Students' difficulties dealing with number line: a qualitative analysis of a question from national standardized assessment

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Abstract:

In this paper we analyse some peculiar students' mistakes in a task selected from the Italian National Mathematics Standardized Tests. In particular we show different types of errors identified in the solutions of grade 6 students who faced an item that involved the management of the number line. We use both the statistical results of the national sample and the qualitative analysis of answers in a smaller sample of 181 students. Our study draws on previous research on difficulties that students face placing rational numbers on the number line. We use an intertwinement of different theoretical lenses to explain the possible causes of failure. We show how some students' answers can be interpreted as results of different misconceptions. The identified mistakes are related both to the management of the rational numbers representations (i.e. decimal representation and fraction) and to the manipulation of the graduate scale of the number line.

Résumé

Dans cet article, nous analysons certains particulières erreurs des élèves dans une tâche sélectionnée à partir des tests standardisés national italien en mathématique. En particulier, nous montrons différents types d'erreurs relevées dans les solutions des élèves de 6e année qui ont repondu à une question qui a impliqué la gestion de la ligne graduée. Nous utilisons à la fois les résultats statistiques de l'échantillon national et l'analyse qualitative des réponses dans un plus petit échantillon de 181 étudiants. Notre étude se fonde sur des recherches antérieures sur les difficultés que les étudiants ont à placer des nombres rationnels sur la ligne graduée. Nous utilisons un combinaison des différentes lentilles théoriques pour expliquer les causes possibles de l'échec. Nous montrons comment les réponses de certains élèves peuvent être interprétés comme des résultats de différentes misconception. Les erreurs identifiées sont liés à la gestion des représentations des numéros rationnels (i.e. représentation décimale et fraction) et à la manipulation de la ligne graduée.

Introduction

This paper presents an ongoing research developed in the *Ideas for the Research* project, funded by the Italian National Institute for the Educational Evaluation of Instruction (INVALSI). The aim of the project is the analysis of the outcomes of the Italian mathematics standardized tests that were collected by INVALSI from 2010 to 2013 in grades 6 and 8. Here we discuss only one of the stages of this research: the identification and analysis of students' errors in answering to an item that will be part of a vertical study over different grades. Our research group is composed by researchers in Mathematics Education (authors of this paper) and Statistics.

In the first part of the research, qualitative and quantitative analyses were intertwined. The quantitative analysis of the national sample was carried out by Mariagiulia Matteucci and Stefania Mignani: the two researchers in Statistics of our group. They used some INVALSI results to investigate possible trends in the data. They implemented a multilevel latent class analysis (Vermunt, 2003, 2008) to classify students and to judge the item characteristics. This Statistical method has given a classification of students in a fixed number of groups characterized by different levels of performance. The researchers identified some items in which the students with low performances had a higher probability to give a wrong answer in comparison with the other students (Branchetti et al., in press). Starting from these results, we carried out a qualitative analysis of the solution strategies developed by students when facing the identified items. We made a longitudinal analysis of the outcomes of the Italian national standardized mathematics tests in grade 6 and 8 (ibidem): in a subsample of the national one, students' answers were analysed to identify the possible mistakes that occurred in linked items. We focused on the mathematical concepts (in the sense of Vergnaud, 2009) involved and the possible difficulties arising from conversion among different semiotic representations (Duval, 2003).

We conjecture that the grade 6 items selected by our study could have a predictive power for the outcomes in the linked items identified in grade 8. Therefore, in the second step of the research, we are carrying out some classroom activities in order to collect new data about the possible students' answers to the items analysed in the first part of the research. For example, we have already re-administered some items to a sample of 181 students from different Italian cities (both from North and South Italy) and from schools with different socio-economical backgrounds. The data collected are coherent with the national sample results.

In this paper we discuss this last part of our ongoing research on the analysis of students' answers and identification of their errors over different grades.. In particular, we show the analysis of the solutions of some grade 6 students who faced a task that involves the management of rational numbers' different representations and the number line. We selected this topic because literature shows how ability to perform well on this task with fractions is highly predictive of later performance in mathematics (Jordan et al., 2013). Our analysis is based on previous research on students' difficulties with natural or rational numbers on the number line.

After a brief overview of the theoretical lenses used to look at students' answers, we describe and analyse the data collected, showing our interpretation of students' difficulties.

The number line

The international Research highlights the crucial role of the number line in mathematics education. For example, in Skoumpourdi (2010) the number line is presented as a didactical tool with high potential, especially since it provides a simple way to picture mathematical concepts: as a matter of fact, the number line is used for counting, for estimations and for representing time, but also for the representation of different number sets. Moreover, number line can be used for providing geometric models of the arithmetical operations, for measuring, and comparing quantities. In the same work it is also pointed out that many studies report difficulties and limitations in the use of the number line and propose educational activities to overcome these difficulties.

> The potential of the number line for organizing thinking about numbers and operations on the one hand, and the difficulties that arise from its use on the other hand, lead many researchers to propose learning strands and teaching sequences for its use in the teaching/learning process of mathematics. (Skoumpourdi, 2010, p. 2)

Skoumpourdi stresses that the number line can be presented in different versions: structured or semi structured, with or without numbers and other symbols, but also empty. For each of these representations, students may use different approaches in finding solutions.

In this paper we will focus on the difficulties that students could have facing tasks in which they have to place rational numbers on the number line.

In the following, we briefly present some results from the educational research about this specific topic.

Concerning the placing of rational numbers on the number line, we initially have to distinguish the difficulties about the management of decimals and fractions. According to Iuculano and Butterworth (2011) both adults and children are more accurate when performing this task with decimals rather than fractions because “decimals afford direct mapping onto a mental number line and, therefore, allow for easier magnitude assessment than do fractions” (De Wolf et al., 2014, p.2136).

Students may also have difficulties in conversion between representations in different semiotic registers (Duval, 1993): they could have troubles in finding strategies to pinpoint numbers on the number line because the number line is a hybrid representation (a line with a scale on it). Every geometric operation can be translated into an arithmetic operation and carried out algorithmically and vice versa (Gagatsis, et al., 2003). Some studies on the number line and fractions highlight the distinction between making partitions and reading pre-marked partitions (Mitchell & Horne, 2008). In fact, the identification of the unit in number lines seems to be problematic; in particular students' strategies may change whether the line is partitioned or not, since marks may act as perceptual distractors (Lesh et al, 1982). Students may have difficulties if one unit of length is divided into parts (Behr & Bright 1984): for instance some students, in order to determine the fraction denominator, ignore the endpoint and count only internal hash marks. If intervals between points already drawn have unequal lengths, students can count the number of points ignoring the distance. Other difficulties may stem from an over-generalization of part-whole partitioning strategies in measurement contexts. For example, Saxe and colleagues (2007) show that a student can progressively divide one unit by 2 and then an half by 2 and find ¾, but this strategy does not work in general: e.g. 2/7 . Therefore many mistakes can be generated by the interlacement of misconception about rational numbers and number line management. Other studies (e.g. Hartnett & Gelman,1998) show the conflict between ordering natural numbers on the number line (when numbers get bigger as values increase) and fractions (when the denominator gets bigger the fraction is smaller). A common error is to put the fraction close to the value of numerator or of the denominator: this can be explained in terms of the “whole number bias”, that means considering fractions as two separated whole numbers (Ni & Zhou, 2005) and comparing them separately (Stafylidou & Vosniadou, 2004).

In the next paragraph we analyse many of the difficulties identified in the quoted literature. Moreover we show how the same mistakes can be interpreted as the product of different misconceptions.

Data analysis

The item that we analyse in this paper was administered at national level to grade 6 italian students in 2011. This item has been selected, according to statistical methods, because it has a high discriminating power for low achievers: i.e. in the Italian national sample the students with low performances had a higher probability to give a wrong answer in comparison with the other students (Branchetti et al., in press).

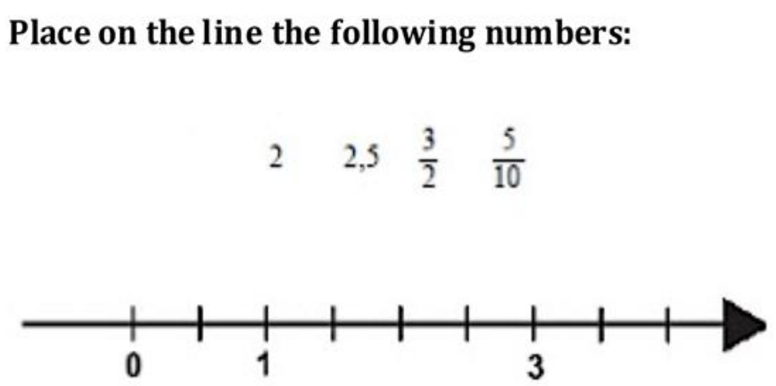

Figure 1: Item D8 from INVALSI test administered in grade 6 (2011)

Item D8 (Figure 1) concerns the placement of rational numbers on an oriented line. The rational numbers are presented in both decimal and fraction representations. The hash marks drawn on the line, refer to a specific unit of measure: the distance between two consecutive hash marks is 0.5.

On national level, only 11% of students places all the numbers in the correct positions (Fig. 2a). In order to identify possible students' difficulties, we administered this item to a sample of 181 students from different Italian cities (both from North and South Italy) and from schools with different socio-economical backgrounds. The data collected are coherent with the national sample results (Fig. 2b).

2a: National results **2b: Sample results**

Figure 2: Results of Item D8

Drawing on the literature results described in the previous paragraph, it sounds reasonable to suppose that students, who faced the D8 item, should:

- identify the correct unit of measure referring to the hash marks (Saxe et al, 2007; Behr & Bright, 1984);
- manage different number representations (Duval, 1993; Gagatsis et al. 2003);
- find out the order relation between these numbers (Hartnett & Gelman, 1998).

First of all, we focus on some difficulties already noticed in putting decimal numbers on the line: identify the unit of measure and the order relation between numbers.

Students' answers analysis allows us to recognize the following possible wrong behaviours.

a) Students do not correctly manage the unit of measure.

Among students who give a wrong answer, almost half of them (46%) fails in the management of the unit of measurement. Students do not consider the size of the intervals (the distance between hash marks): e.g. 2 and 2.5 are placed in two consecutive hash marks, subsequently to 1 (Fig. 3).

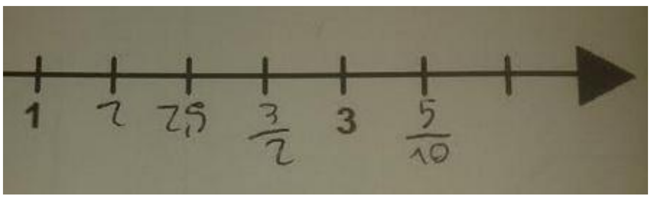

Figure 3: Students who pinpoint 2 and 2.5 subsequent to 1.

b) Students identify correctly just the natural number (2).

The 15% of students correctly uses the hash marks only for the natural number: the most common mistake is to put the 2.5 between the hash mark of 2 and the next one (Fig. 4). The other types of errors are less than 10% frequent.

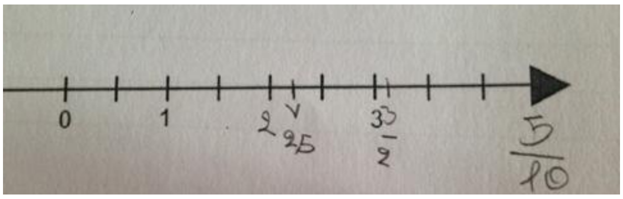

Figure 4: Students who place 2.5 between two consecutive hash marks.

c) Students have many difficulties in placing fractions on the number line

In our sample, 80% of the students correctly places the numbers in decimal form. The main students' difficulty is to identify the correct order between the numbers in different representations: decimal and fraction. In this case, the most frequent approach for numbers comparison is the conversion between registers (Duval, 1993): in particular from fractions to decimals, a conversion that frequently results in errors. Analysing students' errors, we define some common difficulties observed in converting fractions to decimals. Some students try to convert numbers from fractions to decimals dividing numerator and denominator, but they invert the order calculating the division between denominator and numerator. For example, a group of students (8%) places 5/10 as if it were 2 (Fig. 5), probably because they divide 10 by 5.

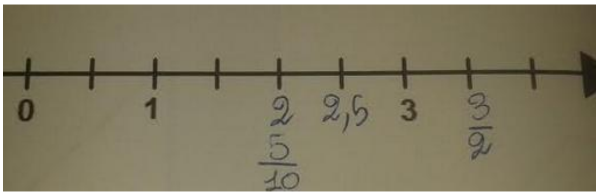

Figure 5: Students place 5/10 as if it were 2.

Other students (41%) pinpoint 3/2 near the hash mark referred to the value 3, probably because they convert 3/2 as if it were 3.2 (Fig. 6).

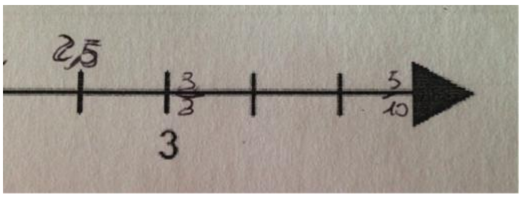

Figure 6: Students place 3/2 near 3.

We can also provide an alternativeinterpretation of this phenomenon:students may consider 3/2 equivalent to 3+½, so they put the number in the hash mark next to 3 (Fig. 7).

Similarly, some students place 5/10 over the end of the line (7%), probably because they interpret 5/10 as 5.10, 5+1/10 or something else greater than 5 (Fig. 7).

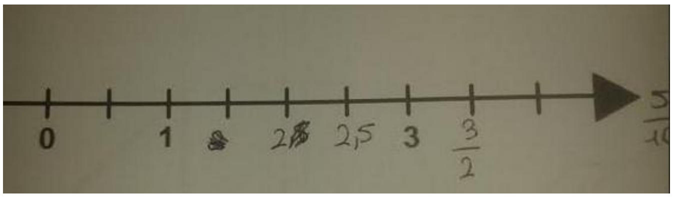

Figure 7: Students place 3/2 as it were 3+1/2 and 5/10 as it were more than 5.

Other errors are less frequent (6% of the students). For example, in Figure 8 we can see one of these unusual answers. A possible interpretation of this excerpt is that the student tries to represent the numbers with brackets that, in two cases (5/10 and 2), are long as the number that they would like to identify.

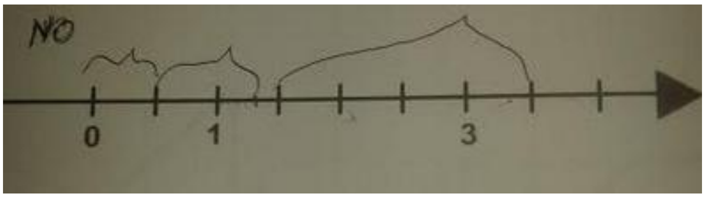

Figure 8: Unusual approach.

Conclusions

In our research we identify and analyse items from Italian national mathematics standardized tests over different years in different grades. We have identified items in which the students with low performances in the tests had a higher probability to give a wrong answer in comparison with the other students and, among those, we have selected items that deals with longitudinal topics. For this reason in this paper we argue an items on rational numbers and number line. Jordan et al (2013) also argue that tasks involving the placing of fractions on a number line, can be high predictive for future outcomes in mathematics.

The data analysis presented in this paper shows some difficulties that students face while dealing with the number line. These difficulties depend both on the number line scale's management and the specific number set that is considered. In particular, according to literature (Iuculano & Butterworth, 2011; Saxe et al., 2007), we observe that most of errors concern the pinpointing of fractions.

Our data confirm that students have fewer difficulties with natural numbers, while errors are more frequent when rational numbers are involved. Students' mistakes concerning decimal representation are related to the management of the placement on hash marks. The analysis carried out in our study show that some students seem to interpret the number line as a list of ordered numbers. They do not take into account that the distance between the hash marks represents the difference between numbers: e.g. many students have difficulty in positioning 2.5 correctly (Fig.3). Furthermore, we find many examples of excerpts as the one shown in Figure 4. These students' answers suggest a kind of *implicit model* (in the sense of Fischbein et al., 1985): hash marks can be employed just for integers.

As already emphasized by many studies, more frequently students show difficulties with fraction and, in particular, in the conversion from fraction to decimals. The explanation of the fact that many students try to convert fractions to decimals can be both didactical and psychological. DeWolf et al. (2014) highlight that in many textbooks decimals are used mostly to refer to continuous models (as the line) while fractions generally represent the discrete ones. Such a result suggests that pupils may try to convert fractions to decimals because decimals are more familiar in the context of a continuous model as the number line is. The same authors suggest, quoting Iuculano & Butterworth (2011), that decimals afford direct mapping onto a mental number line and, therefore, allow easier magnitude assessment than fractions do. The analysis presented in the previous paragraph suggests that most of the errors are not related specifically to the number line representation, but they are connected to the manipulation of fractions in general. For example, students who place 5/10 as if it were 2 can be influenced by what Fischbein et al. (1985) define as "implicit model's rule that the dividend must be larger than the divisor". According to these authors the violation of such rule shows in reversing the order of terms (Fig.5). Moreover, Markovits & Sowder (1991) show that there are students who convert the fraction *a/b* as the decimal *a.b*, as we observe in the answers given by pupils who put *a/b* near the numerator *a* (Fig. 6).

Other errors with fractions could be related to the "whole number bias" (Ni & Zhou, 2005) as, for example, the placing of 5/10 over the position of 5 on the number line. There could be different interpretations about the possible mistakes identified:

- To put 5/10 after 5 because it is transformed in 5.10 or because is seen only as something greater then 5 (5+1/10)

- To put 3/2 after 3 because it is transformed in 3.2 or because is seen only as something greater then 3 (3+1/2)

There are also mistakes that depend on the particular fraction that is considered as, for instance, the identification of 5/10 with 2 (the greater number divided by the smaller one if the bigger is a multiple).

As the difficulties that students face with decimals and fractions seem to be different, it can be questioned if there is a relationship between them. By combining the analysis of the different errors observed in our sample, we have found out an interesting fact: almost all students who place 2.5 wrongly on the number line show difficulties also in placing the two fractions. This evidence leads to conjecture that the inability to pinpoint decimal numbers can be a predictor of the inability of placing fraction, but the question remains still open. In our further research we will deeply investigate this aspect.

This analysis can be useful in teacher education: as a matter of fact, it contains suggestions concerning the most significant students' wrong strategies in a typical item of the Italian national assessment (the management of number line).

Students' strategies can be further investigated in "vertical chains" made of questions on different levels that involve same contents. This kind of analysis may give insights about the predictive power of these tasks.

REFERENCES

Behr, M., & Bright, G. (1984). Identifying fractions on number lines. *Paper presented at the meeting of the American Educational Association*. New Orleans, LA.

Behr, M. J., Lesh, R., Post, T. R., & Silver, E. (1982). Rational-number concepts. In R. Lesh & M. Landau (Eds.), Acquisition of mathematical concepts and processes (pp. 91-126). New York: Academic Press.

Branchetti, L., Ferretti, F., Lemmo, A., Maffia, A., Martignone, F., Matteucci, M., & Mignani, S. (in press). A longitudinal analysis of the Italian standardized mathematics test. *Proceedings of the 9th CERME*. Prague: ERME.

DeWolf, M., Rapp, M., Bassok, M., & Holyoak, K. J. (2014). Semantic alignment of fractions and decimals with discrete versus continuous entities: A textbook analysis. In B. Bello, M. Guarini, M. McShane, & B. Scassellati (Eds.), *Proceedings of the 36th Annual Conference of the Cognitive Science* Society, 2133-2138. Austin: Cognitive Science Society.

Duval, R. (1993). Registres de représentation sémiotique et fonctionnement cognitif de la pensée. *Annales de didactique et de sciences cognitives*, 5, 37-65.

Fischbein, E., Deri, M., Sainati Nello, M., & Sciolis Marino, M. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16(1),* 3-17.

Gagatsis, A., Shiakalli, M., & Panaoura, A. (2003). La droite arithmétique comme modèle géométrique de l'addition et de la soustraction des nombres entiers. *Annales de didactique et de sciences cognitives, 8,* 95-112.

Hartnett, P.M., & Gelman, R. (1998). Early understandings of numbers: Paths or barriers to the construction of new understandings? *Learning and Instruction: The Journal of the European Association for Research in Learning and Instruction, 8(4),* 341-374.

Iuculano, T., & Butterworth, B. (2011). Understanding the real value of fractions and decimals. *Quarterly Journal of Experimental Psychology, 64,* 2088-2098.

Jordan, N.C., Hansen, N., Fuchs, L.S., Siegler, R.S., Gersten, R., & Micklos, D. (2013). Developmental predictors of fractions concepts and procedures. *Journal of experimental psychology, 116*, 45-58.

Markovits, Z., & Sowder, J.T. (1991). Students' understanding of the relationship between fractions and decimals. *Focus on learning problems in mathematics, 13,* 3-11.

Mitchell, A., & Horne, M. (2008). Fraction Number Line Tasks and the Additivity Concept of Length Measurement. *Proceedings of the 31st Annual Conference of the Mathematics Education Research Group of Australasia.* M. Goos, R. Brown, & K. Makar (Eds.)

Ni, Y., & Zhou, Y.D. (2005). Teaching and learning fractions and rational numbers: the origins and implications of whole numbers bias. *Educational psychologist, 40(1*), 27-52.

Saxe, G.B., Shaughnessy, M.M., Shannon, A., Langer-Osuna, J., Chinn, R., & Gearhart, M. (2007). Learning about fractions as points on a number line. In P. Elliot, W.G. Martin, & M.E. Strutchens (Eds.), *NCTM Sixty-ninth Yearbook*. Reston: NCTM.

Skoumpourdi, C. (2010). The number line: an auxiliary means or an obstacle? *International Journal for Mathematics Teaching and Learning*.

Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction, 14,* 503-518.

Vermunt, J.K. (2003). Multilevel latent class models. *Sociological Methodology, 33*, 213–239.

Vermunt, J.K. (2008). Latent class and finite mixture models for multilevel data sets. *Statistical Methods in Medical Research, 17(1)*, 33–51.