sC^*) \cap E_{r+s} \right]$$

so that

$$\begin{aligned} -\operatorname{div} n(E_r \setminus E'_r) &= \lim_{s\to 0} -\operatorname{div} n \left((E'^c_r + sC^*) \cap E_{r+s} \right) \\ &= \lim_{s\to 0} \frac{1}{s} \int 0^S - \operatorname{div} n \left((E'^c_r + tC^*) \cap E_{r+t} \right) dt. \end{aligned}$$

Using Fubini's Theorem, we write this as

$$\begin{aligned} &-\frac{1}{s} \int 0^S \int (\chi_{E'^c_r + tC^*} - \chi_{E^c_{r+t}})\, d(\operatorname{div} n) dt \\ &= - \int \frac{1}{s} \int 0^S (\chi_{E'^c_r + tC^*} - \chi_{E^c_{r+t}})\, dt\, d(\operatorname{div} n) \\ &= - \int \left(1 - \frac{d^*}{s}\right)^+ - \left(\left(\frac{d-r}{s}\right)^+ \wedge 1 \right) d(\operatorname{div} n) \\ &= \int n \cdot \nabla \left(1 - \frac{d^*}{s}\right)^+ - n \cdot \nabla \left(\left(\frac{d-r}{s}\right)^+ \wedge 1 \right) dx \end{aligned}$$

where d^* is defined as in Step 1. Now

$$n \cdot \nabla \left(1 - \frac{d^*}{s}\right)^+ = -n \cdot \nabla d^* \frac{1}{s} \chi_{\{0<d^*<s\}} \leq \frac{1}{s} \chi_{(E'^c_r + sC^*) \setminus E'^c_r}$$

since $-n \cdot \nabla d^* \leq h_{C^*}(\nabla d^*) = 1$, when $d^* > 0$. Next, using $n \cdot \nabla d = 1$ a.e.,

$$n \cdot \nabla \left(\left(\frac{d-r}{s}\right)^+ \wedge 1 \right) = n \cdot \nabla d \frac{1}{s} \chi_{\{r<d\leq r+s\}} = \frac{1}{s} \chi_{E_{r+s} \setminus E_r}.$$

Hence,

$$\begin{aligned} &\lim_{s\to 0} \frac{1}{s} \int 0^S - \operatorname{div} n \left((E'^c_r + tC^*) \cap E_{r+t} \right) dt \leq \lim_{s\to 0} \frac{|(E'^c_r + sC^*) \setminus E'^c_r| - |E_{r+s} \setminus E_r|}{s} \\ &= \mathcal{SM}_{C^*}(E'^c_r) - \mathcal{SM}_C(E_r) = 2 \int_{\partial E'_r \cap E^1_r} \phi_C(\nu_{E'_r})\, d\mathcal{H}^{N-1} \end{aligned}$$

thanks to (5.4), and this ends the proof. □

Proposition 5.3 *For any $r > 0$*

$$\lim_{s\to r^+} \int_{\mathcal{F}E_s} h_C(\nu_{E_s})\, d\mathcal{H}^{N-1} = \int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1} \tag{5.8}$$

and

$$\lim_{s \to r^-} \int_{\mathcal{F}E_s} h_C(\nu_{E_s})\, d\mathcal{H}^{N-1} = \int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1} + 2\int_{\partial E_r' \cap E_r^1} \phi_C(\nu_{E_r'})\, d\mathcal{H}^{N-1}. \tag{5.9}$$

In particular, V_E *is* C^1 *in* $(0, +\infty)\setminus J$.

Proof Let us prove (5.8). The easy part is the estimate from below: since $D\chi_{E_s} \rightharpoonup^* D\chi_{E_r}$, as measures as $s \to r^+$, applying Reshetnyak's lower semicontinuity we have

$$\begin{aligned}\int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1} &= \int h_C\left(\frac{dD\chi_{E_r}}{d|D\chi_{E_r}|}\right) d|D\chi_{E_r}| \\ &\leq \liminf_{s \to r^+} \int h_C\left(\frac{dD\chi_{E_s}}{d|D\chi_{E_s}|}\right) d|D\chi_{E_s}| = \liminf_{s \to r^+} \int_{\mathcal{F}E_s} h_C(\nu_{E_s})\, d\mathcal{H}^{N-1}.\end{aligned}$$

Now we divide the rest of the proof in some steps.

Step 1. We claim that for each continuous function $\psi : [0, 1] \to \mathbb{R}$ we have

$$\lim_{k \to +\infty} \int_0^1 \psi(t) \int_{\mathcal{F}E_{r+t/k}} h_C(\nu_{E_{r+t/k}})\, d\mathcal{H}^{N-1}\, dt = \int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1} \int_0^1 \psi(t)\, dt. \tag{5.10}$$

For simplicity of notation we set

$$f(t) := \int_{\mathcal{F}E_t} h_C(\nu_{E_t})\, d\mathcal{H}^{N-1}.$$

First of all, combining (2.5) with the coarea formula, for any positive integer k we obtain

$$\int_0^1 f(r + t/k)\, dt = \int_0^1 \int_{\{r < d_E < r + t/k\}} h_C(\nabla d_E)\, d\mathcal{H}^{N-1}\, dt = \frac{|E_r + 1/kC| - |E_r|}{1/k}$$

and therefore using Theorem 3.1 we are able to pass to the limit applying [6, Thm. 3.4] and thus

$$\lim_{k \to +\infty} \int_0^1 f(r + t/k)\, dt = f(r).$$

Of course, for any $c > 0$ we also have, by a simple change of variable,

$$\lim_{k \to +\infty} \int_0^c f(r + t/k)\, dt = cf(r)$$

from which, for each bounded open interval I,

$$\lim_{k \to +\infty} \int_I f(r + t/k)\, dt = |I| f(r). \tag{5.11}$$

Now using (5.11) it is easy to get (5.10) whenever $\psi \geq 0$. Indeed, by Fubini's Theorem

$$\begin{aligned}\int_0^1 \psi(t) f(r + t/k)\, dt &= \int_0^1 dt \int_0^{\psi(t)} ds f(r + t/k) = \int_0^{\max \psi} ds \int_{\{\psi > s\}} dt f(r + t/k) \\ &\to \int_0^{\max \psi} ds |\{\psi > s\}| f(r) = f(r) \int_0^1 \psi(t)\, dt.\end{aligned}$$

For a general continuous function ψ it is sufficient to apply the previous argument to ψ^+ and ψ^-.

Step 2: Consider $\eta : \mathbb{R} \to \mathbb{R}$ a smooth nondecreasing function with $\eta \equiv 1$ on $\mathbb{R}_-$ and $\eta(t) = 0$ for $t \geq 1$. Then, letting, for $k \geq 0$, $\psi_k(x) := \eta(k(d_E(x) - r))$ and $\psi_k^\varepsilon(x) := \eta(k(d_E(x) - r - \varepsilon))$, one has, using (4.8),

$$\int n \cdot \nabla(\psi_k - \psi_k^\varepsilon) dx \leq \frac{N-1}{r} \int |\psi_k^\varepsilon - \psi_k|\, dx \to \frac{N-1}{r} |E_{r+\varepsilon} \setminus E_r| \tag{5.12}$$

as $k \to +\infty$. On the other hand, using the definition of n and the coarea formula,

$$\begin{aligned}
\int n \cdot \nabla \psi_k \, dx &= \int k\eta'(k(d_E - r))\, n \cdot \nabla d_E \, dx \\
&= \int k\eta'(k(d_E - r))\, h_C(\nabla d_E)\, dx \\
&= \int_r^{r+1/k} k\eta'(k(s-r)) \int_{\mathcal{F}E_s} h_C(\nu_{E_s})\, d\mathcal{H}^{N-1}\, ds \\
&= \int_0^1 \eta'(s) \int_{\mathcal{F}E_{r+t/k}} h_C(\nu_{E_{r+t/k}})\, d\mathcal{H}^{N-1}\, ds
\end{aligned}$$

and since (5.10) it follows, by definition of η,

$$\lim_{k\to+\infty} \int n \cdot \nabla \psi_k \, dx = - \int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1}. \tag{5.13}$$

Similarly,

$$\lim_{k\to+\infty} \int n \cdot \nabla \psi_k^\varepsilon \, dx = - \int_{\mathcal{F}E_{r+\varepsilon}} h_C(\nu_{E_{r+\varepsilon}})\, d\mathcal{H}^{N-1}.$$

Using (5.13) and definition of $\operatorname{div} n$ we easily get

$$\int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1} - \int_{\mathcal{F}E_s} h_C(\nu_{E_s})\, d\mathcal{H}^{N-1} = \operatorname{div} n\, (E_r \setminus E_s) \tag{5.14}$$

while passing to the limit in (5.12) as $k \to +\infty$ we deduce

$$\int_{\mathcal{F}E_{r+\varepsilon}} h_C(\nu_{E_{r+\varepsilon}})\, d\mathcal{H}^{N-1} \leq \int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1} + \frac{N-1}{r} |E_{r+\varepsilon} \setminus E_r|. \tag{5.15}$$

Passing to the limit in (5.15) as $\varepsilon \to 0^+$ we get

$$\limsup_{\varepsilon\to 0^+} \int_{\mathcal{F}E_{r+\varepsilon}} h_C(\nu_{E_{r+\varepsilon}})\, d\mathcal{H}^{N-1} \leq \int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1}$$

so that the proof of (5.8) is complete. Finally, (5.9) follows from (5.14) and from (5.7) since $\operatorname{div} n$ is a measure, $E_r \setminus E_s \searrow E_r \setminus E_r'$ as $s \to r^-$. □

We next investigate further regularity properties of V_E.

Theorem 5.4 *The second derivative V_E'' is a Radon measure on $(0, +\infty)$ given by*

$$\langle V_E'', \psi \rangle = \int \psi(d_E) d(\operatorname{div} n), \quad \forall \psi \in C_c^\infty(0, +\infty).$$

In particular,

$$V_E'' \leq \frac{N-1}{r} V_E'(r)dr \tag{5.16}$$

in the sense of distributions.

Proof We have, by coarea formula,

$$\begin{aligned}
-\int_0^{+\infty} \psi'(r) V_E'(r)\, dr &= -\int_0^{+\infty} \psi'(r) \int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1}\, dr \\
&= -\int_0^{+\infty} \psi'(d_E) n \cdot \nabla d_E\, dx \\
&= -\int_0^{+\infty} n \cdot \nabla(\psi \circ d_E)\, dx \\
&= \int \psi(d_E) d(\operatorname{div} n)
\end{aligned}$$

from which the conclusion. □

Corollary 5.5 *For any $t, r \in (0, +\infty)\backslash J$ with $t < r$ we have*

$$\frac{1}{r^{N-1}} \int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1} \leq \frac{1}{t^{N-1}} \int_{\mathcal{F}E_t} h_C(\nu_{E_t})\, d\mathcal{H}^{N-1}. \tag{5.17}$$

Moreover,

$$\lim_{r \to +\infty} \frac{1}{r^{N-1}} \int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1} = N|C| = \int_{\partial C} h_C(\nu_C)\, d\mathcal{H}^{N-1}. \tag{5.18}$$

Monotonicity (5.17) follows from (5.16) while (5.18) follows from

$$\lim_{r \to +\infty} \frac{1}{r^{N-1}} \int_{\mathcal{F}E_r} h_C(\nu_{E_r})\, d\mathcal{H}^{N-1} = N \lim_{r \to +\infty} \frac{|E + rC|}{r^N} = N \lim_{r \to +\infty} \left| \frac{E}{r} + C \right|.$$

Remark 5.6 Obviously from Theorem 5.2 the jump part of V_E'' is given by

$$\sum_{r \in J} \left(2 \int_{\partial E_r' \cap E_r^1} \phi_C(\nu_{E_r'})\, d\mathcal{H}^{N-1} \right) \delta_r.$$

In addition we have that for any $r > \varepsilon > 0$

$$(\operatorname{div} n)^-(E_r \backslash E_\varepsilon) \leq \frac{N-1}{\varepsilon} |E_r \backslash E_\varepsilon| + \int_{\mathcal{F}E_\varepsilon} h_C(\nu_{E_\varepsilon})\, d\mathcal{H}^{N-1}.$$

As soon as r_0 is such that $E_{r_0} \supset \overline{\operatorname{conv}}(E) \supset \bigcup_{r \in J} (\partial E_r' \cap E_r^1)$ we have

$$\sum_{r \in J,\, r > \varepsilon} 2 \int_{\partial E_r' \cap E_r^1} \phi_C(\nu_{E_r'})\, d\mathcal{H}^{N-1} \leq \frac{N-1}{\varepsilon} |E_{r_0} \backslash E_\varepsilon| + \int_{\mathcal{F}E_\varepsilon} h_C(\nu_{E_\varepsilon})\, d\mathcal{H}^{N-1}.$$

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