# The Homotopy Fixed Point Theorem and the Quillen–Lichtenbaum conjecture in Hermitian $K$-theory

A.J. Berrick [a,b,*], M. Karoubi [c], M. Schlichting [d], P.A. Østvær [e]

[a] *Department of Mathematics, National University of Singapore, Singapore*
[b] *Yale-NUS College, Singapore*
[c] *UFR de mathématiques, Université Denis Diderot-Paris 7, France*
[d] *Mathematics Institute, University of Warwick, Coventry, United Kingdom*
[e] *Department of Mathematics, University of Oslo, Norway*

A R T I C L E   I N F O

A B S T R A C T

We settle two conjectures for computing higher Grothendieck–Witt groups (also known as Hermitian $K$-groups) of noetherian schemes $X$, under some mild conditions. It is shown that the comparison map from the Hermitian $K$-theory of $X$ to the homotopy fixed points of $K$-theory under the natural $\mathbb{Z}/2$-action is a 2-adic equivalence. We also prove that the mod $2^\nu$ comparison map between the Hermitian $K$-theory of $X$ and its étale version is an isomorphism on homotopy groups in the same range as for the Quillen–Lichtenbaum conjecture in $K$-theory. Applications compute higher Grothendieck–Witt groups of complex algebraic varieties and rings of 2-integers in number fields, and hence values of Dedekind zeta-functions.

© 2015 Published by Elsevier Inc.

* Corresponding author.
 *E-mail addresses:* berrick@math.nus.edu.sg, jon.berrick@yale-nus.edu.sg (A.J. Berrick), max.karoubi@gmail.com (M. Karoubi), m.schlichting@warwick.ac.uk (M. Schlichting), paularne@math.uio.no (P.A. Østvær).

## 1. Introduction

In geometric applications, real topological $K$-groups often yield stronger results than the more easily computable complex topological $K$-groups. This is exemplified by Adams' solution of the vector field problem on spheres, and the image of the $J$-homomorphism in the stable homotopy groups. One can, however, compute the real topological $K$-groups by using the action of the group $C_2$ of order 2 on the groups and spaces underlying the complex theory. More precisely, taking fixed points of the conjugation action yields an inclusion $O = U^{C_2} \subset U$ of the orthogonal group into the unitary group, and there is an induced homotopy equivalence

$$BO \overset{\sim}{\to} BU^{hC_2} \tag{1-a}$$

between the classifying space of $O$ and the homotopy fixed points of $BU$. It leads to the homotopy fixed point spectral sequence relating the complex and real $K$-groups

$$H^{-p}(C_2; \pi_q(BU)) \Rightarrow \pi_{p+q}(BO). \tag{1-b}$$

The algebraic analogs of complex topological $K$-groups are Quillen's algebraic $K$-groups $K_i(R) = \pi_i BGl(R)^+$ of a ring $R$, $i \geq 1$. In motivic lingo, the complex realization functor from $\mathbb{A}^1$-homotopy theory to the ordinary homotopy category sends algebraic $K$-theory to complex topological $K$-theory. Similarly, the algebraic analogs of real topological $K$-groups are the second author's Hermitian $K$-groups $GW_i(R) = \pi_i BO(R)^+$, $i \geq 1$. The Hermitian $K$-groups (also called *higher Grothendieck–Witt groups*) were introduced in [18] at the same time as algebraic $K$-theory and extended to schemes in [39]. Hermitian $K$-theory yields real topological $K$-theory via complex realization; see [40]. As in topology, higher Grothendieck–Witt groups often yield stronger results than algebraic $K$-groups in applications, as exemplified by recent work on projective modules over smooth affine algebras [4,11]. Another major motivation for computing Hermitian $K$-groups is the current quest for understanding motivic stable homotopy groups, a fundamental problem drawing inspiration from topology.

From the homological point of view, Borel's work [9] put on the same footing the general linear group and other classical groups like the orthogonal or symplectic ones. Rationally, or more generally up to 2-torsion, the groups $GW_i(R)$, which deal with the orthogonal and symplectic groups, are well understood thanks to the "fundamental theorem of Hermitian $K$-theory" [19,36]: they can be computed in terms of Witt groups and the symmetric part of Quillen's $K$-theory under the involution induced by the duality functor; see Remark 4.9. On the other hand, much less is known about the 2-torsion in $GW_i(R)$, and our article provides new tools for computing these.

In this paper, under some mild assumptions, we settle two conjectures for computing Hermitian K-groups of commutative rings and more generally of schemes.[1] The first

---

[1] The results of this paper were found independently by the third author in the general case [37] and the other authors in the case of schemes of characteristic 0.

conjecture is the algebraic analog of the homotopy equivalence (1-a) and was formulated by Thomason in [43] as a homotopy limit problem. It was explicitly conjectured by Williams [50, p. 667] in relation with geometric topology.

We prove Williams' conjecture in Theorem 2.4. For commutative rings in which $-1$ is a sum of squares and mild finiteness conditions hold, we obtain a homotopy equivalence

$$BO(R)^+ \sim \left( BGl(R)^+ \right)^{hC_2} \tag{1-c}$$

valid up to connected components, and an associated spectral sequence

$$H^{-p}(C_2; K_q(R)) \Rightarrow GW_{p+q}(R). \tag{1-d}$$

When $-1$ is not a sum of squares in $R$, the homotopy equivalence (1-c) is not valid though its 2-adic version still holds (Theorem 2.2). In fact, we prove our results for schemes – a generality imposed upon us by our use of Nisnevich descent.

Theorems 2.2 and 2.4 below solve Williams' conjecture for noetherian schemes. Our proof uses among other things a result of Hu–Kriz–Ormsby [15] which in turn relies on the solution of Milnor's conjecture by Voevodsky [46]. From our solution of Williams' conjecture, we prove general theorems for higher Grothendieck–Witt groups from their $K$-theory counterparts. As an example of application, we give a conceptual computation of the Hermitian $K$-groups of rings of 2-integers in certain totally real number fields [7] and relate their orders to values of Dedekind zeta-functions; see Theorems 5.6 and 5.9.

Our innovation in the proof of Williams' conjecture is the simultaneous proof of another conjecture: the counterpart, for Hermitian K-theory, of the Quillen–Lichtenbaum conjecture in $K$-theory. The main goal is to compare the higher Grothendieck–Witt groups with mod $2^\nu$ coefficients to their étale analogs. In Theorem 2.7, we show that the étale comparison map for Hermitian $K$-theory is an isomorphism on homotopy groups in the same range and under the same hypotheses as it is for $K$-theory. The Hermitian Quillen–Lichtenbaum conjecture was first explored in [8], where the étale comparison map was shown to be split surjective, and conjectured to be bijective, in sufficiently high degrees.

Since the work of Artin and Grothendieck, it is well known that for complex algebraic varieties, étale homotopy (with finite coefficients) coincides with the topological analog. Therefore, our results enable us to compute higher Grothendieck–Witt groups of complex algebraic varieties in terms of topological data. We also give similar computations for totally imaginary number fields in terms of étale cohomology. See Theorems 5.1 and 5.8 for precise statements.

## 2. Statement of results

Here is a more detailed description of the results in this paper. Most of our arguments take place in the setting of spectra associated to what we shall call a *QL* scheme (in honor

of Quillen and Lichtenbaum). We let $\mathrm{vcd}_2(X)$ be shorthand for $\sup\{\mathrm{vcd}_2(k(x)) \mid x \in X\}$, where, for any field $k$, the virtual mod-2 cohomological dimension $\mathrm{vcd}_2(k)$ is the mod-2 étale cohomological dimension of $k(\sqrt{-1})$.

**Definition 2.1.** We call a scheme $X$ a *QL scheme* if it is noetherian of finite Krull dimension, $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$, $\mathrm{vcd}_2(X) < \infty$ and $X$ has an ample family of line bundles.

Note that $\mathrm{vcd}_2(X) < \infty$ if $X$ is of finite type over $\mathbb{Z}[\frac{1}{2}]$ or $X = \mathrm{Spec}(k)$, where $k$ is a field for which $\mathrm{vcd}_2(k) < \infty$. The ampleness condition means that $X$ is a finite union of open affine subsets of the form $\{f_i \neq 0\}$ with $f_i$ a section of a line bundle $\mathcal{L}_i$ on $X$. Examples include all affine schemes, separated regular noetherian schemes, and quasi-projective schemes over a scheme with an ample family of line bundles. Every QL scheme $X$ is quasi-separated because the underlying topological space of $X$ is noetherian. We are relieved of any regularity conditions in Definition 2.1 because the descent results in [36,45], and the use of Gabber rigidity for $K$-theory and Grothendieck–Witt theory, see Theorem 3.6, apply to QL schemes.

For a fixed line bundle $\mathcal{L}$ on $X$, let $GW^{[n]}(X, \mathcal{L})$ denote the Grothendieck–Witt spectrum of $X$ with coefficients in the $n$th shifted chain complex $\mathcal{L}[n]$. This is the Grothendieck–Witt spectrum of the category of bounded chain complexes of vector bundles over $X$ equipped with the duality functor $E \mapsto \mathrm{Hom}(E, \mathcal{L}[n])$ and quasi-isomorphisms as the weak equivalences [39, §8]. If $X = \mathrm{Spec}(R)$ is affine, $n = 0$ or 2 and $\mathcal{L} = \mathcal{O}_X$, its nonnegative homotopy groups coincide with Karoubi's Hermitian $K$-groups of $R$ [19], with the sign of symmetry $\varepsilon = \pm 1$. If $n = 1$ or 3, we recover the so-called $U$-groups [19]. For $GW^{[n]}(X, \mathcal{L})$, we employ the delooping constructed in [36, Theorem 5.5 and Proposition 5.6], whose $i$th homotopy group $GW_i^{[n]}(X, \mathcal{L})$ is naturally isomorphic to Balmer's triangular Witt group $W^{n-i}(X, \mathcal{L})$ when $i < 0$; see [5], [36, Proposition 6.3].

We write $K^{[n]}(X, \mathcal{L})$ for the connective $K$-theory spectrum $K(X)$ of $X$ equipped with the $C_2 = \mathbb{Z}/2$-action induced by the duality functor $\mathrm{Hom}(-, \mathcal{L}[n])$. Recall from [25], [36, §7.2] the natural map

$$GW^{[n]}(X, \mathcal{L}) \longrightarrow K^{[n]}(X, \mathcal{L})^{hC_2} \tag{2-a}$$

between Hermitian $K$-theory and the homotopy fixed points of $K$-theory.

Throughout the paper we use the term "equivalence" as shorthand for a "map that induces isomorphisms on all homotopy groups."

**Theorem 2.2** (*Homotopy Fixed Point Theorem*). *Let $X$ be a QL scheme as in Definition 2.1 above. Then for all $\nu \geq 1$, the map (2-a) induces an equivalence of spectra mod $2^\nu$:*

$$GW^{[n]}(X, \mathcal{L}; \mathbb{Z}/2^\nu) \xrightarrow{\simeq} K^{[n]}(X, \mathcal{L}; \mathbb{Z}/2^\nu)^{hC_2}.$$

**Remark 2.3.** Williams conjectured this theorem in [50, p. 627] for affine $X$ (and noncommutative rings), but with no restriction on the cohomological dimension. In that generality, however, there are counterexamples; see [15] for fields of infinite virtual mod-2 cohomological dimension and [7, Appendix C] for noncommutative rings.

Most of the results of this paper deal with 2-primary coefficients. For $p$-primary coefficients with $p$ an odd prime see Remark 4.9 below. One exception is the following result, which Proposition 4.7 shows is the best that we may expect integrally.

**Theorem 2.4** *(Integral Homotopy Fixed Point Theorem). Let $X$ be a QL scheme. If $-1$ is a sum of squares in all residue fields of $X$, then the map (2-a) is an equivalence of spectra*

$$GW^{[n]}(X, \mathcal{L}) \xrightarrow{\simeq} K^{[n]}(X, \mathcal{L})^{hC_2}.$$

For example, the map (2-a) is an equivalence when $X$ is of finite type over the Gaussian 2-integers $\mathbb{Z}[\frac{1}{2}, \sqrt{-1}]$ or when $X$ is defined over a field that is not formally real, *e.g.*, an algebraically closed field of characteristic $\neq 2$ or a field of odd characteristic. If $\mathcal{L} = \mathcal{O}_X$ then the converse holds; see Proposition 4.7. For example, the map (2-a) is not an integral equivalence for $X = \mathrm{Spec}(R)$ where $R = \mathbb{Z}[\frac{1}{2}]$, $\mathbb{Q}$ or $\mathbb{R}$.

Recall from [36, Definition 7.1] (for affine $X$; see also [32]) the $L$-theory spectrum $L(X, \mathcal{L})$ of a $\mathbb{Z}[\frac{1}{2}]$-scheme $X$ with coefficients in the line-bundle $\mathcal{L}$. By [36, Proposition 7.2], its homotopy groups $\pi_i L(X, \mathcal{L})$ are naturally isomorphic to the higher Witt-groups $W^{-i}(X, \mathcal{L})$ of Balmer [5]. Further, denote by $\widehat{H}(C_2, F)$ the Tate-spectrum of a spectrum $F$ with $C_2$-action.

**Corollary 2.5.** *For any QL scheme $X$, the map*

$$L(X, \mathcal{L}) \longrightarrow \widehat{H}(C_2, K(X, \mathcal{L}))$$

*is a 2-adic equivalence. It is an integral equivalence under the further hypothesis of Theorem 2.4.*

In the formulation of the next theorem we employ the "non-connective" versions $\mathbb{G}W$ of $GW$ [39, p. 430, Definition 8], [36, Definition 8.6 and Remark 8.8] and $\mathbb{K}$ of $K$ [45, p. 360, Definition 6.4], [38, p. 123, Definition 12.1]. We note the following consequence of our previous results.

**Corollary 2.6.** *Theorems 2.2 and 2.4 remain valid if one replaces $GW$ and $K$ with $\mathbb{G}W$ and $\mathbb{K}$, respectively.*

We write $GW^{[n]}(X_{\text{ét}}, \mathcal{L})$ for the value at $X$ of a globally fibrant replacement of $GW^{[n]}(\ ,\mathcal{L})$ on the small étale site $X_{\text{ét}}$ of $X$; see [8,16], or 3.3 below. Also, recall that a

map is said to be $m$-coconnected when its homotopy fiber is; equivalently, on the $i$th homotopy groups the map induces an isomorphism whenever $i > m$ and a monomorphism when $i = m$.

**Theorem 2.7** *(Hermitian Quillen–Lichtenbaum). Let $X$ be a QL scheme. Then for all $\nu \geq 1$ the natural map*

$$GW^{[n]}(X, \mathcal{L}; \mathbb{Z}/2^\nu) \longrightarrow GW^{[n]}(X_{\text{ét}}, \mathcal{L}; \mathbb{Z}/2^\nu) \tag{2-b}$$

*is $(\mathrm{vcd}_2(X) - 2)$-coconnected.*

Theorem 2.7 is the evident analog for Hermitian $K$-theory of the well known $K$-theoretic Quillen–Lichtenbaum conjecture. In [34], the $K$-theory analog was proved in essentially the same generality as in Theorem 2.7 above; see also Theorem 3.7 below. The statement of Theorem 2.7 was conjectured in [8], where the map was shown to be split surjective in sufficiently high degrees.

## 3. Preliminaries

In this section, we collect a few well-known facts; no originality is claimed.

For a given scheme $X$, fix a line bundle $\mathcal{L}$ on $X$ and set $\ell = 2^\nu$. For legibility we often write $GW(X)$ for the spectrum $GW^{[n]}(X, \mathcal{L})$, $GW/\ell(X)$ for $GW^{[n]}(X, \mathcal{L}; \mathbb{Z}/\ell)$, $K(X)$ for $K^{[n]}(X, \mathcal{L})$, etc. We also sometimes drop the parameter scheme $X$.

**3.1. The $C_2$-action on $K$-theory and homotopy fibrations.** Although it will not be needed in this paper, we note that the $C_2$-action on $K^{[0]}(X, \mathcal{O}_X)$ (resp. $K^{[2]}(X, \mathcal{O}_X)$) coincides up to homotopy with the $C_2$-action defined in [7] and [8] (in the affine case) with the sign of symmetry $\varepsilon = 1$ (resp. $\varepsilon = -1$).

Now suppose that the scheme $X$ has an ample family of line bundles, and $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$. In [36, Theorem 7.6], the following are shown to hold.

(1) There is a homotopy fibration of spectra

$$K^{[n]}(X, \mathcal{L})_{hC_2} \longrightarrow GW^{[n]}(X, \mathcal{L}) \longrightarrow L^{[n]}(X, \mathcal{L}).$$

The first term is the homotopy orbit spectrum for the $C_2$-action on the $K$-theory spectrum $K^{[n]}(X, \mathcal{L})$. The homotopy groups $\pi_j L^{[n]}(X, \mathcal{L})$ are naturally isomorphic to Balmer's Witt groups $W^{n-j}(X, \mathcal{L})$ for all $n, j \in \mathbb{Z}$. Recall from [5] that the groups $W^i$ are 4-periodic in $i$ and coincide with the classical Witt groups in degrees $\equiv 0 \pmod 4$. For a local ring $R$ with $\frac{1}{2} \in R$ we have $W^i(R) = 0$ for $i \not\equiv 0 \pmod 4$.

(2) There is a homotopy cartesian square of spectra

$$
\begin{array}{ccc}
GW^{[n]}(X, \mathcal{L}) & \longrightarrow & L^{[n]}(X, \mathcal{L}) \\
\downarrow & & \downarrow \\
K^{[n]}(X, \mathcal{L})^{hC_2} & \longrightarrow & \hat{H}(C_2, K^{[n]}(X, \mathcal{L}))
\end{array}
$$

where for a spectrum $Y$ with $C_2$-action, the Tate spectrum $\hat{H}(C_2, Y)$ is the cofiber of the hypernorm map $Y_{hC_2} \to Y^{hC_2}$; see [17, Ch. 3].

(3) Let $\eta \in GW^{[-1]}_{-1}(\mathbb{Z}[\frac{1}{2}]) \cong W^0(\mathbb{Z}[\frac{1}{2}])$ correspond to the unit $1 \in W^0(\mathbb{Z}[\frac{1}{2}])$. Then the horizontal maps in (2) induce equivalences of spectra

$$
GW^{[n]}[\eta^{-1}] \xrightarrow{\cong} L^{[n]} \quad \text{and} \quad (K^{[n]})^{hC_2}[\eta^{-1}] \xrightarrow{\cong} \hat{H}(C_2, K^{[n]}).
$$

Both spectra $L^{[n]}(X, \mathcal{L})$ and $\hat{H}(C_2, K^{[n]}(X, \mathcal{L}))$ are 4-periodic and the map $L^{[n]}(X, \mathcal{L}) \to \hat{H}(C_2, K^{[n]}(X, \mathcal{L}))$ commutes with the periodicity maps by [36, Remark 7.7], [49]. Hence the homotopy fiber $\mathcal{F}$ of $GW^{[n]}(X, \mathcal{L}) \to K^{[n]}(X, \mathcal{L})^{hC_2}$ satisfies

$$
\pi_i \mathcal{F} \cong \pi_{i+4} \mathcal{F} \quad \text{for all } i \in \mathbb{Z}.
$$

**Remarks 3.2.** For the affine non-connective analogs of the previous statements, see also [25] and [50, Theorem 13]. A possible generalization to schemes goes via the Mayer–Vietoris principle [39]. An alternate approach is developed in [36].

**3.3. Presheaves of spectra.** Let $X$ be a scheme. Its small étale site $X_{\text{ét}}$ is comprised of finite type étale $X$-schemes $U \to X$ and maps between $X$-schemes, along with étale coverings. If $\mathrm{vcd}_2(X) = n$, then $\mathrm{vcd}_2(U) \le n$ for all $U \in X_{\text{ét}}$.

We denote by $\mathrm{PSp}(X_{\text{ét}})$ the model category of presheaves of spectra on $X_{\text{ét}}$ [16]. Its objects are contravariant functors from $X_{\text{ét}}$ to spectra and maps are natural transformations of such functors. We are mainly interested in the presheaves of spectra $GW^{[n]}(\mathcal{L})$ sending $p : U \to X$ to $GW^{[n]}(U, p^*\mathcal{L})$ and its $K$-theory analog. We shall often suppress $\mathcal{L}$ and $[n]$ in the notation.

A map of presheaves of spectra $\mathcal{F} \to \mathcal{G}$ on $X_{\text{ét}}$ is:

(1) a *pointwise weak equivalence* if for all $U \in X_{\text{ét}}$, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence of spectra;

(2) a *local weak equivalence* if for all points $x \in X$, $\mathcal{F}_x \to \mathcal{G}_x$ is a weak equivalence of spectra, where $\mathcal{F}_x$ is the filtered colimit $\mathcal{F}_x = \mathrm{colim}_{U \to X} \mathcal{F}(U)$ over all étale neighborhoods $U$ of $x$;

(3) a *cofibration* if it is pointwise a cofibration, that is, if for all $U \in X_{\text{ét}}$, the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is a cofibration of spectra in the sense of [10]; and

(4) a *local fibration* if it has the right lifting property with respect to all cofibrations which are also local weak equivalences.

It is proved in [16] that the category $\mathrm{PSp}(X_{\mathrm{ét}})$ together with the local weak equivalences, cofibrations and local fibrations is a proper closed (simplicial) model category.

Note that by Theorem 3.6 below, $(GW^{[n]}/\ell)_x \simeq GW^{[n]}/\ell(k)$, where $k$ is a separable closure of the residue field of $x$. However, there is, *a priori*, no evident equivalence between $(K^{hC_2}/\ell)_x$ and $K/\ell(k)^{hC_2}$ since the homotopy fixed point functor $(\ )^{hC_2}$ does not commute with filtered colimits, in general. Compare Lemma 4.5 below.

From the theory of model categories, there exists a globally fibrant replacement functor

$$\mathrm{PSp}(X_{\mathrm{ét}}) \longrightarrow \mathrm{PSp}(X_{\mathrm{ét}}) : \mathcal{F} \longmapsto \mathcal{F}^{\mathrm{ét}}. \tag{3-a}$$

By definition, this is a functor equipped with a natural local weak equivalence $\mathcal{F} \to \mathcal{F}^{\mathrm{ét}}$ for which the map from $\mathcal{F}^{\mathrm{ét}}$ to the final object is a local fibration. The Hermitian Quillen–Lichtenbaum Theorem 2.7 is a statement about the map $\mathcal{F} \to \mathcal{F}^{\mathrm{ét}}$ when $\mathcal{F}$ is the Hermitian $K$-theory presheaf.

Call a square of presheaves of spectra *pointwise homotopy cartesian* if it becomes a homotopy cartesian square of spectra when evaluated at all finite type étale $X$-schemes. We need the following observations.

**Lemma 3.4.**

(1) *The globally fibrant replacement functor (3-a) sends pointwise homotopy cartesian squares to pointwise homotopy cartesian squares.*
(2) *Let $n$ be an integer. If a presheaf of spectra $\mathcal{F}$ satisfies $\pi_i(\mathcal{F}_x) = 0$ for all $i \geq n$ and all points $x \in X$ then $\pi_i(\mathcal{F}^{\mathrm{ét}}(U)) = 0$ for all $i \geq n$ and $U \in X_{\mathrm{ét}}$.*

**Proof.** Both statements are true for any small Grothendieck site (with enough points so that we can formulate the second part of the lemma). For the first part, recall that in the category of spectra, homotopy cartesian is the same as homotopy co-cartesian. Since cofibrations are pointwise cofibrations and pointwise weak equivalences are local weak equivalences, it is clear that the globally fibrant replacement functor preserves pointwise homotopy co-cartesian squares. The second part is explicitly stated in [17, Proposition 6.12]. $\square$

There are evident analogs for the Nisnevich topology $X_{\mathrm{Nis}}$ on $X$. Details are *mutatis mutandis* the same.

For later reference we include the following results. Recall that $\ell = 2^{\nu}$.

**Lemma 3.5.** *Let $F \subset L$ be a purely inseparable algebraic extension of fields of characteristic $\neq 2$. Then the inclusion $F \subset L$ induces equivalences of spectra*

$$K/\ell(F) \xrightarrow{\simeq} K/\ell(L), \qquad GW^{[n]}/\ell(F) \xrightarrow{\simeq} GW^{[n]}/\ell(L)$$

*and an isomorphism of Witt groups*

$$W(F) \xrightarrow{\cong} W(L).$$

**Proof.** The $K$-theory statement is due to Quillen [30, Proposition 4.8]. For Witt-groups, see [2, p. 456, §2]. The result for $GW^{[n]}/\ell$ now follows from the homotopy fibration (3.1 (1)) and the vanishing of $W^i(k)$ for $k$ a field and $i \not\equiv 0 \pmod 4$. $\quad\square$

**Theorem 3.6** *(Rigidity). Let $R$ be a Henselian local ring with residue field $k$ and $\frac{1}{2} \in R$. Then the map $R \to k$ induces equivalences of spectra*

$$K/\ell(R) \xrightarrow{\simeq} K/\ell(k), \qquad GW^{[n]}/\ell(R) \xrightarrow{\simeq} GW^{[n]}/\ell(k)$$

*and an isomorphism of Witt groups*

$$W(R) \xrightarrow{\cong} W(k).$$

**Proof.** The $K$-theory (resp. Witt-theory) result is due to Gabber [13] (resp. Knebusch [23, Satz 3.3]). The claim for Grothendieck–Witt theory then follows from the homotopy fibration (3.1 (1)). $\quad\square$

The following theorem is implicit in [34] but was formulated only as an equivalence on $(\mathrm{vcd}_2(X) - 2)$-connected covers.

**Theorem 3.7** *($K$-theoretic Quillen–Lichtenbaum). Let $X$ be a QL scheme. Then for all $\nu \geq 1$ the natural map*

$$K(X; \mathbb{Z}/2^\nu) \longrightarrow K^{\text{ét}}(X; \mathbb{Z}/2^\nu)$$

*is $(\mathrm{vcd}_2(X) - 2)$-coconnected.*

**Proof.** With $\ell = 2^\nu$ the map in the theorem factors as

$$K/\ell(X) \to (K/\ell)^{\mathrm{Nis}}(X) \to (K/\ell)^{\text{ét}}(X) \tag{3-b}$$

where $(K/\ell)^{\mathrm{Nis}}$ denotes a globally fibrant model for the Nisnevich topology. We first show that the second map is $(\mathrm{vcd}_2(X) - 2)$-coconnected. For fields, this is [34, (11), §5]. The case of Henselian rings reduces to the case of fields, by rigidity for $K$-theory and its étale version (see *e.g.* the proof of [28, Lemma 4.14] and [29, Proposition 6.1]). For general $X$ as in the theorem, the result follows from the Henselian case in view of the strongly convergent Nisnevich descent spectral sequence applied to the homotopy fiber of the second map in (3-b).

To finish the proof, we note that the first map in (3-b) is always 0-coconnected, so the theorem follows as soon as $\mathrm{vcd}_2(X) \geq 2$. Since $\dim X \leq \mathrm{vcd}_2 X$, we are left with the cases $\dim X = 0, 1$. If $\dim X = 0$ then the first map in (3-b) is an equivalence. If $\dim X = 1$ then this map is $(-1)$-coconnected. This assertion follows from the fact that $K_{-1}$ is torsion free for noetherian schemes of Krull dimension $\leq 1$; see [48, Lemma 2.5 (2)].

Therefore, the maps $K/\ell \to \mathbb{K}/\ell$ and hence $(K/\ell)^{\mathrm{Nis}} \to (\mathbb{K}/\ell)^{\mathrm{Nis}}$ are $(-1)$-coconnected for such schemes. Moreover, $\mathbb{K}/\ell \to (\mathbb{K}/\ell)^{\mathrm{Nis}}$ is a pointwise weak equivalence, by [45]. $\square$

**Lemma 3.8.** *Suppose that $X$ is a quasi-compact scheme with an ample family of line bundles. Then the following are equivalent.*

(1) *There exists an integer $n > 0$ such that $2^n W(X) = 0$.*
(2) *$-1$ is a sum of squares in all the residue fields of $X$.*

**Proof.** In [24, Theorem 3, p. 189] it is shown that (1) is equivalent to the statement that all the residue fields of $X$ have 2-primary torsion Witt groups. The latter is equivalent to (2); see for instance [35, Theorem II.7.1]. $\square$

**Remark 3.9.** If $X$ is affine, the condition that $-1$ is a sum of squares in all the residue fields of $X$ is equivalent to $-1$ being a sum of squares in $\Gamma(X, \mathcal{O}_X)$; see for instance [24, Proposition 4, p. 190]. If $X$ is non-affine, then $-1$ might be a sum of squares in all residue fields without being a sum of squares in $\Gamma(X, \mathcal{O}_X)$. Indeed, every smooth projective real curve $X$ with function field of level $> 1$ has the property that $-1$ is a sum of squares in all of its residue fields, but not in $\Gamma(X, \mathcal{O}_X) = \mathbb{R}$. For example, take the closed subscheme of $\mathbb{P}^2_{\mathbb{R}} = \mathrm{Proj}(\mathbb{R}[X, Y, Z])$ cut out by the equation $X^2 + Y^2 + Z^2 = 0$.

## 4. Proofs

Our proof of the following lemma uses the main result in [15].

**Lemma 4.1.** *Theorem 2.2 holds for fields $k$ with $\mathrm{vcd}_2(k) < \infty$ and $\mathrm{char}(k) \neq 2$.*

**Proof.** We claim that the homotopy fiber $\mathcal{F}$ of $GW/\ell \to K/\ell^{hC_2}$ is contractible. If $\mathrm{char}(k) = 0$, this holds by [15]. If $\mathrm{char}(k) > 0$, we reduce the claim to the case of characteristic 0 by means of "Teichmüller lifting". In effect, we may assume that $k$ is perfect, since $K/\ell$ and $GW/\ell$ are invariant under purely inseparable algebraic field extensions (Lemma 3.5). Then the ring $V$ of Witt-vectors over $k$ is a complete (hence Henselian) dvr with residue field $k$ and fraction field $F$ of characteristic 0. Furthermore,

$$\mathrm{vcd}_2(F) \leq \mathrm{cd}_2(F) = \mathrm{cd}_2(k) + 1 = \mathrm{vcd}_2(k) + 1 < \infty,$$

by [3, Exposé X, Théorème 2.2 (ii)] and [41, II §4.1].

Let $\pi \in V$ be a uniformizer. We claim that $\alpha : V[t, t^{-1}] \to F : t \mapsto \pi$, induces an equivalence $\mathcal{F}(\alpha) : \mathcal{F}(V[t, t^{-1}]) \xrightarrow{\cong} \mathcal{F}(F)$. It is known that $K/\ell(\alpha)$ is an equivalence (one may use the same argument as for Witt groups below), and hence $K/\ell^{hC_2}(\alpha)$ and $K/\ell_{hC_2}(\alpha)$ are equivalences. To show that $GW/\ell(\alpha)$ is an equivalence, we consider the spectrum $L$ defined by the homotopy fibration $K_{hC_2} \to GW \to L$ (3.1 (1)). The groups

$\pi_i L$ are 4-periodic, trivial for local rings in degrees $\not\equiv 0 \pmod 4$ and homotopy invariant for regular rings. Using the localization exact sequence for $V[t] \to V[t, t^{-1}]$, we get $W^i(V[t, t^{-1}]) = 0$ for $i \not\equiv 0 \pmod 4$. Thus, it remains to check that $W^0(V[t, t^{-1}]) \to W^0(F)$ is an isomorphism. This follows by a comparison of the localization exact sequences for $V[t] \to V[t, t^{-1}]$ and $V \to F$, which reduces to a map between short exact sequences:

$$
\begin{array}{ccccccccc}
0 & \to & W^0(V[t]) & \longrightarrow & W^0(V[t, t^{-1}]) & \longrightarrow & W^0(V) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & W^0(V) & \longrightarrow & W^0(F) & \longrightarrow & W^0(k) & \to & 0.
\end{array}
$$

Theorem 3.6 shows that the right vertical map, induced by the reduction map modulo $\pi$, is an isomorphism. The left vertical map is an isomorphism by homotopy invariance of Witt-theory. It follows that $W^0(\alpha)$ is an isomorphism, as claimed.

Because augmentation makes $\mathcal{F}(V)$ a retract of $\mathcal{F}(V[t, t^{-1}])$, combining with the equivalence $\mathcal{F}(\alpha) : \mathcal{F}(V[t, t^{-1}]) \xrightarrow{\simeq} \mathcal{F}(F)$ makes $\mathcal{F}(V)$ also a retract of $\mathcal{F}(F)$. However, since $\mathrm{char}(F) = 0$ and $\mathrm{vcd}_2(F) < \infty$, we have $\mathcal{F}(F) \simeq *$, by [15]; this now implies that $\mathcal{F}(V) \simeq *$. Finally, by rigidity, $\mathcal{F}(V) \to \mathcal{F}(k)$ is an equivalence. Hence, $\mathcal{F}(k) \simeq *$, as sought.  $\square$

**4.2. Bott elements.** Let $p = 8m$ for $m \geq 1$ an integer, and let $\ell$ be the highest power of 2 that divides $3^{4m} - 1$; that is, $\ell = \max\{2^k \mid 2^k \text{ divides } 3^{4m} - 1\}$. An element $\beta \in \pi_p(S^0; \mathbb{Z}/\ell)$ in the mod $\ell$ stable stems is called a *(positive) Bott element* if it maps to the reduction mod $\ell$ of a generator of $KO_p \cong \mathbb{Z}$ under the unit map $S^0 \to KO$, where $KO$ denotes the Bott-periodic real topological $K$-theory spectrum. We require Bott elements in our proof (Lemma 4.5) of the étale version of the Homotopy Fixed Point Theorem 2.2. Bott elements for higher Grothendieck–Witt theory and arbitrary coefficients were first constructed in [8]. In what follows we give a simpler construction that suffices for our purposes.

**Lemma 4.3.** *Let $\ell$ be as in Section 4.2. Then there is an element $\beta \in \pi_p(S^0/\ell)$ in the mod $\ell$ stable stem whose image in $\pi_p(KO/\ell)$ under the unit map $S^0 \to KO$ is the reduction mod $\ell$ of a generator of $KO_p = \mathbb{Z}$.*

**Proof.** The construction of $\beta$, essentially due to Quillen, is based on Adams' work on the image $J(\pi_* O) \subseteq \pi_*(S^0)$ of the $J$-homomorphism [1]. Recall that the 2-primary part $J(\pi_{p-1} O)_{(2)}$ of $J(\pi_{p-1} O)$ is cyclic of order $\ell$. For a spectrum $F$, write $F_{tor}$ for the homotopy fiber of the rationalization map $F \to F_{\mathbb{Q}}$, and note that the natural map $F_{tor}/\ell \to F/\ell$ is an equivalence. Further, recall that $\pi_*(S^0_{tor}) \to \pi_*(S^0)$ is an isomorphism for $* > 0$. So, the $J$-homomorphism has image in $\pi_*(S^0_{tor})$. Consider the commutative diagram

$$\begin{array}{ccc}
\pi_p(S^0_{tor}/\ell) & \longrightarrow & \pi_p(KO_{tor}/\ell) \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\
J(\pi_{p-1}O)_{(2)} \cong \mathbb{Z}/\ell \hookrightarrow \pi_{p-1}(S^0_{tor}) & \longrightarrow & \pi_{p-1}(KO_{tor})
\end{array}$$

in which the diagonal map exists because of the long exact sequence of homotopy groups associated with the fibration $S^0_{tor} \xrightarrow{\cdot\ell} S^0_{tor} \to S^0_{tor}/\ell$. In [31, p. 183, §2], Quillen shows that the composition of the lower two maps is injective. It follows that the composition of the diagonal map with the upper horizontal map in the previous and in the following commutative diagram is injective

$$\begin{array}{ccc}
\pi_p(S^0_{tor}/\ell) & \longrightarrow & \pi_p(KO_{tor}/\ell) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
J(\pi_{p-1}O)_{(2)} \cong \mathbb{Z}/\ell \longrightarrow \pi_p(S^0/\ell) & \longrightarrow & \pi_p(KO/\ell) \cong \mathbb{Z}/\ell.
\end{array}$$

Therefore, the composition of the two lower horizontal maps in the last diagram is an injection of finite groups of the same order. Hence, this composition is an isomorphism. In particular, the map $\pi_p(S^0/\ell) \to \pi_p(KO/\ell)$ is surjective, and we can lift the generator mod $\ell$ of $KO_p$ to an element $\beta \in \pi_p(S^0/\ell)$. $\quad\square$

**Lemma 4.4.** *Let $k$ be a separably closed field of characteristic* $\mathrm{char}(k) \neq 2$, *and let* $\eta \in GW^{[-1]}_{-1}(k) \cong W^0(k)$ *correspond to the unit of the ring $W^0(k)$. Then $\beta\eta^p = 0$ in $GW(k; \mathbb{Z}/\ell)$. In particular, $L/\ell(k)[\beta^{-1}] \simeq *$.*

**Proof.** By [20,21] the ring spectrum $GW/\ell(k)[\beta^{-1}]$ is equivalent to $KO/\ell$ and the natural map $GW(k)/\ell \to GW/\ell(k)[\beta^{-1}]$ is an equivalence on connective covers. Thus, it suffices to check that $\beta\eta^p = 0$ in $\pi_0(KO/\ell)$. In $KO/\ell$, the elements $\beta$ and $\eta$ are reductions of integral classes. More precisely, $\beta$ is the reduction mod $\ell$ of $b^m$, where $b \in KO_8 = \mathbb{Z}$ is a generator, and $\eta^4 \in GW^{[-4]}_{-4}(\mathbb{C})$. However, the map

$$\mathbb{Z}/2 \cong GW^{[-4]}_{-4}(\mathbb{C}) = GW^{[0]}_{-4}(\mathbb{C}) \longrightarrow KO_{-4} = \mathbb{Z}$$

is trivial. Consequently, $\eta^p = 0$ in $KO_{-p}$ and hence $\beta\eta^p = 0$.

For the $L$-theory statement, recall that $L = GW[\eta^{-1}]$; see (3.1 (3)). Therefore, $\beta\eta^p = 0$ implies $L/\ell(k)[\beta^{-1}] \simeq *$. $\quad\square$

**Lemma 4.5.** *Let $\nu > 0$ be an integer and $\ell = 2^\nu$. Let $X$ be a QL scheme. Then the map*

$$GW^{\mathrm{\acute{e}t}}/\ell(X) \longrightarrow (K^{hC_2})^{\mathrm{\acute{e}t}}/\ell(X)$$

*is an equivalence.*

**Proof.** For an integer $\nu > 0$, a map of spectra is an equivalence mod $2^\nu$ if and only if it is an equivalence mod 2. Therefore, we can assume $\ell$ to be as in Section 4.2, and we have Bott elements at our disposal. Consider the commutative diagram

$$
\begin{array}{ccccc}
(GW/\ell)^{\text{ét}}(X) & \longrightarrow & (GW/\ell[\beta^{-1}])^{\text{ét}}(X) & \longrightarrow & (L/\ell[\beta^{-1}])^{\text{ét}}(X) \\
\downarrow & & \downarrow & & \downarrow \\
(K/\ell^{hC_2})^{\text{ét}}(X) & \longrightarrow & (K/\ell^{hC_2}[\beta^{-1}])^{\text{ét}}(X) & \longrightarrow & (\hat{H}/\ell[\beta^{-1}])^{\text{ét}}(X)
\end{array}
$$

in which the right-hand square is obtained from the homotopy cartesian square (3.1 (2)) by reduction mod $\ell$, inverting the positive Bott element constructed in Lemma 4.3, and taking étale globally fibrant replacements. All these operations preserve (pointwise) homotopy cartesian squares. Thus, the right-hand square in the diagram is homotopy cartesian. By Lemma 4.4, Theorem 3.6 and Subsection 3.1 (1), the upper right corner of the diagram is trivial. Since $\hat{H}/\ell[\beta^{-1}]$ is a module spectrum over $L/\ell[\beta^{-1}]$, the lower right corner of the diagram is trivial as well. Hence, the middle vertical arrow is an equivalence. In view of Lemma 3.4 and Theorem 3.6, the upper left horizontal arrow is an equivalence on connective covers since $GW/\ell(F) \to GW/\ell(F)[\beta^{-1}]$ has this property for separably closed fields $F$. By Lemma 3.4, the lower left horizontal map is an equivalence on some connected cover, because, by the solution of the $K$-theoretic Quillen–Lichtenbaum conjecture [44] in the generality of [34], the map $K/\ell^{hC_2} \to K/\ell^{hC_2}[\beta^{-1}]$ is a pointwise (hence local) weak equivalence on $(\text{vcd}_2(X)-2)$-connected covers. Hence, the fiber of the left vertical map has trivial homotopy groups in high degrees. By periodicity (3.1 (3)), we are done. $\square$

**Lemma 4.6.** *If Theorem 2.7 holds for the residue fields of a QL scheme $X$, then the map (2-b) is $n$-coconnected for some integer $n$.*

**Proof.** By definition, $GW^{\text{ét}}/\ell$ satisfies Nisnevich descent. In positive degrees the same holds for $GW/\ell$ [36]. More precisely, $\mathbb{G}W$ satisfies Nisnevich descent [36, Theorem 9.7] and the map $GW \to \mathbb{G}W$ is an equivalence on connective covers [36, Proposition 8.7 or Theorem 8.14]. The map between the $E^2$-pages of the corresponding Nisnevich descent spectral sequences takes the form

$$
H^p_{\text{Nis}}(X; \tilde{\pi}_q(GW/\ell)) \longrightarrow H^p_{\text{Nis}}(X; \tilde{\pi}_q(GW^{\text{ét}}/\ell)).
$$

By rigidity, see Theorem 3.6, the assumption shows that the canonically induced map

$$
\tilde{\pi}_q(GW/\ell) \longrightarrow \tilde{\pi}_q(GW^{\text{ét}}/\ell)
$$

of Nisnevich sheaves is an isomorphism for $q \geq \text{vcd}_2(X) - 1$. The result follows from the fact that $H^p_{\text{Nis}}(X, A) = 0$ for $p > \dim X$ and $p < 0$ and strong convergence of the descent spectral sequences [17, Theorem 7.58]. $\square$

**Proofs of Theorems 2.2 and 2.7.** Consider the commutative diagram:

$$
\begin{array}{ccc}
GW/\ell(X) & \longrightarrow & GW^{\text{ét}}/\ell(X) \\
\downarrow & & \downarrow \\
\left[K/\ell^{hC_2}\right](X) & \longrightarrow & \left[K/\ell^{hC_2}\right]^{\text{ét}}(X).
\end{array}
\tag{4-a}
$$

By the solution of the $K$-theoretic Quillen–Lichtenbaum conjecture (Theorem 3.7), the homotopy fiber of the lower horizontal map is $(\mathrm{vcd}_2(X) - 2)$-coconnected. Lemma 4.5 shows the right vertical map is an equivalence, while Lemma 4.1 shows the left vertical map is an equivalence for fields. This implies the Hermitian Quillen–Lichtenbaum Theorem 2.7 for fields. Using Lemma 4.6, we have that the upper horizontal map is an isomorphism in high degrees. It follows that the homotopy fiber of the left vertical map in (4-a) is trivial in high degrees. By periodicity (3.1 (3)), the homotopy fiber has trivial homotopy in all degrees. Thus the left vertical map in (4-a) is an equivalence. This proves the Homotopy Fixed Point Theorem 2.2. Since both vertical maps are equivalences, the homotopy fiber of the upper horizontal map has trivial homotopy groups in the same range as the homotopy fiber of the lower horizontal map. This proves the Hermitian Quillen–Lichtenbaum Theorem 2.7 for schemes $X$.   □

**Proof of Theorem 2.4.** By Lemma 3.8, we have $2^m W^0(X) = 0$ for some $m > 0$. The homotopy groups of $L^{[n]}(X)$ and the Tate spectrum $\hat{H}(C_2, K^{[n]}(X))$ acquire compatible actions by $W^0$; see [49], [36, Remark 7.7]. It follows that the homotopy groups of the fiber $\mathcal{F}(X)$ of $L^{[n]}(X) \to \hat{H}(C_2, K^{[n]}(X))$ also admit such an action, and are therefore annihilated by $2^m$. However, by the Homotopy Fixed Point Theorem 2.2, the homotopy cofiber of multiplication by $2^m \colon \mathcal{F}(X) \to \mathcal{F}(X)$ is trivial; that is, multiplication by $2^m$ is an isomorphism on the homotopy groups of $\mathcal{F}(X)$. As we have just noticed, this is the zero map. Hence, $\mathcal{F}(X) \simeq *$, and the map (2-a) is an equivalence.   □

**Proof of Corollary 2.5.** This follows from Theorems 2.2 and 2.4 in view of the homotopy cartesian square 3.1 (2).   □

**Proof of Corollary 2.6.** This follows from the fact that the diagram

$$
\begin{array}{ccc}
GW(X) & \longrightarrow & \mathbb{G}W(X) \\
\downarrow & & \downarrow \\
K(X)^{hC_2} & \longrightarrow & \mathbb{K}(X)^{hC_2}
\end{array}
$$

is a homotopy cartesian square; see [36, Theorem 8.14].   □

If $\mathcal{L} = \mathcal{O}_X$ then the converse of the Integral Homotopy Fixed Point Theorem 2.4 holds.

**Proposition 4.7.** *Let $X$ be a scheme with an ample family of line bundles and $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$. If the map (2-a) is an equivalence for $\mathcal{L} = \mathcal{O}_X$ then no residue field of $X$ is*

*formally real. More generally, this conclusion holds if we assume only that the map (2-a) is an equivalence modulo some odd prime.*

**Proof.** For any prime $q$, the map $GW^{[n]}/q(X) \to (K^{[n]}/q(X))^{hC_2}$ is an equivalence if the map (2-a) is an integral equivalence. If $q$ is odd, then by (3.1 (2)), the map $L^{[n]}/q(X) \to \hat{H}(C_2, K^{[n]}/q(X))$ is an equivalence. Now multiplication by 2 is an equivalence on $K^{[n]}/q$, which implies that $\hat{H}(C_2, K^{[n]}/q(X)) \simeq *$, and hence $L^{[n]}/q(X) \simeq *$. Therefore, the Witt ring $W(X)$ is a $\mathbb{Z}[\frac{1}{q}]$-algebra, since multiplication by $q$ on $W(X) = L_n^{[n]}(X)$ is an isomorphism. If $X$ has a formally real residue field $k$, then we obtain ring maps $\mathbb{Z}[\frac{1}{q}] \to W(X) \to W(k) \to W(\bar{k}) = \mathbb{Z}$, where $\bar{k}$ is a real closure of $k$, which leads to a contradiction. $\square$

**Remark 4.8.** We should point out the necessity of our standing assumption that $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$, although the cited results in [39] are proved in greater generality. Without this assumption, the Homotopy Fixed Point Theorem 2.2 cannot hold for the following reason. As proved in [36, §2], the fundamental theorem in Hermitian $K$-theory [19] fails for the $GW^{[n]}$-spectrum whenever $X$ has a residue field of characteristic 2, whereas it does hold for the $(K^{[n]})^{hC_2}$-spectrum; see the proof of [36, Theorem 6.2]. In particular, if $X$ has a residue field of characteristic 2 then (2-a) is not an integral equivalence, in general, even if $\mathrm{vcd}_2(X) < \infty$. Moreover, it is not a 2-adic equivalence for fields of characteristic 2 (in this case the fiber of (2-a) is 2-adically complete).

If $K$-theory of symmetric bilinear forms (that is, $GW$-spectra) is replaced with $K$-theory of quadratic forms, then (2-a) is not an equivalence either, because the latter is not homotopy invariant for regular rings, whereas $K$-theory and its homotopy fixed points are. In particular, the quadratic analog of the map (2-a) is not generally a 2-adic equivalence in characteristic 2.

**Remark 4.9.** The odd-primary analog of the Hermitian Quillen–Lichtenbaum Theorem 2.7 can be read off from the isomorphisms [36, Remark 7.8]

$$GW_i^{[n]}(X, \mathcal{L}) \otimes \mathbb{Z}[1/2] \cong \left[ K_i^{[n]}(X, \mathcal{L})^{C_2} \otimes \mathbb{Z}[1/2] \right] \oplus \left[ W^{n-i}(X) \otimes \mathbb{Z}[1/2] \right].$$

Here, the $K$-summand may be computed by étale techniques thanks to the solution of the Bloch–Kato conjecture by Voevodsky, Rost, and others. Also, $W^r$ denotes Balmer's Witt groups, which coincide up to 2-torsion with the higher Witt groups defined in [19] (for affine schemes). On the other hand, the odd-primary analog of the Homotopy Fixed Point Theorem is false in general, even when $X$ is a QL scheme: see Proposition 4.7.

**Remark 4.10.** As for algebraic $K$-theory [45], higher Grothendieck–Witt theory can be defined via perfect complexes instead of vector bundles. The techniques employed in [45] apply to $GW$-theory of perfect complexes along the lines of [36]. With this definition, the results in this section should remain valid without the "ample family of line bundles"

assumption used in the definition of a QL scheme. To make this precise, one needs to display a strictly functorial model for $GW$-theory of perfect complexes.

## 5. Applications

**Theorem 5.1.** *Let $X$ be a complex algebraic variety of (complex) dimension $d$ which has an ample family of line bundles. Let $X_{\mathbb{C}}$ be the associated analytic topological space of complex points. Then for $\ell = 2^{\nu} > 1$ and $n \in \mathbb{Z}$, the canonical map*

$$GW_i^{[n]}(X; \mathbb{Z}/\ell) \longrightarrow KO^{2n-i}(X_{\mathbb{C}}; \mathbb{Z}/\ell)$$

*is an isomorphism for $i \geq d - 1$ and a monomorphism for $i = d - 2$.*

**Proof.** The theories $GW^{[n]}$ have Bott-periodic topological counterparts $GW_{\mathrm{top}}^{[n]}$ first explored in [18] as

$$GW_{\mathrm{top}}^{[0]}(X_{\mathbb{C}}) = {}_1\mathcal{L}(X_{\mathbb{C}}) = KO(X_{\mathbb{C}}), \qquad GW_{\mathrm{top}}^{[-1]}(X_{\mathbb{C}}) = {}_1\mathcal{U}(X_{\mathbb{C}}) = \Omega^2 KO(X_{\mathbb{C}}),$$
$$GW_{\mathrm{top}}^{[-2]}(X_{\mathbb{C}}) = {}_{-1}\mathcal{L}(X_{\mathbb{C}}) = \Omega^4 KO(X_{\mathbb{C}}), \quad GW_{\mathrm{top}}^{[-3]}(X_{\mathbb{C}}) = {}_{-1}\mathcal{U}(X_{\mathbb{C}}) = \Omega^6 KO(X_{\mathbb{C}}),$$

which induce the maps

$$GW^{[-n]}(X; \mathbb{Z}/\ell) \longrightarrow GW_{\mathrm{top}}^{[-n]}(X_{\mathbb{C}}; \mathbb{Z}/\ell) = \Omega^{2n} KO(X_{\mathbb{C}}; \mathbb{Z}/\ell)$$

in the theorem. In the commutative diagram

$$
\begin{array}{ccc}
GW^{[n]}(X; \mathbb{Z}/\ell) & \longrightarrow & GW_{\mathrm{top}}^{[n]}(X_{\mathbb{C}}; \mathbb{Z}/\ell) \\
\downarrow & & \downarrow \\
\left[K^{[n]}(X; \mathbb{Z}/\ell)\right]^{hC_2} & \longrightarrow & \left[KU^{[n]}(X_{\mathbb{C}}; \mathbb{Z}/\ell)\right]^{hC_2},
\end{array}
$$

the lower horizontal map is $(d-2)$-coconnected, by a theorem of Voevodsky [46, Theorem 7.10]. By Theorem 2.4, the left vertical map is an (integral) equivalence. Finally, it is a classical theorem that the right vertical map is also an (integral) equivalence. Indeed, for $n = 0$, this is the usual homotopy equivalence $KO \simeq KU^{hC_2}$ (see *e.g.* [22]); and for other $n \in \mathbb{Z}$, it follows from the topological version of the fundamental theorem in Hermitian $K$-theory [18]. $\quad\square$

**Remark 5.2.** The proof shows that the theorem also holds for odd prime powers. We simply need to remark that the map $K^{[n]}(X; \mathbb{Z}/\ell) \to KU^{[n]}(X_{\mathbb{C}}; \mathbb{Z}/\ell)$ is also $(d-2)$-coconnected for odd prime powers $\ell$ due to the solution of the Bloch–Kato conjecture by Voevodsky, Rost, Suslin and others. However, the odd prime analog of Theorem 5.1 can be more easily proved using Remark 4.9 in place of the Integral Homotopy Fixed Point Theorem. See also [8] for another argument in that case.

From now on, let $\ell$ again be a power of 2. As a second application, we give new and more conceptual proofs of the main results of [6] and [7]. Let $\mathbb{Z}'$ be short for $\mathbb{Z}[\frac{1}{2}]$. Because $K^{[n]}$ has the same (nonequivariant) homotopy type as $K$, from [33], [47, Corollary 8] we have the existence of a homotopy cartesian square

$$\begin{array}{ccc} K^{[n]}(\mathbb{Z}')/\ell & \longrightarrow & K^{[n]}_{\text{top}}(\mathbb{R})/\ell \\ \downarrow & & \downarrow \\ K^{[n]}(\mathbb{F}_3)/\ell & \longrightarrow & K^{[n]}_{\text{top}}(\mathbb{C})/\ell, \end{array}$$

where $K_{\text{top}}$ stands for connective topological $K$-theory. Now, since the fixed spectrum of $K^{[n]}$ is $GW^{[n]}$, the Homotopy Fixed Point Theorem 2.2 applied to this square yields the following.

**Theorem 5.3.** *For $\ell = 2^\nu > 1$ and $n \in \mathbb{Z}$, the square*

$$\begin{array}{ccc} GW^{[n]}(\mathbb{Z}')/\ell & \longrightarrow & GW^{[n]}_{\text{top}}(\mathbb{R})/\ell \\ \downarrow & & \downarrow \\ GW^{[n]}(\mathbb{F}_3)/\ell & \longrightarrow & GW^{[n]}_{\text{top}}(\mathbb{C})/\ell \end{array}$$

*is homotopy cartesian on connective covers.* $\quad\square$

**Remark 5.4.** According to [42], these results do not depend on whether the fields $\mathbb{R}$ and $\mathbb{C}$ are taken with the discrete or standard Euclidean topology.

This theorem enables complete computation of the groups $GW^{[n]}(\mathbb{Z}')$, up to finite groups of odd order (see [6]). In particular, if $n = 0$, the right vertical map in the above square can be identified with the split surjective map $KO \times KO \to KO$ mod $\ell$. Therefore, using 2-adic completions we get the following corollary.

**Corollary 5.5.** *For $i \geq 0$ and any one-point space* pt, *the natural map*

$$GW^{[0]}_i(\mathbb{Z}') \longrightarrow GW^{[0]}_i(\mathbb{F}_3) \oplus KO^{-i}(\text{pt})$$

*is an isomorphism modulo finite groups of odd order.* $\quad\square$

The groups $KO^{-i}(\text{pt})$ are given by Bott periodicity, and the groups $GW^{[0]}_i(\mathbb{F}_3)$ were computed by Friedlander [12].

Similarly, let $F$ be a number field and $\mathcal{O}'_F = \mathcal{O}_F[\frac{1}{2}]$ be its ring of 2-integers. Assume that $F$ is a 2-regular totally real number field with $r$ real embeddings. Let $q$ be a prime number such that the elements corresponding to the Adams operations $\psi^{-1}$ and $\psi^q$ in the ring of operations of the periodic complex topological $K$-theory spectrum generate the Galois group $F(\mu_{2^\infty})$ over $F$, where $F(\mu_{2^\infty})$ is obtained from $F$ by adjoining all

2-primary roots of unity. From [14] and [29] we have a homotopy cartesian square of connective spectra

$$
\begin{array}{ccc}
K^{[n]}(\mathcal{O}_F')/\ell & \longrightarrow & K_{\text{top}}^{[n]}(\mathbb{R})^r/\ell \\
\downarrow & & \downarrow \\
K^{[n]}(\mathbb{F}_q)/\ell & \longrightarrow & K_{\text{top}}^{[n]}(\mathbb{C})^r/\ell.
\end{array}
$$

After application of the functor $(-)^{hC_2}$ to this homotopy cartesian square, the Homotopy Fixed Point Theorem 2.2 implies the following result, which was first proved in [7] and which allows us to compute completely the groups $GW_i^{[n]}(\mathcal{O}_F')$ up to finite groups of odd order.

**Theorem 5.6.** *Let $\ell = 2^\nu > 1$ and $n \in \mathbb{Z}$. For a 2-regular totally real number field $F$ with $r$ real embeddings, the square of spectra*

$$
\begin{array}{ccc}
GW^{[n]}(\mathcal{O}_F')/\ell & \longrightarrow & GW_{\text{top}}^{[n]}(\mathbb{R})^r/\ell \\
\downarrow & & \downarrow \\
GW^{[n]}(\mathbb{F}_q)/\ell & \longrightarrow & GW_{\text{top}}^{[n]}(\mathbb{C})^r/\ell
\end{array}
$$

*is homotopy cartesian on connective covers. In particular, for $i \geq 0$, $n = 0$ and any one-point space* pt*, the natural map*

$$
GW_i^{[0]}(\mathcal{O}_F') \longrightarrow GW_i^{[0]}(\mathbb{F}_q) \oplus KO^{-i}(\text{pt})^r
$$

*is an isomorphism modulo finite groups of odd order.* □

**Remark 5.7.** Let $X$ be a QL scheme. Our results also give the isomorphism

$$
GW/\ell(X)[\beta^{-1}] \xrightarrow{\cong} GW^{\text{ét}}/\ell(X)[\beta^{-1}] \tag{5-a}
$$

which was first proved in [8]. Note that $GW^{\text{ét}}/\ell(X) \to GW^{\text{ét}}/\ell(X)[\beta^{-1}]$ is an equivalence on connective covers. By the Homotopy Fixed Point Theorem 2.2, cup product with the Bott element is an isomorphism in high degrees, as the same is true for $K$-theory. However, this is also true for étale Hermitian $K$-theory, since by the Hermitian Quillen–Lichtenbaum Theorem 2.7 it coincides with Hermitian $K$-theory in high degrees. Hence the equivalence (5-a).

Let $A \bullet B$ denote an abelian group extension of $B$ by $A$, so that there exists a short exact sequence

$$
0 \to A \to A \bullet B \to B \to 0.
$$

Another convention we follow is that $\mu_{2^\nu}^{\otimes i}$ denotes the $i$th Tate twist of the sheaf of $2^\nu$th roots of unity $\mu_{2^\nu}$ (the kernel of multiplication by $2^\nu$ on the multiplicative group scheme

$\mathbb{G}_m$ over $\mathcal{O}'_F$). At one extreme, when $\nu = 1$ this is independent of the Tate twist; at the other, we use finiteness of the étale cohomology groups of $\mathcal{O}'_F$ to write $\mathbb{Z}_2^{\otimes j}$ for $\varprojlim \mu_{2^\nu}^{\otimes j}$.

By combining the Hermitian Quillen–Lichtenbaum Theorem 2.7 with [8, Lemmas 6.12, 6.16] we deduce our third computational application for higher Grothendieck–Witt groups.

**Theorem 5.8.** *Suppose that $F$ is a totally imaginary number field. The 2-adically completed higher Grothendieck–Witt groups $GW_i^{[n]}(\mathcal{O}'_F)_\#$ of the ring of 2-integers $\mathcal{O}'_F$ of $F$ are computed in terms of étale cohomology groups as follows.*

| $i \bmod 8$ | $GW_i^{[0]}(\mathcal{O}'_F)_\#$ | $GW_i^{[1]}(\mathcal{O}'_F)_\#$ |
|---|---|---|
| $8k > 0$ | $H^2_{\text{ét}}(\mathcal{O}'_F, \mu_2) \bullet H^1_{\text{ét}}(\mathcal{O}'_F, \mu_2)$ | $H^2_{\text{ét}}(\mathcal{O}'_F, \mu_2) \bullet H^1_{\text{ét}}(\mathcal{O}'_F, \mathbb{Z}_2^{\otimes 4k+1})$ |
| $8k + 1$ | $H^1_{\text{ét}}(\mathcal{O}'_F, \mu_2) \bullet H^0_{\text{ét}}(\mathcal{O}'_F, \mu_2)$ | $H^2_{\text{ét}}(\mathcal{O}'_F, \mu_2) \bullet H^1_{\text{ét}}(\mathcal{O}'_F, \mu_2)$ |
| $8k + 2$ | $H^2_{\text{ét}}(\mathcal{O}'_F, \mathbb{Z}_2^{\otimes 4k+2}) \bullet H^0_{\text{ét}}(\mathcal{O}'_F, \mu_2)$ | $H^1_{\text{ét}}(\mathcal{O}'_F, \mu_2) \bullet H^0_{\text{ét}}(\mathcal{O}'_F, \mu_2)$ |
| $8k + 3$ | $H^1_{\text{ét}}(\mathcal{O}'_F, \mathbb{Z}_2^{\otimes 4k+2})$ | $H^2_{\text{ét}}(\mathcal{O}'_F, \mathbb{Z}_2^{\otimes 4k+3}) \bullet H^0_{\text{ét}}(\mathcal{O}'_F, \mu_2)$ |
| $8k + 4$ | $0$ | $H^1_{\text{ét}}(\mathcal{O}'_F, \mathbb{Z}_2^{\otimes 4k+3})$ |
| $8k + 5$ | $0$ | $0$ |
| $8k + 6$ | $H^2_{\text{ét}}(\mathcal{O}'_F, \mathbb{Z}_2^{\otimes 4k+4})$ | $0$ |
| $8k + 7$ | $H^2_{\text{ét}}(\mathcal{O}'_F, \mu_2) \bullet H^1_{\text{ét}}(\mathcal{O}'_F, \mathbb{Z}_2^{\otimes 4k+4})$ | $H^2_{\text{ét}}(\mathcal{O}'_F, \mathbb{Z}_2^{\otimes 4k+5})$ |
| $i \bmod 8$ | $GW_i^{[2]}(\mathcal{O}'_F)_\#$ | $GW_i^{[3]}(\mathcal{O}'_F)_\#$ |
| $8k > 0$ | $0$ | $H^1_{\text{ét}}(\mathcal{O}'_F, \mathbb{Z}_2^{\otimes 4k+1})$ |
| $8k + 1$ | $0$ | $0$ |
| $8k + 2$ | $H^2_{\text{ét}}(\mathcal{O}'_F, \mathbb{Z}_2^{\otimes 4k+2})$ | $0$ |
| $8k + 3$ | $H^2_{\text{ét}}(\mathcal{O}'_F, \mu_2) \bullet H^1_{\text{ét}}(\mathcal{O}'_F, \mathbb{Z}_2^{\otimes 4k+2})$ | $H^2_{\text{ét}}(\mathcal{O}'_F, \mathbb{Z}_2^{\otimes 4k+3})$ |
| $8k + 4$ | $H^2_{\text{ét}}(\mathcal{O}'_F, \mu_2) \bullet H^1_{\text{ét}}(\mathcal{O}'_F, \mu_2)$ | $H^2_{\text{ét}}(\mathcal{O}'_F, \mu_2) \bullet H^1_{\text{ét}}(\mathcal{O}'_F, \mathbb{Z}_2^{\otimes 4k+3})$ |
| $8k + 5$ | $H^1_{\text{ét}}(\mathcal{O}'_F, \mu_2) \bullet H^0_{\text{ét}}(\mathcal{O}'_F, \mu_2)$ | $H^2_{\text{ét}}(\mathcal{O}'_F, \mu_2) \bullet H^1_{\text{ét}}(\mathcal{O}'_F, \mu_2)$ |
| $8k + 6$ | $H^2_{\text{ét}}(\mathcal{O}'_F, \mathbb{Z}_2^{\otimes 4k+4}) \bullet H^0_{\text{ét}}(\mathcal{O}'_F, \mu_2)$ | $H^1_{\text{ét}}(\mathcal{O}'_F, \mu_2) \bullet H^0_{\text{ét}}(\mathcal{O}'_F, \mu_2)$ |
| $8k + 7$ | $H^1_{\text{ét}}(\mathcal{O}'_F, \mathbb{Z}_2^{\otimes 4k+4})$ | $H^2_{\text{ét}}(\mathcal{O}'_F, \mathbb{Z}_2^{\otimes 4k+5}) \bullet H^0_{\text{ét}}(\mathcal{O}'_F, \mu_2)$ |

*In particular, for $0 \le n \le 3$ and $i > 0$, the group $GW_i^{[n]}(\mathcal{O}'_F)_\#$ is trivial when*

$$\left\lfloor \frac{i-n}{2} \right\rfloor \equiv \left\lfloor \frac{4+n}{2} \right\rfloor \pmod 4. \qquad \square$$

Comparing the above with [33, Theorem 0.4], one sees that the cohomology terms involving twisted $\mathbb{Z}_2$-coefficients are detected by the $K$-groups of $\mathcal{O}'_F$.

The Lichtenbaum conjectures relate the orders of $K$-groups to values of Dedekind zeta-functions of totally real number fields [26,27]. We exhibit precise formulas relating the orders of higher Grothendieck–Witt groups to values of Dedekind zeta-functions. If $m$ is even, let $w_m = 2^{a_F + \nu_2(m)}$ where $a_F := (|\mu_{2^\infty}(F(\sqrt{-1}))|)_2$ is the 2-adic valuation

and $2^{\nu_2(m)}$ is the 2-primary part of $m$. If $F = \mathbb{Q}(\zeta_{2^b} + \bar{\zeta}_{2^b})$, then $a_F = b$; and when $F = \mathbb{Q}(\zeta_r + \bar{\zeta}_r)$ (with $r$ odd) or $\mathbb{Q}(\sqrt{d})$ with $d > 2$, then $a_F = 2$. The following theorem applies to these examples, where we write $G_{\mathrm{tor}}$ for the torsion subgroup of an abelian group $G$. Its proof combines our results for $GW^{[n]}(\mathcal{O}'_F)$ with [33, Theorem 0.2].

**Theorem 5.9.** *For every 2-regular totally real abelian number field $F$ with $r$ real embeddings, the Dedekind zeta-function of $F$ takes the values*

$$\zeta_F(-1-4k) = \frac{\#GW^{[0]}_{8k+2}(\mathcal{O}'_F)}{2\#GW^{[0]}_{8k+3}(\mathcal{O}'_F)} = 2^{2r}\frac{\#GW^{[2]}_{8k+2}(\mathcal{O}'_F)}{\#GW^{[2]}_{8k+3}(\mathcal{O}'_F)} = \frac{2^r}{w_{4k+2}}$$

$$\zeta_F(-3-4k) = 2^r\frac{\#GW^{[0]}_{8k+6}(\mathcal{O}'_F)}{\#GW^{[0]}_{8k+7}(\mathcal{O}'_F)} = 2^r\frac{\#GW^{[2]}_{8k+6}(\mathcal{O}'_F)_{\mathrm{tor}}}{\#GW^{[2]}_{8k+7}(\mathcal{O}'_F)} = \frac{2^r}{w_{4k+4}}$$

*up to odd multiples.* □

## Acknowledgments

## References

[1] J.F. Adams, On the groups $J(X)$. IV, Topology 5 (1966) 21–71.

[2] J.K. Arason, Cohomologische invarianten quadratischer Formen, J. Algebra 36 (3) (1975) 448–491.

[3] M. Artin, A. Grothendieck, J.L. Verdier (Eds.), Théorie des topos et cohomologie étale des schémas. Tome 3. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Lecture Notes in Math., vol. 305, Springer-Verlag, Berlin, 1973, Avec la collaboration de P. Deligne et B. Saint-Donat.

[4] A. Asok, J. Fasel, A cohomological classification of vector bundles on smooth affine threefolds, Duke Math. J. 163 (2014) 2561–2601.

[5] P. Balmer, Witt groups, in: Handbook of $K$-Theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 539–576.

[6] A.J. Berrick, M. Karoubi, Hermitian $K$-theory of the integers, Amer. J. Math. 127 (4) (2005) 785–823.

[7] A.J. Berrick, M. Karoubi, P.A. Østvær, Hermitian $K$-theory and 2-regularity for totally real number fields, Math. Ann. 349 (1) (2011) 117–159.

[8] A.J. Berrick, M. Karoubi, P.A. Østvær, Periodicity of Hermitian $K$-groups, J. K-Theory 7 (3) (2011) 429–493.

[9] A. Borel, Stable real cohomology of arithmetic groups, Ann. Sci. Éc. Norm. Supér. 7 (2) (1972) 235–272.

[10] A.K. Bousfield, E.M. Friedlander, Homotopy theory of Γ-spaces, spectra, and bisimplicial sets, in: Geometric Applications of Homotopy Theory, II, Proc. Conf., Evanston, Ill., 1977, in: Lecture Notes in Math., vol. 658, Springer, Berlin, 1978, pp. 80–130.

[11] J. Fasel, R.A. Rao, R.G. Swan, On stably free modules over affine algebras, Publ. Math. Inst. Hautes Études Sci. 116 (2012) 223–243.

[12] E.M. Friedlander, Computations of $K$-theories of finite fields, Topology 15 (1) (1976) 87–109.

[13] O. Gabber, *K*-theory of Henselian local rings and Henselian pairs, in: Algebraic *K*-Theory, Commutative Algebra, and Algebraic Geometry, Santa Margherita Ligure, 1989, in: Contemp. Math., vol. 126, Amer. Math. Soc., Providence, RI, 1992, pp. 59–70.

[14] L. Hodgkin, P.A. Østvær, The homotopy type of two-regular *K*-theory, in: Categorical Decomposition Techniques in Algebraic Topology, Isle of Skye, 2001, in: Progr. Math., vol. 215, Birkhäuser, Basel, 2004, pp. 167–178.

[15] P. Hu, I. Kriz, K. Ormsby, The homotopy limit problem for Hermitian *K*-theory, equivariant motivic homotopy theory and motivic real cobordism, Adv. Math. 228 (1) (2011) 434–480.

[16] J.F. Jardine, Stable homotopy theory of simplicial presheaves, Canad. J. Math. 39 (3) (1987) 733–747.

[17] J.F. Jardine, Generalized Étale Cohomology Theories, Progr. Math., vol. 146, Birkhäuser Verlag, Basel, 1997.

[18] M. Karoubi, Périodicité de la *K*-théorie hermitienne, in: Algebraic *K*-Theory, III: Hermitian *K*-Theory and Geometric Applications, Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972, in: Lecture Notes in Math., vol. 343, Springer, Berlin, 1973, pp. 301–411.

[19] M. Karoubi, Le théorème fondamental de la *K*-théorie hermitienne, Ann. of Math. (2) 112 (2) (1980) 259–282.

[20] M. Karoubi, Homology of the infinite orthogonal and symplectic groups over algebraically closed fields, Invent. Math. 73 (2) (1983) 247–250, an appendix to the paper: "On the *K*-theory of algebraically closed fields" by A. Suslin.

[21] M. Karoubi, Relations between algebraic *K*-theory and Hermitian *K*-theory, in: Proceedings of the Luminy Conference on Algebraic *K*-Theory, Luminy, 1983, J. Pure Appl. Algebra 34 (1984) 259–263.

[22] M. Karoubi, A descent theorem in topological *K*-theory, K-Theory 24 (2) (2001) 109–114.

[23] M. Knebusch, Isometrien über semilokalen Ringen, Math. Z. 108 (1969) 255–268.

[24] M. Knebusch, Symmetric bilinear forms over algebraic varieties, in: Conference on Quadratic Forms—1976, Proc. Conf., Queen's Univ., Kingston, Ont., 1976, in: Queen's Papers in Pure and Appl. Math., vol. 46, Queen's Univ., Kingston, Ont, 1977, pp. 103–283.

[25] D. Kobal, *K*-theory, Hermitian *K*-theory and the Karoubi tower, K-Theory 17 (2) (1999) 113–140.

[26] S. Lichtenbaum, On the values of zeta and *L*-functions. I, Ann. of Math. (2) 96 (1972) 338–360.

[27] S. Lichtenbaum, Values of zeta-functions, étale cohomology, and algebraic *K*-theory, in: Algebraic *K*-Theory, II: "Classical" Algebraic *K*-Theory and Connections with Arithmetic, Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972, in: Lecture Notes in Math., vol. 342, Springer, Berlin, 1973, pp. 489–501.

[28] S.A. Mitchell, Hypercohomology spectra and Thomason's descent theorem, in: Algebraic *K*-Theory, Toronto, ON, 1996, in: Fields Inst. Commun., vol. 16, Amer. Math. Soc., Providence, RI, 1997, pp. 221–277.

[29] S.A. Mitchell, *K*-theory hypercohomology spectra of number rings at the prime 2, in: Une dégustation topologique [Topological morsels]: Homotopy Theory in the Swiss Alps, Arolla, 1999, in: Contemp. Math., vol. 265, Amer. Math. Soc., Providence, RI, 2000, pp. 129–157.

[30] D. Quillen, Higher algebraic *K*-theory. I, in: Algebraic *K*-Theory, I: Higher *K*-Theories, Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972, in: Lecture Notes in Math., vol. 341, Springer, Berlin, 1973, pp. 85–147.

[31] D. Quillen, Letter from Quillen to Milnor on $\mathrm{Im}(\pi_i 0 \to \pi_i^{\mathrm{s}} \to K_i \mathbf{Z})$, in: Algebraic *K*-Theory, Proc. Conf., Northwestern Univ., Evanston, Ill., 1976, in: Lecture Notes in Math., vol. 551, Springer, Berlin, 1976, pp. 182–188.

[32] A.A. Ranicki, Algebraic *L*-Theory and Topological Manifolds, Cambridge Tracts in Math., vol. 102, Cambridge University Press, Cambridge, 1992.

[33] J. Rognes, C.A. Weibel, Two-primary algebraic *K*-theory of rings of integers in number fields, J. Amer. Math. Soc. 13 (1) (2000) 1–54, Appendix A by Manfred Kolster.

[34] A. Rosenschon, P.A. Østvær, The homotopy limit problem for two-primary algebraic *K*-theory, Topology 44 (6) (2005) 1159–1179.

[35] W. Scharlau, Quadratic and Hermitian Forms, Grundlehren Math. Wiss. (Fundamental Principles of Mathematical Sciences), vol. 270, Springer-Verlag, Berlin, 1985.

[36] M. Schlichting, Hermitian *K*-theory, derived equivalences and Karoubi's Fundamental Theorem, arXiv:1209.0848v1.

[37] M. Schlichting, The homotopy limit problem and étale Hermitian *K*-theory, arXiv:1011.4977v1.

[38] M. Schlichting, Negative *K*-theory of derived categories, Math. Z. 253 (1) (2006) 97–134.

[39] M. Schlichting, The Mayer–Vietoris principle for Grothendieck–Witt groups of schemes, Invent. Math. 179 (2) (2010) 349–433.

[40] M. Schlichting, G.S. Tripathi, Geometric models for higher Grothendieck–Witt groups in $\mathbb{A}^1$-homotopy theory, arXiv:1309.5818, to appear in Math. Ann., http://dx.doi.org/10.1007/s00208-014-1154-z.

[41] J.-P. Serre, Cohomologie galoisienne, fifth edition, Lecture Notes in Math., vol. 5, Springer-Verlag, Berlin, 1994.

[42] A.A. Suslin, On the $K$-theory of local fields, in: Proceedings of the Luminy Conference on Algebraic $K$-Theory, Luminy, 1983, J. Pure Appl. Algebra 34 (1984) 301–318.

[43] R.W. Thomason, The homotopy limit problem, in: Proceedings of the Northwestern Homotopy Theory Conference, Evanston, Ill., 1982, in: Contemp. Math., vol. 19, Amer. Math. Soc., Providence, RI, 1983, pp. 407–419.

[44] R.W. Thomason, Algebraic $K$-theory and étale cohomology, Ann. Sci. Éc. Norm. Supér. (4) 18 (3) (1985) 437–552.

[45] R.W. Thomason, T. Trobaugh, Higher algebraic $K$-theory of schemes and of derived categories, in: The Grothendieck Festschrift, Vol. III, in: Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435.

[46] V. Voevodsky, Motivic cohomology with $\mathbf{Z}/2$-coefficients, Publ. Math. Inst. Hautes Études Sci. (98) (2003) 59–104.

[47] C.A. Weibel, The 2-torsion in the $K$-theory of the integers, C. R. Acad. Sci. Paris Ser. I Math. 324 (6) (1997) 615–620.

[48] C.A. Weibel, The negative $K$-theory of normal surfaces, Duke Math. J. 108 (1) (2001) 1–35.

[49] M.S. Weiss, B. Williams, Products and duality in Waldhausen categories, Trans. Amer. Math. Soc. 352 (2) (2000) 689–709.

[50] B. Williams, Quadratic $K$-theory and geometric topology, in: Handbook of $K$-Theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 611–651.