Cellularity of hermitian K-theory and Witt-theory

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Abstract

Hermitian K-theory and Witt-theory are cellular in the sense of stable motivic homotopy theory over any base scheme without points of characteristic two.

1 Introduction

The notion of a cellular object in motivic homotopy theory is intrinsically linked to the geometry of motivic spheres $S^{p,q}$ [4]. Suppose the smooth scheme X admits a filtration by closed subschemes

$$\emptyset \subset X_0 \subset \cdots \subset X_{n-1} \subset X_n = X,$$

where $X_i \smallsetminus X_{i-1}$ is a disjoint union of affine spaces $\mathbf{A}^{n_{ij}}$. Examples of such filtrations arise in the context of Białynicki-Birula decompositions for $\mathbf{G}_m$-action on smooth projective varieties [2], cf. [3] for a more recent implementation. By homotopy purity [7, Theorem 3.2.23] for Thom spaces of normal bundles of closed embeddings, there is a homotopy cofiber sequence

$$X \smallsetminus X_i \longrightarrow X \smallsetminus X_{i-1} \longrightarrow \mathbf{Th}(\mathcal{N}_i).$$

By assumption the normal bundle $\mathcal{N}_i$ is trivial. Thus the splitting $\mathbf{Th}(\mathcal{N}_i) \cong \bigvee_j S^{2n_{ij}, n_{ij}}$ and the two-out-of-three property for stably cellular objects [4, Lemma 2.5] imply inductively that X is stably cellular in the sense of [4, Definition 2.10].

In this paper we employ a similar strategy to prove cellularity for Thom spaces of direct sums of tautological sympletic bundles over quaternionic Grassmannians. This allows us to show cellularity of the motivic spectra representing hermitian K-theory and Witt-theory [5]. By a base scheme we mean any regular noetherian separated scheme of finite Krull dimension.

Theorem 1.1. *Suppose all points on the base scheme have residue characteristic unequal to two. Then hermitian K-theory $\mathbf{KQ}$ and Witt-theory $\mathbf{KW}$ are cellular motivic spectra.*

For a related antecedent result showing cellularity of algebraic K-theory, see [4, Theorem 6.2]. The proof of Theorem 1.1 exploits the geometry of quaternionic Grassmannians and the explicit model for hermitian K-theory from [9].

Recent applications of $\mathbf{KQ}$ and $\mathbf{KW}$ concern computations of stable homotopy groups of motivic spheres [6], [8], [12], and a proof of the Milnor conjecture on quadratic forms [11]. For cellular motivic spectra one has the powerful fact that stable motivic weak equivalences are detected by $\pi_{*,*}$-isomorphisms [4, Corollary 7.2]. Our main motivation for proving Theorem 1.1 is that it is being used in the computation of the slices of $\mathbf{KQ}$ in [12, Theorem 2.14]. In terms of motivic cohomology with integral and mod-2 coefficients, the result is

$$\mathsf{s}_q(\mathbf{KQ}) \cong \begin{cases} \Sigma^{2q,q}\mathbf{MZ} \vee \bigvee_{i<\frac{q}{2}} \Sigma^{2i+q,q}\mathbf{MZ}/2 & q \text{ even} \\ \bigvee_{i<\frac{q+1}{2}} \Sigma^{2i+q,q}\mathbf{MZ}/2 & q \text{ odd.} \end{cases}$$

In turn, this is an essential ingredient in our proof of Morel's π_1-conjecture in [12]. It is an interesting problem to make sense of Theorem 1.1 without any assumptions on the points of the base scheme.

This short paper is organized into Section 2 on basic properties of motivic cellular spectra, Section 3 on the geometry of quaternionic Grassmannians, and Section 4 on hermitian K-theory and Witt-theory.

2   Cellular objects

The subcategory of cellular spectra in the motivic stable homotopy category is the smallest full localizing subcategory that contains all suspensions of the sphere spectrum, cf. [4, §2.8]. For our purposes it suffices to know four basic facts about cellular motivic spectra. First we recall part (3) of Definition 2.1 in [4].

Lemma 2.1. *The homotopy colimit of a diagram of cellular motivic spectra is cellular.*

The second fact is a specialization of [4, Lemma 2.4].

Lemma 2.2. *Let E be a motivic spectrum and let p, q be integers. Then E is cellular if and only if its (p,q)-suspension $\Sigma^{p,q}E$ is cellular.*

The third fact is a specialization of [4, Lemma 2.5].

Lemma 2.3. *If $E \to F \to G$ is a homotopy cofiber sequence of motivic spectra such that any two of E, F, and G are cellular, then so is the third.*

Finally, we recall Lemma 3.2 in [4].

Lemma 2.4. *If E_i is a cellular motivic spectrum for all $i \in I$, then $\coprod_{i \in I} E_i$ is cellular.*

3   Quaternionic Grassmannians

The quaternionic Grassmannian $\mathbf{HGr}(r,n)$ is the open subscheme of the ordinary Grassmannian $\mathbf{Gr}(2r,2n)$ parametrizing $2r$-dimensional subspaces of the trivial vector bundle $\mathcal{O}^{\oplus 2n}$ on which the standard symplectic form is nondegenerate. It is smooth affine of dimension $4r(n-r)$ over the base scheme. Let $\mathcal{U}_{r,n}$ be short for the tautological symplectic subbundle of rank $2r$ on $\mathbf{HGr}(r,n)$. It is the restriction to $\mathbf{HGr}(r,n)$ of the tautological subbundle of $\mathbf{Gr}(2r,2n)$ together with the restriction to $\mathcal{U}_{r,n}$ of the standard symplectic form on $\mathcal{O}^{\oplus 2n}$.

More generally, to every symplectic bundle $(\mathcal{E}, \phi)$ one associates the quaternionic Grassmannian $\mathbf{HGr}(r, \mathcal{E}, \phi)$; it is the open subscheme of the Grassmannian $\mathbf{Gr}(2r, \mathcal{E})$ parametrizing $2r$-dimensional subspaces of the fibers of $\mathcal{E}$ on which ϕ is nondegenerate. Associated to the trivial rank $2n-2$ symplectic bundle $(\mathcal{E}, \psi)$ is the bundle $\mathcal{F} = \mathcal{O} \oplus \mathcal{E} \oplus \mathcal{O}$ equipped with the direct sum of ψ and the hyperbolic symplectic form, i.e.,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & \psi & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

For simplicity we write $\mathbf{HGr}(\mathcal{E})$ for $\mathbf{HGr}(r, \mathcal{E}, \psi)$ and likewise for $\mathcal{F}$.

The normal bundle N of the embedding $\mathbf{HGr}(\mathcal{E}) \subset \mathbf{HGr}(\mathcal{F})$ is the tensor product $\mathcal{U}_{\mathcal{E}}^{\vee} \otimes \mathcal{O}^{\oplus 2}$ for the dual of the tautological symplectic subbundle of rank $2r$ on $\mathbf{HGr}(\mathcal{E})$. Theorem 4.1 in [10] shows that N is naturally isomorphic to an open subscheme of $\mathbf{Gr}(2r, \mathcal{F})$ and there is a decomposition $N = N^+ \oplus N^-$; here, $N^+ = \mathbf{HGr}(\mathcal{F}) \cap \mathbf{Gr}(2r, \mathcal{O} \oplus \mathcal{E})$ and $N^- = \mathbf{HGr}(\mathcal{F}) \cap \mathbf{Gr}(2r, \mathcal{E} \oplus \mathcal{O})$ have

intersection $\mathbf{HGr}(\mathcal{E})$. Thus there are natural vector bundle isomorphisms $N^+ \cong N^- \cong \mathcal{U}_{r,n-1}$ and the normal bundle $\mathcal{N}$ of N^+ in $\mathbf{HGr}(\mathcal{F})$ is isomorphic to $\pi_+^* \mathcal{U}_{r,n-1}$ for the bundle projection $\pi_+ \colon N^+ \to \mathbf{HGr}(\mathcal{E})$. Moreover, there is a vector bundle isomorphism between the restriction $\mathcal{U}_{r,n}|N^+$ of $\mathcal{U}_{r,n}$ to N^+ and $\pi_+^* \mathcal{U}_{r,n-1}$. For $r \leq n-1$, let Y denote the complement of N^+ in $\mathbf{HGr}(\mathcal{F})$ [10, (5.1)].

Proposition 3.1. *For $m \geq 0$ the suspension spectrum of the Thom space of the vector bundle $\mathcal{U}_{r,n}^{\oplus m}$ on $\mathbf{HGr}(r,n)$ is a finite cellular spectrum. In particular, $\Sigma^\infty \mathbf{HGr}(r,n)_+$ is a cellular spectrum.*

Proof. The proof proceeds by a double induction argument on r and $n \geq r$. The base cases $\mathbf{HGr}(0,n)$ and $\mathbf{HGr}(n,n)$ are clear, so we may assume $0 < r < n$. Define the motivic space Z by the homotopy cofiber sequence

$$\mathbf{Th}(\mathcal{U}_{r,n}^{\oplus m}|Y) \longrightarrow \mathbf{Th}(\mathcal{U}_{r,n}^{\oplus m}) \longrightarrow Z. \tag{1}$$

According to [13, Lemma 3.5] there is a canonical isomorphism in the motivic homotopy category

$$Z \cong \mathbf{Th}(\mathcal{U}_{r,n}^{\oplus m}|N^+ \oplus \mathcal{N}).$$

Using the above we note $\mathcal{U}_{r,n}^{\oplus m}|N^+ \oplus \mathcal{N} \cong \pi_+^* \mathcal{U}_{r,n-1}^{\oplus(m+1)}$ and hence there are canonical isomorphisms

$$Z \cong \mathbf{Th}(\pi_+^* \mathcal{U}_{r,n-1}^{\oplus(m+1)}) \cong \mathbf{Th}(\mathcal{U}_{r,n-1}^{\oplus(m+1)}).$$

By induction hypothesis $\Sigma^\infty Z$ is a finite cellular spectrum. Thus Lemma 2.3 and (1) reduce the proof to showing that $\Sigma^\infty \mathbf{Th}(\mathcal{U}_{r,n}^{\oplus m}|Y)$ is a finite cellular spectrum. To this end we recall parts of Theorem 5.1 in [10]: There exists maps

$$Y \xleftarrow{\ g_1\ } Y_1 \xleftarrow{\ g_2\ } Y_2 \xrightarrow{\ q\ } \mathbf{HGr}(r-1,\mathcal{E},\psi),$$

where g_i and q are Zariski locally trivial torsors over vector bundles of rank $2r-i$ and $4n-3$, respectively. Moreover, $g_2^* g_1^* \mathcal{U}_{r,n}$ is isomorphic to $\mathcal{O}_{Y_2}^2 \oplus q^* \mathcal{U}_{r-1,n}$. Invoking [7, §3.2, Example 2.3] this implies the canonical isomorphisms

$$\Sigma^\infty \mathbf{Th}(\mathcal{U}_{r,n}^{\oplus m}|Y) \cong \Sigma^\infty \mathbf{Th}(g_2^* g_1^* \mathcal{U}_{r,n}^{\oplus m}|Y) \cong \Sigma^\infty \mathbf{Th}(\mathcal{O}_{Y_2}^{2m} \oplus q^* \mathcal{U}_{r-1,n}^{\oplus m}) \cong \Sigma^{2m,m} \Sigma^\infty \mathbf{Th}(\mathcal{U}_{r-1,n}^{\oplus m}).$$

Here, the suspension spectrum of $\mathbf{Th}(\mathcal{U}_{r-1,n}^{\oplus m})$ is finite cellular by the induction hypothesis. This finishes the proof using Lemma 2.2. $\qquad\square$

4   Hermitian K-theory and Witt-theory

In this section we finish the proof of Theorem 1.1 stated in the introduction.

The quaternionic plane $\mathbf{HP}^1$ is the first quaternionic Grassmannian $\mathbf{HGr}(1,2)$. In the pointed unstable motivic homotopy category, $(\mathbf{HP}^1, x_0)$ is isomorphic to the two-fold smash product of the Tate object $T \equiv \mathbf{A}^1/\mathbf{A}^1 \smallsetminus \{0\}$. It follows that the $\mathbf{A}^1$-mapping cone $\mathbf{HP}^{1+}$ of the rational point $x_0 \colon S \to \mathbf{HP}^1$ is isomorphic to $T^{\wedge 2}$. Hence the stable homotopy category of $\mathbf{HP}^{1+}$-spectra is equivalent to the standard model for the stable motivic homotopy category [9, Theorem 12.1].

Theorem 12.3 in [9] shows there is an isomorphism between hermitian K-theory $\mathbf{KQ}$ and an $\mathbf{HP}^{1+}$-spectrum $\mathbf{BO}^{\mathrm{geom}}$. For n odd, $\mathbf{BO}_{2n}^{\mathrm{geom}} = \mathbf{Z} \times \mathbf{HGr}$ [9, (12.5)]. Here $\mathbf{HGr}$ denotes the infinite quaternionic Grassmannian, i.e., the sequential colimit

$$\operatorname*{colim}_{n} \mathbf{HGr}(n,2n).$$

We note that the transition maps in the colimit are defined in [9, (8.1)]. The motivic space $\mathbf{Z} \times \mathbf{HGr}$ is pointed by $(0, \mathbf{HGr}(0,0))$. Thus $\mathbf{KQ}$ is isomorphic to the homotopy colimit

$$\operatorname*{hocolim}_{n \text{ odd}} \Sigma^{4n,2n} \Sigma^{\infty} \mathbf{Z} \times \mathbf{HGr}. \tag{2}$$

It remains to show cellularity of (2). Note that $\Sigma^{\infty} \mathbf{Z} \times \mathbf{HGr}$ is a homotopy colimit of cellular spectra by Lemma 2.4 and Proposition 3.1. It follows that $\Sigma^{4n,2n} \Sigma^{\infty} \mathbf{Z} \times \mathbf{HGr}$ is cellular according to Lemmas 2.1 and 2.2. We conclude the proof for $\mathbf{KQ}$ by applying Lemma 2.1.

Cellularity of $\mathbf{KW}$ follows from that of $\mathbf{KQ}$ via Lemma 2.1 and the description of $\mathbf{KW}$ as the homotopy colimit of the diagram

$$\mathbf{KQ} \xrightarrow{\eta} \Sigma^{-1,-1}\mathbf{KQ} \xrightarrow{\Sigma^{-1,-1}\eta} \Sigma^{-2,-2}\mathbf{KQ} \xrightarrow{\Sigma^{-2,-2}\eta} \cdots$$

given in [1, Theorem 6.5]. Here, η is the first stable Hopf map induced by the canonical map $\mathbf{A}^2 \smallsetminus \{0\} \to \mathbf{P}^1$.

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