

Entropic bounds on information backflow

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In the dynamics of open quantum systems, the backflow of information to the reduced system under study has been suggested as the actual physical mechanism inducing memory and thus leading to non-Markovian quantum dynamics. To this aim, the trace-distance or Bures-distance revivals between distinct evolved system states have been shown to be subordinated to the establishment of system-environment correlations or changes in the environmental state. We show that this interpretation can be substantiated also for a class of entropic quantifiers. We exploit a suitably regularized version of Umegaki’s quantum relative entropy, known as telescopic relative entropy, that is tightly connected to the quantum Jensen-Shannon divergence. In particular, we derive general upper bounds on the telescopic relative entropy revivals conditioned and determined by the formation of correlations and changes in the environment. We illustrate our findings by means of examples, considering the Jaynes–Cummings model and a two-qubit dynamics.

I. INTRODUCTION

The notion of quantum non-Markovianity has been attracting a lot of interest for more than a decade now [1–3]. In this time, a wide variety of different definitions of quantum non-Markovianity have been proposed, all of them aimed to reveal the occurrence of memory effects in quantum evolutions. The most widespread are the ones based on the divisibility property of the dynamical map [4–7], the monotonicity of the trace distance (TD) between two distinct reduced states [7–9], the change of the volume of accessible reduced states [10], and the process tensor formalism [11–13]. Furthermore, also entropic quantities have been used to detect non-Markovianity, see [14, 15]. The interest toward non-Markovian quantum dynamics has not only theoretical motivations: non-Markovianity has proven to be beneficial, among others, in quantum control [16, 17] and teleportation tasks [18]. As the different definitions of quantum non-Markovianity are in general not equivalent, it is all the more important to find their corresponding physical interpretations. In the approach based on TD, its increase in time signifies a backflow of information to the open system, resulting in an enhanced reduced-state distinguishability and representing the distinctive trait of memory effects in the dynamics. The revivals of distinguishability are related to the establishment of system-environment correlations and changes in the environmental state [19–23], which then appear to be the basic elements ruling the non-Markovian character of open-system dynamics, though the precise assessment of their role is still under vivid debate [24–30].

The proof of the connection of the distinguishability revivals with correlations and environmental state changes as formulated via the TD essentially relies on the triangular inequality, so that it might be natural to think that it only holds when distance quantifiers are used. Here we show that, to the contrary, such a connection can be maintained also when considering entropic distinguishability quantifiers. Our anal-

ysis extends and strengthens the viewpoint that quantum non-Markovianity can be understood in terms of a backflow of information, which induces an increase of the distinguishability among open-system states and is microscopically motivated by the generation of correlations and changes in the environmental state due to the system-environment interaction. More specifically, we prove an upper bound for the revivals in time of a whole class of entropic distinguishability quantifiers, namely the telescopic relative entropy (TRE), firstly introduced in [31] and providing regularized versions of the quantum relative entropy (QRE). Remarkably, our bound allows to quantitatively link the distinguishability revivals with the establishment of correlations between the system and the environment due to their mutual interaction, as well as to the modification of the state of the environment. We also focus on a special case of the symmetrised version of TRE, which coincides with the quantum Jensen-Shannon divergence (QJSD) [32]. QJSD is a widely used distinguishability measure [33–40], for which it was only recently shown that its square root is a proper distance [41, 42]. We show that the upper bound to distinguishability revivals in this case significantly simplifies and becomes tighter. Finally, we showcase our findings on two examples, namely the paradigmatic Jaynes–Cummings model and a two qubit dynamics, which allows us to compare explicitly the behaviors of the different quantifiers of distinguishability involved in our analysis.

The rest of the paper is organized as follows. In Sec. II, we briefly recall the main features of the TD characterization of non-Markovianity that are relevant to our analysis. In Sec. III, after recalling the main properties of TRE and showing its connection with QJSD, we present the key result of our paper, that is, the upper bound to the TRE revivals in terms of the system-environment correlations and environmental-state changes; the proof of the bound is provided in Appendix A. In Sec. IV, we apply our general analysis to two cases study and finally the conclusions and outlooks of our work are given in Sec. V.

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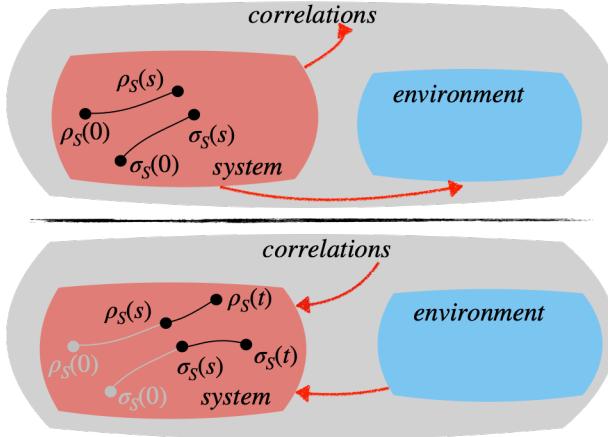


FIG. 1. The concept of information backflow: initially the reduced states ρ_S , σ_S approach each other since the information is flowing outside of the reduced system (top), while information backflow leads the states to diverge from each other (bottom).

II. INFORMATION BACKFLOW AND NON-MARKOVIANITY

Let us start recalling the basic properties of the TD useful for our analysis. The TD is defined as $D(\varrho, \sigma) = 1/2\text{Tr}|\varrho - \sigma|$, and provides a natural distance on the space of statistical operators [43]. Its crucial feature allowing to define and identify memory effects is its contractivity under the action of a (completely) positive trace preserving ((C)PT) map Φ ; namely TD obeys the so-called data processing inequality

$$D(\Phi[\varrho], \Phi[\sigma]) \leq D(\varrho, \sigma) \quad (1)$$

for any pair of states ϱ, σ [44]. Importantly, this property brings with itself invariance under unitary maps and with respect to the tensor product, i.e., $D(\varrho, \sigma) = D(\varrho \otimes \tau, \sigma \otimes \tau)$ for any state τ , as can be readily seen using CPT of both the partial trace and the map $\varrho \mapsto \varrho \otimes \tau$. In particular, TD is a proper quantum f -divergence, which allows us to use it to quantify the distinguishability between quantum states [45]. Finally, we consider two further important properties of the TD, which are an immediate consequence of its being a distance in the mathematical sense. The first is the validity of the triangle inequality, which can be expressed as

$$D(\varrho, \sigma) - D(\varrho, \tau) \leq D(\sigma, \tau) \quad (2)$$

$$D(\varrho, \sigma) - D(\eta, \sigma) \leq D(\varrho, \eta), \quad (3)$$

for arbitrary states ϱ, σ, τ and η . The second is positivity and boundedness according to $0 \leq D(\varrho, \sigma) \leq 1$, with the value 0 iff $\varrho = \sigma$, and 1 iff their supports are orthogonal. Thanks to these properties one can derive an upper bound for the difference of the TD between the open system states ρ_S , σ_S , at different times t, s : $t \geq s$. These states result from two distinct initial conditions given by factorized states with the same environmental marginal, $\varrho(0) = \varrho_S(0) \otimes \varrho_E(0)$, $\sigma(0) = \sigma_S(0) \otimes \sigma_E(0) = \sigma_S(0) \otimes \varrho_E(0)$, so as to ensure the existence

of a reduced dynamics [19, 23, 46]. The bound reads:

$$\begin{aligned} D(\varrho_S(t), \sigma_S(t)) - D(\varrho_S(s), \sigma_S(s)) &\leq D(\varrho_E(s), \sigma_E(s)) + \\ D(\varrho(s), \varrho_S(s) \otimes \varrho_E(s)) + D(\sigma(s), \sigma_S(s) \otimes \sigma_E(s)). \end{aligned} \quad (4)$$

The interpretation of the above inequality is central to our analysis and relies on the TD as a quantifier of distinguishability among quantum states. The terms at the right hand side of Eq. (4) quantify the information about the initial state of the open system, which we call local information, that is outside the open system at time s , i.e., that can be accessed only via measurements involving environmental degrees of freedom. Such information can be encoded in the correlations between the open system and the environment or in the environmental state. Indeed, although the environmental states are initially the same, they will generally differ at later times, due to the initial difference in the reduced states $\varrho_S(0)$ and $\sigma_S(0)$. On the other hand, the left hand side of Eq. (4) concerns the accessibility of the local information via measurements on the open system itself. A positive value of the difference means that the local information at time t is greater than the one at time s , denoting a backflow of information towards the open system. As the global system is closed, this additional information must have an environmental origin, i.e., it was previously contained in the environmental state or in the correlations between system and environment, which is precisely what is shown by Eq. (4), see also Fig. 1.

We stress that, as should be clear from the previous analysis, any distance contractive under CPT maps, e.g. the Bures distance considered in [23, 47], would lead to an inequality analogous to the one in Eq. (4) and would then allow for the same physical interpretation. However, till now no entropic quantifier was used for this purpose, as no entropic distance measure was known, though it was conjectured long ago that the square root of the QJSD is actually a metric [48, 49]. This conjecture was recently proven independently by two authors [41, 42] and, accordingly, also the square root of the QJSD is a suitable quantifier of the information backflow. Here, we show that there is a whole class of entropic quantities that are in general not distance measures but also allow for the interpretation above; such a class includes QJSD as a special case. Thus, we substantiate the fact that the actual physical mechanism behind the occurrence of memory effects in quantum dynamics is the establishment of correlations or changes in the environmental states upper bounding the revivals in local distinguishability.

III. TELESCOPIC RELATIVE ENTROPY

Relative entropy is a fundamental quantity in statistical mechanics and information theory, both at classical and quantum level [50–52]. It also plays a distinguished role in quantum thermodynamics and its foundations, especially in analyzing the formulation of the second law of thermodynamics in the quantum regime [53–56]. The expression of the QRE first introduced by Umegaki [57] reads $S(\varrho, \sigma) = \text{Tr}(\varrho \log \varrho - \varrho \log \sigma)$. As well known, however, the QRE, while being the

most relevant quantum f -divergence distinguishing quantum states [45], is not bounded and can diverge also in finite dimension. To cure this difficulty, regularised versions have been proposed [31, 32, 58]. In particular we will show that the TRE introduced in [31] obeys an analog of the key inequalities Eq. (1)–(4). For a special choice of the telescopic parameter introduced below, the symmetrised version of TRE reduces to the QJSD, and a simplified upper bound for its square root follows. With this, also for this class of entropic quantifiers an interpretation of the distinguishability revivals in terms of information backflow is given, thus fully justifying its use for the description of memory effects in an open quantum system dynamics.

The TRE is defined as

$$S_\mu(\varrho, \sigma) = \log(1/\mu)^{-1} S(\varrho, \mu\varrho + (1-\mu)\sigma) \quad (5)$$

and is actually independent of the logarithm basis used in the definition. The telescopic parameter $\mu \in (0, 1)$ gives the amount of mixing between the two states ϱ and σ , telling how much one state is brought closer to the other moving along the joining line in the convex set of states. For the special choice of telescopic parameter $\mu = 1/2$ the symmetrised TRE $J(\varrho, \sigma) = 1/2(S_{1/2}(\varrho, \sigma) + S_{1/2}(\sigma, \varrho))$ equals the QJSD [32]:

$$J(\varrho, \sigma) = \frac{1}{2} \left(S\left(\varrho, \frac{\varrho + \sigma}{2}\right) + S\left(\sigma, \frac{\varrho + \sigma}{2}\right) \right). \quad (6)$$

The main property of the TRE, which distinguishes it from the standard QRE, is its boundedness. In particular, the prefactor is chosen so that $0 \leq S_\mu(\varrho, \sigma) \leq 1$, assuming the extreme values if and only if the states are identical or have orthogonal support [31]. What is more, TRE inherits from the QRE the joint convexity and the contractivity under (C)PT maps [59]

$$S_\mu(\Phi[\varrho], \Phi[\sigma]) \leq S_\mu(\varrho, \sigma), \quad (7)$$

thus being invariant under unitary transformations and tensor product $S_\mu(\varrho, \sigma) = S_\mu(\varrho \otimes \tau, \sigma \otimes \tau)$. Neither QRE nor TRE are distances as they do not satisfy the triangular inequality and are not symmetric in their arguments. However, it can be shown that TRE obeys the following inequalities [60], similar in spirit to Eq. (2) and Eq. (3):

$$S_\mu(\varrho, \sigma) - S_\mu(\varrho, \tau) \leq 1 - S_\mu(1, D(\sigma, \tau)), \quad (8)$$

$$S_\mu(\varrho, \sigma) - S_\mu(\eta, \sigma) \leq D(\varrho, \eta) - S_\mu(D(\varrho, \eta), 1), \quad (9)$$

where we have generalised the definition of TRE to act on non-negative scalars in the obvious way. The TRE can be bounded from below and above by functions of the TD

$$2(1-\mu)^2 \log(1/\mu)^{-1} D^2(\varrho, \sigma) \leq S_\mu(\varrho, \sigma) \leq D(\varrho, \sigma), \quad (10)$$

where the lower bound is a straightforward generalization of the Pinsker inequality for the QRE [61], but the upper bound is only possible since the TRE is bounded.

Exploiting these properties, we derive, as detailed in Appendix A, the following inequality for the change in TRE

$$S(\varrho_S(t), \sigma_S(t)) - S(\varrho_S(s), \sigma_S(s)) \leq \kappa \left(\sqrt[4]{S(\varrho_E(s), \sigma_E(s))} + \sqrt[4]{S(\varrho(s), \varrho_S(s) \otimes \varrho_E(s))} + \sqrt[4]{S(\sigma(s), \sigma_S(s) \otimes \sigma_E(s))} \right) \quad (11)$$

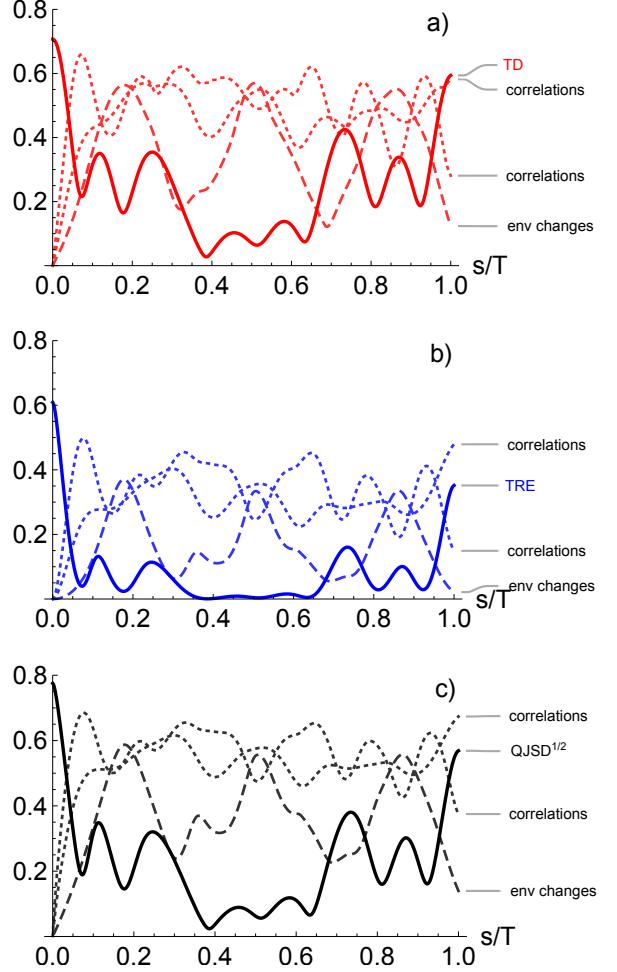


FIG. 2. Different contributions to the bounds for the considered distinguishability quantifiers: a) the solid line is the TD as function of the rescaled time; the dashed line is the first contribution at the r.h.s. of Eq. (4) corresponding to changes in the environmental states, while the two dotted lines correspond to system-environment correlations; b) and c) provide the corresponding quantities relative to TRE and QJSD $^{1/2}$. The initial states of the qubit are given by the up state and a symmetric superposition of up and down state, while the environment starts in a thermal state with $\beta\omega_E = 1$. We set $T = 8.9$ in inverse units of the coupling strength g .

with S the TRE with telescopic parameter $\mu = e^{-3/2}$. While boundedness of the TRE allows to introduce a well defined non-Markovianity measure as for the TD [3], this bound permits a full-fledged interpretation of TRE as a quantifier of information backflow. This is true even if TRE is not a distance, i.e. neither the triangle equality nor the symmetry property are crucial for such an attribution. Note that a similar inequality holds for any telescopic parameter μ , see Appendix A. The value $\mu = e^{-3/2}$ corresponds to the optimal one since it minimizes the prefactor (here, $\kappa = (4e^3/27)^{1/4} \approx 1.31$), and will be taken as reference value. For $\mu = 1/2$, the symmetrised TRE corresponds to the QJSD $J(\varrho, \sigma)$, Eq. (6), and it was recently shown, more than 10 years after the proof for the classi-

cal Jensen-Shannon divergence [62] and the conjecture for the quantum case [48, 49], that its square root is a proper distance [41, 42]. With this and the contractivity under CPT maps one immediately has

$$\sqrt{J(\varrho_S(t), \sigma_S(t))} - \sqrt{J(\varrho_S(s), \sigma_S(s))} \leq \sqrt{J(\varrho_E(s), \sigma_E(s))} + \sqrt{J(\varrho(s), \varrho_S(s) \otimes \varrho_E(s))} + \sqrt{J(\sigma(s), \sigma_S(s) \otimes \sigma_E(s))}. \quad (12)$$

IV. EXAMPLES

We now showcase our findings by examples. Let us consider first the Jaynes–Cummings model, describing the interaction of a qubit with a single bosonic field mode

$$H = \omega_S \sigma_z \otimes \mathbb{I} + g(\sigma_+ \otimes b + \sigma_- \otimes b^\dagger) + \omega_E \mathbb{I} \otimes b^\dagger b, \quad (13)$$

where we introduced the raising and lowering operators $\sigma_{\pm} = \sigma_x \pm i\sigma_y$ expressed in terms of the Pauli matrices, while b, b^\dagger are bosonic creation and annihilation operators, respectively. This model can be solved exactly, see [63, 64], thus allowing for a comparison of the TD and TRE quantities occurring in Eqs. (4), (11) and (12). In Fig. 2 we report in each panel the l.h.s. and the three contributions at the r.h.s. of the bounds for the TD, the TRE and the square root of the QJSD, which we denote as $\text{QJSD}^{1/2}$, respectively. We see that the qualitative behaviour of these quantifiers of quantum-state distinguishability is similar, with respect to both the information contained within the open system and the one outside it, namely the system-environment correlations and environmental states. The quantities referred to the $\text{QJSD}^{1/2}$ in particular mimic very tightly the behaviour of the corresponding TD quantities. The TRE is always smaller than the corresponding TD, which is a general feature for all telescopic parameters μ , see Eq. (10). From this, however, one cannot conclude that the terms appearing in Eq. (11) are always smaller than the corresponding ones in Eq. (4), as can be seen in Fig. 3. Actually, the upper bounds in terms of the entropic quantities are almost always less tight for this model than the corresponding TD one. However, as all three bounds are for most of the time above one, their applicability for estimation of the l.h.s., which is never larger than one, is rather limited. Nonetheless, their existence guarantees the direct relation to information backflow, as the revival of local distinguishability unambiguously originates from establishment of correlations or changes in the environmental states. They thus act as precursors of non-Markovianity [23] and the assessment of the different contributions allows to infer which is the most relevant physical mechanism behind the revivals, whose time dependence is mirrored in the bound.

For better comparison we also consider a simpler model, and set the environment to be a qubit in resonance with the reduced one, see [65], with interaction term $g(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+)$. Fig. 4 shows that the behaviour of TD and entropic quantities is again qualitatively similar, especially the $\text{QJSD}^{1/2}$ follows quite closely the corresponding TD quantities. What is more, the TD bound is once again tighter than the corresponding entropic bounds. However, due to the occurrence of the roots in Eq. (11) and Eq. (12) one clearly notes that the latter

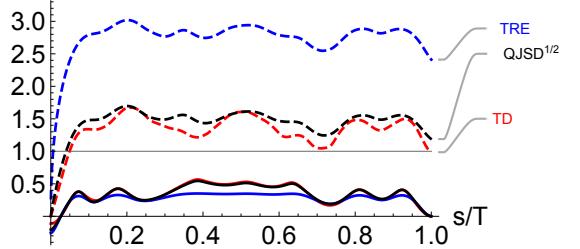


FIG. 3. Revivals of distance and entropic distinguishability quantifiers versus their bounds in terms of correlations and environmental changes for the Jaynes–Cummings model: TD (red), TRE (blue) and $\text{QJSD}^{1/2}$ (black). Solid and dashed lines correspond to l.h.s. and r.h.s. of Eqs. (4), (11) and (12) respectively. The straight line at 1 corresponds to the maximal possible value of the revivals. The value of t at the l.h.s. is set to $T = 8.9$ in inverse units of the coupling strength g , corresponding to a local maximum of the distinguishability as in Fig. 2. The very close behavior of TD and $\text{QJSD}^{1/2}$ clearly appears. All parameters are as in Fig. 2.

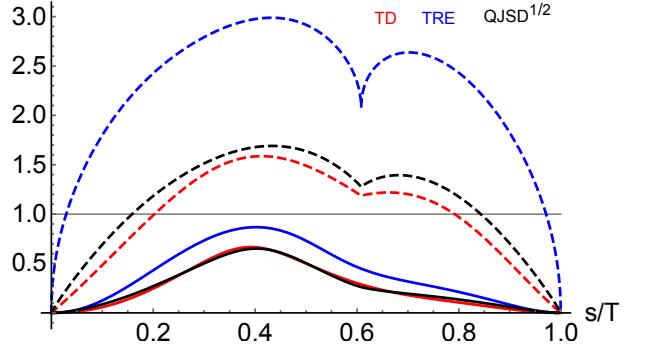


FIG. 4. Revivals in distinguishability and their upper bounds, similarly to Fig. 3 but for the case of the dissipative qubit model. Also in this case solid and dashed lines correspond to l.h.s. and r.h.s. of Eqs. (4), (11) and (12), for TD (red), TRE (blue) and $\text{QJSD}^{1/2}$ (black) respectively. The initial states are chosen to be pure orthogonal states in the yz plane of the Bloch sphere. The reference time t is here set to $T = \pi$ in inverse units of the coupling g . The environmental state is also taken to be pure in the xz plane.

are more sensitive to the changes in the system-environment correlations for times around $t = 0.6T$, when one of the two global states factorizes, as shown in Fig. 5, where we compare the behaviour of the TRE and $\text{QJSD}^{1/2}$ quantities.

V. CONCLUSIONS

We have shown that entropic quantities can be used to consistently define the exchange of information between an open quantum system and its environment. By focusing on a class of regularised versions of the quantum relative entropy, named telescopic relative entropy, we derived an upper bound to the variation of the reduced state distinguishability in terms of the information lying outside the open system,

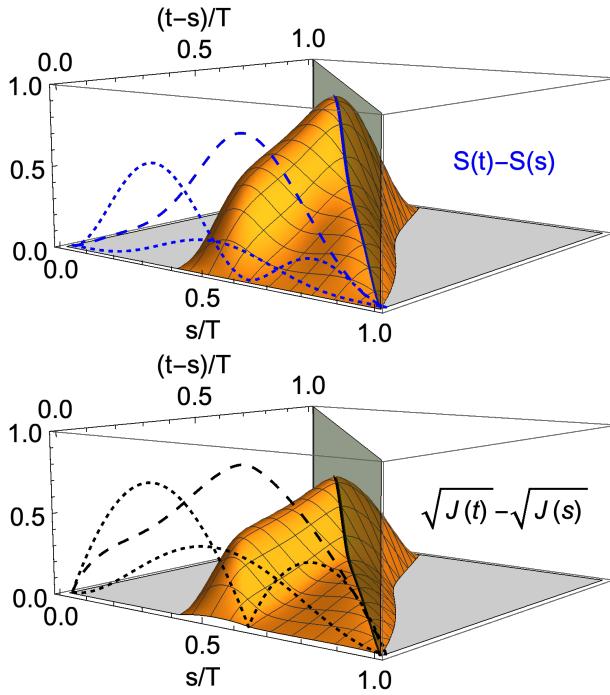


FIG. 5. Comparison of TRE (top) and $\text{QJSD}^{1/2}$ (bottom) for the two qubit dissipative dynamics with initial states as in Fig. 2. The dashed lines in the $s = t$ plane correspond to the difference in the environmental states, the dotted lines to the correlations. Solid lines in the section correspond to TRE and $\text{QJSD}^{1/2}$ as in Fig. 4. The reference time T is set to π in inverse units of the coupling g .

encoded in the system-environment correlations and the environmental states. Besides strengthening the interpretation of non-Markovianity as backflow of information, our results also clarify which are the key mathematical properties behind this picture. Furthermore, we showed that a special case of the telescopic relative entropy can be connected to the quantum Jensen-Shannon divergence. The square root of the latter yields a proper metric on the set of quantum states and it can reproduce both qualitatively and quantitatively the behavior of the trace distance. These features have been highlighted by means of examples.

In future investigations it will be important to understand to what extent the use of entropic quantities to characterize non-Markovian open system dynamics can be further justified, developing a measure of non-Markovianity by the detection of the pair of states maximizing the backflow of information [8, 66, 67], or connecting the revivals of distinguishability with the presence of initial correlations [68].

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- [1] H.-P. Breuer, *Journal of Physics B: Atomic, Molecular and Optical Physics* **45**, 154001 (2012).
 - [2] Á. Rivas, S. F. Huelga, and M. B. Plenio, *Reports on Progress in Physics* **77**, 094001 (2014).
 - [3] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, *Rev. Mod. Phys.* **88**, 021002 (2016).
 - [4] M. M. Wolf, J. Eisert, T. S. Cubitt, and J. I. Cirac, *Phys. Rev. Lett.* **101**, 150402 (2008).
 - [5] A. Rivas, S. F. Huelga, and M. B. Plenio, *Phys. Rev. Lett.* **105**, 050403 (2010).
 - [6] M. J. W. Hall, J. D. Cresser, L. Li, and E. Andersson, *Phys. Rev. A* **89**, 042120 (2014).
 - [7] S. Wißmann, H.-P. Breuer, and B. Vacchini, *Phys. Rev. A* **92**, 042108 (2015).
 - [8] H.-P. Breuer, E.-M. Laine, and J. Piilo, *Phys. Rev. Lett.* **103**, 210401 (2009).
 - [9] D. Chruściński, A. Kossakowski, and A. Rivas, *Phys. Rev. A* **83**, 052128 (2011).
 - [10] S. Lorenzo, F. Plastina, and M. Paternostro, *Phys. Rev. A* **88**, 020102(R) (2013).
 - [11] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, *Phys. Rev. A* **97**, 012127 (2018).
 - [12] S. Milz, M. S. Kim, F. A. Pollock, and K. Modi, *Phys. Rev. Lett.* **123**, 040401 (2019).
 - [13] Y.-Y. Hsieh, Z.-Y. Su, and H.-S. Goan, *Phys. Rev. A* **100**, 012120 (2019).
 - [14] F. F. Fanchini, G. Karpat, B. Çakmak, L. K. Castelano, G. H. Aguilar, O. J. Farias, S. P. Walborn, P. H. S. Ribeiro, and M. C. de Oliveira, *Phys. Rev. Lett.* **112**, 210402 (2014).
 - [15] J. Kołodyński, S. Rana, and A. Streltsov, *Phys. Rev. A* **101**, 020303(R) (2020).
 - [16] B. Bylicka, D. Chruściński, and S. Maniscalco, *Scientific Reports* **4**, 5720 (2014).
 - [17] D. M. Reich, N. Katz, and C. P. Koch, *Scientific Reports* **5**, 12430 (2015).
 - [18] E.-M. Laine, H.-P. Breuer, and J. Piilo, *Sci. Rep.* **4**, 4620 (2014).
 - [19] E.-M. Laine, J. Piilo, and H.-P. Breuer, *EPL* **92**, 60010 (2010).
 - [20] L. Mazzola, C. A. Rodríguez-Rosario, K. Modi, and M. Paternostro, *Phys. Rev. A* **86**, 010102(R) (2012).
 - [21] A. Smirne, L. Mazzola, M. Paternostro, and B. Vacchini, *Phys. Rev. A* **87**, 052129 (2013).
 - [22] S. Cialdi, A. Smirne, M. G. A. Paris, S. Olivares, and B. Vacchini, *Phys. Rev. A* **90**, 050301(R) (2014).
 - [23] S. Campbell, M. Popovic, D. Tamascelli, and B. Vacchini, *New Journal of Physics* **21**, 053036 (2019).
 - [24] N. Megier, D. Chruściński, J. Piilo, and W. T. Strunz, *Scientific Reports* **7**, 6379 (2017).
 - [25] H.-P. Breuer, G. Amato, and B. Vacchini, *New Journal of Physics* **20**, 043007 (2018).
 - [26] A. A. Budini, *Phys. Rev. A* **97**, 052133 (2018).
 - [27] D. De Santis, M. Johansson, B. Bylicka, N. K. Bernardes, and A. Acín, *Phys. Rev. A* **99**, 012303 (2019).
 - [28] D. D. Santos and M. Johansson, *New Journal of Physics* **22**,

- 093034 (2020).
- [29] D. De Santis, M. Johansson, B. Bylicka, N. K. Bernardes, and A. Acín, *Phys. Rev. A* **102**, 012214 (2020).
- [30] M. Banacki, M. Marciniak, K. Horodecki, and P. Horodecki, “Information backflow may not indicate quantum memory,” (2020), arXiv:2008.12638 [quant-ph].
- [31] K. M. R. Audenaert, in *Theory of Quantum Computation, Communication, and Cryptography*, edited by D. Bacon, M. Martin-Delgado, and M. Roetteler (Springer Berlin Heidelberg, Berlin, Heidelberg, 2014) pp. 39–52.
- [32] A. P. Majtey, P. W. Lamberti, and D. P. Prato, *Phys. Rev. A* **72**, 052310 (2005).
- [33] W. Roga, M. Fannes, and K. Życzkowski, *Phys. Rev. Lett.* **105**, 040505 (2010).
- [34] J. Dajka, J. Łuczka, and P. Hägggi, *Phys. Rev. A* **84**, 032120 (2011).
- [35] L. Rossi, A. Torsello, E. R. Hancock, and R. C. Wilson, *Phys. Rev. E* **88**, 032806 (2013).
- [36] M. De Domenico, V. Nicosia, A. Arenas, and V. Latora, *Nature Communications* **6**, 6864 (2015).
- [37] L. Bai, L. Rossi, A. Torsello, and E. R. Hancock, *Pattern Recognition* **48**, 344 (2015).
- [38] C. Radhakrishnan, M. Parthasarathy, S. Jambulingam, and T. Byrnes, *Phys. Rev. Lett.* **116**, 150504 (2016).
- [39] B.-L. Ye, L.-Y. Xue, Y.-L. Fang, S. Liu, Q.-C. Wu, Y.-H. Zhou, and C.-P. Yang, *Physica E* **115**, 113690 (2020).
- [40] A. Slaoui, A. Salah, and M. Daoud, *Physica A: Statistical Mechanics and its Applications* **558**, 124946 (2020).
- [41] D. Virosztek, “The metric property of the quantum Jensen-Shannon divergence,” (2019), arXiv:1910.10447 [math-ph].
- [42] S. Sra, “Metrics induced by Jensen-Shannon and related divergences on positive definite matrices,” (2019), arXiv:1911.02643 [cs.IT].
- [43] A. S. Holevo, *Statistical Structure of Quantum Theory*, Lecture Notes in Physics, Vol. m 67 (Springer, Berlin, 2001).
- [44] Mostly, the data processing inequality is stated for complete positive maps. However, for trace distance and quantum relative entropy the positivity of the map is sufficient [69].
- [45] F. Hiai and M. Mosonyi, *Reviews in Mathematical Physics* **29**, 1750023 (2017).
- [46] G. Amato, H.-P. Breuer, and B. Vacchini, *Phys. Rev. A* **98**, 012120 (2018).
- [47] R. Vasile, S. Maniscalco, M. G. A. Paris, H.-P. Breuer, and J. Piilo, *Phys. Rev. A* **84**, 052118 (2011).
- [48] P. W. Lamberti, A. P. Majtey, A. Borras, M. Casas, and A. Plastino, *Phys. Rev. A* **77**, 052311 (2008).
- [49] J. Briët and P. Harremoës, *Phys. Rev. A* **79**, 052311 (2009).
- [50] A. Wehrl, *Rev. Mod. Phys.* **50**, 221 (1978).
- [51] B. Schumacher and M. Westmoreland, e-print arXiv:quant-ph/0004045 (2000).
- [52] V. Vedral, *Rev. Mod. Phys.* **74**, 197 (2002).
- [53] T. Sagawa, in *Lectures on Quantum Computing, Thermodynamics and Statistical Physics*, edited by M. Nakahara and S. Tanaka (World Scientific, 2012) pp. 125–190.
- [54] M. Esposito, K. Lindenberg, and C. V. den Broeck, *New Journal of Physics* **12**, 013013 (2010).
- [55] K. Ptaszynski and M. Esposito, *Phys. Rev. Lett.* **123**, 200603 (2019).
- [56] S. Floerchinger and T. Haas, *Phys. Rev. E* **102**, 052117 (2020).
- [57] H. Umegaki, *Kodai Math. Sem. Rep.* **14**, 59 (1962).
- [58] K. Lendi, F. Farhadmotamed, and A. J. van Wonderen, *Journal of Statistical Physics* **92**, 1115 (1998).
- [59] A. Müller-Hermes and D. Reeb, *Annales Henri Poincaré* **18**, 1777 (2017).
- [60] K. M. R. Audenaert, “Telescopic relative entropy–II triangle inequalities,” (2011), arXiv:1102.3041 [math-ph].
- [61] M. S. Pinsker, *Izv. Akad. Nauk* (1960).
- [62] D. M. Endres and J. E. Schindelin, *IEEE Transactions on Information Theory* **49**, 1858 (2003).
- [63] R. R. Puri, *Physics and astronomy online library* (Springer, Berlin, 2001).
- [64] A. Smirne and B. Vacchini, *Phys. Rev. A* **82**, 022110 (2010).
- [65] N. Tang, W. Cheng, and H.-S. Zeng, *The European Physical Journal D* **68**, 278 (2014).
- [66] S. Wißmann, A. Karlsson, E.-M. Laine, J. Piilo, and H.-P. Breuer, *Phys. Rev. A* **86**, 062108 (2012).
- [67] B.-H. Liu, S. Wißmann, X.-M. Hu, C. Zhang, Y.-F. Huang, C.-F. Li, G.-C. Guo, A. Karlsson, J. Piilo, and H.-P. Breuer, *Sci. Rep.* **4**, 6327 (2014).
- [68] J. Dajka, J. Łuczka, and P. Hägggi, *Phys. Rev. A* **84**, 032120 (2011).
- [69] S. Khatri and M. M. Wilde, “Principles of quantum communication theory: A modern approach,” (2020), arXiv:2011.04672 [quant-ph].
- [70] F. Topsøe, in *Inequality Theory and Applications*, Vol. 4, edited by Y. J. Cho, J. K. Kim, and S. S. Dragomir (Nova Science Publishers, 2007) p. 137.

APPENDIX A: Proof of Eq. (11)

We prove validity of the inequality Eq. (11), relying on the already introduced properties of the TRE. This inequality provides an upper bound for the revivals of the TRE for the reduced states of the system at different times, showing that this upper bound is due to the establishment of correlations between system and environment, as well as changes in the environmental states. Importantly, in the absence of these modifications the bound is equal to zero.

Fixed a pair of initial system states $\varrho_s(0)$ and $\sigma_s(0)$, we introduce the following notation for the quantity at the l.h.s. of Eq. (11)

$$I_\mu(t, s) \equiv S_\mu(\varrho_s(t), \sigma_s(t)) - S_\mu(\varrho_s(s), \sigma_s(s)). \quad (\text{A1})$$

Exploiting CPT of the partial trace and invariance of TRE under unitaries, with the natural assumption of unitarity of the overall evolution we can write

$$I_\mu(t, s) \leq S_\mu(\varrho(t), \sigma(t)) - S_\mu(\varrho(s), \sigma(s)) = S_\mu(\varrho(s), \sigma(s)) - S_\mu(\varrho(s), \sigma(s)). \quad (\text{A2})$$

Adding and subtracting the quantity $S_\mu(\varrho_s(s) \otimes \varrho_E(s), \sigma(s))$ at the r.h.s. and using invariance under tensor product we are left with

$$I_\mu(t, s) \leq |S_\mu(\varrho(s), \sigma(s)) - S_\mu(\varrho_s(s) \otimes \varrho_E(s), \sigma(s))| + |S_\mu(\varrho_s(s) \otimes \varrho_E(s), \sigma(s)) - S_\mu(\varrho_s(s) \otimes \varrho_E(s), \sigma_s(s) \otimes \varrho_E(s))|. \quad (\text{A3})$$

To proceed further we rearrange the triangle-like inequalities Eqs. (8) and (9) according to

$$S_\mu(\varrho, \sigma) - S_\mu(\varrho, \tau) \leq \frac{1}{\log(1/\mu)} \log \left(1 + D(\sigma, \tau) \frac{1-\mu}{\mu} \right) \quad (\text{A4})$$

and

$$S_\mu(\varrho, \sigma) - S_\mu(\eta, \sigma) \leq \frac{D(\varrho, \eta)}{\log(1/\mu)} \log \left(1 + \frac{1}{D(\varrho, \eta)} \frac{1-\mu}{\mu} \right), \quad (\text{A5})$$

which provide more convenient starting points for upper bounding the different contributions. Indeed we can now exploit the following inequality, valid for non-negative x [70]

$$\log(1+x) \leq \frac{x}{\sqrt{1+x}}, \quad (\text{A6})$$

entailing as special case

$$\log(1+x) \leq \sqrt{x}. \quad (\text{A7})$$

We now exploit Eqs. (A5) and (A6) for the first term at the r.h.s. of Eq. (A3) upon the identification $\sigma \rightarrow \sigma(s)$, as well as Eqs. (A4) and (A7) for the second term at the r.h.s. of Eq. (A3) to obtain

$$I_\mu(t, s) \leq \sqrt{\frac{1-\mu}{\mu}} \frac{1}{\log(1/\mu)} \left(\sqrt{D(\varrho(s), \varrho_s(s) \otimes \varrho_E(s))} + \sqrt{D(\sigma(s), \sigma_s(s) \otimes \varrho_E(s))} \right). \quad (\text{A8})$$

We further use the triangle inequality and the invariance of the TD with respect to tensor product, together with the inequality $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ valid for non-negative x and y , thus finally coming to

$$I_\mu(t, s) \leq \sqrt{\frac{1-\mu}{\mu}} \frac{1}{\log(1/\mu)} \left(\sqrt{D(\varrho(s), \varrho_s(s) \otimes \varrho_E(s))} + \sqrt{D(\sigma(s), \sigma_s(s) \otimes \varrho_E(s))} + \sqrt{D(\sigma(s), \varrho_E(s))} \right). \quad (\text{A9})$$

As a last step we use the generalized Pinsker inequality Eq. (10) and obtain the upper bound

$$I_\mu(t, s) \leq \frac{1}{\sqrt[4]{2\mu^2 \log^3(1/\mu)}} \left(\sqrt[4]{S_\mu(\varrho(s), \varrho_s(s) \otimes \varrho_E(s))} + \sqrt[4]{S_\mu(\sigma(s), \sigma_s(s) \otimes \varrho_E(s))} + \sqrt[4]{S_\mu(\sigma(s), \varrho_E(s))} \right). \quad (\text{A10})$$

Note that the prefactor $1/\sqrt[4]{2\mu^2 \log^3(1/\mu)}$ as a function of μ has a global minimum $(4e^3/27)^{1/4} \approx 1.31$ at $\mu = (1/e)^{3/2}$. This is the choice of telescopic parameter considered in Eq. (11), which is now proven.

Let us mention the fact that different upper bound of the triangle-like inequalities Eqs. (A4) and (A5) can be considered, leading to another bound with different functional dependency on the difference in environmental states and correlations. The bound reads

$$I_\mu(t, s) \leq \frac{1}{\sqrt{2\mu^2 \log(1/\mu)}} \left(\sqrt[4]{S_\mu(\varrho(s), \varrho_s(s) \otimes \varrho_E(s))} + \sqrt[4]{S_\mu(\sigma(s), \sigma_s(s) \otimes \varrho_E(s))} + \sqrt{S_\mu(\sigma(s), \varrho_E(s))} \right), \quad (\text{A11})$$

where the prefactor is obtained by starting from the looser bound $\log(1+x) \leq x$. The prefactor $1/\sqrt{2\mu^2 \log(1/\mu)}$ has its minimum \sqrt{e} at $\mu = 1/\sqrt{e}$. Note that $\sqrt{e} > (4e^3/27)^{1/4}$, accordingly in the most common situations, where during the greatest part of the dynamics of the open quantum system the creation of correlations plays the dominant role rather than the change in the environmental state, the bound given by Eq. (11) is the tighter one.