

The Heavy Quark Form Factors at Two Loops

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Abstract

We compute the two-loop QCD corrections to the heavy quark form factors in case of the vector, axial-vector, scalar and pseudo-scalar currents up to second order in the dimensional parameter $\varepsilon = (4 - D)/2$. These terms are required in the renormalization of the higher order corrections to these form factors.

1 Introduction

Since its discovery [1,2] in 1995, the top quark has been studied extensively both in theoretical and experimental premises. These studies of the heaviest particle of the Standard Model (SM) provide as well a detailed probe onto some aspects of electro-weak symmetry breaking (EWSB). Due to its very short lifetime, the top quark decays before hadronizing, and thus provides a window to study its production dynamics widely without accounting for the hadronization effects and therefore as a quark being directly accessible as such.¹ The experiments carried out at the Tevatron and later at the LHC, have already measured many observables which allow to extract the properties of the top quark with remarkable accuracy. Compared to the Tevatron, the LHC offers an abundant rate of top quark pair and single production, hence providing a perfect ground for precision tests. Due to the combined effort from both the theoretical and experimental sides, a striking accuracy has been achieved in many observables, e.g. the uncertainties on the predictions of the inclusive production cross-section of a top quark pair, now are around 5 % at a fixed top quark mass of $m_t = 172.5$ GeV. While these precise measurements provide a strong ground for testing the predictions within the SM, beyond the Standard Model (BSM) physics scenarios can as well hide under those small uncertainties. To find a hint of BSM physics or to rule some hypotheses out, we need more precision and certainly a future linear or circular e^+e^- collider can achieve that. In order to match the experimental accuracy, precise predictions are required on the theoretical side.

In this paper, we focus on perturbative Quantum Chromodynamics (QCD) corrections to the form factors involving heavy quarks which are basic building blocks of various physical quantities concerning top quark pair production. The massive vector and axial vector form factors play an important role in the forward-backward asymmetry of bottom or top quark production at electron-positron colliders. Likewise the decay of a scalar or pseudo-scalar particle to a pair of heavy quarks could also play a very important role in shedding light on the quantum nature of the Higgs boson. There are also static quantities like the anomalous magnetic moment, which receive contributions from such massive form factors. For these reasons, the phenomenology and the perturbative QCD corrections to these form factors have gained much attention during the last decade.

In Refs. [3,4], the first order QCD corrections were obtained for the vector and axial-vector form factors. A massless approximation was considered to obtain the next-to-next-to-leading order (NNLO) QCD corrections in [5] numerically, later followed by an analytic computation in [6]. Another numerical computation was performed in [7] at NNLO using a different formalism. On the other hand, the next-to-leading order (NLO) contributions to the scalar and pseudo-scalar form factors were known [8–11] for long and NNLO corrections by employing quark mass expansion to various orders, c.f. [12–19]. A series of papers followed obtaining the two-loop QCD corrections for the vector form factor [20], the axial-vector form factor [21], the anomaly contributions [22] and the scalar and pseudo-scalar form factors [23]. An independent cross-check of the vector form factor has been performed in [24] with the addition of the $\mathcal{O}(\varepsilon)$ contribution, where $\varepsilon = (4 - D)/2$ and D is the space-time dimension. Recently, the calculation of a subset of the three-loop master integrals [25] has made it possible to obtain the vector form factor at three loops [26] in the color-planar limit. The large β_0 limit has been considered in [27]. While the main goal is to obtain the complete three-loop corrections for the form factors, the $\mathcal{O}(\varepsilon)$ pieces at two-loop order are important ingredients. In addition, we compute the master integrals to the required order in ε with a different technique.

¹The reconstruction of the t -quark itself, and in particular its mass, however, requires to study the hadronization effects.

In the present paper, we compute the contributions to the massive form factors up to $\mathcal{O}(\varepsilon^2)$ for different currents, namely, vector, axial-vector, scalar and pseudo-scalar currents, which serve as input for ongoing and future 3- and 4-loop calculations. We also perform the expansion of the exact results in different kinematic regions. In Section 2, we briefly describe the theoretical formalism for all the currents and corresponding form factors, followed by their renormalization procedure and a description of the universal infrared (IR) structure. Section 3 contains the details about the computational technique. Here we describe especially how we have computed the master integrals. In Section 4 we present the results, also expanding the complete expressions in regions, which are kinematically relevant. Finally we conclude in Section 5. Various of the expressions are rather voluminous. A part of it is presented in the appendices and the $\mathcal{O}(\varepsilon^2)$ terms are only given in computer readable form in a file attached to this paper.

2 The heavy quark form factors

We consider the decay of a virtual massive boson of momentum q into a pair of heavy quarks of mass m , momenta q_1 and q_2 and color c and d , through a vertex X_{cd} , where $X_{cd} = \Gamma_{V,cd}^\mu, \Gamma_{A,cd}^\mu, \Gamma_{S,cd}$ and $\Gamma_{P,cd}$ indicates the coupling to a vector, an axial-vector, a scalar and a pseudo-scalar boson, respectively. Here $q^2 = (q_1 + q_2)^2$ is the center of mass energy squared and the dimensionless variable s is defined by

$$s = \frac{q^2}{m^2}. \quad (2.1)$$

The amplitude takes the following general form

$$\bar{u}_c(q_1) X_{cd} v_d(q_2), \quad (2.2)$$

where $\bar{u}_c(q_1)$ and $v_d(q_2)$ are the bi-spinors of the quark and the anti-quark, respectively. We denote the corresponding UV renormalized form factors by F_I , with $I = V, A, S, P$. They are expanded in the strong coupling constant $\alpha_s = g_s^2/(4\pi)$ as follows

$$F_I = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^n F_I^{(n)}. \quad (2.3)$$

The unrenormalized form factors are denoted by \hat{F}_I . In the following sub-sections, we discuss the properties of each current, the corresponding renormalization procedure and their universal infrared structure.

2.1 The vector and axial-vector current

In this section, we consider the spin 1 case, i.e. the vertex Γ^μ of a Z -boson or photon coupling to a pair of heavy quarks. The general structure of Γ^μ consists of six form factors, two of which are CP odd. We consider only higher order QCD effects and SM neutral current interactions to lowest order. Since CP invariance holds, we only take into account the four CP even form factors, $F_{V,i}(s), F_{A,i}(s)$ $i = 1, 2$ in the following. They can be cast in the following general form

$$\begin{aligned} \Gamma_{cd}^\mu &= \Gamma_{V,cd}^\mu + \Gamma_{A,cd}^\mu \\ &= -i\delta_{cd} \left[v_Q \left(\gamma^\mu F_{V,1} + \frac{i}{2m} \sigma^{\mu\nu} q_\nu F_{V,2} \right) + a_Q \left(\gamma^\mu \gamma_5 F_{A,1} + \frac{1}{2m} q^\mu \gamma_5 F_{A,2} \right) \right], \end{aligned} \quad (2.4)$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, $q = q_1 + q_2$, and v_Q and a_Q are the SM vector and axial-vector coupling constants as defined by

$$v_Q = \frac{e}{\sin \theta_w \cos \theta_w} \left(\frac{T_3^Q}{2} - \sin^2 \theta_w Q_Q \right), \quad a_Q = -\frac{e}{\sin \theta_w \cos \theta_w} \frac{T_3^Q}{2}. \quad (2.5)$$

e denotes the elementary charge, θ_w the weak mixing angle, T_3^Q the third component of the weak isospin, and Q_Q the charge of the heavy quark.

To extract the form factors $F_{I,i}$, $I = V, A$, we multiply Γ^μ by the following projectors and perform a trace over the spinor and color indices

$$\begin{aligned} P_{V,i} &= \frac{i}{v_Q N_c} \frac{\not{d}_2 - m}{m} \left(\gamma_\mu g_{V,i}^1 + \frac{1}{2m} (q_{2\mu} - q_{1\mu}) g_{V,i}^2 \right) \frac{\not{d}_1 + m}{m}, \\ P_{A,i} &= \frac{i}{a_Q N_c} \frac{\not{d}_2 - m}{m} \left(\gamma_\mu \gamma_5 g_{A,i}^1 + \frac{1}{2m} (q_{1\mu} + q_{2\mu}) \gamma_5 g_{A,i}^2 \right) \frac{\not{d}_1 + m}{m}, \end{aligned} \quad (2.6)$$

where,²

$$\begin{aligned} g_{V,1}^1 &= -\frac{1}{4(1-\varepsilon)} \frac{1}{(s-4)}, & g_{V,1}^2 &= \frac{(3-2\varepsilon)}{(1-\varepsilon)} \frac{1}{(s-4)^2}, \\ g_{V,2}^1 &= \frac{1}{(1-\varepsilon)} \frac{1}{s(s-4)}, & g_{V,2}^2 &= -\frac{1}{(1-\varepsilon)} \frac{1}{(s-4)^2} \left(\frac{4}{s} + 2 - 2\varepsilon \right), \\ g_{A,1}^1 &= -\frac{1}{4(1-\varepsilon)} \frac{1}{(s-4)}, & g_{A,1}^2 &= -\frac{1}{(1-\varepsilon)} \frac{1}{s(s-4)}, \\ g_{A,2}^1 &= \frac{1}{(1-\varepsilon)} \frac{1}{s(s-4)}, & g_{A,2}^2 &= \frac{1}{(1-\varepsilon)} \frac{1}{s^2(s-4)} \left(4(3-2\varepsilon) - 2s(1-\varepsilon) \right), \end{aligned} \quad (2.7)$$

and N_c denotes the number of colors. Later on we will also use the Casimir operators $C_A = N_c$, $C_F = (N_c^2 - 1)/(2N_c)$, $T_F = 1/2$ for $SU(N_c)$, with $N_c = 3$ in the case of QCD.

2.2 The scalar and pseudo-scalar current

We consider the current implied by a general neutral spin 0 particle h that couples to heavy quarks through the Yukawa interaction

$$\mathcal{L}_{int} = -\frac{m}{v} \left[s_Q \bar{Q} Q + ip_Q \bar{Q} \gamma_5 Q \right] h, \quad (2.8)$$

where $v = (\sqrt{2}G_F)^{-1/2}$ is the SM Higgs vacuum expectation value, with G_F being the Fermi constant, s_Q and p_Q are the scalar and pseudo-scalar coupling, respectively, and Q and h are the heavy quark and scalar and pseudo-scalar field, respectively. The vertex for $h \rightarrow \bar{Q} + Q$, $X_{cd} \equiv \Gamma_{cd}$ consists of two form factors with the following general structure

$$\begin{aligned} \Gamma_{cd} &= \Gamma_{S,cd} + \Gamma_{P,cd} \\ &= -\frac{m}{v} \delta_{cd} \left[s_Q F_S + ip_Q \gamma_5 F_P \right], \end{aligned} \quad (2.9)$$

²In [20], the expression for $g_{V,2}^2$ contains a typographical error.

where F_S and F_P denote the renormalized scalar and pseudo-scalar form factors, respectively. As before, the form factors can be obtained from Γ_{cd} through suitable projectors as given below and performing the trace over the spinor and color indices

$$\begin{aligned} P_S &= \frac{v}{2ms_Q} \frac{\delta_{cd}}{N_c} \frac{\not{q}_2 - m}{m} \left(-\frac{1}{(s-4)} \right) \frac{\not{q}_1 + m}{m}, \\ P_P &= \frac{v}{2mp_Q} \frac{\delta_{cd}}{N_c} \frac{\not{q}_2 - m}{m} \left(-\frac{i}{s} \gamma_5 \right) \frac{\not{q}_1 + m}{m}. \end{aligned} \quad (2.10)$$

2.3 Anomaly and Ward identities

Since we use dimensional regularization [28] in $D = 4 - 2\varepsilon$ space-time dimensions, one important point is to define a proper description for the treatment of γ_5 . In the case of the axial-vector and the pseudo-scalar form factors, two types of Feynman diagrams contribute: the non-singlet diagrams containing only open fermion lines, and the singlet diagrams where a fermion loop is attached to the axial-vector or pseudo-scalar vertex. It is convenient to separate the two contributions and write,

$$\Gamma_{A,cd}^\mu = \Gamma_{A,cd}^{\mu,ns} + \Gamma_{A,cd}^{\mu,s}, \quad \Gamma_{P,cd} = \Gamma_{P,cd}^{ns} + \Gamma_{P,cd}^s, \quad (2.11)$$

where ns and s denote the non-singlet and the singlet contributions, respectively.

In the non-singlet case, we use an anticommuting γ_5 in D space-time dimensions, with $\gamma_5^2 = 1$, as it does not lead to any spurious singularities. This approach respects chiral invariance and leaves us with the Ward identity

$$q^\mu \Gamma_{A,cd}^{\mu,ns} = 2m \Gamma_{P,cd}^{ns}, \quad (2.12)$$

which in terms of the form factors, takes the form

$$2F_{A,1}^{ns} + \frac{s}{2} F_{A,2}^{ns} = 2m F_P^{ns}. \quad (2.13)$$

On the other hand, the singlet pieces for the axial-vector and the pseudo-scalar vertex are related to each other through the Adler-Bell-Jackiw (ABJ) anomaly [29,30]. With this constraint, we use the following prescription as presented in [31,32], which mostly followed [28]: For a single γ_5 in a fermion loop, we use

$$\gamma_5 = \frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma, \quad (2.14)$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric Levi-Civita tensor and all Lorentz indices are taken D -dimensional. Finally, the contraction of two ϵ -tensors is expressed in terms of products of D -dimensional metric tensors. This prescription of γ_5 needs a special treatment during renormalization, as discussed later.

The ABJ anomaly involves the truncated matrix element of the gluonic operator between the vacuum and a pair of heavy quark states. The gluonic operator is given by

$$G(x) \tilde{G}(x) \equiv \epsilon_{\mu\nu\rho\sigma} G^{a,\mu\nu}(x) G^{a,\rho\sigma}(x), \quad (2.15)$$

where $G^{a,\mu\nu}$ represents the gluonic field strength tensor. Denoting its contribution by $F_{G,Q}$, we can immediately write down the anomalous Ward identity for the singlet case as follows

$$q_\mu \Gamma_{A,cd}^{\mu,s} = 2m \Gamma_{P,cd}^s - i \left(\frac{\alpha_s}{4\pi} \right) T_F \langle G \tilde{G} \rangle_Q, \quad (2.16)$$

which implies

$$2F_{A,1}^s + \frac{s}{2} F_{A,2}^s = 2m F_P^s - i \left(\frac{\alpha_s}{4\pi} \right) T_F F_{G,Q}. \quad (2.17)$$

2.4 Renormalization

The UV renormalization of the form factors has been performed in a mixed scheme. We renormalize the heavy quark mass and wave function in the on-shell (OS) renormalization scheme, while the strong coupling constant is renormalized in the modified minimal subtraction ($\overline{\text{MS}}$) scheme [33, 34]. The corresponding renormalization constants are well known and are denoted by $Z_{m,\text{OS}}$ [35–39], $Z_{2,\text{OS}}$ [35–37, 40] and Z_{a_s} [41–45] for the heavy quark mass, wave function and strong coupling constant, respectively. All renormalization constants follow a perturbative expansion in α_s

$$Z_I = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^n Z_I^{(n)}. \quad (2.18)$$

For reference, we present $Z_{m,\text{OS}}^{(n)}$ and $Z_{2,\text{OS}}^{(n)}$ in appendix A up to $n = 2$ and $\mathcal{O}(\varepsilon^2)$.

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F(n_l + n_h). \quad (2.19)$$

Here n_l and n_h denote the number of light and heavy quarks, respectively. In the following we will set $n_h = 1$.

While the renormalization of the heavy-quark wave function and the strong coupling constant can be done multiplicatively, the mass renormalization requires the explicit calculation of counterterm diagrams. Hence, the bare and renormalized vector form factors are at two loops related by

$$F_{V,i} = Z_{2,\text{OS}} \hat{F}_{V,i} + \left(\frac{\alpha_s}{4\pi} \right)^2 Z_{m,\text{OS}}^{(1)} \hat{F}_{V,i}^{\text{ct},(1)} + \mathcal{O}(\alpha_s^3), \quad (2.20)$$

where the unrenormalized form factors $\hat{F}_{V,i}$ are expanded in the unrenormalized strong coupling constant $\hat{\alpha}_s = \alpha_s Z_{a_s}$, and $\hat{F}_{V,i}^{\text{ct},(1)}$ denotes the bare contribution from counterterm diagrams at one loop.

The non-singlet contributions to the axial-vector form factor can be renormalized in the same way. The singlet part requires extra care due to the prescription employed for γ_5 . It is infrared finite and the UV pole is renormalized by the multiplicative renormalization constant Z_J as

$$F_{A,i}^s = Z_J Z_5^{\text{fin}} \hat{F}_{A,i}^s, \quad (2.21)$$

where Z_5^{fin} is a finite renormalization constant which restores the anomalous Ward identity Eq. (2.16). We would like to remark that the Ward identities are valid for physical quantities. Therefore, it is not reasonable to study them at higher orders in ε , and neither it is to consider ε -dependent pieces of Z_5^{fin} . In the $\overline{\text{MS}}$ scheme for the form factors,

$$Z_J = 1 + \left(\frac{\alpha_s}{4\pi} \right)^2 \frac{6C_F T_F}{\varepsilon} + \mathcal{O}(\alpha_s^3) \quad (2.22)$$

and

$$Z_5^{\text{fin}} = 1 + \left(\frac{\alpha_s}{4\pi} \right)^2 \left(3C_F T_F \right) + \mathcal{O}(\alpha_s^3). \quad (2.23)$$

The remaining finite renormalization has to be carried out later for the corresponding observables of which the form factors form a part.

The renormalization of the quantity $F_{G,Q}$ appearing in Eq. (2.17) involves the mixing of the gluonic operator $G\tilde{G}$ with another operator, namely, $\partial_\mu \bar{\psi} \gamma^\mu \gamma_5 \psi$, as discussed in [32, 46, 47], where ψ indicates all quark flavors including the massive one. We get,

$$F_{G,Q} = Z_{GG} \hat{F}_{G,Q} + Z_{GJ} \hat{F}_{J,Q}, \quad (2.24)$$

where $\hat{F}_{J,Q}$ indicates the bare contribution from the second operator, while Z_{GG} and Z_{GJ} are the corresponding renormalization constants.

The renormalization of the scalar and pseudo-scalar (non-singlet) vertices also follows a similar procedure, except for the presence of the heavy quark mass in the Yukawa coupling. Thus the renormalized form factors are given by

$$\begin{aligned} F_{S,i} &= Z_{m,\text{OS}} Z_{2,\text{OS}} \hat{F}_{S,i} + \left(\frac{\alpha_s}{4\pi}\right)^2 Z_{m,\text{OS}}^{(1)} \hat{F}_{S,i}^{ct,(1)} + \mathcal{O}(\alpha_s^3), \\ F_{P,i}^{\text{ns}} &= Z_{m,\text{OS}} Z_{2,\text{OS}} \hat{F}_{P,i}^{\text{ns}} + \left(\frac{\alpha_s}{4\pi}\right)^2 Z_{m,\text{OS}}^{(1)} \hat{F}_{P,i}^{ct,\text{ns},(1)} + \mathcal{O}(\alpha_s^3), \end{aligned} \quad (2.25)$$

On the other hand, the singlet piece of the pseudo-scalar vertex is both IR and UV finite, hence no additional renormalization is necessary.

2.5 The infrared structure

The study of the IR behavior of the form factors has attracted a lot of attention in the past few decades. A plethora of works on massless scattering amplitudes [48–52] has already provided a remarkable understanding on the universal IR pattern characterized by soft and collinear dynamics. In [53], the first step was taken to generalize this for the two-loop scattering amplitudes with massive partons. Later, in [54], following a soft-collinear effective theory (SCET) approach, the general IR structures have been presented.

The IR singularities of the massive form factors can be factorized as a multiplicative renormalization factor. Its structure is constrained by the renormalization group equation (RGE), as follows,

$$F_I = Z(\mu) F_I^{\text{fin}}(\mu), \quad (2.26)$$

where F_I^{fin} is finite as $\varepsilon \rightarrow 0$. The RGE for Z reads

$$\frac{d}{d \ln \mu} \ln Z(\varepsilon, x, m, \mu) = -\Gamma(x, m, \mu), \quad (2.27)$$

where Γ is the corresponding anomalous dimension. Notice that Z does not carry any information regarding the vertex. Γ can be identified as the massive cusp anomalous dimension, which is by now available up to the three-loop level [55–58]. Both Z and Γ can be expanded in a perturbative series in α_s as follows

$$Z = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n Z^{(n)}, \quad \Gamma = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^{n+1} \Gamma_n \quad (2.28)$$

and we find the following solution for Eq. (2.27)

$$Z = 1 + \left(\frac{\alpha_s}{4\pi}\right) \left[\frac{\Gamma_0}{2\varepsilon}\right] + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\frac{1}{\varepsilon^2} \left(\frac{\Gamma_0^2}{8} - \frac{\beta_0 \Gamma_0}{4}\right) + \frac{\Gamma_1}{4\varepsilon}\right] + \mathcal{O}(\alpha_s^3). \quad (2.29)$$

Eq. (2.29) correctly predicts the infrared singularities for all massive form factors at the two-loop level.

3 Details of the calculation

The Feynman diagrams were generated using QGRAF [59], the output of which was then processed using Q2e/Exp [60, 61] and FORM [62, 63] in order to express the diagrams in terms of a linear combination of a large set of scalar integrals. These integrals were then reduced to a much smaller set of master integrals (MIs) using integration by parts identities (IBPs) [64–68] with the help of the program Crusher [69]. Since all this is common practice, we refrain from going into any detail.

After performing the reductions, all that remains to be done is to calculate the master integrals. In the following sections, we present the methods we used to achieve this.

3.1 The conventional differential equations method

We computed the two-loop master integrals contributing to the massive fermion form factors as Laurent expansions in the dimensional parameter ε by means of the differential equation method [70–76]. This technique has already been applied to massive form factor integrals at two and three loops in [25, 77, 78]. In this work, we calculate the two-loop master integrals up to a sufficiently high order in ε to obtain $\mathcal{O}(\varepsilon^2)$ accuracy in the form factors.

In this section, we briefly review the main steps of this calculation. The master integrals are classified according to their underlying topology. In particular, we distinguish the non-singlet topologies (Figure 1) from the singlet topology (Figure 2) according to whether the external current does or does not couple to the external massive quark.

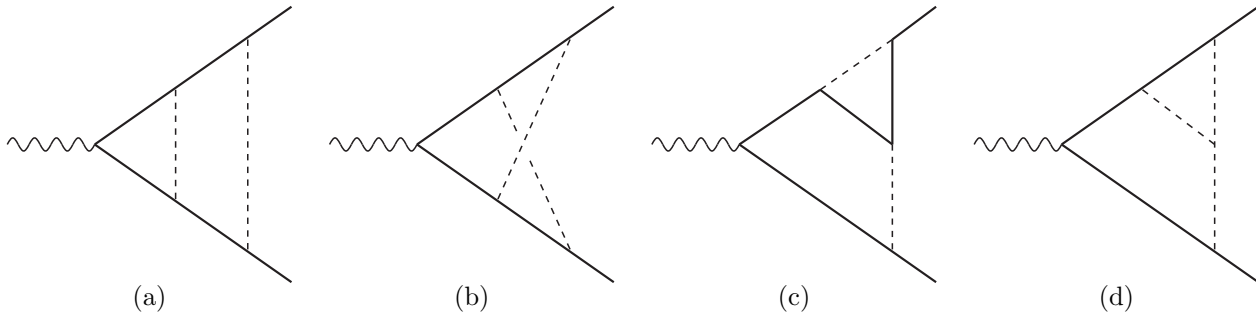


Figure 1: The non-singlet topologies required for the calculation of two-loop form factors. Solid lines represent massive particles in external or internal lines, while dashed lines correspond to massless propagators. The external vector current can also be replaced by an axial-vector, scalar or pseudo-scalar. The master integrals associated to topologies (c) and (d) are a subset of the master integrals required for topologies (a) and (b).

In the case of the non-singlet topologies depicted in Figure 1, it turns out that the master integrals associated to the two topologies on the right of that figure represent a subset of the ones required to calculate the topologies on the left. We therefore concentrate on the topologies in Figures 1(a) and 1(b). The master integrals for both topologies can be expressed in terms of a single integral family with seven propagators given by

$$J(\nu_1, \dots, \nu_7) = \left((4\pi)^{2-\varepsilon} e^{\varepsilon\gamma_E} \right)^2 \int \frac{d^D l_1 d^D l_2}{(2\pi)^{2D}} \frac{1}{D_1^{\nu_1} \dots D_7^{\nu_7}}, \quad (3.1)$$

where

$$\begin{aligned}
 D_1 &= (l_1 + q_1)^2 - m^2, & D_2 &= (l_2 + q_1)^2 - m^2, & D_3 &= (l_1 - q_2)^2 - m^2, \\
 D_4 &= (l_2 - q_2)^2 - m^2, & D_5 &= l_1^2, & D_6 &= (l_1 - l_2)^2, & D_7 &= (l_1 - l_2 + q_2)^2 - m^2.
 \end{aligned}
 \tag{3.2}$$

Here the q_i 's with $i = 1, 2$ are the external momenta, which are taken on-shell ($q_1^2 = q_2^2 = m^2$). The MIs are therefore labeled by the exponents ν_1, \dots, ν_7 of the denominators D_1, \dots, D_7 . The integrals corresponding to the topology in Figure 1(a) will have $\nu_7 = 0$, while the ones corresponding to the topology in Figure 1(b) will have $\nu_3 = 0$. There are several master integrals where ν_3 and ν_7 are both equal to zero, which are therefore common to both topologies³. The list of master integrals required to reduce these topologies is given in Table 1. Notice that there are several sets of master integrals that have the same set of non-vanishing positive powers of propagators. Such a set of integrals is called a *sector*. For example, integrals J_6, J_7 and J_8 belong to the same sector, since for all of them $\nu_1, \nu_4, \nu_6 \geq 1$ and $\nu_2 = \nu_3 = \nu_5 = \nu_7 = 0$. A given sector is said to be a subsector of another sector if the propagators in the first sector are a subset of the propagators in the second.

MI	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6	ν_7	k
J_1	1	1	0	0	0	0	0	6
J_2	0	1	0	0	1	1	0	6
J_3	1	0	0	1	0	0	1	4
J_4	1	1	1	0	0	0	0	4
J_5	1	1	1	1	0	0	0	4
J_6	1	0	0	1	0	1	0	4
J_7	1	0	0	2	0	1	0	3
J_8	1	0	0	1	0	2	0	4
J_9	0	1	1	0	1	1	0	3
J_{10}	0	1	0	1	1	1	0	4
J_{11}	1	1	0	1	0	0	0	3
J_{12}	1	0	0	1	0	1	1	3
J_{13}	1	0	0	2	0	1	1	3
J_{14}	1	0	0	1	0	2	1	3
J_{15}	1	1	0	0	1	0	1	3
J_{16}	1	1	0	0	2	0	1	3
J_{17}	1	1	0	0	1	0	2	3
J_{18}	1	1	0	1	0	0	1	3
J_{19}	1	1	0	2	0	0	1	3
J_{20}	1	1	0	1	1	1	0	2
J_{21}	1	1	0	1	1	2	0	4
J_{22}	1	1	0	1	1	1	1	3
J_{23}	1	1	0	1	1	2	1	3

Table 1: The list of the non-singlet master integrals identified by the indices ν_1 to ν_7 . In the last column, we indicate the order k in ε to which each integral needs to be expanded in order to calculate the form factors to $\mathcal{O}(\varepsilon^2)$.

³This is the reason we chose to group here both topologies within a single integral family, although the actual reductions performed with **Crusher** were done using two separate families.

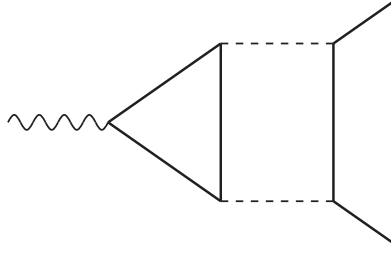


Figure 2: The singlet topology.

In the case of the singlet topology, shown in Figure 2, the master integrals are given by

$$K(\nu_1, \dots, \nu_6) = \left((4\pi)^{2-\varepsilon} e^{\varepsilon\gamma_E} \right)^2 \int \frac{d^D l_1 d^D l_2}{(2\pi)^{2D}} \frac{1}{D_1^{\nu_1} \dots D_6^{\nu_6}}, \quad (3.3)$$

where

$$\begin{aligned} D_1' &= (l_1 + q_1)^2, & D_2' &= (l_2 + q_1)^2 - m^2, & D_3' &= (l_1 - q_2)^2, \\ D_4' &= (l_2 - q_2)^2 - m^2, & D_5' &= l_1^2 - m^2, & D_6' &= (l_1 - l_2)^2 - m^2. \end{aligned} \quad (3.4)$$

The list of master integrals required in this case is given in Table 2. From now on, instead of using the exponents ν_i to identify the master integrals, we will use the single subindex we have assigned to each integral according to the leftmost columns of Tables 1 and 2.

MI	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6	k
K_1	0	1	0	0	1	0	6
K_2	0	1	0	0	1	1	4
K_3	1	1	1	0	0	0	3
K_4	1	1	1	1	0	0	3
K_5	0	1	1	0	0	1	3
K_6	0	1	1	0	0	2	3
K_7	0	1	0	1	1	0	3
K_8	1	1	1	0	1	0	3
K_9	0	1	1	0	1	1	3
K_{10}	0	1	1	0	1	2	3
K_{11}	0	1	0	1	1	1	3
K_{12}	0	1	0	1	1	2	3
K_{13}	1	1	1	1	1	0	2
K_{14}	1	1	1	1	0	1	2
K_{15}	1	1	1	1	1	1	2

Table 2: The list of singlet master integrals identified by the indices ν_1 to ν_6 . In the last column, we indicate the order k in ε to which each integral needs to be expanded in order to calculate the form factors to $\mathcal{O}(\varepsilon^2)$.

All of the master integrals can be expressed in terms of harmonic polylogarithms (HPLs) [79]

in the kinematic variable [80, 81]

$$x = \frac{\sqrt{q^2 - 4m^2} - \sqrt{q^2}}{\sqrt{q^2 - 4m^2} + \sqrt{q^2}} \leftrightarrow \frac{q^2}{m^2} = -\frac{(1-x)^2}{x}. \quad (3.5)$$

In particular, we focus on the Euclidean region, $q^2 < 0$, corresponding to x ranging in $(0, 1)$. A large center of mass energy $|q^2| \gg m^2$ is equivalent to the boundary $x \rightarrow 0$, while the large-mass limit $m^2 \gg |q^2|$ is mapped to the endpoint $x \rightarrow 1$.

We derived a system of coupled linear differential equations for each topology by reducing the derivative with respect to x of each MI to a linear combination of the master integrals themselves, with the help of `Crusher`. The derivative of an integral cannot produce integrals in this linear combination with more propagators than the original one, so they are all either in the same sector as the integral to which one takes the derivative or they belong to a subsector. This means that the integrals in the subsectors need to be solved first, and the differential equations within a given sector will constitute a coupled subsystem. In the case of the singlet master integrals, there are three such coupled subsystems, namely, the subsystems formed by the set of integrals $\{K_5, K_6\}$, $\{K_9, K_{10}\}$ and $\{K_{11}, K_{12}\}$. In all other sectors, only one integral is present. The differential equations in each coupled subsystem will therefore need to be decoupled in order to solve them. The strategy for solving the whole system consists then in solving first the simplest sectors (with fewer propagators), and move up in the chain of subsystems, decoupling and solving each one of them until all integrals are obtained. The starting point are the integrals for which the derivative with respect to x equals zero⁴, which must therefore be obtained not by the differential equations method but by other means.

In the case of the non-singlet topologies, by expanding the MIs in a Laurent series around $\varepsilon = 0$, the systems greatly simplify, assuming an almost complete block-triangular form. Only one 2×2 coupled subsystem remains after the expansion in ε . We therefore solve the system order-by-order in ε by integrating each equation by quadratures. These steps are automated and results are efficiently simplified using a minimal set of independent HPLs by means of the `Mathematica` packages `Sigma` [82, 83] and `HarmonicSums` [84–89]. This procedure is slightly modified for the aforementioned 2×2 coupled system, which does not assume a triangular form after expanding in ε

$$\frac{d}{dx} \begin{pmatrix} J_{22} \\ J_{23} \end{pmatrix} = \begin{bmatrix} \frac{1+x^2}{x(1-x^2)} & \frac{1-x^2}{x^2} \\ -\frac{1}{1-x^2} & \frac{4(1+x^2)}{x(1-x^2)} \end{bmatrix} \begin{pmatrix} J_{22} \\ J_{23} \end{pmatrix} + \begin{pmatrix} R_1(\varepsilon, x) \\ R_2(\varepsilon, x) \end{pmatrix}, \quad (3.6)$$

where the inhomogeneities $R_1(\varepsilon, x)$, $R_2(\varepsilon, x)$ are determined at each order in ε by subsector MIs. The most general solution of the homogeneous system involves only logarithms and rational functions

$$\begin{aligned} J_{22} &= c_1 \frac{x^2}{(1-x^2)^2} + c_2 \frac{x^2 \ln(x)}{(1-x^2)^2}, \\ J_{23} &= c_1 \frac{x^2(1+x^2)}{(1-x^2)^4} + c_2 \frac{x^3(x^2 \ln(x) + \ln(x) + 2)}{(1-x^2)^4}, \end{aligned} \quad (3.7)$$

therefore we integrate the inhomogeneous system order-by-order in ε using the method of variation of constants. For example, at leading order in ε , the inhomogeneous parts of the system

⁴These are J_1 , J_2 and J_3 in the case of the non-singlet topologies, and K_1 and K_2 in the singlet case.

(3.6) are

$$R_1(\varepsilon, x) = \frac{1}{\varepsilon} \left\{ \frac{x^4 + x^3 + 6x^2 + x + 1}{(1-x)^2(x+1)^4} \ln(x) - \frac{x^6 + 2x^5 - 25x^4 - 4x^3 - 25x^2 + 2x + 1}{16x^2(1-x)(x+1)^3} \right\} + \mathcal{O}(\varepsilon^0), \quad (3.8)$$

$$R_2(\varepsilon, x) = \frac{1}{\varepsilon} \left\{ \frac{x \ln(x)}{(1-x)^4(x+1)^6} (2x^6 + x^5 - 4x^4 + 6x^3 - 4x^2 + x + 2) - \frac{x^8 + 2x^7 - 8x^6 - 10x^5 + 22x^4 - 10x^3 - 8x^2 + 2x + 1}{4(1-x)^3x(x+1)^5} \right\} + \mathcal{O}(\varepsilon^0). \quad (3.9)$$

By introducing Eq. (3.7) with $c_i \rightarrow c_i(x)$ into the system (3.6), we get first order differential equations for $c_1(x)$ and $c_2(x)$ that can be solved by quadratures using the same automated tools introduced above. The integration constants are fixed by imposing the regularity of the functions J_{22} and J_{23} in the limit of vanishing space-like momentum $q^2 \rightarrow 0$, corresponding to $x \rightarrow 1$, giving

$$\begin{aligned} J_{22} &= \frac{1}{\varepsilon} \left\{ -\frac{x^2}{3(1-x^2)^2} [H_0^3(x) - 3H_0(x)(2H_{0,-1}(x) - 2H_{0,1}(x) - \zeta_2) \right. \\ &\quad \left. + 12H_{0,0,-1}(x) - 12H_{0,0,1}(x) + 3\zeta_3] \right\} + \mathcal{O}(\varepsilon^0), \\ J_{23} &= \frac{1}{\varepsilon} \left\{ \frac{x^3(1+x^2)}{(1-x^2)^4} \left[2H_0(x)H_{0,-1}(x) - 2H_0(x)H_{0,1}(x) - 4H_{0,0,-1}(x) + 4H_{0,0,1}(x) \right. \right. \\ &\quad \left. - \zeta_2 H_0(x) - \frac{1}{3} H_0^3(x) - \zeta_3 \right] + \frac{x^3}{(1-x^2)^3} [2H_{0,1}(x) - 2H_{0,-1}(x) + 2H_{-1}(x)H_0(x) \\ &\quad - 2H_0(x)H_1(x) - H_0^2(x) - \zeta_2] + \frac{x^6 + 2x^5 - 25x^4 - 4x^3 - 25x^2 + 2x + 1}{16(1-x)^2(x+1)^4} \\ &\quad \left. - \frac{x^2(x^4 + x^3 + 6x^2 + x + 1)}{(1-x)^3(x+1)^5} H_0(x) \right\} + \mathcal{O}(\varepsilon^0), \end{aligned} \quad (3.10)$$

where the harmonic polylogarithms, $H_{a_1, a_2, \dots, a_n}(x)$, are defined by

$$H_{a_1, a_2, \dots, a_n}(x) = \int_0^x dy f_{a_1}(y) H_{a_2, \dots, a_n}(y), \quad H_\emptyset = 1, \quad a_i \in \{0, 1, -1\}, \quad (3.11)$$

with

$$f_0(x) = \frac{1}{x}, \quad f_1(x) = \frac{1}{1-x}, \quad f_{-1}(x) = \frac{1}{1+x}, \quad (3.12)$$

and

$$\underbrace{H_{0, \dots, 0}}_{n \text{ times}}(x) = \frac{1}{n!} \ln^n(x). \quad (3.13)$$

ζ_n denotes the Riemann ζ -function

$$\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n}, \quad n \geq 2, \quad n \in \mathbb{N}.$$

We proceed similarly at higher orders, getting expansions of the MIs J_{22} and J_{23} up to $\mathcal{O}(\varepsilon^3)$. It should be noticed that solving a system of coupled differential equations is in general far from

trivial. When the solution is written in terms of multiple polylogarithms or related iterated integrals, as in the case of the two-loop massive form factor [77, 78] and of the planar three-loop massive form factor [25], it is possible to properly choose the set of MIs such that the system doesn't have any coupled equation, assuming for example the *canonical* form of [74, 75]. In this work we are interested in applying the methods developed in [76, 90, 91], to solve the systems of coupled differential equations algorithmically, as discussed in the next sections.

In order to solve the differential equations, boundary conditions have to be determined. As it was observed in [77, 78], the analytic structure of the master integrals strongly constrains the choice of the integration constants. In particular, boundary conditions of the master integrals of the non-singlet topologies are completely determined by requiring the regularity of the functions in $x = 1$, as will be discussed in the example above. However, we cannot use the same argument for some of the master integrals of the singlet topology, which are characterized by a branch cut at $x = 1$, as occurs, for example, in the case of the integral $K_{14} = K(1, 1, 1, 1, 0, 1)$ depicted in Figure 3. We fixed the boundary conditions of such integrals by matching the general solutions of



Figure 3: The master integral K_{14} . Massive (massless) propagators are represented by solid (dashed) lines. The presence of a massless cut determines the asymptotic behaviour $K_{14} \sim \log(1 - x)$.

the differential equations with the asymptotic expansions of the corresponding integrals around $x \rightarrow 1$. The latter were computed by means of the large-mass expansion $m^2 \gg q^2$ [92], or with a Mellin Barnes representation, as described e.g. in [93].

3.2 Calculation of the master integrals using difference equations

In the following we describe an alternative method for calculating the master integrals. The idea of the method is to write all integrals in terms of series expansions and then use the differential equations obeyed by the MIs to derive difference equations satisfied by the coefficients of these series. In the non-singlet case, this can be done using the fact that the MIs are regular around $x = 1$. In terms of the variable $y = 1 - x$, we can therefore write

$$J_i(y) = \sum_{n=0}^{\infty} \sum_{j=-2}^r \varepsilon^j C_{i,j}(n) y^n, \quad (3.14)$$

where we have included the expansion in ε up to $\mathcal{O}(\varepsilon^r)$. We can introduce Eq. (3.14) in the differential equations after rewriting them in terms of the variable y , leading to a system of coupled difference equations for the different $C_{i,j}$'s.

The method of solving differential equations by introducing series expansions and then finding the solutions to the resulting recursion relations is, of course well-known. However, to the best of our knowledge, this method has not been used before to calculate master integrals in perturbative quantum field theory. We propose this method here since it can be useful also for higher loop calculations, and we can take advantage of the powerful mathematical tools implemented in `Sigma` and `HarmonicSums` to solve the difference equations.

Whenever we are given subsystems of differential equations, it is usually more convenient to uncouple them and then insert Eq. (3.14) in the corresponding uncoupled equations, as supposed to inserting Eq. (3.14) first and then uncouple the resulting coupled difference equations. This is so because usually the later approach will lead to difference equations of higher order, although occasionally, this might not be the case and this later approach might turn out to be preferable.

In the singlet case, some of the master integrals will not be regular at $x = 1$, due to the presence of the logarithm $\ln(1 - x) = \ln(y)$ discussed in the previous section. We therefore include a formal expansion around powers of this logarithm⁵,

$$K_i(y) = \sum_{n=0}^{\infty} \sum_{k=0}^3 \sum_{j=-2}^r \varepsilon^j C_{i,j,k}(n) \ln^k(y) y^n. \quad (3.15)$$

The integrals belonging to the three multi-integral sectors in the singlet case have no logarithmic singularities at $y = 0$, and can therefore all be written as in Eq. (3.14). Only some of the integrals that are the sole representative of their sector in Table 2 need to be written according to Eq. (3.15). Since Eq. (3.14) is a particular case of Eq. (3.15)⁶, we will consider for illustration purposes one of the integrals that require an expansion of the form (3.15). The integral $K_4 = K(1, 1, 1, 1, 0, 0)$ is simple enough (it consists of a product of two one-loop integrals) to be calculated just through Feynman parameters, but precisely because of this simplicity, it allows us to describe the main features of the method without unnecessary complications or long formulae.

The differential equation associated to K_4 is given by

$$\frac{dK_4}{dy} + \left(\frac{2\varepsilon y}{(2-y)(1-y)} + \frac{2(1+2\varepsilon)}{y(2-y)} \right) K_4 = \frac{2-2\varepsilon}{y(2-y)} K_3. \quad (3.16)$$

As we discussed above, in order to solve the differential equations, we must first obtain all of the integrals associated to the subtopologies of the system under consideration. In the case of Eq. (3.16), this means obtaining K_3 . In terms of the variable y , it is given by

$$\begin{aligned} K_3(y) = & -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} [-3 - \ln(1-y) + 2\ln(y)] - 7 - 3\ln(1-y) - \frac{1}{2}\ln^2(1-y) \\ & + [6 + 2\ln(1-y)]\ln(y) - 2\ln^2(y) + \varepsilon \left[-15 + \frac{8}{3}\zeta_3 - 7\ln(1-y) \right. \\ & - \frac{3}{2}\ln^2(1-y) - \frac{1}{6}\ln^3(1-y) + [14 + 6\ln(1-y) + \ln^2(1-y)]\ln(y) \\ & \left. - [6 + 2\ln(1-y)]\ln^2(y) + \frac{4}{3}\ln^2(y) \right] + \mathcal{O}(\varepsilon^2), \end{aligned} \quad (3.17)$$

which, in expanded form, as in Eq. (3.15), can be written as

$$\begin{aligned} K_3(y) = & -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left(-3 + 2\ln(y) + \sum_{n=1}^{\infty} \frac{y^n}{n} \right) - 7 + 6\ln(y) - 2\ln^2(y) \\ & - \sum_{n=1}^{\infty} \frac{2}{n} y^n \ln(y) + \sum_{n=1}^{\infty} \left(\frac{1+3n}{n^2} - \frac{1}{n} S_1(n) \right) y^n \\ & + \varepsilon \left[\frac{4}{3}\ln^3(y) - 6\ln^2(y) + 14\ln(y) - 15 + \frac{8}{3}\zeta_3 \right] \end{aligned}$$

⁵In two particular cases (integrals K_8 and K_{13}), the sum in n actually starts from $n = -1$.

⁶That is, the case where $C_{i,j,k}(n) = 0$ for $k > 0$.

$$\begin{aligned}
& + \sum_{n=1}^{\infty} \frac{2}{n} y^n \ln^2(y) - 2 \sum_{n=1}^{\infty} \left(\frac{1+3n}{n^2} - \frac{1}{n} S_1(n) \right) y^n \ln(y) \\
& + \sum_{n=1}^{\infty} \left(\frac{1+3n+7n^2}{n^3} - \frac{1+3n}{n^2} S_1(n) + \frac{1}{2n} S_1^2(n) - \frac{1}{2n} S_2(n) \right) y^n \Big] \\
& + \mathcal{O}(\varepsilon^2), \tag{3.18}
\end{aligned}$$

where

$$S_k(n) = \sum_{i=1}^n \frac{1}{i^k}. \tag{3.19}$$

We proceed as follows: First, we clear the denominators in Eq. (3.16), which in this case means multiplying by $(2-y)(1-y)y$. After that, we can then insert the expanded version of K_3 given in Eq. (3.18) together with the generic expanded version of K_4 according to Eq. (3.15). The resulting equation will be satisfied if the coefficients in the expansion in y , for each power in $\ln(y)$, are equal on both sides of the equation. This leads to the following system of difference equations

$$0 = (2-n)C_{4,-2,3}(n-2) + (3n-1)C_{4,-2,3}(n-1) - 2(n+1)C_{4,-2,3}(n), \tag{3.20}$$

$$\begin{aligned}
0 &= (2-n)C_{4,-2,k}(n-2) + (3n-1)C_{4,-2,k}(n-1) - 2(n+1)C_{4,-2,k}(n) \\
&\quad - (k+1)C_{4,-2,k+1}(n-2) + 3(k+1)C_{4,-2,k+1}(n-1) - 2(k+1)C_{4,-2,k+1}(n) \\
&\text{for } k = 0, 1, 2, \tag{3.21}
\end{aligned}$$

$$\begin{aligned}
0 &= -2C_{4,j,3}(n-2) + 4C_{4,j,3}(n-1) - 4C_{4,j,3}(n) + (2-n)C_{4,j+1,3}(n-2) \\
&\quad + (3n-1)C_{4,j+1,3}(n-1) - 2(n+1)C_{4,j+1,3}(n) \\
&\text{for } j = -2, -1, 0, \tag{3.22}
\end{aligned}$$

$$\begin{aligned}
0 &= -2C_{4,j,k}(n-2) + 4C_{4,j,k}(n-1) - 4C_{4,j,k}(n) + (2-n)C_{4,j+1,k}(n-2) \\
&\quad + (3n-1)C_{4,j+1,k}(n-1) - 2(n+1)C_{4,j+1,k}(n) - (k+1)C_{4,j+1,k+1}(n-2) \\
&\quad + 3(k+1)C_{4,j+1,k+1}(n-1) - 2(k+1)C_{4,j+1,k+1}(n) \\
&\text{for } j = -2, k = 1, 2 \quad \text{and} \quad j = -1, k = 2, \tag{3.23}
\end{aligned}$$

$$\begin{aligned}
0 &= -2C_{4,j,k}(n-2) + 4C_{4,j,k}(n-1) - 4C_{4,j,k}(n) + (2-n)C_{4,j+1,k}(n-2) \\
&\quad + (3n-1)C_{4,j+1,k}(n-1) - 2(n+1)C_{4,j+1,k}(n) - (k+1)C_{4,j+1,k+1}(n-2) \\
&\quad + 3(k+1)C_{4,j+1,k+1}(n-1) - 2(k+1)C_{4,j+1,k+1}(n) + \frac{j-2k-7jk}{(n-1)n} \\
&\text{for } j = -2, k = 0; j = -1, k = 1 \quad \text{and} \quad j = 0, k = 2, \tag{3.24}
\end{aligned}$$

$$\begin{aligned}
0 &= -2C_{4,j,k}(n-2) + 4C_{4,j,k}(n-1) - 4C_{4,j,k}(n) + (2-n)C_{4,j+1,k}(n-2) \\
&\quad + (3n-1)C_{4,j+1,k}(n-1) - 2(n+1)C_{4,j+1,k}(n) - (k+1)C_{4,j+1,k+1}(n-2) \\
&\quad + 3(k+1)C_{4,j+1,k+1}(n-1) - 2(k+1)C_{4,j+1,k+1}(n) \\
&\quad + \frac{2(-1)^k(k+1)}{(n-1)n} S_1(n) - 2(-1)^k(k+1) \frac{3n^2-n-1}{(n-1)^2 n^2} \\
&\text{for } j = -1, k = 0 \quad \text{and} \quad j = 0, k = 1, \tag{3.25}
\end{aligned}$$

$$\begin{aligned}
0 &= -2C_{4,0,0}(n-2) + 4C_{4,0,0}(n-1) - 4C_{4,0,0}(n) + (2-n)C_{4,1,0}(n-2) \\
&\quad + (3n-1)C_{4,1,0}(n-1) - 2(n+1)C_{4,1,0}(n) - C_{4,1,1}(n-2) \\
&\quad + 3C_{4,1,1}(n-1) - 2C_{4,1,1}(n) + \frac{2(3n^2-n-1)}{(n-1)^2 n^2} S_1(n) - \frac{S_1^2(n)}{(n-1)n}
\end{aligned}$$

$$+\frac{S_2(n)}{(n-1)n} - \frac{2(6n^4 - 6n^3 + 1)}{(n-1)^3 n^3}. \quad (3.26)$$

This system is triangular. So, the coefficients $C_{4,j,k}(n)$ can be obtained successively by solving Eqs. (3.20) to (3.26) one after the other, inserting the results of the $C_{4,j,k}$'s obtained at each step in subsequent equations. We start with the coefficient for which the value of j is the lowest and the value of k is the largest (in this case, $C_{4,-2,3}(n)$), and proceed to obtain the coefficients for lower values of k , keeping j fixed, until all coefficients for $k = 3$ to $k = 0$ are determined. After that, we increase the value of j by one and repeat the procedure until all values of j are exhausted and all coefficients are determined. All of this can be done automatically using **Sigma** and **HarmonicSums**. The results will be given in terms of harmonic sums [94],

$$S_{b,\vec{a}}(n) = \sum_{k=1}^n \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \quad S_{\emptyset} = 1, \quad b, a_i \in \mathbb{Z} \setminus \{0\}, \quad (3.27)$$

and generalized harmonic sums [88, 95],

$$S_{b,\vec{a}}(\{c, \vec{d}\}, n) = \sum_{k=1}^n \frac{c^k}{k^b} S_{\vec{a}}(\{\vec{d}\}, k), \quad b, a_i \in \mathbb{N} \setminus \{0\}, \quad c, d_i \in \mathbb{Z} \setminus \{0\}, \quad S_{\emptyset} = 1. \quad (3.28)$$

Since all difference equations are of second order, we need at least two initial values in order to solve them. The first few expansion coefficients of $K_4(y)$ are given by

$$\begin{aligned} K_4(y) = & -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left(-2 + y + \frac{2}{3}y^2 + \frac{y^3}{2} + 2 \ln(y) \right) - 4 + 2y + \frac{5}{6}y^2 + \frac{y^3}{3} \\ & + \left(4 - 2y - \frac{4}{3}y^2 - y^3 \right) \ln(y) - 2 \ln^2(y) + \varepsilon \left[-8 + \frac{8}{3}\zeta_3 + 4y + \frac{5}{3}y^2 \right. \\ & + \frac{5}{6}y^3 + \left(-4 + 2y + \frac{4}{3}y^2 + y^3 \right) \ln^2(y) + \left(8 - 4y - \frac{5}{3}y^2 - \frac{2}{3}y^3 \right) \ln(y) \\ & \left. + \frac{4}{3} \ln^3(y) \right] + \mathcal{O}(\varepsilon^2, y^4). \end{aligned} \quad (3.29)$$

These were obtained as described in the previous section. In general, the solutions to the difference equations (3.20–3.26) for general values of n will be valid starting from a certain value $n = n_0$, and therefore the initial values used to solve the difference equations must also be taken starting from $n \geq n_0$. There are a few cases above where we can start from $n_0 = 0$, but in most cases we must take the initial values starting from $n_0 = 1$. Later, when we formally perform the sum (3.15), the expansion terms for $n < n_0$ will have to be added separately.

We obtain the following results for the expansion coefficients,

$$C_{4,-2,k}(n) = 0 \quad \text{for } k \geq 1, n \geq 0 \quad \text{and } k = 0, n \geq 1, \quad (3.30)$$

$$C_{4,-1,k}(n) = 0 \quad \text{for } k \geq 2, n \geq 0, \quad \text{and } k = 1, n \geq 1, \quad (3.31)$$

$$C_{4,-1,0}(n) = \frac{2}{n+1} \quad \text{for } n \geq 1, \quad (3.32)$$

$$C_{4,0,k}(n) = 0 \quad \text{for } k = 3, n \geq 0, \quad \text{and } k = 2, n \geq 1, \quad (3.33)$$

$$C_{4,0,1}(n) = -\frac{4}{n+1} \quad \text{for } n \geq 1, \quad (3.34)$$

$$C_{4,0,0}(n) = \frac{8}{n+1} + \frac{2(1-2n)}{n(n+1)} S_1(n) - \frac{2^{1-n}}{n(n+1)} S_1(\{2\}, n) \quad \text{for } n \geq 1, \quad (3.35)$$

$$C_{4,1,3}(n) = 0 \quad \text{for } n \geq 1, \quad (3.36)$$

$$C_{4,1,2}(n) = \frac{4}{n+1} \quad \text{for } n \geq 1, \quad (3.37)$$

$$C_{4,1,1}(n) = -2C_{4,0,0}(n) \quad \text{for } n \geq 1, \quad (3.38)$$

$$\begin{aligned} C_{4,1,0}(n) &= \frac{24}{n+1} + \frac{4n-3}{n(n+1)} S_1^2(n) - \frac{S_2(n)}{n(n+1)} - \frac{8}{n(n+1)2^n} S_1(\{2\}, n) \\ &\quad - \frac{4S_1(n)}{n(n+1)} \left[4n-2 + \frac{1}{2^n} S_1(\{2\}, n) \right] - \frac{6}{n(n+1)2^n} S_2(\{2\}, n) \\ &\quad - \frac{4(n-1)}{n(n+1)} S_{1,1}(\{\frac{1}{2}, 2\}, n) + \frac{10}{n(n+1)2^n} S_{1,1}(\{2, 1\}, n) \quad \text{for } n \geq 1, \end{aligned} \quad (3.39)$$

with the separate values,

$$\begin{aligned} C_{4,-2,0}(0) &= -1, & C_{4,-1,1}(0) &= 2, & C_{4,-1,0}(0) &= -2, & C_{4,0,2}(0) &= -2, \\ C_{4,0,1}(0) &= 4, & C_{4,0,0}(0) &= -4, & C_{4,1,3}(0) &= \frac{4}{3}, & C_{4,1,2}(0) &= -4, \\ C_{4,1,1}(0) &= 8, & C_{4,1,0}(0) &= \frac{8}{3}\zeta_3 - 8, \end{aligned} \quad (3.40)$$

We can insert the results from Eq. (3.30) to Eq. (3.40) into the expansion (3.15) and perform the sums using `Sigma`, `HarmonicSums`, `EvaluateMultiSums` and `SumProduction` [96–98]. We obtain

$$\begin{aligned} K_4(y) &= -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left[\frac{2}{y} H_1(y) + 2 \ln(y) - 4 \right] - 12 - \frac{2}{y} (y-2) H_{2,1}(y) + \frac{(y-3)}{y} H_1^2(y) \\ &\quad + \frac{8}{y} H_1(y) + \left(8 - \frac{4}{y} H_1(y) \right) \ln(y) - 2 \ln^2(y) + \varepsilon \left[\frac{y-2}{y} (-8H_{2,1}(y)) \right. \\ &\quad \left. + 4H_{1,2,1}(y) + 6H_{2,1,1}(y) - 4H_{2,2,1}(y) \right] - \frac{3y-7}{3y} H_1^3(y) + \frac{4}{y} (y-3) H_1^2(y) \\ &\quad + \frac{24}{y} H_1(y) + \frac{8}{3}\zeta_3 - 32 + \left(\frac{4}{y} (y-2) H_{2,1}(y) - \frac{2}{y} (y-3) H_1^2(y) - \frac{16}{y} H_1(y) \right. \\ &\quad \left. + 24 \right) \ln(y) + \left(\frac{4}{y} H_1(y) - 8 \right) \ln^2(y) + \frac{4}{3} \ln^3(y) \Big] + \mathcal{O}(\varepsilon^2). \end{aligned} \quad (3.41)$$

Notice the presence of the letters

$$\frac{1}{2-y}, \quad \frac{1}{1-y} \quad \text{and} \quad \frac{1}{y}. \quad (3.42)$$

After we go back to the original variable $x = 1 - y$, we obtain a representation in terms of the standard harmonic polylogarithms with the letters (3.12),

$$\begin{aligned} K_4(x) &= -\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} \left[\frac{2}{1-x} H_0(x) + 2H_1(x) + 4 \right] - \frac{2(1+x)}{1-x} H_{0,-1}(x) - \frac{2+x}{1-x} H_0^2(x) \\ &\quad + \frac{1}{1-x} (2(1+x)H_{-1}(x) - 4H_1(x) - 8)H_0(x) - 2H_1^2(x) - 8H_1(x) - 12 \\ &\quad + \frac{1+x}{1-x} \zeta_2 + \varepsilon \left[\left(\frac{1+x}{1-x} (2\zeta_2 - 4H_{0,-1}(x)) - 24 \right) H_1(x) - \frac{4}{3} H_1^3(x) \right. \\ &\quad \left. - \frac{1+x}{1-x} [H_{-1}(x) (-4H_{0,-1}(x) - 3H_0^2(x) - (4H_1(x) + 8) H_0(x) + 2\zeta_2)] \right] \end{aligned}$$

$$\begin{aligned}
& +2H_0(x) (H_{0,-1}(x) - \zeta_2) + 8H_{0,-1}(x) + 4H_{0,-1,-1}(x) + 2H_{0,0,-1}(x) \\
& +2H_0(x)H_{-1}^2(x) - 4\zeta_2] - \frac{4+3x}{3(1-x)}H_0^3(x) - \frac{2+x}{1-x}(2H_1(x) + 4)H_0^2(x) \\
& - \frac{2}{1-x}(2H_1^2(x) + 8H_1(x) + 12)H_0(x) - 8H_1^2(x) + \frac{2(7-x)}{3(1-x)}\zeta_3 - 32] \\
& +\mathcal{O}(\varepsilon^2).
\end{aligned} \tag{3.43}$$

4 Results

We calculated the heavy-quark form factors F_I , $I = V, A, S, P$, up to two loops and $\mathcal{O}(\varepsilon^2)$. Due to the length of the expressions we list here only the expansion corresponding to the low-energy ($0 < q^2 \ll m^2$), high-energy ($|q^2| \gg m^2$) and threshold ($|q^2| \approx 4m^2$) region up to $\mathcal{O}(\varepsilon)$. In appendices B, C, D, and E we present the complete analytic expressions up to $\mathcal{O}(\varepsilon)$. The full expressions up to $\mathcal{O}(\varepsilon^2)$ are provided as supplemental material together with this publication. We present renormalized results for all form factors but the singlet contributions to the axial-vector and pseudo-scalar currents $\hat{F}_{A,i}^{s,(2)}$ and $\hat{F}_P^{s,(2)}$ for which we present the bare results as discussed in Section 2.4. The expansions have been obtained with the help of the Mathematica packages `Sigma` and `HarmonicSums`.

For convenience, we collect here the notation used in the presentation of the results. We use the dimensionless variable x , Eq. (3.5). The kinematic regions of interest correspond to $x \rightarrow 1$ ($q^2 = 0$), $x \rightarrow \pm 0$ ($q^2 = \mp\infty$) and $x \rightarrow -1$ ($q^2 = 4m^2$).

Since the region $0 < q^2 < 4m^2$ corresponds to the upper half of the unit circle in the complex plane it is convenient to define the variable ϕ by

$$x = e^{i\phi} \tag{4.1}$$

and to expand around $\phi = 0$ instead.

In the threshold region we use the velocity of the heavy quarks as basic variable

$$\beta = \sqrt{1 - \frac{4m^2}{q^2}} \quad \leftrightarrow \quad x = \frac{\beta - 1}{\beta + 1} \tag{4.2}$$

and expand around $\beta = 0$. This avoids the appearance of square roots.

For the presentation of the complete analytic results in appendices B, C, D and E we introduce the following subsidiary variables

$$x_+ = \frac{1}{1+x}, \quad \eta = \frac{1}{(1-x)(1+x)}, \quad \xi = \frac{(1+x^2)}{(1-x)(1+x)}. \tag{4.3}$$

Furthermore, we use the abbreviations

$$\begin{aligned}
c_1 &= 12\zeta_2 \ln^2(2) + \ln^4(2) + 24\text{Li}_4\left(\frac{1}{2}\right) \\
c_2 &= 26\zeta_2^2 \ln(2) - 20\zeta_2 \ln^3(2) - \ln^5(2) + 120\text{Li}_5\left(\frac{1}{2}\right)
\end{aligned} \tag{4.4}$$

and

$$H_{a_1, \dots, a_n} \equiv H_{a_1, \dots, a_n}(x), \quad L \equiv H_0(x) = \ln(x), \quad \bar{H}_0(\phi) \equiv H_0(\phi) - \frac{i\pi}{2}, \tag{4.5}$$

with the harmonic polylogarithms $H_{a_1, \dots, a_n}(x)$ as defined in Eq. (3.11).

To validate our results we compare them to the existing literature. Up to $\mathcal{O}(\varepsilon^0)$ we agree with all available unrenormalized results for the various form factors. Note that in Refs. [20, 21, 23] a different normalization for the master integrals has been used, resulting in a difference proportional to $(\Gamma(1 + \varepsilon)/\exp(\gamma_E\varepsilon))^2$, where γ_E denotes the Euler-Mascheroni constant.

At $\mathcal{O}(\varepsilon)$ we can compare our results for the vector form factors with the results given in Ref. [24] and find a difference

$$- C_F C_A \left\{ \varepsilon \left[\frac{1037x^3}{(1+x)^6} \right] \right\}, \quad (4.6)$$

which has been reported in [26] already. In addition we compared our analytic results as well as the corresponding expansions with the results for the color-planar limit given in [26] and find agreement.

Comparing the renormalized results, we found that the wave function renormalization in Refs. [20, 21, 23] has been performed incorrectly, resulting in a difference proportional to ζ_2 at $\mathcal{O}(\varepsilon^0)$. For the vector form factor we agree with the renormalized results in the color-planar limit given in Ref. [26] and up to the term mentioned in Eq. (4.6) above with the results given in Ref. [24].

Although we do not present results for the renormalized singlet contributions we like to point out, that we cannot reproduce the result for the singlet axial-vector contribution presented in Ref. [22].

4.1 Low energy region $0 < q^2 \ll m^2$

The low energy limit of the space-like form factors is given by $x \rightarrow 1$. To facilitate the expansion of the HPLs in the region, we use the variable x as defined in Eq. (4.1) and expand around $\phi = 0$. In the following we present the series expansion of the one and two-loop form factors, denoted by \bar{F} , for all the currents up to 4th order in ϕ .

4.1.1 Vector form factor

For the vector form factors we find

$$\begin{aligned} \bar{F}_{V,1}^{(1)} &= C_F \left\{ \frac{1}{\varepsilon} \left[-\frac{2}{3}\phi^2 - \frac{2}{45}\phi^4 \right] + \left[-\frac{1}{2}\phi^2 - \frac{17}{120}\phi^4 \right] + \varepsilon \left[\phi^2 \left(-2 - \frac{\zeta_2}{3} \right) \right. \right. \\ &\quad \left. \left. + \phi^4 \left(-\frac{1}{4} - \frac{\zeta_2}{45} \right) \right] + \varepsilon^2 \left[\phi^2 \left(-4 - \frac{\zeta_2}{4} + \frac{2\zeta_3}{9} \right) + \phi^4 \left(-\frac{2}{3} - \frac{17\zeta_2}{240} + \frac{2\zeta_3}{135} \right) \right] \right\}. \quad (4.7) \\ \bar{F}_{V,1}^{(2)} &= C_F^2 \left\{ \frac{2}{9\varepsilon^2}\phi^4 + \frac{1}{3\varepsilon}\phi^4 + \left[\phi^2 \left(-\frac{47}{36} - \frac{175}{9}\zeta_2 + 48\ln(2)\zeta_2 - 12\zeta_3 \right) + \phi^4 \left(\frac{14473}{6480} \right. \right. \right. \\ &\quad \left. \left. - \frac{34243\zeta_2}{5400} + \frac{68}{5}\ln(2)\zeta_2 - \frac{17}{5}\zeta_3 \right) \right] + \varepsilon \left[\phi^2 \left(\frac{11713}{216} - 8c_1 - \frac{6763}{54}\zeta_2 + \frac{578}{3}\ln(2)\zeta_2 + \frac{504}{5}\zeta_2^2 \right. \right. \\ &\quad \left. \left. - \frac{1567}{18}\zeta_3 \right) + \phi^4 \left(\frac{3508637}{194400} - \frac{34c_1}{15} - \frac{758317\zeta_2}{20250} + \frac{10139}{180}\ln(2)\zeta_2 + \frac{714}{25}\zeta_2^2 - \frac{295457\zeta_3}{10800} \right) \right] \right\} \\ &\quad + C_F C_A \left\{ \frac{1}{\varepsilon^2} \left[\frac{11}{9}\phi^2 + \frac{11}{135}\phi^4 \right] + \frac{1}{\varepsilon} \left[\phi^2 \left(-\frac{94}{27} + \frac{4\zeta_2}{3} \right) + \phi^4 \left(-\frac{91}{405} + \frac{4\zeta_2}{45} \right) \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + \left[\phi^2 \left(-\frac{2579}{324} + \frac{155}{18} \zeta_2 - 24 \ln(2) \zeta_2 + \frac{26}{3} \zeta_3 \right) + \phi^4 \left(-\frac{36239}{19440} + \frac{7447 \zeta_2}{2160} - \frac{34}{5} \ln(2) \zeta_2 \right. \right. \\
& \left. \left. + \frac{169}{90} \zeta_3 \right) \right] + \varepsilon \left[\phi^2 \left(-\frac{134327}{1944} + 4c_1 + \frac{1297}{27} \zeta_2 - \frac{289}{3} \ln(2) \zeta_2 - \frac{608}{15} \zeta_2^2 + \frac{1487}{36} \zeta_3 \right) \right. \\
& \left. + \phi^4 \left(-\frac{278341}{23328} + \frac{17c_1}{15} + \frac{110029 \zeta_2}{6480} - \frac{10139}{360} \ln(2) \zeta_2 - \frac{613}{45} \zeta_2^2 + \frac{12041}{864} \zeta_3 \right) \right] \Big\} \\
& + C_{FnI} T_F \left\{ \frac{1}{\varepsilon^2} \left[-\frac{4}{9} \phi^2 - \frac{4}{135} \phi^4 \right] + \frac{1}{\varepsilon} \left[\frac{20}{27} \phi^2 + \frac{4}{81} \phi^4 \right] + \left[\phi^2 \left(\frac{283}{81} + \frac{16 \zeta_2}{9} \right) \right. \right. \\
& \left. \left. + \phi^4 \left(\frac{3139}{4860} + \frac{16 \zeta_2}{135} \right) \right] + \varepsilon \left[\phi^2 \left(\frac{8827}{486} + \frac{181 \zeta_2}{27} + \frac{32 \zeta_3}{9} \right) + \phi^4 \left(\frac{95527}{29160} + \frac{1777 \zeta_2}{1620} + \frac{32 \zeta_3}{135} \right) \right] \right\} \\
& + C_{FT} T_F \left\{ \left[\phi^2 \left(-\frac{1099}{81} + 9 \zeta_2 \right) + \phi^4 \left(-\frac{21019}{4860} + \frac{53 \zeta_2}{20} \right) \right] + \varepsilon \left[\phi^2 \left(-\frac{635}{18} + \frac{937}{27} \zeta_2 \right. \right. \right. \\
& \left. \left. - \frac{154}{3} \ln(2) \zeta_2 + \frac{1601}{54} \zeta_3 \right) + \phi^4 \left(-\frac{8293}{648} + \frac{9821}{810} \zeta_2 - \frac{283}{18} \ln(2) \zeta_2 + \frac{29651 \zeta_3}{3240} \right) \right] \Big\} \quad (4.8)
\end{aligned}$$

$$\begin{aligned}
\bar{F}_{V,2}^{(1)} & = C_F \left\{ \left[2 + \frac{1}{3} \phi^2 + \frac{7}{180} \phi^4 \right] + \varepsilon \left[8 + \frac{5}{3} \phi^2 + \frac{41}{180} \phi^4 \right] + \varepsilon^2 \left[16 + \zeta_2 + \phi^2 \left(4 + \frac{\zeta_2}{6} \right) \right. \right. \\
& \left. \left. + \phi^4 \left(\frac{19}{30} + \frac{7 \zeta_2}{360} \right) \right] \right\}. \quad (4.9)
\end{aligned}$$

$$\begin{aligned}
\bar{F}_{V,2}^{(2)} & = C_F^2 \left\{ \frac{1}{\varepsilon} \left[-\frac{4}{3} \phi^2 - \frac{14}{45} \phi^4 \right] + \left[-31 + 40 \zeta_2 - 48 \ln(2) \zeta_2 + 12 \zeta_3 + \phi^2 \left(-\frac{77}{5} + \frac{122}{5} \zeta_2 \right. \right. \right. \\
& \left. \left. - \frac{184}{5} \ln(2) \zeta_2 + \frac{46}{5} \zeta_3 \right) + \phi^4 \left(-\frac{4931}{1260} + \frac{2963}{350} \zeta_2 - \frac{1478}{105} \ln(2) \zeta_2 + \frac{739}{210} \zeta_3 \right) \right] + \varepsilon \left[-\frac{1243}{6} \right. \\
& \left. + 8c_1 + \frac{944}{3} \zeta_2 - 384 \ln(2) \zeta_2 - \frac{504}{5} \zeta_2^2 + 176 \zeta_3 + \phi^2 \left(-\frac{19666}{225} + \frac{92c_1}{15} + \frac{3704}{25} \zeta_2 \right. \right. \\
& \left. \left. - \frac{14164}{75} \ln(2) \zeta_2 - \frac{1932}{25} \zeta_2^2 + \frac{7201}{75} \zeta_3 \right) + \phi^4 \left(-\frac{9903863}{396900} + \frac{739c_1}{315} + \frac{10057561 \zeta_2}{220500} \right. \right. \\
& \left. \left. - \frac{125887 \ln(2) \zeta_2}{2205} - \frac{739}{25} \zeta_2^2 + \frac{1376111 \zeta_3}{44100} \right) \right] \Big\} + C_F C_A \left\{ \left[\frac{317}{9} - 12 \zeta_2 + 24 \ln(2) \zeta_2 - 6 \zeta_3 \right. \right. \\
& \left. \left. + \phi^2 \left(\frac{1699}{270} - \frac{137}{15} \zeta_2 + \frac{92}{5} \ln(2) \zeta_2 - \frac{23}{5} \zeta_3 \right) + \phi^4 \left(\frac{11927}{22680} - \frac{21269 \zeta_2}{6300} + \frac{739}{105} \ln(2) \zeta_2 \right. \right. \right. \\
& \left. \left. - \frac{739}{420} \zeta_3 \right) \right] + \varepsilon \left[\frac{12881}{54} - 4c_1 - \frac{313}{3} \zeta_2 + 192 \ln(2) \zeta_2 + \frac{252}{5} \zeta_2^2 - 72 \zeta_3 \right. \\
& \left. + \phi^2 \left(\frac{485453}{8100} - \frac{46c_1}{15} - \frac{8983}{150} \zeta_2 + \frac{7082}{75} \ln(2) \zeta_2 + \frac{966}{25} \zeta_2^2 - \frac{6281}{150} \zeta_3 \right) \right. \\
& \left. \left. + \phi^4 \left(\frac{50620531}{4762800} - \frac{739c_1}{630} - \frac{4335431 \zeta_2}{220500} + \frac{125887 \ln(2) \zeta_2}{4410} + \frac{739}{50} \zeta_2^2 - \frac{1224967 \zeta_3}{88200} \right) \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + C_F n_l T_F \left\{ \left[-\frac{100}{9} - \frac{62}{27}\phi^2 - \frac{253}{810}\phi^4 \right] + \varepsilon \left[-\frac{1922}{27} - 12\zeta_2 \right. \right. \\
& \left. \left. + \phi^2 \left(-\frac{1405}{81} - 2\zeta_2 \right) + \phi^4 \left(-\frac{13147}{4860} - \frac{7\zeta_2}{30} \right) \right] \right\} + C_F T_F \left\{ \left[\frac{476}{9} - 32\zeta_2 + \phi^2 \left(\frac{622}{27} - 14\zeta_2 \right) \right. \right. \\
& \left. \left. + \phi^4 \left(\frac{4841}{810} - \frac{109\zeta_2}{30} \right) \right] + \varepsilon \left[\frac{2254}{27} - \frac{308}{3}\zeta_2 + 192 \ln(2)\zeta_2 - 112\zeta_3 + \phi^2 \left(\frac{4247}{81} - \frac{490}{9}\zeta_2 \right. \right. \right. \\
& \left. \left. \left. + 84 \ln(2)\zeta_2 - 49\zeta_3 \right) + \phi^4 \left(\frac{16753}{972} - \frac{2203}{135}\zeta_2 + \frac{109}{5} \ln(2)\zeta_2 - \frac{763}{60}\zeta_3 \right) \right] \right\}. \tag{4.10}
\end{aligned}$$

Note, that $\bar{F}_{V,2}$ is UV and IR finite in this limit and the leading term agrees with the computation of the anomalous magnetic moment in [99].

4.1.2 Axial-vector form factor

For the axial-vector form factor we present the renormalized results for the non-singlet contributions and the unrenormalized one for the singlet parts.

$$\begin{aligned}
\bar{F}_{A,1}^{(1),\text{ns}} = C_F \left\{ \frac{1}{\varepsilon} \left[-\frac{2}{3}\phi^2 - \frac{2}{45}\phi^4 \right] + \left[-2 - \frac{5}{6}\phi^2 - \frac{13}{72}\phi^4 \right] + \varepsilon \left[\phi^2 \left(-\frac{7}{3} - \frac{\zeta_2}{3} \right) + \phi^4 \left(-\frac{29}{90} \right. \right. \right. \\
\left. \left. \left. - \frac{\zeta_2}{45} \right) \right] + \varepsilon^2 \left[-\zeta_2 + \phi^2 \left(-4 - \frac{5\zeta_2}{12} + \frac{2\zeta_3}{9} \right) + \phi^4 \left(-\frac{7}{10} - \frac{13\zeta_2}{144} + \frac{2\zeta_3}{135} \right) \right] \right\}. \tag{4.11}
\end{aligned}$$

$$\begin{aligned}
\bar{F}_{A,1}^{(2),\text{ns}} = C_F^2 \left\{ \frac{1}{\varepsilon^2} \left[\frac{2}{9}\phi^4 \right] + \frac{1}{\varepsilon} \left[\frac{4}{3}\phi^2 + \frac{29}{45}\phi^4 \right] + \left[-\frac{29}{3} + 32\zeta_2 - 32 \ln(2)\zeta_2 + 8\zeta_3 + \phi^2 \left(-\frac{1121}{180} \right. \right. \right. \\
\left. \left. \left. - \frac{217}{45}\zeta_2 + \frac{128}{5} \ln(2)\zeta_2 - \frac{32}{5}\zeta_3 \right) + \phi^4 \left(\frac{95341}{45360} - \frac{5423\zeta_2}{5400} + \frac{376}{105} \ln(2)\zeta_2 - \frac{94}{105}\zeta_3 \right) \right] \right\} \\
+ C_F C_A \left\{ \frac{1}{\varepsilon^2} \left[\frac{11}{9}\phi^2 + \frac{11}{135}\phi^4 \right] + \frac{1}{\varepsilon} \left[\phi^2 \left(-\frac{94}{27} \right. \right. \right. \\
\left. \left. \left. + \frac{4\zeta_2}{3} \right) + \phi^4 \left(-\frac{91}{405} + \frac{4\zeta_2}{45} \right) \right] + \left[-\frac{143}{9} - 8\zeta_2 + 16 \ln(2)\zeta_2 - 4\zeta_3 + \phi^2 \left(-\frac{19813}{1620} \right. \right. \right. \\
\left. \left. \left. + \frac{317}{90}\zeta_2 - \frac{64}{5} \ln(2)\zeta_2 + \frac{88}{15}\zeta_3 \right) + \phi^4 \left(-\frac{413831}{136080} + \frac{14479\zeta_2}{10800} - \frac{188}{105} \ln(2)\zeta_2 + \frac{197}{315}\zeta_3 \right) \right] \right\} \\
+ \varepsilon \left[-\frac{887}{54} - \frac{8c_1}{3} - \frac{59}{3}\zeta_2 + \frac{80}{3} \ln(2)\zeta_2 + \frac{168}{5}\zeta_2^2 - \frac{68}{3}\zeta_3 + \phi^2 \left(-\frac{3362201}{48600} + \frac{32c_1}{15} \right. \right. \\
\left. \left. + \frac{1753}{150}\zeta_2 - \frac{3479}{75} \ln(2)\zeta_2 - \frac{1276}{75}\zeta_2^2 + \frac{16777}{900}\zeta_3 \right) + \phi^4 \left(-\frac{281936227}{28576800} + \frac{94c_1}{315} + \frac{583651\zeta_2}{378000} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& - \frac{721523 \ln(2)\zeta_2}{88200} - \frac{698}{225}\zeta_2^2 + \frac{5002453\zeta_3}{1058400} \Big) \Big] \Big\} + C_F n_l T_F \left\{ \frac{1}{\varepsilon^2} \left[-\frac{4}{9}\phi^2 - \frac{4}{135}\phi^4 \right] \right. \\
& + \frac{1}{\varepsilon} \left[\frac{20}{27}\phi^2 + \frac{4}{81}\phi^4 \right] + \left[\frac{28}{9} + \phi^2 \left(\frac{361}{81} + \frac{16\zeta_2}{9} \right) + \phi^4 \left(\frac{3901}{4860} + \frac{16\zeta_2}{135} \right) \right] \\
& + \varepsilon \left[-\frac{34}{27} + 12\zeta_2 + \phi^2 \left(\frac{9229}{486} + \frac{235\zeta_2}{27} + \frac{32\zeta_3}{9} \right) + \phi^4 \left(\frac{105253}{29160} + \frac{431\zeta_2}{324} + \frac{32\zeta_3}{135} \right) \right] \Big\} \\
& + C_F T_F \left\{ \left[\frac{460}{9} - 32\zeta_2 + \phi^2 \left(\frac{491}{81} - 3\zeta_2 \right) + \phi^4 \left(\frac{1343}{4860} - \frac{3\zeta_2}{20} \right) \right] + \varepsilon \left[\frac{3998}{27} - \frac{412}{3}\zeta_2 \right. \right. \\
& + 192 \ln(2)\zeta_2 - 112\zeta_3 + \phi^2 \left(\frac{4931}{162} - \frac{653}{27}\zeta_2 + \frac{62}{3} \ln(2)\zeta_2 - \frac{667}{54}\zeta_3 \right) \\
& \left. \left. + \phi^4 \left(\frac{10963}{1944} - \frac{1436}{405}\zeta_2 + \frac{97}{90} \ln(2)\zeta_2 - \frac{2101\zeta_3}{3240} \right) \right] \right\}. \tag{4.12}
\end{aligned}$$

$$\begin{aligned}
\bar{F}_{A,1}^{(2),s} &= C_F T_F \left\{ -\frac{6}{\varepsilon} + \left[-\frac{37}{3} + \frac{64}{3}\zeta_2 + \phi^2 \left(-\frac{85}{18} + \frac{136\zeta_2}{45} \right) + \phi^4 \left(-\frac{341}{540} + \frac{2554\zeta_2}{4725} \right) \right] \right. \\
& + \varepsilon \left[-\frac{817}{18} + \frac{266}{9}\zeta_2 - 128 \ln(2)\zeta_2 + \frac{224}{3}\zeta_3 + \phi^2 \left(-\frac{3659}{540} + \frac{4876}{675}\zeta_2 - \frac{272}{15} \ln(2)\zeta_2 \right. \right. \\
& \left. \left. + \frac{476}{45}\zeta_3 \right) - \frac{1}{2} i \phi^3 \zeta_2 + \phi^4 \left(-\frac{14281}{16200} + \frac{985837\zeta_2}{1984500} - \frac{5108 \ln(2)\zeta_2}{1575} + \frac{1277}{675}\zeta_3 \right) \right] \Big\}. \tag{4.13}
\end{aligned}$$

$$\begin{aligned}
\bar{F}_{A,2}^{(1),ns} &= C_F \left\{ \left[\frac{14}{3} + \frac{11}{15}\phi^2 + \frac{103}{1260}\phi^4 \right] + \varepsilon \left[8 + \frac{31}{15}\phi^2 + \frac{389}{1260}\phi^4 \right] + \varepsilon^2 \left[16 + \frac{7}{3}\zeta_2 \right. \right. \\
& \left. \left. + \phi^2 \left(4 + \frac{11\zeta_2}{30} \right) + \phi^4 \left(\frac{47}{70} + \frac{103\zeta_2}{2520} \right) \right] \right\}. \tag{4.14}
\end{aligned}$$

$$\begin{aligned}
\bar{F}_{A,2}^{(2),ns} &= C_F^2 \left\{ \frac{1}{\varepsilon} \left[-\frac{28}{9}\phi^2 - \frac{94}{135}\phi^4 \right] + \left[-\frac{23}{5} + \frac{176}{5}\zeta_2 - \frac{176}{5} \ln(2)\zeta_2 + \frac{44}{5}\zeta_3 + \phi^2 \left(-\frac{11111}{945} \right. \right. \right. \\
& \left. \left. + \frac{592}{45}\zeta_2 - \frac{88}{7} \ln(2)\zeta_2 + \frac{22}{7}\zeta_3 \right) + \phi^4 \left(-\frac{251113}{56700} + \frac{208091\zeta_2}{66150} - \frac{298}{105} \ln(2)\zeta_2 + \frac{149}{210}\zeta_3 \right) \right] \\
& + \varepsilon \left[\frac{15527}{450} + \frac{88c_1}{15} + \frac{28688}{225}\zeta_2 - \frac{6512}{25} \ln(2)\zeta_2 - \frac{1848}{25}\zeta_2^2 + \frac{3388}{25}\zeta_3 + \phi^2 \left(\frac{376}{11025} + \frac{44c_1}{21} \right. \right. \\
& \left. \left. + \frac{3776}{135}\zeta_2 - \frac{235856 \ln(2)\zeta_2}{3675} - \frac{132}{5}\zeta_2^2 + \frac{466972\zeta_3}{11025} \right) + \phi^4 \left(-\frac{47414267}{17860500} + \frac{149c_1}{315} \right. \right. \\
& \left. \left. + \frac{34009847\zeta_2}{6945750} - \frac{25891 \ln(2)\zeta_2}{2205} - \frac{149}{25}\zeta_2^2 + \frac{1220729\zeta_3}{132300} \right) \right] \Big\} + C_F C_A \left\{ \left[\frac{7663}{135} - \frac{752}{45}\zeta_2 \right. \right. \\
& + \frac{88}{5} \ln(2)\zeta_2 - \frac{22}{5}\zeta_3 + \phi^2 \left(\frac{5039}{378} - \frac{422}{75}\zeta_2 + \frac{44}{7} \ln(2)\zeta_2 - \frac{11}{7}\zeta_3 \right) + \phi^4 \left(\frac{27793}{12600} - \frac{56827\zeta_2}{44100} \right. \\
& \left. \left. + \frac{149}{105} \ln(2)\zeta_2 - \frac{149}{420}\zeta_3 \right) \right] + \varepsilon \left[\frac{871991}{4050} - \frac{44c_1}{15} - \frac{19517}{675}\zeta_2 + \frac{3256}{25} \ln(2)\zeta_2 + \frac{924}{25}\zeta_2^2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{14846}{225}\zeta_3 + \phi^2 \left(\frac{20526143}{396900} - \frac{22c_1}{21} - \frac{142673\zeta_2}{15750} + \frac{117928 \ln(2)\zeta_2}{3675} + \frac{66}{5}\zeta_2^2 - \frac{70838\zeta_3}{3675} \right) \\
& + \phi^4 \left(\frac{548179231}{71442000} - \frac{149c_1}{630} - \frac{5017571\zeta_2}{3087000} + \frac{25891 \ln(2)\zeta_2}{4410} + \frac{149}{50}\zeta_2^2 - \frac{118921\zeta_3}{29400} \right) \Bigg\} \\
& + C_F n_l T_F \left\{ \left[-\frac{412}{27} - \frac{466}{135}\phi^2 - \frac{2761}{5670}\phi^4 \right] + \varepsilon \left[-\frac{5630}{81} - 28\zeta_2 + \phi^2 \left(-\frac{7427}{405} - \frac{22\zeta_2}{5} \right) \right. \right. \\
& \left. \left. + \phi^4 \left(-\frac{104863}{34020} - \frac{103\zeta_2}{210} \right) \right] \right\} + C_F T_F \left\{ \left[-\frac{412}{27} + \frac{32}{3}\zeta_2 + \phi^2 \left(-\frac{14}{27} + \frac{2\zeta_2}{5} \right) + \phi^4 \left(\frac{4601}{5670} \right. \right. \right. \\
& \left. \left. - \frac{103\zeta_2}{210} \right) \right] + \varepsilon \left[-\frac{9230}{81} + \frac{724}{9}\zeta_2 - 64 \ln(2)\zeta_2 + \frac{112}{3}\zeta_3 + \phi^2 \left(-\frac{10703}{405} + \frac{146}{9}\zeta_2 \right. \right. \\
& \left. \left. - \frac{12}{5} \ln(2)\zeta_2 + \frac{7}{5}\zeta_3 \right) + \phi^4 \left(-\frac{119839}{34020} + \frac{1238}{945}\zeta_2 + \frac{103}{35} \ln(2)\zeta_2 - \frac{103}{60}\zeta_3 \right) \right] \Bigg\}. \quad (4.15)
\end{aligned}$$

$$\begin{aligned}
\bar{F}_{A,2}^{(2),s} &= C_F T_F \left\{ \left[\frac{24}{\phi^2} + \frac{20}{3} + \frac{16}{15}\zeta_2 + 4i\phi\zeta_2 + \phi^2 \left(-\frac{8}{15} - \frac{184}{315}\zeta_2 + \frac{4\bar{H}_0(\phi)}{3} \right) + \frac{13}{15}i\phi^3\zeta_2 \right. \right. \\
& \left. \left. + \phi^4 \left(-\frac{461}{6300} - \frac{1174\zeta_2}{4725} + \frac{13\bar{H}_0(\phi)}{45} \right) \right] + \varepsilon \left[\frac{1}{\phi^2} \left(124 - 128\zeta_2 \right) + \frac{3394}{45} - \frac{11264}{225}\zeta_2 \right. \right. \\
& \left. \left. - \frac{32}{5} \ln(2)\zeta_2 + \frac{56}{15}\zeta_3 + i\phi \left(-10\zeta_2 + 8 \ln(2)\zeta_2 - 8\zeta_2\bar{H}_0(\phi) \right) + \phi^2 \left(\frac{3173}{135} - \frac{221449\zeta_2}{33075} \right) \right. \right. \\
& \left. \left. + \frac{368}{105} \ln(2)\zeta_2 - \frac{92}{45}\zeta_3 - 4\bar{H}_0(\phi) - \frac{4}{3}\bar{H}_0^2(\phi) \right) + i\phi^3 \left(-\frac{9}{5}\zeta_2 + \frac{26}{15} \ln(2)\zeta_2 - \frac{26}{15}\zeta_2\bar{H}_0(\phi) \right) \right. \\
& \left. \left. + \phi^4 \left(\frac{5566123}{1701000} - \frac{1063927\zeta_2}{1984500} + \frac{2348 \ln(2)\zeta_2}{1575} - \frac{587}{675}\zeta_3 - \frac{31\bar{H}_0(\phi)}{45} - \frac{13}{45}\bar{H}_0^2(\phi) \right) \right] \Bigg\}. \quad (4.16)
\end{aligned}$$

4.1.3 Scalar form factor

For the scalar form factor in the low energy limit we obtain

$$\begin{aligned}
\bar{F}_S^{(1)} &= C_F \left\{ \frac{1}{\varepsilon} \left[-\frac{2}{3}\phi^2 - \frac{2}{45}\phi^4 \right] + \left[-6 - \frac{1}{3}\phi^2 - \frac{31}{180}\phi^4 \right] + \varepsilon \left[-8 + \phi^2 \left(-\frac{7}{3} - \frac{\zeta_2}{3} \right) \right. \right. \\
& \left. \left. + \phi^4 \left(-\frac{49}{180} - \frac{\zeta_2}{45} \right) \right] + \varepsilon^2 \left[-16 - 3\zeta_2 + \phi^2 \left(-4 - \frac{\zeta_2}{6} + \frac{2\zeta_3}{9} \right) \right. \right. \\
& \left. \left. + \phi^4 \left(-\frac{7}{10} - \frac{31\zeta_2}{360} + \frac{2\zeta_3}{135} \right) \right] \right\}. \quad (4.17)
\end{aligned}$$

$$\begin{aligned}
\bar{F}_S^{(2)} &= C_F^2 \left\{ \frac{2}{9\varepsilon^2}\phi^4 + \frac{1}{\varepsilon} \left[4\phi^2 + \frac{22}{45}\phi^4 \right] + \left[33 + \phi^2 \left(\frac{62}{9} - \frac{71}{9}\zeta_2 + 28 \ln(2)\zeta_2 - 7\zeta_3 \right) \right. \right. \\
& \left. \left. + \phi^4 \left(\frac{5743}{1620} - \frac{7001\zeta_2}{5400} + \frac{11}{3} \ln(2)\zeta_2 - \frac{11}{12}\zeta_3 \right) \right] + \varepsilon \left[\frac{135}{2} - 120\zeta_2 + 192 \ln(2)\zeta_2 - 48\zeta_3 \right. \right. \\
& \left. \left. + \phi^2 \left(\frac{1297}{27} - \frac{14c_1}{3} - \frac{2723}{54}\zeta_2 + 98 \ln(2)\zeta_2 + \frac{294}{5}\zeta_2^2 - \frac{725}{18}\zeta_3 \right) + \phi^4 \left(\frac{193531}{24300} - \frac{11c_1}{18} \right) \right. \right. \\
& \left. \left. + \varepsilon^2 \left[-16 - 3\zeta_2 + \phi^2 \left(-4 - \frac{\zeta_2}{6} + \frac{2\zeta_3}{9} \right) + \phi^4 \left(-\frac{7}{10} - \frac{31\zeta_2}{360} + \frac{2\zeta_3}{135} \right) \right] \right\}.
\end{aligned}$$

$$\begin{aligned}
& - \frac{141179\zeta_2}{20250} + \frac{5971}{300} \ln(2)\zeta_2 + \frac{77}{10}\zeta_2^2 - \frac{29381\zeta_3}{3600} \Big] \Big\} + C_F C_A \left\{ \frac{1}{\varepsilon^2} \left[\frac{11}{9}\phi^2 + \frac{11}{135}\phi^4 \right] \right. \\
& + \frac{1}{\varepsilon} \left[\phi^2 \left(-\frac{94}{27} + \frac{4\zeta_2}{3} \right) + \phi^4 \left(-\frac{91}{405} + \frac{4\zeta_2}{45} \right) \right] + \left[-\frac{185}{3} + \phi^2 \left(-\frac{650}{81} + \frac{47}{18}\zeta_2 \right. \right. \\
& - 14 \ln(2)\zeta_2 + \frac{37}{6}\zeta_3 \Big) + \phi^4 \left(-\frac{2389}{972} + \frac{11897\zeta_2}{10800} - \frac{11}{6} \ln(2)\zeta_2 + \frac{229}{360}\zeta_3 \right) \Big] + \varepsilon \left[-\frac{1463}{6} \right. \\
& + 21\zeta_2 - 96 \ln(2)\zeta_2 + 24\zeta_3 + \phi^2 \left(-\frac{32507}{486} + \frac{7c_1}{3} + \frac{1873}{108}\zeta_2 - 49 \ln(2)\zeta_2 - \frac{293}{15}\zeta_2^2 + \frac{629}{36}\zeta_3 \right) \\
& \left. + \phi^4 \left(-\frac{2663879}{291600} + \frac{11c_1}{36} + \frac{212591\zeta_2}{81000} - \frac{5971}{600} \ln(2)\zeta_2 - \frac{2873}{900}\zeta_2^2 + \frac{101327\zeta_3}{21600} \right) \right] \Big\} \\
& + C_F n_l T_F \left\{ \frac{1}{\varepsilon^2} \left[-\frac{4}{9}\phi^2 - \frac{4}{135}\phi^4 \right] + \frac{1}{\varepsilon} \left[\frac{20}{27}\phi^2 + \frac{4}{81}\phi^4 \right] + \left[\frac{52}{3} + \phi^2 \left(\frac{316}{81} + \frac{16\zeta_2}{9} \right) \right. \right. \\
& \left. + \phi^4 \left(\frac{883}{1215} + \frac{16\zeta_2}{135} \right) \right] + \varepsilon \left[\frac{206}{3} + 36\zeta_2 + \phi^2 \left(\frac{4808}{243} + \frac{154\zeta_2}{27} + \frac{32\zeta_3}{9} \right) + \phi^4 \left(\frac{2560}{729} + \frac{1037\zeta_2}{810} \right. \right. \\
& \left. \left. + \frac{32\zeta_3}{135} \right) \right] \Big\} + C_F T_F \left\{ \left[\frac{52}{3} + 8i\phi\zeta_2 + \phi^2 \left(-\frac{1417}{81} + \frac{91}{9}\zeta_2 + \frac{8}{3}\bar{H}_0(\phi) \right) + \frac{17}{15}i\phi^3\zeta_2 \right. \right. \\
& \left. + \phi^4 \left(-\frac{25076}{6075} + \frac{6071\zeta_2}{2700} + \frac{17}{45}\bar{H}_0(\phi) \right) \right] + \varepsilon \left[\frac{350}{3} - 60\zeta_2 + i\phi \left(16 \ln(2)\zeta_2 - 16\bar{H}_0(\phi)\zeta_2 \right) \right. \\
& \left. + \phi^2 \left(-\frac{787}{162} + 23\zeta_2 - 58 \ln(2)\zeta_2 + \frac{1811}{54}\zeta_3 + \frac{8}{3}\bar{H}_0(\phi) - \frac{8}{3}\bar{H}_0(\phi)^2 \right) + i\phi^3 \left(\frac{17}{15}\zeta_2 \right. \right. \\
& \left. + \frac{34}{15} \ln(2)\zeta_2 - \frac{34}{15}\zeta_2\bar{H}_0(\phi) \right) + \phi^4 \left(-\frac{241487}{121500} + \frac{35681\zeta_2}{6750} - \frac{1997}{150} \ln(2)\zeta_2 + \frac{125491\zeta_3}{16200} \right. \\
& \left. \left. + \frac{6}{5}\bar{H}_0(\phi) - \frac{17}{45}\bar{H}_0(\phi)^2 \right) \right] \Big\}. \tag{4.18}
\end{aligned}$$

4.1.4 Pseudo-scalar form factor

The non-singlet part of the pseudo-scalar form factor can be obtained by using Eq. (2.13)

$$\begin{aligned}
\bar{F}_P^{(1),\text{ns}} &= \bar{F}_{A,1}^{(1)} + \left(\frac{\phi^2}{4} - \frac{\phi^4}{48} \right) \bar{F}_{A,2}^{(1)} + \mathcal{O}(\phi^6), \\
\bar{F}_P^{(2),\text{ns}} &= \bar{F}_{A,1}^{(2)} + \left(\frac{\phi^2}{4} - \frac{\phi^4}{48} \right) \bar{F}_{A,2}^{(2)} + \mathcal{O}(\phi^6). \tag{4.19}
\end{aligned}$$

The unrenormalized singlet contribution to the pseudo-scalar form factor reads

$$\begin{aligned}
\bar{F}_P^{(2),s} &= C_F T_F \left\{ \left[\frac{8}{3} + \frac{64}{3}\zeta_2 + 12i\phi\zeta_2 + \phi^2 \left(-\frac{68}{9} + 4\bar{H}_0(\phi) + \frac{148\zeta_2}{45} \right) + 2i\phi^3\zeta_2 + \phi^4 \left(-\frac{689}{540} \right. \right. \right. \\
& \left. + \frac{1759\zeta_2}{4725} + \frac{2}{3}\bar{H}_0(\phi) \right) \Big] + \varepsilon \left[\frac{88}{9} - \frac{352}{9}\zeta_2 - 128 \ln(2)\zeta_2 + \frac{224}{3}\zeta_3 + i\phi \left(-36\zeta_2 + 24 \ln(2)\zeta_2 \right. \right. \\
& \left. \left. - 24\zeta_2\bar{H}_0(\phi) \right) + \phi^2 \left(\frac{5227}{135} - \frac{3182}{675}\zeta_2 - \frac{296}{15} \ln(2)\zeta_2 + \frac{518}{45}\zeta_3 - 12\bar{H}_0(\phi) - 4\bar{H}_0(\phi)^2 \right) \right] \Big\}
\end{aligned}$$

$$\begin{aligned}
& + i\phi^3 \left(-7\zeta_2 + 4\ln(2)\zeta_2 - 4\zeta_2\bar{H}_0(\phi) \right) + \phi^4 \left(\frac{101609}{16200} - \frac{255167\zeta_2}{496125} - \frac{3518\ln(2)\zeta_2}{1575} \right. \\
& \left. + \frac{1759\zeta_3}{1350} - 2\bar{H}_0(\phi) - \frac{2}{3}\bar{H}_0(\phi)^2 \right) \Bigg\}. \tag{4.20}
\end{aligned}$$

4.2 High energy region $|q^2| \gg m^2$

We now present the expansion of all the form factors in the asymptotic limit i.e. for $x \rightarrow 0^+$ up to $\mathcal{O}(x^2)$. The expanded form factors are denoted by \mathcal{F}_I . We use the abbreviation $L = \ln(x)$ in the following. The correct analytic continuation to negative values of x is given by $L \rightarrow L + i\pi$.

4.2.1 Vector form factor

$$\begin{aligned}
\mathcal{F}_{V,1}^{(1)} = C_F \Bigg\{ & \frac{1}{\varepsilon} \left[-2 - 2L - 4Lx^2 \right] + \left[-4 + 2\zeta_2 - 3L - L^2 + 2(-2 + L)x + x^2(1 - 8L - 2L^2 \right. \\
& \left. + 4\zeta_2) \right] + \varepsilon \left[-8 + 2\zeta_2 + 4\zeta_3 + L(-8 + \zeta_2) - \frac{3L^2}{2} - \frac{L^3}{3} + x(-10 - 2\zeta_2 + 6L + L^2) \right. \\
& \left. + x^2(4 + 8\zeta_2 + 8\zeta_3 + L(-19 + 2\zeta_2) - 4L^2 - \frac{2L^3}{3}) \right] + \varepsilon^2 \left[-16 + 6\zeta_2 + \frac{14}{5}\zeta_2^2 + \frac{20}{3}\zeta_3 \right. \\
& \left. + L(-16 + \frac{3\zeta_2}{2} + \frac{14\zeta_3}{3}) + L^2(-4 + \frac{\zeta_2}{2}) - \frac{L^3}{2} - \frac{L^4}{12} + x(-26 - 8\zeta_2 - 4\zeta_3) \right. \\
& \left. + L(16 - \zeta_2) + 3L^2 + \frac{L^3}{3} \right] + x^2 \left(17 + \frac{39}{2}\zeta_2 + \frac{28}{5}\zeta_2^2 + 16\zeta_3 + L(-46 + 4\zeta_2 + \frac{28\zeta_3}{3}) \right. \\
& \left. + L^2(-\frac{19}{2} + \zeta_2) - \frac{4L^3}{3} - \frac{L^4}{6} \right) \Bigg\}. \tag{4.21}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_{V,1}^{(2)} = C_F^2 \Bigg\{ & \frac{1}{\varepsilon^2} \left[2 + 4L + 2L^2 + 8L(1 + L)x^2 \right] + \frac{1}{\varepsilon} \left[8 + 14L + 8L^2 + 2L^3 - 4\zeta_2 - 4L\zeta_2 - 4(-2 \right. \\
& \left. + L)(1 + L)x + x^2(-2 + 30L + 32L^2 + 8L^3 - 8\zeta_2 - 16L\zeta_2) \right] + \left[46 + \frac{85L}{2} + \frac{55L^2}{2} \right. \\
& \left. + \frac{20L^3}{3} + \frac{7L^4}{6} + 39\zeta_2 - 4L^2\zeta_2 - 48\ln(2)\zeta_2 - \frac{118}{5}\zeta_2^2 - 44\zeta_3 - 32L\zeta_3 + x(-22 + 13L \right. \\
& \left. - 37L^2 - \frac{28L^3}{3} - \frac{L^4}{3} - 30\zeta_2 + 36L\zeta_2 + 8L^2\zeta_2 + 288\ln(2)\zeta_2 + \frac{128}{5}\zeta_2^2 - 88\zeta_3 - 48L\zeta_3) \right. \\
& \left. + x^2 \left(\frac{1307}{2} - 365L + 214L^2 + 32L^3 + \frac{26L^4}{3} - 980\zeta_2 - 376L\zeta_2 - 84L^2\zeta_2 - 576\ln(2)\zeta_2 \right. \right. \\
& \left. \left. - \frac{1756}{5}\zeta_2^2 + 808\zeta_3 + 496L\zeta_3 \right) \right] + \varepsilon \left[4 + 8c_1 + \frac{479L}{4} + \frac{153L^2}{2} + \frac{137L^3}{6} + \frac{11L^4}{3} + \frac{L^5}{2} \right. \\
& \left. + 163\zeta_2 + 17L\zeta_2 - \frac{8}{3}L^3\zeta_2 - 24\ln(2)\zeta_2 - 160\zeta_2^2 - \frac{106}{5}L\zeta_2^2 - \frac{346}{3}\zeta_3 - \frac{284}{3}L\zeta_3 - \frac{112}{3}L^2\zeta_3 \right. \\
& \left. - 12\zeta_2\zeta_3 - 18\zeta_5 + x \left(210 - 48c_1 - \frac{335L}{2} - 96L^2 - \frac{131L^3}{3} - 6L^4 - \frac{L^5}{5} - 580\zeta_2 + 282L\zeta_2 \right. \right. \\
& \left. \left. + 34L^2\zeta_2 + \frac{40}{3}L^3\zeta_2 + 1728\ln(2)\zeta_2 + 44\zeta_2^2 - \frac{668}{5}L\zeta_2^2 - 156\zeta_3 + 300L\zeta_3 + 4L^2\zeta_3 - 208\zeta_2\zeta_3 \right) \right] \Bigg\}
\end{aligned}$$

$$\begin{aligned}
& + 480\zeta_5) + x^2 \left(-\frac{1951}{4} + 96c_1 + 464L + 185L^2 + 178L^3 + \frac{74L^4}{3} + \frac{58L^5}{15} - 1164\zeta_2 \right. \\
& - 2206L\zeta_2 - 72L^2\zeta_2 - \frac{476}{3}L^3\zeta_2 - 4224\ln(2)\zeta_2 + 3212\zeta_2^2 + \frac{5992}{5}L\zeta_2^2 - 2000\zeta_3 - \frac{7792}{3}L\zeta_3 \\
& \left. - \frac{268}{3}L^2\zeta_3 + 1656\zeta_2\zeta_3 - 6252\zeta_5 \right) \Bigg\} + C_F C_A \left\{ \frac{1}{\varepsilon^2} \left[\frac{11}{3} + \frac{11L}{3} + \frac{22Lx^2}{3} \right] + \frac{1}{\varepsilon} \left[-\frac{49}{9} - \frac{67L}{9} \right. \right. \\
& + 2\zeta_2 + 2L\zeta_2 - 2\zeta_3 + x^2 \left(4 - \frac{188L}{9} - 4L^2 - \frac{4L^3}{3} - 4\zeta_2 - 8\zeta_3 \right) \Bigg] + \left[-\frac{1595}{27} - \frac{2545L}{54} \right. \\
& - \frac{233L^2}{18} - \frac{11L^3}{9} - \frac{7}{9}\zeta_2 - \frac{22}{3}L\zeta_2 + 2L^2\zeta_2 + 24\ln(2)\zeta_2 - \frac{3}{5}\zeta_2^2 + \frac{134}{3}\zeta_3 + 26L\zeta_3 + x \left(-\frac{904}{9} \right. \\
& + \frac{341L}{9} - \frac{25L^2}{3} + \frac{8L^3}{3} + \frac{L^4}{6} + \frac{494}{3}\zeta_2 + 44L\zeta_2 + 8L^2\zeta_2 - 144\ln(2)\zeta_2 + 28\zeta_2^2 - 200\zeta_3 \\
& \left. - 72L\zeta_3 \right) + x^2 \left(\frac{8723}{18} - \frac{8968L}{27} + \frac{931L^2}{9} - \frac{94L^3}{9} - \frac{7L^4}{3} - \frac{6848}{9}\zeta_2 - \frac{1076}{3}L\zeta_2 - 72L^2\zeta_2 \right. \\
& \left. + 288\ln(2)\zeta_2 - \frac{1408}{5}\zeta_2^2 + \frac{5188}{3}\zeta_3 + 808L\zeta_3 \right) \Bigg] + \varepsilon \left[-\frac{28745}{162} - 4c_1 - \frac{70165L}{324} - \frac{3337L^2}{54} \right. \\
& - \frac{565L^3}{54} - \frac{11L^4}{12} - \frac{71}{27}\zeta_2 - \frac{575}{18}L\zeta_2 - \frac{11}{2}L^2\zeta_2 + \frac{4}{3}L^3\zeta_2 + 12\ln(2)\zeta_2 + \frac{637}{5}\zeta_2^2 + \frac{88}{5}L\zeta_2^2 \\
& + \frac{1577}{9}\zeta_3 + \frac{260}{3}L\zeta_3 + 26L^2\zeta_3 - 2\zeta_2\zeta_3 - 157\zeta_5 + x \left(-\frac{12683}{27} + 24c_1 + \frac{5639L}{54} + \frac{134L^2}{9} \right. \\
& - \frac{17L^3}{3} + \frac{11L^4}{6} + \frac{L^5}{10} + \frac{4448}{9}\zeta_2 + 115L\zeta_2 + 26L^2\zeta_2 + \frac{16}{3}L^3\zeta_2 - 864\ln(2)\zeta_2 - \frac{2944}{5}\zeta_2^2 \\
& \left. - \frac{482}{5}L\zeta_2^2 + 272\zeta_3 + 44L\zeta_3 - 38L^2\zeta_3 + 8\zeta_2\zeta_3 + 768\zeta_5 \right) + x^2 \left(-\frac{25015}{27} - 48c_1 + \frac{155567L}{162} \right. \\
& - \frac{18413L^2}{54} + \frac{3260L^3}{27} - \frac{53L^4}{6} - \frac{19L^5}{15} - \frac{41933}{54}\zeta_2 - \frac{6002}{9}L\zeta_2 - 125L^2\zeta_2 - 52L^3\zeta_2 \\
& + 2112\ln(2)\zeta_2 + \frac{19996}{5}\zeta_2^2 + 1052L\zeta_2^2 + \frac{154}{9}\zeta_3 - \frac{5168}{3}L\zeta_3 + 476L^2\zeta_3 + 148\zeta_2\zeta_3 \\
& \left. - 8178\zeta_5 \right) \Bigg\} + C_F n_l T_F \left\{ \frac{1}{\varepsilon^2} \left[-\frac{4}{3} - \frac{4L}{3} - \frac{8Lx^2}{3} \right] + \frac{1}{\varepsilon} \left[\frac{20}{9} + \frac{20L}{9} + \frac{40Lx^2}{9} \right] \right. \\
& + \left[\frac{424}{27} + \frac{418L}{27} + \frac{38L^2}{9} + \frac{4L^3}{9} - \frac{28}{9}\zeta_2 + \frac{8}{3}L\zeta_2 - \frac{16}{3}\zeta_3 + x \left(\frac{200}{9} - \frac{148L}{9} - \frac{4L^2}{3} + \frac{8}{3}\zeta_2 \right) \right. \\
& + x^2 \left(-\frac{68}{9} + \frac{1064L}{27} + \frac{88L^2}{9} + \frac{8L^3}{9} - \frac{176}{9}\zeta_2 + \frac{16}{3}L\zeta_2 - \frac{32}{3}\zeta_3 \right) \Bigg] + \varepsilon \left[\frac{5204}{81} + \frac{5813L}{81} \right. \\
& + \frac{562L^2}{27} + \frac{94L^3}{27} + \frac{L^4}{3} - \frac{176}{27}\zeta_2 + \frac{74}{9}L\zeta_2 + 2L^2\zeta_2 - \frac{96}{5}\zeta_2^2 - \frac{280}{9}\zeta_3 - \frac{16}{3}L\zeta_3 + x \left(\frac{3808}{27} \right. \\
& - \frac{2654L}{27} - \frac{184L^2}{9} - \frac{4L^3}{3} + \frac{584}{9}\zeta_2 - 4L\zeta_2 + 16\zeta_3 \Bigg) + x^2 \left(-\frac{2698}{27} + \frac{18238L}{81} + \frac{1406L^2}{27} \right. \\
& \left. + \frac{224L^3}{27} + \frac{2L^4}{3} - \frac{2974}{27}\zeta_2 + \frac{184}{9}L\zeta_2 + 4L^2\zeta_2 - \frac{192}{5}\zeta_2^2 - \frac{896}{9}\zeta_3 - \frac{32}{3}L\zeta_3 \right) \Bigg] \Bigg\} \\
& + C_F T_F \left\{ \left[\frac{1532}{27} + \frac{530L}{27} + \frac{38L^2}{9} + \frac{4L^3}{9} - \frac{8}{3}\zeta_2 + 4L\zeta_2 + x \left(-\frac{784}{9} - \frac{436L}{9} - \frac{52L^2}{3} \right) \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -88\zeta_2) + x^2 \left(\frac{1568}{9} + \frac{9604L}{27} + \frac{808L^2}{9} - \frac{16L^3}{9} + \frac{2096}{3}\zeta_2 - 8L\zeta_2 \right) \Big] + \varepsilon \left[\frac{4138}{27} + \frac{191L}{3} \right. \\
& + \frac{562L^2}{27} + \frac{94L^3}{27} + \frac{L^4}{3} - \frac{1616}{27}\zeta_2 + 6L\zeta_2 + 2L^2\zeta_2 + \frac{224}{3}\ln(2)\zeta_2 - \frac{8}{5}\zeta_2^2 - \frac{184}{3}\zeta_3 - \frac{56}{9}L\zeta_3 \\
& + x \left(-\frac{1148}{27} - \frac{5246L}{27} - \frac{472L^2}{9} - 12L^3 - \frac{2192}{9}\zeta_2 - 4L\zeta_2 + 320\ln(2)\zeta_2 - \frac{352}{3}\zeta_3 \right) \\
& + x^2 \left(\frac{215}{27} + \frac{32696L}{27} + \frac{9452L^2}{27} + \frac{1772L^3}{27} - \frac{4L^4}{3} + \frac{47870}{27}\zeta_2 + 40L\zeta_2 - 4L^2\zeta_2 \right. \\
& \left. - \frac{9344}{3}\ln(2)\zeta_2 - \frac{8}{5}\zeta_2^2 + \frac{13120}{9}\zeta_3 + \frac{176}{9}L\zeta_3 \right) \Big] \Big\}. \tag{4.22}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_{V,2}^{(1)} = C_F \Big\{ & -4Lx + \varepsilon \left[x \left(-16L - 2L^2 + 4\zeta_2 \right) + 8(-1+L)x^2 \right] + \varepsilon^2 \left[x \left(-8L^2 - \frac{2L^3}{3} \right) \right. \\
& \left. + \frac{1}{3}L \left(-96 + 6\zeta_2 \right) + 16\zeta_2 + 8\zeta_3 \right] + x^2 \left(-40 + 32L + 4L^2 - 8\zeta_2 \right) \Big] \Big\}. \tag{4.23}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_{V,2}^{(2)} = C_F^2 \Big\{ & \frac{1}{\varepsilon} \left[8L(1+L)x \right] + \left[x \left(62L + 34L^2 + 8L^3 + 60\zeta_2 - 48L\zeta_2 - 192\ln(2)\zeta_2 + 16\zeta_3 \right) \right. \\
& + x^2 \left(-232 + 232L - 200L^2 - \frac{64L^3}{3} - \frac{4L^4}{3} + 752\zeta_2 + 208L\zeta_2 + 64L^2\zeta_2 + 384\ln(2)\zeta_2 \right. \\
& \left. + \frac{1056}{5}\zeta_2^2 - 864\zeta_3 - 448L\zeta_3 \right) \Big] + \varepsilon \left[x \left(-20 + 32c_1 + 249L + 74L^2 + \frac{94L^3}{3} + \frac{14L^4}{3} \right) \right. \\
& + 492\zeta_2 - 116L\zeta_2 - 16L^2\zeta_2 - 1152\ln(2)\zeta_2 - \frac{1192}{5}\zeta_2^2 + 128\zeta_3 - 192L\zeta_3 \Big] + x^2 \left(-640 \right. \\
& - 64c_1 + 392L - 308L^2 - \frac{728L^3}{3} - \frac{40L^4}{3} - \frac{4L^5}{5} + 2200\zeta_2 + 1296L\zeta_2 + 264L^2\zeta_2 \\
& + \frac{256}{3}L^3\zeta_2 + 3072\ln(2)\zeta_2 - \frac{14176}{5}\zeta_2^2 - \frac{4848}{5}L\zeta_2^2 + 2672\zeta_3 + 1488L\zeta_3 - 80L^2\zeta_3 - 1088\zeta_2\zeta_3 \\
& \left. + 4608\zeta_5 \right) \Big] \Big\} + C_F C_A \Big\{ \left[x \left(12 - \frac{346L}{9} + \frac{2L^2}{3} - \frac{244}{3}\zeta_2 + 96\ln(2)\zeta_2 + 80\zeta_3 \right) + x^2 \left(-\frac{616}{3} \right) \right. \right. \\
& + \frac{232L}{3} - 72L^2 + \frac{32L^3}{3} + \frac{2L^4}{3} + 656\zeta_2 + 368L\zeta_2 + 64L^2\zeta_2 - 192\ln(2)\zeta_2 + \frac{1104}{5}\zeta_2^2 \\
& \left. - 1456\zeta_3 - 544L\zeta_3 \right) \Big] + \varepsilon \left[x \left(78 - 16c_1 - \frac{8057L}{27} - \frac{250L^2}{9} + \frac{2L^3}{3} - \frac{1768}{9}\zeta_2 - 38L\zeta_2 \right) \right. \\
& - 8L^2\zeta_2 + 576\ln(2)\zeta_2 + \frac{1504}{5}\zeta_2^2 - 264\zeta_3 + 48L\zeta_3 \Big] + x^2 \left(\frac{2672}{9} + 32c_1 - \frac{5912L}{9} - 4L^2 \right. \\
& - \frac{152L^3}{3} + \frac{20L^4}{3} + \frac{2L^5}{5} + 664\zeta_2 + 656L\zeta_2 + 160L^2\zeta_2 + \frac{160}{3}L^3\zeta_2 - 1536\ln(2)\zeta_2 - \frac{16032}{5}\zeta_2^2 \\
& \left. - \frac{4104}{5}L\zeta_2^2 + 608\zeta_3 + 1168L\zeta_3 - 248L^2\zeta_3 - 224\zeta_2\zeta_3 + 5760\zeta_5 \right) \Big] \Big\} \\
& + C_F n_l T_F \Big\{ \left[-\frac{32}{3}(-1+L)x^2 + x \left(\frac{200L}{9} + \frac{8L^2}{3} - \frac{16}{3}\zeta_2 \right) \right] + \varepsilon \left[x \left(\frac{3844L}{27} + \frac{296L^2}{9} \right) \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \left. + \frac{8L^3}{3} - \frac{592}{9}\zeta_2 + 8L\zeta_2 - 32\zeta_3 \right) + x^2 \left(\frac{1472}{9} - \frac{1184L}{9} - 16L^2 + 32\zeta_2 \right) \Bigg] \Bigg\} \\
& + C_F T_F \left\{ \left[x \left(\frac{272}{3} + \frac{200L}{9} + \frac{8L^2}{3} - 16\zeta_2 \right) + x^2 \left(-\frac{544}{3} - \frac{992L}{3} - 64L^2 - 512\zeta_2 \right) \right] \right. \\
& + \varepsilon \left[x \left(\frac{1528}{9} + \frac{3844L}{27} + \frac{296L^2}{9} + \frac{8L^3}{3} + \frac{256}{9}\zeta_2 + 8L\zeta_2 + 128 \ln(2)\zeta_2 - \frac{256}{3}\zeta_3 \right) \right. \\
& \left. \left. + x^2 \left(-\frac{2368}{9} - \frac{8672L}{9} - 336L^2 - \frac{128L^3}{3} - \frac{4000}{3}\zeta_2 + 2304 \ln(2)\zeta_2 - 1088\zeta_3 \right) \right] \right\}. \quad (4.24)
\end{aligned}$$

We note that in the asymptotic limit, the magnetic part of the vector form factors vanish.

4.2.2 Axial-vector form factor

$$\begin{aligned}
\mathcal{F}_{A,1}^{(1),\text{ns}} = C_F \left\{ \frac{1}{\varepsilon} \left[-2 - 2L - 4Lx^2 \right] + \left[-4 - 3L - L^2 + 2\zeta_2 + 2(-2 + 3L)x + x^2(1 - 8L \right. \right. \\
\left. - 2L^2 + 4\zeta_2) \right] + \varepsilon \left[-8 - 8L - \frac{3L^2}{2} - \frac{L^3}{3} + 2\zeta_2 + L\zeta_2 + 4\zeta_3 + x \left(-10 + 6L + 3L^2 \right. \right. \\
\left. - 6\zeta_2) + x^2 \left(12 - 27L - 4L^2 - \frac{2L^3}{3} + 8\zeta_2 + 2L\zeta_2 + 8\zeta_3 \right) \right] + \varepsilon^2 \left[-16 - 16L - 4L^2 \right. \\
\left. - \frac{L^3}{2} - \frac{L^4}{12} + 6\zeta_2 + \frac{3}{2}L\zeta_2 + \frac{1}{2}L^2\zeta_2 + \frac{14}{5}\zeta_2^2 + \frac{20}{3}\zeta_3 + \frac{14}{3}L\zeta_3 + x \left(-26 + 16L + 3L^2 \right. \right. \\
\left. \left. + L^3 - 8\zeta_2 - 3L\zeta_2 - 12\zeta_3 \right) + x^2 \left(25 - 46L - \frac{27L^2}{2} - \frac{4L^3}{3} - \frac{L^4}{6} + \frac{55}{2}\zeta_2 + 4L\zeta_2 \right. \right. \\
\left. \left. + L^2\zeta_2 + \frac{28}{5}\zeta_2^2 + 16\zeta_3 + \frac{28}{3}L\zeta_3 \right) \right] \Bigg\} \quad (4.25)
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_{A,1}^{(2),\text{ns}} = C_F^2 \left\{ \frac{1}{\varepsilon^2} \left[2 + 4L + 2L^2 + 8L(1 + L)x^2 \right] + \frac{1}{\varepsilon} \left[8 + 14L + 8L^2 + 2L^3 - 4\zeta_2 - 4L\zeta_2 \right. \right. \\
\left. - 4(1 + L)(-2 + 3L)x + x^2 \left(-2 + 30L + 32L^2 + 8L^3 - 8\zeta_2 - 16L\zeta_2 \right) \right] + \left[46 + \frac{85L}{2} \right. \\
\left. + \frac{55L^2}{2} + \frac{20L^3}{3} + \frac{7L^4}{6} + 39\zeta_2 - 4L^2\zeta_2 - 48 \ln(2)\zeta_2 - \frac{118}{5}\zeta_2^2 - 44\zeta_3 - 32L\zeta_3 \right. \\
\left. + x \left(-22 + 19L - 19L^2 - 12L^3 + \frac{L^4}{3} - 66\zeta_2 + 44L\zeta_2 + 8L^2\zeta_2 + 288 \ln(2)\zeta_2 + \frac{144}{5}\zeta_2^2 \right. \right. \\
\left. - 104\zeta_3 - 80L\zeta_3 \right) + x^2 \left(\frac{1195}{2} - 357L + 182L^2 + \frac{64L^3}{3} + \frac{14L^4}{3} - 228\zeta_2 - 216L\zeta_2 - 52L^2\zeta_2 \right. \\
\left. - 576 \ln(2)\zeta_2 - \frac{1308}{5}\zeta_2^2 + 488\zeta_3 + 432L\zeta_3 \right) \Bigg] + \varepsilon \left[4 + 8c_1 + \frac{479L}{4} + \frac{153L^2}{2} + \frac{137L^3}{6} \right. \\
\left. + \frac{11L^4}{3} + \frac{L^5}{2} + 163\zeta_2 + 17L\zeta_2 - \frac{8}{3}L^3\zeta_2 - 24 \ln(2)\zeta_2 - 160\zeta_2^2 - \frac{106}{5}L\zeta_2^2 - \frac{346}{3}\zeta_3 - \frac{284}{3}L\zeta_3 \right. \\
\left. - \frac{112}{3}L^2\zeta_3 - 12\zeta_2\zeta_3 - 18\zeta_5 + x \left(190 - 48c_1 - \frac{173L}{2} - 26L^2 - \frac{53L^3}{3} - \frac{22L^4}{3} + \frac{L^5}{5} - 856\zeta_2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + 166L\zeta_2 - 2L^2\zeta_2 + \frac{8}{3}L^3\zeta_2 + 1728 \ln(2)\zeta_2 + \frac{1124}{5}\zeta_2^2 - 84L\zeta_2^2 - 628\zeta_3 + 228L\zeta_3 - 52L^2\zeta_3 \\
& + 80\zeta_2\zeta_3 + 864\zeta_5) + x^2\left(-\frac{543}{4} + 96c_1 + 792L - 163L^2 + \frac{518L^3}{3} + \frac{58L^4}{3} + \frac{22L^5}{15} + 1708\zeta_2 \right. \\
& - 510L\zeta_2 + 136L^2\zeta_2 - \frac{188}{3}L^3\zeta_2 - 2688 \ln(2)\zeta_2 + \frac{3596}{5}\zeta_2^2 + \frac{2328}{5}L\zeta_2^2 + 832\zeta_3 - \frac{5008}{3}L\zeta_3 \\
& \left. + \frac{452}{3}L^2\zeta_3 - 328\zeta_2\zeta_3 - 5868\zeta_5\right) \Big] \Big\} + C_F C_A \left\{ \frac{1}{\varepsilon^2} \left[\frac{11}{3} + \frac{11L}{3} + \frac{22Lx^2}{3} \right] + \frac{1}{\varepsilon} \left[-\frac{49}{9} - \frac{67L}{9} \right. \right. \\
& + 2\zeta_2 + 2L\zeta_2 - 2\zeta_3 + x^2\left(4 - \frac{188L}{9} - 4L^2 - \frac{4L^3}{3} - 4\zeta_2 - 8\zeta_3\right) \Big] + \left[-\frac{1595}{27} - \frac{2545L}{54} \right. \\
& - \frac{233L^2}{18} - \frac{11L^3}{9} - \frac{7}{9}\zeta_2 - \frac{22}{3}L\zeta_2 + 2L^2\zeta_2 + 24 \ln(2)\zeta_2 - \frac{3}{5}\zeta_2^2 + \frac{134}{3}\zeta_3 + 26L\zeta_3 + x\left(-\frac{796}{9} \right. \\
& \left. + \frac{241L}{3} - 9L^2 - \frac{L^4}{6} + 158\zeta_2 + 36L\zeta_2 + 8L^2\zeta_2 - 144 \ln(2)\zeta_2 + \frac{132}{5}\zeta_2^2 - 168\zeta_3 - 56L\zeta_3\right) \\
& + x^2\left(\frac{7523}{18} - \frac{9760L}{27} + \frac{283L^2}{9} - \frac{46L^3}{9} - \frac{L^4}{3} - \frac{2528}{9}\zeta_2 - \frac{164}{3}L\zeta_2 - 24L^2\zeta_2 + 288 \ln(2)\zeta_2 \right. \\
& \left. - \frac{544}{5}\zeta_2^2 + \frac{1060}{3}\zeta_3 + 328L\zeta_3\right) \Big] + \varepsilon \left[-\frac{28745}{162} - 4c_1 - \frac{70165L}{324} - \frac{3337L^2}{54} - \frac{565L^3}{54} \right. \\
& - \frac{11L^4}{12} - \frac{71}{27}\zeta_2 - \frac{575}{18}L\zeta_2 - \frac{11}{2}L^2\zeta_2 + \frac{4}{3}L^3\zeta_2 + 12 \ln(2)\zeta_2 + \frac{637}{5}\zeta_2^2 + \frac{88}{5}L\zeta_2^2 + \frac{1577}{9}\zeta_3 \\
& + \frac{260}{3}L\zeta_3 + 26L^2\zeta_3 - 2\zeta_2\zeta_3 - 157\zeta_5 + x\left(-\frac{10577}{27} + 24c_1 + \frac{3799L}{18} + \frac{112L^2}{3} - 9L^3 + \frac{L^4}{6} \right. \\
& \left. - \frac{L^5}{10} + \frac{1744}{3}\zeta_2 + 153L\zeta_2 + 42L^2\zeta_2 + \frac{32}{3}L^3\zeta_2 - 864 \ln(2)\zeta_2 - \frac{2944}{5}\zeta_2^2 - \frac{606}{5}L\zeta_2^2 + 184\zeta_3 \right. \\
& + 20L\zeta_3 - 10L^2\zeta_3 - 136\zeta_2\zeta_3 + 576\zeta_5) + x^2\left(-\frac{12007}{27} - 48c_1 - \frac{19681L}{162} - \frac{22517L^2}{54} \right. \\
& \left. + \frac{956L^3}{27} - \frac{37L^4}{6} - \frac{L^5}{15} - \frac{25085}{54}\zeta_2 - \frac{4418}{9}L\zeta_2 + 43L^2\zeta_2 - 36L^3\zeta_2 + 1344 \ln(2)\zeta_2 + \frac{8004}{5}\zeta_2^2 \right. \\
& \left. + 548L\zeta_2^2 + \frac{4042}{9}\zeta_3 - \frac{2720}{3}L\zeta_3 + 164L^2\zeta_3 + 628\zeta_2\zeta_3 - 2994\zeta_5\right) \Big] \Big\} + C_F n_l T_F \left\{ \frac{1}{\varepsilon^2} \left[-\frac{4}{3} \right. \right. \\
& \left. - \frac{4L}{3} - \frac{8Lx^2}{3} \right] + \frac{1}{\varepsilon} \left[\frac{20}{9} + \frac{20L}{9} + \frac{40Lx^2}{9} \right] + \left[\frac{424}{27} + \frac{418L}{27} + \frac{38L^2}{9} + \frac{4L^3}{9} - \frac{28}{9}\zeta_2 \right. \\
& + \frac{8}{3}L\zeta_2 - \frac{16}{3}\zeta_3 + x\left(\frac{200}{9} - \frac{68L}{3} - 4L^2 + 8\zeta_2\right) + x^2\left(-\frac{164}{9} + \frac{1352L}{27} + \frac{88L^2}{9} + \frac{8L^3}{9} \right. \\
& \left. - \frac{176}{9}\zeta_2 + \frac{16}{3}L\zeta_2 - \frac{32}{3}\zeta_3\right) \Big] + \varepsilon \left[\frac{5204}{81} + \frac{5813L}{81} + \frac{562L^2}{27} + \frac{94L^3}{27} + \frac{L^4}{3} - \frac{176}{27}\zeta_2 + \frac{74}{9}L\zeta_2 \right. \\
& + 2L^2\zeta_2 - \frac{96}{5}\zeta_2^2 - \frac{280}{9}\zeta_3 - \frac{16}{3}L\zeta_3 + x\left(\frac{3808}{27} - \frac{862L}{9} - \frac{80L^2}{3} - 4L^3 + \frac{232}{3}\zeta_2 - 12L\zeta_2 \right. \\
& \left. + 48\zeta_3\right) + x^2\left(-\frac{4234}{27} + \frac{20254L}{81} + \frac{1838L^2}{27} + \frac{224L^3}{27} + \frac{2L^4}{3} - \frac{3838}{27}\zeta_2 + \frac{184}{9}L\zeta_2 + 4L^2\zeta_2 \right. \\
& \left. - \frac{192}{5}\zeta_2^2 - \frac{896}{9}\zeta_3 - \frac{32}{3}L\zeta_3\right) \Big] \Big\} + C_F T_F \left\{ \left[\frac{1532}{27} + \frac{530L}{27} + \frac{38L^2}{9} + \frac{4L^3}{9} - \frac{8}{3}\zeta_2 + 4L\zeta_2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + x \left(-\frac{160}{9} - \frac{164L}{3} - 20L^2 - 136\zeta_2 \right) + x^2 \left(\frac{320}{9} + \frac{2980L}{27} + \frac{520L^2}{9} - \frac{16L^3}{9} + \frac{1136}{3}\zeta_2 \right. \\
& \left. - 8L\zeta_2 \right) + \varepsilon \left[\frac{4138}{27} + \frac{191L}{3} + \frac{562L^2}{27} + \frac{94L^3}{27} + \frac{L^4}{3} - \frac{1616}{27}\zeta_2 + 6L\zeta_2 + 2L^2\zeta_2 \right. \\
& + \frac{224}{3} \ln(2)\zeta_2 - \frac{8}{5}\zeta_2^2 - \frac{184}{3}\zeta_3 - \frac{56}{9}L\zeta_3 + x \left(\frac{3628}{27} - \frac{1726L}{9} - \frac{176L^2}{3} - \frac{44L^3}{3} - 400\zeta_2 \right. \\
& \left. - 12L\zeta_2 + 576 \ln(2)\zeta_2 - 256\zeta_3 \right) + x^2 \left(-\frac{10537}{27} + \frac{10904L}{27} + \frac{2972L^2}{27} + \frac{1196L^3}{27} - \frac{4L^4}{3} \right. \\
& \left. + \frac{25118}{27}\zeta_2 + 40L\zeta_2 - 4L^2\zeta_2 - \frac{4736}{3} \ln(2)\zeta_2 - \frac{8}{5}\zeta_2^2 + \frac{6208}{9}\zeta_3 + \frac{176}{9}L\zeta_3 \right) \left. \right] \Bigg\}. \quad (4.26)
\end{aligned}$$

Here we observe that for $x = 0$, the electric vector and axial-vector form factors are the same, as expected. The bare singlet piece for the electric axial-vector form factor is given by

$$\begin{aligned}
\hat{\mathcal{F}}_{A,1}^{(2),s} = C_F T_F \Bigg\{ & -\frac{6}{\varepsilon} + \left[-29 - 12L + 8\zeta_2 + x \left(8 + 24L + 8L^2 - 48\zeta_2 - 32L\zeta_2 + 64\zeta_3 \right) \right. \\
& \left. + x^2 \left(-4 - 16L - 56L^2 - \frac{16L^3}{3} + 32\zeta_2 + 32L\zeta_2 + 32L^2\zeta_2 - \frac{64}{5}\zeta_2^2 - 128\zeta_3 - 128L\zeta_3 \right) \right] \\
& + \varepsilon \left[-\frac{199}{2} - 58L - 12L^2 + 26\zeta_2 + 16L\zeta_2 + 8\zeta_3 + x \left(-4 + 76L + 56L^2 + 8L^3 - 176\zeta_2 \right. \right. \\
& \left. - 48L\zeta_2 - 48L^2\zeta_2 + 384 \ln(2)\zeta_2 + \frac{224}{5}\zeta_2^2 - 80\zeta_3 + 32L\zeta_3 \right) + x^2 \left(-274 + 88L - 180L^2 \right. \\
& \left. - \frac{136L^3}{3} - \frac{10L^4}{3} + 632\zeta_2 - 16L\zeta_2 + 64L^2\zeta_2 + 32L^3\zeta_2 - 1536 \ln(2)\zeta_2 - 48\zeta_2^2 - \frac{448}{5}L\zeta_2^2 \right. \\
& \left. \left. + 400\zeta_3 + 224L\zeta_3 - 32L^2\zeta_3 + 160\zeta_2\zeta_3 - 144\zeta_5 \right) \right] \Bigg\}. \quad (4.27)
\end{aligned}$$

The magnetic parts of the axial-vector form factor read

$$\begin{aligned}
\mathcal{F}_{A,2}^{(1),ns} = C_F \Bigg\{ & -4(2+3L)x - 16(1+L)x^2 + \varepsilon \left[x \left(-16 - 24L - 6L^2 + 12\zeta_2 \right) + x^2 \left(-56 \right. \right. \\
& \left. \left. - 8L - 8L^2 + 16\zeta_2 \right) \right] + \varepsilon^2 \left[x \left(-32 - 48L - 12L^2 - 2L^3 + 20\zeta_2 + 6L\zeta_2 + 24\zeta_3 \right) \right. \\
& \left. \left. + x^2 \left(-136 - 16L - 4L^2 - \frac{8L^3}{3} + 8L\zeta_2 + 32\zeta_3 \right) \right] \right\}. \quad (4.28)
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_{A,2}^{(2),ns} = C_F^2 \Bigg\{ & \frac{1}{\varepsilon} \left[8(1+L)(2+3L)x + 32(1+L)^2x^2 \right] + \left[x \left(68 + 122L + 86L^2 + 24L^3 + 132\zeta_2 \right. \right. \\
& \left. - 48L\zeta_2 - 192 \ln(2)\zeta_2 + 48\zeta_3 \right) + x^2 \left(-80 + 416L + 48L^2 + 32L^3 + \frac{4L^4}{3} + 16L\zeta_2 \right. \\
& \left. \left. + 384 \ln(2)\zeta_2 + \frac{32}{5}\zeta_2^2 - 160\zeta_3 - 64L\zeta_3 \right) \right] + \varepsilon \left[x \left(242 + 32c_1 + 335L + 250L^2 + \frac{218L^3}{3} \right. \right. \\
& \left. \left. + 14L^4 + 812\zeta_2 - 28L\zeta_2 - 48L^2\zeta_2 - 1152 \ln(2)\zeta_2 - \frac{1944}{5}\zeta_2^2 + 176\zeta_3 - 96L\zeta_3 \right) + x^2 \left(316 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -64c_1 + 548L + 620L^2 + 16L^3 + \frac{56L^4}{3} + \frac{4L^5}{5} - 616\zeta_2 + 384L\zeta_2 - 24L^2\zeta_2 - \frac{64}{3}L^3\zeta_2 \\
& + 3072 \ln(2)\zeta_2 + \frac{96}{5}\zeta_2^2 + \frac{496}{5}L\zeta_2^2 - 2000\zeta_3 - 176L\zeta_3 - 112L^2\zeta_3 + 576\zeta_2\zeta_3 + 768\zeta_5 \Big) \Big] \Big\} \\
& + C_F C_A \left\{ \left[x \left(-\frac{968}{9} - \frac{458L}{3} - 22L^2 - 84\zeta_2 + 96 \ln(2)\zeta_2 + 48\zeta_3 \right) + x^2 \left(-\frac{2800}{9} - \frac{1936L}{9} \right. \right. \right. \\
& \left. \left. \left. - \frac{304L^2}{3} - \frac{2L^4}{3} + \frac{1184}{3}\zeta_2 + 144L\zeta_2 + 32L^2\zeta_2 - 192 \ln(2)\zeta_2 + \frac{528}{5}\zeta_2^2 - 560\zeta_3 - 224L\zeta_3 \right) \right] \right. \\
& \left. + \varepsilon \left[x \left(-\frac{14872}{27} - 16c_1 - \frac{6853L}{9} - \frac{590L^2}{3} - 22L^3 - \frac{908}{3}\zeta_2 - 82L\zeta_2 + 576 \ln(2)\zeta_2 \right. \right. \right. \\
& \left. \left. \left. + \frac{1248}{5}\zeta_2^2 + 232\zeta_3 + 144L\zeta_3 \right) + x^2 \left(-\frac{61028}{27} + 32c_1 - \frac{18980L}{27} - \frac{5092L^2}{9} - 96L^3 - \frac{2L^5}{5} \right. \right. \right. \\
& \left. \left. \left. + \frac{12992}{9}\zeta_2 + 568L\zeta_2 + 176L^2\zeta_2 + \frac{128}{3}L^3\zeta_2 - 1536 \ln(2)\zeta_2 - \frac{7296}{5}\zeta_2^2 - \frac{2424}{5}L\zeta_2^2 + 1312\zeta_3 \right. \right. \right. \\
& \left. \left. \left. + 304L\zeta_3 - 40L^2\zeta_3 - 544\zeta_2\zeta_3 + 2304\zeta_5 \right) \right] \right\} + C_F n_l T_F \left\{ \left[x \left(\frac{304}{9} + \frac{136L}{3} + 8L^2 - 16\zeta_2 \right) \right. \right. \\
& \left. \left. + x^2 \left(\frac{896}{9} + \frac{320L}{9} + \frac{32L^2}{3} - \frac{64}{3}\zeta_2 \right) \right] + \varepsilon \left[x \left(\frac{4664}{27} + \frac{2036L}{9} + \frac{184L^2}{3} + 8L^3 - \frac{224}{3}\zeta_2 \right. \right. \right. \\
& \left. \left. \left. + 24L\zeta_2 - 96\zeta_3 \right) + x^2 \left(\frac{18544}{27} + \frac{2704L}{27} + \frac{368L^2}{9} + \frac{32L^3}{3} + \frac{128}{9}\zeta_2 + 32L\zeta_2 - 128\zeta_3 \right) \right] \right\} \\
& + C_F T_F \left\{ \left[x \left(-\frac{128}{9} + \frac{136L}{3} + 8L^2 + 80\zeta_2 \right) + x^2 \left(-\frac{1024}{9} - \frac{1408L}{9} - \frac{160L^2}{3} - 320\zeta_2 \right) \right] \right. \\
& \left. + \varepsilon \left[x \left(-\frac{304}{27} + \frac{2036L}{9} + \frac{184L^2}{3} + 8L^3 + \frac{1072}{3}\zeta_2 + 24L\zeta_2 - 384 \ln(2)\zeta_2 + 192\zeta_3 \right) \right. \right. \\
& \left. \left. + x^2 \left(\frac{5200}{27} - \frac{5936L}{27} - \frac{1360L^2}{9} - 32L^3 - \frac{6080}{9}\zeta_2 + 32L\zeta_2 + 1280 \ln(2)\zeta_2 - \frac{1600}{3}\zeta_3 \right) \right] \right\}. \tag{4.29}
\end{aligned}$$

The bare singlet piece for the magnetic axial-vector form factor is given by

$$\begin{aligned}
\hat{\mathcal{F}}_{A,2}^{(2),s} = C_F T_F \left\{ \left[x \left(-80 - 48L - 8L^2 - 32\zeta_2 \right) + x^2 \left(-160 + \frac{4L^4}{3} - 256\zeta_2 - 128L\zeta_2 + 32L^2\zeta_2 \right. \right. \right. \\
\left. \left. \left. + \frac{256}{5}\zeta_2^2 + 256\zeta_3 + 192L\zeta_3 \right) \right] + \varepsilon \left[x \left(-360 - 264L - 72L^2 - \frac{32L^3}{3} + 64\zeta_2 + 16L\zeta_2 \right. \right. \right. \\
\left. \left. \left. - 48\zeta_3 \right) + x^2 \left(-768 - 288L - 32L^2 - \frac{16L^3}{3} - \frac{8L^4}{3} + \frac{16L^5}{15} - 896\zeta_2 - 192L\zeta_2 \right. \right. \right. \\
\left. \left. \left. - 192L^2\zeta_2 + 16L^3\zeta_2 + 1536 \ln(2)\zeta_2 + \frac{256}{5}\zeta_2^2 - 32L\zeta_2^2 - 448\zeta_3 - 512L\zeta_3 \right. \right. \right. \\
\left. \left. \left. + 208L^2\zeta_3 + 64\zeta_2\zeta_3 - 704\zeta_5 \right) \right] \right\}. \tag{4.30}
\end{aligned}$$

Similar to the vector form factor, the magnetic part of the axial-vector form factor also vanishes for $x = 0$.

4.2.3 Scalar form factor

Next, we present the scalar form factor in the asymptotic limit.

$$\begin{aligned} \mathcal{F}_S^{(1)} = C_F \left\{ \frac{1}{\varepsilon} \left[-2 - 2L - 4Lx^2 \right] + \left[-2 - L^2 + 2\zeta_2 + 4(-1 + 3L)x + x^2 \left(1 - 2L - 2L^2 \right. \right. \right. \\ \left. \left. \left. + 4\zeta_2 \right) \right] + \varepsilon \left[-4 - 2L - \frac{L^3}{3} - \zeta_2 + L\zeta_2 + 4\zeta_3 + x \left(-4 + 12L + 6L^2 - 12\zeta_2 \right) + x^2 \left(\frac{45}{2} \right. \right. \right. \\ \left. \left. \left. - 24L - L^2 - \frac{2L^3}{3} + 2\zeta_2 + 2L\zeta_2 + 8\zeta_3 \right) \right] + \varepsilon^2 \left[-8 - 4L - L^2 - \frac{L^4}{12} + \frac{1}{2}L^2\zeta_2 + \frac{14}{5}\zeta_2^2 \right. \right. \\ \left. \left. + \frac{2}{3}\zeta_3 + \frac{14}{3}L\zeta_3 + \zeta_2 + x \left(-8 + 28L + 6L^2 + 2L^3 - 14\zeta_2 - 6L\zeta_2 - 24\zeta_3 \right) + x^2 \left(\frac{193}{4} - 34L \right. \right. \right. \\ \left. \left. \left. - 12L^2 - \frac{L^3}{3} - \frac{L^4}{6} + \frac{49}{2}\zeta_2 + L\zeta_2 + L^2\zeta_2 + \frac{28}{5}\zeta_2^2 + 4\zeta_3 + \frac{28}{3}L\zeta_3 \right) \right] \right\}. \quad (4.31) \end{aligned}$$

$$\begin{aligned} \mathcal{F}_S^{(2)} = C_F^2 \left\{ \frac{1}{\varepsilon^2} \left[2 + 4L + 2L^2 + 8L(1 + L)x^2 \right] + \frac{1}{\varepsilon} \left[4 + 4L + 2L^2 + 2L^3 - 4\zeta_2 - 4L\zeta_2 \right. \right. \\ \left. \left. - 8(1 + L)(-1 + 3L)x + x^2 \left(-2 + 10L + 8L^2 + 8L^3 - 8\zeta_2 - 16L\zeta_2 \right) \right] + \left[29 + 12L + 6L^2 \right. \right. \\ \left. \left. + \frac{2L^3}{3} + \frac{7L^4}{6} + 6\zeta_2 + 12L\zeta_2 - 4L^2\zeta_2 - \frac{118}{5}\zeta_2^2 - 56\zeta_3 - 32L\zeta_3 + x \left(32 - 72L + 4L^2 - 24L^3 \right. \right. \right. \\ \left. \left. \left. - 112\zeta_2 + 64L\zeta_2 + 8L^2\zeta_2 + 288 \ln(2)\zeta_2 + \frac{136}{5}\zeta_2^2 - 120\zeta_3 - 64L\zeta_3 \right) + x^2 \left(\frac{1069}{2} - 380L \right. \right. \right. \\ \left. \left. \left. + 166L^2 - \frac{10L^3}{3} + 5L^4 - 36\zeta_2 - 132L\zeta_2 - 36L^2\zeta_2 - 576 \ln(2)\zeta_2 - \frac{724}{5}\zeta_2^2 + 72\zeta_3 \right. \right. \right. \\ \left. \left. \left. + 224L\zeta_3 \right) \right] + \varepsilon \left[-\frac{113}{2} + 36L + 14L^2 + \frac{14L^3}{3} + \frac{L^4}{6} + \frac{L^5}{2} - 40\zeta_2 + 24L\zeta_2 + 12L^2\zeta_2 \right. \right. \\ \left. \left. - \frac{8}{3}L^3\zeta_2 + 264 \ln(2)\zeta_2 - \frac{314}{5}\zeta_2^2 - \frac{106}{5}L\zeta_2^2 - \frac{478}{3}\zeta_3 - \frac{212}{3}L\zeta_3 - \frac{112}{3}L^2\zeta_3 - 12\zeta_2\zeta_3 - 18\zeta_5 \right. \right. \\ \left. \left. + x \left(232 - 48c_1 - 216L - 40L^2 + 12L^3 - \frac{41L^4}{3} - 1016\zeta_2 + 24L\zeta_2 + 12L^2\zeta_2 + 8L^3\zeta_2 \right. \right. \right. \\ \left. \left. \left. + 1536 \ln(2)\zeta_2 + \frac{1552}{5}\zeta_2^2 - \frac{544}{5}L\zeta_2^2 - 616\zeta_3 + 240L\zeta_3 - 24L^2\zeta_3 - 64\zeta_2\zeta_3 + 672\zeta_5 \right) \right. \right. \\ \left. \left. + x^2 \left(-\frac{2149}{4} + 96c_1 + 895L - 221L^2 + \frac{472L^3}{3} + \frac{37L^4}{6} + \frac{5L^5}{3} + 2568\zeta_2 - 366L\zeta_2 \right. \right. \right. \\ \left. \left. \left. + 214L^2\zeta_2 - 52L^3\zeta_2 - 2304 \ln(2)\zeta_2 - \frac{306}{5}\zeta_2^2 + 100L\zeta_2^2 + 604\zeta_3 - \frac{4240}{3}L\zeta_3 + \frac{128}{3}L^2\zeta_3 \right. \right. \right. \\ \left. \left. \left. - 520\zeta_2\zeta_3 - 2916\zeta_5 \right) \right] \right\} + C_F C_A \left\{ \frac{1}{\varepsilon^2} \left[\frac{11}{3} + \frac{11L}{3} + \frac{22Lx^2}{3} \right] + \frac{1}{\varepsilon} \left[-\frac{49}{9} - \frac{67L}{9} + 2\zeta_2 \right. \right. \\ \left. \left. + 2L\zeta_2 - 2\zeta_3 + x^2 \left(4 - \frac{188L}{9} - 4L^2 - \frac{4L^3}{3} - 4\zeta_2 - 8\zeta_3 \right) \right] + \left[-\frac{869}{27} - \frac{242L}{27} - \frac{67L^2}{9} \right. \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{11L^3}{9} + \frac{182}{9}\zeta_2 - \frac{22}{3}L\zeta_2 + 2L^2\zeta_2 - \frac{3}{5}\zeta_2^2 + \frac{98}{3}\zeta_3 + 26L\zeta_3 + x\left(-\frac{580}{9} + \frac{416L}{3} + 12L^2\right. \\
& + 92\zeta_2 - 144\ln(2)\zeta_2 - 44\zeta_3) + x^2\left(\frac{3463}{36} - \frac{5705L}{54} - \frac{37L^2}{18} - \frac{43L^3}{9} - \frac{L^4}{2} - \frac{611}{9}\zeta_2 + \frac{58}{3}L\zeta_2\right. \\
& \left. + 10L^2\zeta_2 + 288\ln(2)\zeta_2 - \frac{122}{5}\zeta_2^2 + \frac{364}{3}\zeta_3 + 96L\zeta_3\right) + \varepsilon\left[-\frac{6437}{162} - \frac{2122L}{81} - \frac{341L^2}{27}\right. \\
& - \frac{134L^3}{27} - \frac{11L^4}{12} + \frac{1972}{27}\zeta_2 - \frac{103}{9}L\zeta_2 - \frac{11}{2}L^2\zeta_2 + \frac{4}{3}L^3\zeta_2 - 132\ln(2)\zeta_2 + 65\zeta_2^2 + \frac{88}{5}L\zeta_2^2 \\
& + \frac{1055}{9}\zeta_3 + \frac{152}{3}L\zeta_3 + 26L^2\zeta_3 - 2\zeta_2\zeta_3 - 157\zeta_5 + x\left(-\frac{2756}{27} + 24c_1 + \frac{3700L}{9} + \frac{476L^2}{3}\right. \\
& \left. + \frac{32L^3}{3} - \frac{L^4}{6} + \frac{746}{3}\zeta_2 + 14L\zeta_2 + 2L^2\zeta_2 - 768\ln(2)\zeta_2 - 360\zeta_2^2 + 220\zeta_3 - 16L\zeta_3\right) \\
& + x^2\left(\frac{40951}{216} - 48c_1 - \frac{85217L}{324} - \frac{24019L^2}{108} + \frac{553L^3}{54} - \frac{79L^4}{12} - \frac{L^5}{6} - \frac{13060}{27}\zeta_2 - \frac{188}{9}L\zeta_2\right. \\
& + 28L^2\zeta_2 + \frac{2}{3}L^3\zeta_2 + 1152\ln(2)\zeta_2 + 841\zeta_2^2 + \frac{798}{5}L\zeta_2^2 - \frac{1916}{9}\zeta_3 - \frac{908}{3}L\zeta_3 + 92L^2\zeta_3 \\
& \left. + 388\zeta_2\zeta_3 - 942\zeta_5\right) \left. \right\} + C_{FM}T_F \left\{ \frac{1}{\varepsilon^2} \left[-\frac{4}{3} - \frac{4L}{3} - \frac{8Lx^2}{3} \right] + \frac{1}{\varepsilon} \left[\frac{20}{9} + \frac{20L}{9} + \frac{40Lx^2}{9} \right] \right. \\
& + \left[\frac{196}{27} + \frac{112L}{27} + \frac{20L^2}{9} + \frac{4L^3}{9} + \frac{8}{9}\zeta_2 + \frac{8}{3}L\zeta_2 - \frac{16}{3}\zeta_3 + x\left(\frac{128}{9} - \frac{112L}{3} - 8L^2 + 16\zeta_2\right) \right. \\
& \left. + x^2\left(-\frac{290}{9} + \frac{1064L}{27} + \frac{52L^2}{9} + \frac{8L^3}{9} - \frac{104}{9}\zeta_2 + \frac{16}{3}L\zeta_2 - \frac{32}{3}\zeta_3\right) \right] + \varepsilon \left[\frac{1706}{81} + \frac{1232L}{81} \right. \\
& + \frac{148L^2}{27} + \frac{40L^3}{27} + \frac{L^4}{3} + \frac{328}{27}\zeta_2 + \frac{20}{9}L\zeta_2 + 2L^2\zeta_2 - \frac{96}{5}\zeta_2^2 - \frac{64}{9}\zeta_3 - \frac{16}{3}L\zeta_3 + x\left(\frac{1504}{27} \right. \\
& \left. - \frac{1328L}{9} - \frac{136L^2}{3} - 8L^3 + \frac{344}{3}\zeta_2 - 24L\zeta_2 + 96\zeta_3\right) + x^2\left(-\frac{7375}{27} + \frac{16600L}{81} + \frac{1496L^2}{27} \right. \\
& \left. + \frac{116L^3}{27} + \frac{2L^4}{3} - \frac{3154}{27}\zeta_2 + \frac{76}{9}L\zeta_2 + 4L^2\zeta_2 - \frac{192}{5}\zeta_2^2 - \frac{464}{9}\zeta_3 - \frac{32}{3}L\zeta_3\right) \left. \right\} \\
& + C_F T_F \left\{ \left[\frac{1628}{27} + \frac{224L}{27} + \frac{20L^2}{9} + \frac{4L^3}{9} - \frac{68}{3}\zeta_2 + 4L\zeta_2 + x\left(-\frac{160}{9} + \frac{128L}{3} + 8L^2 - \frac{L^4}{3} \right. \right. \right. \\
& + 16\zeta_2 - 8L^2\zeta_2 - \frac{64}{5}\zeta_2^2 - 48L\zeta_3) + x^2\left(\frac{2192}{9} - \frac{1088L}{27} - \frac{416L^2}{9} - \frac{136L^3}{9} - \frac{4L^4}{3} + \frac{512}{3}\zeta_2 \right. \\
& \left. + 40L\zeta_2 - 32L^2\zeta_2 - \frac{256}{5}\zeta_2^2 - 256\zeta_3 - 192L\zeta_3\right) \left. \right] + \varepsilon \left[\frac{4214}{27} + \frac{64L}{9} + \frac{148L^2}{27} + \frac{40L^3}{27} + \frac{L^4}{3} \right. \\
& - \frac{4028}{27}\zeta_2 + 2L^2\zeta_2 + \frac{512}{3}\ln(2)\zeta_2 - \frac{8}{5}\zeta_2^2 - \frac{328}{3}\zeta_3 - \frac{56}{9}L\zeta_3 + x\left(\frac{2368}{27} + \frac{1696L}{9} + \frac{344L^2}{3} \right. \\
& + \frac{32L^3}{3} + \frac{L^4}{3} - \frac{4L^5}{15} - \frac{520}{3}\zeta_2 - 88L\zeta_2 - 8L^2\zeta_2 - 4L^3\zeta_2 + 192\ln(2)\zeta_2 + \frac{96}{5}\zeta_2^2 + 8L\zeta_2^2 \\
& + 240\zeta_3 + 112L\zeta_3 - 52L^2\zeta_3 - 16\zeta_2\zeta_3 + 176\zeta_5) + x^2\left(\frac{2621}{27} + \frac{10172L}{27} - \frac{10276L^2}{27} - \frac{2116L^3}{27} \right. \\
& \left. - 14L^4 - \frac{16L^5}{15} + \frac{29510}{27}\zeta_2 + 148L\zeta_2 + 148L^2\zeta_2 - 16L^3\zeta_2 - \frac{7040}{3}\ln(2)\zeta_2 - \frac{1504}{5}\zeta_2^2 + 32L\zeta_2^2 \right.
\end{aligned}$$

$$\left. + \frac{3904}{9}\zeta_3 - \frac{5296}{9}L\zeta_3 - 208L^2\zeta_3 - 64\zeta_2\zeta_3 + 704\zeta_5 \right) \Bigg\}. \quad (4.32)$$

4.2.4 Pseudo-scalar form factor

The non-singlet part of the pseudo-scalar form factor can be obtained in this limit using Eq. (2.13) as

$$\mathcal{F}_P^{(n),\text{ns}} = \mathcal{F}_{A,1}^{(n),\text{ns}} + \left(-\frac{(1-x)^2}{4x} \right) \mathcal{F}_{A,2}^{(n),\text{ns}} \quad \text{for } n = 1, 2. \quad (4.33)$$

The unrenormalized singlet piece is given by

$$\begin{aligned} \hat{\mathcal{F}}_P^{(2),s} = C_F T_F \Bigg\{ & \left[x \left(8L^2 - \frac{L^4}{3} + 32\zeta_2 - 8L^2\zeta_2 - \frac{64}{5}\zeta_2^2 - 48L\zeta_3 \right) + x^2 \left(112 - 64L + 16L^2 + 16L^3 \right. \right. \\ & \left. \left. + 32\zeta_2 + 96L\zeta_2 \right) \right] + \varepsilon \left[x \left(16L^2 + 8L^3 + L^4 - \frac{4L^5}{15} + 16L\zeta_2 + 8L^2\zeta_2 - 4L^3\zeta_2 + \frac{224}{5}\zeta_2^2 \right. \right. \\ & \left. \left. + 8L\zeta_2^2 + 48\zeta_3 + 208L\zeta_3 - 52L^2\zeta_3 - 16\zeta_2\zeta_3 + 176\zeta_5 \right) + x^2 \left(-400 + 480L - 80L^2 \right. \right. \\ & \left. \left. - \frac{56L^3}{3} + \frac{32L^4}{3} - 64\zeta_2 - 160L\zeta_2 - 48L^2\zeta_2 - \frac{176}{5}\zeta_2^2 + 32\zeta_3 - 128L\zeta_3 \right) \right] \Bigg\}. \quad (4.34) \end{aligned}$$

4.3 Threshold region $q^2 \sim 4m^2$

Now we provide the expansion of the form factors in the threshold region $q^2 \sim 4m^2$ or $x \rightarrow -1$. We expand the form factors around $\beta = 0$ up to $\mathcal{O}(\beta^2)$ and denote the form factors by \tilde{F} in this limit.

4.3.1 Vector form factor

First, we present the electric component of the vector form factor

$$\begin{aligned} \tilde{F}_{V,1}^{(1)} = C_F \Bigg\{ & \frac{1}{\varepsilon} \left[\frac{8\beta^2}{3} + i\pi \left(-\frac{1}{\beta} - \beta \right) \right] + \left[\frac{6\zeta_2}{\beta} - 6 + 6\beta\zeta_2 - \frac{10\beta^2}{9} + i\pi \left(\frac{1}{\beta} \left(-1 + 2\ln(2) \right. \right. \right. \\ & \left. \left. + 2\log(\beta) \right) + \beta \left(-1 + 2\ln(2) + 2\log(\beta) \right) \right] + \varepsilon \left[\frac{1}{\beta} \left(6\zeta_2 - 12\zeta_2 \ln(2) - 12\zeta_2 \log(\beta) \right) + 4 \right. \\ & \left. + \beta \left(6\zeta_2 - 12\zeta_2 \ln(2) - 12\zeta_2 \log(\beta) \right) + \beta^2 \left(\frac{344}{27} + \frac{4\zeta_2}{3} \right) + i\pi \left(\frac{1}{\beta} \left(-4 + \frac{3}{2}\zeta_2 + 2\ln(2) \right. \right. \right. \\ & \left. \left. - 2\ln^2(2) + 2\log(\beta) - 4\ln(2)\log(\beta) - 2\log^2(\beta) \right) + \beta \left(-3 + \frac{3}{2}\zeta_2 + 2\ln(2) - 2\ln^2(2) \right. \right. \\ & \left. \left. + 2\log(\beta) - 4\ln(2)\log(\beta) - 2\log^2(\beta) \right) \right] + \varepsilon^2 \left[\frac{1}{\beta} \left(24\zeta_2 + 3\zeta_2^2 - 12\zeta_2 \ln(2) + 12\zeta_2 \ln^2(2) \right. \right. \\ & \left. \left. - 12\zeta_2 \log(\beta) + 24\zeta_2 \ln(2)\log(\beta) + 12\zeta_2 \log^2(\beta) \right) - 24 - 3\zeta_2 + \beta \left(18\zeta_2 + 3\zeta_2^2 - 12\zeta_2 \ln(2) \right. \right. \\ & \left. \left. + 12\zeta_2 \ln^2(2) - 12\zeta_2 \log(\beta) + 24\zeta_2 \ln(2)\log(\beta) + 12\zeta_2 \log^2(\beta) \right) + \beta^2 \left(-\frac{472}{81} - \frac{5\zeta_2}{9} - \frac{8\zeta_3}{9} \right) \right] \Bigg\}. \end{aligned}$$

$$\begin{aligned}
& + i\pi \left(\frac{1}{\beta} \left(-8 + \frac{3}{2}\zeta_2 + \frac{7}{3}\zeta_3 + 8\ln(2) - 3\zeta_2 \ln(2) - 2\ln^2(2) + \frac{4\ln^3(2)}{3} + 8\log(\beta) \right. \right. \\
& - 3\zeta_2 \log(\beta) - 4\ln(2) \log(\beta) + 4\ln^2(2) \log(\beta) - 2\log^2(\beta) + 4\ln(2) \log^2(\beta) + \left. \frac{4\log^3(\beta)}{3} \right) \\
& + \beta \left(-4 + \frac{3}{2}\zeta_2 + \frac{7}{3}\zeta_3 + 6\ln(2) - 3\zeta_2 \ln(2) - 2\ln^2(2) + \frac{4\ln^3(2)}{3} + 6\log(\beta) - 3\zeta_2 \log(\beta) \right. \\
& \left. \left. - 4\ln(2) \log(\beta) + 4\ln^2(2) \log(\beta) - 2\log^2(\beta) + 4\ln(2) \log^2(\beta) + \frac{4\log^3(\beta)}{3} \right) \right) \Bigg\}, \quad (4.35)
\end{aligned}$$

$$\begin{aligned}
\tilde{F}_{V,1}^{(2)} = & C_F^2 \left\{ \frac{1}{\varepsilon^2} \left[-\frac{3\zeta_2}{\beta^2} - 6\zeta_2 - 3\beta^2\zeta_2 + i\pi \left(-\frac{8\beta}{3} \right) \right] + \frac{1}{\varepsilon} \left[\frac{1}{\beta^2} \left(-6\zeta_2 + 12\zeta_2 \ln(2) \right. \right. \right. \\
& + 12\zeta_2 \log(\beta) \left. \left. \right) - 12\zeta_2 + 16\beta\zeta_2 + 24\zeta_2 \ln(2) + 24\zeta_2 \log(\beta) + \beta^2 \left(-16 + 3\zeta_2 + 12\zeta_2 \ln(2) \right. \right. \\
& + 12\zeta_2 \log(\beta) \left. \left. \right) + i\pi \left(-\frac{6\zeta_2}{\beta^2} + \frac{6}{\beta} - 12\zeta_2 + \frac{8}{9}\beta(5 + 6\ln(2) + 6\log(\beta)) - 6\beta^2\zeta_2 \right) \right] \\
& + \left[\frac{1}{\beta^2} \left(-28\zeta_2 + 15\zeta_2^2 + 24\zeta_2 \ln(2) - 24\zeta_2 \ln^2(2) + 24\zeta_2 \log(\beta) - 48\zeta_2 \ln(2) \log(\beta) \right. \right. \\
& - 24\zeta_2 \log^2(\beta) \left. \left. \right) - \frac{36\zeta_2}{\beta} + \frac{421}{15} - \frac{1926}{25}\zeta_2 + 30\zeta_2^2 - \frac{81}{5}\zeta_3 + \frac{156}{5}\zeta_2 \ln(2) - 48\zeta_2 \ln^2(2) \right. \\
& + \frac{56}{5}\zeta_2 \log(\beta) - 96\zeta_2 \ln(2) \log(\beta) - 48\zeta_2 \log^2(\beta) + \beta \left(\frac{256}{3}\zeta_2 - 32\zeta_2 \ln(2) - 32\zeta_2 \log(\beta) \right) \\
& + \beta^2 \left(\frac{691}{35} - \frac{51302\zeta_2}{3675} + 15\zeta_2^2 + \frac{1192}{35}\zeta_3 - \frac{108}{35}\zeta_2 \ln(2) - 24\zeta_2 \ln^2(2) + \frac{412}{35}\zeta_2 \log(\beta) \right. \\
& - 48\zeta_2 \ln(2) \log(\beta) - 24\zeta_2 \log^2(\beta) \left. \right) + i\pi \left(\frac{1}{\beta^2} \left(-12\zeta_2 + 24\zeta_2 \ln(2) + 24\zeta_2 \log(\beta) \right) \right. \\
& - \frac{1}{\beta} \left(12(\ln(2) + \log(\beta)) \right) - \frac{28}{5}\zeta_2 + 48\zeta_2 \ln(2) + 48\zeta_2 \log(\beta) + \beta \left(-\frac{2498}{27} + \frac{8}{3}\zeta_2 \right. \\
& + \frac{448\ln(2)}{9} - \frac{16}{3}\ln^2(2) + \frac{256\log(\beta)}{9} - \frac{32}{3}\ln(2) \log(\beta) - \frac{16}{3}\log^2(\beta) \left. \right) + \beta^2 \left(-\frac{281}{35}\zeta_2 \right. \\
& + 24\zeta_2 \ln(2) + 24\zeta_2 \log(\beta) \left. \right) \Bigg] + \varepsilon \left[\frac{1}{\beta^2} \left(-28\zeta_2 + 30\zeta_2^2 - 106\zeta_2\zeta_3 + 112\zeta_2 \ln(2) \right. \right. \\
& - 60\zeta_2^2 \ln(2) - 48\zeta_2 \ln^2(2) + 32\zeta_2 \ln^3(2) + 112\zeta_2 \log(\beta) - 60\zeta_2^2 \log(\beta) - 96\zeta_2 \ln(2) \log(\beta) \\
& + 96\zeta_2 \ln^2(2) \log(\beta) - 48\zeta_2 \log^2(\beta) + 96\zeta_2 \ln(2) \log^2(\beta) + 32\zeta_2 \log^3(\beta) \left. \right) + \frac{1}{\beta} \left(72\ln(2)\zeta_2 \right. \\
& + 72\log(\beta)\zeta_2 \left. \right) - \frac{36163}{450} - \frac{10c_1}{3} + \frac{157876}{375}\zeta_2 - \frac{306}{5}\zeta_2^2 - \frac{43756}{75}\zeta_3 - 212\zeta_2\zeta_3 \\
& - \frac{32552}{75}\zeta_2 \ln(2) - 120\zeta_2^2 \ln(2) + \frac{32}{5}\zeta_2 \ln^2(2) + 64\zeta_2 \ln^3(2) - \frac{7144}{25}\zeta_2 \log(\beta) \\
& - 120\zeta_2^2 \log(\beta) - \frac{224}{5}\zeta_2 \ln(2) \log(\beta) + 192\zeta_2 \ln^2(2) \log(\beta) - \frac{112}{5}\zeta_2 \log^2(\beta) \\
& + 192\zeta_2 \ln(2) \log^2(\beta) + 64\zeta_2 \log^3(\beta) + \beta \left(\frac{14416}{9}\zeta_2 + 16\zeta_2^2 - \frac{3776}{3}\zeta_2 \ln(2) + 32\zeta_2 \ln^2(2) \right. \\
& \left. - \frac{2624}{3}\zeta_2 \log(\beta) + 64\zeta_2 \ln(2) \log(\beta) + 32\zeta_2 \log^2(\beta) \right) + \beta^2 \left(\frac{1880267}{22050} + \frac{16c_1}{21} + \frac{61140199\zeta_2}{385875} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{26498}{175}\zeta_2^2 + \frac{286234\zeta_3}{3675} - 106\zeta_2\zeta_3 + \frac{800012\zeta_2 \ln(2)}{3675} - 60\zeta_2^2 \ln(2) + \frac{208}{7}\zeta_2 \ln^2(2) \\
& + 32\zeta_2 \ln^3(2) + \frac{175684\zeta_2 \log(\beta)}{1225} - 60\zeta_2^2 \log(\beta) - \frac{2368}{35}\zeta_2 \ln(2) \log(\beta) + 96\zeta_2 \ln^2(2) \log(\beta) \\
& - \frac{824}{35}\zeta_2 \log^2(\beta) + 96\zeta_2 \ln(2) \log^2(\beta) + 32\zeta_2 \log^3(\beta) \Big) + i\pi \left(\frac{1}{\beta^2} \left(-56\zeta_2 - 18\zeta_2^2 \right. \right. \\
& + 48\zeta_2 \ln(2) - 48\zeta_2 \ln^2(2) + 48\zeta_2 \log(\beta) - 96\zeta_2 \ln(2) \log(\beta) - 48\zeta_2 \log^2(\beta) \Big) \\
& + \frac{1}{\beta} \left(36 - 6\zeta_2 + 12 \ln^2(2) + 24 \ln(2) \log(\beta) + 12 \log^2(\beta) \right) + \frac{3572}{25}\zeta_2 - 36\zeta_2^2 + \frac{112}{5}\zeta_2 \ln(2) \\
& - 96\zeta_2 \ln^2(2) + \frac{112}{5}\zeta_2 \log(\beta) - 192\zeta_2 \ln(2) \log(\beta) - 96\zeta_2 \log^2(\beta) + \beta \left(-\frac{49163}{81} + \frac{1064}{9}\zeta_2 \right. \\
& + \frac{64}{9}\zeta_3 + \frac{19408 \ln(2)}{27} - \frac{16}{3}\zeta_2 \ln(2) - \frac{2656}{9} \ln^2(2) + \frac{32 \ln^3(2)}{9} + \frac{14416 \log(\beta)}{27} \\
& - \frac{16}{3}\zeta_2 \log(\beta) - \frac{3776}{9} \ln(2) \log(\beta) + \frac{32}{3} \ln^2(2) \log(\beta) - \frac{1312}{9} \log^2(\beta) + \frac{32}{3} \ln(2) \log^2(\beta) \\
& + \left. \left. \left. \left. \frac{32 \log^3(\beta)}{9} \right) + \beta^2 \left(-\frac{62992\zeta_2}{1225} - 18\zeta_2^2 - \frac{76}{35}\zeta_2 \ln(2) - 48\zeta_2 \ln^2(2) + \frac{674}{35}\zeta_2 \log(\beta) \right. \right. \right. \\
& - 96\zeta_2 \ln(2) \log(\beta) - 48\zeta_2 \log^2(\beta) \Big) \Big) \Big] \Big\} + C_F C_A \left\{ \frac{1}{\varepsilon^2} \left[-\frac{44\beta^2}{9} + i\pi \left(\frac{11}{6\beta} + \frac{11\beta}{6} \right) \right] \right. \\
& + \frac{1}{\varepsilon} \left[-8\beta\zeta_2 + \beta^2 \left(\frac{376}{27} + \frac{32\zeta_2}{3} \right) + i\pi \left(-\frac{31}{18\beta} + \frac{1}{6}\beta(-13 - 32 \ln(2) - 16 \log(\beta)) \right) \right] \\
& + \left[\frac{1}{\beta} \left(\frac{146}{3}\zeta_2 - 44\zeta_2 \ln(2) - 44\zeta_2 \log(\beta) \right) - \frac{379}{15} + \frac{5482}{75}\zeta_2 - \frac{166}{5}\zeta_3 - \frac{312}{5}\zeta_2 \ln(2) \right. \\
& - \frac{144}{5}\zeta_2 \log(\beta) + \beta \left(-\frac{16}{3}\zeta_2 + 84\zeta_2 \ln(2) + 36\zeta_2 \log(\beta) \right) + \beta^2 \left(\frac{26779}{2835} + 88\zeta_2 - \frac{346}{105}\zeta_3 \right. \\
& - \frac{1952}{35}\zeta_2 \ln(2) - 48\zeta_2 \log(\beta) \Big) + i\pi \left(\frac{1}{\beta} \left(-\frac{394}{27} + \frac{146 \ln(2)}{9} - \frac{22 \ln^2(2)}{3} + \frac{146 \log(\beta)}{9} \right. \right. \\
& - \frac{44}{3} \ln(2) \log(\beta) - \frac{22 \log^2(\beta)}{3} \Big) + \frac{72}{5}\zeta_2 + \beta \left(-\frac{139}{9} - \frac{40}{3}\zeta_2 - \frac{152 \ln(2)}{9} + \frac{74 \ln^2(2)}{3} \right. \\
& - \left. \left. \left. \left. \frac{16 \log(\beta)}{9} + 28 \ln(2) \log(\beta) + 6 \log^2(\beta) \right) + \frac{351}{14}\beta^2\zeta_2 \right) \right] + \varepsilon \left[\frac{1}{\beta} \left(\frac{2422}{9}\zeta_2 + 99\zeta_2^2 \right. \right. \\
& - \frac{752}{3}\zeta_2 \ln(2) + 132\zeta_2 \ln^2(2) - \frac{752}{3}\zeta_2 \log(\beta) + 264\zeta_2 \ln(2) \log(\beta) + 132\zeta_2 \log^2(\beta) \Big) \\
& - \frac{28301}{150} + \frac{28c_1}{5} + \frac{228613\zeta_2}{1125} - \frac{7748}{25}\zeta_2^2 + \frac{8539}{75}\zeta_3 - \frac{1076}{5}\zeta_2 \ln(2) - \frac{256}{5}\zeta_2 \ln^2(2) \\
& - \frac{4144}{25}\zeta_2 \log(\beta) + \frac{576}{5}\zeta_2 \ln(2) \log(\beta) + \frac{288}{5}\zeta_2 \log^2(\beta) + \beta \left(\frac{976}{9}\zeta_2 + 59\zeta_2^2 + \frac{1612}{3}\zeta_2 \ln(2) \right. \\
& - 508\zeta_2 \ln^2(2) + \frac{796}{3}\zeta_2 \log(\beta) - 632\zeta_2 \ln(2) \log(\beta) - 172\zeta_2 \log^2(\beta) \Big) + \beta^2 \left(\frac{39802157}{198450} \right. \\
& - \frac{284c_1}{105} + \frac{5048597\zeta_2}{9450} - \frac{209161\zeta_2^2}{1050} - \frac{244703\zeta_3}{7350} - \frac{323804\zeta_2 \ln(2)}{3675} + \frac{264}{7}\zeta_2 \ln^2(2)
\end{aligned}$$

$$\begin{aligned}
& -\frac{1274}{5}\zeta_2 \log(\beta) + 144\zeta_2 \ln(2) \log(\beta) + 96\zeta_2 \log^2(\beta) \Big) + i\pi \left(\frac{1}{\beta} \left(-\frac{13177}{162} + \frac{38}{3}\zeta_2 + \frac{22}{3}\zeta_3 \right. \right. \\
& + \frac{2422 \ln(2)}{27} - 11\zeta_2 \ln(2) - \frac{376}{9} \ln^2(2) + \frac{44 \ln^3(2)}{3} + \frac{2422 \log(\beta)}{27} - 11\zeta_2 \log(\beta) \\
& - \frac{752}{9} \ln(2) \log(\beta) + 44 \ln^2(2) \log(\beta) - \frac{376}{9} \log^2(\beta) + 44 \ln(2) \log^2(\beta) + \frac{44 \log^3(\beta)}{3} \Big) \\
& + \frac{2072}{25}\zeta_2 - \frac{288}{5}\zeta_2 \ln(2) - \frac{288}{5}\zeta_2 \log(\beta) + \beta \left(-\frac{2950}{27} - \frac{596}{9}\zeta_2 - \frac{194}{3}\zeta_3 - \frac{256 \ln(2)}{27} \right. \\
& + \frac{335}{3}\zeta_2 \ln(2) + 150 \ln^2(2) - \frac{764}{9} \ln^3(2) + \frac{976 \log(\beta)}{27} + 77\zeta_2 \log(\beta) + \frac{1612}{9} \ln(2) \log(\beta) \\
& - \frac{508}{3} \ln^2(2) \log(\beta) + \frac{398 \log^2(\beta)}{9} - \frac{316}{3} \ln(2) \log^2(\beta) - \frac{172}{9} \log^3(\beta) \Big) + \beta^2 \left(\frac{792}{5}\zeta_2 \right. \\
& \left. - \frac{786}{7}\zeta_2 \ln(2) - \frac{657}{7}\zeta_2 \log(\beta) \right) \Big) \Big] \Big\} + C_{FTF} n_l \left\{ \frac{1}{\varepsilon^2} \left[\frac{16\beta^2}{9} + i\pi \left(-\frac{2}{3\beta} - \frac{2\beta}{3} \right) \right] \right. \\
& + \frac{1}{\varepsilon} \left[-\frac{80\beta^2}{27} + i\pi \left(\frac{10}{9\beta} + \frac{10\beta}{9} \right) \right] + \left[\frac{1}{\beta} \left(-\frac{64}{3}\zeta_2 + 16\zeta_2 \ln(2) + 16\zeta_2 \log(\beta) \right) + 4 \right. \\
& + \beta \left(-\frac{64}{3}\zeta_2 + 16\zeta_2 \ln(2) + 16\zeta_2 \log(\beta) \right) + \beta^2 \left(-\frac{868}{81} - \frac{64\zeta_2}{9} \right) + i\pi \left(\frac{1}{\beta} \left(\frac{206}{27} - \frac{64 \ln(2)}{9} \right. \right. \\
& + \frac{8 \ln^2(2)}{3} - \frac{64 \log(\beta)}{9} + \frac{16}{3} \ln(2) \log(\beta) + \frac{8 \log^2(\beta)}{3} \Big) + \beta \left(\frac{116}{27} - \frac{64 \ln(2)}{9} + \frac{8 \ln^2(2)}{3} \right. \\
& \left. \left. - \frac{64 \log(\beta)}{9} + \frac{16}{3} \ln(2) \log(\beta) + \frac{8 \log^2(\beta)}{3} \right) \right] + \varepsilon \left[\frac{1}{\beta} \left(-\frac{1112}{9}\zeta_2 - 36\zeta_2^2 + \frac{304}{3}\zeta_2 \ln(2) \right. \right. \\
& - 48\zeta_2 \ln^2(2) + \frac{304}{3}\zeta_2 \log(\beta) - 96\zeta_2 \ln(2) \log(\beta) - 48\zeta_2 \log^2(\beta) \Big) + \frac{574}{9} + 36\zeta_2 \\
& + \beta \left(-\frac{680}{9}\zeta_2 - 36\zeta_2^2 + \frac{304}{3}\zeta_2 \ln(2) - 48\zeta_2 \ln^2(2) + \frac{304}{3}\zeta_2 \log(\beta) - 96\zeta_2 \ln(2) \log(\beta) \right. \\
& \left. - 48\zeta_2 \log^2(\beta) \right) + \beta^2 \left(-\frac{518}{81} - \frac{220\zeta_2}{27} - \frac{128\zeta_3}{9} \right) + i\pi \left(\frac{1}{\beta} \left(\frac{3211}{81} - \frac{16}{3}\zeta_2 - \frac{8}{3}\zeta_3 - \frac{1112 \ln(2)}{27} \right. \right. \\
& + 4\zeta_2 \ln(2) + \frac{152 \ln^2(2)}{9} - \frac{16}{3} \ln^3(2) - \frac{1112 \log(\beta)}{27} + 4\zeta_2 \log(\beta) + \frac{304}{9} \ln(2) \log(\beta) \\
& - 16 \ln^2(2) \log(\beta) + \frac{152 \log^2(\beta)}{9} - 16 \ln(2) \log^2(\beta) - \frac{16}{3} \log^3(\beta) \Big) + \beta \left(\frac{934}{81} - \frac{16}{3}\zeta_2 - \frac{8}{3}\zeta_3 \right. \\
& \left. - \frac{680 \ln(2)}{27} + 4\zeta_2 \ln(2) + \frac{152 \ln^2(2)}{9} - \frac{16}{3} \ln^3(2) - \frac{680 \log(\beta)}{27} + 4\zeta_2 \log(\beta) \right. \\
& \left. \left. + \frac{304}{9} \ln(2) \log(\beta) - 16 \ln^2(2) \log(\beta) + \frac{152 \log^2(\beta)}{9} - 16 \ln(2) \log^2(\beta) - \frac{16}{3} \log^3(\beta) \right) \right] \Big] \Big\} \\
& + C_{FTF} \left\{ \left[\frac{148}{3} - \frac{416}{15}\zeta_2 + \beta^2 \left(-\frac{1372}{81} + \frac{52\zeta_2}{5} \right) + i\pi \left(\frac{2\zeta_2}{3\beta} + \beta \left(\frac{16}{15} + \frac{2\zeta_2}{3} \right) \right) \right] \right. \\
& \left. + \varepsilon \left[-\frac{4\zeta_2^2}{\beta} + \frac{2102}{15} - \frac{28616}{225}\zeta_2 - \frac{1456}{15}\zeta_3 + \frac{832}{5}\zeta_2 \ln(2) + \beta \left(-\frac{32\zeta_2}{5} - 4\zeta_2^2 \right) \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \beta^2 \left(-\frac{47902}{1215} + \frac{475514\zeta_2}{11025} + \frac{5104}{135}\zeta_3 - \frac{6232}{105}\zeta_2 \ln(2) + \frac{24}{7}\zeta_2 \log(\beta) \right) \\
& + i\pi \left(\frac{1}{\beta} \left(\frac{2}{3}\zeta_2 - \frac{4}{9}\zeta_3 - \frac{4}{3}\zeta_2 \ln(2) - \frac{4}{3}\zeta_2 \log(\beta) \right) + \beta \left(\frac{16}{15} + \frac{2}{3}\zeta_2 - \frac{4}{9}\zeta_3 - \frac{32 \ln(2)}{15} \right. \right. \\
& \left. \left. - \frac{4}{3}\zeta_2 \ln(2) - \frac{32 \log(\beta)}{15} - \frac{4}{3}\zeta_2 \log(\beta) \right) \right) \Bigg\}. \tag{4.36}
\end{aligned}$$

Next, we present the magnetic component of the vector form factor

$$\begin{aligned}
\tilde{F}_{V,2}^{(1)} = C_F & \left\{ -2 + \frac{4\beta^2}{3} + i\pi \left(\frac{1}{\beta} - \beta \right) + \varepsilon \left[-\frac{6\zeta_2}{\beta} - 4 + 6\beta\zeta_2 + \frac{28\beta^2}{9} + i\pi \left(-\frac{1}{\beta} \left(2(-2 \right. \right. \right. \right. \\
& \left. \left. \left. + \ln(2) + \log(\beta) \right) \right) + \beta(-5 + 2 \ln(2) + 2 \log(\beta)) \right] + \varepsilon^2 \left[\frac{1}{\beta} \left(-24\zeta_2 + 12\zeta_2 \ln(2) \right. \right. \right. \\
& \left. \left. \left. + 12\zeta_2 \log(\beta) \right) - 8 - \zeta_2 + \beta \left(30\zeta_2 - 12\zeta_2 \ln(2) - 12\zeta_2 \log(\beta) \right) + \beta^2 \left(\frac{160}{27} + \frac{2\zeta_2}{3} \right) \right. \right. \\
& \left. \left. + i\pi \left(\frac{1}{\beta} \left(8 - \frac{3}{2}\zeta_2 - 8 \ln(2) + 2 \ln^2(2) - 8 \log(\beta) + 4 \ln(2) \log(\beta) + 2 \log^2(\beta) \right) \right. \right. \right. \\
& \left. \left. \left. + \beta \left(-12 + \frac{3}{2}\zeta_2 + 10 \ln(2) - 2 \ln^2(2) + 10 \log(\beta) - 4 \ln(2) \log(\beta) - 2 \log^2(\beta) \right) \right) \right] \right\}. \tag{4.37}
\end{aligned}$$

$$\begin{aligned}
\tilde{F}_{V,2}^{(2)} = C_F^2 & \left\{ \frac{1}{\varepsilon} \left[\frac{6\zeta_2}{\beta^2} + \beta^2 \left(-\frac{16}{3} - 6\zeta_2 \right) + i\pi \left(\frac{2}{\beta} + \frac{10\beta}{3} \right) \right] + \left[\frac{1}{\beta^2} \left(28\zeta_2 - 24\zeta_2 \ln(2) \right. \right. \right. \\
& \left. \left. \left. - 24\zeta_2 \log(\beta) \right) + \frac{269}{15} - \frac{4922}{75}\zeta_2 - \frac{12\zeta_2}{\beta} - 60\beta\zeta_2 + \frac{41}{5}\zeta_3 + \frac{404}{5}\zeta_2 \ln(2) + \frac{24}{5}\zeta_2 \log(\beta) \right. \right. \\
& \left. \left. + \beta^2 \left(\frac{3658}{315} + \frac{103928\zeta_2}{3675} - \frac{471}{35}\zeta_3 - \frac{1628}{35}\zeta_2 \ln(2) + \frac{512}{35}\zeta_2 \log(\beta) \right) + i\pi \left(\frac{12\zeta_2}{\beta^2} - \frac{1}{\beta} \left(4(\ln(2) \right. \right. \right. \right. \\
& \left. \left. \left. + \log(\beta) \right) \right) - \frac{12}{5}\zeta_2 - \frac{2}{9}\beta(-73 + 90 \ln(2) + 90 \log(\beta)) - \frac{181}{35}\beta^2\zeta_2 \right] + \varepsilon \left[\frac{1}{\beta^2} \left(28\zeta_2 - 30\zeta_2^2 \right. \right. \right. \\
& \left. \left. \left. - 112\zeta_2 \ln(2) + 48\zeta_2 \ln^2(2) - 112\zeta_2 \log(\beta) + 96\zeta_2 \ln(2) \log(\beta) + 48\zeta_2 \log^2(\beta) \right) \right. \right. \\
& \left. \left. + \frac{1}{\beta} \left(24 \ln(2)\zeta_2 + 24 \log(\beta)\zeta_2 \right) + \frac{20713}{450} - \frac{38c_1}{3} - \frac{529628\zeta_2}{1125} + \frac{1342}{5}\zeta_2^2 - \frac{15944}{75}\zeta_3 \right. \right. \\
& \left. \left. + \frac{27752}{75}\zeta_2 \ln(2) + \frac{448}{5}\zeta_2 \ln^2(2) + \frac{344}{25}\zeta_2 \log(\beta) - \frac{96}{5}\zeta_2 \ln(2) \log(\beta) - \frac{48}{5}\zeta_2 \log^2(\beta) \right. \right. \\
& \left. \left. + \beta \left(-\frac{200}{3}\zeta_2 + 280\zeta_2 \ln(2) + 280\zeta_2 \log(\beta) \right) + \beta^2 \left(-\frac{4863529}{33075} + \frac{250c_1}{21} + \frac{115201589\zeta_2}{385875} \right. \right. \right. \\
& \left. \left. \left. - \frac{42283}{175}\zeta_2^2 + \frac{330121\zeta_3}{1225} - \frac{749248\zeta_2 \ln(2)}{3675} - \frac{3672}{35}\zeta_2 \ln^2(2) + \frac{245324\zeta_2 \log(\beta)}{1225} \right) \right. \right. \\
& \left. \left. - \frac{1328}{35}\zeta_2 \ln(2) \log(\beta) - \frac{1024}{35}\zeta_2 \log^2(\beta) \right) + i\pi \left(\frac{1}{\beta^2} \left(56\zeta_2 - 48\zeta_2 \ln(2) - 48\zeta_2 \log(\beta) \right) \right. \right. \\
& \left. \left. + \frac{1}{\beta} \left(-4 - 2\zeta_2 + 4 \ln^2(2) + 8 \ln(2) \log(\beta) + 4 \log^2(\beta) \right) - \frac{172}{25}\zeta_2 + \frac{48}{5}\zeta_2 \ln(2) + \frac{48}{5}\zeta_2 \log(\beta) \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \beta \left(-\frac{241}{9} - \frac{50}{3}\zeta_2 - \frac{8 \ln(2)}{9} + \frac{140 \ln^2(2)}{3} - \frac{200 \log(\beta)}{9} + \frac{280}{3} \ln(2) \log(\beta) + \frac{140 \log^2(\beta)}{3} \right) \\
& + \beta^2 \left(-\frac{147512\zeta_2}{1225} + \frac{1924}{35}\zeta_2 \ln(2) + \frac{1174}{35}\zeta_2 \log(\beta) \right) \Bigg] \Bigg\} + C_F C_A \left\{ \left[-\frac{28\zeta_2}{\beta} - \frac{373}{45} \right. \right. \\
& + \frac{1156}{25}\zeta_2 - \frac{94}{5}\zeta_3 - \frac{328}{5}\zeta_2 \ln(2) - \frac{96}{5}\zeta_2 \log(\beta) + 60\beta\zeta_2 + \beta^2 \left(-\frac{1234}{315} - \frac{1577}{25}\zeta_2 + \frac{288}{35}\zeta_3 \right. \\
& + \frac{2344}{35}\zeta_2 \ln(2) + \frac{32}{5}\zeta_2 \log(\beta) \Big) + i\pi \left(-\frac{1}{9\beta} \left(4(-25 + 21 \ln(2) + 21 \log(\beta)) \right) + \frac{48}{5}\zeta_2 \right. \\
& + \frac{2}{3}\beta(-49 + 38 \ln(2) + 30 \log(\beta)) - \frac{299}{70}\beta^2\zeta_2 \Big) \Bigg] + \varepsilon \left[\frac{1}{\beta} \left(-\frac{682}{3}\zeta_2 + 168\zeta_2 \ln(2) \right. \right. \\
& + 168\zeta_2 \log(\beta) \Big) - \frac{181841}{1350} + \frac{116c_1}{15} + \frac{104129}{375}\zeta_2 - \frac{6652}{25}\zeta_2^2 + \frac{6236}{75}\zeta_3 - \frac{4112}{15}\zeta_2 \ln(2) \\
& - \frac{224}{5}\zeta_2 \ln^2(2) - \frac{1856}{25}\zeta_2 \log(\beta) + \frac{384}{5}\zeta_2 \ln(2) \log(\beta) + \frac{192}{5}\zeta_2 \log^2(\beta) + \beta \left(\frac{1822}{3}\zeta_2 \right. \\
& - 520\zeta_2 \ln(2) - 424\zeta_2 \log(\beta) \Big) + \beta^2 \left(\frac{16374871}{99225} - \frac{128c_1}{21} - \frac{975547\zeta_2}{5250} + \frac{82843}{350}\zeta_2^2 - \frac{458029\zeta_3}{3675} \right. \\
& - \frac{71626\zeta_2 \ln(2)}{3675} + \frac{4056}{35}\zeta_2 \ln^2(2) - \frac{638}{25}\zeta_2 \log(\beta) + \frac{112}{5}\zeta_2 \ln(2) \log(\beta) - \frac{64}{5}\zeta_2 \log^2(\beta) \Big) \\
& + i\pi \left(\frac{1}{\beta} \left(\frac{4015}{54} - \frac{15}{2}\zeta_2 - \frac{682 \ln(2)}{9} + 28 \ln^2(2) - \frac{682 \log(\beta)}{9} + 56 \ln(2) \log(\beta) + 28 \log^2(\beta) \right) \right. \\
& + \frac{928}{25}\zeta_2 - \frac{192}{5}\zeta_2 \ln(2) - \frac{192}{5}\zeta_2 \log(\beta) + \beta \left(-\frac{12113}{54} + \frac{79}{2}\zeta_2 + \frac{2270 \ln(2)}{9} - 108 \ln^2(2) \right. \\
& + \frac{1822 \log(\beta)}{9} - \frac{520}{3} \ln(2) \log(\beta) - \frac{212}{3} \log^2(\beta) \Big) + \beta^2 \left(-\frac{456}{25}\zeta_2 + \frac{1018}{35}\zeta_2 \ln(2) \right. \\
& + \frac{373}{35}\zeta_2 \log(\beta) \Big) \Bigg] \Bigg\} + C_F T_F n_l \left\{ \left[\frac{8\zeta_2}{\beta} + \frac{52}{9} - 8\beta\zeta_2 - \frac{40\beta^2}{9} + i\pi \left(\frac{1}{9\beta} \left(2(-25 + 12 \ln(2) \right. \right. \right. \right. \\
& + 12 \log(\beta)) \Big) - \frac{2}{9}\beta(-31 + 12 \ln(2) + 12 \log(\beta)) \Big) \Bigg] + \varepsilon \left[\frac{1}{\beta} \left(\frac{296}{3}\zeta_2 - 48\zeta_2 \ln(2) \right. \right. \\
& - 48\zeta_2 \log(\beta) \Big) + \frac{1010}{27} + 12\zeta_2 + \beta \left(-\frac{368}{3}\zeta_2 + 48\zeta_2 \ln(2) + 48\zeta_2 \log(\beta) \right) + \beta^2 \left(-\frac{2228}{81} \right. \\
& - 8\zeta_2 \Big) + i\pi \left(\frac{1}{\beta} \left(-\frac{961}{27} + 2\zeta_2 + \frac{296 \ln(2)}{9} - 8 \ln^2(2) + \frac{296 \log(\beta)}{9} - 16 \ln(2) \log(\beta) \right. \right. \\
& - 8 \log^2(\beta) \Big) + \beta \left(\frac{1405}{27} - 2\zeta_2 - \frac{368 \ln(2)}{9} + 8 \ln^2(2) - \frac{368 \log(\beta)}{9} + 16 \ln(2) \log(\beta) \right. \\
& + 8 \log^2(\beta) \Big) \Bigg] \Bigg\} + C_F T_F \left\{ \left[-\frac{92}{9} + \frac{32}{5}\zeta_2 + \beta^2 \left(\frac{248}{27} - \frac{36\zeta_2}{5} \right) \right] + \varepsilon \left[\frac{82}{135} + \frac{324}{25}\zeta_2 \right. \right. \\
& + \frac{112}{5}\zeta_3 - \frac{192}{5}\zeta_2 \ln(2) + \beta^2 \left(\frac{772}{405} - \frac{171322\zeta_2}{11025} - \frac{96}{5}\zeta_3 + \frac{1032}{35}\zeta_2 \ln(2) - \frac{24}{7}\zeta_2 \log(\beta) \right) \\
& + i\pi \left(-\frac{2\zeta_2}{3\beta} + \beta \left(\frac{16}{15} + \frac{2\zeta_2}{3} \right) \right) \Bigg] \Bigg\}. \tag{4.38}
\end{aligned}$$

4.3.2 Axial-vector form factor

In the following, we provide the non-singlet part of the axial-vector form factors in the threshold region

$$\begin{aligned}
\tilde{F}_{A,1}^{(1),\text{ns}} = & C_F \left\{ \frac{1}{\varepsilon} \left[\frac{8\beta^2}{3} + i\pi \left(-\frac{1}{\beta} - \beta \right) \right] + \left[\frac{6\zeta_2}{\beta} - 4 + 6\beta\zeta_2 - \frac{22\beta^2}{9} + i\pi \left(\frac{1}{\beta} (2(-1 + \ln(2) \right. \right. \right. \\
& + \log(\beta)) \left. \left. \left. + 2\beta(\ln(2) + \log(\beta)) \right) \right] + \varepsilon \left[\frac{1}{\beta} (12\zeta_2 - 12\zeta_2 \ln(2) - 12\zeta_2 \log(\beta)) \right. \right. \\
& + \beta \left(-12 \ln(2)\zeta_2 - 12 \log(\beta)\zeta_2 \right) + \beta^2 \left(\frac{404}{27} + \frac{4\zeta_2}{3} \right) + i\pi \left(\frac{1}{\beta} \left(-4 + \frac{3}{2}\zeta_2 + 4 \ln(2) \right. \right. \\
& - 2 \ln^2(2) + 4 \log(\beta) - 4 \ln(2) \log(\beta) - 2 \log^2(\beta) \left. \left. \right) + \beta \left(-2 + \frac{3}{2}\zeta_2 - 2 \ln^2(2) \right. \right. \\
& \left. \left. - 4 \ln(2) \log(\beta) - 2 \log^2(\beta) \right) \right] + \varepsilon^2 \left[\frac{1}{\beta} (24\zeta_2 + 3\zeta_2^2 - 24\zeta_2 \ln(2) + 12\zeta_2 \ln^2(2) \right. \right. \\
& - 24\zeta_2 \log(\beta) + 24\zeta_2 \ln(2) \log(\beta) + 12\zeta_2 \log^2(\beta) \left. \left. \right) - 16 - 2\zeta_2 + \beta (12\zeta_2 + 3\zeta_2^2 \right. \\
& + 12\zeta_2 \ln^2(2) + 24\zeta_2 \ln(2) \log(\beta) + 12\zeta_2 \log^2(\beta) \left. \right) + \beta^2 \left(-\frac{808}{81} - \frac{11\zeta_2}{9} - \frac{8\zeta_3}{9} \right) \\
& + i\pi \left(\frac{1}{\beta} \left(-8 + 3\zeta_2 + \frac{7}{3}\zeta_3 + 8 \ln(2) - 3\zeta_2 \ln(2) - 4 \ln^2(2) + \frac{4 \ln^3(2)}{3} + 8 \log(\beta) \right. \right. \\
& - 3\zeta_2 \log(\beta) - 8 \ln(2) \log(\beta) + 4 \ln^2(2) \log(\beta) - 4 \log^2(\beta) + 4 \ln(2) \log^2(\beta) + \frac{4 \log^3(\beta)}{3} \left. \left. \right) \right. \\
& + \beta \left(-4 + \frac{7}{3}\zeta_3 + 4 \ln(2) - 3\zeta_2 \ln(2) + \frac{4 \ln^3(2)}{3} + 4 \log(\beta) - 3\zeta_2 \log(\beta) + 4 \ln^2(2) \log(\beta) \right. \\
& \left. \left. + 4 \ln(2) \log^2(\beta) + \frac{4 \log^3(\beta)}{3} \right) \right] \left. \right\}. \tag{4.39}
\end{aligned}$$

$$\begin{aligned}
\tilde{F}_{A,1}^{(2),\text{ns}} = & C_F^2 \left\{ \frac{1}{\varepsilon^2} \left[-\frac{3\zeta_2}{\beta^2} - 6\zeta_2 - 3\beta^2\zeta_2 + i\pi \left(-\frac{8\beta}{3} \right) \right] + \frac{1}{\varepsilon} \left[\frac{1}{\beta^2} \left(-12\zeta_2 + 12\zeta_2 \ln(2) \right. \right. \right. \\
& + 12\zeta_2 \log(\beta) \left. \left. \right) - 12\zeta_2 + 16\beta\zeta_2 + 24\zeta_2 \ln(2) + 24\zeta_2 \log(\beta) + \beta^2 \left(-\frac{32}{3} + 9\zeta_2 + 12\zeta_2 \ln(2) \right. \right. \\
& + 12\zeta_2 \log(\beta) \left. \left. \right) + i\pi \left(-\frac{6\zeta_2}{\beta^2} + \frac{4}{\beta} - 12\zeta_2 + \frac{2}{9}\beta(5 + 24 \ln(2) + 24 \log(\beta)) - 6\beta^2\zeta_2 \right) \right] \\
& + \left[\frac{1}{\beta^2} \left(-24\zeta_2 + 15\zeta_2^2 + 48\zeta_2 \ln(2) - 24\zeta_2 \ln^2(2) + 48\zeta_2 \log(\beta) - 48\zeta_2 \ln(2) \log(\beta) \right. \right. \\
& - 24\zeta_2 \log^2(\beta) \left. \left. \right) + \frac{46}{3} - 32\zeta_2 - \frac{24\zeta_2}{\beta} + 30\zeta_2^2 - 27\zeta_3 + 84\zeta_2 \ln(2) - 48\zeta_2 \ln^2(2) + 8\zeta_2 \log(\beta) \right. \\
& - 96\zeta_2 \ln(2) \log(\beta) - 48\zeta_2 \log^2(\beta) + \beta \left(-\frac{20}{3}\zeta_2 - 32\zeta_2 \ln(2) - 32\zeta_2 \log(\beta) \right) + \beta^2 \left(\frac{247}{9} \right. \\
& \left. - \frac{1508}{75}\zeta_2 + 15\zeta_2^2 + 50\zeta_3 - \frac{172}{5}\zeta_2 \ln(2) - 24\zeta_2 \ln^2(2) - \frac{4}{5}\zeta_2 \log(\beta) - 48\zeta_2 \ln(2) \log(\beta) \right) \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& - 24\zeta_2 \log^2(\beta) \Big) + i\pi \left(\frac{1}{\beta^2} \left(- 24\zeta_2 + 24\zeta_2 \ln(2) + 24\zeta_2 \log(\beta) \right) - \frac{1}{\beta} \left(8(-1 + \ln(2) \right. \right. \\
& \left. \left. + \log(\beta)) \right) - 4\zeta_2 + 48\zeta_2 \ln(2) + 48\zeta_2 \log(\beta) + \beta \left(- \frac{956}{27} + \frac{8}{3}\zeta_2 - \frac{20 \ln(2)}{9} - \frac{16}{3} \ln^2(2) \right. \right. \\
& \left. \left. - \frac{20 \log(\beta)}{9} - \frac{32}{3} \ln(2) \log(\beta) - \frac{16}{3} \log^2(\beta) \right) + \beta^2 \left(\frac{2}{5}\zeta_2 + 24\zeta_2 \ln(2) + 24\zeta_2 \log(\beta) \right) \right) \Big] \\
& + \varepsilon \left[\frac{1}{\beta^2} \left(48\zeta_2 + 60\zeta_2^2 - 106\zeta_2\zeta_3 + 96\zeta_2 \ln(2) - 60\zeta_2^2 \ln(2) - 96\zeta_2 \ln^2(2) + 32\zeta_2 \ln^3(2) \right. \right. \\
& \left. \left. + 96\zeta_2 \log(\beta) - 60\zeta_2^2 \log(\beta) - 192\zeta_2 \ln(2) \log(\beta) + 96\zeta_2 \ln^2(2) \log(\beta) - 96\zeta_2 \log^2(\beta) \right. \right. \\
& \left. \left. + 96\zeta_2 \ln(2) \log^2(\beta) + 32\zeta_2 \log^3(\beta) \right) + \frac{1}{\beta} \left(- 48\zeta_2 + 48\zeta_2 \ln(2) + 48\zeta_2 \log(\beta) \right) - \frac{313}{9} \right. \\
& \left. - \frac{38c_1}{3} + \frac{940}{3}\zeta_2 + \frac{282}{5}\zeta_2^2 - \frac{1202}{3}\zeta_3 - 212\zeta_2\zeta_3 - \frac{424}{3}\zeta_2 \ln(2) - 120\zeta_2^2 \ln(2) + 32\zeta_2 \ln^2(2) \right. \\
& \left. + 64\zeta_2 \ln^3(2) - 304\zeta_2 \log(\beta) - 120\zeta_2^2 \log(\beta) - 32\zeta_2 \ln(2) \log(\beta) + 192\zeta_2 \ln^2(2) \log(\beta) \right. \\
& \left. - 16\zeta_2 \log^2(\beta) + 192\zeta_2 \ln(2) \log^2(\beta) + 64\zeta_2 \log^3(\beta) + \beta \left(\frac{3964}{9}\zeta_2 + 16\zeta_2^2 + \frac{40}{3}\zeta_2 \ln(2) \right. \right. \\
& \left. \left. + 32\zeta_2 \ln^2(2) + \frac{40}{3}\zeta_2 \log(\beta) + 64\zeta_2 \ln(2) \log(\beta) + 32\zeta_2 \log^2(\beta) \right) + \beta^2 \left(- \frac{35237}{270} + \frac{28c_1}{5} \right. \right. \\
& \left. \left. - \frac{117812\zeta_2}{1125} + \frac{3639}{25}\zeta_2^2 + \frac{724}{15}\zeta_3 - 106\zeta_2\zeta_3 - \frac{624}{5}\zeta_2 \ln(2) - 60\zeta_2^2 \ln(2) + \frac{488}{5}\zeta_2 \ln^2(2) \right. \right. \\
& \left. \left. + 32\zeta_2 \ln^3(2) + \frac{3696}{25}\zeta_2 \log(\beta) - 60\zeta_2^2 \log(\beta) + \frac{16}{5}\zeta_2 \ln(2) \log(\beta) + 96\zeta_2 \ln^2(2) \log(\beta) \right. \right. \\
& \left. \left. + \frac{8}{5}\zeta_2 \log^2(\beta) + 96\zeta_2 \ln(2) \log^2(\beta) + 32\zeta_2 \log^3(\beta) \right) + i\pi \left(\frac{1}{\beta^2} \left(- 48\zeta_2 - 18\zeta_2^2 + 96\zeta_2 \ln(2) \right. \right. \right. \\
& \left. \left. - 48\zeta_2 \ln^2(2) + 96\zeta_2 \log(\beta) - 96\zeta_2 \ln(2) \log(\beta) - 48\zeta_2 \log^2(\beta) \right) + \frac{1}{\beta} \left(32 - 4\zeta_2 - 16 \ln(2) \right. \right. \\
& \left. \left. + 8 \ln^2(2) - 16 \log(\beta) + 16 \ln(2) \log(\beta) + 8 \log^2(\beta) \right) + 152\zeta_2 - 36\zeta_2^2 + 16\zeta_2 \ln(2) \right. \\
& \left. - 96\zeta_2 \ln^2(2) + 16\zeta_2 \log(\beta) - 192\zeta_2 \ln(2) \log(\beta) - 96\zeta_2 \log^2(\beta) + \beta \left(- \frac{18635}{81} - \frac{10}{9}\zeta_2 \right. \right. \\
& \left. \left. + \frac{64}{9}\zeta_3 + \frac{5116 \ln(2)}{27} - \frac{16}{3}\zeta_2 \ln(2) + \frac{20 \ln^2(2)}{9} + \frac{32 \ln^3(2)}{9} + \frac{3964 \log(\beta)}{27} - \frac{16}{3}\zeta_2 \log(\beta) \right. \right. \\
& \left. \left. + \frac{40}{9} \ln(2) \log(\beta) + \frac{32}{3} \ln^2(2) \log(\beta) + \frac{20 \log^2(\beta)}{9} + \frac{32}{3} \ln(2) \log^2(\beta) + \frac{32 \log^3(\beta)}{9} \right) \right. \\
& \left. + \beta^2 \left(- \frac{1848}{25}\zeta_2 - 18\zeta_2^2 - \frac{8}{5}\zeta_2 \ln(2) - 48\zeta_2 \ln^2(2) - \frac{8}{5}\zeta_2 \log(\beta) - 96\zeta_2 \ln(2) \log(\beta) \right. \right. \\
& \left. \left. - 48\zeta_2 \log^2(\beta) \right) \right) \Big] \Big\} + C_F C_A \left\{ \frac{1}{\varepsilon^2} \left[- \frac{44\beta^2}{9} + i\pi \left(\frac{11}{6\beta} + \frac{11\beta}{6} \right) \right] + \frac{1}{\varepsilon} \left[- 8\beta\zeta_2 + \beta^2 \left(\frac{376}{27} \right. \right. \right. \\
& \left. \left. + \frac{32\zeta_2}{3} \right) + i\pi \left(- \frac{31}{18\beta} + \frac{1}{6}\beta(-13 - 32 \ln(2) - 16 \log(\beta)) \right) \right] + \left[\frac{1}{\beta} \left(\frac{194}{3}\zeta_2 - 44\zeta_2 \ln(2) \right. \right. \right. \\
& \left. \left. - 44\zeta_2 \log(\beta) \right) - \frac{202}{9} + \frac{178}{3}\zeta_2 - 18\zeta_3 - 72\zeta_2 \ln(2) - 16\zeta_2 \log(\beta) + \beta \left(\frac{8}{3}\zeta_2 + 84\zeta_2 \ln(2) \right. \right. \\
& \left. \left. + 36\zeta_2 \log(\beta) \right) + \beta^2 \left(\frac{1159}{405} + \frac{6263}{75}\zeta_2 - \frac{346}{15}\zeta_3 - \frac{216}{5}\zeta_2 \ln(2) - \frac{336}{5}\zeta_2 \log(\beta) \right) \right] \Big\}
\end{aligned}$$

$$\begin{aligned}
& + i\pi \left(\frac{1}{\beta} \left(-\frac{478}{27} + \frac{194 \ln(2)}{9} - \frac{22 \ln^2(2)}{3} + \frac{194 \log(\beta)}{9} - \frac{44}{3} \ln(2) \log(\beta) - \frac{22 \log^2(\beta)}{3} \right) \right. \\
& + 8\zeta_2 + \beta \left(-\frac{143}{9} - \frac{40}{3} \zeta_2 - \frac{80 \ln(2)}{9} + \frac{74 \ln^2(2)}{3} + \frac{8 \log(\beta)}{9} + 28 \ln(2) \log(\beta) + 6 \log^2(\beta) \right) \\
& \left. + \frac{168}{5} \beta^2 \zeta_2 \right) + \varepsilon \left[\frac{1}{\beta} \left(\frac{2704}{9} \zeta_2 + 99 \zeta_2^2 - \frac{1040}{3} \zeta_2 \ln(2) + 132 \zeta_2 \ln^2(2) - \frac{1040}{3} \zeta_2 \log(\beta) \right. \right. \\
& + 264 \zeta_2 \ln(2) \log(\beta) + 132 \zeta_2 \log^2(\beta) \left. \right) - \frac{4103}{27} + \frac{28c_1}{3} + \frac{2294}{9} \zeta_2 - \frac{1404}{5} \zeta_2^2 + 73 \zeta_3 \\
& - \frac{940}{3} \zeta_2 \ln(2) - 64 \zeta_2 \ln^2(2) - 144 \zeta_2 \log(\beta) + 64 \zeta_2 \ln(2) \log(\beta) + 32 \zeta_2 \log^2(\beta) + \beta \left(\frac{1702}{9} \zeta_2 \right. \\
& + 59 \zeta_2^2 + \frac{988}{3} \zeta_2 \ln(2) - 508 \zeta_2 \ln^2(2) + \frac{460}{3} \zeta_2 \log(\beta) - 632 \zeta_2 \ln(2) \log(\beta) - 172 \zeta_2 \log^2(\beta) \left. \right) \\
& + \beta^2 \left(\frac{361877}{1350} - 4c_1 + \frac{2123531 \zeta_2}{3375} - \frac{17228}{75} \zeta_2^2 - \frac{8419}{150} \zeta_3 - \frac{6966}{25} \zeta_2 \ln(2) + \frac{576}{5} \zeta_2 \ln^2(2) \right. \\
& - \frac{9536}{25} \zeta_2 \log(\beta) + \frac{1344}{5} \zeta_2 \ln(2) \log(\beta) + \frac{672}{5} \zeta_2 \log^2(\beta) \left. \right) + i\pi \left(\frac{1}{\beta} \left(-\frac{6338}{81} + \frac{97}{6} \zeta_2 + \frac{22}{3} \zeta_3 \right. \right. \\
& + \frac{2704 \ln(2)}{27} - 11 \zeta_2 \ln(2) - \frac{520}{9} \ln^2(2) + \frac{44 \ln^3(2)}{3} + \frac{2704 \log(\beta)}{27} - 11 \zeta_2 \log(\beta) \\
& - \frac{1040}{9} \ln(2) \log(\beta) + 44 \ln^2(2) \log(\beta) - \frac{520}{9} \log^2(\beta) + 44 \ln(2) \log^2(\beta) + \frac{44 \log^3(\beta)}{3} \left. \right) \\
& + 72 \zeta_2 - 32 \zeta_2 \ln(2) - 32 \zeta_2 \log(\beta) + \beta \left(-\frac{9149}{54} - \frac{727}{18} \zeta_2 - \frac{194}{3} \zeta_3 + \frac{1238 \ln(2)}{27} + \frac{335}{3} \zeta_2 \ln(2) \right. \\
& + 94 \ln^2(2) - \frac{764}{9} \ln^3(2) + \frac{1702 \log(\beta)}{27} + 77 \zeta_2 \log(\beta) + \frac{988}{9} \ln(2) \log(\beta) - \frac{508}{3} \ln^2(2) \log(\beta) \\
& + \frac{230 \log^2(\beta)}{9} - \frac{316}{3} \ln(2) \log^2(\beta) - \frac{172}{9} \log^3(\beta) \left. \right) + \beta^2 \left(\frac{4768}{25} \zeta_2 - \frac{672}{5} \zeta_2 \ln(2) \right. \\
& \left. - \frac{672}{5} \zeta_2 \log(\beta) \right) \left. \right] + C_F T_F n_l \left\{ \frac{1}{\varepsilon^2} \left[\frac{16\beta^2}{9} + i\pi \left(-\frac{2}{3\beta} - \frac{2\beta}{3} \right) \right] + \frac{1}{\varepsilon} \left[-\frac{80\beta^2}{27} \right. \right. \\
& + i\pi \left(\frac{10}{9\beta} + \frac{10\beta}{9} \right) \left. \right] + \left[\frac{1}{\beta} \left(-\frac{88}{3} \zeta_2 + 16 \zeta_2 \ln(2) + 16 \zeta_2 \log(\beta) \right) + \frac{56}{9} + \beta \left(-\frac{40}{3} \zeta_2 \right. \right. \\
& + 16 \zeta_2 \ln(2) + 16 \zeta_2 \log(\beta) \left. \right) + \beta^2 \left(-\frac{940}{81} - \frac{64 \zeta_2}{9} \right) + i\pi \left(\frac{1}{\beta} \left(\frac{248}{27} - \frac{88 \ln(2)}{9} + \frac{8 \ln^2(2)}{3} \right. \right. \\
& - \frac{88 \log(\beta)}{9} + \frac{16}{3} \ln(2) \log(\beta) + \frac{8 \log^2(\beta)}{3} \left. \right) + \beta \left(\frac{38}{27} - \frac{40 \ln(2)}{9} + \frac{8 \ln^2(2)}{3} - \frac{40 \log(\beta)}{9} \right. \\
& \left. \left. + \frac{16}{3} \ln(2) \log(\beta) + \frac{8 \log^2(\beta)}{3} \right) \right] + \varepsilon \left[\frac{1}{\beta} \left(-\frac{1280}{9} \zeta_2 - 36 \zeta_2^2 + \frac{448}{3} \zeta_2 \ln(2) - 48 \zeta_2 \ln^2(2) \right. \right. \\
& + \frac{448}{3} \zeta_2 \log(\beta) - 96 \zeta_2 \ln(2) \log(\beta) - 48 \zeta_2 \log^2(\beta) \left. \right) + \frac{1228}{27} + 24 \zeta_2 + \beta \left(-\frac{296}{9} \zeta_2 - 36 \zeta_2^2 \right. \\
& + \frac{160}{3} \zeta_2 \ln(2) - 48 \zeta_2 \ln^2(2) + \frac{160}{3} \zeta_2 \log(\beta) - 96 \zeta_2 \ln(2) \log(\beta) - 48 \zeta_2 \log^2(\beta) \left. \right) + \beta^2 \left(\frac{22}{9} \right. \\
& \left. - \frac{4 \zeta_2}{27} - \frac{128 \zeta_3}{9} \right) + i\pi \left(\frac{1}{\beta} \left(\frac{3160}{81} - \frac{22}{3} \zeta_2 - \frac{8}{3} \zeta_3 - \frac{1280 \ln(2)}{27} + 4 \zeta_2 \ln(2) + \frac{224 \ln^2(2)}{9} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{16}{3}\ln^3(2) - \frac{1280\overline{\log(\beta)}}{27} + 4\zeta_2\log(\beta) + \frac{448}{9}\ln(2)\log(\beta) - 16\ln^2(2)\log(\beta) + \frac{224\log^2(\beta)}{9} \\
& - 16\ln(2)\log^2(\beta) - \frac{16}{3}\log^3(\beta) \Big) + \beta \left(\frac{733}{81} - \frac{10}{3}\zeta_2 - \frac{8}{3}\zeta_3 - \frac{296\ln(2)}{27} + 4\zeta_2\ln(2) + \frac{80\ln^2(2)}{9} \right. \\
& \left. - \frac{16}{3}\ln^3(2) - \frac{296\log(\beta)}{27} + 4\zeta_2\log(\beta) + \frac{160}{9}\ln(2)\log(\beta) - 16\ln^2(2)\log(\beta) + \frac{80\log^2(\beta)}{9} \right. \\
& \left. - 16\ln(2)\log^2(\beta) - \frac{16}{3}\log^3(\beta) \right) \Big) \Big] \Big\} + C_F T_F \left\{ \left[\frac{320}{9} - \frac{64}{3}\zeta_2 + \beta^2 \left(-\frac{388}{81} + \frac{16\zeta_2}{5} \right) \right. \right. \\
& \left. \left. + i\pi \left(\frac{2\zeta_2}{3\beta} + \frac{2}{3}\beta\zeta_2 \right) \right] + \varepsilon \left[-\frac{4\zeta_2^2}{\beta} + \frac{2944}{27} - \frac{884}{9}\zeta_2 - \frac{224}{3}\zeta_3 + 128\zeta_2\ln(2) - 4\beta\zeta_2^2 \right. \right. \\
& \left. \left. + \beta^2 \left(-\frac{18106}{1215} + \frac{1256}{75}\zeta_2 + \frac{2512}{135}\zeta_3 - \frac{448}{15}\zeta_2\ln(2) \right) + i\pi \left(\frac{1}{\beta} \left(\frac{4}{3}\zeta_2 - \frac{4}{9}\zeta_3 - \frac{4}{3}\zeta_2\ln(2) \right) \right. \right. \right. \\
& \left. \left. \left. - \frac{4}{3}\zeta_2\log(\beta) \right) + \beta \left(-\frac{4}{9}\zeta_3 - \frac{4}{3}\zeta_2\ln(2) - \frac{4}{3}\zeta_2\log(\beta) \right) \right] \right] \Big\}. \tag{4.40}
\end{aligned}$$

The unrenormalized singlet part is given by

$$\begin{aligned}
\tilde{F}_{A,1}^{(2),s} &= C_F T_F \left\{ -\frac{6}{\varepsilon} - \frac{6}{\beta^2} - \frac{53}{3} - \frac{76}{3}\zeta_2 + 64\zeta_2\ln(2) + \beta^2 \left(\frac{32}{3} + \frac{44}{15}\zeta_2 - \frac{256}{5}\zeta_2\ln(2) \right) \right. \\
& + \varepsilon \left[\frac{1}{\beta^2} \left(-15 + 20\zeta_2 - 42\zeta_3 + 16\ln(2) + 48\zeta_2\ln(2) \right) - \frac{48\zeta_2}{\beta} - \frac{1169}{18} - \frac{32c_1}{3} - \frac{1424}{9}\zeta_2 \right. \\
& + \frac{944}{5}\zeta_2^2 - \frac{98}{3}\zeta_3 + \frac{32\ln(2)}{3} - \frac{88}{3}\zeta_2\ln(2) + 64\zeta_2\ln^2(2) + 48\beta\zeta_2 + \beta^2 \left(\frac{2528}{45} + \frac{128c_1}{15} \right. \\
& \left. \left. + \frac{15374}{225}\zeta_2 - \frac{3776}{25}\zeta_2^2 - \frac{14}{15}\zeta_3 - \frac{176\ln(2)}{15} - \frac{4936}{75}\zeta_2\ln(2) - \frac{256}{5}\zeta_2\ln^2(2) \right) \right. \\
& \left. + i\pi \left(\frac{1}{\beta^2} \left(-8 + 12\zeta_2 \right) - \frac{1}{\beta} \left(16(-1 + \ln(2)) \right) - \frac{16}{3} - 16\zeta_2 + \beta \left(-\frac{64}{3} + 16\ln(2) \right) \right) \right. \\
& \left. \left. + \beta^2 \left(\frac{88}{15} + \frac{16\zeta_2}{5} \right) \right] \right\}, \tag{4.41}
\end{aligned}$$

$$\begin{aligned}
\tilde{F}_{A,2}^{(1),ns} &= C_F \left\{ -2 - \frac{4\beta^2}{3} + i\pi \left(\frac{2}{\beta} - \beta \right) + \varepsilon \left[-\frac{12\zeta_2}{\beta} + 4 + 6\beta\zeta_2 - \frac{28\beta^2}{9} + i\pi \left(-\frac{1}{\beta} \left(4(-1 \right. \right. \right. \right. \\
& \left. \left. \left. + \ln(2) + \log(\beta) \right) \right) + 2\beta(-2 + \ln(2) + \log(\beta)) \right] + \varepsilon^2 \left[\frac{1}{\beta} \left(-24\zeta_2 + 24\zeta_2\ln(2) \right. \right. \right. \\
& \left. \left. \left. + 24\zeta_2\log(\beta) \right) - 8 - \zeta_2 + \beta \left(24\zeta_2 - 12\zeta_2\ln(2) - 12\zeta_2\log(\beta) \right) + \beta^2 \left(-\frac{160}{27} - \frac{2\zeta_2}{3} \right) \right. \right. \\
& \left. \left. + i\pi \left(\frac{1}{\beta} \left(8 - 3\zeta_2 - 8\ln(2) + 4\ln^2(2) - 8\log(\beta) + 8\ln(2)\log(\beta) + 4\log^2(\beta) \right) \right) \right. \right. \\
& \left. \left. \left. + \beta \left(-8 + \frac{3}{2}\zeta_2 + 8\ln(2) - 2\ln^2(2) + 8\log(\beta) - 4\ln(2)\log(\beta) - 2\log^2(\beta) \right) \right] \right] \Big\}. \tag{4.42}
\end{aligned}$$

$$\begin{aligned}
\tilde{F}_{A,2}^{\text{ns}(2)} = & C_F^2 \left\{ \frac{1}{\varepsilon} \left[\frac{12\zeta_2}{\beta^2} + 6\zeta_2 + \beta^2 \left(-\frac{16}{3} - 12\zeta_2 \right) + i\pi \left(\frac{2}{\beta} + \frac{26\beta}{3} \right) \right] + \left[\frac{1}{\beta^2} \left(24\zeta_2 - 48\zeta_2 \ln(2) \right. \right. \right. \\
& - 48\zeta_2 \log(\beta) \left. \left. \left. - \frac{12\zeta_2}{\beta} + \frac{41}{3} - 18\zeta_2 - 45\zeta_3 - 60\zeta_2 \ln(2) - 80\zeta_2 \log(\beta) + 20\beta\zeta_2 + \beta^2 \left(\frac{218}{9} \right. \right. \right. \right. \\
& - \frac{3292}{75}\zeta_2 - 7\zeta_3 - \frac{268}{5}\zeta_2 \ln(2) + \frac{264}{5}\zeta_2 \log(\beta) \left. \left. \left. \right) + i\pi \left(\frac{24\zeta_2}{\beta^2} - \frac{1}{\beta} \left(4(3 + \ln(2) + \log(\beta)) \right) \right) \right. \right. \\
& \left. \left. \left. + 40\zeta_2 + \frac{2}{9}\beta(-265 + 126 \ln(2) + 30 \log(\beta)) - \frac{132}{5}\beta^2\zeta_2 \right] + \varepsilon \left[\frac{1}{\beta^2} \left(-48\zeta_2 - 60\zeta_2^2 \right. \right. \right. \right. \\
& - 96\zeta_2 \ln(2) + 96\zeta_2 \ln^2(2) - 96\zeta_2 \log(\beta) + 192\zeta_2 \ln(2) \log(\beta) + 96\zeta_2 \log^2(\beta) \left. \left. \left. \right) + \frac{1}{\beta} \left(72\zeta_2 \right. \right. \right. \\
& + 24\zeta_2 \ln(2) + 24\zeta_2 \log(\beta) \left. \left. \left. \right) - \frac{823}{18} - \frac{10c_1}{3} + \frac{776}{3}\zeta_2 - \frac{1572}{5}\zeta_2^2 - \frac{673}{3}\zeta_3 - \frac{1052}{3}\zeta_2 \ln(2) \right. \right. \\
& + 112\zeta_2 \ln^2(2) - 128\zeta_2 \log(\beta) + 320\zeta_2 \ln(2) \log(\beta) + 160\zeta_2 \log^2(\beta) + \beta \left(\frac{3608}{3}\zeta_2 \right. \\
& - 968\zeta_2 \ln(2) - 584\zeta_2 \log(\beta) \left. \left. \right) + \beta^2 \left(\frac{6661}{135} + \frac{266c_1}{15} - \frac{193438\zeta_2}{1125} - \frac{6534}{25}\zeta_2^2 + \frac{3811}{15}\zeta_3 \right. \right. \\
& + \frac{9292}{15}\zeta_2 \ln(2) - \frac{1248}{5}\zeta_2 \ln^2(2) + \frac{9704}{25}\zeta_2 \log(\beta) - \frac{1056}{5}\zeta_2 \ln(2) \log(\beta) - \frac{528}{5}\zeta_2 \log^2(\beta) \left. \left. \right) \right. \\
& + i\pi \left(\frac{1}{\beta^2} \left(48\zeta_2 - 96\zeta_2 \ln(2) - 96\zeta_2 \log(\beta) \right) + \frac{1}{\beta} \left(-8 - 2\zeta_2 + 24 \ln(2) + 4 \ln^2(2) + 24 \log(\beta) \right. \right. \\
& + 8 \ln(2) \log(\beta) + 4 \log^2(\beta) \left. \left. \right) + 64\zeta_2 - 160\zeta_2 \ln(2) - 160\zeta_2 \log(\beta) + \beta \left(-\frac{1031}{3} + \frac{302}{3}\zeta_2 \right. \right. \\
& + \frac{5080 \ln(2)}{9} - \frac{740}{3} \ln^2(2) + \frac{3608 \log(\beta)}{9} - \frac{968}{3} \ln(2) \log(\beta) - \frac{292}{3} \log^2(\beta) \left. \left. \right) \right. \\
& \left. \left. \left. + \beta^2 \left(-\frac{4852}{25}\zeta_2 + \frac{528}{5}\zeta_2 \ln(2) + \frac{528}{5}\zeta_2 \log(\beta) \right) \right] \right\} + C_F C_A \left\{ \left[-\frac{44\zeta_2}{\beta} + \frac{151}{9} + \frac{104}{3}\zeta_2 \right. \right. \right. \\
& - 30\zeta_3 - 72\zeta_2 \ln(2) - 32\zeta_2 \log(\beta) + 84\beta\zeta_2 + \beta^2 \left(-\frac{2666}{45} - \frac{5138}{75}\zeta_2 + \frac{252}{5}\zeta_3 + \frac{656}{5}\zeta_2 \ln(2) \right. \\
& + \frac{256}{5}\zeta_2 \log(\beta) \left. \left. \right) + i\pi \left(-\frac{1}{9\beta} \left(4(-32 + 33 \ln(2) + 33 \log(\beta)) \right) + 16\zeta_2 + \frac{14}{9}\beta(-23 + 18 \ln(2) \right. \right. \\
& + 18 \log(\beta)) - \frac{128}{5}\beta^2\zeta_2 \left. \left. \right) \right] + \varepsilon \left[\frac{1}{\beta} \left(-\frac{776}{3}\zeta_2 + 264\zeta_2 \ln(2) + 264\zeta_2 \log(\beta) \right) - \frac{1433}{54} + \frac{20c_1}{3} \right. \\
& + \frac{415}{9}\zeta_2 - \frac{1644}{5}\zeta_2^2 + 46\zeta_3 - \frac{104}{3}\zeta_2 \ln(2) - 32\zeta_2 \ln^2(2) - 96\zeta_2 \log(\beta) + 128\zeta_2 \ln(2) \log(\beta) \\
& + 64\zeta_2 \log^2(\beta) + \beta \left(\frac{1990}{3}\zeta_2 - 504\zeta_2 \ln(2) - 504\zeta_2 \log(\beta) \right) + \beta^2 \left(-\frac{145603}{2025} - \frac{40c_1}{3} \right. \\
& - \frac{202052\zeta_2}{1125} + \frac{14536}{25}\zeta_2^2 - \frac{2001}{25}\zeta_3 + \frac{9148}{75}\zeta_2 \ln(2) + \frac{384}{5}\zeta_2 \ln^2(2) + \frac{4736}{25}\zeta_2 \log(\beta) \\
& - \frac{1024}{5}\zeta_2 \ln(2) \log(\beta) - \frac{512}{5}\zeta_2 \log^2(\beta) \left. \left. \right) + i\pi \left(\frac{1}{\beta} \left(\frac{1924}{27} - 11\zeta_2 - \frac{776 \ln(2)}{9} + 44 \ln^2(2) \right. \right. \right. \\
& \left. \left. \left. - \frac{776 \log(\beta)}{9} + 88 \ln(2) \log(\beta) + 44 \log^2(\beta) \right) + 48\zeta_2 - 64\zeta_2 \ln(2) - 64\zeta_2 \log(\beta) \right. \right. \\
\end{aligned}$$

$$\begin{aligned}
& + \beta \left(-\frac{10195}{54} + \frac{157}{6} \zeta_2 + \frac{2182 \ln(2)}{9} - 84 \ln^2(2) + \frac{1990 \log(\beta)}{9} - 168 \ln(2) \log(\beta) \right. \\
& \left. - 84 \log^2(\beta) \right) + \beta^2 \left(-\frac{2368}{25} \zeta_2 + \frac{512}{5} \zeta_2 \ln(2) + \frac{512}{5} \zeta_2 \log(\beta) \right) \Big] \Big\} \\
& + C_F T_F n_l \left\{ \left[\frac{16 \zeta_2}{\beta} - \frac{44}{9} - 8 \beta \zeta_2 + 8 \beta^2 + i\pi \left(\frac{1}{9\beta} (16(-4 + 3 \ln(2) + 3 \log(\beta))) \right) - \frac{2}{9} \beta (-25 \right. \right. \\
& \left. \left. + 12 \ln(2) + 12 \log(\beta)) \right) \right] + \varepsilon \left[\frac{1}{\beta} \left(\frac{352}{3} \zeta_2 - 96 \zeta_2 \ln(2) - 96 \zeta_2 \log(\beta) \right) + \frac{338}{27} + 12 \zeta_2 \right. \right. \\
& \left. \left. + \beta \left(-\frac{296}{3} \zeta_2 + 48 \zeta_2 \ln(2) + 48 \zeta_2 \log(\beta) \right) + \beta^2 \left(\frac{3044}{81} + 8 \zeta_2 \right) + i\pi \left(\frac{1}{\beta} \left(-\frac{944}{27} + 4 \zeta_2 \right. \right. \right. \right. \\
& \left. \left. \left. + \frac{352 \ln(2)}{9} - 16 \ln^2(2) + \frac{352 \log(\beta)}{9} - 32 \ln(2) \log(\beta) - 16 \log^2(\beta) \right) \right) + \beta \left(\frac{889}{27} - 2 \zeta_2 \right. \right. \right. \\
& \left. \left. \left. - \frac{296 \ln(2)}{9} + 8 \ln^2(2) - \frac{296 \log(\beta)}{9} + 16 \ln(2) \log(\beta) + 8 \log^2(\beta) \right) \right) \right] \Big\} \\
& + C_F T_F \left\{ \left[\frac{196}{9} - \frac{32}{3} \zeta_2 + \beta^2 \left(-\frac{632}{27} + \frac{64 \zeta_2}{5} \right) + i\pi \left(\frac{16\beta}{15} \right) \right] + \varepsilon \left[\frac{1322}{27} - \frac{412}{9} \zeta_2 - \frac{112}{3} \zeta_3 \right. \right. \\
& \left. \left. + 64 \zeta_2 \ln(2) - \frac{32}{5} \beta \zeta_2 + \beta^2 \left(-\frac{13348}{405} + \frac{9232}{225} \zeta_2 + \frac{224}{5} \zeta_3 - \frac{384}{5} \zeta_2 \ln(2) \right) + i\pi \left(-\frac{4 \zeta_2}{3\beta} \right. \right. \right. \\
& \left. \left. \left. + \beta \left(\frac{32}{15} + \frac{2}{3} \zeta_2 - \frac{32 \ln(2)}{15} - \frac{32 \log(\beta)}{15} \right) \right) \right] \Big\}. \tag{4.43}
\end{aligned}$$

The unrenormalized singlet contribution is

$$\begin{aligned}
\tilde{F}_{A,2}^{(2),s} = C_F T_F \left\{ \left[\frac{6}{\beta^2} + \frac{8}{3} + \frac{136}{3} \zeta_2 - 42 \zeta_3 + 16 \ln(2) - 16 \zeta_2 \ln(2) - 48 \beta \zeta_2 + \beta^2 \left(-30 - \frac{904}{15} \zeta_2 \right. \right. \right. \\
\left. \left. \left. + 70 \zeta_3 + \frac{16 \ln(2)}{3} + \frac{176}{5} \zeta_2 \ln(2) \right) + i\pi \left(-8 + 12 \zeta_2 + \beta(16 - 16 \ln(2)) + \beta^2 \left(-\frac{8}{3} \right. \right. \right. \right. \\
\left. \left. \left. - 20 \zeta_2 \right) \right) \right] + \varepsilon \left[\frac{1}{\beta^2} \left(15 - 20 \zeta_2 + 42 \zeta_3 - 16 \ln(2) - 48 \zeta_2 \ln(2) \right) + \frac{48 \zeta_2}{\beta} + \frac{364}{9} - 4c_1 \right. \\
\left. + \frac{2684}{9} \zeta_2 - \frac{976}{5} \zeta_2^2 + \frac{182}{3} \zeta_3 + \frac{112 \ln(2)}{3} - \frac{416}{3} \zeta_2 \ln(2) - 56 \zeta_3 \ln(2) - 16 \ln^2(2) \right. \\
\left. + 200 \zeta_2 \ln^2(2) + \beta \left(-240 \zeta_2 + 288 \zeta_2 \ln(2) + 96 \zeta_2 \log(\beta) \right) + \beta^2 \left(-\frac{427}{5} + \frac{236c_1}{45} \right. \right. \\
\left. \left. - \frac{101524}{225} \zeta_2 + \frac{26288}{75} \zeta_2^2 - \frac{1666}{15} \zeta_3 - \frac{3392 \ln(2)}{45} + \frac{33536}{75} \zeta_2 \ln(2) + \frac{280}{3} \zeta_3 \ln(2) - \frac{16}{3} \ln^2(2) \right. \right. \\
\left. \left. - \frac{1624}{5} \zeta_2 \ln^2(2) \right) + i\pi \left(\frac{1}{\beta^2} (8 - 12 \zeta_2) + \frac{1}{\beta} (16(-1 + \ln(2))) - \frac{56}{3} + 28 \zeta_2 + 28 \zeta_3 + 16 \ln(2) \right. \right. \\
\left. \left. - 24 \zeta_2 \ln(2) + \beta \left(\frac{160}{3} - 80 \zeta_2 - 80 \ln(2) + 48 \ln^2(2) - 32 \log(\beta) + 32 \ln(2) \log(\beta) \right) \right) \right] \Big\}
\end{aligned}$$

$$+ \beta^2 \left(\frac{1696}{45} - \frac{428}{15} \zeta_2 - \frac{140}{3} \zeta_3 + \frac{16 \ln(2)}{3} + 40 \zeta_2 \ln(2) \right) \Big] \Big] \Big\}. \quad (4.44)$$

4.3.3 Scalar form factor

The scalar form factor is in this limit given by

$$\begin{aligned} \tilde{F}_S^{(1)} = & C_F \left\{ \frac{1}{\varepsilon} \left[\frac{8\beta^2}{3} + i\pi \left(-\frac{1}{\beta} - \beta \right) \right] + \left[\frac{6\zeta_2}{\beta} - 2 + 6\beta\zeta_2 - \frac{76\beta^2}{9} + i\pi \left(\frac{1}{\beta} \left(2(-1 + \ln(2)) \right. \right. \right. \right. \\ & \left. \left. \left. + \log(\beta) \right) \right) + \beta(3 + 2 \ln(2) + 2 \log(\beta)) \right] + \varepsilon \left[\frac{1}{\beta} \left(12\zeta_2 - 12\zeta_2 \ln(2) - 12\zeta_2 \log(\beta) \right) + 4 \right. \\ & \left. + \beta \left(-18\zeta_2 - 12\zeta_2 \ln(2) - 12\zeta_2 \log(\beta) \right) + \beta^2 \left(\frac{404}{27} + \frac{4\zeta_2}{3} \right) + i\pi \left(\frac{1}{\beta} \left(-4 + \frac{3}{2} \zeta_2 + 4 \ln(2) \right. \right. \right. \\ & \left. \left. \left. - 2 \ln^2(2) + 4 \log(\beta) - 4 \ln(2) \log(\beta) - 2 \log^2(\beta) \right) + \beta \left(4 + \frac{3}{2} \zeta_2 - 6 \ln(2) - 2 \ln^2(2) \right. \right. \right. \\ & \left. \left. \left. - 6 \log(\beta) - 4 \ln(2) \log(\beta) - 2 \log^2(\beta) \right) \right] + \varepsilon^2 \left[\frac{1}{\beta} \left(24\zeta_2 + 3\zeta_2^2 - 24\zeta_2 \ln(2) + 12\zeta_2 \ln^2(2) \right. \right. \right. \\ & \left. \left. \left. - 24\zeta_2 \log(\beta) + 24\zeta_2 \ln(2) \log(\beta) + 12\zeta_2 \log^2(\beta) \right) - 8 - \zeta_2 + \beta \left(-24\zeta_2 + 3\zeta_2^2 + 36\zeta_2 \ln(2) \right. \right. \right. \\ & \left. \left. \left. + 12\zeta_2 \ln^2(2) + 36\zeta_2 \log(\beta) + 24\zeta_2 \ln(2) \log(\beta) + 12\zeta_2 \log^2(\beta) \right) + \beta^2 \left(-\frac{2752}{81} - \frac{38\zeta_2}{9} - \frac{8\zeta_3}{9} \right) \right. \\ & \left. + i\pi \left(\frac{1}{\beta} \left(-8 + 3\zeta_2 + \frac{7}{3} \zeta_3 + 8 \ln(2) - 3\zeta_2 \ln(2) - 4 \ln^2(2) + \frac{4 \ln^3(2)}{3} + 8 \log(\beta) - 3\zeta_2 \log(\beta) \right. \right. \right. \\ & \left. \left. \left. - 8 \ln(2) \log(\beta) + 4 \ln^2(2) \log(\beta) - 4 \log^2(\beta) + 4 \ln(2) \log^2(\beta) + \frac{4 \log^3(\beta)}{3} \right) + \beta \left(8 - \frac{9}{2} \zeta_2 \right. \right. \right. \\ & \left. \left. \left. + \frac{7}{3} \zeta_3 - 8 \ln(2) - 3\zeta_2 \ln(2) + 6 \ln^2(2) + \frac{4 \ln^3(2)}{3} - 8 \log(\beta) - 3\zeta_2 \log(\beta) + 12 \ln(2) \log(\beta) \right. \right. \right. \\ & \left. \left. \left. + 4 \ln^2(2) \log(\beta) + 6 \log^2(\beta) + 4 \ln(2) \log^2(\beta) + \frac{4 \log^3(\beta)}{3} \right) \right] \Big] \Big\}. \quad (4.45) \end{aligned}$$

$$\begin{aligned} \tilde{F}_S^{(2)} = & C_F^2 \left\{ \frac{1}{\varepsilon^2} \left[-\frac{3\zeta_2}{\beta^2} - 6\zeta_2 - 3\beta^2\zeta_2 + i\pi \left(-\frac{8\beta}{3} \right) \right] + \frac{1}{\varepsilon} \left[\frac{1}{\beta^2} \left(-12\zeta_2 + 12\zeta_2 \ln(2) \right. \right. \right. \\ & \left. \left. \left. + 12\zeta_2 \log(\beta) \right) + 6\zeta_2 + 24\zeta_2 \ln(2) + 24\zeta_2 \log(\beta) + 16\beta\zeta_2 + \beta^2 \left(-\frac{16}{3} + 27\zeta_2 + 12\zeta_2 \ln(2) \right. \right. \right. \\ & \left. \left. \left. + 12\zeta_2 \log(\beta) \right) + i\pi \left(-\frac{6\zeta_2}{\beta^2} + \frac{2}{\beta} - 12\zeta_2 + \frac{2}{9} \beta(23 + 24 \ln(2) + 24 \log(\beta)) - 6\beta^2\zeta_2 \right) \right] \right. \\ & \left. + \left[\frac{1}{\beta^2} \left(-24\zeta_2 + 15\zeta_2^2 + 48\zeta_2 \ln(2) - 24\zeta_2 \ln^2(2) + 48\zeta_2 \log(\beta) - 48\zeta_2 \ln(2) \log(\beta) \right. \right. \right. \\ & \left. \left. \left. - 24\zeta_2 \log^2(\beta) \right) - \frac{12\zeta_2}{\beta} + 5 + 98\zeta_2 + 30\zeta_2^2 - 44\zeta_3 - 40\zeta_2 \ln(2) - 48\zeta_2 \ln^2(2) - 88\zeta_2 \log(\beta) \right. \right. \\ & \left. \left. - 96\zeta_2 \ln(2) \log(\beta) - 48\zeta_2 \log^2(\beta) + \beta \left(-\frac{92}{3} \zeta_2 - 32\zeta_2 \ln(2) - 32\zeta_2 \log(\beta) \right) + \beta^2 \left(\frac{1684}{45} \right. \right. \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{2446}{25}\zeta_2 + 15\zeta_2^2 - \frac{58}{5}\zeta_3 - \frac{964}{5}\zeta_2 \ln(2) - 24\zeta_2 \ln^2(2) - \frac{684}{5}\zeta_2 \log(\beta) - 48\zeta_2 \ln(2) \log(\beta) \\
& - 24\zeta_2 \log^2(\beta) \Big) + i\pi \left(\frac{1}{\beta^2} \left(-24\zeta_2 + 24\zeta_2 \ln(2) + 24\zeta_2 \log(\beta) \right) - \frac{4(\ln(2) + \log(\beta))}{\beta} \right. \\
& + 44\zeta_2 + 48\zeta_2 \ln(2) + 48\zeta_2 \log(\beta) + \beta \left(-\frac{632}{27} + \frac{8}{3}\zeta_2 - \frac{92 \ln(2)}{9} - \frac{16}{3} \ln^2(2) - \frac{92 \log(\beta)}{9} \right. \\
& \left. \left. - \frac{32}{3} \ln(2) \log(\beta) - \frac{16}{3} \log^2(\beta) \right) + \beta^2 \left(\frac{342}{5}\zeta_2 + 24\zeta_2 \ln(2) + 24\zeta_2 \log(\beta) \right) \right) \Big] \\
& + \varepsilon \left[\frac{1}{\beta^2} \left(48\zeta_2 + 60\zeta_2^2 - 106\zeta_2\zeta_3 + 96\zeta_2 \ln(2) - 60\zeta_2^2 \ln(2) - 96\zeta_2 \ln^2(2) + 32\zeta_2 \ln^3(2) \right. \right. \\
& + 96\zeta_2 \log(\beta) - 60\zeta_2^2 \log(\beta) - 192\zeta_2 \ln(2) \log(\beta) + 96\zeta_2 \ln^2(2) \log(\beta) - 96\zeta_2 \log^2(\beta) \\
& + 96\zeta_2 \ln(2) \log^2(\beta) + 32\zeta_2 \log^3(\beta) \Big) + \frac{1}{\beta} \left(24 \ln(2)\zeta_2 + 24 \log(\beta)\zeta_2 \right) - \frac{451}{6} - 8c_1 + \frac{2368}{3}\zeta_2 \\
& - \frac{1278}{5}\zeta_2^2 - 291\zeta_3 - 212\zeta_2\zeta_3 - 740\zeta_2 \ln(2) - 120\zeta_2^2 \ln(2) + 176\zeta_2 \ln^2(2) + 64\zeta_2 \ln^3(2) \\
& - 688\zeta_2 \log(\beta) - 120\zeta_2^2 \log(\beta) + 352\zeta_2 \ln(2) \log(\beta) + 192\zeta_2 \ln^2(2) \log(\beta) + 176\zeta_2 \log^2(\beta) \\
& + 192\zeta_2 \ln(2) \log^2(\beta) + 64\zeta_2 \log^3(\beta) + \beta \left(\frac{3316}{9}\zeta_2 + 16\zeta_2^2 + \frac{184}{3}\zeta_2 \ln(2) + 32\zeta_2 \ln^2(2) \right. \\
& + \frac{184}{3}\zeta_2 \log(\beta) + 64\zeta_2 \ln(2) \log(\beta) + 32\zeta_2 \log^2(\beta) \Big) + \beta^2 \left(-\frac{100966}{675} + \frac{28c_1}{3} - \frac{61918}{125}\zeta_2 \right. \\
& - \frac{10939}{25}\zeta_2^2 - \frac{15298}{75}\zeta_3 - 106\zeta_2\zeta_3 - \frac{5392}{75}\zeta_2 \ln(2) - 60\zeta_2^2 \ln(2) + \frac{1144}{5}\zeta_2 \ln^2(2) + 32\zeta_2 \ln^3(2) \\
& + \frac{4776}{25}\zeta_2 \log(\beta) - 60\zeta_2^2 \log(\beta) + \frac{2736}{5}\zeta_2 \ln(2) \log(\beta) + 96\zeta_2 \ln^2(2) \log(\beta) + \frac{1368}{5}\zeta_2 \log^2(\beta) \\
& + 96\zeta_2 \ln(2) \log^2(\beta) + 32\zeta_2 \log^3(\beta) \Big) + i\pi \left(\frac{1}{\beta^2} \left(-48\zeta_2 - 18\zeta_2^2 + 96\zeta_2 \ln(2) - 48\zeta_2 \ln^2(2) \right. \right. \\
& + 96\zeta_2 \log(\beta) - 96\zeta_2 \ln(2) \log(\beta) - 48\zeta_2 \log^2(\beta) \Big) + \frac{1}{\beta} \left(8 - 2\zeta_2 + 4 \ln^2(2) + 8 \ln(2) \log(\beta) \right. \\
& + 4 \log^2(\beta) \Big) + 344\zeta_2 - 36\zeta_2^2 - 176\zeta_2 \ln(2) - 96\zeta_2 \ln^2(2) - 176\zeta_2 \log(\beta) - 192\zeta_2 \ln(2) \log(\beta) \\
& - 96\zeta_2 \log^2(\beta) + \beta \left(-\frac{11021}{81} - \frac{46}{9}\zeta_2 + \frac{64}{9}\zeta_3 + \frac{4468 \ln(2)}{27} - \frac{16}{3}\zeta_2 \ln(2) + \frac{92 \ln^2(2)}{9} \right. \\
& + \frac{32 \ln^3(2)}{9} + \frac{3316 \log(\beta)}{27} - \frac{16}{3}\zeta_2 \log(\beta) + \frac{184}{9} \ln(2) \log(\beta) + \frac{32}{3} \ln^2(2) \log(\beta) + \frac{92 \log^2(\beta)}{9} \\
& + \frac{32}{3} \ln(2) \log^2(\beta) + \frac{32 \log^3(\beta)}{9} \Big) + \beta^2 \left(-\frac{2388}{25}\zeta_2 - 18\zeta_2^2 - \frac{1368}{5}\zeta_2 \ln(2) - 48\zeta_2 \ln^2(2) \right. \\
& \left. \left. - \frac{1368}{5}\zeta_2 \log(\beta) - 96\zeta_2 \ln(2) \log(\beta) - 48\zeta_2 \log^2(\beta) \right) \right) \Big] \Big\} + C_F C_A \left\{ \frac{1}{\varepsilon^2} \left[-\frac{44\beta^2}{9} + i\pi \left(\frac{11}{6\beta} \right. \right. \right. \\
& + \frac{11\beta}{6} \Big) \right] + \frac{1}{\varepsilon} \left[-8\beta\zeta_2 + \beta^2 \left(\frac{376}{27} + \frac{32\zeta_2}{3} \right) + i\pi \left(-\frac{31}{18\beta} + \frac{1}{6}\beta(-13 - 32 \ln(2) \right. \right. \\
& \left. \left. - 16 \log(\beta) \right) \right] + \left[\frac{1}{\beta} \left(\frac{194}{3}\zeta_2 - 44\zeta_2 \ln(2) - 44\zeta_2 \log(\beta) \right) + \frac{49}{9} + 38\zeta_2 - 20\zeta_3 - 64\zeta_2 \ln(2) \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -16\zeta_2 \log(\beta) + \beta \left(-\frac{100}{3}\zeta_2 + 84\zeta_2 \ln(2) + 36\zeta_2 \log(\beta) \right) + \beta^2 \left(-\frac{16472}{405} + \frac{1679}{15}\zeta_2 - \frac{472}{15}\zeta_3 \right. \\
& - \frac{336}{5}\zeta_2 \ln(2) - 80\zeta_2 \log(\beta) \left. \right) + i\pi \left(\frac{1}{\beta} \left(-\frac{478}{27} + \frac{194 \ln(2)}{9} - \frac{22 \ln^2(2)}{3} + \frac{194 \log(\beta)}{9} \right. \right. \\
& - \frac{44}{3} \ln(2) \log(\beta) - \frac{22 \log^2(\beta)}{3} \left. \right) + 8\zeta_2 + \beta \left(\frac{43}{9} - \frac{40}{3}\zeta_2 - \frac{188 \ln(2)}{9} + \frac{74 \ln^2(2)}{3} \right. \\
& - \left. \frac{100 \log(\beta)}{9} + 28 \ln(2) \log(\beta) + 6 \log^2(\beta) \right) + 40\beta^2 \zeta_2 \left. \right] + \varepsilon \left[\frac{1}{\beta} \left(\frac{2704}{9}\zeta_2 + 99\zeta_2^2 \right. \right. \\
& - \frac{1040}{3}\zeta_2 \ln(2) + 132\zeta_2 \ln^2(2) - \frac{1040}{3}\zeta_2 \log(\beta) + 264\zeta_2 \ln(2) \log(\beta) + 132\zeta_2 \log^2(\beta) \left. \right) \\
& - \frac{383}{54} + 8c_1 + \frac{409}{3}\zeta_2 - 264\zeta_2^2 - \zeta_3 - 220\zeta_2 \ln(2) - 64\zeta_2 \ln^2(2) - 144\zeta_2 \log(\beta) \\
& + 64\zeta_2 \ln(2) \log(\beta) + 32\zeta_2 \log^2(\beta) + \beta \left(-\frac{692}{9}\zeta_2 + 59\zeta_2^2 + \frac{1636}{3}\zeta_2 \ln(2) - 508\zeta_2 \ln^2(2) \right. \\
& + \frac{1108}{3}\zeta_2 \log(\beta) - 632\zeta_2 \ln(2) \log(\beta) - 172\zeta_2 \log^2(\beta) \left. \right) + \beta^2 \left(\frac{17767}{675} - \frac{32c_1}{15} + \frac{450622}{675}\zeta_2 \right. \\
& - \frac{25112}{75}\zeta_2^2 + \frac{2717}{150}\zeta_3 - \frac{26306}{75}\zeta_2 \ln(2) + \frac{608}{5}\zeta_2 \ln^2(2) - 416\zeta_2 \log(\beta) + 320\zeta_2 \ln(2) \log(\beta) \\
& + 160\zeta_2 \log^2(\beta) \left. \right) + i\pi \left(\frac{1}{\beta} \left(-\frac{6338}{81} + \frac{97}{6}\zeta_2 + \frac{22}{3}\zeta_3 + \frac{2704 \ln(2)}{27} - 11\zeta_2 \ln(2) - \frac{520}{9} \ln^2(2) \right. \right. \\
& + \frac{44 \ln^3(2)}{3} + \frac{2704 \log(\beta)}{27} - 11\zeta_2 \log(\beta) - \frac{1040}{9} \ln(2) \log(\beta) + 44 \ln^2(2) \log(\beta) \\
& - \frac{520}{9} \log^2(\beta) + 44 \ln(2) \log^2(\beta) + \frac{44 \log^3(\beta)}{3} \left. \right) + 72\zeta_2 - 32\zeta_2 \ln(2) - 32\zeta_2 \log(\beta) \\
& + \beta \left(-\frac{2317}{27} - \frac{422}{9}\zeta_2 - \frac{194}{3}\zeta_3 - \frac{1156 \ln(2)}{27} + \frac{335}{3}\zeta_2 \ln(2) + 130 \ln^2(2) - \frac{764}{9} \ln^3(2) \right. \\
& - \frac{692 \log(\beta)}{27} + 77\zeta_2 \log(\beta) + \frac{1636}{9} \ln(2) \log(\beta) - \frac{508}{3} \ln^2(2) \log(\beta) + \frac{554 \log^2(\beta)}{9} \\
& - \left. \left. \frac{316}{3} \ln(2) \log^2(\beta) - \frac{172}{9} \log^3(\beta) \right) + \beta^2 \left(208\zeta_2 - 160\zeta_2 \ln(2) - 160\zeta_2 \log(\beta) \right) \right] \left. \right\} \\
& + C_F T_F n_l \left\{ \frac{1}{\varepsilon^2} \left[\frac{16\beta^2}{9} + i\pi \left(-\frac{2}{3\beta} - \frac{2\beta}{3} \right) \right] + \frac{1}{\varepsilon} \left[-\frac{80\beta^2}{27} + i\pi \left(\frac{10}{9\beta} + \frac{10\beta}{9} \right) \right] \right. \\
& + \left[\frac{1}{\beta} \left(-\frac{88}{3}\zeta_2 + 16\zeta_2 \ln(2) + 16\zeta_2 \log(\beta) \right) - \frac{20}{9} + \beta \left(\frac{32}{3}\zeta_2 + 16\zeta_2 \ln(2) + 16\zeta_2 \log(\beta) \right) \right. \\
& + \beta^2 \left(-\frac{400}{81} - \frac{64\zeta_2}{9} \right) + i\pi \left(\frac{1}{\beta} \left(\frac{248}{27} - \frac{88 \ln(2)}{9} + \frac{8 \ln^2(2)}{3} - \frac{88 \log(\beta)}{9} + \frac{16}{3} \ln(2) \log(\beta) \right. \right. \\
& + \frac{8 \log^2(\beta)}{3} \left. \right) + \beta \left(-\frac{268}{27} + \frac{32 \ln(2)}{9} + \frac{8 \ln^2(2)}{3} + \frac{32 \log(\beta)}{9} + \frac{16}{3} \ln(2) \log(\beta) \right. \\
& + \left. \left. \frac{8 \log^2(\beta)}{3} \right) \right] + \varepsilon \left[\frac{1}{\beta} \left(-\frac{1280}{9}\zeta_2 - 36\zeta_2^2 + \frac{448}{3}\zeta_2 \ln(2) - 48\zeta_2 \ln^2(2) + \frac{448}{3}\zeta_2 \log(\beta) \right. \right. \\
& - 96\zeta_2 \ln(2) \log(\beta) - 48\zeta_2 \log^2(\beta) \left. \right) + \frac{62}{27} + 12\zeta_2 + \beta \left(\frac{1360}{9}\zeta_2 - 36\zeta_2^2 - \frac{272}{3}\zeta_2 \ln(2) \right.
\end{aligned}$$

$$\begin{aligned}
& -48\zeta_2 \ln^2(2) - \frac{272}{3}\zeta_2 \log(\beta) - 96\zeta_2 \ln(2) \log(\beta) - 48\zeta_2 \log^2(\beta) \Big) + \beta^2 \left(\frac{800}{9} + \frac{968\zeta_2}{27} \right. \\
& - \frac{128\zeta_3}{9} \Big) + i\pi \left(\frac{1}{\beta} \left(\frac{3160}{81} - \frac{22}{3}\zeta_2 - \frac{8}{3}\zeta_3 - \frac{1280 \ln(2)}{27} + 4\zeta_2 \ln(2) + \frac{224 \ln^2(2)}{9} - \frac{16}{3} \ln^3(2) \right. \right. \\
& - \frac{1280 \log(\beta)}{27} + 4\zeta_2 \log(\beta) + \frac{448}{9} \ln(2) \log(\beta) - 16 \ln^2(2) \log(\beta) + \frac{224 \log^2(\beta)}{9} \\
& - 16 \ln(2) \log^2(\beta) - \frac{16}{3} \log^3(\beta) \Big) + \beta \left(-\frac{3848}{81} + \frac{8}{3}\zeta_2 - \frac{8}{3}\zeta_3 + \frac{1360 \ln(2)}{27} + 4\zeta_2 \ln(2) \right. \\
& - \frac{136}{9} \ln^2(2) - \frac{16}{3} \ln^3(2) + \frac{1360 \log(\beta)}{27} + 4\zeta_2 \log(\beta) - \frac{272}{9} \ln(2) \log(\beta) - 16 \ln^2(2) \log(\beta) \\
& \left. \left. - \frac{136}{9} \log^2(\beta) - 16 \ln(2) \log^2(\beta) - \frac{16}{3} \log^3(\beta) \right) \right) \Big] \Big\} + C_F T_F \left\{ \left[\frac{580}{9} - \frac{212}{3}\zeta_2 - 16 \ln(2) \right. \right. \\
& + 64\zeta_2 \ln(2) + \beta^2 \left(-\frac{712}{81} + \frac{252}{5}\zeta_2 - 42\zeta_3 + \frac{32 \ln(2)}{3} - \frac{336}{5}\zeta_2 \ln(2) \right) + i\pi \left(\frac{2\zeta_2}{3\beta} + 8 + \frac{2}{3}\beta\zeta_2 \right. \\
& \left. \left. + \beta^2 \left(-\frac{16}{3} + 12\zeta_2 \right) \right) \right] + \varepsilon \left[-\frac{4\zeta_2^2}{\beta} + \frac{5186}{27} - \frac{32c_1}{3} - \frac{2800}{9}\zeta_2 + \frac{944}{5}\zeta_2^2 - \frac{616}{3}\zeta_3 \right. \\
& - \frac{224 \ln(2)}{3} + \frac{584}{3}\zeta_2 \ln(2) + 16 \ln^2(2) + 64\zeta_2 \ln^2(2) - 4\beta\zeta_2^2 + \beta^2 \left(-\frac{35116}{1215} + \frac{68c_1}{15} \right. \\
& + \frac{18992}{75}\zeta_2 - \frac{8656}{25}\zeta_2^2 + \frac{5914}{135}\zeta_3 + \frac{3584 \ln(2)}{45} - \frac{8696}{75}\zeta_2 \ln(2) - 56\zeta_3 \ln(2) - \frac{32}{3} \ln^2(2) \\
& + \frac{744}{5}\zeta_2 \ln^2(2) \Big) + i\pi \left(\frac{1}{\beta} \left(\frac{4}{3}\zeta_2 - \frac{4}{9}\zeta_3 - \frac{4}{3}\zeta_2 \ln(2) - \frac{4}{3}\zeta_2 \log(\beta) \right) + \frac{112}{3} - 12\zeta_2 - 16 \ln(2) \right. \\
& + \beta \left(-2\zeta_2 - \frac{4}{9}\zeta_3 - \frac{4}{3}\zeta_2 \ln(2) - \frac{4}{3}\zeta_2 \log(\beta) \right) + \beta^2 \left(-\frac{1792}{45} + 40\zeta_2 + 28\zeta_3 + \frac{32 \ln(2)}{3} \right. \\
& \left. \left. - 24\zeta_2 \ln(2) \right) \right) \Big] \Big\}. \tag{4.46}
\end{aligned}$$

4.3.4 Pseudo-scalar form factor

The non-singlet part of the pseudo-scalar form factor can be obtained using the chiral Ward identity Eq. (2.13) as follows

$$\tilde{F}_P^{(n),\text{ns}} = \tilde{F}_{A,1}^{(n),\text{ns}} + \left(1 + \beta^2 + \beta^4 \right) \tilde{F}_{A,2}^{(n),\text{ns}} + \mathcal{O}(\beta^3). \tag{4.47}$$

The bare singlet piece is given by

$$\begin{aligned}
\tilde{F}_P^{(2),\text{s}} = C_F T_F \Big\{ & \left[20\zeta_2 - 42\zeta_3 + 48\zeta_2 \ln(2) - 48\beta\zeta_2 + \beta^2 \left(-\frac{40}{3} - 12\zeta_2 + 28\zeta_3 + 16 \ln(2) \right. \right. \\
& \left. \left. - 32\zeta_2 \ln(2) \right) + i\pi \left(12\zeta_2 + \beta(16 - 16 \ln(2)) + \beta^2 \left(-8 - 8\zeta_2 \right) \right) \right] + \varepsilon \left[-\frac{44c_1}{3} + 88\zeta_2 \right. \\
& - \frac{32}{5}\zeta_2^2 + 112\zeta_3 - 264\zeta_2 \ln(2) - 56\zeta_3 \ln(2) + 264\zeta_2 \ln^2(2) + \beta \left(-144\zeta_2 + 288\zeta_2 \ln(2) \right. \\
& \left. \left. + 96\zeta_2 \log(\beta) \right) + \beta^2 \left(\frac{256}{9} + \frac{88c_1}{9} - 108\zeta_2 + \frac{64}{15}\zeta_2^2 - \frac{112}{3}\zeta_3 - 48 \ln(2) + \frac{680}{3}\zeta_2 \ln(2) \right) \right] \Big\}.
\end{aligned}$$

$$\begin{aligned}
& + \frac{112}{3}\zeta_3 \ln(2) - 16 \ln^2(2) - 176\zeta_2 \ln^2(2) \Big) + i\pi \Big(- 12\zeta_2 + 28\zeta_3 - 24\zeta_2 \ln(2) \\
& + \beta \Big(16 - 80\zeta_2 - 48 \ln(2) + 48 \ln^2(2) - 32 \log(\beta) + 32 \ln(2) \log(\beta) \Big) \\
& + \beta^2 \Big(24 - \frac{4}{3}\zeta_2 - \frac{56}{3}\zeta_3 + 16 \ln(2) + 16\zeta_2 \ln(2) \Big) \Big) \Big] \Big] \Big\}. \tag{4.48}
\end{aligned}$$

5 Conclusion

The massive form factors are basic building blocks to many observables in heavy quark physics. The precision study of these objects will both shed light on the physical structure of the top quark itself and also on important aspects of the mechanism to create the fermion masses. A future electron-positron collider can achieve high precision and hence an equal or better theory prediction is indispensable. In a similar way this also applies to the LHC for its high luminosity phase. In the present paper, we have computed the heavy quark form factors for vector, axial-vector, scalar and pseudo-scalar currents at two-loop level up to the $\mathcal{O}(\varepsilon^2)$ contributions. These contributions constitute important ingredients to renormalize the three-loop form factors and do also contribute to potential future 4-loop calculations. In addition, they serve as a cross-check of earlier results available in the literature. In the calculation we have used both traditional techniques in solving the differential equations for the master integrals, as well as a more recent automated method, based on coupled difference equations. Both methods play a role in computing higher than second order corrections to the different form factors.

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A Renormalization Constants

In this appendix, we present corresponding renormalization constants up to relevant order in ε . The wave function renormalization constants in OS scheme up to two-loop [35–37, 40] are

$$Z_{2,OS}^{(1)} = C_F \left[-\frac{3}{\varepsilon} - 4 + \varepsilon \left\{ -8 - \frac{3}{2}\zeta_2 \right\} + \varepsilon^2 \left\{ -16 - 2\zeta_2 + \zeta_3 \right\} + \varepsilon^3 \left\{ -32 - 4\zeta_2 - \frac{27}{40}\zeta_2^2 + \frac{4}{3}\zeta_3 \right\} \right], \quad (\text{A.1})$$

$$\begin{aligned} Z_{2,OS}^{(2)} = & C_F^2 \left[\frac{1}{\varepsilon^2} \left\{ \frac{9}{2} \right\} + \frac{1}{\varepsilon} \left\{ \frac{51}{4} \right\} + \left\{ \frac{433}{8} - \frac{147}{2}\zeta_2 - 24\zeta_3 + 96\zeta_2 \ln(2) \right\} + \varepsilon \left\{ \frac{211}{16} - 16c_1 \right. \right. \\ & \left. \left. - \frac{1017}{4}\zeta_2 + \frac{1008}{5}\zeta_2^2 - 297\zeta_3 + 552\zeta_2 \ln(2) \right\} + \varepsilon^2 \left\{ \frac{4889}{32} - 92c_1 - \frac{96c_2}{5} - \frac{8851}{8}\zeta_2 \right. \right. \\ & \left. \left. + \frac{11703}{20}\zeta_2^2 - \frac{2069}{2}\zeta_3 + 264\zeta_2\zeta_3 + 2436\zeta_5 + 1968\zeta_2 \ln(2) \right\} \right] + C_A C_F \left[\frac{1}{\varepsilon^2} \left\{ \frac{11}{2} \right\} \right. \\ & \left. + \frac{1}{\varepsilon} \left\{ -\frac{127}{12} \right\} + \left\{ -\frac{1705}{24} + 30\zeta_2 + 12\zeta_3 - 48\zeta_2 \ln(2) \right\} + \varepsilon \left\{ -\frac{9907}{48} + 8c_1 + \frac{769}{12}\zeta_2 \right. \right. \\ & \left. \left. - \frac{504}{5}\zeta_2^2 + 129\zeta_3 - 276\zeta_2 \ln(2) \right\} + \varepsilon^2 \left\{ -\frac{79225}{96} + 46c_1 + \frac{48c_2}{5} + \frac{6367}{24}\zeta_2 - \frac{14359}{40}\zeta_2^2 \right. \right. \\ & \left. \left. + \frac{7595}{18}\zeta_3 - 132\zeta_2\zeta_3 - 1218\zeta_5 - 984\zeta_2 \ln(2) \right\} \right] + C_F T_F n_l \left[\frac{1}{\varepsilon^2} \left\{ -2 \right\} + \frac{1}{\varepsilon} \left\{ \frac{11}{3} \right\} \right. \\ & \left. + \left\{ \frac{113}{6} + 8\zeta_2 \right\} + \varepsilon \left\{ \frac{851}{12} + \frac{127}{3}\zeta_2 + 16\zeta_3 \right\} + \varepsilon^2 \left\{ \frac{5753}{24} + \frac{853}{6}\zeta_2 + \frac{597}{10}\zeta_2^2 + \frac{610}{9}\zeta_3 \right\} \right] \\ & + C_F T_F \left[\frac{1}{\varepsilon} + \left\{ \frac{947}{18} - 30\zeta_2 \right\} + \varepsilon \left\{ \frac{17971}{108} - \frac{445}{3}\zeta_2 - \frac{340}{3}\zeta_3 + 192\zeta_2 \ln(2) \right\} \right. \\ & \left. + \varepsilon^2 \left\{ \frac{422747}{648} - 32c_1 - \frac{8605}{18}\zeta_2 + \frac{1683}{10}\zeta_2^2 - \frac{4810}{9}\zeta_3 + 912\zeta_2 \ln(2) \right\} \right]. \quad (\text{A.2}) \end{aligned}$$

The heavy quark mass renormalization constants in OS scheme are [35–39]

$$Z_{m,OS}^{(1)} = C_F \left[-\frac{3}{\varepsilon} - 4 + \varepsilon \left\{ -8 - \frac{3}{2}\zeta_2 \right\} + \varepsilon^2 \left\{ -16 - 2\zeta_2 + \zeta_3 \right\} + \varepsilon^3 \left\{ -32 - 4\zeta_2 - \frac{27}{40}\zeta_2^2 + \frac{4}{3}\zeta_3 \right\} \right], \quad (\text{A.3})$$

$$\begin{aligned} Z_{m,OS}^{(2)} = & C_F^2 \left[\frac{1}{\varepsilon^2} \left\{ \frac{9}{2} \right\} + \frac{1}{\varepsilon} \left\{ \frac{45}{4} \right\} + \left\{ \frac{199}{8} - \frac{51}{2}\zeta_2 - 12\zeta_3 + 48\zeta_2 \ln(2) \right\} + \varepsilon \left\{ \frac{677}{16} - 8c_1 - \frac{615}{4}\zeta_2 \right. \right. \\ & \left. \left. + \frac{504}{5}\zeta_2^2 - 135\zeta_3 + 288\zeta_2 \ln(2) \right\} + \varepsilon^2 \left\{ \frac{1167}{32} - 48c_1 - \frac{48c_2}{5} - \frac{4821}{8}\zeta_2 + \frac{7719}{20}\zeta_2^2 \right. \right. \end{aligned}$$

$$\begin{aligned}
& - \frac{1203}{2} \zeta_3 + 132 \zeta_2 \zeta_3 + 1218 \zeta_5 + 1056 \zeta_2 \ln(2) \Bigg] + C_A C_F \left[\frac{1}{\varepsilon^2} \left\{ \frac{11}{2} \right\} + \frac{1}{\varepsilon} \left\{ -\frac{97}{12} \right\} \right. \\
& + \left\{ -\frac{1111}{24} + 8 \zeta_2 + 6 \zeta_3 - 24 \zeta_2 \ln(2) \right\} + \varepsilon \left\{ -\frac{8581}{48} + 4c_1 + \frac{271}{12} \zeta_2 - \frac{252}{5} \zeta_2^2 + 52 \zeta_3 \right. \\
& - 144 \zeta_2 \ln(2) \Bigg\} + \varepsilon^2 \left\{ -\frac{58543}{96} + 24c_1 + \frac{24c_2}{5} + \frac{1537}{24} \zeta_2 - \frac{9783}{40} \zeta_2^2 + \frac{3929}{18} \zeta_3 - 66 \zeta_2 \zeta_3 \right. \\
& - 609 \zeta_5 - 528 \zeta_2 \ln(2) \Bigg\} \Bigg] + C_F T_F n_l \left[\frac{1}{\varepsilon^2} \left\{ -2 \right\} + \frac{1}{\varepsilon} \left\{ \frac{5}{3} \right\} + \left\{ \frac{71}{6} + 8 \zeta_2 \right\} \right. \\
& + \varepsilon \left\{ \frac{581}{12} + \frac{97}{3} \zeta_2 + 16 \zeta_3 \right\} + \varepsilon^2 \left\{ \frac{4079}{24} + \frac{643}{6} \zeta_2 + \frac{597}{10} \zeta_2^2 + \frac{478}{9} \zeta_3 \right\} \Bigg] + C_F T_F \left[\frac{1}{\varepsilon^2} \left\{ -2 \right\} \right. \\
& + \frac{1}{\varepsilon} \left\{ \frac{5}{3} \right\} + \left\{ \frac{143}{6} - 16 \zeta_2 \right\} + \varepsilon \left\{ \frac{1133}{12} - \frac{227}{3} \zeta_2 - 56 \zeta_3 + 96 \zeta_2 \ln(2) \right\} \\
& + \varepsilon^2 \left\{ \frac{8135}{24} - 16c_1 - \frac{1553}{6} \zeta_2 + \frac{837}{10} \zeta_2^2 - \frac{2546}{9} \zeta_3 + 480 \zeta_2 \ln(2) \right\} \Bigg]. \tag{A.4}
\end{aligned}$$

B The vector form factors up to two-loop

In this appendix, we present the vector form factors $F_{V,1}^{(n)}$ and $F_{V,2}^{(n)}$ up to two loops and $\mathcal{O}(\varepsilon)$.

$$\begin{aligned}
F_{V,1}^{(1)} = & C_F \left[\frac{1}{\varepsilon} \left\{ -2 - 2\xi H_0 \right\} + \left\{ -4 + \xi \left(4H_{-1}H_0 - H_0^2 - 4H_{0,-1} + 2\zeta_2 \right) + \left(-3 - 2x - 3x^2 \right) \eta H_0 \right\} \right. \\
& + \varepsilon \left\{ -8 + \eta \left(2 \left(3 + 2x + 3x^2 \right) H_{-1}H_0 + \frac{1}{2} \left(-3 - 2x - 3x^2 \right) H_0^2 - 2 \left(3 + 2x + 3x^2 \right) H_{0,-1} + 2 \left(1 \right. \right. \right. \\
& + \left. \left. \left. x + 2x^2 \right) \zeta_2 \right) + \xi \left(\left(-8 - 4H_{-1}^2 + \zeta_2 \right) H_0 + 2H_{-1}H_0^2 - \frac{1}{3}H_0^3 + 8H_{-1}H_{0,-1} - 4H_{0,0,-1} - 8H_{0,-1,-1} \right. \right. \\
& \left. \left. - 4H_{-1}\zeta_2 + 4\zeta_3 \right) \right\} + \varepsilon^2 \left\{ -16 + \eta \left(\left(-2 \left(3 + 2x + 3x^2 \right) H_{-1}^2 + \frac{1}{2} \left(3 + 2x + 3x^2 \right) \zeta_2 \right) H_0 + \left(3 + 2x \right. \right. \right. \\
& + \left. \left. \left. 3x^2 \right) H_{-1}H_0^2 + \frac{1}{6} \left(-3 - 2x - 3x^2 \right) H_0^3 + 4 \left(3 + 2x + 3x^2 \right) H_{-1}H_{0,-1} - 2 \left(3 + 2x + 3x^2 \right) H_{0,0,-1} \right. \right. \\
& - 4 \left(3 + 2x + 3x^2 \right) H_{0,-1,-1} + 2 \left(3 + 5x^2 \right) \zeta_2 - 2 \left(3 + 2x + 3x^2 \right) H_{-1}\zeta_2 + \frac{4}{3} \left(5 + 3x + 4x^2 \right) \zeta_3 \right) + \xi \left(\left(\right. \right. \\
& - 16 + \left(16 - 2\zeta_2 \right) H_{-1} + \frac{8}{3}H_{-1}^3 + \frac{14}{3}\zeta_3 \right) H_0 + \left(-4 - 2H_{-1}^2 + \frac{1}{2}\zeta_2 \right) H_0^2 + \frac{2}{3}H_{-1}H_0^3 - \frac{1}{12}H_0^4 + \left(-16 \right. \\
& \left. \left. - 8H_{-1}^2 - 2\zeta_2 \right) H_{0,-1} + 8H_{-1}H_{0,0,-1} + 16H_{-1}H_{0,-1,-1} - 4H_{0,0,0,-1} - 8H_{0,0,-1,-1} - 16H_{0,-1,-1,-1} \right. \\
& \left. \left. + 4H_{-1}^2\zeta_2 + \frac{14}{5}\zeta_2^2 - 8H_{-1}\zeta_3 \right) \right\} + \varepsilon^3 \left\{ -32 + \eta \left(\left(\frac{4}{3} \left(3 + 2x + 3x^2 \right) H_{-1}^3 + \left(-3 - 2x - 3x^2 \right) H_{-1}\zeta_2 \right. \right. \right. \\
& + \frac{7}{3} \left(3 + 2x + 3x^2 \right) \zeta_3 \right) H_0 + \left(\left(-3 - 2x - 3x^2 \right) H_{-1}^2 + \frac{1}{4} \left(3 + 2x + 3x^2 \right) \zeta_2 + \left(-1 - x^2 \right) H_{-1}\zeta_2 \right) H_0^2 \\
& + \frac{1}{3} \left(3 + 2x + 3x^2 \right) H_{-1}H_0^3 + \frac{1}{24} \left(-3 - 2x - 3x^2 \right) H_0^4 + \left(-4 \left(3 + 2x + 3x^2 \right) H_{-1}^2 + \left(-3 \right. \right. \\
& \left. \left. - 2x - 3x^2 \right) \zeta_2 \right) H_{0,-1} + 4 \left(3 + 2x + 3x^2 \right) H_{-1}H_{0,0,-1} + 8 \left(3 + 2x + 3x^2 \right) H_{-1}H_{0,-1,-1} - 2 \left(3 + 2x \right. \\
& + \left. \left. 3x^2 \right) H_{0,0,0,-1} - 4 \left(3 + 2x + 3x^2 \right) H_{0,0,-1,-1} - 8 \left(3 + 2x + 3x^2 \right) H_{0,-1,-1,-1} + 4 \left(3 + 5x^2 \right) \zeta_2 + 2 \left(3 \right. \right. \\
& + \left. \left. 2x + 3x^2 \right) H_{-1}^2\zeta_2 + \frac{1}{20} \left(75 + 56x + 93x^2 \right) \zeta_2^2 + \frac{4}{3} \left(13 + 11x^2 \right) \zeta_3 - 4 \left(3 + 2x + 3x^2 \right) H_{-1}\zeta_3 \right) \\
& + \xi \left(\left(-32 + \left(32 - \frac{28\zeta_3}{3} \right) H_{-1} + \left(-16 + 2\zeta_2 \right) H_{-1}^2 - \frac{4}{3}H_{-1}^4 + 4\zeta_2 + \frac{47}{20}\zeta_2^2 \right) H_0 + \left(-8 + 8H_{-1} \right. \right. \\
& + \frac{4}{3}H_{-1}^3 + \frac{7}{3}\zeta_3 \right) H_0^2 + \left(\frac{1}{6} \left(-8 + \zeta_2 \right) - \frac{2}{3}H_{-1}^2 \right) H_0^3 + \frac{1}{6}H_{-1}H_0^4 - \frac{1}{60}H_0^5 + \left(-16\zeta_2 - \frac{28\zeta_2^2}{5} \right) H_{-1} \\
& + \left(-32 + \left(32 + 4\zeta_2 \right) H_{-1} + \frac{16}{3}H_{-1}^3 + \frac{4}{3}\zeta_3 \right) H_{0,-1} + \left(-16 - 8H_{-1}^2 - 2\zeta_2 \right) H_{0,0,-1} + \left(-32 \right. \\
& \left. \left. - 16H_{-1}^2 - 4\zeta_2 \right) H_{0,-1,-1} + 8H_{-1}H_{0,0,0,-1} + 16H_{-1}H_{0,0,-1,-1} + 32H_{-1}H_{0,-1,-1,-1} - 4H_{0,0,0,0,-1} \right. \\
& \left. \left. - 8H_{0,0,0,-1,-1} - 16H_{0,0,-1,-1,-1} - 32H_{0,-1,-1,-1,-1} - \frac{8}{3}H_{-1}^3\zeta_2 + 8H_{-1}^2\zeta_3 - \frac{8}{3}\zeta_2\zeta_3 + 12\zeta_5 \right) \right\} \Bigg]. \tag{B.1}
\end{aligned}$$

$$F_{V,1}^{(2)} = C_F^2 \left[\frac{1}{\varepsilon^2} \left\{ 2 + 4\xi H_0 + 2\xi^2 H_0^2 \right\} + \frac{1}{\varepsilon} \left\{ 8 + \xi \left(-8H_{-1}H_0 + 4 \left(2 + x + x^2 \right) \eta H_0^2 + 8H_{0,-1} - 4\zeta_2 \right) \right\} \right]$$

$$\begin{aligned}
& + \xi^2 \left(-8H_{-1}H_0^2 + 2H_0^3 + 8H_0H_{0,-1} - 4H_0\zeta_2 \right) + 2 \left(7 + 2x + 7x^2 \right) \eta H_0 \left. \right\} + \left\{ 46 + \eta^2 \left(\left(\right. \right. \right. \\
& - 4H_{-1}H_0^2P_{103} + 8H_0H_{0,-1}P_{121} - 8H_0H_{0,1}P_{124} + 8H_{0,0,1}P_{135} - 8H_{0,0,-1}P_{136} + \frac{1}{2}H_0^2P_{161} \\
& - 4P_{115}\zeta_3 \left. \right) x_+^2 + \left(\frac{16}{3}H_0^3H_1P_{176} - 4H_0^2H_{0,-1}P_{180} - 8H_0H_{0,0,-1}P_{193} + 48H_{0,0,0,-1}P_{194} \right. \\
& - 8H_0^2H_{0,1}P_{195} - \frac{4}{3}H_0^3P_{204} + 16H_0H_{0,0,1}P_{216} - 16H_{0,0,0,1}P_{242} + \frac{1}{6}H_0^4P_{274} + \left(-4xH_0P_{134} \right. \\
& + 16H_0H_1P_{174} - 16H_{0,1}P_{174} - 8H_{0,-1}P_{218} + 4H_0^2P_{219} \left. \right) \zeta_2 + \frac{2}{5}P_{308}\zeta_2^2 \left. \right) x_+^3 \left. \right\} + \xi \left(8H_{-1}^2H_0 \right. \\
& - 16H_{-1}H_{0,-1} + 16H_{0,-1,-1} + 48(-1+x)\ln(2)x_+\zeta_2 \left. \right) + \xi^2 \left(16H_{-1}^2H_0^2 - 8H_{-1}H_0^3 + \left(32H_0H_1 \right. \right. \\
& - 32H_{0,-1} \left. \right) H_{0,1} + 16H_{0,1}^2 - 32H_{-1}H_0H_{0,-1} - 32H_0H_1H_{0,-1} + 8H_{0,-1}^2 - 64H_1H_{0,0,1} + 64H_1H_{0,0,-1} \\
& - 64H_0H_{0,1,1} + 32H_0H_{0,1,-1} + 32H_0H_{0,-1,1} + 16H_0H_{0,-1,-1} + 64H_{0,0,1,1} + 32H_{0,-1,0,1} \\
& + 16H_{-1}H_0\zeta_2 + 16H_1\zeta_3 \left. \right) + \eta \left(x_+^3\zeta_2 \left(39 + 7x - 750x^2 + 1074x^3 - 185x^4 - 57x^5 - 8H_{-1}P_{51} \right) \right. \\
& + \frac{1}{2} \left(85 - 18x + 85x^2 \right) H_0 + \left(2H_{-1}H_0P_{37} - 2H_{0,-1}P_{37} - 32 \left(2 + x + 10x^2 + x^3 + 2x^4 \right) H_0H_1 \right. \\
& + 32 \left(2 + x + 10x^2 + x^3 + 2x^4 \right) H_{0,1} \left. \right) x_+^2 - 16H_0P_{99}x_+^4\zeta_3 \left. \right) + 4H_0^2H_1P_{24}x_+^4 + 192x\ln(2)x_+^2\zeta_2 \left. \right\} \\
& + \varepsilon \left\{ \ln(2)x_+^4\zeta_2 \left(-24P_4 + 48H_1P_{12} + 48H_{-1}P_{12} \right) + \xi \left(\eta \left(\ln(2)x_+^3\zeta_2 \left(-192(-1+x)x^2H_{0,1} \right. \right. \right. \right. \\
& - 192(-1+x)x^2H_{0,-1} \left. \right) - \frac{4}{3}H_0^3P_{66}x_+^3\zeta_2 \left. \right) - \frac{16}{3}H_{-1}^3H_0 - 32x \left(2 - 29x + 2x^2 \right) x_+^4H_0H_{0,1}^2 \\
& + 16H_{-1}^2H_{0,-1} - 32H_{-1}H_{0,-1,-1} + 32H_{0,-1,-1,-1} - 4(-1+x)x_+ \left. \right) + \eta^2 \left(16H_0^2H_{0,1,1}P_{53}x_+ \right. \\
& + \left(H_{-1}H_0^2P_{83} + 16H_0H_{0,1,1}P_{95} + 4H_{-1}^2H_0^2P_{100} + 8H_{0,1}^2P_{119} + 16H_0H_{0,1,-1}P_{129} + 16H_0H_{0,-1,1}P_{129} \right. \\
& + 16H_{0,-1,0,1}P_{132} + 16H_{0,0,1,-1}P_{133} + 16H_{0,0,-1,1}P_{133} - 16H_{0,0,-1,-1}P_{137} + 8H_{0,-1}^2P_{141} + \frac{1}{2}H_0^2P_{163} \\
& + \left(32H_0P_{88} + 32H_{-1}H_0P_{109} - 16H_{0,-1}P_{140} \right) H_{0,1} + \left(-16H_{-1}H_0P_{125} - 8H_0P_{149} \right) H_{0,-1} + \left(\right. \\
& - 16H_{-1}P_{133} - 4P_{156} \left. \right) H_{0,0,1} + \left(16H_{-1}P_{137} + 2P_{164} \right) H_{0,0,-1} + \left(4H_{-1}P_{145} - \frac{2P_{165}}{3} \right) \zeta_3 \left. \right) x_+^2 + \left(\right. \\
& - 16H_0^2H_{0,1,-1}P_{111} - 16H_0^2H_{0,-1,1}P_{111} + \frac{16}{3}H_0^3H_1^2P_{186} + 32H_{0,0,1,0,1}P_{189} - 16H_0H_{0,0,-1,-1}P_{199} \\
& - 32H_{0,0,1,0,-1}P_{203} - 32H_0H_{0,0,1,-1}P_{213} - 32H_0H_{0,0,-1,1}P_{213} + 32H_{0,0,-1,0,1}P_{220} + 8H_0^2H_{0,-1,-1}P_{230} \\
& - 32H_{0,0,0,1,-1}P_{238} - 32H_{0,0,0,-1,1}P_{238} + \frac{1}{3}H_0^4P_{241} + 32H_{0,0,-1,0,-1}P_{249} + 96H_{0,0,0,-1,-1}P_{250} \\
& - 32H_0H_{0,0,1,1}P_{256} + 32H_{0,0,0,1,1}P_{287} + \frac{1}{30}H_0^5P_{290} - 16H_{0,0,0,0,1}P_{305} + 16H_{0,0,0,0,-1}P_{306} \\
& + \frac{1}{6}H_0^3P_{309} + \left(2H_0^4P_{175} - \frac{32}{3}H_{-1}H_0^3P_{186} \right) H_1 + \left(\frac{4}{3}H_0^3P_{196} - \frac{2}{3}H_0^4P_{231} \right) H_{-1} + \left(\frac{8}{3}H_0^3P_{190} \right. \\
& + 16H_{-1}H_0^2P_{198} - 4H_0^2P_{284} \left. \right) H_{0,1} + \left(-16xH_{-1}H_0^2P_{98} + 16H_0^2H_1P_{178} - \frac{4}{3}H_0^3P_{257} \right. \\
& + 4H_0^2P_{292} \left. \right) H_{0,-1} + \left(32H_{0,1}P_{170} - 64H_{-1}H_0P_{179} + 256H_0H_1P_{185} - 8H_0P_{228} - 8H_0^2P_{232} \right) H_{0,0,1}
\end{aligned}$$

$$\begin{aligned}
& + \left(32H_{-1}H_0P_{117} - 8H_0P_{225} - 32H_{0,1}P_{227} + 4H_0^2P_{239} - 16H_{0,-1}P_{278} \right) H_{0,0,-1} + \left(32H_{-1}P_{182} \right. \\
& - 64H_1P_{248} + 16H_0P_{293} + 8P_{300} \left. \right) H_{0,0,0,1} + \left(-96H_{-1}P_{112} + 32H_1P_{285} - 8H_0P_{297} - 8P_{303} \right) H_{0,0,0,-1} \\
& + \left(2xH_0^2P_{154} + H_0P_{166} + 32H_0H_1^2P_{177} + 64H_{0,1,1}P_{186} - 32H_{0,1,-1}P_{223} - 32H_{0,-1,1}P_{224} \right. \\
& - 16H_{0,0,1}P_{254} + 16H_{0,-1,-1}P_{275} + 8H_{0,0,-1}P_{294} + \left(-64H_{-1}H_0P_{177} - 8H_0^2P_{236} \right) H_1 + \left(\right. \\
& - 8H_0^2P_{188} + 8H_0P_{266} \left. \right) H_{-1} + \left(64H_{-1}P_{177} - 32H_1P_{197} - 8P_{268} \right) H_{0,1} + \left(-32H_{-1}P_{184} - 8P_{200} \right. \\
& - 16H_0P_{207} + 32H_1P_{224} \left. \right) H_{0,-1} - 4P_{289}\zeta_3 \left. \right) \zeta_2 + \left(-\frac{8}{5}H_{-1}P_{183} - \frac{8}{5}H_1P_{296} + \frac{4P_{307}}{5} - \frac{2}{5}H_0P_{310} \right) \zeta_2^2 \\
& + \left(-16H_0H_1P_{209} - 16H_{0,-1}P_{245} + 16H_{0,1}P_{253} - \frac{4}{3}H_0^2P_{291} - \frac{4}{3}H_0P_{295} \right) \zeta_3 + 6P_{299}\zeta_5 \left. \right) x_+^3 \\
& + \eta \left(\ln(2)x_+^4\zeta_2 \left(-3072x^3H_{0,1} - 3072x^3H_{0,-1} \right) + \left(4H_{-1}H_0P_{40} - 2H_{-1}^2H_0P_{41} - 4H_{0,-1,-1}P_{41} \right. \right. \\
& + \frac{1}{4}H_0P_{43} + 32 \left(2 + x + 10x^2 + x^3 + 2x^4 \right) \zeta_2 H_1 + \left(-4H_0P_{36} + 192 \left(2 + x + 10x^2 + x^3 \right. \right. \\
& + 2x^4 \left. \right) H_{-1}H_0 \left. \right) H_1 - 32 \left(2 + x + 10x^2 + x^3 + 2x^4 \right) H_0H_1^2 + \left(4P_{36} + 64 \left(2 + x + 10x^2 + x^3 \right. \right. \\
& + 2x^4 \left. \right) H_1 - 192 \left(2 + x + 10x^2 + x^3 + 2x^4 \right) H_{-1} \left. \right) H_{0,1} + \left(-4P_{40} + 4H_{-1}P_{41} - 192 \left(2 + x + 10x^2 \right. \right. \\
& + x^3 + 2x^4 \left. \right) H_1 \left. \right) H_{0,-1} - 64 \left(2 + x + 10x^2 + x^3 + 2x^4 \right) H_{0,1,1} + 192 \left(2 + x + 10x^2 + x^3 + 2x^4 \right) H_{0,1,-1} \\
& + 192 \left(2 + x + 10x^2 + x^3 + 2x^4 \right) H_{0,-1,1} \left. \right) x_+^2 + \left(-2H_0^2H_1P_{71} + \left(163 + 251x - 3084x^2 + 4532x^3 \right. \right. \\
& - 847x^4 - 183x^5 + 8H_{-1}^2P_{63} - 2H_{-1}P_{72} \left. \right) \zeta_2 \left. \right) x_+^3 + \left(384H_0H_1H_{0,0,-1}P_{92} - 48H_0H_{0,-1}^2P_{97} \right. \\
& + 32H_0H_{0,-1,0,1}P_{122} - 32H_{0,-1}H_{0,0,1}P_{127} - \frac{4}{3}H_0^3H_1P_{151} + \left(32H_0^2H_1P_{104} - 32H_0H_{0,-1}P_{106} \right) H_{0,1} \\
& + \left(-16H_0H_{0,1}P_{128} - 8H_0H_1P_{142} \right) \zeta_2 - 32H_{-1}H_0P_{99}\zeta_3 \left. \right) x_+^4 + \xi^2 \left(-\frac{64}{3}H_{-1}^3H_0^2 + 16H_{-1}^2H_0^3 \right. \\
& + \left(128H_{-1}H_0H_{0,-1} - 96H_{0,-1}^2 \right) H_1 + \left(\left(-128H_{-1}H_0 + 64H_{0,-1} \right) H_1 + 64H_0H_1^2 \right. \\
& + 128H_{-1}H_{0,-1} \left. \right) H_{0,1} + \left(32H_1 - 64H_{-1} \right) H_{0,1}^2 + 64H_{-1}^2H_0H_{0,-1} - 64H_0H_1^2H_{0,-1} - 32H_{-1}H_{0,-1}^2 \\
& + \left(256H_{-1}H_1 - 128H_1^2 \right) H_{0,0,1} + \left(-256H_{-1}H_1 + 128H_1^2 \right) H_{0,0,-1} + \left(256H_{-1}H_0 - 192H_0H_1 \right. \\
& - 256H_{0,-1} \left. \right) H_{0,1,1} + \left(-128H_{-1}H_0 + 64H_0H_1 - 64H_{0,1} + 192H_{0,-1} \right) H_{0,1,-1} + \left(-128H_{-1}H_0 \right. \\
& + 64H_0H_1 + 192H_{0,1} + 192H_{0,-1} \left. \right) H_{0,-1,1} + \left(-64H_{-1}H_0 + 64H_0H_1 - 320H_{0,1} + 32H_{0,-1} \right) H_{0,-1,-1} \\
& + \left(256H_1 - 256H_{-1} \right) H_{0,0,1,1} - 256H_1H_{0,0,1,-1} - 256H_1H_{0,0,-1,1} + 256H_1H_{0,0,-1,-1} + 192H_0H_{0,1,1,1} \\
& - 64H_0H_{0,1,-1,-1} - 128H_{-1}H_{0,-1,0,1} - 64H_0H_{0,-1,1,-1} - 64H_0H_{0,-1,-1,1} + 32H_0H_{0,-1,-1,-1} \\
& - 384H_{0,0,1,1,1} + 768H_{0,0,1,1,-1} + 256H_{0,0,1,-1,1} - 256H_{0,0,-1,1,1} - 64H_{0,1,0,1,1} + 256H_{0,1,0,1,-1} \\
& - 256H_{0,-1,0,1,1} + 128H_{0,-1,0,1,-1} + 128H_{0,-1,0,-1,1} - 256H_{0,-1,1,0,1} + 320H_{0,-1,-1,0,1} - 32H_{-1}^2H_0\zeta_2 \\
& + \left(-64H_{-1}H_1 + 32H_1^2 \right) \zeta_3 \left. \right) + 28xx_+^2 + 8c_1 \left(1 - 4x + x^2 \right) x_+^2 + \left(-8H_{-1}H_0^2H_1P_{12} \right. \\
& - 32H_0H_{0,-1,-1}P_{19} + 16H_1H_{0,0,1}P_{20} - 16H_{0,0,1,1}P_{20} - 16H_0H_1H_{0,-1}P_{21} - 16H_0H_1H_{0,1}P_{27} \\
& \left. \left. + 4H_0^2H_1^2P_{29} + 16H_1H_{0,0,-1}P_{30} + \left(-48H_{-1}H_1P_{12} + 48H_{-1,1}P_{12} \right) \zeta_2 - 4H_1P_{39}\zeta_3 \right) x_+^4 \right\} \Bigg]
\end{aligned}$$

$$\begin{aligned}
& + C_F C_A \left[\frac{1}{\varepsilon^2} \left\{ \frac{11}{3} + \frac{11\xi H_0}{3} \right\} + \frac{1}{\varepsilon} \left\{ -\frac{49}{9} + \xi \left(\eta \left(-\frac{4}{3} x^2 H_0^3 - 2(-1 + 3x^2) H_0 \zeta_2 \right) \right. \right. \right. \\
& - \frac{67}{9} H_0 + 4H_{-1} H_0 - 4H_0 H_1 + 4H_{0,1} - 4H_{0,-1} \left. \left. \left. \right) + \eta \left(-4x^2 H_0^2 - 2(-1 + 3x^2) \zeta_2 \right) \right. \right. \\
& \left. \left. \left. + \xi^2 \left(-4H_0 H_{0,1} + 4H_0 H_{0,-1} + 8H_{0,0,1} - 8H_{0,0,-1} - 2\zeta_3 \right) \right\} + \left\{ \frac{1}{27} \left(-1595 - 3514x \right. \right. \right. \\
& - 1595x^2 \left. \left. \left. \right) x_+^2 + \eta^2 x_+^3 \left(-32x H_0 H_{0,0,1} P_{94} - \frac{1}{6} x H_0^4 P_{123} + \frac{4}{3} H_0^3 H_1 P_{169} + 4H_0^2 H_{0,-1} P_{211} \right. \right. \\
& - 8H_{0,0,0,-1} P_{237} + 8H_{0,0,0,1} P_{251} + \left(16H_0 H_1 P_{172} + 2H_0^2 P_{173} - 4H_{0,1} P_{192} + 4H_{0,-1} P_{217} \right) \zeta_2 \\
& + \frac{1}{5} P_{168} \zeta_2^2 - 2H_0 P_{244} \zeta_3 \left. \right) + \xi^2 \left(\left(-8H_1 + 8H_{-1} \right) \zeta_3 + \left(16H_{-1} H_0 - 16H_0 H_1 - 8H_{0,-1} \right) H_{0,1} \right. \\
& - 4H_{0,1}^2 - 16H_{-1} H_0 H_{0,-1} + 16H_0 H_1 H_{0,-1} + 12H_{0,-1}^2 + \left(32H_1 - 32H_{-1} \right) H_{0,0,1} + \left(-32H_1 \right. \\
& \left. \left. \left. + 32H_{-1} \right) H_{0,0,-1} + 24H_0 H_{0,1,1} - 8H_0 H_{0,1,-1} - 8H_0 H_{0,-1,1} - 8H_0 H_{0,-1,-1} - 32H_{0,0,1,1} \right. \right. \\
& \left. \left. \left. + 32H_{0,0,1,-1} + 32H_{0,0,-1,1} - 32H_{0,0,-1,-1} \right) + \eta \left(\left(\frac{1}{54} H_0 P_3 + 4H_0 H_1 P_{11} - 4H_{0,1} P_{11} + \frac{4}{9} H_{-1} H_0 P_{35} \right. \right. \right. \\
& - \frac{4}{9} H_{0,-1} P_{35} \left. \left. \left. \right) x_+^2 + \left(4H_0^2 H_1 P_{49} + 8H_{0,0,1} P_{57} + \frac{2}{3} H_{-1} H_0^2 P_{67} - \frac{4}{3} H_{0,0,-1} P_{73} + \frac{1}{18} H_0^2 P_{77} \right. \right. \\
& + \left(\frac{4}{3} H_{-1} P_{56} + \frac{P_{79}}{9} \right) \zeta_2 - \frac{2}{3} P_{50} \zeta_3 \left. \right) x_+^3 + \left(\frac{1}{9} H_0^3 P_{84} + 4H_0^2 H_{0,1} P_{96} - \frac{2}{3} H_0 P_{143} \zeta_2 \right) x_+^4 + \xi \left(\eta \left(\frac{4}{3} \left(3 \right. \right. \right. \\
& \left. \left. \left. + x^2 \right) H_{-1} H_0^3 + 16H_{-1} H_0 \zeta_2 \right) - \frac{104}{3} H_{-1}^2 H_0 + 24H_{-1} H_0 H_1 - 4H_0 H_1^2 + \left(8H_1 - 24H_{-1} \right) H_{0,1} \right. \\
& \left. \left. \left. + \left(-24H_1 + \frac{208H_{-1}}{3} \right) H_{0,-1} - 8H_{0,1,1} + 24H_{0,1,-1} + 24H_{0,-1,1} - \frac{208}{3} H_{0,-1,-1} + 4H_1 \zeta_2 \right. \right. \right. \\
& - 24(-1+x) \ln(2)x_+ \zeta_2 \left. \left. \left. \right) + \left(24H_0 H_{0,1} P_9 - 4H_0 H_{0,-1} P_{17} \right) x_+^4 - 8(-1+x) H_0 H_{0,0,-1} P_8 x_+^5 \right. \right. \\
& \left. \left. \left. - 96x \ln(2) x_+^2 \zeta_2 \right\} + \varepsilon \left\{ \frac{1}{162} \left(-28745 - 70126x - 28745x^2 \right) x_+^2 + \ln(2) x_+^4 \zeta_2 \left(12P_4 - 24H_1 P_{12} \right. \right. \right. \\
& - 24H_{-1} P_{12} \left. \left. \left. \right) + \eta \left(\ln(2) x_+^4 \zeta_2 \left(1536x^3 H_{0,1} + 1536x^3 H_{0,-1} \right) + \left(\frac{1}{324} H_0 P_1 + 4H_0 H_1^2 P_{11} + 8H_{0,1,1} P_{11} \right. \right. \right. \\
& - 24H_{0,1,-1} P_{11} - 24H_{0,-1,1} P_{11} + \frac{4}{9} H_{-1}^2 H_0 P_{34} + \frac{8}{9} H_{0,-1,-1} P_{34} + \frac{1}{27} H_{-1} H_0 P_{45} + \left(-24H_{-1} H_0 P_{11} \right. \\
& \left. \left. \left. + 4H_0 P_{15} \right) H_1 + \left(-8H_1 P_{11} + 24H_{-1} P_{11} - 4P_{15} \right) H_{0,1} + \left(\frac{P_2}{27} + 24H_1 P_{11} - \frac{8}{9} H_{-1} P_{34} \right) H_{0,-1} \right. \right. \\
& - 4H_1 P_{11} \zeta_2 \left. \left. \left. \right) x_+^2 + \left(-8H_{0,0,1,1} P_{58} + 2H_0^2 H_1^2 P_{59} + 8H_{0,0,1,-1} P_{62} + 8H_{0,0,-1,1} P_{62} - 2H_{-1}^2 H_0^2 P_{68} \right. \right. \\
& - 8H_{0,0,-1,-1} P_{70} + \frac{1}{9} H_{-1} H_0^2 P_{74} + \frac{1}{54} H_0^2 P_{78} + \left(4H_{-1} H_0^2 P_{55} + 2H_0^2 P_{64} \right) H_1 + \left(8H_1 P_{58} - 4P_{60} \right. \\
& \left. \left. \left. - 8H_{-1} P_{62} \right) H_{0,0,1} + \left(8H_1 P_{54} + 8H_{-1} P_{70} + \frac{2P_{80}}{9} \right) H_{0,0,-1} + \left(-48H_{-1} H_1 P_{48} - 12H_{-1}^2 P_{52} \right. \right. \\
& - \frac{2}{9} H_{-1} P_{76} + \frac{P_{81}}{27} \left. \left. \left. \right) \zeta_2 + \left(\frac{P_{46}}{9} - 2H_{-1} P_{65} - 2H_1 P_{69} \right) \zeta_3 \right) x_+^3 + \left(\frac{1}{12} H_0^4 P_{85} + 32H_0 H_{0,0,1,-1} P_{87} \right. \\
& + 32H_0 H_{0,0,-1,1} P_{87} + 48H_0 H_{0,-1}^2 P_{93} + 16H_0 H_{0,1}^2 P_{101} - 16H_0 H_{0,-1,0,1} P_{110} + 4H_0 H_{0,0,1} P_{116} \\
& - 16H_0 H_{0,0,-1,-1} P_{120} + \frac{2}{3} H_0^3 H_1 P_{146} + 2H_0^2 H_{0,-1} P_{150} + 4H_{0,0,0,1} P_{152} + \frac{2}{3} H_{-1} H_0^3 P_{153}
\end{aligned}$$

$$\begin{aligned}
& -4H_{0,0,0,-1}P_{155} + \frac{1}{54}H_0^3P_{162} + \left(8H_{-1}H_0^2P_{96} - 16H_0^2H_1P_{104} - 16H_0H_{0,-1}P_{108} - 2H_0^2P_{147}\right)H_{0,1} \\
& + \left(-32H_0H_1P_{105} - 4H_0P_{148}\right)H_{0,0,-1} + \left(\frac{1}{18}H_0P_{82} + 2H_{-1}H_0P_{126} - 2H_{0,-1}P_{130} - 4H_{0,1}P_{131}\right. \\
& + 4H_0H_1P_{139} + \frac{1}{2}H_0^2P_{159}\left.)\zeta_2 + \frac{1}{5}P_{160}\zeta_2^2 + \frac{4}{3}H_0P_{144}\zeta_3\right)x_+^4 + \xi\left(\eta\left(\ln(2)x_+^3\zeta_2\left(96(-1+x)x^2H_{0,1}\right.\right.\right. \\
& + 96(-1+x)x^2H_{0,-1}\left.)\right) - \frac{4}{3}\left(3+5x^2\right)H_{-1}^2H_0^3 - 8\left(1+3x^2\right)H_{-1}^2H_0\zeta_2\left.) + 80H_{-1}^3H_0 - 72H_{-1}^2H_0H_1\right. \\
& + 24H_{-1}H_0H_1^2 - \frac{8}{3}H_0H_1^3 + \left(-48H_{-1}H_1 + 8H_1^2 + 72H_{-1}^2\right)H_{0,1} + \left(144H_{-1}H_1 - 24H_1^2\right. \\
& - 240H_{-1}^2\left.)H_{0,-1} + \left(-16H_1 + 48H_{-1}\right)H_{0,1,1} + \left(48H_1 - 144H_{-1}\right)H_{0,1,-1} + \left(48H_1\right. \\
& - 144H_{-1}\left.)H_{0,-1,1} + \left(-144H_1 + 480H_{-1}\right)H_{0,-1,-1} + 16H_{0,1,1,1} - 48H_{0,1,1,-1} - 48H_{0,1,-1,1}\right. \\
& + 144H_{0,1,-1,-1} - 48H_{0,-1,1,1} + 144H_{0,-1,1,-1} + 144H_{0,-1,-1,1} - 480H_{0,-1,-1,-1} + 4H_1^2\zeta_2\left.)\right) \\
& + \eta^2\left(\left(-\frac{1}{30}xH_0^5P_{138} + 128H_0H_{0,0,1,1}P_{187} - 8H_0^2H_{0,1,1}P_{202} + 8H_0^2H_{0,1,-1}P_{205} + 8H_0^2H_{0,-1,1}P_{205}\right.\right. \\
& + \frac{1}{3}H_{-1}H_0^4P_{212} - \frac{4}{3}H_0^3H_1^2P_{215} - \frac{8}{3}H_0^3H_{0,1}P_{229} - 16H_{0,0,1,0,1}P_{246} - 8H_0^2H_{0,-1,-1}P_{247} \\
& - 32H_{0,0,-1,0,-1}P_{252} + 16H_{0,0,-1,0,1}P_{255} + 16H_{0,0,1,0,-1}P_{262} + 32H_{0,0,0,1,-1}P_{283} + 32H_{0,0,0,-1,1}P_{283} \\
& - 16H_{0,0,0,1,1}P_{298} + 8H_{0,0,0,0,1}P_{301} - 8H_{0,0,0,0,-1}P_{302} - 16H_{0,0,0,-1,-1}P_{304} + \left.\left(\frac{1}{3}H_0^4P_{86}\right.\right. \\
& + \left.\left.\frac{8}{3}H_{-1}H_0^3P_{215}\right)H_1 + \left(-8H_{-1}H_0^2P_{118} - 8H_0^2H_1P_{201} + \frac{4}{3}H_0^3P_{288}\right)H_{0,-1} + \left(32H_{-1}H_0P_{171}\right. \\
& - 32H_0H_1P_{210} + 16H_{0,1}P_{221} - 16H_{0,-1}P_{263} + 4H_0^2P_{269}\left.)H_{0,0,1} + \left(-16H_{0,1}P_{233} + 32H_{0,-1}P_{243}\right.\right. \\
& - 4H_0^2P_{281}\left.)H_{0,0,-1} + \left(-16H_{-1}P_{214} + 16H_1P_{259} - 8H_0P_{286}\right)H_{0,0,0,1} + \left(16H_{-1}P_{226} - 16H_1P_{240}\right. \\
& + 8H_0P_{280}\left.)H_{0,0,0,-1} + \left(-8x^2H_{-1}H_0^2P_{47} - 8H_0H_1^2P_{206} - 8H_{0,1,1}P_{234} + \frac{4}{3}H_0^3P_{261} - 16H_{0,0,-1}P_{264}\right. \\
& + 8H_{0,1,-1}P_{270} + 8H_{0,-1,1}P_{271} - 8H_{0,-1,-1}P_{276} + 8H_{0,0,1}P_{282} + \left(4xH_0^2P_{107} + 16H_{-1}H_0P_{206}\right)H_1 \\
& + \left(32xH_1P_{102} - 32xH_{-1}P_{102} - 4H_0P_{279}\right)H_{0,1} + \left(16H_{-1}P_{208} - 16H_1P_{235} + 4H_0P_{277}\right)H_{0,-1} \\
& + 2P_{267}\zeta_3\left.)\zeta_2 + \left(\frac{8}{5}H_{-1}P_{222} - \frac{2}{5}H_0P_{260} - \frac{4}{5}H_1P_{265}\right)\zeta_2^2 + \left(16H_0H_1P_{181} - 16H_{-1}H_0P_{191}\right. \\
& - 4H_{0,1}P_{258} + 2H_0^2P_{272} + 4H_{0,-1}P_{273}\left.)\zeta_3 + P_{167}\zeta_5\right)x_+^3 - 8H_{-1}H_0^2x_+^2\zeta_2\left.) + \xi^2\left(\left(-64H_{-1}H_0H_{0,-1}\right.\right. \\
& + 48H_{0,-1}^2\left.)H_1 + \left(-32H_{-1}^2H_0 + \left(64H_{-1}H_0 - 32H_{0,-1}\right)H_1 - 32H_0H_1^2 + 32H_{-1}H_{0,-1}\right)H_{0,1} \\
& + \left(-16H_1 + 16H_{-1}\right)H_{0,1}^2 + 32H_{-1}^2H_0H_{0,-1} + 32H_0H_1^2H_{0,-1} - 48H_{-1}H_{0,-1}^2 + \left(-128H_{-1}H_1\right. \\
& + 64H_1^2 + 64H_{-1}^2\left.)H_{0,0,1} + \left(128H_{-1}H_1 - 64H_1^2 - 64H_{-1}^2\right)H_{0,0,-1} + \left(-96H_{-1}H_0 + 96H_0H_1\right. \\
& + 16H_{0,1} + 80H_{0,-1}\left.)H_{0,1,1} + \left(32H_{-1}H_0 - 32H_0H_1 + 16H_{0,1} - 48H_{0,-1}\right)H_{0,1,-1} + \left(32H_{-1}H_0\right. \\
& - 32H_0H_1 - 48H_{0,1} - 48H_{0,-1}\left.)H_{0,-1,1} + \left(32H_{-1}H_0 - 32H_0H_1 + 16H_{0,1} - 48H_{0,-1}\right)H_{0,-1,-1} \\
& + \left(-128H_1 + 128H_{-1}\right)H_{0,0,1,1} + \left(128H_1 - 128H_{-1}\right)H_{0,0,1,-1} + \left(128H_1 - 128H_{-1}\right)H_{0,0,-1,1} \\
& + \left(-128H_1 + 128H_{-1}\right)H_{0,0,-1,-1} - 112H_0H_{0,1,1,1} + 16H_0H_{0,1,1,-1} + 16H_0H_{0,1,-1,1}
\end{aligned}$$

$$\begin{aligned}
& + 16H_0H_{0,1,-1,-1} + 16H_0H_{0,-1,1,1} + 16H_0H_{0,-1,1,-1} + 16H_0H_{0,-1,-1,1} + 16H_0H_{0,-1,-1,-1} \\
& + 128H_{0,0,1,1,1} - 256H_{0,0,1,1,-1} - 128H_{0,0,1,-1,1} + 128H_{0,0,1,-1,-1} + 128H_{0,0,-1,1,-1} \\
& + 128H_{0,0,-1,-1,1} + 256H_{0,0,-1,-1,-1} - 64H_{0,1,0,1,-1} + 64H_{0,-1,0,1,1} + 192H_{0,-1,0,-1,-1} \\
& + 64H_{0,-1,1,0,1} + \left(32H_{-1}H_1 - 16H_1^2 - 16H_{-1}^2\right)\zeta_3 - 4c_1\left(1 - 4x + x^2\right)x_+^2 + \left(8H_{-1}H_0H_{0,-1}P_5 \right. \\
& - 8H_0H_{0,1,1}P_{10} - 8H_{0,1}^2P_{14} - 24H_{0,-1,0,1}P_{16} - 16H_0H_{0,1,-1}P_{22} - 16H_0H_{0,-1,1}P_{22} \\
& + 8H_0H_1H_{0,-1}P_{23} + 16H_0H_{0,-1}P_{25} + 8H_0H_{0,-1,-1}P_{31} - 4H_{0,-1}^2P_{33} + \left(-16H_{-1}H_0P_{13} + 4H_0P_{18} \right. \\
& \left. + 8H_0H_1P_{26} + 16H_{0,-1}P_{28}\right)H_{0,1} - 24H_{-1,1}P_{12}\zeta_2)x_+^4 - 16(-1+x)H_{-1}H_0H_{0,0,-1}P_8x_+^5 \left. \right] \\
& + C_F n_l T_F \left[\frac{1}{\varepsilon^2} \left\{ -\frac{4}{3} - \frac{4\xi H_0}{3} \right\} + \frac{1}{\varepsilon} \left\{ \frac{20}{9} + \frac{20\xi H_0}{9} \right\} + \left\{ \frac{424}{27} + \eta \left(\frac{2}{27} (209 + 6x + 209x^2) H_0 - \frac{8}{9} (19 \right. \right. \right. \\
& + 6x + 19x^2) H_{-1}H_0 + \frac{2}{9} (19 + 6x + 19x^2) H_0^2 + \frac{8}{9} (19 + 6x + 19x^2) H_{0,-1} - \frac{4}{9} (7 + 6x + 31x^2) \zeta_2) \\
& + \xi \left(\frac{16}{3} H_{-1}^2 H_0 - \frac{8}{3} H_{-1} H_0^2 + \frac{4}{9} H_0^3 - \frac{32}{3} H_{-1} H_{0,-1} + \frac{16}{3} H_{0,0,-1} + \frac{32}{3} H_{0,-1,-1} + \left(\frac{8}{3} H_0 + \frac{16}{3} H_{-1} \right) \zeta_2 \right. \\
& \left. \left. - \frac{16\zeta_3}{3} \right) \right\} + \varepsilon \left\{ \frac{5204}{81} + \xi \left(\left(-\frac{16H_0}{3} + 32H_{-1} \right) \zeta_3 - \frac{32}{3} H_{-1}^3 H_0 + 8H_{-1}^2 H_0^2 - \frac{8}{3} H_{-1} H_0^3 + 32H_{-1}^2 H_{0,-1} \right. \right. \\
& - 32H_{-1} H_{0,0,-1} - 64H_{-1} H_{0,-1,-1} + 16H_{0,0,0,-1} + 32H_{0,0,-1,-1} + 64H_{0,-1,-1,-1} + \left(-8H_{-1}H_0 + 2H_0^2 \right. \\
& \left. \left. - 16H_{-1}^2 + 24H_{0,-1} \right) \zeta_2 - \frac{96\zeta_2^2}{5} \right) + \eta \left(\frac{1}{81} (5813 - 1218x + 5813x^2) H_0 + \frac{8}{9} (47 + 18x + 47x^2) H_{-1}^2 H_0 \right. \\
& + \frac{2}{27} (281 + 6x + 281x^2) H_0^2 + \frac{2}{27} (47 + 18x + 47x^2) H_0^3 + \frac{1}{3} (1 + x^2) H_0^4 + \left(-\frac{8}{27} (281 + 6x \right. \\
& + 281x^2) H_0 - \frac{4}{9} (47 + 18x + 47x^2) H_0^2) H_{-1} + \left(\frac{8}{27} (281 + 6x + 281x^2) - \frac{16}{9} (47 + 18x \right. \\
& + 47x^2) H_{-1}) H_{0,-1} + \frac{8}{9} (47 + 18x + 47x^2) H_{0,0,-1} + \frac{16}{9} (47 + 18x + 47x^2) H_{0,-1,-1} + \left(-\frac{8}{27} (22 + 3x \right. \\
& \left. \left. + 259x^2) + \frac{2}{9} (37 + 18x + 37x^2) H_0 + \frac{8}{9} (47 + 18x + 47x^2) H_{-1} \right) \zeta_2 - \frac{8}{9} (35 + 18x + 59x^2) \zeta_3 \right) \left. \right\} \\
& + C_F T_F \left[\left\{ \frac{4}{27} (383 + 178x + 383x^2) x_+^2 + \eta \left(\frac{2}{27} H_0 P_{42} x_+^2 + \left(\frac{8}{3} x H_0^3 P_6 + 4H_0 P_{90} \zeta_2 \right) x_+^4 \right) \right. \right. \\
& \left. \left. - \frac{4}{9} (-1+x) \xi \eta H_0^3 P_6 x_+^3 + \left(\frac{2}{9} H_0^2 P_{32} - \frac{8}{3} P_7 \zeta_2 \right) x_+^4 \right\} + \varepsilon \left\{ \frac{2}{27} (2069 + 1990x + 2069x^2) x_+^2 \right. \right. \\
& + \frac{32}{3} (-1+x)^2 (7 + 72x + 7x^2) \ln(2) x_+^4 \zeta_2 + \xi \eta x_+^3 \left(-\frac{8}{9} (-1+x) H_{-1} H_0^3 P_6 - \frac{1}{3} (-1+x) H_0^4 P_6 \right. \\
& - \frac{16}{3} (-1+x) H_0^2 H_{0,1} P_6 + \frac{16}{3} (-1+x) H_0^2 H_{0,-1} P_6 + \frac{64}{3} (-1+x) H_0 H_{0,0,1} P_6 \\
& \left. \left. - \frac{32}{3} (-1+x) H_0 H_{0,0,-1} P_6 - 32(-1+x) H_{0,0,0,1} P_6 + \frac{16}{3} (-1+x) H_{0,0,0,-1} P_6 \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \eta \left(\left(-\frac{4}{27}H_{-1}H_0P_{42} + \frac{4}{27}H_{0,-1}P_{42} + \frac{1}{27}H_0P_{44} \right) x_+^2 + \left(-\frac{2}{27}H_0^2P_{61} + \frac{4P_{75}\zeta_2}{27} \right) x_+^3 \right. \\
& + \left(\frac{16}{3}xH_{-1}H_0^3P_6 + 2xH_0^4P_6 + 32xH_0^2H_{0,1}P_6 - 32xH_0^2H_{0,-1}P_6 - 128xH_0H_{0,0,1}P_6 \right. \\
& + 64xH_0H_{0,0,-1}P_6 + 192xH_{0,0,0,1}P_6 - 32xH_{0,0,0,-1}P_6 - \frac{2}{27}H_0^3P_{157} + \left(\frac{8}{3}H_{-1}H_0P_{89} \right. \\
& + 2H_0^2P_{90} - \frac{8}{3}H_{0,-1}P_{113} - \frac{2}{9}H_0P_{158} \Big) \zeta_2 - \frac{8}{5}P_{91}\zeta_2^2 - \frac{8}{9}H_0P_{114}\zeta_3 \Big) x_+^4 \Big) + \left(-\frac{4}{9}H_{-1}H_0^2P_{32} \right. \\
& \left. \left. + \frac{8}{9}H_0^2H_1P_{32} - \frac{16}{9}H_0H_{0,1}P_{32} + \frac{16}{9}H_{0,0,1}P_{32} + \frac{8}{9}H_{0,0,-1}P_{32} - \frac{8}{3}H_{-1}P_{32}\zeta_2 - \frac{8}{9}P_{38}\zeta_3 \right) x_+^4 \right) \Big].
\end{aligned}$$

The polynomials are defined as

$$\begin{aligned}
P_1 &= -70165x^4 - 177188x^3 - 77966x^2 - 177188x - 70165, \\
P_2 &= -3623x^4 - 23056x^3 + 6062x^2 - 23056x - 3623, \\
P_3 &= -2545x^4 - 6812x^3 - 4646x^2 - 6812x - 2545, \\
P_4 &= x^4 - 56x^3 - 42x^2 - 56x + 1, \\
P_5 &= x^4 - 16x^3 + 26x^2 - 16x + 1, \\
P_6 &= x^4 - 2x^3 + 12x^2 - 2x + 1, \\
P_7 &= x^4 + 37x^3 - 124x^2 + 37x + 1, \\
P_8 &= 2x^4 - 9x^3 + 122x^2 - 9x + 2, \\
P_9 &= 2x^4 + 9x^3 + 18x^2 + 9x + 2, \\
P_{10} &= 3x^4 - 94x^3 + 322x^2 - 94x + 3, \\
P_{11} &= 3x^4 - 8x^3 + 26x^2 - 8x + 3, \\
P_{12} &= 3x^4 + 16x^3 + 62x^2 + 16x + 3, \\
P_{13} &= 4x^4 - 13x^3 + 140x^2 - 13x + 4, \\
P_{14} &= 4x^4 + 27x^3 + 10x^2 + 27x + 4, \\
P_{15} &= 5x^4 - 90x^3 + 138x^2 - 90x + 5, \\
P_{16} &= 5x^4 + 26x^3 + 46x^2 + 26x + 5, \\
P_{17} &= 5x^4 + 79x^3 - 146x^2 + 79x + 5, \\
P_{18} &= 7x^4 - 42x^3 + 234x^2 - 42x + 7, \\
P_{19} &= 7x^4 - 3x^3 + 349x^2 - 3x + 7, \\
P_{20} &= 7x^4 + 7x^3 - 48x^2 + 7x + 7, \\
P_{21} &= 7x^4 + 11x^3 + 284x^2 + 11x + 7, \\
P_{22} &= 7x^4 + 85x^3 - 96x^2 + 85x + 7, \\
P_{23} &= 7x^4 + 104x^3 - 142x^2 + 104x + 7, \\
P_{24} &= 8x^4 + 15x^3 + 194x^2 + 15x + 8, \\
P_{25} &= 10x^4 + 46x^3 + 29x^2 + 46x + 10, \\
P_{26} &= 11x^4 - 40x^3 + 342x^2 - 40x + 11, \\
P_{27} &= 11x^4 + 22x^3 + 58x^2 + 22x + 11, \\
P_{28} &= 11x^4 + 72x^3 + 44x^2 + 72x + 11, \\
P_{29} &= 15x^4 + 37x^3 + 164x^2 + 37x + 15, \\
P_{30} &= 17x^4 + 38x^3 + 630x^2 + 38x + 17,
\end{aligned}$$

$$\begin{aligned}
P_{31} &= 18x^4 + 227x^3 - 272x^2 + 227x + 18, \\
P_{32} &= 19x^4 - 2x^3 + 206x^2 - 2x + 19, \\
P_{33} &= 19x^4 + 211x^3 - 246x^2 + 211x + 19, \\
P_{34} &= 28x^4 - 187x^3 + 110x^2 - 187x + 28, \\
P_{35} &= 31x^4 + 59x^3 + 164x^2 + 59x + 31, \\
P_{36} &= 43x^4 + 20x^3 + 130x^2 + 20x + 43, \\
P_{37} &= 55x^4 - 48x^3 + 562x^2 - 48x + 55, \\
P_{38} &= 69x^4 + 408x^3 - 698x^2 + 408x + 69, \\
P_{39} &= 79x^4 + 142x^3 + 1698x^2 + 142x + 79, \\
P_{40} &= 81x^4 - 104x^3 + 766x^2 - 104x + 81, \\
P_{41} &= 257x^4 - 48x^3 + 2078x^2 - 48x + 257, \\
P_{42} &= 265x^4 - 124x^3 + 3494x^2 - 124x + 265, \\
P_{43} &= 479x^4 + 704x^3 - 30x^2 + 704x + 479, \\
P_{44} &= 1719x^4 + 452x^3 + 20170x^2 + 452x + 1719, \\
P_{45} &= 3623x^4 + 23056x^3 - 6062x^2 + 23056x + 3623, \\
P_{46} &= -1045x^5 - 8139x^4 + 3518x^3 + 1634x^2 + 8151x + 1577, \\
P_{47} &= x^5 + 5x^4 - 9x^3 + 60x^2 - 46x + 19, \\
P_{48} &= 2x^5 + 8x^4 + 25x^3 - 21x^2 - 5x - 1, \\
P_{49} &= 2x^5 + 13x^4 - 5x^3 - 11x^2 - 25x - 6, \\
P_{50} &= 5x^5 - 285x^4 + 1592x^3 - 1840x^2 + 99x - 67, \\
P_{51} &= 5x^5 - 30x^4 + 53x^3 - 61x^2 + 24x - 7, \\
P_{52} &= 5x^5 - 26x^4 + 87x^3 - 119x^2 + 2x - 13, \\
P_{53} &= 5x^5 + 8x^4 + 6x^3 - 22x^2 - 16x - 13, \\
P_{54} &= 5x^5 + 171x^4 - 494x^3 + 398x^2 - 243x - 29, \\
P_{55} &= 9x^5 + 23x^4 + 2x^3 + 94x^2 + 49x + 15, \\
P_{56} &= 10x^5 - 105x^4 + 139x^3 - 203x^2 + 57x - 26, \\
P_{57} &= 10x^5 + 29x^4 + 59x^3 - 43x^2 - 17x - 6, \\
P_{58} &= 11x^5 - 81x^4 + 480x^3 - 448x^2 + 105x - 3, \\
P_{59} &= 11x^5 - 21x^4 + 284x^3 - 316x^2 - 3x - 19, \\
P_{60} &= 13x^5 + 61x^4 - 316x^3 + 460x^2 - 101x + 11, \\
P_{61} &= 16x^5 - 120x^4 - 287x^3 - 3083x^2 - 21x - 281, \\
P_{62} &= 25x^5 - 45x^4 + 614x^3 - 518x^2 + 117x - 1, \\
P_{63} &= 26x^5 - 45x^4 + 305x^3 - 313x^2 + 39x - 28, \\
P_{64} &= 27x^5 - 37x^4 + 236x^3 - 92x^2 - 3x - 3, \\
P_{65} &= 41x^5 + 423x^4 - 482x^3 + 1106x^2 + 45x + 115, \\
P_{66} &= 42x^5 + 136x^4 + 87x^3 + 89x^2 - 4x + 2, \\
P_{67} &= 53x^5 - 12x^4 + 851x^3 - 571x^2 + 222x + 17, \\
P_{68} &= 56x^5 + 43x^4 + 695x^3 - 383x^2 + 191x + 22, \\
P_{69} &= 57x^5 + 183x^4 + 430x^3 - 606x^2 - 315x - 101, \\
P_{70} &= 58x^5 + 9x^4 + 779x^3 - 467x^2 + 225x + 20,
\end{aligned}$$

$$\begin{aligned}
P_{71} &= 77x^5 + 139x^4 + 406x^3 + 298x^2 + 53x + 51, \\
P_{72} &= 83x^5 - 217x^4 + 2594x^3 - 646x^2 + 59x + 47, \\
P_{73} &= 83x^5 + 432x^4 - 499x^3 + 779x^2 - 222x - 13, \\
P_{74} &= 97x^5 + 1533x^4 + 3340x^3 - 4976x^2 - 357x - 533, \\
P_{75} &= 139x^5 + 2247x^4 - 9395x^3 + 6025x^2 - 2388x - 404, \\
P_{76} &= 187x^5 + 399x^4 - 5192x^3 + 9388x^2 - 843x + 349, \\
P_{77} &= 217x^5 + 81x^4 - 860x^3 + 616x^2 - 621x - 233, \\
P_{78} &= 254x^5 - 25098x^4 - 1801x^3 - 10009x^2 - 10761x - 3337, \\
P_{79} &= 347x^5 - 1737x^4 + 5228x^3 - 3040x^2 + 1737x - 7, \\
P_{80} &= 1343x^5 + 3651x^4 - 5788x^3 + 7424x^2 - 4827x - 907, \\
P_{81} &= 4774x^5 - 4044x^4 + 30811x^3 - 3449x^2 + 12363x - 71, \\
P_{82} &= -1367x^6 + 1822x^5 - 25085x^4 + 19772x^3 - 1829x^2 + 2110x - 575, \\
P_{83} &= -399x^6 - 124x^5 - 2335x^4 - 1448x^3 + 2075x^2 - 124x + 307, \\
P_{84} &= -89x^6 + 154x^5 - 1547x^4 + 1262x^3 - 29x^2 + 4x - 11, \\
P_{85} &= -29x^6 + 188x^5 - 335x^4 + 974x^3 + 67x^2 + 2x - 11, \\
P_{86} &= -9x^6 + 55x^5 - 422x^4 + 254x^3 - 175x^2 - 63x - 24, \\
P_{87} &= x^6 - 31x^5 + 213x^4 - 428x^3 + 213x^2 - 31x + 1, \\
P_{88} &= x^6 - 12x^5 + 14x^4 - 38x^3 + 14x^2 - 12x + 1, \\
P_{89} &= x^6 + 4x^5 - 5x^4 + 128x^3 - 5x^2 + 4x + 1, \\
P_{90} &= x^6 + 4x^5 + 3x^4 + 48x^3 + 3x^2 + 4x + 1, \\
P_{91} &= x^6 + 4x^5 + 6x^4 + 18x^3 + 6x^2 + 4x + 1, \\
P_{92} &= x^6 + 4x^5 + 11x^4 + 11x^2 + 4x + 1, \\
P_{93} &= x^6 + 6x^5 - 4x^4 + 21x^3 - 4x^2 + 6x + 1, \\
P_{94} &= x^6 + 7x^5 - 23x^4 + 57x^3 - 50x^2 + 28x - 4, \\
P_{95} &= 2x^6 - 11x^5 - 114x^4 + 182x^3 - 114x^2 - 11x + 2, \\
P_{96} &= 2x^6 - 5x^5 + 92x^4 - 162x^3 + 92x^2 - 5x + 2, \\
P_{97} &= 2x^6 + 6x^5 + 33x^4 - 68x^3 + 33x^2 + 6x + 2, \\
P_{98} &= 2x^6 + 9x^5 - 21x^4 + 102x^3 - 88x^2 + 31x - 3, \\
P_{99} &= 2x^6 + 11x^5 - 9x^4 + 48x^3 - 9x^2 + 11x + 2, \\
P_{100} &= 2x^6 + 13x^5 + 246x^4 - 274x^3 + 248x^2 + 17x + 4, \\
P_{101} &= 3x^6 - 4x^5 + 115x^4 - 302x^3 + 115x^2 - 4x + 3, \\
P_{102} &= 3x^6 + 4x^5 + 47x^4 - 116x^3 + 137x^2 - 32x + 5, \\
P_{103} &= 3x^6 + 5x^5 + 153x^4 - 202x^3 + 155x^2 + 9x + 5, \\
P_{104} &= 3x^6 + 13x^5 + 22x^4 + 12x^3 + 22x^2 + 13x + 3, \\
P_{105} &= 3x^6 + 24x^5 - 37x^4 + 196x^3 - 37x^2 + 24x + 3, \\
P_{106} &= 4x^6 + 6x^5 + 141x^4 - 220x^3 + 141x^2 + 6x + 4, \\
P_{107} &= 4x^6 + 41x^5 - 173x^4 + 608x^3 - 580x^2 + 193x - 29, \\
P_{108} &= 5x^6 - 24x^5 + 291x^4 - 698x^3 + 291x^2 - 24x + 5, \\
P_{109} &= 5x^6 + 14x^5 - 19x^4 + 64x^3 - 19x^2 + 14x + 5, \\
P_{110} &= 5x^6 + 59x^5 - 175x^4 + 508x^3 - 175x^2 + 59x + 5,
\end{aligned}$$

$$\begin{aligned}
P_{111} &= 6x^6 - 57x^5 + 167x^4 - 223x^3 + 17x^2 - 30x - 8, \\
P_{112} &= 6x^6 - 53x^5 + 267x^4 - 239x^3 + 73x^2 + 6x + 4, \\
P_{113} &= 7x^6 + 28x^5 + x^4 + 536x^3 + x^2 + 28x + 7, \\
P_{114} &= 7x^6 + 28x^5 + 13x^4 + 416x^3 + 13x^2 + 28x + 7, \\
P_{115} &= 7x^6 + 40x^5 - 163x^4 + 280x^3 - 159x^2 + 48x + 11, \\
P_{116} &= 7x^6 + 132x^5 + 71x^4 - 2312x^3 + 1121x^2 - 396x - 15, \\
P_{117} &= 9x^6 - 84x^5 + 355x^4 - 341x^3 + 94x^2 - 3x + 2, \\
P_{118} &= 9x^6 - 57x^5 + 232x^4 - 176x^3 + 97x^2 + 15x + 8, \\
P_{119} &= 9x^6 + 11x^5 + 139x^4 - 254x^3 + 139x^2 + 11x + 9, \\
P_{120} &= 10x^6 - 25x^5 + 470x^4 - 874x^3 + 470x^2 - 25x + 10, \\
P_{121} &= 11x^6 - 8x^5 + 339x^4 - 556x^3 + 339x^2 - 8x + 11, \\
P_{122} &= 11x^6 + 35x^5 + 195x^4 - 324x^3 + 195x^2 + 35x + 11, \\
P_{123} &= 12x^6 + 37x^5 + 49x^4 + 42x^3 + 42x^2 + 11x - 1, \\
P_{124} &= 13x^6 + 3x^5 + 231x^4 - 430x^3 + 231x^2 + 3x + 13, \\
P_{125} &= 13x^6 + 14x^5 + 111x^4 - 20x^3 + 111x^2 + 14x + 13, \\
P_{126} &= 13x^6 + 34x^5 - 61x^4 + 908x^3 - 873x^2 + 138x - 63, \\
P_{127} &= 13x^6 + 57x^5 + 76x^4 - 56x^3 + 76x^2 + 57x + 13, \\
P_{128} &= 14x^6 + 52x^5 + 153x^4 - 76x^3 + 153x^2 + 52x + 14, \\
P_{129} &= 15x^6 - 9x^5 + 517x^4 - 982x^3 + 517x^2 - 9x + 15, \\
P_{130} &= 15x^6 + 706x^5 - 55x^4 + 2888x^3 - 159x^2 + 198x - 41, \\
P_{131} &= 17x^6 - 150x^5 + 1153x^4 - 1852x^3 + 883x^2 - 174x + 11, \\
P_{132} &= 17x^6 + 25x^5 + 231x^4 - 418x^3 + 231x^2 + 25x + 17, \\
P_{133} &= 17x^6 + 46x^5 - 109x^4 + 348x^3 - 109x^2 + 46x + 17, \\
P_{134} &= 18x^6 - x^5 - 85x^4 + 150x^3 + 76x^2 + 99x - 1, \\
P_{135} &= 18x^6 + 7x^5 + 290x^4 - 502x^3 + 290x^2 + 7x + 18, \\
P_{136} &= 19x^6 - 21x^5 + 525x^4 - 910x^3 + 523x^2 - 25x + 17, \\
P_{137} &= 24x^6 + 15x^5 - 24x^4 + 234x^3 - 26x^2 + 11x + 22, \\
P_{138} &= 24x^6 + 75x^5 + 91x^4 + 118x^3 + 50x^2 + 29x - 3, \\
P_{139} &= 25x^6 - 126x^5 + 1233x^4 - 1820x^3 + 859x^2 - 166x + 11, \\
P_{140} &= 25x^6 + 19x^5 + 479x^4 - 854x^3 + 479x^2 + 19x + 25, \\
P_{141} &= 27x^6 - 20x^5 + 835x^4 - 1428x^3 + 835x^2 - 20x + 27, \\
P_{142} &= 28x^6 + 129x^5 - 244x^4 + 892x^3 - 334x^2 + 27x + 14, \\
P_{143} &= 29x^6 - 352x^5 + 839x^4 - 1502x^3 + 317x^2 - 46x + 11, \\
P_{144} &= 29x^6 + 665x^5 - 2515x^4 + 4174x^3 - 715x^2 + 269x + 65, \\
P_{145} &= 37x^6 - 94x^5 + 223x^4 - 108x^3 + 231x^2 - 78x + 45, \\
P_{146} &= 37x^6 - 10x^5 + 1175x^4 - 1924x^3 + 735x^2 - 386x - 27, \\
P_{147} &= 39x^6 + 20x^5 + 1021x^4 - 2324x^3 + 865x^2 - 376x - 21, \\
P_{148} &= 44x^6 + 265x^5 + 704x^4 - 2880x^3 + 1696x^2 - 529x - 20, \\
P_{149} &= 45x^6 - 80x^5 + 239x^4 - 792x^3 + 239x^2 - 80x + 45, \\
P_{150} &= 50x^6 + 119x^5 + 610x^4 - 2060x^3 + 1254x^2 - 435x - 2,
\end{aligned}$$

$$\begin{aligned}
P_{151} &= 53x^6 + 173x^5 + 357x^4 + 892x^3 - 935x^2 - 17x - 11, \\
P_{152} &= 59x^6 - 326x^5 + 1675x^4 + 1888x^3 - 1503x^2 + 446x + 9, \\
P_{153} &= 63x^6 + 9x^5 + 1145x^4 + 762x^3 - 1073x^2 + 385x - 11, \\
P_{154} &= 72x^6 + 381x^5 - 659x^4 + 1986x^3 - 786x^2 + 9x + 21, \\
P_{155} &= 81x^6 - 429x^5 + 863x^4 + 3222x^3 - 2399x^2 + 667x + 43, \\
P_{156} &= 93x^6 - 130x^5 + 491x^4 - 716x^3 - 21x^2 - 194x - 35, \\
P_{157} &= 105x^6 - 30x^5 + 1065x^4 - 56x^3 - 431x^2 - 14x - 47, \\
P_{158} &= 125x^6 - 94x^5 + 1295x^4 - 220x^3 - 201x^2 - 78x - 27, \\
P_{159} &= 133x^6 - 332x^5 + 4847x^4 - 2420x^3 + 159x^2 + 24x - 11, \\
P_{160} &= 145x^6 - 738x^5 + 15433x^4 - 15038x^3 + 12285x^2 - 404x + 637, \\
P_{161} &= 229x^6 - 60x^5 + 1517x^4 - 968x^3 + 247x^2 + 4x + 55, \\
P_{162} &= 443x^6 + 2186x^5 - 12793x^4 + 14416x^3 + 2471x^2 - 766x - 565, \\
P_{163} &= 649x^6 - 416x^5 + 4063x^4 - 2272x^3 + 975x^2 - 80x + 153, \\
P_{164} &= 759x^6 - 516x^5 + 4247x^4 - 4888x^3 - 163x^2 - 516x + 53, \\
P_{165} &= 1043x^6 - 224x^5 + 10999x^4 - 14384x^3 + 2761x^2 + 400x + 173, \\
P_{166} &= -337x^7 - 125x^6 - 3325x^5 + 2351x^4 - 431x^3 - 1203x^2 - 19x + 17, \\
P_{167} &= -117x^7 - 1119x^6 + 4661x^5 - 17313x^4 + 15395x^3 - 6031x^2 + 297x - 157, \\
P_{168} &= -71x^7 - 353x^6 + 621x^5 - 2761x^4 + 2243x^3 - 991x^2 + 131x - 3, \\
P_{169} &= x^7 - 7x^6 + 69x^5 - 267x^4 + 239x^3 - 89x^2 - 5x - 5, \\
P_{170} &= x^7 - 6x^6 + 115x^5 - 150x^4 + 206x^3 - 75x^2 + 30x + 7, \\
P_{171} &= x^7 - 5x^6 + 61x^5 - 93x^4 + 121x^3 - 41x^2 + 17x + 3, \\
P_{172} &= x^7 - 2x^6 + 37x^5 - 130x^4 + 123x^3 - 42x^2 - x - 2, \\
P_{173} &= x^7 - x^6 + 33x^5 - 99x^4 + 113x^3 - 23x^2 + 7x + 1, \\
P_{174} &= x^7 + x^6 + 29x^5 - 39x^4 + 53x^3 - 19x^2 + 5x + 1, \\
P_{175} &= x^7 + 2x^6 + 16x^5 + 41x^4 + 15x^3 + 24x^2 + 22x + 7, \\
P_{176} &= x^7 + 2x^6 + 17x^5 - 16x^4 + 30x^3 - 7x^2 + 4x + 1, \\
P_{177} &= x^7 + 4x^6 - 7x^5 + 30x^4 - 16x^3 + 17x^2 + 2x + 1, \\
P_{178} &= x^7 + 6x^6 - 26x^5 - 17x^4 - 39x^3 - 14x^2 - 30x - 9, \\
P_{179} &= x^7 + 7x^6 - 31x^5 + 215x^4 - 173x^3 + 61x^2 + 11x + 5, \\
P_{180} &= x^7 + 9x^6 - 57x^5 + 183x^4 - 197x^3 + 47x^2 - 15x - 3, \\
P_{181} &= x^7 + 9x^6 - 33x^5 + 113x^4 - 120x^3 + 28x^2 - 12x - 2, \\
P_{182} &= x^7 + 14x^6 - 91x^5 + 692x^4 - 552x^3 + 191x^2 + 46x + 19, \\
P_{183} &= x^7 + 35x^6 - 220x^5 + 1700x^4 - 846x^3 + 830x^2 + 331x + 121, \\
P_{184} &= 2x^7 + 3x^6 + 19x^5 - 76x^4 + 62x^3 - 29x^2 - 9x - 4, \\
P_{185} &= 2x^7 + 6x^6 + 13x^5 - 13x^4 + 20x^3 - 8x^2 - 3x - 1, \\
P_{186} &= 2x^7 + 7x^6 - 2x^5 + 37x^4 - 9x^3 + 22x^2 + 5x + 2, \\
P_{187} &= 2x^7 + 7x^6 + 5x^5 + 14x^4 - 7x^3 - 4x - 1, \\
P_{188} &= 2x^7 + 8x^6 - x^5 + 83x^4 + x^3 + 61x^2 + 28x + 10, \\
P_{189} &= 2x^7 + 23x^6 - 172x^5 + 363x^4 - 433x^3 + 122x^2 - 53x - 12, \\
P_{190} &= 2x^7 + 24x^6 - 157x^5 + 447x^4 - 587x^3 + 57x^2 - 84x - 22,
\end{aligned}$$

$$\begin{aligned}
P_{191} &= 3x^7 + 76x^5 - 247x^4 + 240x^3 - 81x^2 - 3x - 4, \\
P_{192} &= 3x^7 - 11x^6 + 143x^5 - 527x^4 + 485x^3 - 173x^2 - 7x - 9, \\
P_{193} &= 3x^7 - 9x^6 + 183x^5 - 689x^4 + 703x^3 - 173x^2 + 15x - 1, \\
P_{194} &= 3x^7 + 3x^6 + 68x^5 - 246x^4 + 260x^3 - 58x^2 + 3x - 1, \\
P_{195} &= 3x^7 + 4x^6 + 63x^5 - 182x^4 + 210x^3 - 43x^2 + 8x + 1, \\
P_{196} &= 3x^7 + 6x^6 - 400x^5 + 317x^4 + 389x^3 - 590x^2 + 19, \\
P_{197} &= 3x^7 + 11x^6 - 9x^5 + 67x^4 - 25x^3 + 39x^2 + 7x + 3, \\
P_{198} &= 3x^7 + 14x^6 - 33x^5 + 224x^4 - 168x^3 + 73x^2 + 10x + 5, \\
P_{199} &= 3x^7 + 59x^6 - 477x^5 + 1807x^4 - 1849x^3 + 447x^2 - 77x - 9, \\
P_{200} &= 3x^7 + 177x^6 + 507x^5 + 365x^4 - 1153x^3 + 413x^2 - 93x + 37, \\
P_{201} &= 4x^7 + 3x^6 + 77x^5 - 316x^4 + 260x^3 - 117x^2 - 27x - 12, \\
P_{202} &= 4x^7 + 33x^6 - 113x^5 + 462x^4 - 490x^3 + 93x^2 - 45x - 8, \\
P_{203} &= 4x^7 + 35x^6 - 212x^5 + 557x^4 - 655x^3 + 142x^2 - 77x - 18, \\
P_{204} &= 5x^7 - 7x^6 - 94x^5 + 45x^4 + 7x^3 - 5x^2 - 10x - 5, \\
P_{205} &= 5x^7 + 3x^6 + 103x^5 - 269x^4 + 241x^3 - 123x^2 - 15x - 9, \\
P_{206} &= 5x^7 + 5x^6 + 89x^5 - 239x^4 + 267x^3 - 69x^2 + 7x - 1, \\
P_{207} &= 5x^7 + 9x^6 + 32x^5 - 314x^4 + 188x^3 - 122x^2 - 63x - 23, \\
P_{208} &= 5x^7 + 12x^6 + 32x^5 - 61x^4 + 75x^3 - 22x^2 - 6x - 3, \\
P_{209} &= 5x^7 + 15x^6 + 37x^5 - 73x^4 + 59x^3 - 47x^2 - 21x - 7, \\
P_{210} &= 5x^7 + 27x^6 - 51x^5 + 247x^4 - 219x^3 + 71x^2 - 15x - 1, \\
P_{211} &= 6x^7 + 9x^6 + 87x^5 - 190x^4 + 218x^3 - 67x^2 + 3x - 2, \\
P_{212} &= 6x^7 + 17x^6 + 41x^5 + 84x^4 + 84x^3 + 79x^2 + 55x + 18, \\
P_{213} &= 7x^7 - 5x^6 + 293x^5 - 1119x^4 + 1133x^3 - 283x^2 + 11x - 5, \\
P_{214} &= 7x^7 + 2x^6 + 174x^5 - 21x^4 + 245x^3 - 14x^2 + 94x + 25, \\
P_{215} &= 7x^7 + 11x^6 + 99x^5 - 225x^4 + 281x^3 - 59x^2 + 13x + 1, \\
P_{216} &= 7x^7 + 13x^6 + 107x^5 - 367x^4 + 409x^3 - 77x^2 + 5x - 1, \\
P_{217} &= 7x^7 + 15x^6 + 49x^5 - 143x^4 + 129x^3 - 59x^2 - 21x - 9, \\
P_{218} &= 7x^7 + 15x^6 + 53x^5 - 131x^4 + 145x^3 - 43x^2 - 9x - 5, \\
P_{219} &= 7x^7 + 19x^6 + 46x^5 - 20x^4 + 62x^3 - 16x^2 - x - 1, \\
P_{220} &= 8x^7 - 5x^6 + 351x^5 - 878x^4 + 1004x^3 - 261x^2 + 59x + 10, \\
P_{221} &= 10x^7 - 15x^6 + 345x^5 - 1186x^4 + 1158x^3 - 365x^2 + 3x - 14, \\
P_{222} &= 10x^7 - 5x^6 + 294x^5 - 496x^4 + 755x^3 - 109x^2 + 116x + 27, \\
P_{223} &= 10x^7 + 29x^6 + 29x^5 - 40x^4 + 54x^3 - 19x^2 - 23x - 8, \\
P_{224} &= 10x^7 + 29x^6 + 35x^5 - 142x^4 + 156x^3 - 25x^2 - 23x - 8, \\
P_{225} &= 11x^7 - 15x^6 + 131x^5 + 1575x^4 - 3873x^3 + 2265x^2 - 13x + 47, \\
P_{226} &= 11x^7 - 3x^6 + 317x^5 - 487x^4 + 711x^3 - 157x^2 + 99x + 21, \\
P_{227} &= 11x^7 + 12x^6 + 294x^5 - 665x^4 + 777x^3 - 214x^2 + 36x + 5, \\
P_{228} &= 11x^7 + 13x^6 + 403x^5 - 1715x^4 + 2781x^3 - 1669x^2 - 43x - 37, \\
P_{229} &= 11x^7 + 15x^6 + 174x^5 - 413x^4 + 420x^3 - 169x^2 - 12x - 10, \\
P_{230} &= 11x^7 + 55x^6 - 183x^5 + 653x^4 - 639x^3 + 193x^2 - 49x - 9,
\end{aligned}$$

$$\begin{aligned}
P_{231} &= 12x^7 + 35x^6 + 69x^5 + 120x^4 + 118x^3 + 101x^2 + 67x + 22, \\
P_{232} &= 12x^7 + 71x^6 - 253x^5 + 1158x^4 - 1200x^3 + 223x^2 - 89x - 18, \\
P_{233} &= 13x^7 - 48x^6 + 636x^5 - 1993x^4 + 1965x^3 - 656x^2 + 36x - 17, \\
P_{234} &= 13x^7 + 19x^6 + 193x^5 - 457x^4 + 555x^3 - 123x^2 + 23x + 1, \\
P_{235} &= 13x^7 + 26x^6 + 138x^5 - 441x^4 + 455x^3 - 128x^2 - 20x - 11, \\
P_{236} &= 13x^7 + 32x^6 + 136x^5 - 289x^4 + 317x^3 - 116x^2 - 20x - 9, \\
P_{237} &= 13x^7 + 75x^6 - 197x^5 + 655x^4 - 543x^3 + 277x^2 - 27x + 3, \\
P_{238} &= 13x^7 + 119x^6 - 727x^5 + 2363x^4 - 2517x^3 + 617x^2 - 185x - 35, \\
P_{239} &= 13x^7 + 121x^6 - 755x^5 + 2461x^4 - 2755x^3 + 545x^2 - 247x - 55, \\
P_{240} &= 13x^7 + 128x^6 - 498x^5 + 2121x^4 - 1897x^3 + 658x^2 - 32x + 19, \\
P_{241} &= 13x^7 + 132x^6 + 185x^5 + 187x^4 - 215x^3 - 10x^2 + 17x + 11, \\
P_{242} &= 14x^7 + 31x^6 + 166x^5 - 587x^4 + 657x^3 - 116x^2 - x - 4, \\
P_{243} &= 14x^7 + 40x^6 + 71x^5 - 165x^4 + 207x^3 - 41x^2 - 22x - 8, \\
P_{244} &= 15x^7 + 9x^6 + 319x^5 - 967x^4 + 981x^3 - 309x^2 - 3x - 13, \\
P_{245} &= 15x^7 + 49x^6 - 2x^5 + 34x^4 - 20x^3 + 12x^2 - 43x - 13, \\
P_{246} &= 16x^7 - 13x^6 + 487x^5 - 1568x^4 + 1540x^3 - 507x^2 + x - 20, \\
P_{247} &= 16x^7 + 11x^6 + 319x^5 - 862x^4 + 890x^3 - 299x^2 + x - 12, \\
P_{248} &= 17x^7 + 49x^6 + 127x^5 - 89x^4 + 201x^3 - 47x^2 - x - 1, \\
P_{249} &= 17x^7 + 63x^6 - 63x^5 + 167x^4 - 209x^3 + 33x^2 - 81x - 23, \\
P_{250} &= 17x^7 + 69x^6 - 116x^5 + 434x^4 - 448x^3 + 106x^2 - 75x - 19, \\
P_{251} &= 17x^7 + 70x^6 - 54x^5 + 189x^4 - 77x^3 + 134x^2 - 22x - 1, \\
P_{252} &= 18x^7 + 30x^6 + 241x^5 - 635x^4 + 677x^3 - 211x^2 - 12x - 12, \\
P_{253} &= 18x^7 + 43x^6 + 185x^5 - 710x^4 + 752x^3 - 155x^2 - 25x - 12, \\
P_{254} &= 18x^7 + 53x^6 + 81x^5 - 204x^4 + 190x^3 - 91x^2 - 59x - 20, \\
P_{255} &= 19x^7 - 46x^6 + 778x^5 - 2375x^4 + 2347x^3 - 798x^2 + 34x - 23, \\
P_{256} &= 19x^7 + 65x^6 + 47x^5 + 109x^4 - 67x^3 - 17x^2 - 47x - 13, \\
P_{257} &= 19x^7 + 91x^6 - 279x^5 + 593x^4 - 859x^3 + 89x^2 - 205x - 57, \\
P_{258} &= 19x^7 + 95x^6 - 173x^5 + 7x^4 - 77x^3 + 123x^2 - 125x - 29, \\
P_{259} &= 19x^7 + 113x^6 - 261x^5 + 1317x^4 - 1093x^3 + 421x^2 - 17x + 13, \\
P_{260} &= 20x^7 - 181x^6 + 1831x^5 - 5450x^4 + 5282x^3 - 1951x^2 + 109x - 44, \\
P_{261} &= 20x^7 + 56x^6 + 131x^5 - 118x^4 + 265x^3 - 26x^2 + 7x + 1, \\
P_{262} &= 21x^7 - 4x^6 + 554x^5 - 1661x^4 + 1633x^3 - 574x^2 - 8x - 25, \\
P_{263} &= 21x^7 + 50x^6 + 188x^5 - 565x^4 + 593x^3 - 168x^2 - 38x - 17, \\
P_{264} &= 21x^7 + 55x^6 + 131x^5 - 375x^4 + 445x^3 - 81x^2 - 25x - 11, \\
P_{265} &= 22x^7 - 179x^6 + 1753x^5 - 4744x^4 + 5262x^3 - 1383x^2 + 401x + 52, \\
P_{266} &= 22x^7 - x^6 + 405x^5 - 124x^4 - 50x^3 + 221x^2 - x + 40, \\
P_{267} &= 23x^7 + 65x^6 + 25x^5 - 129x^4 + 283x^3 + 85x^2 + x - 1, \\
P_{268} &= 23x^7 + 92x^6 - 436x^5 + 1149x^4 - 1213x^3 + 298x^2 - 22x - 19, \\
P_{269} &= 24x^7 + 11x^6 + 543x^5 - 1222x^4 + 1236x^3 - 533x^2 - 5x - 22, \\
P_{270} &= 25x^7 + 49x^6 + 259x^5 - 685x^4 + 699x^3 - 249x^2 - 43x - 23,
\end{aligned}$$

$$\begin{aligned}
P_{271} &= 25x^7 + 49x^6 + 271x^5 - 889x^4 + 903x^3 - 261x^2 - 43x - 23, \\
P_{272} &= 25x^7 + 94x^6 - 4x^5 + 567x^4 - 301x^3 + 194x^2 + 20x + 13, \\
P_{273} &= 27x^7 + 17x^6 + 525x^5 - 1825x^4 + 1671x^3 - 635x^2 - 83x - 49, \\
P_{274} &= 27x^7 + 83x^6 + 117x^5 + 117x^4 + 121x^3 + 53x^2 + 19x + 7, \\
P_{275} &= 29x^7 + 57x^6 + 247x^5 - 901x^4 + 887x^3 - 257x^2 - 63x - 31, \\
P_{276} &= 31x^7 + 63x^6 + 253x^5 - 899x^4 + 885x^3 - 263x^2 - 69x - 33, \\
P_{277} &= 31x^7 + 69x^6 + 263x^5 - 783x^4 + 825x^3 - 233x^2 - 51x - 25, \\
P_{278} &= 31x^7 + 85x^6 + 183x^5 - 763x^4 + 749x^3 - 193x^2 - 91x - 33, \\
P_{279} &= 31x^7 + 121x^6 - 53x^5 + 505x^4 - 379x^3 + 143x^2 - 67x - 13, \\
P_{280} &= 32x^7 - 57x^6 + 1229x^5 - 3252x^4 + 3154x^3 - 1299x^2 + 15x - 46, \\
P_{281} &= 34x^7 - 7x^6 + 899x^5 - 2460x^4 + 2446x^3 - 909x^2 + x - 36, \\
P_{282} &= 34x^7 + 101x^6 + 153x^5 - 116x^4 + 270x^3 - 43x^2 - 35x - 12, \\
P_{283} &= 35x^7 - 5x^6 + 918x^5 - 2502x^4 + 2572x^3 - 868x^2 + 35x - 25, \\
P_{284} &= 37x^7 + 78x^6 - 64x^5 + 1211x^4 - 2191x^3 + 1360x^2 + 50x + 31, \\
P_{285} &= 37x^7 + 104x^6 + 326x^5 - 271x^4 + 495x^3 - 166x^2 - 8x - 5, \\
P_{286} &= 38x^7 + 29x^6 + 807x^5 - 1678x^4 + 1748x^3 - 757x^2 + x - 28, \\
P_{287} &= 40x^7 + 167x^6 - 262x^5 + 911x^4 - 897x^3 + 272x^2 - 161x - 38, \\
P_{288} &= 42x^7 + 71x^6 + 557x^5 - 1378x^4 + 1490x^3 - 477x^2 - 23x - 26, \\
P_{289} &= 43x^7 + 77x^6 + 485x^5 - 1245x^4 + 1567x^3 - 255x^2 + 61x + 3, \\
P_{290} &= 43x^7 + 135x^6 + 177x^5 + 261x^4 + 145x^3 + 113x^2 + 39x + 15, \\
P_{291} &= 46x^7 + 141x^6 + 284x^5 + 193x^4 + 325x^3 + 86x^2 + 81x + 28, \\
P_{292} &= 47x^7 + 3x^6 + 543x^5 + 259x^4 - 1821x^3 + 1571x^2 + 15x + 23, \\
P_{293} &= 50x^7 + 209x^6 - 257x^5 + 2042x^4 - 1776x^3 + 447x^2 - 95x - 12, \\
P_{294} &= 51x^7 + 121x^6 + 375x^5 - 1683x^4 + 1613x^3 - 425x^2 - 151x - 61, \\
P_{295} &= 55x^7 + 216x^6 - 2082x^5 + 6547x^4 - 5893x^3 + 1446x^2 + 24x + 71, \\
P_{296} &= 56x^7 + 101x^6 + 925x^5 - 3030x^4 + 2512x^3 - 1295x^2 - 323x - 130, \\
P_{297} &= 65x^7 + 345x^6 - 1061x^5 + 5123x^4 - 4969x^3 + 1171x^2 - 279x - 43, \\
P_{298} &= 67x^7 + 74x^6 + 1200x^5 - 3387x^4 + 3527x^3 - 1100x^2 - 14x - 47, \\
P_{299} &= 77x^7 + 151x^6 + 1175x^5 - 2699x^4 + 3217x^3 - 805x^2 + 71x - 3, \\
P_{300} &= 91x^7 + 153x^6 + 833x^5 - 2047x^4 + 3597x^3 - 1845x^2 + 15x - 29, \\
P_{301} &= 96x^7 + 179x^6 + 1307x^5 - 2262x^4 + 2682x^3 - 1007x^2 + x - 36, \\
P_{302} &= 98x^7 + 115x^6 + 1791x^5 - 3676x^4 + 3984x^3 - 1571x^2 + 17x - 54, \\
P_{303} &= 111x^7 + 60x^6 + 836x^5 - 3631x^4 + 6545x^3 - 2672x^2 + 84x - 53, \\
P_{304} &= 119x^7 + 177x^6 + 1763x^5 - 4297x^4 + 4773x^3 - 1423x^2 + 27x - 51, \\
P_{305} &= 139x^7 + 512x^6 - 90x^5 + 3131x^4 - 2207x^3 + 750x^2 - 116x - 7, \\
P_{306} &= 141x^7 + 544x^6 - 346x^5 + 4169x^4 - 3273x^3 + 986x^2 - 160x - 13, \\
P_{307} &= 146x^7 + 1297x^6 - 1553x^5 + 8418x^4 - 9754x^3 + 3745x^2 - 563x - 200, \\
P_{308} &= 181x^7 + 479x^6 + 1355x^5 - 2119x^4 + 2973x^3 - 745x^2 - 113x - 59, \\
P_{309} &= 543x^7 + 587x^6 + 2403x^5 + 3623x^4 - 3279x^3 + 277x^2 - 195x + 137, \\
P_{310} &= 649x^7 + 1613x^6 + 5451x^5 - 6985x^4 + 11899x^3 - 1941x^2 + 493x + 53.
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
F_{V,2}^{(1)} = C_F \left[\left\{ -4x\eta H_0 \right\} + \varepsilon \left\{ \eta \left(-16xH_0 + 8xH_{-1}H_0 - 2xH_0^2 - 8xH_{0,-1} + 4x\zeta_2 \right) \right\} \right. \\
+ \varepsilon^2 \left\{ \eta \left(\left(-32x + 32xH_{-1} - 8xH_{-1}^2 + 2x\zeta_2 \right) H_0 + \left(-8x + 4xH_{-1} \right) H_0^2 - \frac{2}{3}xH_0^3 + \left(\right. \right. \\
- 32x + 16xH_{-1} \left. \left. \right) H_{0,-1} - 8xH_{0,0,-1} - 16xH_{0,-1,-1} + 16x\zeta_2 - 8xH_{-1}\zeta_2 + 8x\zeta_3 \right) \right\} + \varepsilon^3 \left\{ \eta \left(\left(\right. \right. \right. \\
- 64x + \left(64x - 4x\zeta_2 \right) H_{-1} - 32xH_{-1}^2 + \frac{16}{3}xH_{-1}^3 + 8x\zeta_2 + \frac{28}{3}x\zeta_3 \left. \right) H_0 + \left(-16x + 16xH_{-1} \right. \\
- 4xH_{-1}^2 + x\zeta_2 \left. \right) H_0^2 + \left(-\frac{8x}{3} + \frac{4xH_{-1}}{3} \right) H_0^3 - \frac{1}{6}xH_0^4 + \left(-32x\zeta_2 - 16x\zeta_3 \right) H_{-1} + \left(-64x \right. \\
+ 64xH_{-1} - 16xH_{-1}^2 - 4x\zeta_2 \left. \right) H_{0,-1} + \left(-32x + 16xH_{-1} \right) H_{0,0,-1} + \left(-64x + 32xH_{-1} \right) H_{0,-1,-1} \\
\left. \left. \left. - 8xH_{0,0,0,-1} - 16xH_{0,0,-1,-1} - 32xH_{0,-1,-1,-1} + 32x\zeta_2 + 8xH_{-1}^2\zeta_2 + \frac{28}{5}x\zeta_2^2 + 32x\zeta_3 \right) \right\} \right]. \tag{B.3}
\end{aligned}$$

$$\begin{aligned}
F_{V,2}^{(2)} = C_F^2 \left[\frac{1}{\varepsilon} \left\{ 8x\eta H_0 + 8x\xi\eta H_0^2 \right\} + \left\{ \xi^2\eta^3 \left(-\frac{4}{3}(-1+x)^2x^2H_0^4 - 192(-1+x)^2x^2H_{0,-1}\zeta_2 \right) \right. \right. \\
+ \eta^3x_+^2 \left(384x^2H_{0,0,0,-1}P_{358} + 32x^2H_0^2H_{0,-1}P_{362} + \frac{16}{3}x^4H_0^4 + \left(32x^2H_0^2P_{334} - 1536x^4H_{0,-1} \right) \zeta_2 \right. \\
+ \frac{96}{5}x^2P_{372}\zeta_2^2 \left. \right) + \xi \left(\eta^3x_+ \left(4(-1+x)x^3H_0^4 - 1728(-1+x)x^3H_{0,-1}\zeta_2 \right) + \eta \left(\left(128(-1+x)xH_0H_1 \right. \right. \right. \\
- 128(-1+x)xH_{0,1} \left. \left. \right) x_+ + \left(-64(-1+x)x^2H_0^3H_1 + 1344(-1+x)x^2H_{0,0,0,1} + \left(\right. \right. \right. \\
- 384(-1+x)x^2H_0H_1 + 384(-1+x)x^2H_{0,1} \left. \left. \right) \zeta_2 + 448(-1+x)x^2H_0\zeta_3 \right) x_+^3 \left. \right) \left. \right. \\
+ \eta^2 \left(\left(32xH_{0,0,1}P_{313} - 8xH_{-1}H_0^2P_{314} + 48xH_0H_{0,-1}P_{315} - 80xH_{0,0,-1}P_{317} - 16xH_0^2H_1P_{327} \right. \right. \\
+ 2xH_0^2P_{450} + 48xH_{-1}P_{355}\zeta_2 + 16xP_{316}\zeta_3 \left. \right) x_+^2 + \left(-\frac{8}{3}xH_0^3P_{471} - 16xH_0P_{473}\zeta_2 \right) x_+^3 \left. \right) \\
+ \eta \left(-4x \left(-15 - 173x + 323x^2 + x^3 \right) x_+^3\zeta_2 + 62xH_0 + \left(24x \left(13 - 38x + 13x^2 \right) H_{-1}H_0 \right. \right. \\
+ 128x^2H_0H_1 - 128x^2H_{0,1} - 24x \left(13 - 38x + 13x^2 \right) H_{0,-1} \left. \right) x_+^2 + \left(-64x^3H_0^3H_1 \right. \\
- 32x^2 \left(11 - 41x + 11x^2 \right) H_0^2H_{0,1} + 512x^2 \left(2 - 11x + 2x^2 \right) H_0H_{0,0,1} - 64x^2 \left(19 - 73x \right. \\
+ 19x^2 \left. \right) H_0H_{0,0,-1} + 9408x^3H_{0,0,0,1} + \left(-384x^3H_0H_1 + 384x^3H_{0,1} \right) \zeta_2 + 448x^3H_0\zeta_3 \left. \right) x_+^4 \left. \right) \\
+ 1824x^2H_0H_{0,1}x_+^4 - 192x \log(2)x_+^2\zeta_2 \left. \right\} + \varepsilon \left\{ \xi^2\eta^3 \left(-\frac{8}{3}(-1+x)^2x^2H_{-1}H_0^4 \right. \right. \\
- 4096(-1+x)^2x^2H_{0,0,-1,0,-1} + \left(-384(-1+x)^2x^2H_{-1}H_{0,-1} + 384(-1+x)^2x^2H_{0,1,-1} \right) \zeta_2 \left. \right) \\
+ \xi \left(\eta^2x_+ \left(-64(-1+x)x \left(3 + 22x + 3x^2 \right) H_1H_{0,0,1} - 32(-1+x)x \left(23 - 20x + 23x^2 \right) H_{-1}H_{0,0,-1} \right. \right. \\
+ 64(-1+x)x \left(3 + 22x + 3x^2 \right) H_{0,0,1,1} + 32(-1+x)x \left(23 - 20x + 23x^2 \right) H_{0,0,-1,-1} \left. \right) \left. \right)
\end{aligned}$$

$$\begin{aligned}
& + \eta^3 x_+ \left(8(-1+x)x^3 H_{-1} H_0^4 + \frac{4}{15}(-1+x)x^2(3-4x+3x^2)H_0^5 + 256(-1+x)x^2(29 \right. \\
& - 61x + 29x^2)H_0 H_{0,0,1,-1} + 256(-1+x)x^2(29-61x+29x^2)H_0 H_{0,0,-1,1} \\
& - 8192(-1+x)x^3 H_{0,0,-1,0,-1} + \left. (32(-1+x)x^2(19-31x+19x^2)H_0^2 H_1 \right. \\
& - 3456(-1+x)x^3 H_{-1} H_{0,-1} - 2688(-1+x)x^3 H_{0,1,-1})\zeta_2 \left. + \eta \left((-768(-1+x)x H_{-1} H_0 H_1 \right. \right. \\
& + 128(-1+x)x H_0 H_1^2 + \left. (-256(-1+x)x H_1 + 768(-1+x)x H_{-1})H_{0,1} + 768(-1+x)x H_1 H_{0,-1} \right. \\
& + 256(-1+x)x H_{0,1,1} - 768(-1+x)x H_{0,1,-1} - 768(-1+x)x H_{0,-1,1} - 128(-1+x)x H_1 \zeta_2 \left. \right) x_+ \\
& + \left(-128(-1+x)x^2 H_{-1} H_0^3 H_1 + 64(-1+x)x^2 H_0^3 H_1^2 - 128(-1+x)x^2 H_0^2 H_1 H_{0,1} \right. \\
& - 1024(-1+x)x^2 H_0 H_1 H_{0,0,1} + 1024(-1+x)x^2 H_0 H_1 H_{0,0,-1} + 2688(-1+x)x^2 H_{-1} H_{0,0,0,1} \\
& - 1024(-1+x)x^2 H_0 H_{0,0,1,1} + \left. (-768(-1+x)x^2 H_{-1} H_0 H_1 + 384(-1+x)x^2 H_0 H_1^2 + \left(\right. \right. \\
& - 768(-1+x)x^2 H_1 + 768(-1+x)x^2 H_{-1})H_{0,1} + 768(-1+x)x^2 H_{0,1,1})\zeta_2 + \left. (896(-1+x)x^2 H_{-1} H_0 \right. \\
& + 256(-1+x)x^2 H_0 H_1)\zeta_3 \left. \right) x_+^3 \left. \right) + \eta^3 \left(\log(2)x_+^2 \zeta_2 \left(1536(-2+x)x^3(-1+2x)H_{0,1} \right. \right. \\
& + 1536(-2+x)x^3(-1+2x)H_{0,-1} \left. \right) + 64x^2(3-5x+3x^2)H_0^2 H_{0,1,1} + \left(384x^2 H_0 H_{0,-1}^2 P_{333} \right. \\
& + 384x^2 H_0^2 H_{0,1,-1} P_{335} + 384x^2 H_0^2 H_{0,-1,1} P_{335} - 256x^2 H_0 H_{0,1}^2 P_{349} + \frac{8}{3}x^2 H_0^4 H_1 P_{364} \\
& + 1536x^2 H_{0,0,0,1,-1} P_{383} + 1536x^2 H_{0,0,0,-1,1} P_{383} - 768x^2 H_{0,0,0,-1,-1} P_{396} - 128x^2 H_0 H_{0,-1,0,1} P_{400} \\
& - 64x^2 H_0^2 H_{0,-1,-1} P_{404} - 128x^2 H_{0,0,1,0,1} P_{415} + 128x^2 H_{0,0,1,0,-1} P_{423} + 128x^2 H_{0,0,-1,0,1} P_{436} \\
& + 64x^2 H_{0,0,0,0,1} P_{455} - 64x^2 H_{0,0,0,0,-1} P_{460} + \frac{2}{3}x H_0^4 P_{492} - \frac{2}{3}x H_0^3 P_{506} + \frac{32}{3}x^4 H_{-1} H_0^4 + \frac{4}{15}x^3(3 \\
& - 4x + 3x^2)H_0^5 + \left. (256x^2 H_0 H_{0,-1} P_{380} - \frac{64}{3}x^2 H_0^3 P_{391} + 16x H_0^2 P_{485})H_{0,1} + (64x^2 H_{-1} H_0^2 P_{362} \right. \\
& - 64x^2 H_0^2 H_1 P_{363} + \frac{32}{3}x^2 H_0^3 P_{414} - 8x H_0^2 P_{495})H_{0,-1} + \left. (-128x^2 H_{0,-1} P_{375} + 128x^2 H_{0,1} P_{405} \right. \\
& + 32x^2 H_0^2 P_{430} - 64x H_0 P_{478})H_{0,0,1} + \left. (-512x^2 H_{0,-1} P_{318} - 128x^2 H_{0,1} P_{427} - 32x^2 H_0^2 P_{441} \right. \\
& - 16x H_0 P_{487})H_{0,0,-1} + \left. (-64x^2 H_0 P_{443} - 32x P_{491})H_{0,0,0,1} + (768x^2 H_{-1} P_{358} + 384x^2 H_1 P_{378} \right. \\
& + 64x^2 H_0 P_{452} + 16x P_{502})H_{0,0,0,-1} + 1024x^3(29-61x+29x^2)H_0 H_{0,0,1,-1} + 1024x^3(29 \\
& - 61x + 29x^2)H_0 H_{0,0,-1,1} + 128x^2(5-11x+5x^2)(11-34x+11x^2)H_0 H_{0,0,-1,-1} \\
& - 384x^2(5-6x+5x^2)(7-13x+7x^2)H_{0,0,0,1,1} - 4096x^4 H_{0,0,-1,0,-1} + \left. (-64x^2 H_{0,0,1} P_{312} \right. \\
& + 384x^2 H_{0,-1,1} P_{325} + 512x^2 H_{0,0,-1} P_{332} + 64x^2 H_{-1} H_0^2 P_{334} + 384x^2 H_{0,-1,-1} P_{353} \\
& + 128x^2 H_0 H_{0,1} P_{354} + \frac{32}{3}x^2 H_0^3 P_{369} - 8x H_0^2 P_{484} + 4x H_0 P_{507} + 128x^3(19-31x+19x^2)H_0^2 H_1 \\
& + \left. (-384x^2 H_1 P_{325} - 128x^2 H_0 P_{331} - 16x P_{499} - 3072x^4 H_{-1})H_{0,-1} - 3072x^4 H_{0,1,-1} \right. \\
& - 64x^2 P_{386} \zeta_3 \left. \right) \zeta_2 + \left(\frac{192}{5}x^2 H_{-1} P_{372} - \frac{96}{5}x^2 H_1 P_{426} - \frac{16}{5}x^2 H_0 P_{462} + \frac{8x P_{496}}{5} \right) \zeta_2^2 + \left(\right. \\
& - 16x^2 H_0^2 P_{356} + 128x^2 H_{0,-1} P_{370} + 64x^2 H_{0,1} P_{394} + 16x H_0 P_{479} \left. \right) \zeta_3 + 96x^2 P_{419} \zeta_5 \left. \right) x_+^2 \left. \right)
\end{aligned}$$

$$\begin{aligned}
& + \eta^2 \left(\log(2)x_+^2 \zeta_2 \left(-384xH_1P_{324} - 384xH_{-1}P_{324} - 384xP_{344} \right) + \left(8xH_{-1}^2H_0^2P_{311} \right. \right. \\
& + 16xH_0^2H_1^2P_{319} - 64xH_{0,-1,0,1}P_{328} + 128xH_{0,0,1,-1}P_{336} + 128xH_{0,0,-1,1}P_{336} + 64xH_0H_{0,1,1}P_{337} \\
& - 64xH_0H_{0,1,-1}P_{339} - 64xH_0H_{0,-1,1}P_{339} + 16xH_{0,-1}^2P_{340} + 32xH_0H_{0,-1,-1}P_{371} - 4xH_{-1}H_0^2P_{459} \\
& + 2xH_0^2P_{466} + \left(64xH_{-1}H_0^2P_{324} + 8xH_0^2P_{401} \right) H_1 + \left(-128xH_0H_1P_{322} + 64xH_{-1}H_0P_{326} \right. \\
& + 64xH_{0,-1}P_{330} + 64xH_0P_{343} \left. \right) H_{0,1} + \left(-384xH_{-1}H_0P_{323} + 64xH_0H_1P_{338} - 16xH_0P_{413} \right) H_{0,-1} \\
& + \left(-128xH_{-1}P_{336} - 16xP_{420} - 64x^2(3+22x+3x^2)H_1 \right) H_{0,0,1} + \left(-128xH_1P_{346} + 8xP_{464} \right. \\
& + 128x^2(23-20x+23x^2)H_{-1} \left. \right) H_{0,0,-1} + 64x^2(3+22x+3x^2)H_{0,0,1,1} - 128x^2(23-20x \\
& + 23x^2)H_{0,0,-1,-1} + \left(384xH_{-1}H_1P_{324} - 384xH_{-1,1}P_{324} - 48xH_{-1}^2P_{373} + 24xH_{-1}P_{425} \right. \\
& + 4xP_{428} \left. \right) \zeta_2 + \left(32xH_1P_{347} + 32xH_{-1}P_{390} - 16xP_{440} \right) \zeta_3 \left. \right) x_+^2 + \left(-\frac{16}{3}xH_0^3H_1P_{470} \right. \\
& + \frac{8}{3}xH_{-1}H_0^3P_{472} + \left(16xH_{-1}H_0P_{476} - 32(-2+x)x(5+39x-129x^2+101x^3)H_{0,1} \right) \zeta_2 \left. \right) x_+^3 \\
& + \eta \left(\left(3x(83+206x+83x^2)H_0 + 32x(41-60x+41x^2)H_{-1}H_0 - 24x(51-122x \right. \right. \\
& + 51x^2)H_{-1}^2H_0 + \left(-176x(3+2x+3x^2)H_0 - 768x^2H_{-1}H_0 \right) H_1 + 128x^2H_0H_1^2 + \left(176x(3 \right. \\
& + 2x+3x^2) - 256x^2H_1 + 768x^2H_{-1} \left. \right) H_{0,1} + \left(-32x(41-60x+41x^2) + 768x^2H_1 \right. \\
& + 48x(51-122x+51x^2)H_{-1} \left. \right) H_{0,-1} + 256x^2H_{0,1,1} - 768x^2H_{0,1,-1} - 768x^2H_{0,-1,1} - 48x(51 \\
& - 122x+51x^2)H_{0,-1,-1} - 128x^2H_1\zeta_2 \left. \right) x_+^2 + \left(-128x^3H_{-1}H_0^3H_1 + 64x^3H_0^3H_1^2 + \left(\right. \right. \\
& - 64x^2(11-41x+11x^2)H_{-1}H_0^2 + 640x^3H_0^2H_1 \left. \right) H_{0,1} + \left(1024x^2(2-11x+2x^2)H_{-1}H_0 \right. \\
& - 4096x^3H_0H_1 \left. \right) H_{0,0,1} + \left(-128x^2(19-73x+19x^2)H_{-1}H_0 + 4096x^3H_0H_1 \right) H_{0,0,-1} + \left(\right. \\
& - 1536(-2+x)x^2(-1+2x)H_1 + 18816x^3H_{-1} \left. \right) H_{0,0,0,1} - 1024x^3H_0H_{0,0,1,1} + \left(-32xH_0H_1P_{329} \right. \\
& - 768x^3H_{-1}H_0H_1 + 384x^3H_0H_1^2 + \left(-768x^3H_1 + 768x^3H_{-1} \right) H_{0,1} + 768x^3H_{0,1,1} \left. \right) \zeta_2 \\
& + \left. \left(896x^3H_{-1}H_0 + 1024x^3H_0H_1 \right) \zeta_3 \right) x_+^4 - 20xx_+^2 + 32c_1xx_+^2 - 1056x^2H_{0,1}^2x_+^4 \left. \right\} \\
& + C_F C_A \left[\left\{ \eta^2 x_+^2 \left(-4xH_{-1}H_0^2P_{382} + 8xH_0H_{0,-1}P_{392} - 8xH_{0,0,-1}P_{397} - \frac{2}{3}xH_0^2P_{402} + \left(\right. \right. \right. \right. \\
& - 24xH_{-1}P_{355} - \frac{4xP_{451}}{3} \left. \right) \zeta_2 + \xi^2 \eta^3 \left(\frac{2}{3}(-1+x)^2x^2H_0^4 + 96(-1+x)^2x^2H_{0,-1}\zeta_2 \right) + \xi \eta^3 x_+ \left(\right. \\
& - 2(-1+x)x^3H_0^4 + 864(-1+x)x^3H_{0,-1}\zeta_2 \left. \right) + \eta^3 x_+^2 \left(16x^2H_0^2H_{0,-1}P_{389} + 96x^2H_{0,0,0,-1}P_{398} \right. \\
& - \frac{4}{3}x^2H_0^3P_{474} - \frac{8}{3}x^4H_0^4 + \left(8x^2H_0^2P_{367} - 8x^2H_0P_{475} + 768x^4H_{0,-1} \right) \zeta_2 + \frac{24}{5}x^2P_{417}\zeta_2 \left. \right) + \eta \left(\left(\right. \right. \\
& - \frac{2}{9}x(173+670x+173x^2)H_0 + \frac{8}{3}x(5-8x+5x^2)H_{-1}H_0 - \frac{8}{3}x(5-8x+5x^2)H_{0,-1} \left. \right) x_+^2 \\
& + \left(-32x^2(17-47x+17x^2)\zeta_3H_0 + 32x^2(3-8x+3x^2)H_0^3H_1 - 80x^2(5-8x+5x^2)H_0^2H_{0,1} \right. \\
& + 128x^2(8-5x+8x^2)H_0H_{0,0,1} - 32x^2(41-62x+41x^2)H_0H_{0,0,-1} - 96x^2(13+4x+13x^2)H_{0,0,0,1} \\
\end{aligned}$$

$$\begin{aligned}
& + \left(192x^2(3 - 8x + 3x^2)H_0H_1 - 192x^2(3 - 8x + 3x^2)H_{0,1} \right) \zeta_2 x_+^4 + 12xx_+^2 + \left(\right. \\
& - 48(-1 + x)xH_0H_1 + 48(-1 + x)xH_{0,1} \left. \right) x_+^3 + \left(16x(2 + x + 2x^2)H_0^2H_1 - 32x(2 + 7x \right. \\
& \left. + 2x^2)H_0H_{0,1} + 32x(2 + 13x + 2x^2)H_{0,0,1} + 16x(5 - 71x + 5x^2)\zeta_3 \right) x_+^4 + 96x \log(2)x_+^2 \zeta_2 \left. \right\} \\
& + \varepsilon \left\{ \xi^2 \left(\eta^3 \left(\frac{4}{3}(-1 + x)^2 x^2 H_{-1} H_0^4 + 192(-1 + x)^2 x^2 H_{-1} H_{0,-1} \zeta_2 \right) + 64(-1 + x)^2 x \eta^2 H_{-1} H_0 H_{0,-1} \right) \right. \\
& + \eta^2 \left(\log(2)x_+^2 \zeta_2 \left(192xH_1P_{324} + 192xH_{-1}P_{324} + 192xP_{344} \right) + \left(-32xH_0^2H_1^2P_{320} \right. \right. \\
& + 128xH_{0,0,1,1}P_{341} - 128xH_0H_{0,1,1}P_{342} + 64xH_{0,-1,0,1}P_{345} - 64xH_{0,0,1,-1}P_{357} - 64xH_{0,0,-1,1}P_{357} \\
& + 8xH_{-1}H_0^2P_{384} + 16xH_{0,0,-1,-1}P_{411} + 8xH_{0,-1}^2P_{418} - 16xH_0H_{0,-1,-1}P_{421} - \frac{2}{9}xH_0^2P_{469} \\
& - 32x^2(31 - 64x + 31x^2)H_{-1}^2H_0^2 + \left(-32xH_{-1}H_0^2P_{324} - 32x^2(-4 + 23x - 16x^2 + 6x^3)H_0^2 \right) H_1 \\
& + \left(64xH_0H_1P_{348} - 32xH_{-1}H_0P_{352} + 32xH_0P_{361} - 32xH_{0,-1}P_{379} \right) H_{0,1} + \left(-8xH_0P_{408} \right. \\
& + 768x^3H_{-1}H_0 \left. \right) H_{0,-1} + \left(-64xP_{321} - 128xH_1P_{341} + 64xH_{-1}P_{357} \right) H_{0,0,1} + \left(64xH_1P_{376} \right. \\
& + 16xP_{381} - 16xH_{-1}P_{411} \left. \right) H_{0,0,-1} + \left(-192xH_{-1}H_1P_{324} + 192xH_{-1,1}P_{324} + 24xH_{-1}^2P_{373} \right. \\
& \left. - 16xH_{-1}P_{374} - \frac{4xP_{468}}{9} \right) \zeta_2 + \left(-16xH_1P_{377} - 16xH_{-1}P_{395} - 8xP_{429} \right) \zeta_3 \left. \right) x_+^2 + \eta^3 \left(\log(2)x_+^2 \zeta_2 \left(\right. \right. \\
& - 768(-2 + x)x^3(-1 + 2x)H_{0,1} - 768(-2 + x)x^3(-1 + 2x)H_{0,-1} \left. \right) + \left(96x^2H_0^2H_{0,1,-1}P_{366} \right. \\
& + 96x^2H_0^2H_{0,-1,1}P_{366} + 96x^2H_0H_{0,-1}^2P_{368} - 64x^2H_0H_{0,1}^2P_{407} + 32x^2H_0^2H_{0,1,1}P_{410} \\
& - 128x^2H_{0,0,-1,0,-1}P_{416} - 128x^2H_0H_{0,0,1,-1}P_{431} - 128x^2H_0H_{0,0,-1,1}P_{431} - 32x^2H_0^2H_{0,-1,-1}P_{432} \\
& + 384x^2H_{0,0,0,1,-1}P_{433} + 384x^2H_{0,0,0,-1,1}P_{433} - 64x^2H_0H_{0,-1,0,1}P_{434} - 192x^2H_{0,0,0,1,1}P_{439} \\
& - 192x^2H_{0,0,0,-1,-1}P_{444} - 64x^2H_{0,0,1,0,1}P_{445} + 64x^2H_0H_{0,0,-1,-1}P_{447} + 64x^2H_{0,0,1,0,-1}P_{449} \\
& + 64x^2H_{0,0,-1,0,1}P_{457} + 32x^2H_{0,0,0,0,1}P_{461} - 32x^2H_{0,0,0,0,-1}P_{465} - \frac{1}{3}x^2H_0^4P_{477} + \frac{2}{3}xH_0^3P_{481} - \frac{2}{15}x^3(3 \\
& - 4x + 3x^2)H_0^5 + \left(\frac{4}{3}x^2H_0^4P_{387} + \frac{16}{3}xH_0^3P_{480} \right) H_1 + \left(\frac{4}{3}xH_0^3P_{500} - \frac{16}{3}x^4H_0^4 \right) H_{-1} + \left(-\frac{16}{3}x^2H_0^3P_{435} \right. \\
& + 64x^2H_0H_{0,-1}P_{437} - 16xH_0^2P_{482} \left. \right) H_{0,1} + \left(-32x^2H_0^2H_1P_{388} + 32x^2H_{-1}H_0^2P_{389} + \frac{16}{3}x^2H_0^3P_{442} \right. \\
& \left. - 4xH_0^2P_{488} \right) H_{0,-1} + \left(-64x^2H_{0,-1}P_{399} + 64x^2H_{0,1}P_{438} + 16x^2H_0^2P_{448} - 32xH_0P_{483} \right) H_{0,0,1} \\
& + \left(256x^2H_{0,-1}P_{318} - 64x^2H_{0,1}P_{454} - 16x^2H_0^2P_{458} + 8xH_0P_{497} \right) H_{0,0,-1} + \left(-32x^2H_0P_{456} \right. \\
& + 32xP_{494} \left. \right) H_{0,0,0,1} + \left(192x^2H_{-1}P_{398} + 192x^2H_1P_{422} + 32x^2H_0P_{463} - 8xP_{504} \right) H_{0,0,0,-1} \\
& + \left(32x^2H_{0,0,1}P_{312} - 256x^2H_{0,0,-1}P_{332} - 192x^2H_{0,-1,-1}P_{353} + 192x^2H_{0,-1,1}P_{359} + 192x^2H_{0,1,-1}P_{360} \right. \\
& + \frac{8}{3}x^2H_0^3P_{393} + 4xH_0^2P_{498} - 2xH_0P_{505} + \left(-16x^2H_0^2P_{424} + 32xH_0P_{489} \right) H_1 + \left(16x^2H_0^2P_{367} \right. \\
& + 8xH_0P_{486} \left. \right) H_{-1} + \left(32x^2H_0P_{403} - 32xP_{490} \right) H_{0,1} + \left(-192x^2H_1P_{359} + 32x^2H_0P_{385} + 8xP_{501} \right. \\
& \left. + 1536x^4H_{-1} \right) H_{0,-1} - 32x^2P_{365}\zeta_3 \left. \right) \zeta_2 + \left(\frac{48}{5}x^2H_{-1}P_{417} - \frac{48}{5}x^2H_1P_{453} - \frac{8}{5}x^2H_0P_{467} + \frac{4xP_{503}}{5} \right) \zeta_2^2
\end{aligned}$$

$$\begin{aligned}
& + \left(64x^2 H_{0,-1} P_{406} - 8x^2 H_0^2 P_{409} + 32x^2 H_{0,1} P_{412} - 16x H_0 P_{493} \right) \zeta_3 + 48x^2 P_{446} \zeta_5 \Big) x_+^2 \\
& + \xi \left(\eta^2 x_+ \left(-4(-1+x)x(31-64x+31x^2) H_{-1}^2 H_0^2 + 448(-1+x)x^2 H_{-1} H_0 H_{0,-1} \right) \right. \\
& + \eta x_+^3 \left(64(-1+x)x^2 H_0^2 H_1 H_{0,1} + 512(-1+x)x^2 H_0 H_{0,0,1,1} \right) + \eta^3 x_+ \left(-4(-1+x)x^3 H_{-1} H_0^4 \right. \\
& - \frac{2}{15}(-1+x)x^2(3-4x+3x^2) H_0^5 + 1728(-1+x)x^3 H_{-1} H_{0,-1} \zeta_2 \Big) + \left(-576(-1+x)x H_0 H_{0,1,-1} \right. \\
& - 576(-1+x)x H_0 H_{0,-1,1} \Big) x_+^3 + \eta \left(\left(-\frac{7}{27}x(1151+3922x+1151x^2) H_0 - \frac{4}{9}x(749-2246x \right. \right. \\
& + 749x^2) H_{-1} H_0 + 8x(7-16x+7x^2) H_{-1}^2 H_0 + 32x(11-19x+11x^2) H_0 H_1 - 32x(11 \\
& - 19x+11x^2) H_{0,1} + \left(\frac{4}{9}x(749-2246x+749x^2) - 16x(7-16x+7x^2) H_{-1} \right) H_{0,-1} \\
& + 16x(7-16x+7x^2) H_{0,-1,-1} \Big) x_+^2 + \left(64x^2(3-8x+3x^2) H_{-1} H_0^3 H_1 - 32x^2(3-8x \right. \\
& + 3x^2) H_0^3 H_1^2 + \left(-160x^2(5-8x+5x^2) H_{-1} H_0^2 - 320x^3 H_0^2 H_1 \right) H_{0,1} + \left(256x^2(8-5x \right. \\
& + 8x^2) H_{-1} H_0 + 256x^2(10-19x+10x^2) H_0 H_1 \Big) H_{0,0,1} + \left(-64x^2(41-62x+41x^2) H_{-1} H_0 \right. \\
& - 256x^2(10-19x+10x^2) H_0 H_1 \Big) H_{0,0,-1} + \left(-384x^2(16-35x+16x^2) H_1 - 192x^2(13+4x \right. \\
& + 13x^2) H_{-1} \Big) H_{0,0,0,1} + 512x^3 H_0 H_{0,0,1,1} + \left(384x^2(3-8x+3x^2) H_{-1} H_0 H_1 - 192x^2(3-8x \right. \\
& + 3x^2) H_0 H_1^2 + \left(384x^2(3-8x+3x^2) H_1 - 384x^2(3-8x+3x^2) H_{-1} \right) H_{0,1} - 384x^2(3-8x \\
& + 3x^2) H_{0,1,1} \Big) \zeta_2 + \left(-64x^2(17-47x+17x^2) H_{-1} H_0 - 64x^2(10-19x+10x^2) H_0 H_1 \right) \zeta_3 \Big) x_+^4 \\
& + 78x x_+^2 - 16c_1 x x_+^2 + \left(288(-1+x)x H_{-1} H_0 H_1 - 48(-1+x)x H_0 H_1^2 + \left(96(-1+x)x H_1 \right. \right. \\
& - 288(-1+x)x H_{-1} \Big) H_{0,1} - 288(-1+x)x H_1 H_{0,-1} - 96(-1+x)x H_{0,1,1} + 288(-1+x)x H_{0,1,-1} \\
& + 288(-1+x)x H_{0,-1,1} + 48(-1+x)x H_1 \zeta_2 \Big) x_+^3 + \left(32(-2+x)x(-1+2x) H_{0,1}^2 \right. \\
& \left. - 32x(11-62x+11x^2) H_0 H_1 H_{0,-1} - 2880x^2 H_0 H_{0,1,-1} - 2880x^2 H_0 H_{0,-1,1} \right) x_+^4 \Big] \\
& + C_F n_i T_F \left[\eta \left\{ \frac{200}{9} x H_0 - \frac{32}{3} x H_{-1} H_0 + \frac{8}{3} x H_0^2 + \frac{32}{3} x H_{0,-1} - \frac{16x \zeta_2}{3} \right\} + \varepsilon \eta \left\{ \frac{3844}{27} x H_0 \right. \right. \\
& + 32x H_{-1}^2 H_0 + \frac{296}{9} x H_0^2 + \frac{8}{3} x H_0^3 + \left(-\frac{1184}{9} x H_0 - 16x H_0^2 \right) H_{-1} + \left(\frac{1184x}{9} \right. \\
& \left. - 64x H_{-1} \right) H_{0,-1} + 32x H_{0,0,-1} + 64x H_{0,-1,-1} + \left(-\frac{592x}{9} + 8x H_0 + 32x H_{-1} \right) \zeta_2 - 32x \zeta_3 \Big] \\
& + C_F T_F \left[\left\{ \xi x_+^3 \left(-\frac{8}{3}(-1+x)x H_0^2 + 16(-1+x)x \zeta_2 \right) + \eta \left(\frac{8}{9}x(25-322x+25x^2) x_+^2 H_0 \right. \right. \right. \\
& \left. \left. + \left(-32x^3 H_0^3 - 192x^3 H_0 \zeta_2 \right) x_+^4 \right) + \frac{272x x_+^2}{3} + \left(-\frac{160}{3}x^2 H_0^2 - 576x^2 \zeta_2 \right) x_+^4 \right\} \right]
\end{aligned}$$

$$\begin{aligned}
& + \varepsilon \left\{ \eta \left(\left(\frac{4}{27}x(961 - 4138x + 961x^2)H_0 - \frac{16}{9}x(25 - 322x + 25x^2)H_{-1}H_0 + \frac{16}{9}x(25 \right. \right. \\
& - 322x + 25x^2)H_{0,-1})x_+^2 + \left(-\frac{16}{9}x(-16 + 684x - 981x^2 + 41x^3)\zeta_2 - \frac{8}{9}x(-37 \right. \\
& + 273x + 24x^2 + 12x^3)H_0^2)x_+^3 + \left(-\frac{8}{9}xH_0^3P_{351} - 64x^3H_{-1}H_0^3 - 24x^3H_0^4 - 384x^3H_0^2H_{0,1} \right. \\
& + 384x^3H_0^2H_{0,-1} + 1536x^3H_0H_{0,0,1} - 768x^3H_0H_{0,0,-1} - 2304x^3H_{0,0,0,1} + 384x^3H_{0,0,0,-1} \\
& + \left. \left. \left(-\frac{8}{3}xH_0P_{350} - 384x^3H_{-1}H_0 - 96x^3H_0^2 + 1536x^3H_{0,-1} \right)\zeta_2 + \frac{96}{5}x^3\zeta_2^2 + 384x^3H_0\zeta_3 \right)x_+^4 \right) \\
& + \xi \left(\left(\frac{16}{3}(-1+x)xH_{-1}H_0^2 - \frac{32}{3}(-1+x)xH_0^2H_1 + \frac{64}{3}(-1+x)xH_0H_{0,1} - \frac{64}{3}(-1+x)xH_{0,0,1} \right. \right. \\
& - \frac{32}{3}(-1+x)xH_{0,0,-1} + 32(-1+x)xH_{-1}\zeta_2)x_+^3 - 128(-1+x)x \log(2)x_+^3\zeta_2) \\
& + \frac{1528x_+^2}{9} + \left(\frac{320}{3}x^2H_{-1}H_0^2 - \frac{640}{3}x^2H_0^2H_1 + \frac{1280}{3}x^2H_0H_{0,1} - \frac{1280}{3}x^2H_{0,0,1} \right. \\
& \left. \left. - \frac{640}{3}x^2H_{0,0,-1} + 640x^2H_{-1}\zeta_2 - \frac{64}{3}x(4 + 67x + 4x^2)\zeta_3 \right)x_+^4 + 2816x^2 \log(2)x_+^4\zeta_2 \right\}.
\end{aligned}$$

The polynomials are given by

$$\begin{aligned}
P_{311} &= x^4 - 96x^3 + 178x^2 - 96x + 1, \\
P_{312} &= x^4 - 69x^3 + 106x^2 - 69x + 1, \\
P_{313} &= x^4 - 67x^3 + 140x^2 - 67x + 1, \\
P_{314} &= x^4 - 64x^3 + 122x^2 - 64x + 1, \\
P_{315} &= x^4 - 54x^3 + 102x^2 - 54x + 1, \\
P_{316} &= x^4 - 52x^3 + 62x^2 - 52x + 1, \\
P_{317} &= x^4 - 52x^3 + 98x^2 - 52x + 1, \\
P_{318} &= x^4 - 29x^3 + 51x^2 - 29x + 1, \\
P_{319} &= x^4 - 27x^3 + 64x^2 - 27x + 1, \\
P_{320} &= x^4 - 20x^3 + 39x^2 - 20x + 1, \\
P_{321} &= x^4 - 18x^3 + 61x^2 - 30x + 7, \\
P_{322} &= x^4 - 2x^3 + 12x^2 - 2x + 1, \\
P_{323} &= x^4 - x^3 + 6x^2 - x + 1, \\
P_{324} &= x^4 + 5x^3 - 10x^2 + 5x + 1, \\
P_{325} &= x^4 + 15x^3 - 26x^2 + 15x + 1, \\
P_{326} &= x^4 + 22x^3 - 18x^2 + 22x + 1, \\
P_{327} &= x^4 + 47x^3 - 88x^2 + 47x + 1, \\
P_{328} &= x^4 + 53x^3 - 112x^2 + 53x + 1, \\
P_{329} &= x^4 + 93x^3 - 222x^2 + 91x + 9, \\
P_{330} &= x^4 + 115x^3 - 228x^2 + 115x + 1, \\
P_{331} &= 2x^4 - 44x^3 + 61x^2 - 44x + 2, \\
P_{332} &= 2x^4 - 33x^3 + 53x^2 - 33x + 2, \\
P_{333} &= 2x^4 - 16x^3 + 27x^2 - 16x + 2, \\
P_{334} &= 2x^4 - 10x^3 + 9x^2 - 10x + 2,
\end{aligned}$$

$$\begin{aligned}
P_{335} &= 2x^4 - 10x^3 + 17x^2 - 10x + 2, \\
P_{336} &= 2x^4 + 27x^3 - 28x^2 + 27x + 2, \\
P_{337} &= 2x^4 + 29x^3 - 42x^2 + 29x + 2, \\
P_{338} &= 2x^4 + 75x^3 - 130x^2 + 75x + 2, \\
P_{339} &= 2x^4 + 137x^3 - 246x^2 + 137x + 2, \\
P_{340} &= 3x^4 - 410x^3 + 770x^2 - 410x + 3, \\
P_{341} &= 3x^4 - 35x^3 + 65x^2 - 35x + 3, \\
P_{342} &= 3x^4 - 32x^3 + 59x^2 - 32x + 3, \\
P_{343} &= 3x^4 - 26x^3 - 12x^2 - 26x + 3, \\
P_{344} &= 3x^4 - 4x^3 + 4x^2 - 4x + 3, \\
P_{345} &= 3x^4 + 2x^3 - 8x^2 + 2x + 3, \\
P_{346} &= 3x^4 + 80x^3 - 140x^2 + 80x + 3, \\
P_{347} &= 3x^4 + 202x^3 - 388x^2 + 202x + 3, \\
P_{348} &= 4x^4 - 55x^3 + 104x^2 - 55x + 4, \\
P_{349} &= 4x^4 - 8x^3 + 9x^2 - 8x + 4, \\
P_{350} &= 5x^4 - 172x^3 - 54x^2 - 12x - 3, \\
P_{351} &= 5x^4 - 124x^3 - 24x^2 + 36x - 3, \\
P_{352} &= 5x^4 - 88x^3 + 162x^2 - 88x + 5, \\
P_{353} &= 5x^4 - 53x^3 + 70x^2 - 53x + 5, \\
P_{354} &= 5x^4 - 41x^3 + 61x^2 - 41x + 5, \\
P_{355} &= 5x^4 - 12x^3 + 26x^2 - 12x + 5, \\
P_{356} &= 5x^4 - 9x^3 + 20x^2 - 9x + 5, \\
P_{357} &= 6x^4 - 83x^3 + 152x^2 - 83x + 6, \\
P_{358} &= 6x^4 - 40x^3 + 67x^2 - 40x + 6, \\
P_{359} &= 7x^4 - 49x^3 + 78x^2 - 49x + 7, \\
P_{360} &= 7x^4 - 41x^3 + 58x^2 - 41x + 7, \\
P_{361} &= 7x^4 - 34x^3 + 84x^2 - 34x + 7, \\
P_{362} &= 7x^4 - 31x^3 + 50x^2 - 31x + 7, \\
P_{363} &= 7x^4 - 7x^3 + 2x^2 - 7x + 7, \\
P_{364} &= 7x^4 + 9x^3 - 26x^2 + 9x + 7, \\
P_{365} &= 7x^4 + 21x^3 - 11x^2 + 21x + 7, \\
P_{366} &= 8x^4 - 31x^3 + 44x^2 - 31x + 8, \\
P_{367} &= 8x^4 - 31x^3 + 60x^2 - 31x + 8, \\
P_{368} &= 8x^4 - 19x^3 + 24x^2 - 19x + 8, \\
P_{369} &= 8x^4 - 12x^3 + 11x^2 - 12x + 8, \\
P_{370} &= 8x^4 - 4x^3 + 9x^2 - 4x + 8, \\
P_{371} &= 9x^4 + 398x^3 - 698x^2 + 398x + 9, \\
P_{372} &= 11x^4 - 77x^3 + 109x^2 - 77x + 11, \\
P_{373} &= 11x^4 - 56x^3 + 118x^2 - 56x + 11, \\
P_{374} &= 11x^4 - 35x^3 + 267x^2 - 91x + 40,
\end{aligned}$$

$$\begin{aligned}
P_{375} &= 11x^4 + 29x^3 - 84x^2 + 29x + 11, \\
P_{376} &= 12x^4 - 79x^3 + 136x^2 - 79x + 12, \\
P_{377} &= 12x^4 + 43x^3 - 112x^2 + 43x + 12, \\
P_{378} &= 13x^4 - 49x^3 + 74x^2 - 49x + 13, \\
P_{379} &= 13x^4 - 38x^3 + 54x^2 - 38x + 13, \\
P_{380} &= 14x^4 - 62x^3 + 97x^2 - 62x + 14, \\
P_{381} &= 14x^4 - 55x^3 + 325x^2 - 107x + 39, \\
P_{382} &= 15x^4 - 140x^3 + 254x^2 - 140x + 15, \\
P_{383} &= 15x^4 - 67x^3 + 106x^2 - 67x + 15, \\
P_{384} &= 15x^4 + 17x^3 - 115x^2 + 69x - 10, \\
P_{385} &= 16x^4 - 139x^3 + 200x^2 - 139x + 16, \\
P_{386} &= 17x^4 - 123x^3 + 167x^2 - 123x + 17, \\
P_{387} &= 17x^4 - 111x^3 + 182x^2 - 111x + 17, \\
P_{388} &= 17x^4 - 95x^3 + 154x^2 - 95x + 17, \\
P_{389} &= 17x^4 - 71x^3 + 106x^2 - 71x + 17, \\
P_{390} &= 17x^4 - 9x^3 + 50x^2 - 9x + 17, \\
P_{391} &= 18x^4 - 82x^3 + 133x^2 - 82x + 18, \\
P_{392} &= 19x^4 - 148x^3 + 254x^2 - 148x + 19, \\
P_{393} &= 20x^4 - 129x^3 + 212x^2 - 129x + 20, \\
P_{394} &= 21x^4 - 221x^3 + 398x^2 - 221x + 21, \\
P_{395} &= 21x^4 - 119x^3 + 230x^2 - 119x + 21, \\
P_{396} &= 22x^4 - 72x^3 + 115x^2 - 72x + 22, \\
P_{397} &= 23x^4 - 156x^3 + 254x^2 - 156x + 23, \\
P_{398} &= 24x^4 - 73x^3 + 100x^2 - 73x + 24, \\
P_{399} &= 25x^4 - 182x^3 + 318x^2 - 182x + 25, \\
P_{400} &= 25x^4 - 165x^3 + 286x^2 - 165x + 25, \\
P_{401} &= 25x^4 - 24x^3 + 54x^2 + 40x - 39, \\
P_{402} &= 25x^4 + 8x^3 - 240x^2 + 64x - 1, \\
P_{403} &= 26x^4 - 71x^3 + 112x^2 - 71x + 26, \\
P_{404} &= 27x^4 - 107x^3 + 170x^2 - 107x + 27, \\
P_{405} &= 27x^4 - 71x^3 + 92x^2 - 71x + 27, \\
P_{406} &= 28x^4 - 149x^3 + 225x^2 - 149x + 28, \\
P_{407} &= 28x^4 - 137x^3 + 216x^2 - 137x + 28, \\
P_{408} &= 29x^4 - 38x^3 + 210x^2 - 38x + 29, \\
P_{409} &= 31x^4 - 144x^3 + 214x^2 - 144x + 31, \\
P_{410} &= 33x^4 - 154x^3 + 238x^2 - 154x + 33, \\
P_{411} &= 39x^4 - 368x^3 + 686x^2 - 368x + 39, \\
P_{412} &= 39x^4 - 34x^3 - 8x^2 - 34x + 39, \\
P_{413} &= 41x^4 - 274x^3 + 130x^2 - 274x + 41, \\
P_{414} &= 41x^4 - 121x^3 + 186x^2 - 121x + 41,
\end{aligned}$$

$$\begin{aligned}
P_{415} &= 43x^4 - 143x^3 + 204x^2 - 143x + 43, \\
P_{416} &= 44x^4 - 223x^3 + 342x^2 - 223x + 44, \\
P_{417} &= 46x^4 - 135x^3 + 224x^2 - 135x + 46, \\
P_{418} &= 47x^4 - 316x^3 + 534x^2 - 316x + 47, \\
P_{419} &= 48x^4 - 250x^3 + 391x^2 - 250x + 48, \\
P_{420} &= 49x^4 - 232x^3 - 42x^2 - 168x - 15, \\
P_{421} &= 51x^4 - 344x^3 + 590x^2 - 344x + 51, \\
P_{422} &= 51x^4 - 223x^3 + 342x^2 - 223x + 51, \\
P_{423} &= 53x^4 - 205x^3 + 312x^2 - 205x + 53, \\
P_{424} &= 53x^4 - 199x^3 + 306x^2 - 199x + 53, \\
P_{425} &= 53x^4 - 174x^3 + 356x^2 + 6x + 15, \\
P_{426} &= 57x^4 - 325x^3 + 490x^2 - 325x + 57, \\
P_{427} &= 57x^4 - 245x^3 + 386x^2 - 245x + 57, \\
P_{428} &= 59x^4 + 1366x^3 - 2290x^2 + 710x + 123, \\
P_{429} &= 61x^4 - 128x^3 + 134x^2 - 76x + 33, \\
P_{430} &= 67x^4 - 367x^3 + 614x^2 - 367x + 67, \\
P_{431} &= 67x^4 - 231x^3 + 322x^2 - 231x + 67, \\
P_{432} &= 69x^4 - 301x^3 + 454x^2 - 301x + 69, \\
P_{433} &= 70x^4 - 291x^3 + 438x^2 - 291x + 70, \\
P_{434} &= 71x^4 - 243x^3 + 338x^2 - 243x + 71, \\
P_{435} &= 72x^4 - 295x^3 + 436x^2 - 295x + 72, \\
P_{436} &= 73x^4 - 317x^3 + 498x^2 - 317x + 73, \\
P_{437} &= 80x^4 - 335x^3 + 508x^2 - 335x + 80, \\
P_{438} &= 81x^4 - 388x^3 + 610x^2 - 388x + 81, \\
P_{439} &= 81x^4 - 386x^3 + 606x^2 - 386x + 81, \\
P_{440} &= 89x^4 - 453x^3 + 539x^2 - 143x - 8, \\
P_{441} &= 91x^4 - 419x^3 + 684x^2 - 419x + 91, \\
P_{442} &= 103x^4 - 491x^3 + 750x^2 - 491x + 103, \\
P_{443} &= 107x^4 - 591x^3 + 986x^2 - 591x + 107, \\
P_{444} &= 112x^4 - 519x^3 + 784x^2 - 519x + 112, \\
P_{445} &= 113x^4 - 520x^3 + 810x^2 - 520x + 113, \\
P_{446} &= 120x^4 - 464x^3 + 701x^2 - 464x + 120, \\
P_{447} &= 125x^4 - 474x^3 + 686x^2 - 474x + 125, \\
P_{448} &= 125x^4 - 449x^3 + 634x^2 - 449x + 125, \\
P_{449} &= 127x^4 - 560x^3 + 858x^2 - 560x + 127, \\
P_{450} &= 127x^4 - 314x^3 + 256x^2 - 54x + 17, \\
P_{451} &= 143x^4 - 620x^3 + 732x^2 - 460x + 61, \\
P_{452} &= 151x^4 - 795x^3 + 1318x^2 - 795x + 151, \\
P_{453} &= 151x^4 - 559x^3 + 862x^2 - 559x + 151, \\
P_{454} &= 159x^4 - 673x^3 + 1018x^2 - 673x + 159,
\end{aligned}$$

$$\begin{aligned}
P_{455} &= 163x^4 - 827x^3 + 1350x^2 - 827x + 163, \\
P_{456} &= 181x^4 - 633x^3 + 886x^2 - 633x + 181, \\
P_{457} &= 191x^4 - 805x^3 + 1218x^2 - 805x + 191, \\
P_{458} &= 209x^4 - 856x^3 + 1266x^2 - 856x + 209, \\
P_{459} &= 215x^4 - 306x^3 - 148x^2 + 522x - 187, \\
P_{460} &= 221x^4 - 1153x^3 + 1914x^2 - 1153x + 221, \\
P_{461} &= 245x^4 - 907x^3 + 1302x^2 - 907x + 245, \\
P_{462} &= 303x^4 - 1763x^3 + 2306x^2 - 1763x + 303, \\
P_{463} &= 305x^4 - 1143x^3 + 1646x^2 - 1143x + 305, \\
P_{464} &= 379x^4 - 1402x^3 + 372x^2 - 574x - 23, \\
P_{465} &= 379x^4 - 1397x^3 + 1986x^2 - 1397x + 379, \\
P_{466} &= 497x^4 - 686x^3 + 638x^2 - 294x + 37, \\
P_{467} &= 513x^4 - 1705x^3 + 2998x^2 - 1705x + 513, \\
P_{468} &= 1277x^4 - 3170x^3 + 1683x^2 - 2680x + 442, \\
P_{469} &= 1666x^4 - 3184x^3 - 189x^2 + 430x + 125, \\
P_{470} &= x^5 + 64x^4 + 185x^3 - 443x^2 + 238x + 11, \\
P_{471} &= 8x^5 + 46x^4 - 23x^3 + x^2 - x - 3, \\
P_{472} &= 9x^5 + 237x^4 - 112x^3 - 160x^2 + 331x + 31, \\
P_{473} &= 14x^5 + 43x^4 - 2x^3 + 22x^2 - 4x + 3, \\
P_{474} &= 19x^5 - 176x^4 + 382x^3 - 356x^2 + 91x - 8, \\
P_{475} &= 19x^5 - 138x^4 + 314x^3 - 356x^2 + 159x - 46, \\
P_{476} &= 23x^5 - 137x^4 + 208x^3 + 160x^2 - 43x + 45, \\
P_{477} &= 31x^5 - 220x^4 + 488x^3 - 560x^2 + 153x - 20, \\
P_{478} &= x^6 + 33x^5 - 265x^4 + 592x^3 - 557x^2 + 199x + 5, \\
P_{479} &= 2x^6 + 159x^5 - 718x^4 + 1124x^3 - 616x^2 + 69x - 12, \\
P_{480} &= 3x^6 - 158x^5 + 520x^4 - 742x^3 + 507x^2 - 172x + 18, \\
P_{481} &= 3x^6 + 360x^5 - 1283x^4 + 1288x^3 - 5x^2 - 12x + 1, \\
P_{482} &= 4x^6 - 121x^5 + 529x^4 - 846x^3 + 594x^2 - 177x + 21, \\
P_{483} &= 4x^6 - 51x^5 - 262x^4 + 760x^3 - 684x^2 + 203x - 26, \\
P_{484} &= 5x^6 + 323x^5 - 654x^4 + 738x^3 - 185x^2 - 29x + 2, \\
P_{485} &= 6x^6 + 7x^5 - 316x^4 + 906x^3 - 906x^2 + 303x + 8, \\
P_{486} &= 7x^6 + 44x^5 - 207x^4 + 712x^3 - 739x^2 + 324x - 45, \\
P_{487} &= 9x^6 + 210x^5 + 697x^4 - 2900x^3 + 3103x^2 - 1134x - 17, \\
P_{488} &= 9x^6 + 232x^5 - 1443x^4 + 3096x^3 - 2669x^2 + 1032x - 113, \\
P_{489} &= 10x^6 - 187x^5 + 555x^4 - 742x^3 + 472x^2 - 143x + 11, \\
P_{490} &= 10x^6 - 181x^5 + 543x^4 - 742x^3 + 484x^2 - 149x + 11, \\
P_{491} &= 13x^6 - 114x^5 + 763x^4 - 1462x^3 + 1305x^2 - 512x - 17, \\
P_{492} &= 16x^6 - 14x^5 + 33x^4 - 180x^3 + 64x^2 - 6x + 7, \\
P_{493} &= 18x^6 - 301x^5 + 883x^4 - 980x^3 + 490x^2 - 79x - 3, \\
P_{494} &= 21x^6 - 358x^5 + 281x^4 + 484x^3 - 777x^2 + 250x - 33,
\end{aligned}$$

$$\begin{aligned}
P_{495} &= 23x^6 - 342x^5 + 199x^4 + 1128x^3 - 1799x^2 + 726x + 17, \\
P_{496} &= 38x^6 - 1856x^5 + 4771x^4 - 9562x^3 + 6836x^2 - 2070x - 149, \\
P_{497} &= 39x^6 + 12x^5 - 1857x^4 + 4240x^3 - 3727x^2 + 1204x - 135, \\
P_{498} &= 45x^6 - 704x^5 + 1622x^4 - 1268x^3 + 327x^2 + 36x - 2, \\
P_{499} &= 47x^6 + 66x^5 - 173x^4 - 816x^3 + 697x^2 - 378x + 29, \\
P_{500} &= 65x^6 - 604x^5 + 891x^4 + 712x^3 - 1837x^2 + 972x - 103, \\
P_{501} &= 77x^6 - 464x^5 + 909x^4 - 1576x^3 + 703x^2 - 288x + 15, \\
P_{502} &= 105x^6 - 168x^5 + 2339x^4 - 5364x^3 + 4403x^2 - 1524x - 31, \\
P_{503} &= 147x^6 - 3672x^5 + 12192x^4 - 15040x^3 + 9997x^2 - 3256x + 376, \\
P_{504} &= 155x^6 - 1264x^5 - 351x^4 + 4144x^3 - 5011x^2 + 1488x - 169, \\
P_{505} &= 159x^6 - 750x^5 + 1921x^4 - 2228x^3 + 469x^2 - 294x + 19, \\
P_{506} &= 165x^6 - 710x^5 - 709x^4 + 1724x^3 - 665x^2 + 306x - 47, \\
P_{507} &= 327x^6 - 458x^5 + 1457x^4 - 1020x^3 - 499x^2 + 158x - 29
\end{aligned} \tag{B.4}$$

C The axial-vector form factor up to two-loop

In this appendix, we present the axial-vector form factors $F_{A,1}^{(n)}$ and $F_{A,2}^{(n)}$ up to two loops and $\mathcal{O}(\varepsilon)$.

$$\begin{aligned}
F_{A,1}^{(1)} = & C_F \left[\frac{1}{\varepsilon} \left\{ -2 - 2\xi H_0 \right\} + \left\{ -4 + \xi \left(4H_{-1}H_0 - H_0^2 - 4H_{0,-1} + 2\zeta_2 \right) + \left(-3 + 2x - 3x^2 \right) \eta H_0 \right\} \right. \\
& + \varepsilon \left\{ -8 + \eta \left(2 \left(3 - 2x + 3x^2 \right) H_{-1}H_0 + \frac{1}{2} \left(-3 + 2x - 3x^2 \right) H_0^2 - 2 \left(3 - 2x + 3x^2 \right) H_{0,-1} \right. \right. \\
& + 2 \left(1 - x + 2x^2 \right) \zeta_2 \left. \right\} + \xi \left(\left(-8 - 4H_{-1}^2 + \zeta_2 \right) H_0 + 2H_{-1}H_0^2 - \frac{1}{3}H_0^3 + 8H_{-1}H_{0,-1} \right. \\
& \left. \left. - 4H_{0,0,-1} - 8H_{0,-1,-1} - 4H_{-1}\zeta_2 + 4\zeta_3 \right) \right\} + \varepsilon^2 \left\{ -16 + \eta \left(\left(-2 \left(3 - 2x + 3x^2 \right) H_{-1}^2 \right. \right. \right. \\
& + \frac{1}{2} \left(3 - 2x + 3x^2 \right) \zeta_2 \left. \right) H_0 + \left(3 - 2x + 3x^2 \right) H_{-1}H_0^2 + \frac{1}{6} \left(-3 + 2x - 3x^2 \right) H_0^3 + 4 \left(3 - 2x \right. \\
& + 3x^2 \left. \right) H_{-1}H_{0,-1} - 2 \left(3 - 2x + 3x^2 \right) H_{0,0,-1} - 4 \left(3 - 2x + 3x^2 \right) H_{0,-1,-1} + 2 \left(3 + 5x^2 \right) \zeta_2 - 2 \left(3 \right. \\
& \left. \left. - 2x + 3x^2 \right) H_{-1}\zeta_2 + \frac{4}{3} \left(5 - 3x + 4x^2 \right) \zeta_3 \right) + \xi \left(\left(-16 + \left(16 - 2\zeta_2 \right) H_{-1} + \frac{8}{3}H_{-1}^3 + \frac{14}{3}\zeta_3 \right) H_0 \right. \\
& + \left(-4 - 2H_{-1}^2 + \frac{1}{2}\zeta_2 \right) H_0^2 + \frac{2}{3}H_{-1}H_0^3 - \frac{1}{12}H_0^4 + \left(-16 - 8H_{-1}^2 - 2\zeta_2 \right) H_{0,-1} + 8H_{-1}H_{0,0,-1} \\
& \left. \left. + 16H_{-1}H_{0,-1,-1} - 4H_{0,0,0,-1} - 8H_{0,0,-1,-1} - 16H_{0,-1,-1,-1} + 4H_{-1}^2\zeta_2 + \frac{14}{5}\zeta_2^2 - 8H_{-1}\zeta_3 \right) \right\} \\
& + \varepsilon^3 \left\{ -32 + \eta \left(\left(\frac{4}{3} \left(3 - 2x + 3x^2 \right) H_{-1}^3 + \left(-3 + 2x - 3x^2 \right) H_{-1}\zeta_2 + \frac{7}{3} \left(3 - 2x + 3x^2 \right) \zeta_3 \right) H_0 + \left(\left(\right. \right. \right. \\
& \left. \left. -3 + 2x - 3x^2 \right) H_{-1}^2 + \frac{1}{4} \left(3 - 2x + 3x^2 \right) \zeta_2 + \left(-1 - x^2 \right) H_{-1}\zeta_2 \right) H_0^2 + \frac{1}{3} \left(3 - 2x + 3x^2 \right) H_{-1}H_0^3 \\
& + \frac{1}{24} \left(-3 + 2x - 3x^2 \right) H_0^4 + \left(-4 \left(3 - 2x + 3x^2 \right) H_{-1}^2 + \left(-3 + 2x - 3x^2 \right) \zeta_2 \right) H_{0,-1} + 4 \left(3 \right. \\
& \left. \left. - 2x + 3x^2 \right) H_{-1}H_{0,0,-1} + 8 \left(3 - 2x + 3x^2 \right) H_{-1}H_{0,-1,-1} - 2 \left(3 - 2x + 3x^2 \right) H_{0,0,0,-1} - 4 \left(3 \right. \right. \\
& \left. \left. - 2x + 3x^2 \right) H_{0,0,-1,-1} - 8 \left(3 - 2x + 3x^2 \right) H_{0,-1,-1,-1} + 4 \left(3 + 5x^2 \right) \zeta_2 + 2 \left(3 - 2x + 3x^2 \right) H_{-1}^2\zeta_2 \right. \\
& + \frac{1}{20} \left(75 - 56x + 93x^2 \right) \zeta_2^2 + \frac{4}{3} \left(13 + 11x^2 \right) \zeta_3 - 4 \left(3 - 2x + 3x^2 \right) H_{-1}\zeta_3 \left. \right) + \xi \left(\left(-32 + \left(32 \right. \right. \right. \\
& \left. \left. - \frac{28\zeta_3}{3} \right) H_{-1} + \left(-16 + 2\zeta_2 \right) H_{-1}^2 - \frac{4}{3}H_{-1}^4 + 4\zeta_2 + \frac{47}{20}\zeta_2^2 \right) H_0 + \left(-8 + 8H_{-1} + \frac{4}{3}H_{-1}^3 \right. \\
& + \frac{7}{3}\zeta_3 \left. \right) H_0^2 + \left(\frac{1}{6} \left(-8 + \zeta_2 \right) - \frac{2}{3}H_{-1}^2 \right) H_0^3 + \frac{1}{6}H_{-1}H_0^4 - \frac{1}{60}H_0^5 + \left(-16\zeta_2 - \frac{28\zeta_2^2}{5} \right) H_{-1} + \left(\right. \\
& \left. \left. -32 + \left(32 + 4\zeta_2 \right) H_{-1} + \frac{16}{3}H_{-1}^3 + \frac{4}{3}\zeta_3 \right) H_{0,-1} + \left(-16 - 8H_{-1}^2 - 2\zeta_2 \right) H_{0,0,-1} + \left(-32 \right. \right. \\
& \left. \left. - 16H_{-1}^2 - 4\zeta_2 \right) H_{0,-1,-1} + 8H_{-1}H_{0,0,0,-1} + 16H_{-1}H_{0,0,-1,-1} + 32H_{-1}H_{0,-1,-1,-1} - 4H_{0,0,0,0,-1} \right. \\
& \left. \left. - 8H_{0,0,0,-1,-1} - 16H_{0,0,-1,-1,-1} - 32H_{0,-1,-1,-1,-1} - \frac{8}{3}H_{-1}^3\zeta_2 + 8H_{-1}^2\zeta_3 - \frac{8}{3}\zeta_2\zeta_3 + 12\zeta_5 \right) \right\} \left. \right]. \tag{C.1}
\end{aligned}$$

$$\begin{aligned}
F_{A,1}^{(2),\text{ns}} = & C_F^2 \left[\frac{1}{\varepsilon^2} \left\{ 2 + 4\xi H_0 + 2\xi^2 H_0^2 \right\} + \frac{1}{\varepsilon} \left\{ 8 + \xi \left(-8H_{-1}H_0 + 4(2-x+x^2)\eta H_0^2 + 8H_{0,-1} - 4\zeta_2 \right) \right. \right. \\
& + \xi^2 \left(-8H_{-1}H_0^2 + 2H_0^3 + 8H_0H_{0,-1} - 4H_0\zeta_2 \right) + 2(7-2x+7x^2)\eta H_0 \left. \left. \right\} + \left\{ 46 + \eta \left(\frac{1}{2} (85 \right. \right. \right. \\
& - 6x + 85x^2) H_0 + 2(55-82x+55x^2) H_{-1}H_0 - 32(2-x+2x^2) H_0H_1 + 32(2-x \\
& + 2x^2) H_{0,1} - 2(55-82x+55x^2) H_{0,-1} \left. \left. \right\} + \xi \left(8H_{-1}^2H_0 - 16H_{-1}H_{0,-1} + 16H_{0,-1,-1} \right. \right. \\
& + 48(-1+x) \log(2)x_+\zeta_2 \left. \left. \right) + \eta^2 \left(-4H_{-1}H_0^2P_{528} + 4H_0^2H_1P_{552} + 8H_0H_{0,-1}P_{559} - 8H_0H_{0,1}P_{563} \right. \right. \\
& + 8H_{0,0,1}P_{574} - 8H_{0,0,-1}P_{576} + \frac{1}{2}H_0^2P_{603} + \left(\frac{16}{3}H_0^3H_1P_{614} + (16H_0H_1P_{612} - 16H_{0,1}P_{612})\zeta_2 \right) x_+ \\
& + (39-146x-46x^3+57x^4+8H_{-1}P_{536})\zeta_2 - 4P_{547}\zeta_3 \left. \left. \right) + \eta^3 \left(\frac{1}{6}H_0^4P_{643} + 4H_0^2H_{0,-1}P_{658} \right. \right. \\
& + 8H_0H_{0,0,-1}P_{670} + 8H_0^2H_{0,1}P_{673} - 48H_{0,0,0,-1}P_{674} + \frac{4}{3}H_0^3P_{690} - 16H_0H_{0,0,1}P_{698} + 16H_{0,0,0,1}P_{727} \\
& + (4xH_0P_{632} - 4H_0^2P_{699} + 8H_{0,-1}P_{701})\zeta_2 - \frac{2}{5}P_{811}\zeta_2^2 - 16H_0P_{667}\zeta_3 \left. \left. \right) + \xi^2 \left(16H_{-1}^2H_0^2 - 8H_{-1}H_0^3 \right. \right. \\
& + (32H_0H_1 - 32H_{0,-1})H_{0,1} + 16H_{0,1}^2 - 32H_{-1}H_0H_{0,-1} - 32H_0H_1H_{0,-1} + 8H_{0,-1}^2 - 64H_1H_{0,0,1} \\
& + 64H_1H_{0,0,-1} - 64H_0H_{0,1,1} + 32H_0H_{0,1,-1} + 32H_0H_{0,-1,1} + 16H_0H_{0,-1,-1} + 64H_{0,0,1,1} \\
& + 32H_{0,-1,0,1} + 16H_{-1}H_0\zeta_2 + 16H_1\zeta_3 \left. \left. \right) + 192x \log(2)x_+^2\zeta_2 \right\} + \varepsilon \left\{ \xi \left(\eta^2 \log(2)\zeta_2 \left(192x^2H_{0,1} \right. \right. \right. \\
& + 192x^2H_{0,-1} \left. \left. \right) - \frac{16}{3}H_{-1}^3H_0 + (32(-1+x^2)x(10-21x+10x^2)\eta^3)H_0H_{0,1}^2 + 16H_{-1}^2H_{0,-1} \right. \\
& - 32H_{-1}H_{0,-1,-1} + 32H_{0,-1,-1,-1} - 4(-1+x)x_+ \left. \left. \right) + \eta \left(\frac{1}{4} (479 + 70x + 479x^2) H_0 + 4(81 \right. \right. \\
& - 146x + 81x^2) H_{-1}H_0 - 2(257 - 286x + 257x^2) H_{-1}^2H_0 + \left(-4(43 - 18x + 43x^2) H_0 \right. \\
& + 192(2-x+2x^2) H_{-1}H_0) H_1 - 32(2-x+2x^2) H_0H_1^2 + \left(4(43 - 18x + 43x^2) + 64(2 \right. \\
& - x + 2x^2) H_1 - 192(2-x+2x^2) H_{-1} \left. \left. \right) H_{0,1} + \left(-4(81 - 146x + 81x^2) - 192(2-x \right. \right. \\
& + 2x^2) H_1 + 4(257 - 286x + 257x^2) H_{-1} \left. \left. \right) H_{0,-1} - 64(2-x+2x^2) H_{0,1,1} + 192(2-x \right. \\
& + 2x^2) H_{0,1,-1} + 192(2-x+2x^2) H_{0,-1,1} - 4(257 - 286x + 257x^2) H_{0,-1,-1} + 32(2 \\
& - x + 2x^2) H_1\zeta_2 \left. \left. \right) + \eta^2 \left(H_{-1}H_0^2P_{509} + 4H_{-1}^2H_0^2P_{522} + 16H_0H_{0,1,1}P_{525} - 32H_0H_{0,-1,-1}P_{542} \right. \right. \\
& - 16H_{0,0,1,1}P_{544} + 8H_{0,1}^2P_{553} + 16H_0H_{0,1,-1}P_{567} + 16H_0H_{0,-1,1}P_{567} + 4H_0^2H_1^2P_{568} + 16H_{0,-1,0,1}P_{573} \\
& - 16H_{0,0,-1,-1}P_{581} + 8H_{0,-1}^2P_{585} + \frac{1}{2}H_0^2P_{605} + \log(2)\zeta_2 \left(-24P_{512} + 48(3-2x^2+3x^4)H_1 + 48(3 \right. \\
& - 2x^2+3x^4)H_{-1} \left. \left. \right) + (2H_0^2P_{596} - 8(3-2x^2+3x^4)H_{-1}H_0^2)H_1 + (32H_0P_{520} - 16H_0H_1P_{560} \right. \\
& - 16H_{0,-1}P_{583} + 160(1-4x^2+x^4)H_{-1}H_0)H_{0,1} + \left(-16H_0H_1P_{543} - 16H_{-1}H_0P_{564} \right. \\
& - 8H_0P_{590} \left. \left. \right) H_{0,-1} + (16H_1P_{544} - 4P_{600} - 16(17-78x^2+17x^4)H_{-1})H_{0,0,1} + (16H_1P_{571} \right. \\
& + 16H_{-1}P_{581} + 2P_{606} \left. \left. \right) H_{0,0,-1} + 16(17-78x^2+17x^4)H_{0,0,1,-1} + 16(17-78x^2+17x^4)H_{0,0,-1,1} \right.
\end{aligned}$$

$$\begin{aligned}
& + \left(-\frac{32}{3}H_{-1}H_0^3H_1P_{618} + \frac{16}{3}H_0^3H_1^2P_{618} + \left(-64H_{-1}H_0H_1P_{616} + 32H_0H_1^2P_{616} + 64H_{0,1,1}P_{618} \right. \right. \\
& + \left. \left. \left(64H_{-1}P_{616} - 32H_1P_{620} \right) H_{0,1} \right) \zeta_2 \right) x_+ + \left(163 - 514x + 82x^2 - 74x^3 + 183x^4 - 8H_{-1}^2P_{584} \right. \\
& + 2H_{-1}P_{598} - 48 \left(3 - 2x^2 + 3x^4 \right) H_{-1}H_1 + 48 \left(3 - 2x^2 + 3x^4 \right) H_{-1,1} \left. \right) \zeta_2 + \left(4H_{-1}P_{587} - 4H_1P_{597} \right. \\
& - \left. \frac{2P_{607}}{3} \right) \zeta_3 + \xi^2 \left(-\frac{64}{3}H_{-1}^3H_0^2 + 16H_{-1}^2H_0^3 + \left(128H_{-1}H_0H_{0,-1} - 96H_{0,-1}^2 \right) H_1 + \left(\left(-128H_{-1}H_0 \right. \right. \right. \\
& + 64H_{0,-1} \left. \right) H_1 + 64H_0H_1^2 + 128H_{-1}H_{0,-1} \left. \right) H_{0,1} + \left(32H_1 - 64H_{-1} \right) H_{0,1}^2 + 64H_{-1}^2H_0H_{0,-1} \\
& - 64H_0H_1^2H_{0,-1} - 32H_{-1}H_{0,-1}^2 + \left(256H_{-1}H_1 - 128H_1^2 \right) H_{0,0,1} + \left(-256H_{-1}H_1 + 128H_1^2 \right) H_{0,0,-1} \\
& + \left(256H_{-1}H_0 - 192H_0H_1 - 256H_{0,-1} \right) H_{0,1,1} + \left(-128H_{-1}H_0 + 64H_0H_1 - 64H_{0,1} \right. \\
& + 192H_{0,-1} \left. \right) H_{0,1,-1} + \left(-128H_{-1}H_0 + 64H_0H_1 + 192H_{0,1} + 192H_{0,-1} \right) H_{0,-1,1} + \left(-64H_{-1}H_0 \right. \\
& + 64H_0H_1 - 320H_{0,1} + 32H_{0,-1} \left. \right) H_{0,-1,-1} + \left(256H_1 - 256H_{-1} \right) H_{0,0,1,1} - 256H_1H_{0,0,1,-1} \\
& - 256H_1H_{0,0,-1,1} + 256H_1H_{0,0,-1,-1} + 192H_0H_{0,1,1,1} - 64H_0H_{0,1,-1,-1} - 128H_{-1}H_{0,-1,0,1} \\
& - 64H_0H_{0,-1,1,-1} - 64H_0H_{0,-1,-1,1} + 32H_0H_{0,-1,-1,-1} - 384H_{0,0,1,1,1} + 768H_{0,0,1,1,-1} \\
& + 256H_{0,0,1,-1,1} - 256H_{0,0,-1,1,1} - 64H_{0,1,0,1,1} + 256H_{0,1,0,1,-1} - 256H_{0,-1,0,1,1} + 128H_{0,-1,0,1,-1} \\
& + 128H_{0,-1,0,-1,1} - 256H_{0,-1,1,0,1} + 320H_{0,-1,-1,0,1} - 32H_{-1}^2H_0\zeta_2 + \left(-64H_{-1}H_1 + 32H_1^2 \right) \zeta_3 \left. \right) \\
& + \eta^3 \left(16H_0^2H_{0,1,-1}P_{623} + 16H_0^2H_{0,-1,1}P_{623} + \frac{1}{6}H_0^3P_{639} + \frac{1}{30}H_0^5P_{641} + \frac{1}{3}H_0^4P_{644} - 48H_0H_{0,-1}^2P_{663} \right. \\
& - 32H_{0,0,1,0,1}P_{669} + 16H_0H_{0,0,-1,-1}P_{682} + 32H_{0,0,1,0,-1}P_{686} - 16H_0^2H_{0,1,1}P_{688} + 32H_0H_{0,0,1,-1}P_{696} \\
& + 32H_0H_{0,0,-1,1}P_{696} - 32H_{0,0,-1,0,1}P_{703} + 32H_0H_{0,-1,0,1}P_{712} - 8H_0^2H_{0,-1,-1}P_{715} + 32H_{0,0,0,1,-1}P_{726} \\
& + 32H_{0,0,0,-1,1}P_{726} - 32H_{0,0,-1,0,-1}P_{738} - 96H_{0,0,0,-1,-1}P_{739} + 32H_0H_{0,0,1,1}P_{744} - 32H_{0,0,0,1,1}P_{780} \\
& + 16H_{0,0,0,0,1}P_{807} - 16H_{0,0,0,0,-1}P_{808} + \left(-\frac{2}{3}H_0^4P_{675} - \frac{4}{3}H_0^3P_{790} \right) H_1 + \left(-\frac{4}{3}H_0^3P_{680} \right. \\
& + \left. \frac{2}{3}H_0^4P_{716} \right) H_{-1} + \left(-\frac{8}{3}H_0^3P_{668} + 32H_0^2H_1P_{677} - 16H_{-1}H_0^2P_{679} - 32H_0H_{0,-1}P_{683} + 4H_0^2P_{776} \right) H_{0,1} \\
& + 128x^2 \left(10 - 21x + 10x^2 \right) H_0H_{0,1}^2 + \left(16xH_{-1}H_0^2P_{617} - 16H_0^2H_1P_{655} + \frac{4}{3}H_0^3P_{747} - 4H_0^2P_{786} \right) H_{0,-1} \\
& + \left(-32H_{0,1}P_{647} + 64H_{-1}H_0P_{654} - 256H_0H_1P_{662} + 8H_0P_{714} + 8H_0^2P_{717} - 32H_{0,-1}P_{722} \right) H_{0,0,1} \\
& + \left(-32H_{-1}H_0P_{629} + 128H_0H_1P_{676} + 8H_0P_{709} + 32H_{0,1}P_{710} - 4H_0^2P_{725} + 16H_{0,-1}P_{768} \right) H_{0,0,-1} \\
& + \left(-32H_{-1}P_{657} + 64H_1P_{736} - 16H_0P_{788} - 8P_{800} \right) H_{0,0,0,1} + \left(96H_{-1}P_{624} - 32H_1P_{777} + 8H_0P_{795} \right. \\
& + 8P_{803} \left. \right) H_{0,0,0,-1} + \left(-2xH_0^2P_{635} + 32H_{0,1,-1}P_{707} + 32H_{0,-1,1}P_{708} + 16H_{0,0,1}P_{742} - 16H_{0,-1,-1}P_{765} \right. \\
& + \frac{4}{3}H_0^3P_{782} - 8H_{0,0,-1}P_{789} + H_0P_{812} + \left(8H_0^2P_{719} - 8H_0P_{763} \right) H_1 + \left(8H_0^2P_{665} - 8H_0P_{754} \right) H_{-1} + \left(\right. \\
& - 16H_0P_{728} + 8P_{755} \left. \right) H_{0,1} + \left(32H_{-1}P_{666} + 8P_{681} + 16H_0P_{694} - 32H_1P_{708} \right) H_{0,-1} + 4P_{783}\zeta_3 \left. \right) \zeta_2 \\
& + \left(\frac{8}{5}H_{-1}P_{659} + \frac{8}{5}H_1P_{792} - \frac{4P_{810}}{5} + \frac{2}{5}H_0P_{814} \right) \zeta_2^2 + \left(-32H_{-1}H_0P_{667} + 16H_0H_1P_{691} + 16H_{0,-1}P_{731} \right. \\
& \left. - 16H_{0,1}P_{741} + \frac{4}{3}H_0^2P_{785} + \frac{4}{3}H_0P_{791} \right) \zeta_3 - 6P_{798}\zeta_5 + 8xx_+^2 + 8c_1 \left(1 - 4x + x^2 \right) x_+^2 \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& + C_F C_A \left[\frac{1}{\varepsilon^2} \left\{ \frac{11}{3} + \frac{11\xi H_0}{3} \right\} + \frac{1}{\varepsilon} \left\{ -\frac{49}{9} + \xi \left(\eta \left(-\frac{4}{3} x^2 H_0^3 - 2(-1+3x^2) H_0 \zeta_2 \right) - \frac{67}{9} H_0 + 4H_{-1} H_0 \right. \right. \right. \\
& - 4H_0 H_1 + 4H_{0,1} - 4H_{0,-1} \left. \left. \right) + \eta \left(-4x^2 H_0^2 - 2(-1+3x^2) \zeta_2 \right) + \xi^2 \left(-4H_0 H_{0,1} + 4H_0 H_{0,-1} + 8H_{0,0,1} \right. \right. \\
& \left. \left. - 8H_{0,0,-1} - 2\zeta_3 \right) \right\} + \left\{ -\frac{1595}{27} + \eta \left(-\frac{5}{54} (509 - 114x + 509x^2) H_0 + \frac{4}{9} (31 - 105x + 31x^2) H_{-1} H_0 \right. \right. \\
& + 4(3 - 2x + 3x^2) H_0 H_1 - 4(3 - 2x + 3x^2) H_{0,1} - \frac{4}{9} (31 - 105x + 31x^2) H_{0,-1} \left. \left. \right) + \xi^2 \left((-8H_1 \right. \right. \\
& + 8H_{-1}) \zeta_3 + (16H_{-1} H_0 - 16H_0 H_1 - 8H_{0,-1}) H_{0,1} - 4H_{0,1}^2 - 16H_{-1} H_0 H_{0,-1} + 16H_0 H_1 H_{0,-1} \\
& + 12H_{0,-1}^2 + (32H_1 - 32H_{-1}) H_{0,0,1} + (-32H_1 + 32H_{-1}) H_{0,0,-1} + 24H_0 H_{0,1,1} - 8H_0 H_{0,1,-1} \\
& - 8H_0 H_{0,-1,1} - 8H_0 H_{0,-1,-1} - 32H_{0,0,1,1} + 32H_{0,0,1,-1} + 32H_{0,0,-1,1} - 32H_{0,0,-1,-1} \left. \right) + \xi \left(\eta \left(\frac{4}{3} (3 \right. \right. \\
& + x^2) H_{-1} H_0^3 + 16H_{-1} H_0 \zeta_2 \right) - \frac{104}{3} H_{-1}^2 H_0 + 24H_{-1} H_0 H_1 - 4H_0 H_1^2 + (8H_1 - 24H_{-1}) H_{0,1} \\
& + \left(-24H_1 + \frac{208H_{-1}}{3} \right) H_{0,-1} - 8H_{0,1,1} + 24H_{0,1,-1} + 24H_{0,-1,1} - \frac{208}{3} H_{0,-1,-1} + 4H_1 \zeta_2 \\
& - 24(-1+x) \log(2) x_+ \zeta_2 \left. \right) + \eta^2 \left(-4H_0^2 H_1 P_{526} - 4H_0 H_{0,-1} P_{538} - 8H_{0,0,1} P_{555} - \frac{2}{3} H_{-1} H_0^2 P_{591} \right. \\
& + \frac{4}{3} H_{0,0,-1} P_{599} - \frac{1}{18} (1+x)(233 - 299x + 65x^2 + 217x^3) H_0^2 + 8(1+x)^2 (6 - 11x + 6x^2) H_0 H_{0,1} \\
& + \left(\frac{4}{3} H_0^3 H_1 P_{611} + (16H_0 H_1 P_{613} - 4H_{0,1} P_{619}) \zeta_2 \right) x_+ + \left(\frac{P_{510}}{9} - \frac{4}{3} H_{-1} P_{554} \right) \zeta_2 + \frac{2P_{534}\zeta_3}{3} \left. \right) \\
& + \eta^3 \left(32x H_0 H_{0,0,1} P_{615} + \frac{1}{6} x H_0^4 P_{630} + \frac{1}{9} H_0^3 P_{640} + 8H_0 H_{0,0,-1} P_{660} + 4H_0^2 H_{0,1} P_{661} - 4H_0^2 H_{0,-1} P_{695} \right. \\
& + 8H_{0,0,0,-1} P_{723} - 8H_{0,0,0,1} P_{737} + \left(-2H_0^2 P_{650} - 4H_{0,-1} P_{700} - \frac{2}{3} H_0 P_{764} \right) \zeta_2 + \frac{1}{5} P_{797} \zeta_2^2 \\
& \left. \left. + 2H_0 P_{730} \zeta_3 \right) - 96x \log(2) x_+^2 \zeta_2 \right\} + \varepsilon \left\{ \eta \left(-\frac{5}{324} (14033 + 462x + 14033x^2) H_0 + \frac{1}{27} (3623 + 534x \right. \right. \\
& + 3623x^2) H_{-1} H_0 + \frac{4}{9} (28 + 279x + 28x^2) H_{-1}^2 H_0 + \frac{1}{9} (-533 - 659x - 677x^2 + 97x^3) x_+ H_{-1} H_0^2 \\
& + (4(5 - 32x + 5x^2) H_0 - 24(3 - 2x + 3x^2) H_{-1} H_0) H_1 + 4(3 - 2x + 3x^2) H_0 H_1^2 + (-4(5 - 32x \\
& + 5x^2) - 8(3 - 2x + 3x^2) H_1 + 24(3 - 2x + 3x^2) H_{-1}) H_{0,1} + \left(\frac{1}{27} (-3623 - 534x - 3623x^2) \right. \\
& + 24(3 - 2x + 3x^2) H_1 - \frac{8}{9} (28 + 279x + 28x^2) H_{-1}) H_{0,-1} + 8(3 - 2x + 3x^2) H_{0,1,1} - 24(3 - 2x \\
& + 3x^2) H_{0,1,-1} - 24(3 - 2x + 3x^2) H_{0,-1,1} + \frac{8}{9} (28 + 279x + 28x^2) H_{0,-1,-1} - 4(3 - 2x + 3x^2) H_1 \zeta_2 \left. \right) \\
& + \xi \left(\eta^2 \log(2) \zeta_2 (-96x^2 H_{0,1} - 96x^2 H_{0,-1}) + \eta \left(-\frac{4}{3} (3 + 5x^2) H_{-1}^2 H_0^3 - 8(1 + 3x^2) H_{-1}^2 H_0 \zeta_2 \right) \right. \\
& + 80H_{-1}^3 H_0 - 72H_{-1}^2 H_0 H_1 + 24H_{-1} H_0 H_1^2 - \frac{8}{3} H_0 H_1^3 + (-48H_{-1} H_1 + 8H_1^2 + 72H_{-1}^2) H_{0,1} \\
& + (144H_{-1} H_1 - 24H_1^2 - 240H_{-1}^2) H_{0,-1} + (-16H_1 + 48H_{-1}) H_{0,1,1} + (48H_1 - 144H_{-1}) H_{0,1,-1} \\
& \left. \left. + (48H_1 - 144H_{-1}) H_{0,-1,1} + (-144H_1 + 480H_{-1}) H_{0,-1,-1} + 16H_{0,1,1,1} - 48H_{0,1,1,-1} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -48H_{0,1,-1,1} + 144H_{0,1,-1,-1} - 48H_{0,-1,1,1} + 144H_{0,-1,1,-1} + 144H_{0,-1,-1,1} - 480H_{0,-1,-1,-1} \\
& + 4H_1^2\zeta_2) + \eta^2\left(\frac{1}{54}H_0^2P_{511} - 24H_0H_{0,1,1}P_{513} - 8H_{0,1}^2P_{533} - 16H_0H_{0,1,-1}P_{548} - 16H_0H_{0,-1,1}P_{548} \right. \\
& + 8H_{0,0,1,1}P_{556} - 2H_0^2H_1^2P_{558} - 8H_{0,-1,0,1}P_{569} + 8H_0H_{0,-1,-1}P_{575} - 4H_{0,-1}^2P_{578} - 8H_{0,0,1,-1}P_{582} \\
& - 8H_{0,0,-1,1}P_{582} + 2H_{-1}^2H_0^2P_{592} + 8H_{0,0,-1,-1}P_{594} + \log(2)\zeta_2\left(12P_{512} - 24\left(3 - 2x^2 + 3x^4\right)H_1 \right. \\
& \left. - 24\left(3 - 2x^2 + 3x^4\right)H_{-1}\right) + \left(-2H_0^2P_{586} - 4\left(-15 + 2x^2 + 9x^4\right)H_{-1}H_0^2\right)H_1 + \left(\right. \\
& \left. -16H_{-1}H_0P_{531} + 4H_0P_{541} + 8H_0H_1P_{557} + 16H_{0,-1}P_{561}\right)H_{0,1} + \left(8H_{-1}H_0P_{514} + 8H_0H_1P_{549} \right. \\
& \left. + 8H_0P_{580}\right)H_{0,-1} + \left(-8H_1P_{556} + 4P_{565} + 8H_{-1}P_{582}\right)H_{0,0,1} + \left(-8H_1P_{539} - 8H_{-1}P_{594} \right. \\
& \left. - \frac{2P_{609}}{9}\right)H_{0,0,-1} + \left(\frac{8}{3}H_{-1}H_0^3H_1P_{627} - \frac{4}{3}H_0^3H_1^2P_{627} + \left(16H_{-1}H_0H_1P_{622} - 8H_0H_1^2P_{622} \right. \right. \\
& \left. - 8H_{0,1,1}P_{631} + \left(32x\left(3 - 8x + 14x^2 + 3x^4\right)H_1 - 32x\left(3 - 8x + 14x^2 + 3x^4\right)H_{-1}\right)H_{0,1}\right)\zeta_2)x_+ \\
& + \left(\frac{P_{508}}{27} + 12H_{-1}^2P_{535} + \frac{2}{9}H_{-1}P_{602} + 48\left(1 - x^2 + 2x^4\right)H_{-1}H_1 - 24\left(3 - 2x^2 + 3x^4\right)H_{-1,1}\right)\zeta_2 \\
& + \left(2H_{-1}P_{589} + 2H_1P_{593} + \frac{P_{608}}{9}\right)\zeta_3) + \xi^2\left(\left(-64H_{-1}H_0H_{0,-1} + 48H_{0,-1}^2\right)H_1 + \left(-32H_{-1}^2H_0 \right. \right. \\
& \left. + \left(64H_{-1}H_0 - 32H_{0,-1}\right)H_1 - 32H_0H_1^2 + 32H_{-1}H_{0,-1}\right)H_{0,1} + \left(-16H_1 + 16H_{-1}\right)H_{0,1}^2 \\
& + 32H_{-1}^2H_0H_{0,-1} + 32H_0H_1^2H_{0,-1} - 48H_{-1}H_{0,-1}^2 + \left(-128H_{-1}H_1 + 64H_1^2 + 64H_{-1}^2\right)H_{0,0,1} \\
& + \left(128H_{-1}H_1 - 64H_1^2 - 64H_{-1}^2\right)H_{0,0,-1} + \left(-96H_{-1}H_0 + 96H_0H_1 + 16H_{0,1} + 80H_{0,-1}\right)H_{0,1,1} \\
& + \left(32H_{-1}H_0 - 32H_0H_1 + 16H_{0,1} - 48H_{0,-1}\right)H_{0,1,-1} + \left(32H_{-1}H_0 - 32H_0H_1 - 48H_{0,1} \right. \\
& \left. - 48H_{0,-1}\right)H_{0,-1,1} + \left(32H_{-1}H_0 - 32H_0H_1 + 16H_{0,1} - 48H_{0,-1}\right)H_{0,-1,-1} + \left(-128H_1 \right. \\
& \left. + 128H_{-1}\right)H_{0,0,1,1} + \left(128H_1 - 128H_{-1}\right)H_{0,0,1,-1} + \left(128H_1 - 128H_{-1}\right)H_{0,0,-1,1} + \left(-128H_1 \right. \\
& \left. + 128H_{-1}\right)H_{0,0,-1,-1} - 112H_0H_{0,1,1,1} + 16H_0H_{0,1,1,-1} + 16H_0H_{0,1,-1,1} + 16H_0H_{0,1,-1,-1} \\
& + 16H_0H_{0,-1,1,1} + 16H_0H_{0,-1,1,-1} + 16H_0H_{0,-1,-1,1} + 16H_0H_{0,-1,-1,-1} + 128H_{0,0,1,1,1} \\
& - 256H_{0,0,1,1,-1} - 128H_{0,0,1,-1,1} + 128H_{0,0,1,-1,-1} + 128H_{0,0,-1,1,-1} + 128H_{0,0,-1,-1,1} \\
& + 256H_{0,0,-1,-1,-1} - 64H_{0,1,0,1,-1} + 64H_{0,-1,0,1,1} + 192H_{0,-1,0,-1,-1} + 64H_{0,-1,1,0,1} + \left(32H_{-1}H_1 \right. \\
& \left. - 16H_1^2 - 16H_{-1}^2\right)\zeta_3) + \eta^3\left(\frac{1}{30}xH_0^5P_{634} + \frac{1}{12}H_0^4P_{642} + 32H_0H_{0,0,1,-1}P_{646} + 32H_0H_{0,0,-1,1}P_{646} \right. \\
& + 48H_0H_{0,-1}^2P_{651} - 80H_0H_{0,-1,0,1}P_{656} - 128H_0H_{0,0,1,1}P_{664} + 16H_0H_{0,1}^2P_{671} + 8H_0^2H_{0,1,1}P_{685} \\
& - 8H_0^2H_{0,1,-1}P_{689} - 8H_0^2H_{0,-1,1}P_{689} - 16H_0H_{0,0,-1,-1}P_{704} + 16H_{0,0,1,0,1}P_{733} + 8H_0^2H_{0,-1,-1}P_{734} \\
& + 32H_{0,0,-1,0,-1}P_{740} - 16H_{0,0,-1,0,1}P_{743} - 16H_{0,0,1,0,-1}P_{750} - 32H_{0,0,0,1,-1}P_{774} - 32H_{0,0,0,-1,1}P_{774} \\
& + 16H_{0,0,0,1,1}P_{796} - 8H_{0,0,0,0,1}P_{801} + 8H_{0,0,0,0,-1}P_{802} + 16H_{0,0,0,-1,-1}P_{805} + \frac{1}{54}H_0^3P_{813} + \left(\frac{1}{3}H_0^4P_{626} \right. \\
& \left. + \frac{2}{3}H_0^3P_{775}\right)H_1 + \left(\frac{1}{3}H_0^4P_{645} + \frac{2}{3}H_0^3P_{794}\right)H_{-1} + \left(8H_{-1}H_0^2P_{661} - 16H_0^2H_1P_{677} - 16H_0H_{0,-1}P_{687} \right. \\
& \left. + \frac{8}{3}H_0^3P_{713} - 2H_0^2P_{779}\right)H_{0,1} + \left(8H_{-1}H_0^2P_{625} + 8H_0^2H_1P_{684} - \frac{4}{3}H_0^3P_{781} + 2H_0^2P_{787}\right)H_{0,-1} \\
& + \left(-32H_{-1}H_0P_{649} + 32H_0H_1P_{693} + 4H_0P_{702} - 16H_{0,1}P_{706} + 16H_{0,-1}P_{751} - 4H_0^2P_{757}\right)H_{0,0,1}
\end{aligned}$$

$$\begin{aligned}
& + \left(16H_{-1}H_0P_{660} - 32H_0H_1P_{678} + 16H_{0,1}P_{718} - 32H_{0,-1}P_{729} + 4H_0^2P_{772} - 4H_0P_{784} \right) H_{0,0,-1} \\
& + \left(16H_{-1}P_{697} - 16H_1P_{745} + 8H_0P_{778} + 4P_{793} \right) H_{0,0,0,1} + \left(-16H_{-1}P_{711} + 16H_1P_{724} \right. \\
& - 8H_0P_{771} - 4P_{799} \left. \right) H_{0,0,0,-1} + \left(\frac{1}{18}H_0P_{638} - \frac{4}{3}H_0^3P_{749} + 16H_{0,0,-1}P_{752} - 8H_{0,1,-1}P_{759} \right. \\
& - 8H_{0,-1,1}P_{760} + 8H_{0,-1,-1}P_{770} - 8H_{0,0,1}P_{773} + \frac{1}{2}H_0^2P_{806} + \left(-4xH_0^2P_{621} + 4H_0P_{758} \right) H_1 \\
& + \left(8H_0^2P_{653} + 2H_0P_{721} \right) H_{-1} + \left(-4P_{735} + 4H_0P_{769} \right) H_{0,1} + \left(-16H_{-1}P_{692} \right. \\
& + 16H_1P_{720} - 2P_{732} - 4H_0P_{767} \left. \right) H_{0,-1} - 2P_{756}\zeta_3 \left. \right) \zeta_2 + \left(-\frac{8}{5}H_{-1}P_{705} + \frac{2}{5}H_0P_{748} \right. \\
& + \frac{4}{5}H_1P_{753} + \frac{P_{809}}{5} \left. \right) \zeta_2^2 + \left(-16H_0H_1P_{652} + 16H_{-1}H_0P_{672} + 4H_{0,1}P_{746} - 2H_0^2P_{761} \right. \\
& - 4H_{0,-1}P_{762} + \frac{4}{3}H_0P_{766} \left. \right) \zeta_3 + P_{804}\zeta_5 \left. \right) - \frac{28745}{162}(1+x)^2x_+^2 - 4c_1(1-4x+x^2)x_+^2 \left. \right\} \\
& + C_F n_l T_F \left[\frac{1}{\varepsilon^2} \left\{ -\frac{4}{3} - \frac{4\xi H_0}{3} \right\} + \frac{1}{\varepsilon} \left\{ \frac{20}{9} + \frac{20\xi H_0}{9} \right\} + \left\{ \frac{424}{27} + \eta \left(\frac{2}{27} (209 - 78x + 209x^2) \right) H_0 \right. \right. \\
& - \frac{8}{9} (19 - 6x + 19x^2) H_{-1}H_0 + \frac{2}{9} (19 - 6x + 19x^2) H_0^2 + \frac{8}{9} (19 - 6x + 19x^2) H_{0,-1} - \frac{4}{9} (7 \\
& - 6x + 31x^2) \zeta_2 \left. \right\} + \xi \left(\frac{16}{3} H_{-1}^2 H_0 - \frac{8}{3} H_{-1} H_0^2 + \frac{4}{9} H_0^3 - \frac{32}{3} H_{-1} H_{0,-1} + \frac{16}{3} H_{0,0,-1} + \frac{32}{3} H_{0,-1,-1} \right. \\
& + \left. \left(\frac{8}{3} H_0 + \frac{16}{3} H_{-1} \right) \zeta_2 - \frac{16\zeta_3}{3} \right) \left. \right\} + \varepsilon \left\{ \frac{5204}{81} + \xi \left(\left(-\frac{16H_0}{3} + 32H_{-1} \right) \zeta_3 - \frac{32}{3} H_{-1}^3 H_0 + 8H_{-1}^2 H_0^2 \right. \right. \\
& - \frac{8}{3} H_{-1} H_0^3 + 32H_{-1}^2 H_{0,-1} - 32H_{-1} H_{0,0,-1} - 64H_{-1} H_{0,-1,-1} + 16H_{0,0,0,-1} + 32H_{0,0,-1,-1} \\
& + 64H_{0,-1,-1,-1} + \left(-8H_{-1}H_0 + 2H_0^2 - 16H_{-1}^2 + 24H_{0,-1} \right) \zeta_2 - \frac{96\zeta_2^2}{5} \left. \right\} + \eta \left(\frac{1}{81} (5813 \right. \\
& - 1014x + 5813x^2) H_0 + \frac{8}{9} (47 - 18x + 47x^2) H_{-1}^2 H_0 + \frac{2}{27} (281 - 78x + 281x^2) H_0^2 \\
& + \frac{2}{27} (47 - 18x + 47x^2) H_0^3 + \frac{1}{3} (1 + x^2) H_0^4 + \left(-\frac{8}{27} (281 - 78x + 281x^2) H_0 - \frac{4}{9} (47 \right. \\
& - 18x + 47x^2) H_0^2 \left. \right) H_{-1} + \left(\frac{8}{27} (281 - 78x + 281x^2) - \frac{16}{9} (47 - 18x + 47x^2) H_{-1} \right) H_{0,-1} \\
& + \frac{8}{9} (47 - 18x + 47x^2) H_{0,0,-1} + \frac{16}{9} (47 - 18x + 47x^2) H_{0,-1,-1} + \left(-\frac{8}{27} (22 - 39x + 259x^2) \right. \\
& + \frac{2}{9} (37 - 18x + 37x^2) H_0 + \frac{8}{9} (47 - 18x + 47x^2) H_{-1} \left. \right) \zeta_2 - \frac{8}{9} (35 - 18x + 59x^2) \zeta_3 \left. \right\} \left. \right] \\
& + C_F T_F \left[\left\{ \frac{4}{27} (383 + 646x + 383x^2) x_+^2 + \eta x_+^2 \left(\frac{4}{9} H_0^3 P_{517} + \frac{2}{27} H_0 P_{604} + 4H_0 P_{518} \zeta_2 \right) \right. \right. \\
& + \left. \left(\frac{2}{9} H_0^2 P_{577} - \frac{8}{3} P_{521} \zeta_2 \right) x_+^4 \right\} + \varepsilon \left\{ \frac{2}{27} (2069 + 4378x + 2069x^2) x_+^2 + \eta \left(\frac{1}{27} (1719 \right. \right. \\
& - 2918x + 1719x^2) H_0 + \left(\frac{8}{9} H_{-1} H_0^3 P_{517} + \frac{1}{3} H_0^4 P_{517} + \frac{16}{3} H_0^2 H_{0,1} P_{517} - \frac{64}{3} H_0 H_{0,0,1} P_{517} \right. \\
& + \frac{32}{3} H_0 H_{0,0,-1} P_{517} + 32H_{0,0,0,1} P_{517} - \frac{16}{3} H_{0,0,0,-1} P_{517} - \frac{2}{27} H_0^2 P_{570} - \frac{4}{27} H_{-1} H_0 P_{604} \left. \right. \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& -\frac{8}{5}x(2+2x^2+x^3)\zeta_2^2 + \left(-\frac{16}{3}H_0^2P_{517} + \frac{4P_{604}}{27}\right)H_{0,-1} + \left(\frac{8}{3}H_{-1}H_0P_{516} + 2H_0^2P_{518}\right. \\
& -\frac{8}{3}H_{0,-1}P_{545} + \frac{4P_{601}}{27}\zeta_2 - \frac{8}{9}H_0P_{546}\zeta_3\left.)x_+^2 + \left(-\frac{2}{27}H_0^3P_{636} - \frac{2}{9}H_0P_{637}\zeta_2\right)x_+^3\right) \\
& + \left(-\frac{4}{9}H_{-1}H_0^2P_{577} + \frac{8}{9}H_0^2H_1P_{577} - \frac{16}{9}H_0H_{0,1}P_{577} + \frac{16}{9}H_{0,0,1}P_{577} + \frac{8}{9}H_{0,0,-1}P_{577}\right. \\
& \left. - \frac{8}{3}H_{-1}P_{577}\zeta_2 - \frac{8}{9}P_{595}\zeta_3\right)x_+^4 + \frac{32}{3}\log(2)P_{550}x_+^4\zeta_2 + \frac{8}{5}(-1+x)\xi\eta x_+\zeta_2^2\left.\right\}. \tag{C.2}
\end{aligned}$$

$$\begin{aligned}
\hat{F}_{A,1}^{(2),s} = & C_F T_F \left[\frac{1}{\varepsilon} \left\{ -6 \right\} + \left\{ \eta^3 \left(-32x\zeta_2(1-x+4x^2-x^3+x^4)H_0 - \frac{16}{3}x^2(1+4x-3x^2+2x^3)H_0^3 \right) \right. \right. \\
& + \eta \left(-4(3-2x+3x^2)H_0 + \left(\frac{64}{3}x^2H_0^3H_1 + 64x^2H_0^2H_{0,1} - 512x^2H_0H_{0,0,1} + 256x^2H_0H_{0,0,-1} \right. \right. \\
& + 1024x^2H_{0,0,0,1} - 768x^2H_{0,0,0,-1} + \left. \left. \left(32x^2H_0^2 + 128x^2H_0H_1 - 128x^2H_{0,1} \right) \zeta_2 - \frac{64}{5}x^2\zeta_2^2 \right. \right. \\
& - 128x^2H_0\zeta_3 \left. \left. \right) x_+^2 + \left(-32(-1+x)H_{-1}H_0 + 16(-1+x)H_0H_1 - 16(-1+x)H_{0,1} \right. \right. \\
& + 32(-1+x)H_{0,-1} \left. \left. \right) x_+ + \left(-29 - 34x - 29x^2 + 8x(1+3x)H_0^2 - 64xH_0^2H_1 \right. \right. \\
& + 192xH_0H_{0,1} - 128xH_0H_{0,-1} - 256xH_{0,0,1} + 256xH_{0,0,-1} + 64x\zeta_3 \left. \left. \right) x_+^2 - 48x\xi\eta\zeta_2 \right. \\
& \left. + 8\xi^2\zeta_2 \right\} + \varepsilon \left\{ \log(2)x_+^2\zeta_2(384x - 384xH_1 - 384xH_{-1}) + \eta \left(\log(2)x_+^2\zeta_2(768x^2H_{0,1} \right. \right. \\
& + 768x^2H_{0,-1}) - 2(29 + 3x + 15x^2 + 25x^3)x_+H_0 + 8(19 - 42x + 19x^2)H_{-1}H_0 \\
& - 32(-2+x)(-1+2x)H_0H_1 - 8(19 - 42x + 19x^2)H_{0,-1} + \left(\left(\frac{128}{3}x^2H_{-1}H_0^3 + 16x^2H_0^4 \right) H_1 \right. \\
& - \frac{64}{3}x^2H_0^3H_1^2 + \left(16P_{532} + 128x^2H_{-1}H_0^2 + \frac{64}{3}x^2H_0^3 + 128x^2H_0^2H_1 + 256x^2H_0H_{0,-1} \right) H_{0,1} \\
& - 512x^2H_0H_{0,1}^2 + \left(\frac{256}{3}x^2H_0^3 - 256x^2H_0^2H_1 \right) H_{0,-1} - 384x^2H_0H_{0,-1}^2 + \left(\left(-64x(5-14x \right. \right. \\
& + 3x^2) - 1024x^2H_{-1} \left. \left. \right) H_0 - 448x^2H_0^2 + 1280x^2H_{0,1} - 1792x^2H_{0,-1} \right) H_{0,0,1} + \left(\left(-128x(-5 \right. \right. \\
& + 3x + 3x^2) + 512x^2H_{-1} \left. \left. \right) H_0 + 128x^2H_0^2 - 512x^2H_{0,1} + 1792x^2H_{0,-1} \right) H_{0,0,-1} + 256x^2H_0^2H_{0,1,1} \\
& - 128x^2H_0^2H_{0,1,-1} - 128x^2H_0^2H_{0,-1,1} - 128x^2H_0^2H_{0,-1,-1} + \left(1536x^2H_0 - 512x^2H_1 \right. \\
& + 2048x^2H_{-1} \left. \right) H_{0,0,0,1} + \left(-1536x^2H_0 + 1280x^2H_1 - 1536x^2H_{-1} \right) H_{0,0,0,-1} + 2048x^2H_0H_{0,0,1,-1} \\
& + 2048x^2H_0H_{0,0,-1,1} - 1024x^2H_0H_{0,0,-1,-1} + 1024x^2H_0H_{0,-1,0,1} - 1664x^2H_{0,0,0,0,1} \\
& + 2560x^2H_{0,0,0,0,-1} - 3328x^2H_{0,0,0,1,1} + 256x^2H_{0,0,0,1,-1} + 256x^2H_{0,0,0,-1,1} - 2304x^2H_{0,0,0,-1,-1} \\
& - 1280x^2H_{0,0,1,0,1} + 768x^2H_{0,0,1,0,-1} + 512x^2H_{0,0,-1,0,1} - 1280x^2H_{0,0,-1,0,-1} + \left(64x^2H_{-1}H_0^2 \right. \\
& + 32x^2H_0^3 + 256x^2H_{-1}H_0H_1 - 128x^2H_0H_1^2 + \left(-128x^2H_0 + 256x^2H_1 - 256x^2H_{-1} \right) H_{0,1} \\
& + \left(384x^2H_0 - 1024x^2H_1 \right) H_{0,-1} + 384x^2H_{0,0,1} - 1536x^2H_{0,0,-1} - 256x^2H_{0,1,1} + 256x^2H_{0,1,-1} \\
& \left. + 1024x^2H_{0,-1,1} - 768x^2H_{0,-1,-1} + 160x^2\zeta_3 \right) \zeta_2 + \left(-\frac{448}{5}x^2H_0 + \frac{64}{5}x^2H_1 - \frac{128}{5}x^2H_{-1} \right) \zeta_2^2
\end{aligned}$$

$$\begin{aligned}
& + \left(-32x \left(-1 - 9x + 5x^2 \right) H_0 - 256x^2 H_{-1} H_0 - 32x^2 H_0^2 - 1344x^2 H_{0,1} + 448x^2 H_{0,-1} \right) \zeta_3 \\
& - 144x^2 \zeta_5 \left. x_+^2 \right) + \eta^3 \left(\frac{32}{3} x H_{-1} H_0^3 P_{524} - 64x H_0^2 H_{0,1} P_{530} + 64x H_0^2 H_{0,-1} P_{537} + \frac{32}{3} x H_0^3 H_1 P_{551} \right. \\
& - 64x H_{0,0,0,-1} P_{566} + 32x H_{0,0,0,1} P_{588} - \frac{8}{3} x H_0^3 P_{628} - \frac{2}{3} x^2 \left(5 + 26x - 19x^2 + 12x^3 \right) H_0^4 + \left(\right. \\
& - 64x H_{0,-1} P_{515} - 64x H_{0,1} P_{523} + 64x H_{-1} H_0 P_{524} + 64x H_0 H_1 P_{529} + 16H_0 P_{648} \left. \right) \zeta_2 - \frac{16}{5} x P_{579} \zeta_2^2 \left. \right) \\
& + \eta^2 \left(16H_0 H_{0,1} P_{527} - 16H_{-1} H_0^2 P_{540} + 32H_{0,0,-1} P_{562} + 32 \left(-1 + 2x^2 - 4x^3 + x^4 \right) H_0^2 H_1 - 32 \left(3 \right. \right. \\
& - x - x^3 + 3x^4 \left. \right) H_0 H_{0,-1} - 16x \left(21 - 26x + 5x^2 + 8x^3 \right) H_{0,0,1} + \left(4H_0^2 P_{633} + 2P_{610} \zeta_2 \right) x_+ \\
& - 16H_{-1} P_{519} \zeta_2 - 8P_{572} \zeta_3 \left. \right) + \left(128(-1+x) H_{-1}^2 H_0 - 96(-1+x) H_{-1} H_0 H_1 + 16(-1+x) H_0 H_1^2 \right. \\
& + \left(-32(-1+x) H_1 + 96(-1+x) H_{-1} \right) H_{0,1} + \left(96(-1+x) H_1 - 256(-1+x) H_{-1} \right) H_{0,-1} \\
& + 32(-1+x) H_{0,1,1} - 96(-1+x) H_{0,1,-1} - 96(-1+x) H_{0,-1,1} + 256(-1+x) H_{0,-1,-1} \\
& - 16(-1+x) H_1 \zeta_2 \left. \right) x_+ + \left(\frac{1}{2} \left(-199 - 278x - 199x^2 \right) + \left(672x H_1 - 96x H_{-1} \right) \zeta_3 + 32x H_{-1}^2 H_0^2 + \left(\right. \right. \\
& - 8(-1+x)^2 + 64x H_{-1} H_0^2 \left. \right) H_1 - 96x H_0^2 H_1^2 + \left(-256x H_{-1} H_0 + 384x H_0 H_1 + 640x H_{0,-1} \right) H_{0,1} \\
& - 64x H_{0,1}^2 + \left(128x H_{-1} H_0 + 128x H_0 H_1 \right) H_{0,-1} - 384x H_{0,-1}^2 + \left(-384x H_1 + 384x H_{-1} \right) H_{0,0,1} \\
& + \left(-384x H_1 - 384x H_{-1} \right) H_{0,0,-1} - 256x H_0 H_{0,1,1} - 384x H_0 H_{0,1,-1} - 384x H_0 H_{0,-1,1} \\
& + 640x H_0 H_{0,-1,-1} + 384x H_{0,0,1,1} - 384x H_{0,0,1,-1} - 384x H_{0,0,-1,1} + 384x H_{0,0,-1,-1} \\
& - 384x H_{0,-1,0,1} + \left. \left(384x H_{-1} H_1 + 192x H_{-1}^2 - 384x H_{-1,1} \right) \zeta_2 \right. x_+^2 - 16x(3+5x) H_0^2 x_+^3 \zeta_2 \left. \right\} . \\
\end{aligned} \tag{C.3}$$

The polynomials are defined as

$$\begin{aligned}
P_{508} &= -4774x^4 + 21306x^3 - 28365x^2 + 14928x - 71, \\
P_{509} &= -399x^4 + 486x^3 - 660x^2 - 182x + 307, \\
P_{510} &= -347x^4 + 2262x^3 - 3174x^2 + 1698x - 7, \\
P_{511} &= -254x^4 + 4452x^3 - 4779x^2 + 462x - 3337, \\
P_{512} &= x^4 - 60x^3 + 110x^2 - 60x + 1, \\
P_{513} &= x^4 - 30x^3 + 54x^2 - 30x + 1, \\
P_{514} &= x^4 - 16x^3 + 26x^2 - 16x + 1, \\
P_{515} &= x^4 - 6x^3 - 16x^2 + 14x - 9, \\
P_{516} &= x^4 + 2x^3 - 10x^2 + 2x + 1, \\
P_{517} &= x^4 + 2x^3 - 4x^2 + 2x + 1, \\
P_{518} &= x^4 + 2x^3 - 2x^2 + 2x + 1, \\
P_{519} &= x^4 + 10x^3 - 42x^2 + 2x + 5, \\
P_{520} &= x^4 + 10x^3 - 35x^2 + 10x + 1, \\
P_{521} &= x^4 + 55x^3 + 68x^2 + 55x + 1, \\
P_{522} &= 2x^4 - 21x^3 + 142x^2 - 21x + 4, \\
P_{523} &= 2x^4 - 7x^3 + 14x^2 - 7x + 2,
\end{aligned}$$

$$\begin{aligned}
P_{524} &= 2x^4 - 5x^3 - 4x^2 + 7x - 4, \\
P_{525} &= 2x^4 - x^3 - 70x^2 - x + 2, \\
P_{526} &= 2x^4 + 5x^3 - 14x^2 + 5x + 6, \\
P_{527} &= 2x^4 + 11x^3 - 18x^2 + 11x + 2, \\
P_{528} &= 3x^4 - 15x^3 + 76x^2 - 15x + 5, \\
P_{529} &= 3x^4 - 9x^3 + 14x^2 - 5x + 1, \\
P_{530} &= 3x^4 - 4x^3 - 2x^2 + 8x - 3, \\
P_{531} &= 4x^4 - 27x^3 + 44x^2 - 27x + 4, \\
P_{532} &= 4x^4 - 3x^3 - 10x^2 - 3x + 4, \\
P_{533} &= 4x^4 + 9x^3 - 22x^2 + 9x + 4, \\
P_{534} &= 5x^4 - 252x^3 + 408x^2 - 252x + 67, \\
P_{535} &= 5x^4 - 27x^3 + 34x^2 - 27x + 13, \\
P_{536} &= 5x^4 - 27x^3 + 36x^2 - 27x + 7, \\
P_{537} &= 5x^4 - 9x^3 + 4x^2 + 7x - 3, \\
P_{538} &= 5x^4 + 49x^3 - 98x^2 + 49x + 5, \\
P_{539} &= 5x^4 + 128x^3 - 254x^2 + 128x + 29, \\
P_{540} &= 6x^4 - 10x^3 - 7x^2 + 12x - 5, \\
P_{541} &= 7x^4 - 102x^3 + 162x^2 - 102x + 7, \\
P_{542} &= 7x^4 - 39x^3 + 135x^2 - 39x + 7, \\
P_{543} &= 7x^4 - 23x^3 + 96x^2 - 23x + 7, \\
P_{544} &= 7x^4 - 11x^3 - 52x^2 - 11x + 7, \\
P_{545} &= 7x^4 + 14x^3 - 34x^2 + 14x + 7, \\
P_{546} &= 7x^4 + 14x^3 - 22x^2 + 14x + 7, \\
P_{547} &= 7x^4 + 30x^3 - 142x^2 + 30x + 11, \\
P_{548} &= 7x^4 + 47x^3 - 96x^2 + 47x + 7, \\
P_{549} &= 7x^4 + 64x^3 - 126x^2 + 64x + 7, \\
P_{550} &= 7x^4 + 82x^3 + 110x^2 + 82x + 7, \\
P_{551} &= 8x^4 - 19x^3 + 14x^2 + 5x - 4, \\
P_{552} &= 8x^4 - 19x^3 + 46x^2 - 19x + 8, \\
P_{553} &= 9x^4 - 13x^3 + 40x^2 - 13x + 9, \\
P_{554} &= 10x^4 - 81x^3 + 108x^2 - 81x + 26, \\
P_{555} &= 10x^4 - 3x^3 - 6x^2 - 3x + 6, \\
P_{556} &= 11x^4 - 98x^3 + 170x^2 - 98x + 3, \\
P_{557} &= 11x^4 - 72x^3 + 118x^2 - 72x + 11, \\
P_{558} &= 11x^4 - 46x^3 + 66x^2 - 46x + 19, \\
P_{559} &= 11x^4 - 42x^3 + 136x^2 - 42x + 11, \\
P_{560} &= 11x^4 - 14x^3 - 30x^2 - 14x + 11, \\
P_{561} &= 11x^4 + 20x^3 - 52x^2 + 20x + 11, \\
P_{562} &= 12x^4 - 12x^3 - 7x^2 + 10x + 1, \\
P_{563} &= 13x^4 - 29x^3 + 64x^2 - 29x + 13,
\end{aligned}$$

$$\begin{aligned}
P_{564} &= 13x^4 - 24x^3 + 50x^2 - 24x + 13, \\
P_{565} &= 13x^4 + 126x^3 - 218x^2 + 142x - 11, \\
P_{566} &= 14x^4 - 41x^3 + 68x^2 - 29x + 8, \\
P_{567} &= 15x^4 - 53x^3 + 188x^2 - 53x + 15, \\
P_{568} &= 15x^4 - 17x^3 - 8x^2 - 17x + 15, \\
P_{569} &= 15x^4 + 10x^3 - 38x^2 + 10x + 15, \\
P_{570} &= 16x^4 - 136x^3 - 7x^2 + 344x - 281, \\
P_{571} &= 17x^4 - 46x^3 + 190x^2 - 46x + 17, \\
P_{572} &= 17x^4 - 26x^3 + 24x^2 + 10x - 1, \\
P_{573} &= 17x^4 - 23x^3 + 56x^2 - 23x + 17, \\
P_{574} &= 18x^4 - 39x^3 + 82x^2 - 39x + 18, \\
P_{575} &= 18x^4 + 125x^3 - 256x^2 + 125x + 18, \\
P_{576} &= 19x^4 - 69x^3 + 196x^2 - 69x + 17, \\
P_{577} &= 19x^4 - 14x^3 + 14x^2 - 14x + 19, \\
P_{578} &= 19x^4 + 109x^3 - 230x^2 + 109x + 19, \\
P_{579} &= 20x^4 - 53x^3 + 36x^2 + 15x - 14, \\
P_{580} &= 20x^4 + 23x^3 - 58x^2 + 23x + 20, \\
P_{581} &= 24x^4 - 27x^3 - 42x^2 - 27x + 22, \\
P_{582} &= 25x^4 - 108x^3 + 178x^2 - 108x + 1, \\
P_{583} &= 25x^4 - 53x^3 + 148x^2 - 53x + 25, \\
P_{584} &= 26x^4 - 81x^3 + 102x^2 - 81x + 28, \\
P_{585} &= 27x^4 - 102x^3 + 320x^2 - 102x + 27, \\
P_{586} &= 27x^4 - 78x^3 + 106x^2 - 62x + 3, \\
P_{587} &= 37x^4 - 108x^3 + 66x^2 - 108x + 45, \\
P_{588} &= 38x^4 - 127x^3 + 158x^2 - 55x + 2, \\
P_{589} &= 41x^4 + 216x^3 - 322x^2 + 216x - 115, \\
P_{590} &= 45x^4 - 8x^3 - 186x^2 - 8x + 45, \\
P_{591} &= 53x^4 - 141x^3 + 228x^2 - 141x - 17, \\
P_{592} &= 56x^4 - 103x^3 + 162x^2 - 103x - 22, \\
P_{593} &= 57x^4 - 8x^3 - 82x^2 - 8x + 101, \\
P_{594} &= 58x^4 - 135x^3 + 214x^2 - 135x - 20, \\
P_{595} &= 69x^4 + 564x^3 + 790x^2 + 564x + 69, \\
P_{596} &= 77x^4 - 132x^3 + 230x^2 - 68x - 51, \\
P_{597} &= 79x^4 - 182x^3 + 362x^2 - 182x + 79, \\
P_{598} &= 83x^4 - 70x^3 - 180x^2 + 310x - 47, \\
P_{599} &= 83x^4 + 153x^3 - 360x^2 + 153x + 13, \\
P_{600} &= 93x^4 + 28x^3 - 330x^2 + 92x - 35, \\
P_{601} &= 139x^4 + 3248x^3 - 7x^2 - 3040x - 404, \\
P_{602} &= 187x^4 - 1314x^3 + 810x^2 + 234x - 349, \\
P_{603} &= 229x^4 - 266x^3 + 244x^2 - 70x + 55,
\end{aligned}$$

$$\begin{aligned}
P_{604} &= 265x^4 - 208x^3 + 14x^2 - 208x + 265, \\
P_{605} &= 649x^4 - 902x^3 + 1050x^2 - 246x + 153, \\
P_{606} &= 759x^4 - 550x^3 - 828x^2 + 118x + 53, \\
P_{607} &= 1043x^4 - 618x^3 - 1024x^2 + 762x + 173, \\
P_{608} &= 1045x^4 + 5364x^3 - 8166x^2 + 2628x + 1577, \\
P_{609} &= 1343x^4 + 2430x^3 - 4194x^2 + 1530x + 907, \\
P_{610} &= x^5 - 15x^4 - 70x^3 + 18x^2 - 43x + 13, \\
P_{611} &= x^5 - 5x^4 + 18x^3 - 26x^2 + x - 5, \\
P_{612} &= x^5 - 5x^4 + 34x^3 - 30x^2 + 7x + 1, \\
P_{613} &= x^5 - 2x^4 + 10x^3 - 12x^2 + x - 2, \\
P_{614} &= x^5 - 2x^4 + 18x^3 - 14x^2 + 4x + 1, \\
P_{615} &= x^5 + 4x^4 - 15x^3 + 26x^2 - 16x + 4, \\
P_{616} &= x^5 + 4x^4 - 14x^3 + 18x^2 - 2x + 1, \\
P_{617} &= 2x^5 + 5x^4 - 34x^3 + 52x^2 - 36x + 5, \\
P_{618} &= 2x^5 + 5x^4 - 12x^3 + 20x^2 - x + 2, \\
P_{619} &= 3x^5 - 9x^4 + 38x^3 - 50x^2 + 3x - 9, \\
P_{620} &= 3x^5 + 9x^4 - 26x^3 + 38x^2 - 3x + 3, \\
P_{621} &= 4x^5 + 19x^4 - 72x^3 + 96x^2 - 76x + 19, \\
P_{622} &= 5x^5 - x^4 + 26x^3 - 18x^2 + 5x - 1, \\
P_{623} &= 6x^5 - 51x^4 + 56x^3 - 43x^2 + 6x + 8, \\
P_{624} &= 6x^5 - 47x^4 + 80x^3 - 51x^2 + 6x - 4, \\
P_{625} &= 7x^5 - 24x^4 + 56x^3 - 32x^2 + 7x - 8, \\
P_{626} &= 7x^5 - 22x^4 + 88x^3 - 46x^2 + 7x - 24, \\
P_{627} &= 7x^5 + x^4 + 30x^3 - 14x^2 + 7x + 1, \\
P_{628} &= 10x^5 - 5x^4 - 57x^3 + 66x^2 - 15x - 3, \\
P_{629} &= 11x^5 - 83x^4 + 132x^3 - 85x^2 + 11x - 2, \\
P_{630} &= 12x^5 - x^4 + 10x^3 - 28x^2 - 2x - 1, \\
P_{631} &= 13x^5 + x^4 + 58x^3 - 30x^2 + 13x + 1, \\
P_{632} &= 18x^5 - 37x^4 - 72x^3 + 118x^2 - 78x + 3, \\
P_{633} &= 24x^5 - 34x^4 - 43x^3 + 25x^2 - x - 3, \\
P_{634} &= 24x^5 - 3x^4 + 22x^3 - 52x^2 - 2x - 3, \\
P_{635} &= 72x^5 + 115x^4 - 762x^3 + 644x^2 - 90x - 3, \\
P_{636} &= 105x^5 - 195x^4 + 156x^3 - 68x^2 + 69x - 47, \\
P_{637} &= 125x^5 - 279x^4 + 182x^3 - 42x^2 - 15x - 27, \\
P_{638} &= -1367x^6 + 2934x^5 - 4861x^4 + 8604x^3 - 9253x^2 + 5094x - 575, \\
P_{639} &= -543x^6 + 702x^5 - 157x^4 - 660x^3 + 1035x^2 - 450x + 137, \\
P_{640} &= -89x^6 + 402x^5 - 499x^4 + 354x^3 - 37x^2 + 24x - 11, \\
P_{641} &= -43x^6 + 6x^5 - 59x^4 + 104x^3 - x^2 + 6x + 15, \\
P_{642} &= -29x^6 + 240x^5 - 287x^4 + 170x^3 + 19x^2 + 26x - 11, \\
P_{643} &= -27x^6 + 2x^5 - 27x^4 + 56x^3 + 7x^2 + 2x + 7,
\end{aligned}$$

$$\begin{aligned}
P_{644} &= -13x^6 - 59x^5 + 14x^4 + 33x^3 - 30x^2 - 20x + 11, \\
P_{645} &= -6x^6 - x^5 - 8x^4 - 28x^3 + 16x^2 - x + 18, \\
P_{646} &= x^6 - 29x^5 + 117x^4 - 204x^3 + 117x^2 - 29x + 1, \\
P_{647} &= x^6 - 27x^5 + 174x^4 - 276x^3 + 166x^2 - 27x - 7, \\
P_{648} &= x^6 - 13x^5 + 52x^4 - 42x^3 + 8x^2 - 3x + 1, \\
P_{649} &= x^6 - 8x^5 + 33x^4 - 52x^3 + 29x^2 - 8x - 3, \\
P_{650} &= x^6 - 4x^5 + 17x^4 - 14x^3 + 15x^2 - 4x - 1, \\
P_{651} &= x^6 + 2x^5 - 8x^4 + 11x^3 - 8x^2 + 2x + 1, \\
P_{652} &= x^6 + 2x^5 - 7x^4 + 2x^3 - 6x^2 + 2x + 2, \\
P_{653} &= x^6 + 2x^5 - 7x^4 + 7x^3 - 9x^2 + 2x - 1, \\
P_{654} &= x^6 + 4x^5 - 31x^4 + 56x^3 - 37x^2 + 4x - 5, \\
P_{655} &= x^6 + 5x^5 - 35x^4 + 36x^3 - 27x^2 + 5x + 9, \\
P_{656} &= x^6 + 5x^5 - 19x^4 + 28x^3 - 19x^2 + 5x + 1, \\
P_{657} &= x^6 + 9x^5 - 76x^4 + 148x^3 - 96x^2 + 9x - 19, \\
P_{658} &= x^6 + 10x^5 - 71x^4 + 104x^3 - 69x^2 + 10x + 3, \\
P_{659} &= x^6 + 36x^5 - 208x^4 + 664x^3 - 330x^2 + 36x - 121, \\
P_{660} &= 2x^6 - 19x^5 + 78x^4 - 138x^3 + 78x^2 - 19x + 2, \\
P_{661} &= 2x^6 - 11x^5 + 44x^4 - 78x^3 + 44x^2 - 11x + 2, \\
P_{662} &= 2x^6 - 2x^5 + 13x^4 - 26x^3 + 12x^2 - 2x + 1, \\
P_{663} &= 2x^6 - 2x^5 + 17x^4 - 32x^3 + 17x^2 - 2x + 2, \\
P_{664} &= 2x^6 - x^5 + 6x^4 - 16x^3 + 5x^2 - x + 1, \\
P_{665} &= 2x^6 + 2x^5 - 7x^4 + 40x^3 - 19x^2 + 2x - 10, \\
P_{666} &= 2x^6 + 3x^5 - 16x^4 + 12x^3 - 14x^2 + 3x + 4, \\
P_{667} &= 2x^6 + 5x^5 - 33x^4 + 48x^3 - 33x^2 + 5x + 2, \\
P_{668} &= 2x^6 + 18x^5 - 147x^4 + 168x^3 - 127x^2 + 18x + 22, \\
P_{669} &= 2x^6 + 35x^5 - 235x^4 + 356x^3 - 225x^2 + 35x + 12, \\
P_{670} &= 3x^6 - 22x^5 + 169x^4 - 264x^3 + 167x^2 - 22x + 1, \\
P_{671} &= 3x^6 - 8x^5 + 35x^4 - 58x^3 + 35x^2 - 8x + 3, \\
P_{672} &= 3x^6 - 7x^5 + 31x^4 - 60x^3 + 32x^2 - 7x + 4, \\
P_{673} &= 3x^6 - 7x^5 + 54x^4 - 84x^3 + 50x^2 - 7x - 1, \\
P_{674} &= 3x^6 - 6x^5 + 50x^4 - 80x^3 + 48x^2 - 6x + 1, \\
P_{675} &= 3x^6 - 5x^5 + 41x^4 - 20x^3 + 17x^2 - 5x - 21, \\
P_{676} &= 3x^6 - 4x^5 + 25x^4 - 52x^3 + 25x^2 - 4x + 3, \\
P_{677} &= 3x^6 - x^5 + 6x^4 - 20x^3 + 6x^2 - x + 3, \\
P_{678} &= 3x^6 + 4x^5 - 13x^4 + 4x^3 - 13x^2 + 4x + 3, \\
P_{679} &= 3x^6 + 7x^5 - 48x^4 + 84x^3 - 56x^2 + 7x - 5, \\
P_{680} &= 3x^6 + 35x^5 - 207x^4 - 118x^3 + 355x^2 - x - 19, \\
P_{681} &= 3x^6 + 68x^5 + 139x^4 - 76x^3 - 81x^2 + 152x - 37, \\
P_{682} &= 3x^6 + 70x^5 - 519x^4 + 776x^3 - 513x^2 + 70x + 9, \\
P_{683} &= 4x^6 - 26x^5 + 173x^4 - 296x^3 + 173x^2 - 26x + 4,
\end{aligned}$$

$$\begin{aligned}
P_{684} &= 4x^6 - 7x^5 + 28x^4 - 72x^3 + 36x^2 - 7x + 12, \\
P_{685} &= 4x^6 + 3x^5 - 12x^4 - 18x^3 - 8x^2 + 3x + 8, \\
P_{686} &= 4x^6 + 37x^5 - 261x^4 + 372x^3 - 247x^2 + 37x + 18, \\
P_{687} &= 5x^6 - 28x^5 + 115x^4 - 190x^3 + 115x^2 - 28x + 5, \\
P_{688} &= 5x^6 - 15x^5 + 82x^4 - 180x^3 + 90x^2 - 15x + 13, \\
P_{689} &= 5x^6 - 12x^5 + 47x^4 - 106x^3 + 51x^2 - 12x + 9, \\
P_{690} &= 5x^6 - 10x^5 - 60x^4 + 59x^3 - 16x^2 - 7x + 5, \\
P_{691} &= 5x^6 - 8x^5 + 49x^4 - 104x^3 + 51x^2 - 8x + 7, \\
P_{692} &= 5x^6 + 3x^5 - 15x^4 + 12x^3 - 17x^2 + 3x + 3, \\
P_{693} &= 5x^6 + 4x^5 - 11x^4 + 4x^3 - 15x^2 + 4x + 1, \\
P_{694} &= 5x^6 + 10x^5 - 58x^4 + 32x^3 - 40x^2 + 10x + 23, \\
P_{695} &= 6x^6 - 7x^5 + 30x^4 - 56x^3 + 26x^2 - 7x + 2, \\
P_{696} &= 7x^6 - 38x^5 + 287x^4 - 456x^3 + 285x^2 - 38x + 5, \\
P_{697} &= 7x^6 - 21x^5 + 95x^4 - 122x^3 + 63x^2 - 21x - 25, \\
P_{698} &= 7x^6 - 8x^5 + 71x^4 - 112x^3 + 65x^2 - 8x + 1, \\
P_{699} &= 7x^6 - 2x^5 + 16x^4 - 40x^3 + 10x^2 - 2x + 1, \\
P_{700} &= 7x^6 + 6x^5 - 33x^4 + 24x^3 - 31x^2 + 6x + 9, \\
P_{701} &= 7x^6 + 6x^5 - 29x^4 + 24x^3 - 31x^2 + 6x + 5, \\
P_{702} &= 7x^6 + 76x^5 + 231x^4 - 952x^3 + 961x^2 - 292x - 15, \\
P_{703} &= 8x^6 - 59x^5 + 410x^4 - 644x^3 + 392x^2 - 59x - 10, \\
P_{704} &= 10x^6 - 55x^5 + 230x^4 - 422x^3 + 230x^2 - 55x + 10, \\
P_{705} &= 10x^6 - 33x^5 + 143x^4 - 127x^3 + 106x^2 - 33x - 27, \\
P_{706} &= 10x^6 - 27x^5 + 116x^4 - 210x^3 + 120x^2 - 27x + 14, \\
P_{707} &= 10x^6 + 9x^5 - 52x^4 + 52x^3 - 54x^2 + 9x + 8, \\
P_{708} &= 10x^6 + 9x^5 - 46x^4 + 52x^3 - 48x^2 + 9x + 8, \\
P_{709} &= 11x^6 - 96x^5 - 53x^4 + 946x^3 - 911x^2 + 166x - 47, \\
P_{710} &= 11x^6 - 51x^5 + 349x^4 - 564x^3 + 333x^2 - 51x - 5, \\
P_{711} &= 11x^6 - 36x^5 + 165x^4 - 246x^3 + 133x^2 - 36x - 21, \\
P_{712} &= 11x^6 - 23x^5 + 163x^4 - 292x^3 + 163x^2 - 23x + 11, \\
P_{713} &= 11x^6 - 18x^5 + 76x^4 - 159x^3 + 75x^2 - 18x + 10, \\
P_{714} &= 11x^6 + 32x^5 + 107x^4 - 728x^3 + 597x^2 - 72x + 37, \\
P_{715} &= 11x^6 + 42x^5 - 285x^4 + 424x^3 - 287x^2 + 42x + 9, \\
P_{716} &= 12x^6 + x^5 + 12x^4 + 28x^3 - 22x^2 + x - 22, \\
P_{717} &= 12x^6 + 29x^5 - 242x^4 + 300x^3 - 236x^2 + 29x + 18, \\
P_{718} &= 13x^6 - 57x^5 + 233x^4 - 408x^3 + 237x^2 - 57x + 17, \\
P_{719} &= 13x^6 - 17x^5 + 117x^4 - 228x^3 + 113x^2 - 17x + 9, \\
P_{720} &= 13x^6 - 3x^5 + 17x^4 - 56x^3 + 15x^2 - 3x + 11, \\
P_{721} &= 13x^6 + 2x^5 - 37x^4 + 236x^3 - 337x^2 + 282x - 63, \\
P_{722} &= 13x^6 + 7x^5 - 36x^4 + 28x^3 - 36x^2 + 7x + 13, \\
P_{723} &= 13x^6 + 36x^5 - 141x^4 + 246x^3 - 157x^2 + 36x - 3,
\end{aligned}$$

$$\begin{aligned}
P_{724} &= 13x^6 + 39x^5 - 133x^4 + 196x^3 - 165x^2 + 39x - 19, \\
P_{725} &= 13x^6 + 118x^5 - 869x^4 + 1208x^3 - 827x^2 + 118x + 55, \\
P_{726} &= 13x^6 + 120x^5 - 859x^4 + 1264x^3 - 837x^2 + 120x + 35, \\
P_{727} &= 14x^6 - 9x^5 + 91x^4 - 148x^3 + 81x^2 - 9x + 4, \\
P_{728} &= 14x^6 - 8x^5 + 57x^4 - 144x^3 + 57x^2 - 8x + 14, \\
P_{729} &= 14x^6 + 2x^5 - 3x^4 - 16x^3 - 9x^2 + 2x + 8, \\
P_{730} &= 15x^6 - 28x^5 + 127x^4 - 240x^3 + 125x^2 - 28x + 13, \\
P_{731} &= 15x^6 + 20x^5 - 122x^4 + 152x^3 - 124x^2 + 20x + 13, \\
P_{732} &= 15x^6 + 426x^5 - 543x^4 + 320x^3 - 71x^2 + 230x - 41, \\
P_{733} &= 16x^6 - 43x^5 + 186x^4 - 346x^3 + 190x^2 - 43x + 20, \\
P_{734} &= 16x^6 - 27x^5 + 114x^4 - 220x^3 + 110x^2 - 27x + 12, \\
P_{735} &= 17x^6 - 166x^5 + 489x^4 - 716x^3 + 475x^2 - 158x + 11, \\
P_{736} &= 17x^6 - 18x^5 + 121x^4 - 220x^3 + 105x^2 - 18x + 1, \\
P_{737} &= 17x^6 + 21x^5 - 71x^4 + 122x^3 - 87x^2 + 21x + 1, \\
P_{738} &= 17x^6 + 28x^5 - 183x^4 + 224x^3 - 177x^2 + 28x + 23, \\
P_{739} &= 17x^6 + 34x^5 - 230x^4 + 304x^3 - 228x^2 + 34x + 19, \\
P_{740} &= 18x^6 - 16x^5 + 73x^4 - 158x^3 + 67x^2 - 16x + 12, \\
P_{741} &= 18x^6 - 9x^5 + 94x^4 - 172x^3 + 88x^2 - 9x + 12, \\
P_{742} &= 18x^6 + x^5 - 68x^3 + 2x^2 + x + 20, \\
P_{743} &= 19x^6 - 73x^5 + 303x^4 - 544x^3 + 307x^2 - 73x + 23, \\
P_{744} &= 19x^6 - 8x^5 + 43x^4 - 128x^3 + 37x^2 - 8x + 13, \\
P_{745} &= 19x^6 + 24x^5 - 73x^4 + 100x^3 - 105x^2 + 24x - 13, \\
P_{746} &= 19x^6 + 42x^5 - 147x^4 + 196x^3 - 137x^2 + 42x + 29, \\
P_{747} &= 19x^6 + 62x^5 - 433x^4 + 536x^3 - 395x^2 + 62x + 57, \\
P_{748} &= 20x^6 - 303x^5 + 1214x^4 - 1692x^3 + 1238x^2 - 303x + 44, \\
P_{749} &= 20x^6 - 8x^5 + 51x^4 - 101x^3 + 30x^2 - 8x - 1, \\
P_{750} &= 21x^6 - 53x^5 + 227x^4 - 438x^3 + 231x^2 - 53x + 25, \\
P_{751} &= 21x^6 - 11x^5 + 59x^4 - 134x^3 + 55x^2 - 11x + 17, \\
P_{752} &= 21x^6 + 8x^5 - 25x^4 + 8x^3 - 35x^2 + 8x + 11, \\
P_{753} &= 22x^6 - 171x^5 + 708x^4 - 892x^3 + 634x^2 - 171x - 52, \\
P_{754} &= 22x^6 - 61x^5 + 162x^4 - 118x^3 - 12x^2 + 95x - 40, \\
P_{755} &= 23x^6 + 37x^5 - 429x^4 + 596x^3 - 277x^2 - 17x + 19, \\
P_{756} &= 23x^6 + 68x^5 - 295x^4 + 448x^3 - 317x^2 + 68x + 1, \\
P_{757} &= 24x^6 - 67x^5 + 282x^4 - 564x^3 + 280x^2 - 67x + 22, \\
P_{758} &= 25x^6 - 174x^5 + 489x^4 - 716x^3 + 467x^2 - 150x + 11, \\
P_{759} &= 25x^6 - 6x^5 + 21x^4 - 112x^3 + 19x^2 - 6x + 23, \\
P_{760} &= 25x^6 - 6x^5 + 33x^4 - 112x^3 + 31x^2 - 6x + 23, \\
P_{761} &= 25x^6 + 5x^5 - 5x^4 + 22x^3 - 43x^2 + 5x - 13, \\
P_{762} &= 27x^6 - 32x^5 + 129x^4 - 344x^3 + 151x^2 - 32x + 49, \\
P_{763} &= 28x^6 + 27x^5 - 416x^4 + 596x^3 - 290x^2 - 7x + 14,
\end{aligned}$$

$$\begin{aligned}
P_{764} &= 29x^6 - 348x^5 + 511x^4 - 354x^3 + 85x^2 - 78x + 11, \\
P_{765} &= 29x^6 + 30x^5 - 139x^4 + 120x^3 - 137x^2 + 30x + 31, \\
P_{766} &= 29x^6 + 435x^5 - 1475x^4 + 1674x^3 - 731x^2 - 9x + 65, \\
P_{767} &= 31x^6 + 8x^5 - 29x^4 - 44x^3 - 35x^2 + 8x + 25, \\
P_{768} &= 31x^6 + 8x^5 - 13x^4 - 64x^3 - 11x^2 + 8x + 33, \\
P_{769} &= 31x^6 + 20x^5 - 69x^4 + 36x^3 - 87x^2 + 20x + 13, \\
P_{770} &= 31x^6 + 30x^5 - 141x^4 + 120x^3 - 139x^2 + 30x + 33, \\
P_{771} &= 32x^6 - 151x^5 + 636x^4 - 1240x^3 + 650x^2 - 151x + 46, \\
P_{772} &= 34x^6 - 91x^5 + 382x^4 - 746x^3 + 384x^2 - 91x + 36, \\
P_{773} &= 34x^6 + x^5 + 8x^4 - 68x^3 - 14x^2 + x + 12, \\
P_{774} &= 35x^6 - 90x^5 + 388x^4 - 718x^3 + 378x^2 - 90x + 25, \\
P_{775} &= 37x^6 - 118x^5 + 343x^4 - 716x^3 + 639x^2 - 206x - 27, \\
P_{776} &= 37x^6 - 59x^5 - 233x^4 + 714x^3 - 541x^2 + 73x - 31, \\
P_{777} &= 37x^6 - 57x^5 + 371x^4 - 668x^3 + 339x^2 - 57x + 5, \\
P_{778} &= 38x^6 - 107x^5 + 466x^4 - 932x^3 + 456x^2 - 107x + 28, \\
P_{779} &= 39x^6 - 112x^5 + 413x^4 - 900x^3 + 769x^2 - 228x - 21, \\
P_{780} &= 40x^6 + 69x^5 - 463x^4 + 628x^3 - 465x^2 + 69x + 38, \\
P_{781} &= 42x^6 - 41x^5 + 182x^4 - 380x^3 + 166x^2 - 41x + 26, \\
P_{782} &= 42x^6 + 2x^5 + 3x^4 - 40x^3 - 41x^2 + 2x - 2, \\
P_{783} &= 43x^6 + 20x^5 - 27x^4 + 16x^3 - 73x^2 + 20x - 3, \\
P_{784} &= 44x^6 + 67x^5 + 304x^4 - 1272x^3 + 1264x^2 - 371x - 20, \\
P_{785} &= 46x^6 - 39x^5 + 271x^4 - 420x^3 + 197x^2 - 39x - 28, \\
P_{786} &= 47x^6 - 154x^5 + 53x^4 + 580x^3 - 701x^2 + 110x - 23, \\
P_{787} &= 50x^6 - 55x^5 + 170x^4 - 732x^3 + 814x^2 - 333x - 2, \\
P_{788} &= 50x^6 + 37x^5 - 310x^4 + 364x^3 - 348x^2 + 37x + 12, \\
P_{789} &= 51x^6 + 32x^5 - 117x^4 + 32x^3 - 107x^2 + 32x + 61, \\
P_{790} &= 53x^6 - 33x^5 - 279x^4 + 596x^3 - 427x^2 + 53x - 11, \\
P_{791} &= 55x^6 + 105x^5 - 1711x^4 + 2400x^3 - 889x^2 + 135x - 71, \\
P_{792} &= 56x^6 - 141x^5 + 930x^4 - 1916x^3 + 1004x^2 - 141x + 130, \\
P_{793} &= 59x^6 - 446x^5 + 203x^4 + 872x^3 - 1215x^2 + 398x + 9, \\
P_{794} &= 63x^6 - 205x^5 + 273x^4 + 118x^3 - 537x^2 + 347x - 11, \\
P_{795} &= 65x^6 + 154x^5 - 1187x^4 + 1624x^3 - 1209x^2 + 154x + 43, \\
P_{796} &= 67x^6 - 105x^5 + 485x^4 - 938x^3 + 465x^2 - 105x + 47, \\
P_{797} &= 71x^6 + 132x^5 - 461x^4 + 508x^3 - 535x^2 + 132x - 3, \\
P_{798} &= 77x^6 - 144x^5 + 1043x^4 - 1760x^3 + 969x^2 - 144x + 3, \\
P_{799} &= 81x^6 - 571x^5 - 129x^4 + 1738x^3 - 1887x^2 + 461x + 43, \\
P_{800} &= 91x^6 - 48x^5 - 99x^4 - 638x^3 + 595x^2 - 50x + 29, \\
P_{801} &= 96x^6 - 163x^5 + 758x^4 - 1528x^3 + 698x^2 - 163x + 36, \\
P_{802} &= 98x^6 - 221x^5 + 988x^4 - 1976x^3 + 944x^2 - 221x + 54, \\
P_{803} &= 111x^6 - 139x^5 + 111x^4 - 1216x^3 + 985x^2 - 169x + 53,
\end{aligned}$$

$$\begin{aligned}
P_{804} &= 117x^6 + 576x^5 - 2249x^4 + 3792x^3 - 2523x^2 + 576x - 157, \\
P_{805} &= 119x^6 - 132x^5 + 603x^4 - 1194x^3 + 535x^2 - 132x + 51, \\
P_{806} &= 133x^6 - 760x^5 + 1463x^4 - 1204x^3 + 231x^2 + 100x - 11, \\
P_{807} &= 139x^6 + 41x^5 - 335x^4 + 308x^3 - 467x^2 + 41x + 7, \\
P_{808} &= 141x^6 + 79x^5 - 621x^4 + 724x^3 - 749x^2 + 79x + 13, \\
P_{809} &= 145x^6 - 634x^5 + 3441x^4 - 7530x^3 + 6725x^2 - 2952x + 637, \\
P_{810} &= 146x^6 + 877x^5 - 3462x^4 + 3876x^3 - 1458x^2 - 263x + 200, \\
P_{811} &= 181x^6 - 72x^5 + 599x^4 - 1328x^3 + 477x^2 - 72x + 59, \\
P_{812} &= 337x^6 - 154x^5 - 165x^4 - 68x^3 + 283x^2 - 186x + 17, \\
P_{813} &= 443x^6 + 2898x^5 - 5681x^4 + 4104x^3 - 2225x^2 + 1314x - 565, \\
P_{814} &= 649x^6 - 210x^5 + 2025x^4 - 3960x^3 + 1323x^2 - 210x - 53.
\end{aligned} \tag{C.4}$$

$$\begin{aligned}
F_{A,2}^{(1)} = C_F \left[\right. & \left\{ -8x(1+x)^2\eta^2 - 4x(1+x)^2(3-2x+3x^2)\eta^3 H_0 \right\} + \varepsilon \left\{ -16x(1+x)^2\eta^2 \right. \\
& + \eta^3 \left(-8x(1+x)^2(3-2x+3x^2)H_0 + 8x(1+x)^2(3-2x+3x^2)H_{-1}H_0 - 2x(1+x)^2(3-2x \right. \\
& + 3x^2)H_0^2 - 8x(1+x)^2(3-2x+3x^2)H_{0,-1} + 4x(1+x)^2(3-2x+3x^2)\zeta_2 \left. \right\} + \varepsilon^2 \left\{ -32x(1+x)^2\eta^2 \right. \\
& + \eta^3 \left(\left(-16x(1+x)^2(3-2x+3x^2) + 16x(1+x)^2(3-2x+3x^2)H_{-1} - 8x(1+x)^2(3-2x \right. \right. \\
& + 3x^2)H_{-1}^2 + 2x(1+x)^2(3-2x+3x^2)\zeta_2 \left. \right) H_0 + \left(-4x(1+x)^2(3-2x+3x^2) + 4x(1+x)^2(3-2x \right. \\
& + 3x^2)H_{-1} \left. \right) H_0^2 - \frac{2}{3}x(1+x)^2(3-2x+3x^2)H_0^3 + \left(-16x(1+x)^2(3-2x+3x^2) + 16x(1+x)^2(3 \right. \\
& - 2x+3x^2)H_{-1} \left. \right) H_{0,-1} - 8x(1+x)^2(3-2x+3x^2)H_{0,0,-1} - 16x(1+x)^2(3-2x+3x^2)H_{0,-1,-1} \\
& + 4x(1+x)^2(5-4x+7x^2)\zeta_2 - 8x(1+x)^2(3-2x+3x^2)H_{-1}\zeta_2 + 8x(1+x)^2(3-2x+3x^2)\zeta_3 \left. \right\} \\
& + \varepsilon^3 \left\{ -64x(1+x)^2\eta^2 + \eta^3 \left(\left(-32x(1+x)^2(3-2x+3x^2) + (32x(1+x)^2(3-2x+3x^2) \right. \right. \right. \\
& - 4x(1+x)^2(3-2x+3x^2)\zeta_2 \left. \right) H_{-1} - 16x(1+x)^2(3-2x+3x^2)H_{-1}^2 + \frac{16}{3}x(1+x)^2(3-2x \\
& + 3x^2)H_{-1}^3 + 4x(1+x)^2(3-2x+3x^2)\zeta_2 + \frac{28}{3}x(1+x)^2(3-2x+3x^2)\zeta_3 \left. \right) H_0 + \left(-8x(1+x)^2(3 \right. \\
& - 2x+3x^2) + 8x(1+x)^2(3-2x+3x^2)H_{-1} - 4x(1+x)^2(3-2x+3x^2)H_{-1}^2 + x(1+x)^2(3 \\
& - 2x+3x^2)\zeta_2 \left. \right) H_0^2 + \left(-\frac{4}{3}x(1+x)^2(3-2x+3x^2) + \frac{4}{3}x(1+x)^2(3-2x+3x^2)H_{-1} \right) H_0^3 \\
& - \frac{1}{6}x(1+x)^2(3-2x+3x^2)H_0^4 + \left(-16x(1+x)^2(3-2x+3x^2)\zeta_2 - 16x(1+x)^2(3-2x \right. \\
& + 3x^2)\zeta_3 \left. \right) H_{-1} + \left(-32x(1+x)^2(3-2x+3x^2) + 32x(1+x)^2(3-2x+3x^2)H_{-1} - 16x(1+x)^2(3 \right. \\
& - 2x+3x^2)H_{-1}^2 - 4x(1+x)^2(3-2x+3x^2)\zeta_2 \left. \right) H_{0,-1} + \left(-16x(1+x)^2(3-2x+3x^2) \right. \\
& + 16x(1+x)^2(3-2x+3x^2)H_{-1} \left. \right) H_{0,0,-1} + \left(-32x(1+x)^2(3-2x+3x^2) + 32x(1+x)^2(3-2x \right. \\
& + 3x^2)H_{-1} \left. \right) H_{0,-1,-1} - 8x(1+x)^2(3-2x+3x^2)H_{0,0,0,-1} - 16x(1+x)^2(3-2x+3x^2)H_{0,0,-1,-1} \\
& - 32x(1+x)^2(3-2x+3x^2)H_{0,-1,-1,-1} + 8x(1+x)^2(5-4x+7x^2)\zeta_2 + 8x(1+x)^2(3 \\
& - 2x+3x^2)H_{-1}^2\zeta_2 + \frac{28}{5}x(1+x)^2(3-2x+3x^2)\zeta_2^2 + \frac{8}{3}x(1+x)^2(19-12x+17x^2)\zeta_3 \left. \right\} \left. \right].
\end{aligned} \tag{C.5}$$

$$F_{A,2}^{(2),\text{ns}} = C_F^2 \left[\frac{1}{\varepsilon} \left\{ 16x(1+x)^2\eta^2 + 8x(1+x)^2(5-2x+5x^2)\eta^3 H_0 + 8x(1+x)^2(3-2x+3x^2)\xi\eta^3 H_0^2 \right\} \right]$$

$$\begin{aligned}
& + \left\{ 68x(1+x)^2\eta^2 + \eta^3 \left(2x(1+x)^2(61-22x+61x^2)H_0 + 8x(1+x)^2(25-86x+25x^2)H_{-1}H_0 \right. \right. \\
& + \left(128x^2(1+x)^2H_0 - \frac{1088}{3}x^3H_0^3 \right)H_1 - 128x^2(1+x)^2H_{0,1} - 8x(1+x)^2(25-86x \\
& + 25x^2)H_{0,-1} + \left(-2176x^3H_0H_1 + 2176x^3H_{0,1} \right)\zeta_2 \left. \right\} + \xi^2 \left(-192(-1+x)x^2(1+x)^3\eta^4\zeta_2H_{0,-1} \right. \\
& - 144(-1+x)x(1+x)^3\eta^3H_{-1}\zeta_2 \left. \right) + \xi \left(\eta^3 \left(\left(128(-1+x)x(1+x)^3H_0 - \frac{64}{3}(-1+x)x^2(1+x)H_0^3 \right)H_1 \right. \right. \\
& - 128(-1+x)x(1+x)^3H_{0,1} + \left(-128(-1+x)x^2(1+x)H_0H_1 - 576x^2(1+x)^2H_{-1} \right. \\
& + 128(-1+x)x^2(1+x)H_{0,1} \left. \right)\zeta_2 \left. \right) - 960x^3(1+x)^2\eta^4H_{0,-1}\zeta_2 \left. \right) + \eta^5 \left(-64x^2(1+x)^2H_0H_{0,0,-1}P_{817} \right. \\
& - 32x^2(1+x)^2H_0^2H_{0,1}P_{818} + 32x^2(1+x)^2H_0^2H_{0,-1}P_{820} + \frac{4}{3}x^2(1+x)^2H_0^4P_{835} \\
& + 64x^2(1+x)^2H_{0,0,0,1}P_{843} - \frac{8}{3}x(1+x)^2H_0^3P_{1018} + \frac{32}{5}x^2(1+x)^2P_{816}\zeta_2^2 - 1024x^3(1+x)^2 \left(5 \right. \\
& - 7x + 5x^2 \left. \right)H_0H_{0,0,1} - 384x^3(1+x)^2(28-41x+28x^2)H_{0,0,0,-1} + \left(16x(1+x)^2H_0P_{1037} \right. \\
& - 96x^3(1+x)^2(2-9x+2x^2)H_0^2 \left. \right)\zeta_2 - 64x^2(1+x)^2H_0P_{821}\zeta_3 \left. \right) + \eta^4 \left(16x(1+x)^2H_0^2H_1P_{868} \right. \\
& - 32x(1+x)^2H_0H_{0,1}P_{900} + 32x(1+x)^2H_{0,0,1}P_{930} - 8x(1+x)^2H_{-1}H_0^2P_{948} \\
& + 16x(1+x)^2H_0H_{0,-1}P_{962} - 16x(1+x)^2H_{0,0,-1}P_{968} + 2x(1+x)^2H_0^2P_{1002} + 4x(1+x)^2P_{985}\zeta_2 \\
& + 16x(1+x)^2P_{862}\zeta_3 \left. \right) - 192x \log(2)x_+^2\zeta_2 \left. \right\} + \varepsilon \left\{ \eta^2 \left(32c_1x(x-1)^2 + 242x(x+1)^2 \right) \right. \\
& + \xi \left(\eta^4 \left(\log(2)\zeta_2 \left(-384(-1+x)x^2(1+x)^3H_1 - 384(-1+x)x^2(1+x)^3H_{-1} \right) \right. \right. \\
& + 64(-1+x)x^2(1+x)^3H_{-1}H_0^2H_1 + \left(384(-1+x)x^2(1+x)^3H_{-1}H_1 - 1920x^3(1+x)^2H_{-1}H_{0,-1} \right. \\
& - 384(-1+x)x^2(1+x)^3H_{-1,1} \left. \right)\zeta_2 \left. \right) + \eta^3 \left(\left(-768(-1+x)x(1+x)^3H_{-1}H_0 \right. \right. \\
& - \frac{128}{3}(-1+x)x^2(1+x)H_{-1}H_0^3 \left. \right)H_1 + \left(128(-1+x)x(1+x)^3H_0 + \frac{64}{3}(-1+x)x^2(1+x)H_0^3 \right)H_1^2 \\
& + \left(-256(-1+x)x(1+x)^3H_1 + 768(-1+x)x(1+x)^3H_{-1} \right)H_{0,1} + 768(-1+x)x(1+x)^3H_1H_{0,-1} \\
& + 256(-1+x)x(1+x)^3H_{0,1,1} - 768(-1+x)x(1+x)^3H_{0,1,-1} - 768(-1+x)x(1+x)^3H_{0,-1,1} + \left(\left(\right. \right. \\
& - 256(-1+x)x^2(1+x)H_{-1}H_1 + 128(-1+x)x^2(1+x)H_1^2 \left. \right)H_0 - 128(-1+x)x(1+x)^3H_1 + \left(\right. \\
& - 256(-1+x)x^2(1+x)H_1 + 256(-1+x)x^2(1+x)H_{-1} \left. \right)H_{0,1} + 256(-1+x)x^2(1+x)H_{0,1,1} \left. \right)\zeta_2 \left. \right) \\
& + 384(-1+x)x^2(1+x)^3(5-x+5x^2)\eta^5\zeta_2H_{0,-1,-1} \left. \right) + \eta^3 \left(5x(1+x)^2(67-18x+67x^2)H_0 \right. \\
& + 16x(1+x)^2(75-166x+75x^2)H_{-1}H_0 - 8x(1+x)^2(127-298x+127x^2)H_{-1}^2H_0 \\
& + \left(-208x(1+x)^2(3-2x+3x^2)H_0 + \left(-768x^2(1+x)^2H_0 - \frac{2176}{3}x^3H_0^3 \right)H_{-1} \right)H_1 \\
& + \left(128x^2(1+x)^2H_0 + \frac{1088}{3}x^3H_0^3 \right)H_1^2 + \left(208x(1+x)^2(3-2x+3x^2) - 256x^2(1+x)^2H_1 \right. \\
& + 768x^2(1+x)^2H_{-1} \left. \right)H_{0,1} + \left(-16x(1+x)^2(75-166x+75x^2) + 768x^2(1+x)^2H_1 \right.
\end{aligned}$$

$$\begin{aligned}
& + 16x(1+x)^2 \left(127 - 298x + 127x^2 \right) H_{-1} \Big) H_{0,-1} + 256x^2(1+x)^2 H_{0,1,1} - 768x^2(1+x)^2 H_{0,1,-1} \\
& - 768x^2(1+x)^2 H_{0,-1,1} - 16x(1+x)^2 \left(127 - 298x + 127x^2 \right) H_{0,-1,-1} + \left(\left(-4352x^3 H_{-1} H_1 \right. \right. \\
& \left. \left. + 2176x^3 H_1^2 \right) H_0 - 128x^2(1+x)^2 H_1 + \left(-4352x^3 H_1 + 4352x^3 H_{-1} \right) H_{0,1} + 4352x^3 H_{0,1,1} \right) \zeta_2 \Big) \\
& + \eta^4 \left(384x(1+x)^2 H_{0,0,1,-1} P_{837} + 384x(1+x)^2 H_{0,0,-1,1} P_{837} + 192x(1+x)^2 H_{0,0,1,1} P_{840} \right. \\
& + 16x(1+x)^2 H_0^2 H_1^2 P_{867} + 192x(1+x)^2 H_{0,-1,0,1} P_{869} + 192x(1+x)^2 H_0 H_{0,1,-1} P_{883} \\
& + 192x(1+x)^2 H_0 H_{0,-1,1} P_{883} + 32x(1+x)^2 H_{0,1}^2 P_{903} - 64x(1+x)^2 H_0 H_{0,1,1} P_{904} \\
& - 96x(1+x)^2 H_{0,0,-1,-1} P_{916} - 32x(1+x)^2 H_0 H_{0,-1,-1} P_{960} + 8x(1+x)^2 H_{-1}^2 H_0^2 P_{969} \\
& + 16x(1+x)^2 H_{0,-1}^2 P_{983} - 4x(1+x)^2 H_{-1} H_0^2 P_{1008} + 2x(1+x)^2 H_0^2 P_{1012} + \log(2) \zeta_2 \Big(\\
& - 384x(1+x)^2 P_{865} + 1536x^3(1+x)^2 H_1 + 1536x^3(1+x)^2 H_{-1} \Big) + \left(8x(1+x)^2 H_0^2 P_{984} \right. \\
& - 256x^3(1+x)^2 H_{-1} H_0^2 \Big) H_1 + \left(64x(1+x)^2 H_{-1} H_0 P_{870} - 32x(1+x)^2 H_0 P_{911} \right. \\
& - 64x(1+x)^2 H_{0,-1} P_{947} + 256x^2(1+x)^2 \left(2 + 15x + 2x^2 \right) H_0 H_1 \Big) H_{0,1} + \left(-128x(1+x)^2 H_{-1} H_0 P_{901} \right. \\
& - 64x(1+x)^2 H_0 H_1 P_{902} - 16x(1+x)^2 H_0 P_{949} \Big) H_{0,-1} + \left(-384x(1+x)^2 H_{-1} P_{837} \right. \\
& - 192x(1+x)^2 H_1 P_{840} - 16x(1+x)^2 P_{957} \Big) H_{0,0,1} + \left(768x(1+x)^2 H_1 P_{838} + 96x(1+x)^2 H_{-1} P_{916} \right. \\
& + 8x(1+x)^2 P_{1009} \Big) H_{0,0,-1} + \left(-48x(1+x)^2 H_{-1}^2 P_{929} + 8x(1+x)^2 H_{-1} P_{997} + 4x(1+x)^2 P_{1006} \right. \\
& - 1536x^3(1+x)^2 H_{-1} H_1 + 1536x^3(1+x)^2 H_{-1,1} \Big) \zeta_2 + \left(-192x(1+x)^2 H_1 P_{850} \right. \\
& + 96x(1+x)^2 H_{-1} P_{879} - 16x(1+x)^2 P_{992} \Big) \zeta_3 + \eta^5 \left(-768x^2(1+x)^2 H_0 H_{0,0,1,-1} P_{819} \right. \\
& - 768x^2(1+x)^2 H_0 H_{0,0,-1,1} P_{819} - 1024x^2(1+x)^2 H_0 H_{0,0,1,1} P_{833} + \frac{4}{5} x^2(1+x)^2 H_0^5 P_{834} \\
& - 512x^2(1+x)^2 H_{0,0,-1,0,-1} P_{845} - 256x^2(1+x)^2 H_0 H_{0,1}^2 P_{846} - 768x^2(1+x)^2 H_{0,0,0,-1,-1} P_{874} \\
& + 128x^2(1+x)^2 H_0 H_{0,0,-1,-1} P_{886} + 512x^2(1+x)^2 H_{0,0,0,1,-1} P_{887} + 512x^2(1+x)^2 H_{0,0,0,-1,1} P_{887} \\
& - 64x^2(1+x)^2 H_0^2 H_{0,-1,-1} P_{888} + 128x^2(1+x)^2 H_{0,0,1,0,-1} P_{905} - 128x^2(1+x)^2 H_0 H_{0,-1,0,1} P_{907} \\
& - 128x^2(1+x)^2 H_{0,0,1,0,1} P_{921} + 64x^2(1+x)^2 H_0^2 H_{0,1,1} P_{927} - 128x^2(1+x)^2 H_{0,0,0,1,1} P_{934} \\
& + 128x^2(1+x)^2 H_{0,0,-1,0,1} P_{943} + 64x^2(1+x)^2 H_{0,0,0,0,-1} P_{963} - 64x^2(1+x)^2 H_{0,0,0,0,1} P_{970} \\
& - \frac{2}{3} x(1+x)^2 H_0^4 P_{1045} - \frac{2}{3} x(1+x)^2 H_0^3 P_{1060} + \log(2) \zeta_2 \left(1536x^3(1+x)^2 \left(1+x+x^2 \right) H_{0,1} \right. \\
& + 1536x^3(1+x)^2 \left(1+x+x^2 \right) H_{0,-1} \Big) + \left(\frac{8}{3} x^2(1+x)^2 H_0^4 P_{825} - \frac{16}{3} x(1+x)^2 H_0^3 P_{1043} \right) H_1 \\
& + \left(\frac{8}{3} x^2(1+x)^2 H_0^4 P_{835} + \frac{8}{3} x(1+x)^2 H_0^3 P_{1050} \right) H_{-1} + \left(-64x^2(1+x)^2 H_{-1} H_0^2 P_{818} \right. \\
& - 128x^2(1+x)^2 H_0^2 H_1 P_{832} + 256x^2(1+x)^2 H_0 H_{0,-1} P_{875} + 16x(1+x)^2 H_0^2 P_{1044} \\
& + \frac{64}{3} x^3(1+x)^2 \left(74 - 79x + 74x^2 \right) H_0^3 \Big) H_{0,1} + \left(64x^2(1+x)^2 H_{-1} H_0^2 P_{820} - 64x^2(1+x)^2 H_0^2 H_1 P_{823} \right. \\
& \left. + \frac{32}{3} x^2(1+x)^2 H_0^3 P_{906} - 8x(1+x)^2 H_0^2 P_{1054} \right) H_{0,-1} - 384x^3(1+x)^2 \left(10 - 17x + 10x^2 \right) H_0 H_{0,-1}^2
\end{aligned}$$

$$\begin{aligned}
& + \left(-128x^2(1+x)^2 H_{0,-1} P_{826} + 1024x^2(1+x)^2 H_0 H_1 P_{831} - 32x^2(1+x)^2 H_0^2 P_{873} \right. \\
& + 128x^2(1+x)^2 H_{0,1} P_{922} - 64x(1+x)^2 H_0 P_{1031} - 2048x^3(1+x)^2 (5-7x+5x^2) H_{-1} H_0 \left. \right) H_{0,0,1} \\
& + \left(-128x^2(1+x)^2 H_{-1} H_0 P_{817} - 1024x^2(1+x)^2 H_0 H_1 P_{831} + 512x^2(1+x)^2 H_{0,-1} P_{839} \right. \\
& - 96x^2(1+x)^2 H_0^2 P_{853} - 128x^2(1+x)^2 H_{0,1} P_{944} + 16x(1+x)^2 H_0 P_{1049} \left. \right) H_{0,0,-1} - 128x^3(1+x)^2 (26 \\
& - 25x + 26x^2) H_0^2 H_{0,1,-1} - 128x^3(1+x)^2 (26-25x+26x^2) H_0^2 H_{0,-1,1} + (128x^2(1+x)^2 H_{-1} P_{843} \\
& - 1024x^2(1+x)^2 H_1 P_{847} + 64x^2(1+x)^2 H_0 P_{937} - 32x(1+x)^2 P_{1038} \left. \right) H_{0,0,0,1} + (64x^2(1+x)^2 H_0 P_{815} \\
& + 128x^2(1+x)^2 H_1 P_{965} + 16x(1+x)^2 P_{1055} - 768x^3(1+x)^2 (28-41x+28x^2) H_{-1} \left. \right) H_{0,0,0,-1} \\
& + (64x^2(1+x)^2 H_{0,0,1} P_{841} - 512x^2(1+x)^2 H_{0,0,-1} P_{851} - \frac{32}{3}x^2(1+x)^2 H_0^3 P_{852} \\
& + 128x^2(1+x)^2 H_{0,1,-1} P_{890} + 128x^2(1+x)^2 H_{0,-1,1} P_{891} - 8x(1+x)^2 H_0^2 P_{1047} + 4x(1+x)^2 H_0 P_{1059} \\
& + (-32x^2(1+x)^2 H_0^2 P_{889} - 32x(1+x)^2 H_0 P_{1035}) H_1 + (-16x(1+x)^2 H_0 P_{1048} \\
& - 192x^3(1+x)^2 (2-9x+2x^2) H_0^2) H_{-1} + (128x^2(1+x)^2 H_0 P_{822} + 32x(1+x)^2 P_{1032}) H_{0,1} \\
& + (128x^2(1+x)^2 H_0 P_{877} - 128x^2(1+x)^2 H_1 P_{891} + 16x(1+x)^2 P_{1026}) H_{0,-1} + 1536x^3(1+x)^2 (5 \\
& - x + 5x^2) H_{0,-1,-1} + 64x^2(1+x)^2 P_{931} \zeta_3 \left. \right) \zeta_2 + \left(\frac{64}{5}x^2(1+x)^2 H_{-1} P_{816} + \frac{16}{5}x^2(1+x)^2 H_0 P_{975} \right. \\
& - \frac{32}{5}x^2(1+x)^2 H_1 P_{977} - \frac{8}{5}x(1+x)^2 P_{1056} \left. \right) \zeta_2^2 + (-128x^2(1+x)^2 H_{-1} H_0 P_{821} \\
& - 256x^2(1+x)^2 H_0 H_1 P_{831} - 64x^2(1+x)^2 H_{0,1} P_{844} + 128x^2(1+x)^2 H_{0,-1} P_{876} - 16x^2(1+x)^2 H_0^2 P_{909} \\
& - 16x(1+x)^2 H_0 P_{1017}) \zeta_3 + 96x^2(1+x)^2 P_{918} \zeta_5 - 384(-1+x)x^2(1+x)^3 \xi^2 \eta^4 \zeta_2 H_{-1} H_{0,-1} \left. \right\} \\
& + C_F C_A \left[\left\{ -\frac{968}{9}x(1+x)^2 \eta^2 + \eta (16x H_0 H_1 - 16x H_{0,1}) + \eta^3 \left(-\frac{2}{9}x(1+x)^2 (687 \right. \right. \right. \\
& + 26x + 687x^2) H_0 + \frac{8}{3}x(1+x)^2 (3-88x+3x^2) H_{-1} H_0 + \frac{32}{3}x^2 (3-8x+3x^2) H_0^3 H_1 \\
& - \frac{8}{3}x(1+x)^2 (3-88x+3x^2) H_{0,-1} + (64x^2 (3-8x+3x^2) H_0 H_1 - 64x^2 (3-8x+3x^2) H_{0,1}) \zeta_2 \left. \right. \\
& + \xi^2 (96(-1+x)x^2(1+x)^3 \eta^4 \zeta_2 H_{0,-1} + 72(-1+x)x(1+x)^3 \eta^3 H_{-1} \zeta_2) + \xi (288x^2(1+x)^2 \eta^3 H_{-1} \zeta_2 \\
& + 480x^3(1+x)^2 \eta^4 H_{0,-1} \zeta_2) + \eta^4 (-8x(1+x)^2 H_{0,0,-1} P_{923} + 8x(1+x)^2 H_0 H_{0,-1} P_{924} \\
& - 4x(1+x)^2 H_{-1} H_0^2 P_{926} - \frac{2}{3}x(1+x)^2 H_0^2 P_{978} - 48x^2(1+x)^2 (3-2x+3x^2) H_0^2 H_1 + 32x^2(1+x)^2 (7 \\
& - 2x+7x^2) H_0 H_{0,1} - 32x^2(1+x)^2 (5+2x+5x^2) H_{0,0,1} - \frac{4}{3}x(1+x)^2 P_{1000} \zeta_2 + 16x(1+x)^2 P_{858} \zeta_3 \left. \right. \\
& + \eta^5 \left(-\frac{2}{3}x^2(1+x)^2 H_0^4 P_{835} + 128x^2(1+x)^2 H_0 H_{0,0,1} P_{881} + 16x^2(1+x)^2 H_0^2 H_{0,-1} P_{915} \right. \\
& - \frac{4}{3}x^3(1+x)^2 H_0^3 P_{925} - 16x^2(1+x)^2 H_0^2 H_{0,1} P_{936} + 96x^2(1+x)^2 H_{0,0,0,-1} P_{941} \\
& - 32x^2(1+x)^2 H_0 H_{0,0,-1} P_{956} - 32x^2(1+x)^2 H_{0,0,0,1} P_{961} + \frac{8}{5}x^2(1+x)^2 P_{994} \zeta_2^2 \\
& \left. \left. \left. + (8x^2(1+x)^2 H_0^2 P_{878} - 8x^2(1+x)^2 H_0 P_{1016}) \zeta_2 - 32x^2(1+x)^2 H_0 P_{914} \zeta_3 \right) + 96x \log(2)x_+^2 \zeta_2 \right\} \right]
\end{aligned}$$

$$\begin{aligned}
& + \varepsilon \left\{ \eta^2 \left(-16c_1x(x-1)^2 - \frac{14872}{27}x(x+1)^2 \right) + \xi \left(\eta^4 \left(\log(2)\zeta_2 \left(192(-1+x)x^2(1+x)^3 H_1 \right. \right. \right. \\
& + 192(-1+x)x^2(1+x)^3 H_{-1} \left. \left. \left. - 32(-1+x)x^2(1+x)^3 H_{-1} H_0^2 H_1 - 64(-1+x)x(1+x)^3 (3-5x \right. \right. \right. \\
& + 3x^2) H_0 H_1 H_{0,1} + \left(-192(-1+x)x^2(1+x)^3 H_{-1} H_1 + 960x^3(1+x)^2 H_{-1} H_{0,-1} \right. \\
& + 192(-1+x)x^2(1+x)^3 H_{-1,1} \left. \left. \right) \zeta_2 \right) + \eta^3 \left(-8(-1+x)x(1+x)^3 H_{-1}^2 H_0 \right. \\
& + 16(-1+x)x(1+x)^3 H_{-1} H_{0,-1} - 16(-1+x)x(1+x)^3 H_{0,-1,-1} + 624x^2(1+x)^2 H_1 \zeta_3 \left. \right) \\
& - 192(-1+x)x^2(1+x)^3 \left(5-x+5x^2 \right) \eta^5 \zeta_2 H_{0,-1,-1} \left. \right) + \eta^3 \left(-\frac{1}{27}x(1+x)^2 \left(20559 \right. \right. \\
& + 1586x + 20559x^2 \left. \left. \right) H_0 + \frac{4}{9}x(1+x)^2 \left(753 + 634x + 753x^2 \right) H_{-1} H_0 + 640x^2(1+x)^2 H_{-1}^2 H_0 \right. \\
& + \left(32x(1+x)^2 \left(3-19x+3x^2 \right) H_0 + \frac{64}{3}x^2 \left(3-8x+3x^2 \right) H_{-1} H_0^3 \right) H_1 - \frac{32}{3}x^2 \left(3-8x \right. \\
& + 3x^2 \left. \right) H_0^3 H_1^2 - 32x(1+x)^2 \left(3-19x+3x^2 \right) H_{0,1} + \left(-\frac{4}{9}x(1+x)^2 \left(753 + 634x + 753x^2 \right) \right. \\
& - 1280x^2(1+x)^2 H_{-1} \left. \right) H_{0,-1} + 1280x^2(1+x)^2 H_{0,-1,-1} + \left(128x^2 \left(3-8x+3x^2 \right) H_{-1} H_0 H_1 \right. \\
& - 64x^2 \left(3-8x+3x^2 \right) H_0 H_1^2 + \left(128x^2 \left(3-8x+3x^2 \right) H_1 - 128x^2 \left(3-8x+3x^2 \right) H_{-1} \right) H_{0,1} \\
& - 128x^2 \left(3-8x+3x^2 \right) H_{0,1,1} \left. \right) \zeta_2 \left. \right) + \eta \left(-96x H_{-1} H_0 H_1 + 16x H_0 H_1^2 + \left(-32x H_1 + 96x H_{-1} \right) H_{0,1} \right. \\
& + 96x H_1 H_{0,-1} + 32x H_{0,1,1} - 96x H_{0,1,-1} - 96x H_{0,-1,1} - 16x H_1 \zeta_2 \left. \right) + \eta^4 \left(192x(1+x)^2 H_{0,0,1,1} P_{830} \right. \\
& + 64x(1+x)^2 H_0 H_{0,1,-1} P_{856} + 64x(1+x)^2 H_0 H_{0,-1,1} P_{856} - 64x(1+x)^2 H_{0,0,1,-1} P_{857} \\
& - 64x(1+x)^2 H_{0,0,-1,1} P_{857} - 64x(1+x)^2 H_0 H_{0,1,1} P_{859} - 16x(1+x)^2 H_0^2 H_1^2 P_{861} \\
& - 48x(1+x)^2 H_0 H_{0,-1,-1} P_{908} + 12x(1+x)^2 H_{-1}^2 H_0^2 P_{910} - 8x(1+x)^2 H_{-1} H_0^2 P_{946} \\
& + 8x(1+x)^2 H_{0,-1}^2 P_{959} - \frac{2}{9}x(1+x)^2 H_0^2 P_{1011} + \log(2)\zeta_2 \left(192x(1+x)^2 P_{865} - 768x^3(1+x)^2 H_1 \right. \\
& - 768x^3(1+x)^2 H_{-1} \left. \right) + \left(-16x(1+x)^2 H_0^2 P_{892} + 128x^3(1+x)^2 H_{-1} H_0^2 \right) H_1 + \left(64x(1+x)^2 H_0 P_{824} \right. \\
& - 96x(1+x)^2 H_{-1} H_0 P_{828} - 32x(1+x)^2 H_{0,-1} P_{855} - 512x^2(1+x)^2 \left(3-5x+3x^2 \right) H_0 H_1 \left. \right) H_{0,1} \\
& - 32x^2(1+x)^2 \left(11-10x+11x^2 \right) H_{0,1}^2 + \left(-32x(1+x)^2 H_0 H_1 P_{854} + 8x(1+x)^2 H_0 P_{991} \right. \\
& - 64x^2(1+x)^2 \left(5-4x+5x^2 \right) H_{-1} H_0 \left. \right) H_{0,-1} + \left(-192x(1+x)^2 H_1 P_{830} + 32x(1+x)^2 P_{842} \right. \\
& + 64x(1+x)^2 H_{-1} P_{857} \left. \right) H_{0,0,1} + \left(192x(1+x)^2 H_1 P_{827} - 16x(1+x)^2 P_{980} - 16x(1+x)^2 \left(3-4x \right. \right. \\
& + 3x^2 \left. \left. \right) \left(7-68x+7x^2 \right) H_{-1} \right) H_{0,0,-1} + 16x(1+x)^2 \left(3-4x+3x^2 \right) \left(7-68x+7x^2 \right) H_{0,0,-1,-1} \\
& - 128x^2(1+x)^2 \left(5-x+5x^2 \right) H_{0,-1,0,1} + \left(-16x(1+x)^2 H_{-1} P_{899} + 24x(1+x)^2 H_{-1}^2 P_{929} \right. \\
& - \frac{4}{9}x(1+x)^2 P_{1014} + 768x^3(1+x)^2 H_{-1} H_1 - 768x^3(1+x)^2 H_{-1,1} \left. \right) \zeta_2 + \left(-16x(1+x)^2 H_{-1} P_{938} \right. \\
& + 8x(1+x)^2 P_{954} \left. \right) \zeta_3 \left. \right) + \xi^2 \left(192(-1+x)x^2(1+x)^3 \eta^4 \zeta_2 H_{-1} H_{0,-1} - 48(-1+x)x(1+x)^3 \eta^3 H_1 \zeta_3 \right)
\end{aligned}$$

$$\begin{aligned}
& + \eta^5 \left(512x^2(1+x)^2 H_0 H_{0,0,1,1} P_{833} - \frac{2}{5} x^2(1+x)^2 H_0^5 P_{834} + 32x^2(1+x)^2 H_0^2 H_{0,1,1} P_{863} \right. \\
& + \frac{1}{3} x^3(1+x)^2 H_0^4 P_{872} + 96x^2(1+x)^2 H_0 H_{0,-1}^2 P_{880} - 320x^2(1+x)^2 H_0 H_{0,-1,0,1} P_{894} \\
& - 64x^2(1+x)^2 H_0 H_{0,1}^2 P_{919} + 32x^2(1+x)^2 H_0^2 H_{0,1,-1} P_{940} + 32x^2(1+x)^2 H_0^2 H_{0,-1,1} P_{940} \\
& - 448x^2(1+x)^2 H_{0,0,0,1,1} P_{945} - 128x^2(1+x)^2 H_{0,0,-1,0,-1} P_{951} - 32x^2(1+x)^2 H_0^2 H_{0,-1,-1} P_{967} \\
& - 128x^2(1+x)^2 H_0 H_{0,0,1,-1} P_{972} - 128x^2(1+x)^2 H_0 H_{0,0,-1,1} P_{972} - 64x^2(1+x)^2 H_{0,0,1,0,1} P_{981} \\
& - 192x^2(1+x)^2 H_{0,0,0,-1,-1} P_{982} + 64x^2(1+x)^2 H_{0,0,1,0,-1} P_{988} + 64x^2(1+x)^2 H_0 H_{0,0,-1,-1} P_{989} \\
& + 64x^2(1+x)^2 H_{0,0,-1,0,1} P_{996} + 128x^2(1+x)^2 H_{0,0,0,1,-1} P_{998} + 128x^2(1+x)^2 H_{0,0,0,-1,1} P_{998} \\
& + 32x^2(1+x)^2 H_{0,0,0,0,1} P_{1004} - 32x^2(1+x)^2 H_{0,0,0,0,-1} P_{1007} - \frac{2}{3} x(1+x)^2 H_0^3 P_{1051} + \log(2) \zeta_2 \left(\right. \\
& - 768x^3(1+x)^2 \left(1+x+x^2 \right) H_{0,1} - 768x^3(1+x)^2 \left(1+x+x^2 \right) H_{0,-1} \left. \right) + \left(\frac{4}{3} x^2(1+x)^2 H_0^4 P_{912} \right. \\
& + \frac{16}{3} x(1+x)^2 H_0^3 P_{1030} \left. \right) H_1 + \left(-\frac{4}{3} x^2(1+x)^2 H_0^4 P_{835} + \frac{4}{3} x(1+x)^2 H_0^3 P_{1052} \right) H_{-1} \\
& + \left(64x^2(1+x)^2 H_0^2 H_1 P_{832} - 32x^2(1+x)^2 H_{-1} H_0^2 P_{936} + 64x^2(1+x)^2 H_0 H_{0,-1} P_{971} \right. \\
& - \frac{16}{3} x^2(1+x)^2 H_0^3 P_{976} + 16x(1+x)^2 H_0^2 P_{1022} \left. \right) H_{0,1} + \left(-32x^2(1+x)^2 H_0^2 H_1 P_{913} \right. \\
& + 32x^2(1+x)^2 H_{-1} H_0^2 P_{915} + \frac{16}{3} x^2(1+x)^2 H_0^3 P_{979} - 4x(1+x)^2 H_0^2 P_{1039} \left. \right) H_{0,-1} \\
& + \left(256x^2(1+x)^2 H_0 H_1 P_{849} + 256x^2(1+x)^2 H_{-1} H_0 P_{881} - 64x^2(1+x)^2 H_{0,-1} P_{935} \right. \\
& + 64x^2(1+x)^2 H_{0,1} P_{966} + 16x^2(1+x)^2 H_0^2 P_{995} - 32x(1+x)^2 H_0 P_{1033} \left. \right) H_{0,0,1} + \left(\right. \\
& - 256x^2(1+x)^2 H_{0,-1} P_{839} - 256x^2(1+x)^2 H_0 H_1 P_{849} - 64x^2(1+x)^2 H_{-1} H_0 P_{956} \\
& - 64x^2(1+x)^2 H_{0,1} P_{990} - 16x^2(1+x)^2 H_0^2 P_{999} + 8x(1+x)^2 H_0 P_{1053} \left. \right) H_{0,0,-1} + \left(\right. \\
& - 128x^2(1+x)^2 H_1 P_{939} - 64x^2(1+x)^2 H_{-1} P_{961} - 32x^2(1+x)^2 H_0 P_{1001} + 32x(1+x)^2 P_{1042} \left. \right) H_{0,0,0,1} \\
& + \left(192x^2(1+x)^2 H_{-1} P_{941} + 32x^2(1+x)^2 H_0 P_{1003} - 8x(1+x)^2 P_{1057} + 64x^2(1+x)^2 \left(3 - 11x \right. \right. \\
& + 3x^2 \left. \right) \left(13 - 8x + 13x^2 \right) H_1 \left. \right) H_{0,0,0,-1} + \left(-32x^2(1+x)^2 H_{0,0,1} P_{841} + 256x^2(1+x)^2 H_{0,0,-1} P_{851} \right. \\
& + 64x^2(1+x)^2 H_{0,-1,1} P_{860} + 64x^2(1+x)^2 H_{0,1,-1} P_{866} + \frac{8}{3} x^2(1+x)^2 H_0^3 P_{950} + 4x^2(1+x)^2 H_0^2 P_{1023} \\
& - 2x(1+x)^2 H_0 P_{1058} + \left(-16x^2(1+x)^2 H_0^2 P_{955} + 32x(1+x)^2 H_0 P_{1028} \right) H_1 \\
& + \left(16x^2(1+x)^2 H_0^2 P_{878} + 8x(1+x)^2 H_0 P_{1036} \right) H_{-1} + \left(-32x(1+x)^2 P_{1029} + 32x^2(1+x)^2 \left(10 \right. \right. \\
& - 37x - 37x^3 + 10x^4 \left. \right) H_0 \left. \right) H_{0,1} + \left(-64x^2(1+x)^2 H_1 P_{860} - 32x^2(1+x)^2 H_0 P_{882} \right. \\
& + 8x(1+x)^2 P_{1046} \left. \right) H_{0,-1} - 768x^3(1+x)^2 \left(5 - x + 5x^2 \right) H_{0,-1,-1} - 32x^2(1+x)^2 P_{953} \zeta_3 \left. \right) \zeta_2 \\
& + \left(\frac{16}{5} x^2(1+x)^2 H_{-1} P_{994} - \frac{16}{5} x^2(1+x)^2 H_1 P_{1005} - \frac{8}{5} x^2(1+x)^2 H_0 P_{1010} + \frac{4}{5} x(1+x)^2 P_{1041} \right) \zeta_2^2 + \left(\right. \\
& - 64x^2(1+x)^2 H_0 H_1 P_{849} - 8x^2(1+x)^2 H_0^2 P_{893} - 64x^2(1+x)^2 H_{-1} H_0 P_{914} + 96x^2(1+x)^2 H_{0,1} P_{917} \\
& \left. + 64x^2(1+x)^2 H_{0,-1} P_{920} - 16x(1+x)^2 H_0 P_{1027} \right) \zeta_3 + 144x^2(1+x)^2 P_{952} \zeta_5 \left. \right) \left. \right]
\end{aligned}$$

$$\begin{aligned}
& + C_F n_l T_F \left[\left\{ \frac{304}{9} x(1+x)^2 \eta^2 + \eta^3 \left(-\frac{16}{3} x(1+x)^2 (3-2x+3x^2) \zeta_2 + \frac{8}{9} x(1+x)^2 (51 \right. \right. \right. \\
& - 26x + 51x^2) H_0 - \frac{32}{3} x(1+x)^2 (3-2x+3x^2) H_{-1} H_0 + \frac{8}{3} x(1+x)^2 (3-2x+3x^2) H_0^2 \\
& + \left. \left. \left. \frac{32}{3} x(1+x)^2 (3-2x+3x^2) H_{0,-1} \right) \right\} + \varepsilon \left\{ \frac{4664}{27} x(1+x)^2 \eta^2 + \eta^3 \left(-32x(1+x)^2 (3 \right. \right. \right. \\
& - 2x + 3x^2) \zeta_3 + \frac{4}{27} x(1+x)^2 (1527 - 722x + 1527x^2) H_0 + 32x(1+x)^2 (3-2x \\
& + 3x^2) H_{-1} H_0 + \frac{8}{9} x(1+x)^2 (69 - 38x + 69x^2) H_0^2 + \frac{8}{3} x(1+x)^2 (3-2x+3x^2) H_0^3 \\
& + \left(-\frac{32}{9} x(1+x)^2 (69 - 38x + 69x^2) H_0 - 16x(1+x)^2 (3-2x+3x^2) H_0^2 \right) H_{-1} \\
& + \left(\frac{32}{9} x(1+x)^2 (69 - 38x + 69x^2) - 64x(1+x)^2 (3-2x+3x^2) H_{-1} \right) H_{0,-1} + 32x(1+x)^2 (3 \\
& - 2x + 3x^2) H_{0,0,-1} + 64x(1+x)^2 (3-2x+3x^2) H_{0,-1,-1} + \left(-\frac{32}{9} x(1+x)^2 (21 - 19x \right. \\
& + 48x^2) + 8x(1+x)^2 (3-2x+3x^2) H_0 + 32x(1+x)^2 (3-2x+3x^2) H_{-1} \left. \right) \zeta_2 \left. \right\} \Big] \\
& + C_F T_F \left[\left\{ \frac{128}{9} (-1+x)x(1+x) \xi \eta^2 + \eta^2 \left(-\frac{1024x^2}{9} + \left(\frac{8}{3} x H_0^2 P_{864} + 16x P_{895} \zeta_2 \right) x_+^2 \right) \right. \right. \\
& + \left. \left. \eta^3 \left(\frac{8}{9} x H_0 P_{987} - \frac{32}{3} x^3 H_0^3 - 64x^3 H_0 \zeta_2 \right) \right\} + \varepsilon \left\{ -128x (3+2x+3x^2) \log(2) x_+^4 \zeta_2 + \eta^2 \left(-\frac{16}{27} x (19 \right. \right. \right. \\
& - 10x + 19x^2) + \left(-\frac{16}{3} x H_{-1} H_0^2 P_{864} + \frac{32}{3} x H_0^2 H_1 P_{864} - \frac{64}{3} x H_0 H_{0,1} P_{864} + \frac{64}{3} x H_{0,0,1} P_{864} \right. \\
& + \left. \frac{32}{3} x H_{0,0,-1} P_{864} - 32x H_{-1} P_{864} \zeta_2 + \frac{64}{3} x P_{932} \zeta_3 \right) x_+^2 + \eta^3 \left(\frac{4}{27} x H_0 P_{1013} - 8x^3 H_0^4 + \left(\right. \right. \\
& - \frac{16}{9} x H_0 P_{987} - \frac{64}{3} x^3 H_0^3 \left. \right) H_{-1} - 128x^3 H_0^2 H_{0,1} + \left(\frac{16x P_{987}}{9} + 128x^3 H_0^2 \right) H_{0,-1} + 512x^3 H_0 H_{0,0,1} \\
& - 256x^3 H_0 H_{0,0,-1} - 768x^3 H_{0,0,0,1} + 128x^3 H_{0,0,0,-1} + \left(-\frac{8}{9} x H_0^3 P_{1020} - \frac{8}{9} x H_0^2 P_{1021} + \left(-\frac{8}{3} x H_0 P_{1019} \right. \right. \\
& \left. \left. - \frac{16x P_{1025}}{9} \right) \zeta_2 \right) x_+ + \left(-128x^3 H_{-1} H_0 - 32x^3 H_0^2 + 512x^3 H_{0,-1} \right) \zeta_2 + \frac{32}{5} x^3 \zeta_2^2 + 128x^3 H_0 \zeta_3 \left. \right\} \Big]. \tag{C.6}
\end{aligned}$$

$$\begin{aligned}
\hat{F}_{A,2}^{(2),s} = C_F T_F & \left[\left\{ \xi \eta^3 \left(-256(-1+x)x^2(1+x) H_0 H_{0,0,1} - 32(-1+x)x^2(1+x) H_0^2 \zeta_2 \right) \right. \right. \\
& + \eta^4 \left(32x(1+x)^2 \zeta_2 \left(-1 - 6x - 6x^2 - 14x^3 + 3x^4 \right) + 8x(1+x)^2 \left(-1 + 2x - 4x^2 - 26x^3 \right. \right. \\
& + \left. \left. 13x^4 \right) H_0^2 \right) + \eta^5 \left(-128x^2(1+x)^2 \left(1+x + 8x^2 + x^3 + x^4 \right) \zeta_2 H_0 - \frac{64}{3} x^3(1+x)^2 \left(3 + 8x \right. \right. \\
& - \left. \left. x^2 + 2x^3 \right) H_0^3 \right) + \eta \left(128x H_{-1} H_0 - 64x H_0 H_1 + 32x H_{0,1} - 128x H_{0,-1} + \left(-\frac{64}{3} x^2 H_0^3 H_1 \right. \right. \\
& + \left. \left(-128x^2 H_0 H_1 + 128x^2 H_{0,1} \right) \zeta_2 \right) x_+^2 + \eta^3 \left(-16x(1+x)^2 (3-2x+3x^2) H_0 + 64x^2 (3 \right. \\
& - 2x + 3x^2) \zeta_3 H_0 + \frac{4}{3} x^2(1+x)^2 H_0^4 + 256x^3 H_0^2 H_{0,1} - 1536x^3 H_0 H_{0,0,1} + 1024x^3 H_0 H_{0,0,-1}
\end{aligned}$$

$$\begin{aligned}
& -64(-7+x)x^2(-1+7x)H_{0,0,0,1} - 3072x^3H_{0,0,0,-1} + 192x^3H_0^2\zeta_2 + \frac{256}{5}x^2\left(1+x\right. \\
& \left.+ x^2\right)\zeta_2^2) + \eta^2\left(-16x\left(5+4x+5x^2\right) - 256x^2H_0^2H_1 + 768x^2H_0H_{0,1} - 512x^2H_0H_{0,-1}\right. \\
& \left.- 1024x^2H_{0,0,1} + 1024x^2H_{0,0,-1} + 256x^2\zeta_3\right) \left. \vphantom{\frac{256}{5}x^2}\right\} + \varepsilon\left\{\xi\eta^3\left(-256(-1+x)x^2(1+x)H_0H_{0,1}^2 + \left(\right.\right.\right. \\
& \left.\left.\left.-512(-1+x)x^2(1+x)H_{-1}H_0 - 128(-1+x)x^2(1+x)H_0^2\right)H_{0,0,1} - 128(-1+x)x^2(1+x)H_0^2H_{0,0,-1}\right.\right. \\
& \left.\left.+ 128(-1+x)x^2(1+x)H_0^2H_{0,1,1} + 128(-1+x)x^2(1+x)H_0^2H_{0,1,-1}\right.\right. \\
& \left.\left.+ 128(-1+x)x^2(1+x)H_0^2H_{0,-1,1} + 1024(-1+x)x^2(1+x)H_0H_{0,0,1,-1}\right.\right. \\
& \left.\left.+ 1024(-1+x)x^2(1+x)H_0H_{0,0,-1,1} + 256(-1+x)x^2(1+x)H_0H_{0,-1,0,1}\right.\right. \\
& \left.\left.- 128(-1+x)x^2(1+x)H_{0,0,1,0,-1} + \left(-64(-1+x)x^2(1+x)H_{-1}H_0^2 - 16(-1+x)x^2(1+x)H_0^3\right.\right.\right. \\
& \left.\left.\left.+ 256(-1+x)x^2(1+x)H_0H_{0,-1} - 1024(-1+x)x^2(1+x)H_{0,0,-1} - 64(-1+x)x^2(1+x)\zeta_3\right)\zeta_2\right) \\
& + \eta^5\left(\frac{128}{3}x^2(1+x)^2H_{-1}H_0^3P_{848} + 256x^2(1+x)^2H_0^2H_{0,-1}P_{896} - 128x^2(1+x)^2H_0^2H_{0,1}P_{898}\right. \\
& \left.+ \frac{128}{3}x^2(1+x)^2H_0^3H_1P_{933} - \frac{8}{3}x^2(1+x)^2H_0^4P_{942} - 256x^2(1+x)^2H_{0,0,0,-1}P_{958}\right. \\
& \left.+ 128x^2(1+x)^2H_{0,0,0,1}P_{993} - \frac{16}{3}x(1+x)^2H_0^3P_{1040} - \frac{64}{5}x^2(1+x)^2P_{974}\zeta_2^2 + \left(\right.\right. \\
& \left.\left.-256x^2(1+x)^2H_{0,-1}P_{829} + 256x^2(1+x)^2H_{-1}H_0P_{848} - 256x^2(1+x)^2H_{0,1}P_{884}\right.\right. \\
& \left.\left.+ 256x^2(1+x)^2H_0H_1P_{897} + 16x(1+x)^2H_0P_{1034}\right)\zeta_2) + \eta\left(-512xH_{-1}^2H_0 + 384xH_{-1}H_0H_1\right. \\
& \left.- 64xH_0H_1^2 + \left(64xH_1 - 448xH_{-1}\right)H_{0,1} + \left(-384xH_1 + 1024xH_{-1}\right)H_{0,-1} - 64xH_{0,1,1}\right. \\
& \left.+ 448xH_{0,1,-1} + 448xH_{0,-1,1} - 1024xH_{0,-1,-1} + \left(-\frac{128}{3}x^2H_{-1}H_0^3H_1 + \frac{64}{3}x^2H_0^3H_1^2\right.\right. \\
& \left.\left.- 256x^2H_0H_{0,-1}H_{0,1} + 256x^2H_0^2H_1H_{0,-1} - 512x^2H_{0,0,-1,0,1} + \left(-256x^2H_{-1}H_0H_1\right.\right.\right. \\
& \left.\left.\left.+ 128x^2H_0H_1^2 + \left(-256x^2H_1 + 256x^2H_{-1}\right)H_{0,1} + 1024x^2H_1H_{0,-1} + 256x^2H_{0,1,1}\right.\right.\right. \\
& \left.\left.\left.- 1024x^2H_{0,-1,1}\right)\zeta_2\right)x_+^2 - 64xH_1\zeta_2) + \eta^4\left(32x(1+x)^2H_0H_{0,1}P_{871} + 16x(1+x)^2H_0^2H_1P_{928}\right. \\
& \left.- 16x(1+x)^2H_{-1}H_0^2P_{964} + 32x(1+x)^2H_{0,0,-1}P_{986} + 8x(1+x)H_0^2P_{1024} + \left(64x(1+x)^2H_{-1}P_{836}\right.\right. \\
& \left.\left.- 32x(1+x)P_{1015}\right)\zeta_2 - 16x(1+x)^2P_{973}\zeta_3) + \eta^2\left(-120x\left(3+4x+3x^2\right) + \log(2)\zeta_2\left(1536x^2\right.\right. \\
& \left.\left.- 1536x^2H_1 - 1536x^2H_{-1}\right) + 128x^2H_{-1}^2H_0^2 + 256x^2H_{-1}H_0^2H_1 - 384x^2H_0^2H_1^2 + \left(\right.\right. \\
& \left.\left.- 1024x^2H_{-1}H_0 + 1536x^2H_0H_1 + 2560x^2H_{0,-1}\right)H_{0,1} - 256x^2H_{0,1}^2 + \left(\left(-32x\left(11+16x\right.\right.\right. \\
& \left.\left.\left.+ 13x^2\right) + 512x^2H_{-1}\right)H_0 + 512x^2H_0H_1\right)H_{0,-1} - 1536x^2H_{0,-1}^2 + \left(-64x^2(19+8x)\right. \\
& \left.- 1536x^2H_1 + 1536x^2H_{-1}\right)H_{0,0,1} + \left(-1536x^2H_1 - 1536x^2H_{-1}\right)H_{0,0,-1} - 1024x^2H_0H_{0,1,1} \\
& \left.- 1536x^2H_0H_{0,1,-1} - 1536x^2H_0H_{0,-1,1} + 2560x^2H_0H_{0,-1,-1} + 1536x^2H_{0,0,1,1}\right. \\
& \left.- 1536x^2H_{0,0,1,-1} - 1536x^2H_{0,0,-1,1} + 1536x^2H_{0,0,-1,-1} - 1536x^2H_{0,-1,0,1} + \left(1536x^2H_{-1}H_1\right.\right. \\
& \left.\left.+ 768x^2H_{-1}^2 - 1536x^2H_{-1,1}\right)\zeta_2 - 64x^2(3+5x)H_0^2x_+\zeta_2 + \left(2688x^2H_1 - 384x^2H_{-1}\right)\zeta_3)
\end{aligned}$$

$$\begin{aligned}
& + \eta^3 \left(-64x^2(11 + 31x + 11x^2)\zeta_5 + \log(2)\zeta_2(3072x^3H_{0,1} + 3072x^3H_{0,-1}) - 8x(1+x) \left(33 \right. \right. \\
& + 15x + 27x^2 + 29x^3 \Big) H_0 + \frac{16}{15}x^2(1+x)^2H_0^5 + \left(-128(-2+x)x(1+x)^2(-1+2x)H_0 \right. \\
& - \frac{8}{3}x^2(5-14x+5x^2)H_0^4 \Big) H_1 + \left(32x(1+x)^2(19-42x+19x^2)H_0 + \frac{8}{3}x^2(1+x)^2H_0^4 \right) H_{-1} \\
& + \left(64xP_{885} + 512x^3H_{-1}H_0^2 + \frac{64}{3}x^2(3+x)(1+3x)H_0^3 + 512x^3H_0^2H_1 \right) H_{0,1} - 1536x^3H_0H_{0,1}^2 + \left(\right. \\
& - 32x(1+x)^2(19-42x+19x^2) - \frac{32}{3}x^2(3-26x+3x^2)H_0^3 \Big) H_{0,-1} - 1536x^3H_0H_{0,-1}^2 + \left(\right. \\
& - 256x^2(11-2x+9x^2)H_0 - 3072x^3H_{-1}H_0 - 1536x^3H_0^2 - 512x^2(1+x)^2H_0H_1 - 256x^2(3-14x \\
& + 3x^2)H_{0,1} + 128x^2(3-50x+3x^2)H_{0,-1} \Big) H_{0,0,1} + \left(-512x^2(2+x)(-3+2x)H_0 + 2048x^3H_{-1}H_0 \right. \\
& + 768x^3H_0^2 + 512x^2(1+x)^2H_0H_1 + 256x^2(3-2x+3x^2)H_{0,1} + 7168x^3H_{0,-1} \Big) H_{0,0,-1} \\
& + 768x^3H_0^2H_{0,1,1} - 768x^3H_0^2H_{0,1,-1} - 768x^3H_0^2H_{0,-1,1} - 512x^3H_0^2H_{0,-1,-1} + \left(-128x^2(11-26x \right. \\
& + 11x^2)H_0 + 128x^2(13+10x+13x^2)H_1 - 128(-7+x)x^2(-1+7x)H_{-1} \Big) H_{0,0,0,1} + \left(128x^2(5 \right. \\
& - 38x+5x^2)H_0 - 256x^2(11+2x+11x^2)H_1 - 6144x^3H_{-1} \Big) H_{0,0,0,-1} + 512x^2(1+x)^2H_0H_{0,0,1,1} \\
& + 6144x^3H_0H_{0,0,1,-1} + 6144x^3H_0H_{0,0,-1,1} - 4096x^3H_0H_{0,0,-1,-1} + 3584x^3H_0H_{0,-1,0,1} + 64x^2(55 \\
& + 6x+55x^2)H_{0,0,0,0,1} - 64x^2(53-54x+53x^2)H_{0,0,0,0,-1} + 128x^2(5-94x+5x^2)H_{0,0,0,1,1} \\
& + 256x^2(5+14x+5x^2)H_{0,0,0,1,-1} + 256x^2(5+14x+5x^2)H_{0,0,0,-1,1} - 9216x^3H_{0,0,0,-1,-1} \\
& + 256x^2(3-14x+3x^2)H_{0,0,1,0,1} + 3328x^3H_{0,0,1,0,-1} - 5120x^3H_{0,0,-1,0,-1} + \left(384x^3H_{-1}H_0^2 \right. \\
& + 160x^3H_0^3 + 128x^2(1+x)^2H_0^2H_1 + 128x^2(3+2x+3x^2)H_0H_{0,1} + 1024x^3H_0H_{0,-1} - 128x^2(7+2x \\
& + 7x^2)H_{0,0,1} - 4096x^3H_{0,0,-1} - 1024x^2(1+x+x^2)H_{0,1,-1} - 3072x^3H_{0,-1,-1} + 768x^3\zeta_3 \Big) \zeta_2 + \left(\right. \\
& - \frac{32}{5}x^2(5+66x+5x^2)H_0 + \frac{64}{5}x^2(5+14x+5x^2)H_1 + \frac{512}{5}x^2(1+x+x^2)H_{-1} \Big) \zeta_2^2 + \left(-128x^2(4 \right. \\
& + x+10x^2)H_0 + 128x^2(3-2x+3x^2)H_{-1}H_0 + 16x^2(13+18x+13x^2)H_0^2 + 128x^2(1+x)^2H_0H_1 \\
& \left. \left. + 128x^2(5-32x+5x^2)H_{0,1} - 128x^2(3-8x+3x^2)H_{0,-1} \right) \zeta_3 \right) - 32xH_1x_+^2 \Big] .
\end{aligned}$$

(C.7)

The polynomials are given by

$$\begin{aligned}
P_{815} &= x^4 - 661x^3 + 786x^2 - 661x + 1, \\
P_{816} &= x^4 - 145x^3 + 405x^2 - 145x + 1, \\
P_{817} &= x^4 - 93x^3 + 136x^2 - 93x + 1, \\
P_{818} &= x^4 - 57x^3 + 88x^2 - 57x + 1, \\
P_{819} &= x^4 - 53x^3 + 78x^2 - 53x + 1, \\
P_{820} &= x^4 - 37x^3 + 54x^2 - 37x + 1, \\
P_{821} &= x^4 - 33x^3 + 52x^2 - 33x + 1, \\
P_{822} &= x^4 - 31x^3 + 87x^2 - 31x + 1,
\end{aligned}$$

$$\begin{aligned}
P_{823} &= x^4 - 29x^3 + 38x^2 - 29x + 1, \\
P_{824} &= x^4 - 26x^3 + 41x^2 - 26x + 1, \\
P_{825} &= x^4 - 25x^3 + 18x^2 - 25x + 1, \\
P_{826} &= x^4 - 25x^3 + 36x^2 - 25x + 1, \\
P_{827} &= x^4 - 23x^3 + 26x^2 - 23x + 1, \\
P_{828} &= x^4 - 16x^3 + 26x^2 - 16x + 1, \\
P_{829} &= x^4 - 14x^3 - 32x^2 + 6x - 9, \\
P_{830} &= x^4 - 14x^3 + 20x^2 - 14x + 1, \\
P_{831} &= x^4 - 13x^3 + 30x^2 - 13x + 1, \\
P_{832} &= x^4 - 7x^3 + 24x^2 - 7x + 1, \\
P_{833} &= x^4 - 5x^3 + 20x^2 - 5x + 1, \\
P_{834} &= x^4 - 5x^3 + 22x^2 - 5x + 1, \\
P_{835} &= x^4 - 5x^3 + 38x^2 - 5x + 1, \\
P_{836} &= x^4 - 2x^3 + 66x^2 + 14x - 7, \\
P_{837} &= x^4 - x^3 - 22x^2 - x + 1, \\
P_{838} &= x^4 - x^3 + 25x^2 - x + 1, \\
P_{839} &= x^4 + 2x^3 - 9x^2 + 2x + 1, \\
P_{840} &= x^4 + 3x^3 + 36x^2 + 3x + 1, \\
P_{841} &= x^4 + 7x^3 - 94x^2 + 7x + 1, \\
P_{842} &= x^4 + 70x^3 - 108x^2 + 78x - 3, \\
P_{843} &= x^4 + 107x^3 - 144x^2 + 107x + 1, \\
P_{844} &= x^4 + 117x^3 - 170x^2 + 117x + 1, \\
P_{845} &= 2x^4 - 45x^3 + 56x^2 - 45x + 2, \\
P_{846} &= 2x^4 - 30x^3 + 59x^2 - 30x + 2, \\
P_{847} &= 2x^4 - 29x^3 + 63x^2 - 29x + 2, \\
P_{848} &= 2x^4 - 7x^3 - 8x^2 + 5x - 4, \\
P_{849} &= 2x^4 - 7x^3 - 2x^2 - 7x + 2, \\
P_{850} &= 2x^4 - 6x^3 + 39x^2 - 6x + 2, \\
P_{851} &= 2x^4 - 4x^3 + x^2 - 4x + 2, \\
P_{852} &= 2x^4 + 8x^3 + 31x^2 + 8x + 2, \\
P_{853} &= 3x^4 - 151x^3 + 204x^2 - 151x + 3, \\
P_{854} &= 3x^4 - 68x^3 + 82x^2 - 68x + 3, \\
P_{855} &= 3x^4 - 58x^3 + 50x^2 - 58x + 3, \\
P_{856} &= 3x^4 - 53x^3 + 64x^2 - 53x + 3, \\
P_{857} &= 3x^4 - 49x^3 + 74x^2 - 49x + 3, \\
P_{858} &= 3x^4 - 41x^3 + 64x^2 - 41x + 3, \\
P_{859} &= 3x^4 - 40x^3 + 56x^2 - 40x + 3, \\
P_{860} &= 3x^4 - 25x^3 + 62x^2 - 25x + 3, \\
P_{861} &= 3x^4 - 16x^3 + 32x^2 - 16x + 3, \\
P_{862} &= 3x^4 - 16x^3 + 170x^2 - 16x + 3,
\end{aligned}$$

$$\begin{aligned}
P_{863} &= 3x^4 - 14x^3 - 38x^2 - 14x + 3, \\
P_{864} &= 3x^4 - 14x^3 - 2x^2 - 14x + 3, \\
P_{865} &= 3x^4 - 14x^3 + 16x^2 - 14x + 3, \\
P_{866} &= 3x^4 - 13x^3 + 74x^2 - 13x + 3, \\
P_{867} &= 3x^4 - 7x^3 - 12x^2 - 7x + 3, \\
P_{868} &= 3x^4 - 7x^3 + 64x^2 - 7x + 3, \\
P_{869} &= 3x^4 - 3x^3 + 28x^2 - 3x + 3, \\
P_{870} &= 3x^4 - 2x^3 - 62x^2 - 2x + 3, \\
P_{871} &= 3x^4 + 22x^3 - 34x^2 + 22x + 3, \\
P_{872} &= 3x^4 + 176x^3 - 160x^2 + 408x - 3, \\
P_{873} &= 3x^4 + 269x^3 - 274x^2 + 269x + 3, \\
P_{874} &= 4x^4 - 118x^3 + 153x^2 - 118x + 4, \\
P_{875} &= 4x^4 - 90x^3 + 163x^2 - 90x + 4, \\
P_{876} &= 4x^4 - 61x^3 + 75x^2 - 61x + 4, \\
P_{877} &= 4x^4 - 22x^3 + 9x^2 - 22x + 4, \\
P_{878} &= 4x^4 - 21x^3 + 4x^2 - 21x + 4, \\
P_{879} &= 4x^4 - 13x^3 - 16x^2 - 13x + 4, \\
P_{880} &= 4x^4 - 13x^3 + 24x^2 - 13x + 4, \\
P_{881} &= 4x^4 - 13x^3 + 30x^2 - 13x + 4, \\
P_{882} &= 4x^4 - 11x^3 - 40x^2 - 11x + 4, \\
P_{883} &= 4x^4 - 7x^3 + 86x^2 - 7x + 4, \\
P_{884} &= 4x^4 - 5x^3 + 14x^2 - 5x + 4, \\
P_{885} &= 4x^4 - 3x^3 - 10x^2 - 3x + 4, \\
P_{886} &= 5x^4 - 281x^3 + 396x^2 - 281x + 5, \\
P_{887} &= 5x^4 - 224x^3 + 330x^2 - 224x + 5, \\
P_{888} &= 5x^4 - 149x^3 + 222x^2 - 149x + 5, \\
P_{889} &= 5x^4 - 121x^3 + 262x^2 - 121x + 5, \\
P_{890} &= 5x^4 - 53x^3 + 42x^2 - 53x + 5, \\
P_{891} &= 5x^4 - 41x^3 + 54x^2 - 41x + 5, \\
P_{892} &= 5x^4 - 34x^3 + 56x^2 - 26x + 1, \\
P_{893} &= 5x^4 - 20x^3 + 18x^2 - 20x + 5, \\
P_{894} &= 5x^4 - 17x^3 + 30x^2 - 17x + 5, \\
P_{895} &= 5x^4 - 10x^3 + 2x^2 - 10x + 5, \\
P_{896} &= 5x^4 - 7x^3 + 8x^2 + 9x - 3, \\
P_{897} &= 5x^4 - 7x^3 + 14x^2 - 3x + 3, \\
P_{898} &= 5x^4 - 6x^3 + 2x^2 + 18x - 7, \\
P_{899} &= 5x^4 + 49x^3 + 69x^2 - 55x + 16, \\
P_{900} &= 6x^4 - 13x^3 + 78x^2 - 13x + 6, \\
P_{901} &= 6x^4 - 9x^3 + 16x^2 - 9x + 6, \\
P_{902} &= 6x^4 - 7x^3 + 146x^2 - 7x + 6,
\end{aligned}$$

$$\begin{aligned}
P_{903} &= 6x^4 - 5x^3 + 62x^2 - 5x + 6, \\
P_{904} &= 6x^4 + 3x^3 + 122x^2 + 3x + 6, \\
P_{905} &= 7x^4 - 263x^3 + 392x^2 - 263x + 7, \\
P_{906} &= 7x^4 - 211x^3 + 270x^2 - 211x + 7, \\
P_{907} &= 7x^4 - 179x^3 + 314x^2 - 179x + 7, \\
P_{908} &= 7x^4 - 96x^3 + 118x^2 - 96x + 7, \\
P_{909} &= 7x^4 - 79x^3 + 156x^2 - 79x + 7, \\
P_{910} &= 7x^4 - 64x^3 + 94x^2 - 64x + 7, \\
P_{911} &= 7x^4 - 54x^3 + 90x^2 - 54x + 7, \\
P_{912} &= 7x^4 - 41x^3 + 98x^2 - 41x + 7, \\
P_{913} &= 7x^4 - 37x^3 + 78x^2 - 37x + 7, \\
P_{914} &= 7x^4 - 33x^3 + 64x^2 - 33x + 7, \\
P_{915} &= 7x^4 - 29x^3 + 62x^2 - 29x + 7, \\
P_{916} &= 7x^4 - 16x^3 - 82x^2 - 16x + 7, \\
P_{917} &= 7x^4 - 16x^3 + 40x^2 - 16x + 7, \\
P_{918} &= 8x^4 - 267x^3 + 485x^2 - 267x + 8, \\
P_{919} &= 8x^4 - 39x^3 + 56x^2 - 39x + 8, \\
P_{920} &= 8x^4 - 38x^3 + 99x^2 - 38x + 8, \\
P_{921} &= 9x^4 - 233x^3 + 388x^2 - 233x + 9, \\
P_{922} &= 9x^4 - 169x^3 + 308x^2 - 169x + 9, \\
P_{923} &= 9x^4 - 140x^3 + 178x^2 - 140x + 9, \\
P_{924} &= 9x^4 - 116x^3 + 154x^2 - 116x + 9, \\
P_{925} &= 9x^4 - 112x^3 + 146x^2 - 196x + 9, \\
P_{926} &= 9x^4 - 92x^3 + 130x^2 - 92x + 9, \\
P_{927} &= 9x^4 - 85x^3 + 212x^2 - 85x + 9, \\
P_{928} &= 9x^4 - 38x^3 + 8x^2 - 2x - 9, \\
P_{929} &= 9x^4 - 32x^3 + 34x^2 - 32x + 9, \\
P_{930} &= 9x^4 - 19x^3 + 92x^2 - 19x + 9, \\
P_{931} &= 9x^4 - 8x^3 - 5x^2 - 8x + 9, \\
P_{932} &= 9x^4 - 7x^3 + 8x^2 - 7x + 9, \\
P_{933} &= 10x^4 - 17x^3 + 14x^2 + 7x - 2, \\
P_{934} &= 11x^4 - 467x^3 + 660x^2 - 467x + 11, \\
P_{935} &= 11x^4 - 74x^3 + 138x^2 - 74x + 11, \\
P_{936} &= 11x^4 - 42x^3 + 86x^2 - 42x + 11, \\
P_{937} &= 11x^4 + 377x^3 - 302x^2 + 377x + 11, \\
P_{938} &= 12x^4 - 85x^3 + 92x^2 - 85x + 12, \\
P_{939} &= 12x^4 - 49x^3 + 38x^2 - 49x + 12, \\
P_{940} &= 12x^4 - 47x^3 + 124x^2 - 47x + 12, \\
P_{941} &= 12x^4 - 43x^3 + 92x^2 - 43x + 12, \\
P_{942} &= 13x^4 - 7x^3 + 48x^2 + 17x + 1,
\end{aligned}$$

$$\begin{aligned}
P_{943} &= 15x^4 - 419x^3 + 694x^2 - 419x + 15, \\
P_{944} &= 15x^4 - 355x^3 + 614x^2 - 355x + 15, \\
P_{945} &= 15x^4 - 70x^3 + 146x^2 - 70x + 15, \\
P_{946} &= 15x^4 - 65x^3 + 37x^2 + 3x + 22, \\
P_{947} &= 15x^4 - 23x^3 + 196x^2 - 23x + 15, \\
P_{948} &= 15x^4 - 16x^3 + 198x^2 - 16x + 15, \\
P_{949} &= 15x^4 + 70x^3 - 114x^2 + 70x + 15, \\
P_{950} &= 16x^4 - 83x^3 + 236x^2 - 83x + 16, \\
P_{951} &= 16x^4 - 75x^3 + 178x^2 - 75x + 16, \\
P_{952} &= 16x^4 - 65x^3 + 109x^2 - 65x + 16, \\
P_{953} &= 17x^4 - 74x^3 + 111x^2 - 74x + 17, \\
P_{954} &= 17x^4 + 312x^3 - 154x^2 + 124x + 29, \\
P_{955} &= 19x^4 - 77x^3 + 86x^2 - 77x + 19, \\
P_{956} &= 19x^4 - 72x^3 + 154x^2 - 72x + 19, \\
P_{957} &= 19x^4 + 92x^3 - 182x^2 + 156x - 45, \\
P_{958} &= 20x^4 - 31x^3 + 76x^2 - 19x + 14, \\
P_{959} &= 21x^4 - 268x^3 + 338x^2 - 268x + 21, \\
P_{960} &= 21x^4 - 62x^3 + 710x^2 - 62x + 21, \\
P_{961} &= 21x^4 - 58x^3 + 146x^2 - 58x + 21, \\
P_{962} &= 21x^4 - 42x^3 + 326x^2 - 42x + 21, \\
P_{963} &= 21x^4 + 775x^3 - 590x^2 + 775x + 21, \\
P_{964} &= 23x^4 - 46x^3 - 44x^2 + 38x - 19, \\
P_{965} &= 25x^4 - 361x^3 + 762x^2 - 361x + 25, \\
P_{966} &= 27x^4 - 128x^3 + 214x^2 - 128x + 27, \\
P_{967} &= 27x^4 - 115x^3 + 242x^2 - 115x + 27, \\
P_{968} &= 27x^4 - 68x^3 + 454x^2 - 68x + 27, \\
P_{969} &= 27x^4 - 24x^3 + 374x^2 - 24x + 27, \\
P_{970} &= 27x^4 + 461x^3 - 178x^2 + 461x + 27, \\
P_{971} &= 28x^4 - 117x^3 + 196x^2 - 117x + 28, \\
P_{972} &= 29x^4 - 105x^3 + 230x^2 - 105x + 29, \\
P_{973} &= 29x^4 - 30x^3 + 72x^2 + 22x + 3, \\
P_{974} &= 30x^4 - 51x^3 + 20x^2 + 17x - 4, \\
P_{975} &= 31x^4 + 877x^3 - 2470x^2 + 877x + 31, \\
P_{976} &= 36x^4 - 149x^3 + 364x^2 - 149x + 36, \\
P_{977} &= 37x^4 - 1021x^3 + 2202x^2 - 1021x + 37, \\
P_{978} &= 39x^4 + 184x^3 - 48x^2 + 32x + 33, \\
P_{979} &= 41x^4 - 185x^3 + 426x^2 - 185x + 41, \\
P_{980} &= 42x^4 + 191x^3 - 219x^2 + 123x + 35, \\
P_{981} &= 43x^4 - 196x^3 + 366x^2 - 196x + 43, \\
P_{982} &= 44x^4 - 193x^3 + 448x^2 - 193x + 44,
\end{aligned}$$

$$\begin{aligned}
P_{983} &= 45x^4 - 98x^3 + 774x^2 - 98x + 45, \\
P_{984} &= 47x^4 - 124x^3 + 178x^2 - 60x - 17, \\
P_{985} &= 47x^4 + 6x^3 - 272x^2 - 102x + 33, \\
P_{986} &= 49x^4 - 66x^3 - 60x^2 + 26x + 3, \\
P_{987} &= 51x^4 - 176x^3 - 70x^2 - 176x + 51, \\
P_{988} &= 53x^4 - 232x^3 + 478x^2 - 232x + 53, \\
P_{989} &= 55x^4 - 214x^3 + 474x^2 - 214x + 55, \\
P_{990} &= 57x^4 - 239x^3 + 430x^2 - 239x + 57, \\
P_{991} &= 57x^4 + 126x^3 - 182x^2 + 126x + 57, \\
P_{992} &= 60x^4 - 95x^3 + 219x^2 + 147x - 11, \\
P_{993} &= 61x^4 - 119x^3 + 128x^2 - 47x + 25, \\
P_{994} &= 66x^4 - 271x^3 + 176x^2 - 271x + 66, \\
P_{995} &= 67x^4 - 259x^3 + 654x^2 - 259x + 67, \\
P_{996} &= 73x^4 - 307x^3 + 582x^2 - 307x + 73, \\
P_{997} &= 73x^4 - 214x^3 + 276x^2 + 190x + 11, \\
P_{998} &= 90x^4 - 377x^3 + 790x^2 - 377x + 90, \\
P_{999} &= 91x^4 - 372x^3 + 838x^2 - 372x + 91, \\
P_{1000} &= 93x^4 - 700x^3 + 588x^2 - 476x + 63, \\
P_{1001} &= 107x^4 - 415x^3 + 1090x^2 - 415x + 107, \\
P_{1002} &= 125x^4 - 230x^3 + 312x^2 - 26x + 43, \\
P_{1003} &= 151x^4 - 593x^3 + 1418x^2 - 593x + 151, \\
P_{1004} &= 163x^4 - 661x^3 + 1794x^2 - 661x + 163, \\
P_{1005} &= 171x^4 - 695x^3 + 814x^2 - 695x + 171, \\
P_{1006} &= 215x^4 - 98x^3 - 54x^2 - 554x + 203, \\
P_{1007} &= 221x^4 - 875x^3 + 2310x^2 - 875x + 221, \\
P_{1008} &= 233x^4 - 510x^3 + 356x^2 + 182x - 117, \\
P_{1009} &= 293x^4 - 230x^3 - 100x^2 + 462x - 57, \\
P_{1010} &= 303x^4 - 1367x^3 + 1474x^2 - 1367x + 303, \\
P_{1011} &= 348x^4 - 1604x^3 + 1989x^2 + 398x + 885, \\
P_{1012} &= 581x^4 - 910x^3 + 1242x^2 - 142x + 125, \\
P_{1013} &= 1527x^4 - 440x^3 - 2398x^2 - 440x + 1527, \\
P_{1014} &= 1866x^4 - 7378x^3 + 7947x^2 - 5276x + 681, \\
P_{1015} &= x^5 + 7x^4 + 21x^3 - x^2 + 14x - 2, \\
P_{1016} &= 9x^5 - 94x^4 + 110x^3 - 196x^2 + 45x - 18, \\
P_{1017} &= 13x^5 + 620x^4 - 884x^3 + 254x^2 - x + 6, \\
P_{1018} &= 14x^5 + 175x^4 - 56x^3 + 14x^2 + 6x - 9, \\
P_{1019} &= 15x^5 - 157x^4 + 18x^3 - 78x^2 - 21x - 9, \\
P_{1020} &= 15x^5 - 109x^4 + 36x^3 - 60x^2 + 27x - 9, \\
P_{1021} &= 18x^5 + 6x^4 + 87x^3 + 159x^2 + 119x - 69, \\
P_{1022} &= 25x^5 - 251x^4 + 518x^3 - 318x^2 + 105x - 3,
\end{aligned}$$

$$\begin{aligned}
P_{1023} &= 33x^5 - 376x^4 + 764x^3 - 604x^2 + 107x + 44, \\
P_{1024} &= 45x^5 - 81x^4 - 110x^3 + 26x^2 - 15x - 9, \\
P_{1025} &= 252x^5 - 250x^4 - 345x^3 + 99x^2 + 125x - 201, \\
P_{1026} &= 3x^6 + 86x^5 + 575x^4 + 416x^3 - 131x^2 + 146x - 39, \\
P_{1027} &= 6x^6 - 131x^5 + 441x^4 - 572x^3 + 262x^2 - x - 9, \\
P_{1028} &= 6x^6 - 77x^5 + 251x^4 - 402x^3 + 212x^2 - 65x + 3, \\
P_{1029} &= 6x^6 - 75x^5 + 247x^4 - 402x^3 + 216x^2 - 67x + 3, \\
P_{1030} &= 6x^6 - 38x^5 + 215x^4 - 402x^3 + 248x^2 - 104x + 3, \\
P_{1031} &= 6x^6 + 3x^5 - 8x^4 + 352x^3 - 370x^2 + 21x - 12, \\
P_{1032} &= 6x^6 + 9x^5 - 410x^4 + 588x^3 - 320x^2 - 29x + 12, \\
P_{1033} &= 9x^6 - 65x^5 - 181x^4 + 536x^3 - 409x^2 + 121x - 3, \\
P_{1034} &= 9x^6 - 14x^5 + 259x^4 - 156x^3 + 27x^2 - 14x + 1, \\
P_{1035} &= 9x^6 + 3x^5 - 399x^4 + 588x^3 - 331x^2 - 23x + 9, \\
P_{1036} &= 9x^6 + 28x^5 - 89x^4 + 392x^3 - 221x^2 + 196x - 27, \\
P_{1037} &= 12x^6 - 27x^5 - 129x^4 + 56x^3 - 60x^2 + 7x - 3, \\
P_{1038} &= 15x^6 - 86x^5 - 307x^4 - 678x^3 + 815x^2 - 44x + 21, \\
P_{1039} &= 15x^6 - 40x^5 - 517x^4 + 1832x^3 - 1323x^2 + 648x - 39, \\
P_{1040} &= 19x^6 - 35x^5 - 150x^4 + 130x^3 - 19x^2 - 3x + 2, \\
P_{1041} &= 21x^6 - 936x^5 + 4040x^4 - 7744x^3 + 6563x^2 - 2448x + 312, \\
P_{1042} &= 21x^6 - 232x^5 - 5x^4 + 456x^3 - 521x^2 + 152x - 3, \\
P_{1043} &= 21x^6 - 19x^5 - 179x^4 + 588x^3 - 551x^2 - x - 3, \\
P_{1044} &= 24x^6 - 29x^5 - 178x^4 + 674x^3 - 652x^2 + 27x - 18, \\
P_{1045} &= 24x^6 + 70x^5 + 173x^4 + 156x^3 + 8x^2 + 14x - 21, \\
P_{1046} &= 27x^6 - 336x^5 + 219x^4 - 920x^3 + 25x^2 - 80x + 9, \\
P_{1047} &= 27x^6 + 39x^5 - 786x^4 + 722x^3 - 31x^2 - 9x + 6, \\
P_{1048} &= 33x^6 - 80x^5 + 279x^4 - 112x^3 + 99x^2 + 120x - 51, \\
P_{1049} &= 33x^6 - 74x^5 + 177x^4 + 1940x^3 - 2185x^2 + 134x - 57, \\
P_{1050} &= 33x^6 - 20x^5 + 627x^4 + 112x^3 - 1005x^2 - 20x - 15, \\
P_{1051} &= 35x^6 - 448x^5 + 653x^4 - 400x^3 + 235x^2 - 12x + 33, \\
P_{1052} &= 39x^6 - 396x^5 + 501x^4 + 392x^3 - 811x^2 + 620x - 57, \\
P_{1053} &= 45x^6 - 284x^5 - 1011x^4 + 2752x^3 - 2165x^2 + 716x - 21, \\
P_{1054} &= 69x^6 - 166x^5 + 205x^4 + 952x^3 - 1677x^2 + 86x - 45, \\
P_{1055} &= 75x^6 - 256x^5 - 543x^4 - 3076x^3 + 2529x^2 - 124x + 51, \\
P_{1056} &= 90x^6 + 968x^5 - 6009x^4 + 8498x^3 - 3484x^2 - 498x + 243, \\
P_{1057} &= 129x^6 - 1128x^5 - 981x^4 + 3152x^3 - 3337x^2 + 824x - 3, \\
P_{1058} &= 173x^6 - 402x^5 + 963x^4 - 892x^3 + 735x^2 - 426x + 41, \\
P_{1059} &= 245x^6 - 222x^5 + 531x^4 + 92x^3 - 41x^2 - 22x - 7, \\
P_{1060} &= 247x^6 - 522x^5 + 225x^4 + 516x^3 - 1091x^2 + 158x - 109.
\end{aligned} \tag{C.8}$$

D The scalar form factors up to two-loop

In this appendix, we present the scalar form factors $F_S^{(n)}$ up to two loops and $\mathcal{O}(\varepsilon)$.

$$\begin{aligned}
F_S^{(1)} = & C_F \left[\frac{1}{\varepsilon} \left\{ -2 - 2\xi H_0 \right\} + \left\{ -2 + \xi \left(4H_{-1}H_0 - H_0^2 - 4H_{0,-1} + 2\zeta_2 \right) + 8x\eta H_0 \right\} \right. \\
& + \varepsilon \left\{ -4 + \eta \left(12xH_0 - 16xH_{-1}H_0 + 4xH_0^2 + 16xH_{0,-1} + \left(-1 - 8x + x^2 \right) \zeta_2 \right) + \xi \left(\left(-2 - 4H_{-1}^2 \right. \right. \right. \\
& \left. \left. \left. + \zeta_2 \right) H_0 + 2H_{-1}H_0^2 - \frac{1}{3}H_0^3 + 8H_{-1}H_{0,-1} - 4H_{0,0,-1} - 8H_{0,-1,-1} - 4H_{-1}\zeta_2 + 4\zeta_3 \right) \right\} + \varepsilon^2 \left\{ -8 \right. \\
& + \eta \left(\left(24x - 24xH_{-1} + 16xH_{-1}^2 - 4x\zeta_2 \right) H_0 + \left(-1 + 6x - x^2 - 8xH_{-1} \right) H_0^2 + \frac{4}{3}xH_0^3 + \left(24x \right. \right. \\
& \left. \left. - 32xH_{-1} \right) H_{0,-1} + 16xH_{0,0,-1} + 32xH_{0,-1,-1} + \left(1 - 12x + 3x^2 \right) \zeta_2 + 16xH_{-1}\zeta_2 - \frac{2}{3} \left(-1 + 24x \right. \right. \\
& \left. \left. + x^2 \right) \zeta_3 \right) + \xi \left(\left(-4 + \left(4 - 2\zeta_2 \right) H_{-1} + \frac{8}{3}H_{-1}^3 + \frac{14}{3}\zeta_3 \right) H_0 + \left(-2H_{-1}^2 + \frac{\zeta_2}{2} \right) H_0^2 + \frac{2}{3}H_{-1}H_0^3 \right. \\
& \left. - \frac{1}{12}H_0^4 + \left(-4 - 8H_{-1}^2 - 2\zeta_2 \right) H_{0,-1} + 8H_{-1}H_{0,0,-1} + 16H_{-1}H_{0,-1,-1} - 4H_{0,0,0,-1} - 8H_{0,0,-1,-1} \right. \\
& \left. - 16H_{0,-1,-1,-1} + 4H_{-1}^2\zeta_2 + \frac{14}{5}\zeta_2^2 - 8H_{-1}\zeta_3 \right) \right\} + \varepsilon^3 \left\{ -16 + \eta \left(\left(48x + \left(-48x + 8x\zeta_2 \right) H_{-1} \right. \right. \right. \\
& \left. \left. \left. + 24xH_{-1}^2 - \frac{32}{3}xH_{-1}^3 - 6x\zeta_2 - \frac{56}{3}x\zeta_3 \right) H_0 + \left(12x + \left(-12x + \left(-1 - x^2 \right) \zeta_2 \right) H_{-1} + 8xH_{-1}^2 \right. \right. \right. \\
& \left. \left. \left. - 2x\zeta_2 \right) H_0^2 + \left(\frac{1}{3} \left(-1 + 6x - x^2 \right) - \frac{8}{3}xH_{-1} \right) H_0^3 + \frac{1}{3}xH_0^4 + \left(24x\zeta_2 + 32x\zeta_3 \right) H_{-1} + \left(48x - 48xH_{-1} \right. \right. \right. \\
& \left. \left. \left. + 32xH_{-1}^2 + 8x\zeta_2 \right) H_{0,-1} + \left(24x - 32xH_{-1} \right) H_{0,0,-1} + \left(48x - 64xH_{-1} \right) H_{0,-1,-1} + 16xH_{0,0,0,-1} \right. \right. \\
& \left. \left. + 32xH_{0,0,-1,-1} + 64xH_{0,-1,-1,-1} + 2 \left(1 - 12x + 3x^2 \right) \zeta_2 - 16xH_{-1}^2\zeta_2 + \frac{1}{20} \left(-9 - 224x + 9x^2 \right) \zeta_2^2 \right. \right. \\
& \left. \left. + \frac{2}{3} \left(-7 + x \right) \left(-1 + 5x \right) \zeta_3 \right) + \xi \left(\left(-8 + \left(8 - \frac{28\zeta_3}{3} \right) H_{-1} + \left(-4 + 2\zeta_2 \right) H_{-1}^2 - \frac{4}{3}H_{-1}^4 + \frac{47}{20}\zeta_2^2 \right. \right. \right. \\
& \left. \left. \left. + \zeta_2 \right) H_0 + \left(-2 + 2H_{-1} + \frac{4}{3}H_{-1}^3 + \frac{7}{3}\zeta_3 \right) H_0^2 + \left(-\frac{2}{3}H_{-1}^2 + \frac{1}{6}\zeta_2 \right) H_0^3 + \frac{1}{6}H_{-1}H_0^4 - \frac{1}{60}H_0^5 + \left(-4\zeta_2 \right. \right. \right. \\
& \left. \left. \left. - \frac{28\zeta_2^2}{5} \right) H_{-1} + \left(-8 + \left(8 + 4\zeta_2 \right) H_{-1} + \frac{16}{3}H_{-1}^3 + \frac{4}{3}\zeta_3 \right) H_{0,-1} + \left(-4 - 8H_{-1}^2 - 2\zeta_2 \right) H_{0,0,-1} + \left(\right. \right. \\
& \left. \left. -8 - 16H_{-1}^2 - 4\zeta_2 \right) H_{0,-1,-1} + 8H_{-1}H_{0,0,0,-1} + 16H_{-1}H_{0,0,-1,-1} + 32H_{-1}H_{0,-1,-1,-1} - 4H_{0,0,0,0,-1} \right. \\
& \left. \left. - 8H_{0,0,0,-1,-1} - 16H_{0,0,-1,-1,-1} - 32H_{0,-1,-1,-1,-1} - \frac{8}{3}H_{-1}^3\zeta_2 + 8H_{-1}^2\zeta_3 - \frac{8}{3}\zeta_2\zeta_3 + 12\zeta_5 \right) \right\} \right]. \tag{D.1}
\end{aligned}$$

$$\begin{aligned}
F_S^{(2)} = & C_F^2 \left[\frac{1}{\varepsilon^2} \left\{ 2 + 4\xi H_0 + 2\xi^2 H_0^2 \right\} + \frac{1}{\varepsilon} \left\{ 4 + \xi \left(\eta \left(-4(-1+x)(1+x)H_0 - 2 \left(-1 + 8x + x^2 \right) H_0^2 \right) \right. \right. \right. \\
& \left. \left. \left. - 8H_{-1}H_0 + 8H_{0,-1} - 4\zeta_2 \right) + \xi^2 \left(-8H_{-1}H_0^2 + 2H_0^3 + 8H_0H_{0,-1} - 4H_0\zeta_2 \right) - 16x\eta H_0 \right\} \right]
\end{aligned}$$

$$\begin{aligned}
& + \left\{ 29 + \eta^2 \left(4H_0H_{0,-1}P_{1063} + 4H_0^2H_1P_{1100} - 8H_0H_{0,1}P_{1115} + 8H_{0,0,1}P_{1121} + 2H_{-1}H_0^2P_{1122} \right. \right. \\
& - 4H_{0,0,-1}P_{1123} + 2H_0^2P_{1142} + \left. \left(\frac{16}{3}H_0^3H_1P_{1171} - 4H_0^2H_{0,-1}P_{1178} - 8H_0H_{0,0,-1}P_{1194} \right. \right. \\
& + 48H_{0,0,0,-1}P_{1195} - 8H_0^2H_{0,1}P_{1196} + 16H_0H_{0,0,1}P_{1214} - 32H_{0,0,0,1}P_{1215} - \frac{2}{3}H_0^3P_{1223} + \frac{1}{6}H_0^4P_{1281} \\
& + \left. \left. \left(16H_0H_1P_{1170} - 16H_{0,1}P_{1170} - 4H_0P_{1210} + 4H_0^2P_{1216} - 8H_{0,-1}P_{1218} \right) \zeta_2 + \frac{2}{5}P_{1314}\zeta_2^2 \right) x_+ \right. \\
& - 8P_{1105}\zeta_3 \left. \right) + \xi^2 \left(-32(-1+x)^2(1+x)^2\eta x_+^2\zeta_3H_0 + 16H_{-1}^2H_0^2 - 8H_{-1}H_0^3 + \left(32H_0H_1 \right. \right. \\
& - 32H_{0,-1} \left. \left. \right) H_{0,1} + 16H_{0,1}^2 - 32H_{-1}H_0H_{0,-1} - 32H_0H_1H_{0,-1} + 8H_{0,-1}^2 - 64H_1H_{0,0,1} + 64H_1H_{0,0,-1} \right. \\
& - 64H_0H_{0,1,1} + 32H_0H_{0,1,-1} + 32H_0H_{0,-1,1} + 16H_0H_{0,-1,-1} + 64H_{0,0,1,1} + 32H_{0,-1,0,1} + 16H_{-1}H_0\zeta_2 \\
& + 16H_1\zeta_3 \left. \right) + \eta \left(x_+\zeta_2 \left(-2 \left(-3 + 75x - 53x^2 + 5x^3 \right) - 4 \left(-5 + 49x - 53x^2 + x^3 \right) H_{-1} \right) \right. \\
& + 4(-3+x)(-1+3x)H_0 + 12 \left(5 - 14x + 5x^2 \right) H_{-1}H_0 - 12 \left(5 - 14x + 5x^2 \right) H_{0,-1} + 160x^2H_0x_+^2\zeta_3 \left. \right) \\
& + \xi \left(8H_{-1}^2H_0 - 16H_{-1}H_{0,-1} + 16H_{0,-1,-1} + 128(-1+x)x(1+x)\eta H_0x_+^2\zeta_3 \right) + \left(32(-1+x)H_0H_1 \right. \\
& - 32(-1+x)H_{0,1} \left. \right) x_+ + 288x \log(2)x_+^2\zeta_2 \left. \right\} + \varepsilon \left\{ \log(2)x_+^2\zeta_2 \left(24 \left(11 + 74x + 11x^2 \right) + 48 \left(3 + 8x \right. \right. \right. \\
& + 3x^2 \left. \left. \right) H_1 + 48 \left(3 + 8x + 3x^2 \right) H_{-1} \right) + \xi \left(\eta \left(16(-1+x)(1+x)H_0H_1 - 16(-1+x)(1+x)H_{0,1} \right. \right. \right. \\
& + 256(-1+x)x(1+x)H_{-1}H_0x_+^2\zeta_3 \left. \right) - \frac{16}{3}H_{-1}^3H_0 + 16H_{-1}^2H_{0,-1} - 32H_{-1}H_{0,-1,-1} \\
& + 32H_{0,-1,-1,-1} \left. \right) + \eta \left(\log(2)x_+^2\zeta_2 \left(-192x^2H_{0,1} - 192x^2H_{0,-1} \right) + 12(-3+x)(-1+3x)H_0 \right. \\
& + 8 \left(3 - 86x + 3x^2 \right) H_{-1}H_0 - 4 \left(65 - 174x + 65x^2 \right) H_{-1}^2H_0 + 256xH_0H_1 - 256xH_{0,1} + \left(-8 \left(3 \right. \right. \\
& - 86x + 3x^2 \left. \right) + 8 \left(65 - 174x + 65x^2 \right) H_{-1} \left. \right) H_{0,-1} - 8 \left(65 - 174x + 65x^2 \right) H_{0,-1,-1} + \left(2 \left(95 \right. \right. \\
& - 61x - 205x^2 + 83x^3 \left. \right) H_{-1}H_0^2 - 4 \left(-21 + 93x - 101x^2 + 13x^3 \right) \zeta_3H_{-1} - 8 \left(-23 + 43x \right. \\
& - 39x^2 + 27x^3 \left. \right) H_{-1}H_{0,0,-1} + 8 \left(-23 + 43x - 39x^2 + 27x^3 \right) H_{0,0,-1,-1} + \left(8 \left(-5 - 90x + 67x^2 \right. \right. \\
& + 4x^3 \left. \right) - 4 \left(29 - 115x - 7x^2 + 5x^3 \right) H_{-1} + 4 \left(-29 + 121x - 125x^2 + 25x^3 \right) H_{-1}^2 \left. \right) \zeta_2 \left. \right) x_+ \\
& + \left(-48H_0H_{0,-1}^2P_{1081} + 128H_0H_1H_{0,0,-1}P_{1085} + 32H_0H_{0,-1,0,1}P_{1125} - 32H_{0,-1}H_{0,0,1}P_{1131} \right. \\
& + \left. \left(96H_0^2H_1P_{1067} - 32H_0H_{0,-1}P_{1095} \right) H_{0,1} - 32x \left(6 - 29x + 6x^2 \right) H_0H_{0,1}^2 - 16H_0H_{0,1}P_{1134}\zeta_2 \right. \\
& + 320x^2H_{-1}H_0\zeta_3 \left. \right) x_+^2 + \eta^2 \left(48H_0H_{0,1,-1}P_{1066} + 48H_0H_{0,-1,1}P_{1066} + 8H_{0,1}^2P_{1084} + 32H_{0,-1,0,1}P_{1094} \right. \\
& + 32H_0H_{0,1,1}P_{1096} - 8H_0H_{0,-1,-1}P_{1114} + 4H_{0,-1}^2P_{1120} + 4H_0^2H_1P_{1135} + 2H_0^2P_{1136} - 2H_{-1}^2H_0^2P_{1139} \\
& + 8 \left(2 - 3x + 2x^2 \right) \left(3 + 2x + 3x^2 \right) H_0^2H_1^2 + \left(-32H_{0,-1}P_{1099} - 16H_0H_1P_{1124} + 8H_0P_{1126} \right) H_{0,1} \\
& + \left(-16H_{-1}H_0P_{1064} - 16H_0H_1P_{1065} - 4H_0P_{1151} \right) H_{0,-1} + \left(-8P_{1144} + 32 \left(1 + x + x^2 \right) \left(5 \right. \right. \\
& - 2x + 5x^2 \left. \right) H_1 \left. \right) H_{0,0,1} + \left(16H_1P_{1098} + 4P_{1157} \right) H_{0,0,-1} - 32 \left(1 + x + x^2 \right) \left(5 - 2x + 5x^2 \right) H_{0,0,1,1}
\end{aligned}$$

$$\begin{aligned}
& + \left(-16H_0^2H_{0,1,-1}P_{1112} - 16H_0^2H_{0,-1,1}P_{1112} + \frac{32}{3}H_0^3H_1^2P_{1174} + 64H_{0,0,1,0,1}P_{1180} \right. \\
& - 48H_0H_{0,0,-1,-1}P_{1181} - 64H_{0,0,1,0,-1}P_{1191} + \frac{1}{6}H_0^4P_{1193} + 16H_0^2H_{0,1,1}P_{1203} - 32H_0H_{0,0,1,-1}P_{1213} \\
& - 32H_0H_{0,0,-1,1}P_{1213} + 32H_{0,0,-1,0,1}P_{1220} + 8H_0^2H_{0,-1,-1}P_{1230} - 32H_{0,0,0,1,-1}P_{1240} \\
& - 32H_{0,0,0,-1,1}P_{1240} + 32H_{0,0,-1,0,-1}P_{1250} + 96H_{0,0,0,-1,-1}P_{1252} - 32H_0H_{0,0,1,1}P_{1258} \\
& + 64H_{0,0,0,1,1}P_{1265} + \frac{1}{30}H_0^5P_{1298} + \frac{2}{3}H_0^3P_{1306} - 16H_{0,0,0,0,1}P_{1311} + 16H_{0,0,0,0,-1}P_{1312} \\
& + \left(-\frac{64}{3}H_{-1}H_0^3P_{1174} + \frac{2}{3}H_0^4P_{1197} + \frac{8}{3}H_0^3P_{1245} \right) H_1 + \left(-\frac{2}{3}H_0^4P_{1232} + \frac{2}{3}H_0^3P_{1293} \right) H_{-1} \\
& + \left(\frac{8}{3}H_0^3P_{1192} + 16H_{-1}H_0^2P_{1200} - 4H_0^2P_{1234} \right) H_{0,1} + \left(-16xH_{-1}H_0^2P_{1083} + 16H_0^2H_1P_{1176} \right. \\
& - \frac{4}{3}H_0^3P_{1261} + 2H_0^2P_{1276} \left. \right) H_{0,-1} + \left(32H_{0,1}P_{1164} - 64H_{-1}H_0P_{1177} + 256H_0H_1P_{1184} \right. \\
& - 8H_0^2P_{1233} - 8H_0P_{1274} \left. \right) H_{0,0,1} + \left(64H_{-1}H_0P_{1101} - 32H_{0,1}P_{1228} + 4H_0P_{1231} + 4H_0^2P_{1241} \right. \\
& - 16H_{0,-1}P_{1285} \left. \right) H_{0,0,-1} + \left(32H_{-1}P_{1179} - 64H_1P_{1247} + 16P_{1292} + 16H_0P_{1300} \right) H_{0,0,0,1} + \left(\right. \\
& - 96H_{-1}P_{1113} + 32H_1P_{1290} - 8H_0P_{1304} - 4P_{1313} \left. \right) H_{0,0,0,-1} + \left(128H_{0,1,1}P_{1174} + 32H_0H_1^2P_{1175} \right. \\
& - 32H_{0,-1,1}P_{1224} - 32H_{0,1,-1}P_{1225} - 16H_{0,0,1}P_{1255} - 4H_0P_{1271} + 16H_{0,-1,-1}P_{1282} - \frac{4}{3}H_0^3P_{1296} \\
& + 2H_0^2P_{1299} + 8H_{0,0,-1}P_{1301} + \left(-64H_{-1}H_0P_{1175} - 8H_0^2P_{1235} + 8H_0P_{1260} \right) H_1 + \left(-8H_0^2P_{1190} \right. \\
& + 4H_0P_{1227} \left. \right) H_{-1} + \left(64H_{-1}P_{1175} - 32H_1P_{1199} - 8P_{1251} \right) H_{0,1} + \left(-32H_{-1}P_{1185} - 4P_{1201} \right. \\
& - 16H_0P_{1209} + 32H_1P_{1224} \left. \right) H_{0,-1} - 4P_{1297}\zeta_3 \zeta_2 + \left(-\frac{8}{5}H_{-1}P_{1182} - \frac{8}{5}H_1P_{1303} + \frac{2P_{1315}}{5} \right. \\
& - \frac{2}{5}H_0P_{1316} \left. \right) \zeta_2^2 + \left(-16H_0H_1P_{1204} - 16H_{0,-1}P_{1244} + 16H_{0,1}P_{1254} - \frac{8}{3}H_0^2P_{1272} - \frac{4}{3}H_0P_{1302} \right) \zeta_3 \\
& + 6P_{1307}\zeta_5 \left. \right) x_+ + \left(-4H_1P_{1148} - \frac{2P_{1159}}{3} \right) \zeta_3 + \xi^2 \left(-64(-1+x)^2(1+x)^2\eta x_+^2\zeta_3H_{-1}H_0 \right. \\
& - \frac{64}{3}H_{-1}^3H_0^2 + 16H_{-1}^2H_0^3 + \left(128H_{-1}H_0H_{0,-1} - 96H_{0,-1}^2 \right) H_1 + \left(\left(-128H_{-1}H_0 + 64H_{0,-1} \right) H_1 \right. \\
& + 64H_0H_1^2 + 128H_{-1}H_{0,-1} \left. \right) H_{0,1} + \left(32H_1 - 64H_{-1} \right) H_{0,1}^2 + 64H_{-1}^2H_0H_{0,-1} - 64H_0H_1^2H_{0,-1} \\
& - 32H_{-1}H_{0,-1}^2 + \left(256H_{-1}H_1 - 128H_1^2 \right) H_{0,0,1} + \left(-256H_{-1}H_1 + 128H_1^2 \right) H_{0,0,-1} \\
& + \left(256H_{-1}H_0 - 192H_0H_1 - 256H_{0,-1} \right) H_{0,1,1} + \left(-128H_{-1}H_0 + 64H_0H_1 - 64H_{0,1} \right. \\
& + 192H_{0,-1} \left. \right) H_{0,1,-1} + \left(-128H_{-1}H_0 + 64H_0H_1 + 192H_{0,1} + 192H_{0,-1} \right) H_{0,-1,1} + \left(-64H_{-1}H_0 \right. \\
& + 64H_0H_1 - 320H_{0,1} + 32H_{0,-1} \left. \right) H_{0,-1,-1} + \left(256H_1 - 256H_{-1} \right) H_{0,0,1,1} - 256H_1H_{0,0,1,-1} \\
& - 256H_1H_{0,0,-1,1} + 256H_1H_{0,0,-1,-1} + 192H_0H_{0,1,1,1} - 64H_0H_{0,1,-1,-1} - 128H_{-1}H_{0,-1,0,1} \\
& - 64H_0H_{0,-1,1,-1} - 64H_0H_{0,-1,-1,1} + 32H_0H_{0,-1,-1,-1} - 384H_{0,0,1,1,1} + 768H_{0,0,1,1,-1} \\
& + 256H_{0,0,1,-1,1} - 256H_{0,0,-1,1,1} - 64H_{0,1,0,1,1} + 256H_{0,1,0,1,-1} - 256H_{0,-1,0,1,1} + 128H_{0,-1,0,1,-1} \\
& + 128H_{0,-1,0,-1,1} - 256H_{0,-1,1,0,1} + 320H_{0,-1,-1,0,1} - 32H_{-1}^2H_0\zeta_2 + \left(-64H_{-1}H_1 + 32H_1^2 \right) \zeta_3 \left. \right)
\end{aligned}$$

$$\begin{aligned}
& + \left(-192(-1+x)H_{-1}H_0H_1 + 32(-1+x)H_0H_1^2 + \left(-64(-1+x)H_1 + 192(-1+x)H_{-1} \right) H_{0,1} \right. \\
& + 192(-1+x)H_1H_{0,-1} + 64(-1+x)H_{0,1,1} - 192(-1+x)H_{0,1,-1} - 192(-1+x)H_{0,-1,1} \\
& - 32(-1+x)H_1\zeta_2 \Big) x_+ - 48c_1x_+^2 + \left(-\frac{113}{2}(1+x)^2 - 8(3+8x+3x^2)H_{-1}H_0^2H_1 + 16(7+12x \right. \\
& + 7x^2)H_{-1}H_0H_{0,1} - 16(11+16x+11x^2)H_{-1}H_{0,0,1} + 16(11+16x+11x^2)H_{0,0,1,-1} + 16(11 \\
& + 16x+11x^2)H_{0,0,-1,1} + \left. \left(-48(3+8x+3x^2)H_{-1}H_1 + 48(3+8x+3x^2)H_{-1,1} \right) \zeta_2 \right) x_+^2 \Big\} \\
& + C_F C_A \left[\frac{1}{\varepsilon^2} \left\{ \frac{11}{3} + \frac{11\xi H_0}{3} \right\} + \frac{1}{\varepsilon} \left\{ -\frac{49}{9} + \xi \left(\eta \left(-\frac{4}{3}x^2H_0^3 - 2(-1+3x^2)H_0\zeta_2 \right) - \frac{67}{9}H_0 \right. \right. \right. \\
& + 4H_{-1}H_0 - 4H_0H_1 + 4H_{0,1} - 4H_{0,-1} \Big) + \eta \left(-4x^2H_0^2 - 2(-1+3x^2)\zeta_2 \right) + \xi^2 \left(-4H_0H_{0,1} \right. \\
& + 4H_0H_{0,-1} + 8H_{0,0,1} - 8H_{0,0,-1} - 2\zeta_3 \Big) \Big\} + \left\{ -\frac{869}{27} + \eta^2x_+ \left(4H_0^2H_{0,-1}P_{1211} - 8H_{0,0,0,-1}P_{1236} \right. \right. \\
& + 8H_{0,0,0,1}P_{1248} - \frac{1}{2}x^2(1+7x+4x^2+4x^3)H_0^4 - 32x^2(7-5x+x^2+x^3)H_0H_{0,0,1} + \left(4H_{0,-1}P_{1217} \right. \\
& + 2(1+x+3x^2+x^3+x^4+x^5)H_0^2 \Big) \zeta_2 + \frac{1}{5}P_{1162}\zeta_2^2 - 2H_0P_{1243}\zeta_3 \Big) + \xi^2 \left((-8H_1+8H_{-1})\zeta_3 \right. \\
& + (16H_{-1}H_0 - 16H_0H_1 - 8H_{0,-1})H_{0,1} - 4H_{0,1}^2 - 16H_{-1}H_0H_{0,-1} + 16H_0H_1H_{0,-1} + 12H_{0,-1}^2 \\
& + (32H_1 - 32H_{-1})H_{0,0,1} + (-32H_1+32H_{-1})H_{0,0,-1} + 24H_0H_{0,1,1} - 8H_0H_{0,1,-1} - 8H_0H_{0,-1,1} \\
& - 8H_0H_{0,-1,-1} - 32H_{0,0,1,1} + 32H_{0,0,1,-1} + 32H_{0,0,-1,1} - 32H_{0,0,-1,-1} \Big) + \eta \left(-\frac{22}{27}(11-114x \right. \\
& + 11x^2)H_0 + \frac{2}{9}(53-102x+53x^2)H_{-1}H_0 + 8(-3+x^2)H_0^2H_1 - \frac{2}{9}(53-102x+53x^2)H_{0,-1} \\
& + \left(\frac{2}{3}(49-17x+79x^2+13x^3)\zeta_3 + \frac{1}{9}(-67+74x+11x^2+50x^3)H_0^2 + \frac{1}{3}(61+175x-35x^2 \right. \\
& + 79x^3)H_{-1}H_0^2 + 16(-3-x+3x^2+5x^3)H_{0,0,1} - \frac{2}{3}(-53-95x+235x^2+193x^3)H_{0,0,-1} + \left(\frac{2}{9}(91 \right. \\
& + 562x-539x^2+34x^3) - \frac{2}{3}(25-137x+169x^2+7x^3)H_{-1} \Big) \zeta_2 \Big) x_+ + \left(\frac{1}{9}H_0^3P_{1061} + 8H_0^2H_{0,1}P_{1069} \right. \\
& + 16H_0H_{0,0,-1}P_{1070} - \frac{2}{3}H_0P_{1079}\zeta_2 \Big) x_+^2 + \xi \left(8H_0H_1(1+3H_{-1}) + \eta \left(\frac{4}{3}(3+x^2)H_{-1}H_0^3 + \frac{4}{3}(-5 \right. \right. \\
& + x^2)H_0^3H_1 + \left(16H_{-1}H_0 + 16(-2+x^2)H_0H_1 - 12(-3+x^2)H_{0,1} \right) \zeta_2 \right) - \frac{104}{3}H_{-1}^2H_0 - 4H_0H_1^2 \\
& + (-8+8H_1-24H_{-1})H_{0,1} + \left(-24H_1 + \frac{208H_{-1}}{3} \right) H_{0,-1} - 8H_{0,1,1} + 24H_{0,1,-1} + 24H_{0,-1,1} \\
& - \frac{208}{3}H_{0,-1,-1} + 4H_1\zeta_2 \Big) + \left(16(3+5x+3x^2)H_0H_{0,1} - 2(19+64x+19x^2)H_0H_{0,-1} \right) x_+^2 \\
& - 144x \log(2)x_+^2\zeta_2 \Big\} + \varepsilon \left\{ \log(2)x_+^2\zeta_2 \left(-12(11+74x+11x^2) - 24(3+8x+3x^2)H_1 - 24(3 \right. \right. \\
& + 8x+3x^2)H_{-1} \Big) + \eta^2x_+ \left(H_{-1}H_0^4P_{1187} + 128H_0H_{0,0,1,1}P_{1188} - 16H_0^2H_{0,1,1}P_{1189} + 8H_0^2H_{0,1,-1}P_{1208} \right. \\
& + 8H_0^2H_{0,-1,1}P_{1208} - 32H_{0,0,1,0,1}P_{1221} - 64H_{0,0,-1,0,-1}P_{1222} - 8H_0^2H_{0,-1,-1}P_{1246} + 16H_{0,0,-1,0,1}P_{1257}
\end{aligned}$$

$$\begin{aligned}
& + 16H_{0,0,1,0,-1}P_{1268} - \frac{4}{3}H_0^3H_{0,1}P_{1269} + 32H_{0,0,0,1,-1}P_{1289} + 32H_{0,0,0,-1,1}P_{1289} - 16H_{0,0,0,1,1}P_{1305} \\
& + 8H_{0,0,0,0,1}P_{1308} - 8H_{0,0,0,0,-1}P_{1309} - 16H_{0,0,0,-1,-1}P_{1310} - \frac{1}{30}x^2(5+43x+24x^2+24x^3)H_0^5 + (-8 \\
& - 8x - 11x^2 - 5x^3)H_0^4H_1 + (-8H_0^2H_1P_{1202} + \frac{4}{3}H_0^3P_{1295} - 8(8+8x+15x^2+x^3)H_{-1}H_0^2)H_{0,-1} \\
& + (32H_{-1}H_0P_{1173} + 32H_{0,1}P_{1206} - 32H_0H_1P_{1207} - 16H_{0,-1}P_{1267} + 4H_0^2P_{1275})H_{0,0,1} \\
& + (-16H_{0,1}P_{1238} + 32H_{0,-1}P_{1242} - 8H_0^2P_{1249})H_{0,0,-1} + (-16H_{-1}P_{1219} + 16H_1P_{1259} \\
& - 8H_0P_{1291})H_{0,0,0,1} + (16H_{-1}P_{1229} - 16H_1P_{1239} + 8H_0P_{1287})H_{0,0,0,-1} + (-4H_{-1}H_0^2P_{1186} \\
& - 16H_{0,0,-1}P_{1266} + 8H_{0,-1,1}P_{1277} + 8H_{0,1,-1}P_{1278} - 8H_{0,-1,-1}P_{1283} - 4H_0H_{0,1}P_{1286} + 8H_{0,0,1}P_{1288} \\
& + \frac{2}{3}H_0^3P_{1294} + 4x^2(3+5x+4x^2+4x^3)H_0^2H_1 + (16H_{-1}P_{1205} - 16H_1P_{1237} + 4H_0P_{1284})H_{0,-1} \\
& + 2P_{1273}\zeta_3)\zeta_2 + (\frac{4}{5}H_{-1}P_{1263} - \frac{2}{5}H_0P_{1264} - \frac{4}{5}H_1P_{1270})\zeta_2^2 + (16H_0H_1P_{1172} - 16H_{-1}H_0P_{1198} \\
& - 4H_{0,1}P_{1256} + 2H_0^2P_{1279} + 4H_{0,-1}P_{1280})\zeta_3 + P_{1161}\zeta_5) + \xi(8H_0H_1^2(1+3H_{-1}) - 24H_0H_1H_{-1}(2 \\
& + 3H_{-1}) + 4(-2+H_1)H_1\zeta_2 + \eta(-\frac{4}{3}(3+5x^2)H_{-1}^2H_0^3 + (4(-1+x)(1+x)H_0 + \frac{8}{3}(1 \\
& + 7x^2)H_{-1}H_0^3)H_1 - \frac{4}{3}(1+7x^2)H_0^3H_1^2 - 4(-1+x)(1+x)H_{0,1} + (-8(1+3x^2)H_{-1}^2H_0 + 16(- \\
& - 1+5x^2)H_{-1}H_0H_1 - 8(-1+5x^2)H_0H_1^2 + (96x^2H_1 - 96x^2H_{-1})H_{0,1} - 8(1+13x^2)H_{0,1,1})\zeta_2) \\
& + 80H_{-1}^3H_0 - \frac{8}{3}H_0H_1^3 + (-16H_1(1+3H_{-1}) + 8H_1^2 + 48H_{-1} + 72H_{-1}^2)H_{0,1} + (48H_1(1+3H_{-1}) \\
& - 24H_1^2 - 240H_{-1}^2)H_{0,-1} + (16 - 16H_1 + 48H_{-1})H_{0,1,1} + (-48 + 48H_1 - 144H_{-1})H_{0,1,-1} \\
& + (-48 + 48H_1 - 144H_{-1})H_{0,-1,1} + (-144H_1 + 480H_{-1})H_{0,-1,-1} + 16H_{0,1,1,1} - 48H_{0,1,1,-1} \\
& - 48H_{0,1,-1,1} + 144H_{0,1,-1,-1} - 48H_{0,-1,1,1} + 144H_{0,-1,1,-1} + 144H_{0,-1,-1,1} - 480H_{0,-1,-1,-1} + (\\
& - 12(-1+x)(1+x)H_0H_{0,1} - 8(-1+x)(1+x)H_{-1}H_0H_{0,-1} - 80(-1+x)(1+x)H_0H_1H_{0,-1})x_+^2) \\
& + \eta(\log(2)x_+^2\zeta_2(96x^2H_{0,1} + 96x^2H_{0,-1}) - \frac{2}{81}(1061 - 16710x + 1061x^2)H_0 + \frac{4}{27}(341 - 1266x \\
& + 341x^2)H_{-1}H_0 + \frac{2}{9}(47 + 306x + 47x^2)H_{-1}^2H_0 - 24xH_0H_1 + 24xH_{0,1} + (-\frac{4}{27}(341 - 1266x \\
& + 341x^2) - \frac{4}{9}(47 + 306x + 47x^2)H_{-1})H_{0,-1} + \frac{4}{9}(47 + 306x + 47x^2)H_{0,-1,-1} + (\frac{1}{27}(-341 \\
& + 2599x - 1127x^2 - 395x^3)H_0^2 + \frac{1}{9}(-137 - 1037x + 115x^2 - 173x^3)H_{-1}H_0^2 + (-65 - 155x \\
& - x^2 - 91x^3)H_{-1}^2H_0^2 + (2(-1 + 61x - 45x^2 + 17x^3)H_0^2 + 4(15 + 17x + 7x^2 + 9x^3)H_{-1}H_0^2)H_1 \\
& + 2(-13 + 13x - 21x^2 + 5x^3)H_0^2H_1^2 + (-4(5 - 65x + 81x^2 + 11x^3) + 8(3 + 45x - 37x^2 \\
& + 5x^3)H_1 - 8(5 + 43x - 19x^2 + 19x^3)H_{-1})H_{0,0,1} + (\frac{2}{9}(-277 - 1285x + 2207x^2 + 587x^3) \\
& + 8(-35 - 77x + 53x^2 + 11x^3)H_1 + 4(61 + 191x - 35x^2 + 95x^3)H_{-1})H_{0,0,-1} - 8(3 + 45x \\
& - 37x^2 + 5x^3)H_{0,0,1,1} + 8(5 + 43x - 19x^2 + 19x^3)H_{0,0,1,-1} + 8(5 + 43x - 19x^2 + 19x^3)H_{0,0,-1,1}
\end{aligned}$$

$$\begin{aligned}
& -4(61 + 191x - 35x^2 + 95x^3)H_{0,0,-1,-1} + \left(-\frac{4}{27}(-493 - 1480x + 2783x^2 + 206x^3) - 48(-1 - 2x + 3x^2 + 2x^3)H_{-1}H_1 - \frac{2}{9}(61 + 349x - 623x^2 + 277x^3)H_{-1} - 6(-17 + 33x - 49x^2 + x^3)H_{-1}^2\right)\zeta_2 + \left(\frac{1}{9}(1055 + 3683x - 4591x^2 - 739x^3) - 2(-95 - 53x + 9x^2 + 51x^3)H_1 - 2(91 - 55x + 211x^2 + 65x^3)H_{-1}\right)\zeta_3)x_+ + \left(\frac{1}{12}H_0^4P_{1062} + 32H_0H_{0,0,1,-1}P_{1071} + 32H_0H_{0,0,-1,1}P_{1071} + 24H_0H_{0,-1}^2P_{1082} + 16H_0H_{0,1}^2P_{1086} - 16H_0H_{0,-1,0,1}P_{1102} - 32H_0H_{0,0,-1,-1}P_{1104} + 4H_{0,0,0,1}P_{1137} + \frac{2}{3}H_0^3H_1P_{1140} + 4H_0H_{0,0,1}P_{1141} - 2H_{0,0,0,-1}P_{1143} + \frac{1}{3}H_{-1}H_0^3P_{1152} + H_0^2H_{0,-1}P_{1154} + \frac{1}{27}H_0^3P_{1158} + (-48H_0^2H_1P_{1067} + 16H_{-1}H_0^2P_{1069} - 16H_0H_{0,-1}P_{1103} - 2H_0^2P_{1145})H_{0,1} + (-96H_0H_1P_{1068} + 32H_{-1}H_0P_{1070} - 2H_0P_{1156})H_{0,0,-1} + (4H_{-1}H_0P_{1080} - 4H_{0,1}P_{1097} + 4H_0H_1P_{1130} - 4H_{0,-1}P_{1138} + \frac{1}{2}H_0^2P_{1150} + \frac{1}{9}H_0P_{1153})\zeta_2 + \frac{1}{5}P_{1155}\zeta_2^2 + \frac{4}{3}H_0P_{1146}\zeta_3)x_+^2\right) + \xi^2\left(\left(-64H_{-1}H_0H_{0,-1} + 48H_{0,-1}^2\right)H_1 + \left(-32H_{-1}^2H_0 + (64H_{-1}H_0 - 32H_{0,-1})H_1 - 32H_0H_1^2 + 32H_{-1}H_{0,-1}\right)H_{0,1} + \left(-16H_1 + 16H_{-1}\right)H_{0,1}^2 + 32H_{-1}^2H_0H_{0,-1} + 32H_0H_1^2H_{0,-1} - 48H_{-1}H_{0,-1}^2 + \left(-128H_{-1}H_1 + 64H_1^2 + 64H_{-1}^2\right)H_{0,0,1} + \left(128H_{-1}H_1 - 64H_1^2 - 64H_{-1}^2\right)H_{0,0,-1} + \left(-96H_{-1}H_0 + 96H_0H_1 + 16H_{0,1} + 80H_{0,-1}\right)H_{0,1,1} + \left(32H_{-1}H_0 - 32H_0H_1 + 16H_{0,1} - 48H_{0,-1}\right)H_{0,1,-1} + \left(32H_{-1}H_0 - 32H_0H_1 - 48H_{0,1} - 48H_{0,-1}\right)H_{0,-1,1} + \left(32H_{-1}H_0 - 32H_0H_1 + 16H_{0,1} - 48H_{0,-1}\right)H_{0,-1,-1} + \left(-128H_1 + 128H_{-1}\right)H_{0,0,1,1} + \left(128H_1 - 128H_{-1}\right)H_{0,0,1,-1} + \left(128H_1 - 128H_{-1}\right)H_{0,0,-1,1} + \left(-128H_1 + 128H_{-1}\right)H_{0,0,-1,-1} - 112H_0H_{0,1,1,1} + 16H_0H_{0,1,1,-1} + 16H_0H_{0,1,-1,1} + 16H_0H_{0,1,-1,-1} + 16H_0H_{0,-1,1,1} + 16H_0H_{0,-1,1,-1} + 16H_0H_{0,-1,-1,-1} + 128H_{0,0,1,1,1} - 256H_{0,0,1,1,-1} - 128H_{0,0,1,-1,1} + 128H_{0,0,1,-1,-1} + 128H_{0,0,-1,1,-1} + 128H_{0,0,-1,-1,1} + 256H_{0,0,-1,-1,-1} - 64H_{0,1,0,1,-1} + 64H_{0,-1,0,1,1} + 192H_{0,-1,0,-1,-1} + 64H_{0,-1,1,0,1} + \left(32H_{-1}H_1 - 16H_1^2 - 16H_{-1}^2\right)\zeta_3) + 24c_1xx_+^2 + \left(-\frac{6437}{162}(1+x)^2 + (-240xH_0 - 8(5-8x+5x^2)H_{-1}H_0 + 8(5-24x+5x^2)H_0H_1 + 40(5+12x+5x^2)H_{0,-1})H_{0,1} - 16(2+x)(1+2x)H_{0,1}^2 + 2(23+152x+23x^2)H_0H_{0,-1} - 64xH_{-1}H_0H_{0,-1} + 320xH_0H_1H_{0,-1} - 2(59+176x+59x^2)H_{0,-1}^2 + 8(3+44x+3x^2)H_0H_{0,1,1} - 32(5+17x+5x^2)H_0H_{0,1,-1} - 32(5+17x+5x^2)H_0H_{0,-1,1} + 12(19+64x+19x^2)H_0H_{0,-1,-1} - 8(15+32x+15x^2)H_{0,-1,0,1} - 24(3+8x+3x^2)H_{-1,1}\zeta_2)x_+^2\right)\left.\right\} \\
& + C_F n_l T_F \left[\frac{1}{\varepsilon^2} \left\{ -\frac{4}{3} - \frac{4\xi H_0}{3} \right\} + \frac{1}{\varepsilon} \left\{ \frac{20}{9} + \frac{20\xi H_0}{9} \right\} + \left\{ \frac{196}{27} + \eta \left(\frac{16}{27} (7 - 48x + 7x^2) H_0 - \frac{16}{9} (5 - 12x + 5x^2) H_{-1} H_0 + \frac{4}{9} (5 - 12x + 5x^2) H_0^2 + \frac{16}{9} (5 - 12x + 5x^2) H_{0,-1} - \frac{8}{9} (-1 - 12x + 11x^2) \zeta_2 \right) \right\} \right]
\end{aligned}$$

$$\begin{aligned}
& + \xi \left(\frac{16}{3} H_{-1}^2 H_0 - \frac{8}{3} H_{-1} H_0^2 + \frac{4}{9} H_0^3 - \frac{32}{3} H_{-1} H_{0,-1} + \frac{16}{3} H_{0,0,-1} + \frac{32}{3} H_{0,-1,-1} + \left(\frac{8}{3} H_0 + \frac{16}{3} H_{-1} \right) \zeta_2 \right. \\
& \left. - \frac{16\zeta_3}{3} \right) \Bigg\} + \varepsilon \left\{ \frac{1706}{81} + \xi \left(\left(-\frac{16H_0}{3} + 32H_{-1} \right) \zeta_3 - \frac{32}{3} H_{-1}^3 H_0 + 8H_{-1}^2 H_0^2 - \frac{8}{3} H_{-1} H_0^3 \right. \right. \\
& + 32H_{-1}^2 H_{0,-1} - 32H_{-1} H_{0,0,-1} - 64H_{-1} H_{0,-1,-1} + 16H_{0,0,0,-1} + 32H_{0,0,-1,-1} + 64H_{0,-1,-1,-1} + \left(\right. \\
& - 8H_{-1} H_0 + 2H_0^2 - 16H_{-1}^2 + 24H_{0,-1} \Big) \zeta_2 - \frac{96\zeta_2^2}{5} \Big) + \eta \left(\frac{16}{81} (77 - 636x + 77x^2) H_0 + \frac{32}{9} (5 - 18x \right. \\
& + 5x^2) H_{-1}^2 H_0 + \frac{4}{27} (37 - 246x + 37x^2) H_0^2 + \frac{8}{27} (5 - 18x + 5x^2) H_0^3 + \frac{1}{3} (1 + x^2) H_0^4 + \left(-\frac{16}{27} (37 \right. \\
& - 246x + 37x^2) H_0 - \frac{16}{9} (5 - 18x + 5x^2) H_0^2 \Big) H_{-1} + \left(\frac{16}{27} (37 - 246x + 37x^2) - \frac{64}{9} (5 - 18x \right. \\
& + 5x^2) H_{-1} \Big) H_{0,-1} + \frac{32}{9} (5 - 18x + 5x^2) H_{0,0,-1} + \frac{64}{9} (5 - 18x + 5x^2) H_{0,-1,-1} + \left(-\frac{8}{27} (-41 \right. \\
& - 246x + 115x^2) + \frac{4}{9} (5 - 36x + 5x^2) H_0 + \frac{32}{9} (5 - 18x + 5x^2) H_{-1} \Big) \zeta_2 - \frac{64}{9} (1 - 9x + 4x^2) \zeta_3 \Big) \Bigg\} \\
& + C_F T_F \left[\left\{ \frac{4}{27} (407 + 694x + 407x^2) x_+^2 + \frac{64}{5} (-1 + x)x(1 + x) \xi^2 \eta^2 \zeta_2^2 + \eta \left(-128x H_{-1} H_0 \right. \right. \right. \\
& + 64x H_0 H_1 - 32x H_{0,1} + 128x H_{0,-1} + \left(\frac{32}{27} H_0 P_{1118} + 32x^2 H_0^3 H_1 + (192x^2 H_0 H_1 \right. \\
& - 192x^2 H_{0,1}) \zeta_2 \Big) x_+^2 + \frac{4}{3} P_{1253} x_+^3 \zeta_2 \Big) + \xi \left(\eta x_+^2 \left(-\frac{16}{3} (-1 + x)x(1 + x) H_0^3 H_1 + \left(\right. \right. \right. \\
& - 32(-1 + x)x(1 + x) H_0 H_1 + 32(-1 + x)x(1 + x) H_{0,1} \Big) \zeta_2 \Big) + 4H_0 \zeta_2 - \frac{256}{5} x^2 \eta^2 \zeta_2^2 \Big) \\
& + \eta^2 \left(-\frac{4}{9} (-1 + 48x^2 + x^4) H_0^3 + 256x^2 H_0^2 H_1 - 768x^2 H_0 H_{0,1} + 512x^2 H_0 H_{0,-1} \right. \\
& + 1024x^2 H_{0,0,1} - 1024x^2 H_{0,0,-1} + \frac{4}{9} H_0^2 P_{1319} x_+^2 - 256x^2 \zeta_3 \Big) + \eta^3 \left(-64x H_0 H_{0,0,1} P_{1073} \right. \\
& + 16x H_{0,0,0,1} P_{1116} - 256x^3 H_0^2 H_{0,1} - 1024x^3 H_0 H_{0,0,-1} + 3072x^3 H_{0,0,0,-1} - \frac{xH_0^4}{3x_+^4} \\
& \left. \left. - 8x H_0^2 P_{1075} \zeta_2 - 16x H_0 P_{1091} \zeta_3 \right) \right\} + \varepsilon \left\{ \xi \left(\eta x_+^2 \left(-\frac{32}{3} (-1 + x)x(1 + x) H_{-1} H_0^3 H_1 \right. \right. \right. \\
& + \frac{16}{3} (-1 + x)x(1 + x) H_0^3 H_1^2 - 64(-1 + x)x(1 + x) H_0 H_{0,-1} H_{0,1} + 64(-1 + x)x(1 + x) H_0^2 H_1 H_{0,-1} \\
& - 128(-1 + x)x(1 + x) H_{0,0,-1,0,1} + \left(-64(-1 + x)x(1 + x) H_{-1} H_0 H_1 + 32(-1 + x)x(1 + x) H_0 H_1^2 \right. \\
& + \left(-64(-1 + x)x(1 + x) H_1 + 64(-1 + x)x(1 + x) H_{-1} \right) H_{0,1} + 256(-1 + x)x(1 + x) H_1 H_{0,-1} \\
& + 64(-1 + x)x(1 + x) H_{0,1,1} - 256(-1 + x)x(1 + x) H_{0,-1,1} \Big) \zeta_2 \Big) + \eta^2 \left(1024x^2 H_{0,1,-1} \zeta_2 \right. \\
& - \frac{512}{5} x^2 H_{-1} \zeta_2^2 \Big) - 256(-1 + x)x^2(1 + x) \eta^3 H_0 H_{0,-1,0,1} \Big) + \xi^2 \left(\eta^2 \left(-256(-1 + x)x(1 + x) H_{0,1,-1} \zeta_2 \right. \right. \\
& + \frac{128}{5} (-1 + x)x(1 + x) H_{-1} \zeta_2^2 \Big) + 64(-1 + x)^2 x(1 + x)^2 \eta^3 H_0 H_{0,-1,0,1} \Big) + \eta \left(\left(64x H_1 + 96x H_{0,1} \right) \zeta_2 \right. \\
& \left. + \frac{32}{27} (6 + 203x + 257x^2 + 6x^3) x_+ H_0 + 512x H_{-1}^2 H_0 + (512x - 384x H_{-1}) H_0 H_1 + 64x H_0 H_1^2 \right.
\end{aligned}$$

$$\begin{aligned}
& + \left(-64xH_1 + 448xH_{-1} \right) H_{0,1} + \left(384xH_1 - 1024xH_{-1} \right) H_{0,-1} + 64xH_{0,1,1} - 448xH_{0,1,-1} \\
& - 448xH_{0,-1,1} + 1024xH_{0,-1,-1} + \left(-\frac{64}{27}H_{-1}H_0P_{1119} + 64x^2H_{-1}H_0^3H_1 - 32x^2H_0^3H_1^2 + \left(\right. \right. \\
& - 64x \left(7 + 12x + 7x^2 \right) + 384x^2H_0H_{0,-1} \left. \right) H_{0,1} + \left(\frac{64P_{1119}}{27} - 384x^2H_0^2H_1 \right) H_{0,-1} + 768x^2H_{0,0,-1,0,1} \\
& + \left(\frac{4P_{1160}}{27} + 384x^2H_{-1}H_0H_1 - 192x^2H_0H_1^2 + \left(384x^2H_1 - 384x^2H_{-1} \right) H_{0,1} - 1536x^2H_1H_{0,-1} \right. \\
& - 384x^2H_{0,1,1} + 1536x^2H_{0,-1,1} \left. \right) \zeta_2 \left. \right) x_+^2 + \eta^2 \left(-128x^2H_{-1}^2H_0^2 - \frac{8}{9} \left(-1 - 144x^2 + x^4 \right) H_{-1}H_0^3 + \left(\right. \right. \\
& - 256x^2H_{-1}H_0^2 + 16x \left(-1 + 16x + x^2 \right) H_0^3 \left. \right) H_1 + 384x^2H_0^2H_1^2 + \left(1024x^2H_{-1}H_0 - 1536x^2H_0H_1 \right. \\
& - 2560x^2H_{0,-1} \left. \right) H_{0,1} + 256x^2H_{0,1}^2 + \left(\left(64x \left(1 + 27x + 2x^2 \right) - 512x^2H_{-1} \right) H_0 + \frac{16}{3} \left(-1 \right. \right. \\
& + 192x^2 + x^4 \left. \right) H_0^2 - 512x^2H_0H_1 \left. \right) H_{0,-1} + 1536x^2H_{0,-1}^2 + \left(1536x^2H_1 - 1536x^2H_{-1} \right) H_{0,0,1} \\
& + \left(1536x^2H_1 + 1536x^2H_{-1} \right) H_{0,0,-1} + 1024x^2H_0H_{0,1,1} + 1536x^2H_0H_{0,1,-1} + 1536x^2H_0H_{0,-1,1} \\
& - 2560x^2H_0H_{0,-1,-1} - 1536x^2H_{0,0,1,1} + 1536x^2H_{0,0,1,-1} + 1536x^2H_{0,0,-1,1} - 1536x^2H_{0,0,-1,-1} \\
& + 1536x^2H_{0,-1,0,1} + \left(\left(1536x^2H_1 + 1536x^2H_{-1} \right) \zeta_2 + \frac{64}{3}P_{1324}x_+^2\zeta_2 \right) \log(2) - \frac{4}{27}H_0^2P_{1262}x_+ \\
& + \left(\frac{16}{9}H_{0,0,-1}P_{1317} - \frac{8}{9}H_{-1}H_0^2P_{1318} + \frac{16}{9}H_0^2H_1P_{1320} - \frac{32}{9}H_0H_{0,1}P_{1321} + \frac{32}{9}H_{0,0,1}P_{1322} \right. \\
& + \frac{8}{27}H_0^3P_{1325} + \left(\frac{16}{9}xH_0P_{1226} - \frac{16}{3}H_{-1}P_{1323} \right) \zeta_2 - \frac{8}{9}P_{1326}\zeta_3 \left. \right) x_+^2 + \left(-\frac{8}{3} \left(-1 - 288x^2 + x^4 \right) H_{-1}H_0 \right. \\
& + \left(32x(3+x)(-1+3x)H_0 - 1536x^2H_{-1} \right) H_1 - 768x^2H_{-1}^2 + \frac{8}{3} \left(-7 - 480x^2 + 7x^4 \right) H_{0,-1} \\
& + 1536x^2H_{-1,1} \left. \right) \zeta_2 + \left(-2688x^2H_1 + 384x^2H_{-1} \right) \zeta_3 + \eta^3 \left(-64xH_0H_{0,1}^2P_{1073} + 32xH_0^2H_{0,1,1}P_{1073} \right. \\
& + 256xH_0H_{0,0,1,-1}P_{1073} + 256xH_0H_{0,0,-1,1}P_{1073} + 32xH_0^2H_{0,1,-1}P_{1075} + 32xH_0^2H_{0,-1,1}P_{1075} \\
& - 32xH_{0,0,1,0,-1}P_{1078} + \frac{8}{3}xH_0^3H_{0,-1}P_{1088} - 64xH_{0,0,1,0,1}P_{1089} - 32xH_{0,0,0,1,1}P_{1106} + \frac{2}{3}xH_0^4H_1P_{1109} \\
& - 64xH_{0,0,0,1,-1}P_{1110} - 64xH_{0,0,0,-1,1}P_{1110} + 16xH_{0,0,0,0,-1}P_{1147} - 16xH_{0,0,0,0,1}P_{1149} + \log(2)\zeta_2 \left(\right. \\
& - 3072x^3H_{0,1} - 3072x^3H_{0,-1} \left. \right) + \frac{1}{3}(1+x) \left(1 - 53x^2 + 43x^3 + x^5 \right) H_0^4 + \left(-\frac{16}{3}xH_0^3P_{1093} \right. \\
& + \frac{16}{3}(1+x)H_0^2P_{1168} - 512x^3H_0^2H_1 \left. \right) H_{0,1} - 512x^3H_{-1}H_0^2H_{0,1} + 1536x^3H_0H_{0,-1}^2 + \left(\right. \\
& - 32xH_0^2P_{1072} - 128xH_{-1}H_0P_{1073} - 32xH_{0,-1}P_{1087} + 64xH_{0,1}P_{1089} - \frac{64}{3}(1+x)H_0P_{1165} \left. \right) H_{0,0,1} \\
& + \left(-32xH_0^2P_{1075} - 64xH_{0,1}P_{1091} + \frac{32}{3}(1+x)H_0P_{1167} - 2048x^3H_{-1}H_0 - 7168x^3H_{0,-1} \right) H_{0,0,-1} \\
& + 512x^3H_0^2H_{0,-1,-1} + \left(32xH_{-1}P_{1116} + 32xH_0P_{1127} - 32xH_1P_{1132} + 16(1+x)P_{1183} \right) H_{0,0,0,1} + \left(\right. \\
& - 32xH_0P_{1107} + 64xH_1P_{1128} - \frac{16}{3}(1+x)P_{1163} + 6144x^3H_{-1} \left. \right) H_{0,0,0,-1} + 4096x^3H_0H_{0,0,-1,-1} \\
& - 3840x^3H_0H_{0,-1,0,1} + 9216x^3H_{0,0,0,-1,-1} + 5120x^3H_{0,0,-1,0,-1} + (1+x)^4 \left(-\frac{2}{3}xH_{-1}H_0^4 \right. \\
& \left. - \frac{4}{15}xH_0^5 + 128xH_0H_1H_{0,0,1} - 128xH_0H_1H_{0,0,-1} - 128xH_0H_{0,0,1,1} - 32xH_0^2H_1\zeta_2 - 32xH_0H_1\zeta_3 \right)
\end{aligned}$$

$$\begin{aligned}
& + \left(64xH_0H_{0,-1}P_{1074} - 256xH_{0,0,-1}P_{1074} - 16xH_{-1}H_0^2P_{1075} - 4xH_0^3P_{1076} - 32xH_0H_{0,1}P_{1092} \right. \\
& + 32xH_{0,0,1}P_{1117} + 2(1+x)H_0^2P_{1169} + 3072x^3H_{0,-1,-1} - 16xP_{1077}\zeta_3 \Big) \zeta_2 + \left(-\frac{16}{5}xH_1P_{1110} \right. \\
& + \frac{8}{5}xH_0P_{1111} - \frac{8}{5}(1+x)P_{1166} \Big) \zeta_2^2 + \left(32xH_{0,-1}P_{1090} - 32xH_{-1}H_0P_{1091} - 32xH_{0,1}P_{1108} - 4xH_0^2P_{1133} \right. \\
& \left. - \frac{8}{9}(1+x)H_0P_{1212} \right) \zeta_3 + 16xP_{1129}\zeta_5 \Big) + \left(\frac{2}{27}(2107 + 4454x + 2107x^2) - 64xH_1 \right) x_+^2 \Bigg\} . \tag{D.2}
\end{aligned}$$

The polynomials are listed below

$$\begin{aligned}
P_{1061} &= -62x^4 + 74x^3 + 53x^2 + 2x - 11, \\
P_{1062} &= -32x^4 + 56x^3 + 49x^2 - 11, \\
P_{1063} &= x^4 - 102x^3 + 42x^2 - 102x + 1, \\
P_{1064} &= x^4 - 42x^3 + 18x^2 - 42x + 1, \\
P_{1065} &= x^4 - 28x^3 + 6x^2 - 28x + 1, \\
P_{1066} &= x^4 - 20x^3 + 6x^2 - 20x + 1, \\
P_{1067} &= x^4 + 2x^3 + 4x^2 + 2x + 1, \\
P_{1068} &= x^4 + 2x^3 + 6x^2 + 2x + 1, \\
P_{1069} &= x^4 + 2x^3 + 8x^2 + 2x + 1, \\
P_{1070} &= x^4 + 2x^3 + 14x^2 + 2x + 1, \\
P_{1071} &= x^4 + 2x^3 + 35x^2 + 2x + 1, \\
P_{1072} &= x^4 + 4x^3 - 50x^2 + 4x + 1, \\
P_{1073} &= x^4 + 4x^3 - 26x^2 + 4x + 1, \\
P_{1074} &= x^4 + 4x^3 - 18x^2 + 4x + 1, \\
P_{1075} &= x^4 + 4x^3 + 22x^2 + 4x + 1, \\
P_{1076} &= x^4 + 4x^3 + 38x^2 + 4x + 1, \\
P_{1077} &= x^4 + 4x^3 + 46x^2 + 4x + 1, \\
P_{1078} &= x^4 + 4x^3 + 102x^2 + 4x + 1, \\
P_{1079} &= 2x^4 - 194x^3 - 113x^2 - 2x + 11, \\
P_{1080} &= 2x^4 - 19x^3 + 26x^2 + 61x - 18, \\
P_{1081} &= 2x^4 + 2x^3 + 7x^2 + 2x + 2, \\
P_{1082} &= 2x^4 + 4x^3 - x^2 + 4x + 2, \\
P_{1083} &= 2x^4 + 6x^3 - 11x^2 + 15x - 4, \\
P_{1084} &= 3x^4 - 20x^3 + 2x^2 - 20x + 3, \\
P_{1085} &= 3x^4 + 4x^3 + 20x^2 + 4x + 3, \\
P_{1086} &= 3x^4 + 6x^3 + x^2 + 6x + 3, \\
P_{1087} &= 3x^4 + 12x^3 - 206x^2 + 12x + 3, \\
P_{1088} &= 3x^4 + 12x^3 - 110x^2 + 12x + 3, \\
P_{1089} &= 3x^4 + 12x^3 - 62x^2 + 12x + 3, \\
P_{1090} &= 3x^4 + 12x^3 - 38x^2 + 12x + 3, \\
P_{1091} &= 3x^4 + 12x^3 - 14x^2 + 12x + 3,
\end{aligned}$$

$$\begin{aligned}
P_{1092} &= 3x^4 + 12x^3 + 2x^2 + 12x + 3, \\
P_{1093} &= 3x^4 + 12x^3 + 34x^2 + 12x + 3, \\
P_{1094} &= 4x^4 - 15x^3 - 2x^2 - 15x + 4, \\
P_{1095} &= 4x^4 - 10x^3 + 73x^2 - 10x + 4, \\
P_{1096} &= 4x^4 + 9x^3 + 6x^2 + 9x + 4, \\
P_{1097} &= 5x^4 - 62x^3 + 74x^2 - 70x + 5, \\
P_{1098} &= 5x^4 - 54x^3 + 2x^2 - 54x + 5, \\
P_{1099} &= 5x^4 - 31x^3 + 4x^2 - 31x + 5, \\
P_{1100} &= 5x^4 - 16x^3 + 6x^2 - 16x + 5, \\
P_{1101} &= 5x^4 - 12x^3 + 14x^2 - 4x + 1, \\
P_{1102} &= 5x^4 + 10x^3 + x^2 + 10x + 5, \\
P_{1103} &= 5x^4 + 10x^3 + 17x^2 + 10x + 5, \\
P_{1104} &= 5x^4 + 10x^3 + 43x^2 + 10x + 5, \\
P_{1105} &= 5x^4 + 17x^3 - 22x^2 + 17x + 7, \\
P_{1106} &= 5x^4 + 20x^3 - 386x^2 + 20x + 5, \\
P_{1107} &= 5x^4 + 20x^3 - 162x^2 + 20x + 5, \\
P_{1108} &= 5x^4 + 20x^3 - 138x^2 + 20x + 5, \\
P_{1109} &= 5x^4 + 20x^3 - 66x^2 + 20x + 5, \\
P_{1110} &= 5x^4 + 20x^3 + 46x^2 + 20x + 5, \\
P_{1111} &= 5x^4 + 20x^3 + 254x^2 + 20x + 5, \\
P_{1112} &= 6x^4 - 17x^3 + x^2 - 14x - 8, \\
P_{1113} &= 6x^4 - 9x^3 + 17x^2 - 2x + 4, \\
P_{1114} &= 7x^4 - 146x^3 + 54x^2 - 146x + 7, \\
P_{1115} &= 7x^4 - 28x^3 + 10x^2 - 28x + 7, \\
P_{1116} &= 7x^4 + 28x^3 - 214x^2 + 28x + 7, \\
P_{1117} &= 7x^4 + 28x^3 - 6x^2 + 28x + 7, \\
P_{1118} &= 7x^4 + 50x^3 + 146x^2 + 50x + 7, \\
P_{1119} &= 7x^4 + 482x^3 + 1010x^2 + 482x + 7, \\
P_{1120} &= 9x^4 - 230x^3 + 90x^2 - 230x + 9, \\
P_{1121} &= 9x^4 - 40x^3 + 14x^2 - 40x + 9, \\
P_{1122} &= 9x^4 + 58x^3 - 2x^2 + 58x + 5, \\
P_{1123} &= 11x^4 - 146x^3 + 82x^2 - 146x + 7, \\
P_{1124} &= 11x^4 - 2x^3 + 14x^2 - 2x + 11, \\
P_{1125} &= 11x^4 + 6x^3 + 77x^2 + 6x + 11, \\
P_{1126} &= 11x^4 + 22x^3 - 82x^2 + 22x + 11, \\
P_{1127} &= 11x^4 + 44x^3 - 126x^2 + 44x + 11, \\
P_{1128} &= 11x^4 + 44x^3 - 14x^2 + 44x + 11, \\
P_{1129} &= 11x^4 + 44x^3 + 102x^2 + 44x + 11, \\
P_{1130} &= 13x^4 - 50x^3 + 82x^2 - 66x + 5, \\
P_{1131} &= 13x^4 + 32x^3 + 12x^2 + 32x + 13,
\end{aligned}$$

$$\begin{aligned}
P_{1132} &= 13x^4 + 52x^3 + 14x^2 + 52x + 13, \\
P_{1133} &= 13x^4 + 52x^3 + 46x^2 + 52x + 13, \\
P_{1134} &= 14x^4 + 22x^3 + 57x^2 + 22x + 14, \\
P_{1135} &= 15x^4 - 60x^3 + 66x^2 + 4x - 17, \\
P_{1136} &= 17x^4 - 254x^3 + 232x^2 - 18x + 7, \\
P_{1137} &= 17x^4 - 64x^3 - 188x^2 + 256x + 15, \\
P_{1138} &= 21x^4 + 185x^3 + 131x^2 + 73x - 16, \\
P_{1139} &= 23x^4 + 102x^3 + 10x^2 + 102x + 19, \\
P_{1140} &= 25x^4 - 20x^3 + 56x^2 - 148x - 33, \\
P_{1141} &= 25x^4 + 24x^3 + 148x^2 - 200x - 21, \\
P_{1142} &= 26x^4 - 73x^3 + 51x^2 - 15x + 3, \\
P_{1143} &= 33x^4 - 344x^3 - 1006x^2 + 616x + 89, \\
P_{1144} &= 37x^4 - 16x^3 - 98x^2 + 48x + 5, \\
P_{1145} &= 39x^4 - 4x^3 + 104x^2 - 164x - 27, \\
P_{1146} &= 47x^4 + 268x^3 - 263x^2 + 40x + 38, \\
P_{1147} &= 53x^4 + 212x^3 - 322x^2 + 212x + 53, \\
P_{1148} &= 55x^4 - 138x^3 + 70x^2 - 138x + 55, \\
P_{1149} &= 55x^4 + 220x^3 - 86x^2 + 220x + 55, \\
P_{1150} &= 67x^4 - 98x^3 + 88x^2 - 2x - 11, \\
P_{1151} &= 75x^4 - 50x^3 - 98x^2 - 50x + 75, \\
P_{1152} &= 87x^4 + 4x^3 + 206x^2 + 388x + 35, \\
P_{1153} &= 95x^4 - 404x^3 + 496x^2 + 820x - 103, \\
P_{1154} &= 115x^4 + 100x^3 + 116x^2 - 364x - 43, \\
P_{1155} &= 124x^4 + 570x^3 + 801x^2 - 1158x + 325, \\
P_{1156} &= 133x^4 + 216x^3 + 520x^2 - 440x - 61, \\
P_{1157} &= 233x^4 - 388x^3 - 52x^2 + 56x + 55, \\
P_{1158} &= 379x^4 + 263x^3 + 569x^2 + 515x - 134, \\
P_{1159} &= 683x^4 - 672x^3 - 1138x^2 + 744x + 239, \\
P_{1160} &= 895x^4 + 3806x^3 + 2288x^2 - 2878x - 1007, \\
P_{1161} &= -117x^5 - 117x^4 + 80x^3 - 628x^2 - 157x - 157, \\
P_{1162} &= -71x^5 - 71x^4 - 32x^3 - 116x^2 - 3x - 3, \\
P_{1163} &= x^5 - 73x^4 - 216x^3 + 72x^2 - 73x + 1, \\
P_{1164} &= x^5 - 17x^4 + 96x^3 - 80x^2 + 25x + 7, \\
P_{1165} &= x^5 - 16x^4 + 15x^3 - 9x^2 - 16x + 1, \\
P_{1166} &= x^5 - 13x^4 - 100x^3 + 172x^2 - 13x + 1, \\
P_{1167} &= x^5 - 13x^4 + 180x^3 - 204x^2 - 13x + 1, \\
P_{1168} &= x^5 - 7x^4 + 138x^3 - 150x^2 - 7x + 1, \\
P_{1169} &= x^5 - 5x^4 - 156x^3 + 100x^2 - 5x + 1, \\
P_{1170} &= x^5 - 3x^4 + 22x^3 - 18x^2 + 5x + 1, \\
P_{1171} &= x^5 - x^4 + 12x^3 - 8x^2 + 3x + 1,
\end{aligned}$$

$$\begin{aligned}
P_{1172} &= x^5 + x^4 + 5x^3 - 7x^2 - 2x - 2, \\
P_{1173} &= x^5 + x^4 + 16x^3 - 8x^2 + 3x + 3, \\
P_{1174} &= x^5 + 2x^4 - 3x^3 + 7x^2 + 1, \\
P_{1175} &= x^5 + 3x^4 - 8x^3 + 12x^2 - x + 1, \\
P_{1176} &= x^5 + 5x^4 - 21x^3 + 5x^2 - 13x - 9, \\
P_{1177} &= x^5 + 5x^4 - 2x^3 + 14x^2 + x + 5, \\
P_{1178} &= x^5 + 9x^4 - 28x^3 + 24x^2 - 11x - 3, \\
P_{1179} &= x^5 + 11x^4 + 6x^3 + 34x^2 + 9x + 19, \\
P_{1180} &= x^5 + 14x^4 - 57x^3 + 47x^2 - 19x - 6, \\
P_{1181} &= x^5 + 21x^4 - 58x^3 + 54x^2 - 23x - 3, \\
P_{1182} &= x^5 + 35x^4 - 27x^3 + 271x^2 + 87x + 121, \\
P_{1183} &= 2x^5 - 41x^4 - 141x^3 + 147x^2 - 41x + 2, \\
P_{1184} &= 2x^5 + x^4 + 9x^3 - 7x^2 - 1, \\
P_{1185} &= 2x^5 + 2x^4 - 11x^3 + 7x^2 - 4x - 4, \\
P_{1186} &= 2x^5 + 2x^4 + 5x^3 + 3x^2 + 2x + 2, \\
P_{1187} &= 2x^5 + 2x^4 + 5x^3 + 11x^2 + 6x + 6, \\
P_{1188} &= 2x^5 + 2x^4 + 7x^3 - 5x^2 - x - 1, \\
P_{1189} &= 2x^5 + 2x^4 + 15x^3 - 19x^2 - 4x - 4, \\
P_{1190} &= 2x^5 + 4x^4 + x^3 + 23x^2 + 8x + 10, \\
P_{1191} &= 2x^5 + 17x^4 - 56x^3 + 42x^2 - 24x - 9, \\
P_{1192} &= 2x^5 + 20x^4 - 51x^3 + 11x^2 - 40x - 22, \\
P_{1193} &= 2x^5 + 34x^4 - 39x^3 - 145x^2 - 77x + 1, \\
P_{1194} &= 3x^5 - 17x^4 + 54x^3 - 50x^2 + 19x - 1, \\
P_{1195} &= 3x^5 - 3x^4 + 15x^3 - 11x^2 + 5x - 1, \\
P_{1196} &= 3x^5 - 3x^4 + 22x^3 - 14x^2 + 7x + 1, \\
P_{1197} &= 3x^5 - x^4 + 35x^3 + 13x^2 + 25x + 21, \\
P_{1198} &= 3x^5 + 3x^4 + 5x^3 - 7x^2 - 4x - 4, \\
P_{1199} &= 3x^5 + 7x^4 - 14x^3 + 26x^2 - x + 3, \\
P_{1200} &= 3x^5 + 9x^4 - 10x^3 + 26x^2 - x + 5, \\
P_{1201} &= 3x^5 + 129x^4 - 40x^3 - 176x^2 - 271x + 35, \\
P_{1202} &= 4x^5 + 4x^4 - x^3 - 15x^2 - 12x - 12, \\
P_{1203} &= 5x^5 - x^4 + 56x^3 - 72x^2 - 7x - 13, \\
P_{1204} &= 5x^5 + x^4 + 30x^3 - 34x^2 - 3x - 7, \\
P_{1205} &= 5x^5 + 5x^4 - 7x^3 + 11x^2 - 3x - 3, \\
P_{1206} &= 5x^5 + 5x^4 - x^3 - 3x^2 - 7x - 7, \\
P_{1207} &= 5x^5 + 5x^4 + 16x^3 - 8x^2 - x - 1, \\
P_{1208} &= 5x^5 + 5x^4 + 17x^3 - 25x^2 - 9x - 9, \\
P_{1209} &= 5x^5 + 7x^4 - 47x^3 + 11x^2 - 25x - 23, \\
P_{1210} &= 6x^5 - 44x^4 - 27x^3 + 47x^2 - 11x - 3, \\
P_{1211} &= 6x^5 + 6x^4 + 11x^3 - 3x^2 - 2x - 2,
\end{aligned}$$

$$\begin{aligned}
P_{1212} &= 7x^5 - 133x^4 - 234x^3 + 630x^2 - 133x + 7, \\
P_{1213} &= 7x^5 - 25x^4 + 96x^3 - 92x^2 + 27x - 5, \\
P_{1214} &= 7x^5 - x^4 + 22x^3 - 10x^2 + 7x - 1, \\
P_{1215} &= 7x^5 + 2x^4 + 12x^3 - 2x^2 + 3x - 2, \\
P_{1216} &= 7x^5 + 5x^4 + 17x^3 - 5x^2 + x - 1, \\
P_{1217} &= 7x^5 + 7x^4 - 20x^3 + 16x^2 - 9x - 9, \\
P_{1218} &= 7x^5 + 7x^4 - 16x^3 + 20x^2 - 5x - 5, \\
P_{1219} &= 7x^5 + 7x^4 + 68x^3 - 4x^2 + 25x + 25, \\
P_{1220} &= 8x^5 - 36x^4 + 191x^3 - 155x^2 + 54x + 10, \\
P_{1221} &= 8x^5 + 8x^4 + 11x^3 - 15x^2 - 10x - 10, \\
P_{1222} &= 9x^5 + 9x^4 + 10x^3 - 4x^2 - 6x - 6, \\
P_{1223} &= 10x^5 - 9x^3 + 65x^2 + 31x - 1, \\
P_{1224} &= 10x^5 + 14x^4 - 27x^3 + 31x^2 - 12x - 8, \\
P_{1225} &= 10x^5 + 14x^4 - 21x^3 + 25x^2 - 12x - 8, \\
P_{1226} &= 10x^5 + 197x^4 - 200x^3 - 442x^2 + 54x - 51, \\
P_{1227} &= 11x^5 - 131x^4 + 90x^3 - 58x^2 - 197x + 29, \\
P_{1228} &= 11x^5 - 25x^4 + 173x^3 - 141x^2 + 41x + 5, \\
P_{1229} &= 11x^5 + 11x^4 + 83x^3 - 19x^2 + 21x + 21, \\
P_{1230} &= 11x^5 + 43x^4 - 112x^3 + 116x^2 - 41x - 9, \\
P_{1231} &= 11x^5 + 245x^4 + 736x^3 - 384x^2 + 261x - 37, \\
P_{1232} &= 12x^5 + 12x^4 + 25x^3 + 43x^2 + 22x + 22, \\
P_{1233} &= 12x^5 + 44x^4 - 45x^3 + 33x^2 - 50x - 18, \\
P_{1234} &= 13x^5 - 81x^4 - 280x^3 + 136x^2 - 57x + 13, \\
P_{1235} &= 13x^5 + x^4 + 65x^3 - 57x^2 + 3x - 9, \\
P_{1236} &= 13x^5 + 13x^4 - 35x^3 + 67x^2 + 3x + 3, \\
P_{1237} &= 13x^5 + 13x^4 - 7x^3 + 11x^2 - 11x - 11, \\
P_{1238} &= 13x^5 + 13x^4 + 19x^3 - 27x^2 - 17x - 17, \\
P_{1239} &= 13x^5 + 13x^4 + 83x^3 - 19x^2 + 19x + 19, \\
P_{1240} &= 13x^5 + 113x^4 - 330x^3 + 286x^2 - 135x - 35, \\
P_{1241} &= 13x^5 + 113x^4 - 322x^3 + 238x^2 - 155x - 55, \\
P_{1242} &= 14x^5 + 14x^4 - x^3 + 13x^2 - 8x - 8, \\
P_{1243} &= 15x^5 + 15x^4 + 26x^3 - 22x^2 - 13x - 13, \\
P_{1244} &= 15x^5 + 27x^4 - 53x^3 + 57x^2 - 25x - 13, \\
P_{1245} &= 16x^5 - 7x^4 - 101x^3 + 55x^2 - 15x + 4, \\
P_{1246} &= 16x^5 + 16x^4 + 27x^3 - 19x^2 - 12x - 12, \\
P_{1247} &= 17x^5 + 7x^4 + 84x^3 - 52x^2 + 9x - 1, \\
P_{1248} &= 17x^5 + 17x^4 - 20x^3 + 52x^2 - x - 1, \\
P_{1249} &= 17x^5 + 17x^4 + 54x^3 - 56x^2 - 18x - 18, \\
P_{1250} &= 17x^5 + 37x^4 - 78x^3 + 66x^2 - 43x - 23, \\
P_{1251} &= 17x^5 + 41x^4 - 162x^3 + 150x^2 + 25x - 7,
\end{aligned}$$

$$\begin{aligned}
P_{1252} &= 17x^5 + 43x^4 - 87x^3 + 83x^2 - 45x - 19, \\
P_{1253} &= 17x^5 + 135x^4 + 134x^3 + 154x^2 - 39x - 17, \\
P_{1254} &= 18x^5 + 8x^4 + 21x^3 - 9x^2 - 2x - 12, \\
P_{1255} &= 18x^5 + 18x^4 + 7x^3 - 11x^2 - 20x - 20, \\
P_{1256} &= 19x^5 + 19x^4 - 80x^3 + 60x^2 - 29x - 29, \\
P_{1257} &= 19x^5 + 19x^4 + 43x^3 - 51x^2 - 23x - 23, \\
P_{1258} &= 19x^5 + 19x^4 + 54x^3 - 42x^2 - 13x - 13, \\
P_{1259} &= 19x^5 + 19x^4 + 68x^3 - 4x^2 + 13x + 13, \\
P_{1260} &= 19x^5 + 31x^4 - 170x^3 + 142x^2 + 15x - 5, \\
P_{1261} &= 19x^5 + 67x^4 - 192x^3 + 116x^2 - 105x - 57, \\
P_{1262} &= 19x^5 + 4963x^4 + 5538x^3 - 1062x^2 - 781x - 37, \\
P_{1263} &= 20x^5 + 20x^4 + 95x^3 + 53x^2 + 54x + 54, \\
P_{1264} &= 20x^5 + 20x^4 + 263x^3 - 311x^2 - 44x - 44, \\
P_{1265} &= 20x^5 + 49x^4 - 87x^3 + 89x^2 - 48x - 19, \\
P_{1266} &= 21x^5 + 21x^4 - 20x^3 + 40x^2 - 11x - 11, \\
P_{1267} &= 21x^5 + 21x^4 + 14x^3 - 6x^2 - 17x - 17, \\
P_{1268} &= 21x^5 + 21x^4 + 48x^3 - 56x^2 - 25x - 25, \\
P_{1269} &= 22x^5 + 22x^4 + 61x^3 - 57x^2 - 20x - 20, \\
P_{1270} &= 22x^5 + 22x^4 + 95x^3 + 53x^2 + 52x + 52, \\
P_{1271} &= 23x^5 - 52x^4 + 50x^3 - 74x^2 + 43x - 6, \\
P_{1272} &= 23x^5 + 14x^4 + 109x^3 - 35x^2 + 23x + 14, \\
P_{1273} &= 23x^5 + 23x^4 - 152x^3 + 196x^2 - x - 1, \\
P_{1274} &= 23x^5 + 113x^4 + 300x^3 - 156x^2 + 53x - 13, \\
P_{1275} &= 24x^5 + 24x^4 + 123x^3 - 119x^2 - 22x - 22, \\
P_{1276} &= 25x^5 - 329x^4 - 524x^3 + 164x^2 - 233x + 1, \\
P_{1277} &= 25x^5 + 25x^4 - 16x^3 + 20x^2 - 23x - 23, \\
P_{1278} &= 25x^5 + 25x^4 - 4x^3 + 8x^2 - 23x - 23, \\
P_{1279} &= 25x^5 + 25x^4 + 56x^3 + 20x^2 + 13x + 13, \\
P_{1280} &= 27x^5 + 27x^4 - 12x^3 - 32x^2 - 49x - 49, \\
P_{1281} &= 27x^5 + 27x^4 + 52x^3 + 16x^2 + 7x + 7, \\
P_{1282} &= 29x^5 + 29x^4 - 104x^3 + 100x^2 - 31x - 31, \\
P_{1283} &= 31x^5 + 31x^4 - 104x^3 + 100x^2 - 33x - 33, \\
P_{1284} &= 31x^5 + 31x^4 - 32x^3 + 44x^2 - 25x - 25, \\
P_{1285} &= 31x^5 + 31x^4 - 30x^3 + 26x^2 - 33x - 33, \\
P_{1286} &= 31x^5 + 31x^4 + 28x^3 + 8x^2 - 13x - 13, \\
P_{1287} &= 32x^5 + 32x^4 + 223x^3 - 251x^2 - 46x - 46, \\
P_{1288} &= 34x^5 + 34x^4 + 31x^3 + 13x^2 - 12x - 12, \\
P_{1289} &= 35x^5 + 35x^4 + 106x^3 - 86x^2 - 25x - 25, \\
P_{1290} &= 37x^5 + 5x^4 + 243x^3 - 179x^2 + 27x - 5, \\
P_{1291} &= 38x^5 + 38x^4 + 229x^3 - 209x^2 - 28x - 28,
\end{aligned}$$

$$\begin{aligned}
P_{1292} &= 38x^5 + 55x^4 + 131x^3 - 85x^2 + 9x - 4, \\
P_{1293} &= 39x^5 + 189x^4 + 258x^3 + 110x^2 + 119x + 53, \\
P_{1294} &= 40x^5 + 40x^4 + 87x^3 - 3x^2 + 2x + 2, \\
P_{1295} &= 42x^5 + 42x^4 + 59x^3 - 27x^2 - 26x - 26, \\
P_{1296} &= 42x^5 + 48x^4 + 59x^3 + 29x^2 - 4x + 2, \\
P_{1297} &= 43x^5 + 27x^4 - 48x^3 + 140x^2 + 19x + 3, \\
P_{1298} &= 43x^5 + 43x^4 + 96x^3 + 20x^2 + 15x + 15, \\
P_{1299} &= 45x^5 + 101x^4 - 211x^3 + 107x^2 + 16x + 6, \\
P_{1300} &= 50x^5 + 98x^4 + 17x^3 + 59x^2 - 60x - 12, \\
P_{1301} &= 51x^5 + 51x^4 - 130x^3 + 110x^2 - 61x - 61, \\
P_{1302} &= 55x^5 + 211x^4 - 576x^3 + 732x^2 - 91x + 53, \\
P_{1303} &= 56x^5 - 48x^4 + 425x^3 - 573x^2 - 26x - 130, \\
P_{1304} &= 65x^5 + 217x^4 - 264x^3 + 308x^2 - 195x - 43, \\
P_{1305} &= 67x^5 + 67x^4 + 134x^3 - 94x^2 - 47x - 47, \\
P_{1306} &= 74x^5 - 99x^4 - 152x^3 + 192x^2 - 70x + 7, \\
P_{1307} &= 77x^5 - 35x^4 + 516x^3 - 368x^2 + 109x - 3, \\
P_{1308} &= 96x^5 + 96x^4 + 437x^3 - 317x^2 - 36x - 36, \\
P_{1309} &= 98x^5 + 98x^4 + 495x^3 - 407x^2 - 54x - 54, \\
P_{1310} &= 119x^5 + 119x^4 + 203x^3 - 67x^2 - 51x - 51, \\
P_{1311} &= 139x^5 + 207x^4 + 169x^3 + 95x^2 - 75x - 7, \\
P_{1312} &= 141x^5 + 241x^4 + 79x^3 + 177x^2 - 113x - 13, \\
P_{1313} &= 147x^5 - 63x^4 + 894x^3 - 550x^2 + 203x - 55, \\
P_{1314} &= 181x^5 + 113x^4 + 420x^3 - 176x^2 + 9x - 59, \\
P_{1315} &= 202x^5 + 1240x^4 - 3283x^3 + 679x^2 + 583x - 157, \\
P_{1316} &= 649x^5 + 377x^4 + 1488x^3 - 84x^2 + 325x + 53, \\
P_{1317} &= 5x^6 - 530x^5 - 3241x^4 - 4744x^3 - 1585x^2 + 298x + 5, \\
P_{1318} &= 5x^6 - 386x^5 - 1009x^4 - 640x^3 + 503x^2 + 370x + 5, \\
P_{1319} &= 5x^6 - 224x^5 - 577x^4 - 604x^3 - 73x^2 + 28x + 5, \\
P_{1320} &= 5x^6 - 170x^5 - 19x^4 + 314x^3 + 305x^2 - 8x + 5, \\
P_{1321} &= 5x^6 - 116x^5 + 323x^4 + 728x^3 + 323x^2 - 116x + 5, \\
P_{1322} &= 5x^6 - 44x^5 + 755x^4 + 1304x^3 + 467x^2 - 188x + 5, \\
P_{1323} &= 5x^6 + 40x^5 - 157x^4 - 640x^3 - 349x^2 - 56x + 5, \\
P_{1324} &= 8x^6 + 25x^5 - 100x^4 - 154x^3 - 100x^2 + 25x + 8, \\
P_{1325} &= 15x^6 - 183x^5 - 1323x^4 - 1604x^3 - 121x^2 + 43x + 5, \\
P_{1326} &= 123x^6 - 492x^5 - 2159x^4 - 1820x^3 - 1223x^2 - 24x + 123.
\end{aligned} \tag{D.3}$$

E The pseudo-scalar form factors up to two-loop

The non-singlet part of the pseudo-scalar form factor can be obtained using Eq. (2.13). The unrenormalized singlet contribution is given up to $\mathcal{O}(\varepsilon)$ by

$$\begin{aligned}
\hat{F}_P^{(2),s} = C_F T_F \left[\right. & \left. \left\{ \eta^2 (8x(1+x)^2 H_0^2 + 32x(1+x)^2 \zeta_2) + \eta^3 \left(\frac{32}{3} x^2 (1+x)^2 H_0^3 + 64x^2 (1+x)^2 H_0 \zeta_2 \right) + \eta \left(\right. \right. \right. \\
& - \frac{1}{3} x H_0^4 + \frac{16}{3} x H_0^3 H_1 - 64x H_0 H_{0,0,1} + 112x H_{0,0,0,1} + (-8x H_0^2 + 32x H_0 H_1 - 32x H_{0,1}) \zeta_2 \\
& \left. \left. \left. - \frac{64}{5} x \zeta_2^2 - 48x H_0 \zeta_3 \right) \right\} + \varepsilon \left\{ \xi \eta^3 \left(-(-1+x)x(1+x)^3 H_0^4 + 32(-1+x)x(1+x)^3 H_0^2 H_{0,1} \right) \right. \right. \\
& + \eta^3 \left(\frac{32}{5} x(1+x)^2 (7-13x+7x^2) \zeta_2^2 - \frac{8}{3} x(1+x)^2 (-3+11x^2) H_0^3 + \frac{64}{3} x^2 (1+x)^2 H_{-1} H_0^3 \right. \\
& + 6x^2 (1+x)^2 H_0^4 - \frac{16}{3} x(1+x)^2 (5-6x+5x^2) H_0^3 H_1 + 128x^2 (1+x)^2 H_0^2 H_{0,1} \\
& - 128x^2 (1+x)^2 H_0^2 H_{0,-1} - 16x(1+x)^2 (53-90x+53x^2) H_{0,0,0,1} + 128x(1+x)^2 (3 \\
& - x+3x^2) H_{0,0,0,-1} + (-16x(1+x)^2 (-1+3x)(1+3x) H_0 + 128x^2 (1+x)^2 H_{-1} H_0 \\
& - 32x(1+x)^2 (5-6x+5x^2) H_0 H_1 + 32x(1+x)^2 (5-6x+5x^2) H_{0,1} - 512x^2 (1+x)^2 H_{0,-1}) \zeta_2 \\
& + \eta^2 (16x(1+x)^2 H_0^2 - 32x(1+x)^2 H_{-1} H_0^2 + 16x(1+x)^2 H_0^2 H_1 - 32x(1+x)^2 H_0 H_{0,-1} \\
& - 32x(1+x)^2 H_{0,0,1} + 128x(1+x)^2 H_{0,0,-1} - 192x(1+x)^2 H_{-1} \zeta_2 + 48x(1+x)^2 \zeta_3) \\
& + \eta \left(-\frac{2}{3} x H_{-1} H_0^4 - \frac{4}{15} x H_0^5 + \left(\frac{32}{3} x H_{-1} H_0^3 + \frac{10}{3} x H_0^4 \right) H_1 - \frac{16}{3} x H_0^3 H_1^2 + (-16x H_0^3 \right. \\
& + 64x H_0 H_{0,-1}) H_{0,1} - 64x H_0 H_{0,1}^2 + (8x H_0^3 - 64x H_0^2 H_1) H_{0,-1} + ((448x - 128x H_{-1}) H_0 \\
& - 32x H_0^2 + 128x H_0 H_1 + 192x H_{0,1} - 96x H_{0,-1}) H_{0,0,1} + (-128x H_0 - 32x H_0^2 - 128x H_0 H_1 \\
& - 192x H_{0,1}) H_{0,0,-1} + 32x H_0^2 H_{0,1,1} + 32x H_0^2 H_{0,1,-1} + 32x H_0^2 H_{0,-1,1} + (352x H_0 - 416x H_1 \\
& + 224x H_{-1}) H_{0,0,0,1} + (-160x H_0 + 704x H_1) H_{0,0,0,-1} - 128x H_0 H_{0,0,1,1} + 256x H_0 H_{0,0,1,-1} \\
& + 256x H_0 H_{0,0,-1,1} + 64x H_0 H_{0,-1,0,1} - 880x H_{0,0,0,0,1} + 848x H_{0,0,0,0,-1} - 160x H_{0,0,0,0,1,1} \\
& - 320x H_{0,0,0,0,-1} - 320x H_{0,0,0,-1,1} - 192x H_{0,0,1,0,1} - 32x H_{0,0,1,0,-1} + 128x H_{0,0,-1,0,1} \\
& + (8x H_0^2 - 16x H_{-1} H_0^2 - 4x H_0^3 + (64x H_{-1} H_0 - 32x H_0^2) H_1 - 32x H_0 H_1^2 + (-96x H_0 \\
& + 64x H_1 - 64x H_{-1}) H_{0,1} + (64x H_0 - 256x H_1) H_{0,-1} + 224x H_{0,0,1} - 256x H_{0,0,-1} \\
& - 64x H_{0,1,1} + 256x H_{0,1,-1} + 256x H_{0,-1,1} - 16x \zeta_3) \zeta_2 + (8x H_0 - 16x H_1 - \frac{128}{5} x H_{-1}) \zeta_2^2 \\
& \left. \left. \left. + (208x H_0 - 96x H_{-1} H_0 - 52x H_0^2 - 32x H_0 H_1 - 160x H_{0,1} + 96x H_{0,-1}) \zeta_3 + 176x \zeta_5 \right) \right\} \right]. \tag{E.1}
\end{aligned}$$

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