Milan J. Math. Vol. 89 (2021) 241–261
https://doi.org/10.1007/s00032-021-00332-z
Published online June 8, 2021

Milan Journal of Mathematics

Heegner Points and Exceptional Zeros of Garrett p-Adic L-Functions

Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci

Abstract. This article proves a case of the p-adic Birch and Swinnerton–Dyer conjecture for Garrett p-adic L-functions of (Bertolini et al. in On p-adic analogues of the Birch and Swinnerton–Dyer conjecture for Garrett L-functions, 2021), in the imaginary dihedral exceptional zero setting of extended analytic rank 4.

1. Statement of the Main Result

Let A be an elliptic curve defined over the field $\mathbf{Q}$ of rational numbers, having multiplicative reduction at a rational prime $p > 3$. Let K be a quadratic imaginary field of discriminant d_K coprime to the conductor N_A of A, and let

$$\nu_g : G_K \longrightarrow \bar{\mathbf{Q}}^* \quad \text{and} \quad \nu_h : G_K \longrightarrow \bar{\mathbf{Q}}^*$$

be finite order characters of the absolute Galois group $G_K = \mathrm{Gal}(\bar{\mathbf{Q}}/K)$ of K, where $\bar{\mathbf{Q}}$ is the field of algebraic complex numbers. Write $N_A = N_A^+ \cdot N_A^-$, where each prime divisor of N_A^+ (resp., N_A^-) splits (resp., is inert) in K. We make the following

Assumption 1.1. 1. (Heegner assumption) The prime p is *inert* in K (id est divides N_A^-) and N_A^- is a square-free product of an *even* number of primes.
2. (Self-duality) The central characters of ν_g and ν_h are inverse to each other.
3. (Cuspidality) The characters ν_g and ν_h are not induced by Dirichlet characters.
4. (Local signs) The conductors of ν_g and ν_h are coprime to $d_K \cdot N_A$.

Let $f = \sum_{n \geqslant 1} a_n(f) \cdot q^n$ in $S_2(\Gamma_0(N_f))$ be the newform of conductor $N_f = N_A$ attached to A by the modularity theorem. For $\nu_\xi = \nu_g, \nu_h$, let $\varrho_\xi : G_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(\mathbf{C})$ be the odd irreducible (cf. Assumption 1.1.(3)) Artin representation of $G_{\mathbf{Q}}$ induced by ν_ξ, corresponding by modularity to the cuspidal weight one theta series

$$\xi = \sum_{(\mathfrak{a}, \mathfrak{f}_\xi) = \mathcal{O}_K} \nu_\xi(\mathfrak{a}) \cdot q^{\mathbf{N}\mathfrak{a}} \in S_1(N_\xi, \chi_\xi).$$

Here $\mathfrak{a}$ runs the set of non-zero ideals of $\mathcal{O}_K$ coprime to the conductor $\mathfrak{f}_\xi$ of ν_ξ, $\mathbf{N}\mathfrak{a} = |\mathcal{O}_K/\mathfrak{a}|$, $N_\xi = d_K \cdot \mathbf{N}\mathfrak{f}_\xi$ and $\chi_\xi = \varepsilon_K \cdot \nu_\xi^{\mathrm{cen}}$, where $\varepsilon_K : (\mathbf{Z}/d_K\mathbf{Z})^* \longrightarrow \mu_2$ is

Birkhäuser

the quadratic character of K and $\nu_\xi^{\mathrm{cen}} : G_{\mathbf{Q}} \longrightarrow \bar{\mathbf{Q}}^*$ is the central character of ν_ξ. Since p is inert in K by Assumption 1.1.(1), the p-th Hecke polynomial of ξ equals $X^2 + \chi_\xi(p)$ (id est the p-th Fourier coefficient of ξ is equal to zero). In addition $\chi_\xi(p)$ is non-zero by Assumption 1.1.(4), hence $X^2 + \chi_\xi(p) = (X - \alpha_\xi) \cdot (X - \beta_\xi)$ has distinct roots α_ξ and $\beta_\xi = -\alpha_\xi$. According to Assumption 1.1.(2) one has $\alpha_g \cdot \alpha_h = \beta_g \cdot \beta_h = \pm 1$ and $\alpha_g \cdot \beta_h = \beta_g \cdot \alpha_h = -\alpha_g \cdot \alpha_h$, hence we can, and will, assume

$$\alpha_f = \alpha_g \cdot \alpha_h = \beta_g \cdot \beta_h \text{ and } -\alpha_f = \beta_g \cdot \alpha_h = \alpha_g \cdot \beta_h \tag{1}$$

by reordering the roots (α_ξ, β_ξ) of $X^2 + \chi_\xi(p)$ if necessary, where $\alpha_f = a_p(f) = \pm 1$.

Fix an algebraic closure $\bar{\mathbf{Q}}_p$ of $\mathbf{Q}_p$, an embedding $i_p : \bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$, and a finite extension L of $\mathbf{Q}_p$ containing (the images under i_p of) the values of ν_ξ and α_ξ, for $\xi = g, h$. Denote by ξ_α in $S_1(pN_\xi, \chi_\xi)$ the p-stabilisation of ξ with U_p-eigenvalue α_ξ. According to [1,7], there exist unique Hida families

$$\boldsymbol{f} = \sum_{n \geqslant 1} a_n(\boldsymbol{f}) \cdot q^n \in \mathscr{O}_{\boldsymbol{f}}[\![q]\!] \quad \text{and} \quad \boldsymbol{\xi}_\alpha = \sum_{n \geqslant 1} a_n(\boldsymbol{\xi}_\alpha) \cdot q^n \in \mathscr{O}_{\boldsymbol{\xi}_\alpha}$$

specialising to $f = \boldsymbol{f}_2$ and $\xi_\alpha = \boldsymbol{\xi}_{\alpha,1}$ in weights two and one respectively. Here $\mathscr{O}_{\boldsymbol{f}}$ is the ring of bounded analytic functions on a (small) connected open disc $U_{\boldsymbol{f}}$ centred at 2 in the weight space $\mathcal{W} = \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p^*, \mathbf{C}_p^*)$ over $\mathbf{Q}_p$. For each k in $U_{\boldsymbol{f}} \cap \mathbf{Z}_{\geqslant 4}$, the weight-$k$ specialisation $\boldsymbol{f}_k$ of $\boldsymbol{f}$ is the ordinary p-stabilisation of a p-ordinary newform f_k of weight k and level $\Gamma_0(N_f/p)$. Similarly $\mathscr{O}_{\boldsymbol{\xi}_\alpha}$ is the ring of bounded analytic functions on a connected open disc $U_{\boldsymbol{\xi}_\alpha}$ centred at 1 in $\mathcal{W}_L = \mathcal{W} \otimes_{\mathbf{Q}_p} L$, and $\boldsymbol{\xi}_{\alpha,u}$ is the p-stabilisation of a newform ξ_u of weight u and level $\Gamma_1(N_\xi)$ for each l in $U_{\boldsymbol{\xi}_\alpha} \cap \mathbf{Z}_{\geqslant 1}$, with $\xi_1 = \xi$. In order to lighten the notation, we write $U_{\boldsymbol{\xi}} = U_{\boldsymbol{\xi}_\alpha}$ and $\mathscr{O}_{\boldsymbol{\xi}} = \mathscr{O}_{\boldsymbol{\xi}_\alpha}$.

Set $\varrho = \varrho_g \otimes \varrho_h$ and $\mathscr{O}_{\boldsymbol{fgh}} = \mathscr{O}_{\boldsymbol{f}} \hat{\otimes}_{\mathbf{Q}_p} \mathscr{O}_{\boldsymbol{g}} \hat{\otimes}_L \mathscr{O}_{\boldsymbol{h}}$. Under Assumption 1.1, Theorem A of [8] associates with $(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ a Garrett–Hida square root p-adic L-function

$$\mathscr{L}_p^{\alpha\alpha}(A, \varrho) = \mathscr{L}_p(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha) \in \mathscr{O}_{\boldsymbol{fgh}}$$

(denoted $\mathcal{L}_{\boldsymbol{F}}^{\boldsymbol{f}}$ in loc. cit., where $\boldsymbol{F} = (\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$), whose square

$$L_p^{\alpha\alpha}(A, \varrho) = L_p(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha) = \mathscr{L}_p(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)^2$$

interpolates the central critical values $L(f_k \otimes g_l \otimes h_m, (k+l+m-2)/2)$ of the Garrett L-functions attached to (f_k, g_l, h_m) for classical triples (k, l, m) in the f-*unbalanced* region, viz. triples (k, l, m) in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}} \cap \mathbf{Z}_{\geqslant 1}^3$ satisfying $k \geqslant l + m$. The first equality in (1) implies that $L_p^{\alpha\alpha}(A, \varrho)$ has an exceptional zero in the sense of [9] at the "Birch and Swinnerton–Dyer point" $w_o = (2, 1, 1)$ (cf. [5, Section 1.2]).

Fix a number field $\mathbf{Q}(\varrho)$ containing the values of ν_g and ν_h, and for $\xi = g, h$ fix a $\mathbf{Q}(\varrho)[G_{\mathbf{Q}}]$-module V_ξ, two-dimensional over $\mathbf{Q}(\varrho)$, affording the Artin representation ϱ_ξ. Define $A(K_\varrho)^\varrho = H^0(\mathrm{Gal}(K_\varrho/\mathbf{Q}), A(K_\varrho) \otimes_{\mathbf{Z}} V_{gh})$, where $V_{gh} = V_g \otimes_{\mathbf{Q}(\varrho)} V_h$ and K_ϱ is the number field cut-out by $\varrho = \varrho_g \otimes \varrho_h$. Following [9] one exploits Tate's p-adic uniformisation to define an extended Mordell–Weil group

$$A^\dagger(K_\varrho)^\varrho = A(K_\varrho)^\varrho \oplus \mathcal{Q}_p(A, \varrho),$$

where $\mathcal{Q}_p(A,\varrho)$ is a two-dimensional $\mathbf{Q}(\varrho)$-vector space depending only on the base change of A to $\mathbf{Q}_p$ and on the restriction of V_{gh} to $G_{\mathbf{Q}_p}$ (cf. Sect. 2.1.3 below). Moreover, Section 2 of [4] constructs a *Garrett–Nekovář* height-pairing

$$\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} : A^\dagger(K_\varrho)^\varrho \otimes_{\mathbf{Q}(\varrho)} A^\dagger(K_\varrho)^\varrho \longrightarrow \mathscr{I}/\mathscr{I}^2,$$

where $\mathscr{I}$ is the kernel of evaluation at w_o on $\mathscr{O}_{\boldsymbol{f}\boldsymbol{g}\boldsymbol{h}}$. It is a skew-symmetric bilinear form, arising from an application of Nekovář's theory of Selmer complexes to the big self-dual Galois representation associated with $(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$. After setting

$$r^\dagger = \dim_{\mathbf{Q}(\varrho)} A(K_\varrho)^\varrho,$$

Conjecture 1.1 of [4] predicts that $L_p^{\alpha\alpha}(A,\varrho)$ belongs to $\mathscr{I}^{r^\dagger} - \mathscr{I}^{r^\dagger+1}$, and that its image in $(\mathscr{I}^{r^\dagger}/\mathscr{I}^{r^\dagger+1})/\mathbf{Q}(\varrho)^{*2}$ is equal to the discriminant

$$R_p^{\alpha\alpha}(A,\varrho) = \det\left(\langle\!\langle P_i,P_j\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}\right)_{1\leqslant i,j\leqslant r^\dagger}$$

of the p-adic height $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$, where $P_1,\ldots,P_{r^\dagger}$ is any $\mathbf{Q}(\varrho)$-basis of $A^\dagger(K_\varrho)^\varrho$.

The following theorem is the main result of this note.

Theorem. *Assume that Assumptions 1.1 and 1.2 (stated below) are satisfied. If the complex L-function $L(f\otimes g\otimes h,s)$ has order of vanishing 2 at $s=1$, then*

$$\dim_{\mathbf{Q}(\varrho)} A^\dagger(K_\varrho)^\varrho = 4, \qquad L_p^{\alpha\alpha}(A,\varrho) \in \mathscr{I}^4 - \mathscr{I}^5$$

and the equality

$$L_p^{\alpha\alpha}(A,\varrho) \pmod{\mathscr{I}^5} = R_p^{\alpha\alpha}(A,\varrho)$$

holds in the quotient of $\mathscr{I}^4/\mathscr{I}^5$ by the multiplicative action of $\mathbf{Q}(\varrho)^{*2}$.

In the present setting, the Garrett L-function $L(f\otimes g\otimes h,s)$ factors as the product of the Rankin–Selberg L-functions $L(A/K,\varphi,s)$ and $L(A/K,\psi,s)$, where $\varphi=\nu_g\cdot\nu_h$ and $\psi=\nu_g\cdot\nu_h^c$, and ν_h^c is the conjugate of ν_h by the nontrivial element of $\mathrm{Gal}(K/\mathbf{Q})$. Note that φ and ψ are *dihedral* by Assumption 1.1.(2), and that both $L(A/K,\varphi,s)$ and $L(A/K,\psi,s)$ have sign -1 in their functional equation by Assumption 1.1.(1). In particular the assumptions of the Theorem imply that $L(A/K,\chi,s)$ has a simple zero at $s=1$ for $\chi=\varphi$ and $\chi=\psi$, hence $A(K_\varrho)^\varrho$ is two-dimensional over $\mathbf{Q}(\varrho)$ and generated by Heegner points by the Kolyvagin–Gross–Zagier–Zhang theorem.

If $\chi=\varphi,\psi$ is quadratic, $\bar{\mathbf{Q}}^{\ker(\chi)}=\mathbf{Q}(\sqrt{cd_1},\sqrt{cd_2})$, where c, d_1 and d_2 are fundamental discriminants such that $d_K=d_1\cdot d_2$. (We consider 1 as a fundamental discriminant). In this case $L(A/K,\chi,s)$ further factors as the product of the Hasse-Weil L-functions $L(A/\mathbf{Q},\chi_1,s)$ and $L(A/\mathbf{Q},\chi_2,s)$ of the twists of A by the quadratic characters χ_i of $\mathbf{Q}(\sqrt{cd_i})$. By Assumptions 1.1.(1) and 1.1.(4), we can order χ_1 and χ_2 in such a way that $\mathrm{sign}(A,\chi_1)=-1$ and $\mathrm{sign}(A,\chi_2)=+1$, where $\mathrm{sign}(A,\chi_i)$ is the sign in the functional equation satisfied by $L(A/\mathbf{Q},\chi_i,s)$.

Assumption 1.2. If $\chi=\varphi$ or $\chi=\psi$ is quadratic, then $\chi_1(p)=\alpha_f$.

Under the assumptions of the Theorem, the results of [2,5] imply that $L_p^{\alpha\alpha}(A,\varrho)$ belongs to $\mathscr{I}^4-\mathscr{I}^5$. The actual contribution of this note is the proof of the identity $L_p^{\alpha\alpha}(A,\varrho) \pmod{\mathscr{I}^5} = R_p^{\alpha\alpha}(A,\varrho)$, which grounds on the results of loc. cit. and an extension of the techniques of [10–12].

2. Proof of the Main Result

2.1. Preliminaries

2.1.1. Galois Representations. To lighten the notation, set $(\boldsymbol{g},\boldsymbol{h})=(\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$. For $\boldsymbol{\xi}=\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}$ let $V(\boldsymbol{\xi})$ be the big Galois representation attached to $\boldsymbol{\xi}$ (cf. Section 5 of [5]). Under the current assumptions, it is a free $\mathscr{O}_{\boldsymbol{\xi}}$-module of rank two, equipped with a continuous $\mathscr{O}_{\boldsymbol{\xi}}$-linear action of $G_{\mathbf{Q}}$. For each u in $U_{\boldsymbol{\xi}}\cap\mathbf{Z}_{\geqslant 2}$, evaluation at u on $U_{\boldsymbol{\xi}}$ induces a natural specialisation isomorphism

$$\rho_u : V(\boldsymbol{\xi})\otimes_u E \simeq V(\boldsymbol{\xi}_u),$$

where $E=\mathbf{Q}_p$ if $\boldsymbol{\xi}=\boldsymbol{f}$ and $E=L$ if $\boldsymbol{\xi}=\boldsymbol{g},\boldsymbol{h}$, where $\cdot\otimes_u E$ denotes the base change along evaluation at u on $\mathscr{O}_{\boldsymbol{\xi}}$, and where $V(\boldsymbol{\xi}_u)$ is the homological p-adic Deligne representation of $\boldsymbol{\xi}_u$ with coefficients in E (cf. Section 2.4 of [5]).

When $\boldsymbol{\xi}=\boldsymbol{f}$ and $u=2$, the representation $V(f)=V(\boldsymbol{f}_2)$ is equal to the f-isotypic component of the cohomology group $H^1_{\text{ét}}(X_1(N_f)_{\bar{\mathbf{Q}}},\mathbf{Q}_p(1))$, where $X_1(N_f)_{\bar{\mathbf{Q}}}$ is the base change to $\bar{\mathbf{Q}}$ of the compact modular curve $X_1(N_f)$ of level $\Gamma_1(N_f)$ defined over $\mathbf{Q}$. Fix a modular parametrisation (viz. a non-constant morphism of $\mathbf{Q}$-curves)

$$\wp_\infty : X_1(N_f)\longrightarrow A,$$

which induces an isomorphism of $\mathbf{Q}_p[G_{\mathbf{Q}}]$-modules between $V(f)$ and the p-adic Tate module $V_p(A)=H^1_{\text{ét}}(A_{\bar{\mathbf{Q}}},\mathbf{Q}_p(1))$ of A with $\mathbf{Q}_p$-coefficients.

When $\boldsymbol{\xi}=\boldsymbol{g},\boldsymbol{h}$ and $u=1$, the $L[G_{\mathbf{Q}}]$-module

$$V(\xi)=V(\boldsymbol{\xi})\otimes_1 L$$

affords the dual of the Deligne–Serre representation of ξ, id est the induced from G_K to $G_{\mathbf{Q}}$ of the character ν_ξ with coefficients in L. (Recall that $\boldsymbol{\xi}_1=\xi_\alpha$. Here we favour the lighter notation $V(\xi)$ for $V(\boldsymbol{\xi})\otimes_1 L$ over the more consistent one $V(\xi_\alpha)$.)

There exists a perfect $G_{\mathbf{Q}}$-equivariant and skew-symmetric pairing

$$\pi_{\boldsymbol{\xi}} : V(\boldsymbol{\xi})\otimes_{\mathscr{O}_{\boldsymbol{\xi}}} V(\boldsymbol{\xi})\longrightarrow \mathscr{O}_{\boldsymbol{\xi}}(\chi_\xi\cdot\chi_{\text{cyc}}^{\boldsymbol{u}-1}),$$

where $\chi_{\text{cyc}}:G_{\mathbf{Q}}\longrightarrow\mathbf{Z}_p^*$ is the p-adic cyclotomic character and $\chi_{\text{cyc}}^{\boldsymbol{u}-1}:G_{\mathbf{Q}}\longrightarrow\mathscr{O}_{\boldsymbol{\xi}}^*$ satisfies $\chi_{\text{cyc}}^{\boldsymbol{u}-1}(\sigma)(u)=\chi_{\text{cyc}}(\sigma)^{u-1}$ for each σ in $G_{\mathbf{Q}}$ and each u in $U_{\boldsymbol{\xi}}\cap\mathbf{Z}$. (With the notations of [5, Section 5], the pairing $\pi_{\boldsymbol{\xi}}$ is the composition of the twist by $\chi_\xi\cdot\chi_{\text{cyc}}^{\boldsymbol{u}-1}$ of the $\mathscr{O}_{\boldsymbol{\xi}}$-adic Poincaré duality $\langle\cdot,\cdot\rangle_{\boldsymbol{f}} : V(\boldsymbol{\xi})\otimes_{\mathscr{O}_{\boldsymbol{\xi}}} V^*(\boldsymbol{\xi})\longrightarrow\mathscr{O}_{\boldsymbol{\xi}}$ defined in [5, Equation (103)] with $\text{id}_{V(\boldsymbol{\xi})}\otimes w_{N_\xi p}^{-1}$, where $w_{N_\xi p}: V^*(\boldsymbol{\xi})(\chi_\xi\cdot\chi_{\text{cyc}}^{\boldsymbol{u}-1})\simeq V(\boldsymbol{\xi})$ is the $\mathscr{O}_{\boldsymbol{\xi}}$-adic Atkin–Lehner isomorphism defined in [5, Equation (114)].) Up to sign, the pairing $\pi_f : V(f)\otimes_{\mathbf{Q}_p} V(f)\longrightarrow\mathbf{Q}_p(1)$ arising from the base change of $\pi_{\boldsymbol{f}}$ along evaluation at $k=2$ on $\mathscr{O}_{\boldsymbol{f}}$ and the specialisation isomorphism ρ_2 is the one induced on the f-isotypic components by the Poincaré duality on $H^1_{\text{ét}}(X_1(N_f)_{\bar{\mathbf{Q}}},\mathbf{Q}_p(1))$. If $\boldsymbol{\xi}=\boldsymbol{g},\boldsymbol{h}$, the weight-one specialisation of $\pi_{\boldsymbol{\xi}}$ yields a perfect skew-symmetric duality

$$\pi_\xi : V(\xi)\otimes_L V(\xi)\longrightarrow L(\chi_\xi).$$

Identify $G_{\mathbf{Q}_p}$ with a subgroup of $G_{\mathbf{Q}}$ via the embedding $i_p:\bar{\mathbf{Q}}\hookrightarrow\bar{\mathbf{Q}}_p$ fixed at the outset, and let $\check{a}_p(\boldsymbol{\xi}):G_{\mathbf{Q}_p}\longrightarrow\mathscr{O}_{\boldsymbol{\xi}}^*$ be the unramified character sending an arithmetic Frobenius to the p-th Fourier coefficient $a_p(\boldsymbol{\xi})$ of $\boldsymbol{\xi}$. In the present setting there is a natural short exact sequence of $\mathscr{O}_{\boldsymbol{\xi}}[G_{\mathbf{Q}_p}]$-modules

$$V(\boldsymbol{\xi})^+ \hookrightarrow V(\boldsymbol{\xi}) \twoheadrightarrow V(\boldsymbol{\xi})^-,$$

where $V(\boldsymbol{\xi})^+$ and $V(\boldsymbol{\xi})^-$ are free $\mathscr{O}_{\boldsymbol{\xi}}$-modules of rank one and $G_{\mathbf{Q}_p}$ acts on them via the characters $\chi_\xi \cdot \chi_{\mathrm{cyc}}^{\boldsymbol{u}-1} \cdot \breve{a}_p(\boldsymbol{\xi})^{-1}$ and $\breve{a}_p(\boldsymbol{\xi})$ respectively (cf. Section 5 of [5]). If $\boldsymbol{\xi} = \boldsymbol{f}$, the specialisation isomorphism $\rho_2 : V(\boldsymbol{f}) \otimes_2 \mathbf{Q}_p \simeq V(f)$ identifies $V(\boldsymbol{f})^- \otimes_2 \mathbf{Q}_p$ with the maximal p-unramified quotient of $V(f)$ and $V(\boldsymbol{\xi})^+ \otimes_2 \mathbf{Q}_p$ with the kernel $V(f)^+$ of the projection $V(f) \longrightarrow V(f)^-$. If $\boldsymbol{\xi} = \boldsymbol{g}, \boldsymbol{h}$ define

$$V(\xi)_\alpha = V(\boldsymbol{\xi})^- \otimes_1 L \quad \text{and} \quad V(\xi)_\beta = V(\boldsymbol{\xi})^+ \otimes_1 L,$$

so that $V(\xi)_\gamma$ (for $\gamma = \alpha, \beta$) is the submodule of $V(\xi)$ on which an arithmetic Frobenius in $G_{\mathbf{Q}_p}$ acts as multiplication by γ_ξ, and (as $L[G_{\mathbf{Q}_p}]$-modules)

$$V(\xi) = V(\xi)_\alpha \oplus V(\xi)_\beta.$$

Define

$$\boldsymbol{V} = V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = V(\boldsymbol{f}) \hat{\otimes}_{\mathbf{Q}_p} V(\boldsymbol{g}) \hat{\otimes}_L V(\boldsymbol{h})(\Xi_{\boldsymbol{fgh}}),$$

where $\Xi_{\boldsymbol{fgh}} = \chi_{\mathrm{cyc}}^{(4-\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m})/2} : G_{\mathbf{Q}} \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}^*$ satisfies $\Xi_{\boldsymbol{fgh}}(\sigma)(w) = \chi_{\mathrm{cyc}}(\sigma)^{\frac{4-k-l-m}{2}}$ for each σ in $G_{\mathbf{Q}}$ and each $w = (k, l, m)$ in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}} \cap \mathbf{Z}^3$, and

$$V = V(f, g, h) = V(f) \otimes_{\mathbf{Q}_p} V(g) \otimes_L V(h).$$

Evaluation at $w_o = (2, 1, 1)$ on $\mathscr{O}_{\boldsymbol{fgh}}$ induces a specialisation isomorphism

$$\rho_{w_o} : \boldsymbol{V} \otimes_{w_o} L \simeq V.$$

The product of the pairing $\pi_{\boldsymbol{\xi}}$ for $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$ yields a perfect, $G_{\mathbf{Q}}$-equivariant and skew-symmetric duality (cf. Assumption 1.1.(2))

$$\pi_{\boldsymbol{fgh}} : \boldsymbol{V} \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \boldsymbol{V} \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}(1),$$

whose base change along evaluation at w_o on $\mathscr{O}_{\boldsymbol{fgh}}$ recasts (via ρ_{w_o}) the perfect duality

$$\pi_{fgh} : V \otimes_L V \longrightarrow L(1)$$

defined by the product of the perfect pairings π_ξ for $\xi = f, g, h$.

For $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$ let $\mathscr{F}^\bullet V(\boldsymbol{\xi})$ be the decreasing filtration on the $\mathscr{O}_{\boldsymbol{fgh}}[G_{\mathbf{Q}_p}]$-module $V(\boldsymbol{\xi})$ defined by $\mathscr{F}^1 V(\boldsymbol{\xi}) = V(\boldsymbol{\xi})^+$, $\mathscr{F}^i V(\boldsymbol{\xi}) = V(\boldsymbol{\xi})$ for each $i \leqslant 0$ and $\mathscr{F}^i V(\boldsymbol{\xi}) = 0$ for each $i \geqslant 2$. Define the *balanced* submodule $\mathscr{F}^2 \boldsymbol{V}$ of $\boldsymbol{V}$ by

$$\mathscr{F}^2 \boldsymbol{V} = \left[\sum_{a+b+c=2} \mathscr{F}^a V(\boldsymbol{f}) \hat{\otimes}_{\mathbf{Q}_p} \mathscr{F}^b V(\boldsymbol{g}) \hat{\otimes}_L \mathscr{F}^c V(\boldsymbol{h}) \right] \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}},$$

and the f-*unbalanced* submodule $\boldsymbol{V}^+$ of $\boldsymbol{V}$ by

$$\boldsymbol{V}^+ = V(\boldsymbol{f})^+ \hat{\otimes}_{\mathbf{Q}_p} V(\boldsymbol{g}) \hat{\otimes}_L V(\boldsymbol{h}) \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}}.$$

These are $G_{\mathbf{Q}_p}$-invariant free $\mathscr{O}_{\boldsymbol{fgh}}$-submodules of $\boldsymbol{V}$ of rank $4 = \frac{1}{2}\mathrm{rank}_{\mathscr{O}_{\boldsymbol{fgh}}}\boldsymbol{V}$, which are maximal isotropic with respect to the skew-symmetric duality $\pi_{\boldsymbol{fgh}}$. After setting

$$\boldsymbol{V}^- = \boldsymbol{V}/\boldsymbol{V}^+ \quad \text{and} \quad \boldsymbol{V}_f = V(\boldsymbol{f})^- \hat{\otimes}_{\mathbf{Q}_p} V(\boldsymbol{g})^+ \hat{\otimes}_L V(\boldsymbol{h})^+ \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}},$$

one has a commutative diagram of $\mathscr{O}_{\boldsymbol{fgh}}[G_{\mathbf{Q}_p}]$-modules

$$\begin{array}{ccc} \mathscr{F}^2\boldsymbol{V} & \xrightarrow{\;i_{\mathscr{F}}\;} & \boldsymbol{V} \\ {\scriptstyle p_f}\big\downarrow & & \big\downarrow{\scriptstyle p^-} \\ \boldsymbol{V}_f & \xrightarrow{\;i_f\;} & \boldsymbol{V}^- \end{array} \tag{2}$$

with $i_{\mathscr{F}}$ and i_f the natural inclusions and p^- the natural projection. Note that $p^- \circ i_{\mathscr{F}}$ and i_f have the same image, hence the morphism p_f is defined by the commutativity of the diagram. One defines the *balanced local subspace* $H^1_{\mathrm{bal}}(\mathbf{Q}_p, \boldsymbol{V})$ of $H^1(\mathbf{Q}_p, \boldsymbol{V})$ to be the image of the morphism induced in cohomology by $i_{\mathscr{F}}$. This morphism is injective (cf. Section 7.2 of [5]), hence gives a natural identification

$$H^1_{\mathrm{bal}}(\mathbf{Q}_p, \boldsymbol{V}) = H^1(\mathbf{Q}_p, \mathscr{F}^2\boldsymbol{V}) \tag{3}$$

Set $V^{\pm} = V(f)^{\pm} \otimes_{\mathbf{Q}_p} V(g) \otimes_L V(h)$. For each pair (i, j) of elements of $\{\alpha, \beta\}$ define $V^{\cdot}_{ij} = V(f)^{\cdot} \otimes_{\mathbf{Q}_p} V(g)_i \otimes_L V(h)_j$, where $\cdot$ is one of symbols $\emptyset, +$ and $-$. Then

$$V^{\cdot} = V^{\cdot}_{\alpha\alpha} \oplus V^{\cdot}_{\alpha\beta} \oplus V^{\cdot}_{\beta\alpha} \oplus V^{\cdot}_{\beta\beta}$$

as $L[G_{\mathbf{Q}_p}]$-modules, and Eq. (1) implies

$$H^0(\mathbf{Q}_p, V^-) = V^-_{\alpha\alpha} \oplus V^-_{\beta\beta} \quad \text{and} \quad H^0(\mathbf{Q}_p, V^+(-1)) = V^+_{\alpha\alpha}(-1) \oplus V^+_{\beta\beta}(-1). \tag{4}$$

The specialisation isomorphism ρ_{w_o} identifies $\boldsymbol{V}^{\pm} \otimes_{w_o} L$, $\mathscr{F}^2\boldsymbol{V} \otimes_{w_o} L$ and $\boldsymbol{V}_f \otimes_{w_o} L$ with $V^{\pm}$, $\mathscr{F}^2 V = V_{\beta\beta} + V^+_{\alpha\beta} + V^+_{\beta\alpha}$ and $V^-_{\beta\beta}$ respectively. In particular the base change of the commutative diagram (2) along evaluation at w_o on $\mathscr{O}_{\boldsymbol{fgh}}$ is equal to

$$\begin{array}{ccc} \mathscr{F}^2 V & \xrightarrow{\;i_{\mathscr{F}}\;} & V \\ {\scriptstyle p_f}\big\downarrow & & \big\downarrow{\scriptstyle p^-} \\ V^-_{\beta\beta} & \xrightarrow{\;i_f\;} & V^- \end{array} \tag{5}$$

with $i_{\mathscr{F}}$ and i_f the natural inclusions and p^- the natural projection.

The Bloch–Kato finite subspace of $H^1(\mathbf{Q}_p, V)$ is equal to the kernel of the map $p^- : H^1(\mathbf{Q}_p, V) \longrightarrow H^1(\mathbf{Q}_p, V^-)$, cf. Section 9.1 of [5]. (With a slight abuse of notation, we denote by the same symbol a morphism of $G_{\mathbf{Q}_p}$-modules and the maps it induces in cohomology.) By construction (cf. Eqs. (2) and (5)), the specialisation $\kappa = \rho_{w_o}(\boldsymbol{\kappa})$ in $H^1(\mathbf{Q}_p, V)$ at w_o of a local balanced class $\boldsymbol{\kappa}$ in $H^1_{\mathrm{bal}}(\mathbf{Q}_p, \boldsymbol{V})$ belongs to the kernel of the map $H^1(\mathbf{Q}_p, V) \longrightarrow H^1(\mathbf{Q}_p, V^-_{ij})$ for $ij = \alpha\alpha, \alpha\beta, \beta\alpha$. Then κ is crystalline precisely if it belongs to the kernel of $H^1(\mathbf{Q}_p, V) \longrightarrow H^1(\mathbf{Q}_p, V^-_{\beta\beta})$, id est if $p_f(\boldsymbol{\kappa})$ in $H^1(\mathbf{Q}_p, \boldsymbol{V}_f)$ (cf. Eq. (3)) belongs to the kernel of the specialisation map $\rho_{w_o} : H^1(\mathbf{Q}_p, \boldsymbol{V}_f) \longrightarrow H^1(\mathbf{Q}_p, V^-_{\beta\beta})$. Since the ideal $\mathscr{I}$ of $\mathscr{O}_{\boldsymbol{fgh}}$ is generated by a regular sequence and $H^2(\mathbf{Q}_p, V_{\beta\beta}^-) = 0$, the specialisation map ρ_{w_o} induces an isomorphism $H^1(\mathbf{Q}_p, \boldsymbol{V}_f) \otimes_{w_o} L \simeq H^1(\mathbf{Q}_p, V_{\beta\beta}^-)$. We have proved the following

Lemma 2.1. *Let $\boldsymbol{\kappa}$ be a local balanced class in $H^1_{\mathrm{bal}}(\mathbf{Q}_p, \boldsymbol{V})$ and set $\kappa = \rho_{w_o}(\boldsymbol{\kappa})$ in $H^1(\mathbf{Q}_p, V)$. Then κ is crystalline if and only if $p_f(\boldsymbol{\kappa})$ belongs to $\mathscr{I} \cdot H^1(\mathbf{Q}_p, \boldsymbol{V}_f)$.*

2.1.2. *p*-Adic Periods. Let $\hat{\mathbf{Q}}_p^{\mathrm{nr}}$ be the p-adic completion of the maximal unramified extension of $\mathbf{Q}_p$, let $c = c(\chi_g)$ be the conductor of χ_g, and for $\xi = g, h$ define

$$G(\chi_\xi) = (-c)^{i_\xi} \cdot \sum_{a \in (\mathbf{Z}/c\mathbf{Z})^*} \chi_\xi(a)^{-1} \otimes e^{2\pi i a/c} \in D_{\mathrm{cris}}(\chi_\xi),$$

where $i_g = 0$, $i_h = -1$ and $D_{\mathrm{cris}}(\chi_\xi)$ is a shorthand for $H^0(\mathbf{Q}_p, L(\chi_\xi) \otimes_{\mathbf{Q}_p} \hat{\mathbf{Q}}_p^{\mathrm{nr}})$.

As explained in Section 3.1 of [4], for $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$ the module $D(\boldsymbol{\xi})^-$ of $G_{\mathbf{Q}_p}$-invariants of $V(\boldsymbol{\xi})^- \hat{\otimes}_{\mathbf{Q}_p} \hat{\mathbf{Q}}_p^{\mathrm{nr}}$ is free of rank one over $\mathscr{O}_{\boldsymbol{\xi}}$, and its base change

$$D(\boldsymbol{\xi})_u^- = D(\boldsymbol{\xi})^- \otimes_u L$$

along evaluation at a classical weight u in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}_{\geqslant 2}$ on $\mathscr{O}_{\boldsymbol{\xi}}$ is canonically isomorphic to the $\boldsymbol{\xi}_u$-isotypic component $L \cdot \boldsymbol{\xi}_u$ of $S_u(pN_\xi, \chi_\xi)_L$. Moreover there exists an $\mathscr{O}_{\boldsymbol{\xi}}$-basis

$$\omega_{\boldsymbol{\xi}} \in D(\boldsymbol{\xi})^-$$

whose image $\omega_{\boldsymbol{\xi}_u}$ in $D(\boldsymbol{\xi})_u^-$ corresponds to $\boldsymbol{\xi}_u$ under the aforementioned isomorphism for each u in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}_{\geqslant 2}$. (We refer to loc. cit. and the references therein for the details.) The weight-two specialisation of $\omega_{\boldsymbol{f}}$ equals the de Rham class

$$\omega_f \in D_{\mathrm{cris}}(V(f)^-) \simeq \mathrm{Fil}^0 D_{\mathrm{dR}}(V(f))$$

associated with f under the Faltings–Tsuji comparison isomorphism between the étale and de Rham cohomology of $X_1(N_f)_{\mathbf{Q}_p}$. (The isomorphism in the previous equation arises from the projection $V(f) \longrightarrow V(f)^-$.) Denote by

$$\langle \cdot, \cdot \rangle_f : D_{\mathrm{dR}}(V(f)) \otimes_L D_{\mathrm{dR}}(V(f)) \longrightarrow L$$

the perfect duality induced by π_f, and define η_f in $D_{\mathrm{dR}}(V(f))/\mathrm{Fil}^0$ by the identity

$$\langle \eta_f, \omega_f \rangle_f = 1.$$

For $\boldsymbol{\xi} = \boldsymbol{g}, \boldsymbol{h}$, the weight-one specialisation of $\omega_{\boldsymbol{\xi}}$ yields a class

$$\omega_{\xi_\alpha} \in D_{\mathrm{cris}}(V(\xi)_\alpha) = D_{\mathrm{cris}}(V(\xi))^{\varphi = \alpha_\xi^{-1}}$$

(with φ the crystalline Frobenius). The pairing $\pi_\xi = \pi_{\boldsymbol{\xi}} \otimes_1 L$ induces a perfect duality

$$\langle \cdot, \cdot \rangle_\xi : D_{\mathrm{cris}}(V(\xi)) \otimes_L D_{\mathrm{cris}}(V(\xi)) \longrightarrow D_{\mathrm{cris}}(\chi_\xi)$$

and one defines η_{ξ_α} in $D_{\mathrm{cris}}(V(\xi)_\beta) = D_{\mathrm{cris}}(V(\xi))^{\varphi = \beta_\xi^{-1}}$ by the identity

$$\langle \eta_{\xi_\alpha}, \omega_{\xi_\alpha} \rangle_\xi = G(\chi_\xi).$$

Along with ω_f, it is important to consider another p-adic period

$$q(f) \in D_{\mathrm{cris}}(V(f)^-) = \mathrm{Fil}^0 D_{\mathrm{dR}}(V(f))$$

arising from the Tate uniformisation of $A_{\mathbf{Q}_p}$, cf. Section 2 of [3]. Let K_p be the completion of K at p (namely the quadratic unramified extension of $\mathbf{Q}_p$). Tate's theory gives a rigid analytic uniformisation $\wp_{\mathrm{Tate}} : \mathbf{G}^{\mathrm{rig}}_{m,K_p} \longrightarrow A_{K_p}$, unique up to sign, with kernel the lattice generated by the Tate period q_A in $p\mathbf{Z}_p$ of $A_{\mathbf{Q}_p}$. One sets

$$q(A) = p^{-}\big(\wp_{\mathrm{Tate}}(\sqrt[p^\infty]{q_A})\big) \in V_p(A)^{-} \quad \text{and} \quad q(f) = \sqrt{m_p}\cdot \wp_\infty^{-1}(q(A)), \tag{6}$$

where $\sqrt[p^\infty]{q_A}$ is any compatible system of p^nth roots of q_A, $\wp_\infty : V(f)^{-} \simeq V_p(A)^{-}$ is the isomorphism arising from the fixed modular parametrisation $\wp_\infty$, $m_p = 1$ if $\alpha_f = 1$ and $m_p = d_K$ if $\alpha_f = -1$. As in loc. cit., define the generators

$$q_{\alpha\alpha} = q(f) \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha} \quad \text{and} \quad q_{\beta\beta} = q(f) \otimes \eta_{g_\alpha} \otimes \eta_{h_\alpha}$$

of the subspaces $V^{-}_{\alpha\alpha}$ and $V^{-}_{\beta\beta}$ respectively of $H^0(\mathbf{Q}_p, V^{-}) = D_{\mathrm{cris}}(V^{-})^{\varphi=1}$.

2.1.3. The Garrett–Nekovář *p*-Adic Height Pairing. Section 2 of [4] constructs a canonical skew-symmetric p-adic height pairing

$$\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{fgh}} : \tilde{H}^1_f(\mathbf{Q}, V) \otimes_L \tilde{H}^1_f(\mathbf{Q}, V) \longrightarrow \mathscr{I}/\mathscr{I}^2$$

on the extended Selmer group $\tilde{H}^1_f(\mathbf{Q}, V)$ associated with the Greenberg local condition at p arising from the inclusion $i^{+} : V^{+} \hookrightarrow V$. Let $\mathrm{Sel}(\mathbf{Q}, V)$ denote the Bloch–Kato Selmer group of V, which is equal to the kernel of $H^1(\mathbf{Q}, V) \longrightarrow H^1(\mathbf{Q}_p, V^{-})$ in the present setting (cf. [5, Section 9.1]). One has a commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathbf{Q}_p, V^{-}) & \xrightarrow{\jmath} & \tilde{H}^1_f(\mathbf{Q}, V) & \xrightarrow{\pi} & \mathrm{Sel}(\mathbf{Q}, V) & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \cdot^{+}} & & \downarrow{\scriptstyle \mathrm{res}_p} & & \\ & & & & H^1(\mathbf{Q}_p, V^{+}) & \xrightarrow{i^{+}} & H^1(\mathbf{Q}_p, V) & & \end{array} \tag{7}$$

and there exists a unique section $\imath_{\mathrm{ur}} : \mathrm{Sel}(\mathbf{Q}, V) \hookrightarrow \tilde{H}^1_f(\mathbf{Q}, V)$ of π such that the composition $\imath_{\mathrm{ur}}(\cdot)^{+}$ takes values in the finite subspace $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V^{+})$ of $H^1(\mathbf{Q}_p, V^{+})$ (cf. Section 2.3 of [4]). As in loc. cit. we use the maps $\jmath$ and $\imath_{\mathrm{ur}}$ to identify Nekovář's extended Selmer group $\tilde{H}^1_f(\mathbf{Q}, V)$ with the *naive* extended Selmer group

$$\mathrm{Sel}^{\dagger}(\mathbf{Q}, V) = H^0(\mathbf{Q}_p, V^{-}) \oplus \mathrm{Sel}(\mathbf{Q}, V).$$

Enlarging L if necessary, for $\xi = g, h$ fix an isomorphism of $L[G_{\mathbf{Q}}]$-modules

$$\gamma_\xi : V_\xi \otimes_{\mathbf{Q}(\varrho)} L \simeq V(\xi) \ \text{ such that } \ \pi_\xi(\gamma_\xi(x) \otimes \gamma_\xi(y)) \in \mathbf{Q}(\varrho)(\chi_\xi) \tag{8}$$

for each x and y in V_ξ (cf. Eq. (4) of [4]). Set (cf. Eq. (6))

$$\mathcal{Q}_p(A, \varrho) = H^0(\mathbf{Q}_p, \mathbf{Q}(\varrho) \cdot q(A) \otimes_{\mathbf{Q}(\varrho)} V_{gh}). \tag{9}$$

The modular parametrisation $\wp_\infty : X_1(N_f) \longrightarrow A$ fixed in Sect. 2.1.1, the global Kummer map on $A(K_\varrho)\hat{\otimes}\mathbf{Q}_p$ and the isomorphisms γ_g and γ_h induce an embedding

$$\gamma_{gh} : A^{\dagger}(K_\varrho)^{\varrho} \hookrightarrow \mathrm{Sel}^{\dagger}(\mathbf{Q}, V) = \tilde{H}^1_f(\mathbf{Q}, V), \tag{10}$$

and one defines the Garrett–Nekovář p-adic pairing (cf. Sect. 1)

$$\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{fgh}} : A^{\dagger}(K_\varrho)^{\varrho} \otimes_{\mathbf{Q}(\varrho)} A^{\dagger}(K_\varrho)^{\varrho} \longrightarrow \mathscr{I}/\mathscr{I}^2$$

to be the restriction of the canonical height $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{fgh}}$ on $\tilde{H}^1_f(\mathbf{Q}, V)$ along γ_{gh}. Note that the discriminant $R_p^{\alpha\alpha}(A, \varrho)$ of $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{fgh}}$ on $A^\dagger(K_\varrho)^\varrho$ (cf. Sect. 1) is independent of the choice of the modular parametrisation $\wp_\infty$ and the isomorphisms γ_g and γ_h.

2.1.4. Logarithms. Let $V_{\mathrm{dR}} = D_{\mathrm{dR}}(V)$ be the de Rham module of $V = V(f, g, h)$. The duality $\pi_{fgh} : V \otimes_L V \longrightarrow L(1)$ induces a perfect pairing

$$\langle \cdot, \cdot \rangle_{fgh} : V_{\mathrm{dR}} \otimes_L V_{\mathrm{dR}} \longrightarrow L.$$

After identifying V_{dR} with $D_{\mathrm{dR}}(V(f)) \otimes_{\mathbf{Q}_p} D_{\mathrm{cris}}(V(g)) \otimes_L D_{\mathrm{cris}}(V(h))$ and L with $D_{\mathrm{cris}}(\chi_g) \otimes_L D_{\mathrm{cris}}(\chi_h)$ under the natural isomorphisms (cf. Assumption 1.1.(2)), the pairing $\langle \cdot, \cdot \rangle_{fgh}$ agrees with the product of the pairings $\langle \cdot, \cdot \rangle_\xi$ for $\xi = f, g, h$.

The Bloch–Kato exponential map $\exp_p$ gives an isomorphism between the tangent space $V_{\mathrm{dR}}/\mathrm{Fil}^0$ of V and the finite (viz. crystalline) subspace $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V)$ of $H^1(\mathbf{Q}_p, V)$. Denote by $\log_p$ the inverse of $\exp_p$ and define the $\alpha\alpha$-logarithm

$$\log_{\alpha\alpha} = \langle \log_p, \omega_f \otimes \eta_{g_\alpha} \otimes \eta_{h_\alpha} \rangle_{fgh} : H^1_{\mathrm{fin}}(\mathbf{Q}_p, V) \longrightarrow L$$

to be the composition of $\log_p$ with evaluation at $\omega_f \otimes \eta_{g_\alpha} \otimes \eta_{h_\alpha}$ in $\mathrm{Fil}^0 V_{\mathrm{dR}}$ under the perfect duality $\langle \cdot, \cdot \rangle_{fgh}$. Similarly define the $\beta\beta$-logarithm

$$\log_{\beta\beta} = \langle \log_p, \omega_f \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha} \rangle : H^1_{\mathrm{fin}}(\mathbf{Q}_p, V) \longrightarrow L.$$

(Note that $\log_{ii}$ factors through the projection $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V) \longrightarrow H^1(\mathbf{Q}_p, V_{ii})$.)

Set $\mathrm{tg}_{\mathrm{dR}, K_p}(f) = H^0(K_p, V(f) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})/\mathrm{Fil}^0$ and consider the composition

$$\log_{A,p} : A(K_p) \hat{\otimes} \mathbf{Q}_p \simeq H^1_{\mathrm{fin}}(K_p, V_p(A)) \simeq H^1_{\mathrm{fin}}(K_p, V(f)) \simeq \mathrm{tg}_{\mathrm{dR}, K_p}(f),$$

where the first isomorphism is the local Kummer map, the second is induced by the fixed modular parametrisation $\wp_\infty : X_1(N_f) \longrightarrow A$ (cf. Sect. 2.1.1), and the third is the inverse of the Bloch–Kato exponential map. For $\chi = \varphi, \psi$ (cf. Sect. 1) define

$$\log_{\omega_f} = \langle \log_{A,p}, \omega_f \rangle_f : A(K_\chi) \longrightarrow K_p,$$

where K_χ is the ring class field of K cut-out by χ and $A(K_\chi)$ is viewed as a subgroup of $A(K_p)$ via the embedding $i_p : \bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$ fixed at the outset. (Recall that p is inert in K and that χ is dihedral, hence $p\mathcal{O}_K$ splits completely in K_χ.)

2.2. Big Logarithms and Diagonal Classes

Let

$$\mathscr{L}_{\boldsymbol{f}} : H^1(\mathbf{Q}_p, \boldsymbol{V}_f) \longrightarrow \mathscr{I}$$

be the big logarithm map constructed in Proposition 7.3 of [5] using the work of Coleman, Perrin-Riou et alii. (Note that the tame character $\chi_{\boldsymbol{f}}$ of $\boldsymbol{f}$ is trivial in the present setting and that the logarithm $\mathscr{L}_{\boldsymbol{f}}$ takes values in $\mathscr{I}$ by the exceptional zero condition $\alpha_f = \alpha_g \cdot \alpha_h$.) With a slight abuse of notation denote by

$$\mathscr{L}_{\boldsymbol{f}} : H^1_{\mathrm{bal}}(\mathbf{Q}_p, \boldsymbol{V}) \longrightarrow \mathscr{I}$$

also the composition $\mathscr{L}_{\boldsymbol{f}} \circ p_f$ (cf. Eq. (3)).

Let $H^1_{\rm bal}(\mathbf{Q},\boldsymbol{V})$ be the group of global classes in $H^1(\mathbf{Q},\boldsymbol{V})$ whose restriction at p belongs to the balanced local condition $H^1_{\rm bal}(\mathbf{Q}_p,\boldsymbol{V})$. According to Theorem A of [5] (cf. [2, Section 2.1]) there exists a canonical *big diagonal class*

$$\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})=\kappa(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)\in H^1_{\rm bal}(\mathbf{Q},\boldsymbol{V})$$

such that

$$\mathscr{L}_{\boldsymbol{f}}(\mathrm{res}_p(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})))=\mathscr{L}_p^{\alpha\alpha}(A,\varrho). \tag{11}$$

Define the *diagonal class*

$$\kappa(f,g_\alpha,h_\alpha)=\rho_{w_o}(\kappa(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha))$$

to be the image in $H^1(\mathbf{Q},V)$ of $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ under the map induced in cohomology by the specialisation isomorphism $\rho_{w_o}:\boldsymbol{V}\otimes_{w_o}L\simeq V$. Since by assumption the complex Garrett L-function $L(A,\varrho,s)=L(f\otimes g\otimes h,s)$ vanishes at $s=1$, Theorem B of [5] implies that $\kappa(f,g_\alpha,h_\alpha)$ is crystalline at p, hence a Selmer class:

$$\kappa(f,g_\alpha,h_\alpha)\in \mathrm{Sel}(\mathbf{Q},V). \tag{12}$$

Identify $\mathscr{O}_{\boldsymbol{fgh}}$ with a subring of the power series ring $L[\![\boldsymbol{k}-2,\boldsymbol{l}-1,\boldsymbol{m}-1]\!]$, where $\boldsymbol{k}-2$ in $\mathscr{O}_{\boldsymbol{f}}$ is a uniformiser at the centre 2 of $U_{\boldsymbol{f}}$, and $\boldsymbol{l}-1$ and $\boldsymbol{m}-1$ are defined similarly. In light of Eq. (12) and Lemma 2.1 there exist local classes $\mathfrak{Y}_{\boldsymbol{k}},\mathfrak{Y}_{\boldsymbol{l}}$ and $\mathfrak{Y}_{\boldsymbol{m}}$ in $H^1(\mathbf{Q}_p,\boldsymbol{V}_f)$ satisfying the identity

$$p_f(\mathrm{res}_p(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})))=\sum_{\boldsymbol{u}}\mathfrak{Y}_{\boldsymbol{u}}\cdot(\boldsymbol{u}-u_o). \tag{13}$$

Equation (11) gives

$$\mathscr{L}_p^{\alpha\alpha}(A,\varrho)=\sum_{\boldsymbol{u}}\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Y}_{\boldsymbol{u}})\cdot(\boldsymbol{u}-u_o)\in\mathscr{I}^2. \tag{14}$$

The following key lemma, proved in Part 1 of Proposition 9.3 of [5], gives an explicit description of the linear term of $\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Y}_{\boldsymbol{u}})$ at w_o. Identify the p-adic completion of the Galois group of the maximal abelian extension of $\mathbf{Q}_p$ with that of $\mathbf{Q}_p^*$ via the local Artin map, normalised in such a way that p^{-1} corresponds to the arithmetic Frobenius. This identifies $H^1(\mathbf{Q}_p,\mathbf{Q}_p)$ with $\mathrm{Hom}_{\rm cont}(\mathbf{Q}_p^*,\mathbf{Q}_p)$, hence (recalling that $G_{\mathbf{Q}_p}$ acts trivially on $V^-_{\beta\beta}$, cf. Eq. (4))

$$H^1(\mathbf{Q}_p,V^-_{\beta\beta})=\mathrm{Hom}_{\rm cont}(\mathbf{Q}_p^*,\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}V^-_{\beta\beta}, \tag{15}$$

and the Bloch–Kato dual exponential $\exp_p^*$ on $H^1(\mathbf{Q}_p,V^-_{\beta\beta})$ satisfies

$$\exp_p^*(\varphi\otimes v)=\varphi(e(1))\cdot v$$

in $D_{\rm cris}(V^-_{\beta\beta})=V^-_{\beta\beta}$ for each φ in $\mathrm{Hom}_{\rm cont}(\mathbf{Q}_p^*,\mathbf{Q}_p)$ and v in $V^-_{\beta\beta}$, where

$$e(1)=(1+p)\hat{\otimes}\log_p(1+p)^{-1}\in\mathbf{Z}_p^*\hat{\otimes}\mathbf{Q}_p.$$

For $x=\varphi\otimes v$ in $H^1(\mathbf{Q}_p,V^-_{\beta\beta})$ (with φ and v as above) and q in $\mathbf{Q}_p^*\hat{\otimes}\mathbf{Q}_p$, set

$$x(q)=\varphi(q)\cdot v \quad\text{and}\quad x(q)_f=\langle x(q),\eta_f\otimes\omega_{g_\alpha}\otimes\omega_{h_\alpha}\rangle_{fgh}.$$

If $(\boldsymbol{\xi},\boldsymbol{u})$ denotes one of the pairs $(\boldsymbol{f},\boldsymbol{k}),(\boldsymbol{g},\boldsymbol{l})$ and $(\boldsymbol{h},\boldsymbol{m})$, define

$$\tilde{D}_{\boldsymbol{u}}:H^1(\mathbf{Q}_p,\boldsymbol{V}_f)\longrightarrow L$$

to be the linear map which on $\mathfrak{Y}$ in $H^1(\mathbf{Q}_p, \boldsymbol{V}_f)$ takes the value

$$\tilde{D}_{\boldsymbol{u}}(\mathfrak{Y}) = \frac{(-1)^{u_o}}{2\,(1-p^{-1})} \cdot \big(\mathfrak{y}(p^{-1})_f - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{\xi}} \cdot \mathfrak{y}(e(1))_f\big). \tag{16}$$

Here $\mathfrak{y} = \rho_{w_o}(\mathfrak{Y})$ in $H^1(\mathbf{Q}_p, V_{\beta\beta}^-)$ is the w_o-specialisation of $\mathfrak{Y}$, $u_o = 2$ if $\boldsymbol{u} = \boldsymbol{k}$ and $u_o = 1$ if $\boldsymbol{u} = \boldsymbol{l}, \boldsymbol{m}$, and $\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{\xi}}$ in L is the *analytic* $\mathscr{L}$*-invariant of* $\boldsymbol{\xi}$, defined by

$$\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{\xi}} = -2 \cdot d\log a_p(\boldsymbol{\xi})(u_o)$$

(where $d\log a = a'/a$ for a in $\mathscr{O}^*_{\boldsymbol{\xi}}$). We can finally state the aforementioned key lemma.

Lemma 2.2. *For each local class $\mathfrak{Y}$ in $H^1(\mathbf{Q}_p, \boldsymbol{V}_f)$ one has*

$$\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Y}) \pmod{\mathscr{I}^2} = \sum_{\boldsymbol{u}} \tilde{D}_{\boldsymbol{u}}(\mathfrak{Y}) \cdot (\boldsymbol{u} - u_o).$$

For each pair $(\boldsymbol{u}, \boldsymbol{v})$ of distinct elements of $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}\}$, define (cf. Eq. (13))

$$\tilde{D}_{\boldsymbol{u},\boldsymbol{u}}(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) = \tilde{D}_{\boldsymbol{u}}(\mathfrak{Y}_{\boldsymbol{u}}) \quad \text{and} \quad \tilde{D}_{\boldsymbol{u},\boldsymbol{v}}(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) = \tilde{D}_{\boldsymbol{u}}(\mathfrak{Y}_{\boldsymbol{v}}) + \tilde{D}_{\boldsymbol{v}}(\mathfrak{Y}_{\boldsymbol{u}}).$$

Equation (14) and Lemma 2.2 give the following lemma (which implies that the *derivatives* $\tilde{D}_{\cdot}(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$ are independent of the choice of the classes $\mathfrak{Y}_{\boldsymbol{u}}$ satisfying (13)).

Lemma 2.3. *One has the following equality in $\mathscr{I}^2/\mathscr{I}^3$.*

$$\mathscr{L}_p^{\alpha\alpha}(A, \varrho) \pmod{\mathscr{I}^3} = \sum_{\boldsymbol{u},\boldsymbol{v}} \tilde{D}_{\boldsymbol{u},\boldsymbol{v}}(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) \cdot (\boldsymbol{u} - u_o)(\boldsymbol{v} - v_o)$$

2.3. An Exceptional Zero Formula *à la* Rubin–Perrin–Riou

For a positive integer n and each $2n$-tuple $\boldsymbol{y} = (y_1, \ldots, y_{2n})$ of elements of $\tilde{H}^1_f(\mathbf{Q}, V)$ denote by

$$\mathscr{R}_p^{\alpha\alpha}(\boldsymbol{y}) = \mathrm{Pf}\big(\big(\langle\!\langle y_i, y_j\rangle\!\rangle_{\boldsymbol{fgh}}\big)_{1 \leqslant i,j \leqslant 2n}\big) \in \mathscr{I}^n/\mathscr{I}^{n+1}$$

the Pfaffian of the skew-symmetric $2n \times 2n$ matrix whose ij-entry is $\langle\!\langle y_i, y_j\rangle\!\rangle_{\boldsymbol{fgh}}$, and define the *extended Garrett*–Nekovář *p-adic height pairing*

$$\tilde{h}_p^{\alpha\alpha} : \mathrm{Sel}(\mathbf{Q}, V) \otimes_L \mathrm{Sel}(\mathbf{Q}, V) \longrightarrow \mathscr{I}^2/\mathscr{I}^3$$

to be the bilinear form which on $y \otimes y'$ in $\mathrm{Sel}(\mathbf{Q}, V)^{\otimes 2}$ takes the value

$$\tilde{h}_p^{\alpha\alpha}(y \otimes y') = \mathscr{R}_p^{\alpha\alpha}(q_{\alpha\alpha}, q_{\beta\beta}, y, y').$$

The aim of this section is to prove the following proposition.

Proposition 2.4. *Up to sign, one has the equality*

$$\tilde{h}_p^{\alpha\alpha}(\kappa(f, g_\alpha, h_\alpha) \otimes \cdot) = c_A \cdot \log_{\alpha\alpha}(\mathrm{res}_p(\cdot)) \cdot \mathscr{L}_p^{\alpha\alpha}(A, \varrho) \pmod{\mathscr{I}^3}$$

of $\mathscr{I}^2/\mathscr{I}^3$-valued L-linear forms on $\mathrm{Sel}(\mathbf{Q}, V)$*, where* $c_A = \frac{m_p \cdot (1-p^{-1}) \cdot \mathrm{ord}_p(q_A)}{\deg(\wp_\infty)}$.

We divide the proof of Proposition 2.4 in a series of lemmas. Define

$$c_p(f) = \langle q(f), \eta_f \rangle_f$$

in L^* (cf. Sect. 2.1.2). As in Sect. 2.2, identify $H^1(\mathbf{Q}_p, \mathbf{Q}_p)$ with $\mathrm{Hom}_{\mathrm{cont}}(\mathbf{Q}_p^*, \mathbf{Q}_p)$ via the local Artin map (sending p^{-1} to an arithmetic Frobenius), and set

$$\log_{\boldsymbol{\xi}} = \log_p - \mathfrak{L}_{\boldsymbol{\xi}}^{\mathrm{an}} \cdot \mathrm{ord}_p \in H^1(\mathbf{Q}_p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} L,$$

where $\log_p : \mathbf{Q}_p^* \longrightarrow \mathbf{Q}_p$ is the (branch of the) p-adic logarithm (vanishing at p) and $\mathrm{ord}_p : \mathbf{Q}_p^* \longrightarrow \mathbf{Z}$ is the p-adic valuation normalised by $\mathrm{ord}_p(p) = 1$.

Lemma 2.5. *For each Selmer class y in $\mathrm{Sel}(\mathbf{Q}, V)$ one has*

$$-2 \cdot \langle\!\langle q_{\beta\beta}, y \rangle\!\rangle_{\boldsymbol{fgh}} = c_p(f) \cdot \log_{\alpha\alpha}(\mathrm{res}_p(y)) \cdot (\boldsymbol{k} - \boldsymbol{l} - \boldsymbol{m})$$

and

$$-\frac{2 \cdot \deg(\wp_\infty)}{m_p \cdot \mathrm{ord}_p(q_A)} \cdot \langle\!\langle q_{\beta\beta}, q_{\alpha\alpha} \rangle\!\rangle_{\boldsymbol{fgh}} = (\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} - \mathfrak{L}_{\boldsymbol{g}}^{\mathrm{an}}) \cdot (\boldsymbol{l} - 1) + (\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} - \mathfrak{L}_{\boldsymbol{h}}^{\mathrm{an}}) \cdot (\boldsymbol{m} - 1)$$

Proof. See Equations (17) and (27) of [3]. (Note that the p-adic logarithm denoted by $\log_{\alpha\alpha}$ in [3] is equal to $\langle \log_p, q_{\beta\beta} \rangle_{fgh} = -c_p(f) \cdot \log_{\alpha\alpha}$.) □

Let $\mathrm{C}^\bullet_{\mathrm{cont}}(\mathbf{Q}_p, \boldsymbol{V}^-)$ be the complex of (inhomogeneous) continuous cochains of $G_{\mathbf{Q}_p}$ with values in the quotient $p^- : \boldsymbol{V} \longrightarrow \boldsymbol{V}^-$ of $\boldsymbol{V}$ (cf. Sect. 2.1.1), and let

$$\langle \cdot, \cdot \rangle_{\mathrm{Tate}} : H^1(\mathbf{Q}_p, V^-) \otimes_L H^1(\mathbf{Q}_p, V^+) \longrightarrow L$$

the local Tate pairing arising from the perfect duality $\pi_{fgh} : V \otimes_L V \longrightarrow L(1)$. Recall the morphism $\cdot^+ : \tilde{H}^1_f(\mathbf{Q}, V) \longrightarrow H^1(\mathbf{Q}_p, V^+)$ introduced in Diagram (7).

Lemma 2.6. *There exist 1-cochains $X_{\boldsymbol{k}}, X_{\boldsymbol{l}}$ and $X_{\boldsymbol{m}}$ in $\mathrm{C}^1_{\mathrm{cont}}(\mathbf{Q}_p, \boldsymbol{V}^-)$ such that*

$$p^-(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))) = \mathrm{cl}\Big(\sum_{\boldsymbol{u}} X_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o)\Big), \tag{17}$$

id est $\sum_{\boldsymbol{u}} X_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o)$ is a 1-cocycle representing $p^-(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})))$, and

$$\langle\!\langle \kappa(f, g_\alpha, h_\alpha), y \rangle\!\rangle_{\boldsymbol{fgh}} = \sum_{\boldsymbol{u}} \langle \mathfrak{x}_{\boldsymbol{u}}, y^+ \rangle_{\mathrm{Tate}} \cdot (\boldsymbol{u} - u_o)$$

for each extended Selmer class y in $\tilde{H}^1_f(\mathbf{Q}, V)$, where

$$\mathfrak{x}_{\boldsymbol{u}} = \mathrm{cl}(\rho_{w_o}(X_{\boldsymbol{u}}))$$

is the local class in $H^1(\mathbf{Q}_p, V^-)$ represented by the 1-cocycle $\rho_{w_o}(X_{\boldsymbol{u}})$.

Proof. This follows from Equations (30)–(37) in Section 3.4 of [4]. (The paragraphs containing the aforementioned equations do not use the non-exceptionality assumption [4, Equation (26)] imposed in [4, Section 3].) □

Fix in what follows 1-cochains $X_{\boldsymbol{k}}, X_{\boldsymbol{l}}$ and $X_{\boldsymbol{m}}$ satisfying the conclusions of Lemma 2.6. For $i = \alpha\alpha, \beta\beta$ let $\mathrm{pr}_i : H^1(\mathbf{Q}_p, V^-) \longrightarrow H^1(\mathbf{Q}_p, V_i^-)$ be the natural projection.

Lemma 2.7. *For $\boldsymbol{u}$ equal to one of $\boldsymbol{k}$, $\boldsymbol{l}$ and $\boldsymbol{m}$, one has*

$$\mathrm{pr}_{\alpha\alpha}(\mathfrak{x}_{\boldsymbol{u}}) = \mu_{\boldsymbol{u}} \cdot \log_{\boldsymbol{f}} \otimes q_{\alpha\alpha}$$

in $H^1(\mathbf{Q}_p, V^-_{\alpha\alpha}) = H^1(\mathbf{Q}_p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} V^-_{\alpha\alpha}$ for some $\mu_{\boldsymbol{u}}$ in L.

Proof. Set $\kappa_{\alpha\alpha} = \kappa(f, g_\alpha, h_\alpha)$. As explained in Section 3.3 of [3] (cf. Equation (15) of loc. cit.) one has (cf. Diagram (7))

$$q^+_{\beta\beta} = \frac{m_p}{\deg(\wp_\infty)} \cdot (q_A \hat{\otimes} 1) \otimes q^*_{\alpha\alpha}$$

in the direct summand

$$H^1(\mathbf{Q}_p, V^+_{\beta\beta}) = H^1(\mathbf{Q}_p, \mathbf{Q}_p(1)) \otimes_{\mathbf{Q}_p} V^+_{\beta\beta}(-1)$$

of $H^1(\mathbf{Q}_p, V^+)$, where $q^*_{\alpha\alpha}$ in $V^+_{\beta\beta}(-1)$ is the dual basis of $q_{\alpha\alpha}$ under the pairing $\pi_{\boldsymbol{fgh}}(-1)$. It then follows from Lemma 2.6 and local class field theory that

$$\langle\!\langle \kappa_{\alpha\alpha}, q_{\beta\beta} \rangle\!\rangle_{\boldsymbol{fgh}} = \sum_{\boldsymbol{u}} \mathfrak{x}^{\alpha\alpha}_{\boldsymbol{u}}(q_A) \cdot (\boldsymbol{u} - u_o),$$

where the class $\mathfrak{x}^{\alpha\alpha}_{\boldsymbol{u}}$ in $H^1(\mathbf{Q}_p, \mathbf{Q}_p)$ is defined by the identity

$$\mathrm{pr}_{\alpha\alpha}(\mathfrak{x}_{\boldsymbol{u}}) = \mathfrak{x}^{\alpha\alpha}_{\boldsymbol{u}} \otimes q_{\alpha\alpha}.$$

On the other hand, since $\log_{\alpha\alpha}(\mathrm{res}_p(\kappa_{\alpha\alpha})) = 0$ (because $\kappa_{\alpha\alpha}$ is a balanced class, cf. Section 6.1 of [5]), Lemma 2.5 and the skew-symmetry of $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{fgh}}$ yield

$$\langle\!\langle \kappa_{\alpha\alpha}, q_{\beta\beta} \rangle\!\rangle_{\boldsymbol{fgh}} = -\langle\!\langle q_{\beta\beta}, \kappa_{\alpha\alpha} \rangle\!\rangle_{\boldsymbol{fgh}} = 0,$$

hence $\mathfrak{x}^{\alpha\alpha}_{\boldsymbol{u}}(q_A) = 0$, id est $\mathfrak{x}^{\alpha\alpha}_{\boldsymbol{u}}$ is a multiple of $\log_{q_A}$. The lemma follows from this and Theorem 3.18 of [6], according to which $\log_{q_A}$ equals $\log_{\boldsymbol{f}}$. □

Lemma 2.8. *Assume that either $\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \neq \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}}$ or $\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \neq \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}}$. Then the local classes $\mathfrak{x}_{\boldsymbol{k}}$, $\mathfrak{x}_{\boldsymbol{l}}$ and $\mathfrak{x}_{\boldsymbol{m}}$ belong to the direct summand $H^1(\mathbf{Q}_p, V^-_{\beta\beta})$ of $H^1(\mathbf{Q}_p, V^-)$.*

Proof. The proof uses the main properties of the Bockstein map

$$\beta^-_{\boldsymbol{fgh}} : H^0(\mathbf{Q}_p, V^-) \longrightarrow H^1(\mathbf{Q}_p, V^-) \otimes_L \mathscr{I}/\mathscr{I}^2$$

introduced in [3, Section 3.1.1]. As $\kappa(f, g_\alpha, h_\alpha) = \rho_{w_o}(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$ is crystalline at p, Lemma 2.1 shows that there exist $\mathfrak{Z}_{\boldsymbol{k}}$, $\mathfrak{Z}_{\boldsymbol{l}}$ and $\mathfrak{Z}_{\boldsymbol{m}}$ in $H^1(\mathbf{Q}_p, \boldsymbol{V}_f)$ such that

$$p_f(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))) = \sum_{\boldsymbol{u}} \mathfrak{Z}_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o). \tag{18}$$

Recall the specialisation isomorphism $\rho_{w_o} : \boldsymbol{V}_f \otimes_{w_o} L \simeq V^-_{\beta\beta}$ arising from evaluation at w_o on $\mathscr{O}_{\boldsymbol{fgh}}$ (cf. Sect. 2.1.1), set $\mathfrak{z}_{\boldsymbol{u}} = \rho_{w_o}(\mathfrak{Z}_{\boldsymbol{u}})$ in $H^1(\mathbf{Q}_p, V^-_{\beta\beta})$ and define

$$\nabla_f = \sum_{\boldsymbol{u}} \mathfrak{z}_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o)$$

in $H^1(\mathbf{Q}_p, V^-_{\beta\beta}) \otimes \mathscr{I}/\mathscr{I}^2$. It follows from Eqs. (17) and (18) and Lemma 3.2 of [3] that the difference $\sum_{\boldsymbol{u}} \mathfrak{x}_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o) - \nabla_f$ belongs to the image of the Bockstein map $\beta^-_{\boldsymbol{fgh}}$. There exist then μ and ν in L such that

$$\sum_{\boldsymbol{u}} \mathfrak{x}_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o) - \nabla_f - \nu \cdot \beta^-_{\boldsymbol{fgh}}(q_{\beta\beta}) = \mu \cdot \beta^-_{\boldsymbol{fgh}}(q_{\alpha\alpha}). \tag{19}$$

Equation (8) of [3] shows that $\beta^{-}_{\boldsymbol{fgh}}(q_{\beta\beta})$ belongs to $H^1(\mathbf{Q}_p, V^{-}_{\beta\beta}) \otimes_L \mathscr{I}/\mathscr{I}^2$, hence Lemma 2.7 and the previous equation give

$$\sum_{\boldsymbol{u}} \mu_{\boldsymbol{u}} \cdot \log_{\boldsymbol{f}} \otimes q_{\alpha\alpha} \cdot (\boldsymbol{u} - u_o) = \sum_{\boldsymbol{u}} \mathrm{pr}_{\alpha\alpha}(\mathfrak{x}_{\boldsymbol{u}}) \cdot (\boldsymbol{u} - u_o) = \mu \cdot \mathrm{pr}_{\alpha\alpha}(\beta^{-}_{\boldsymbol{fgh}}(q_{\alpha\alpha})) \quad (20)$$

(where in the right-most term we write again $\mathrm{pr}_{\alpha\alpha}$ to denote the $\mathscr{I}/\mathscr{I}^2$-base change of the projection $\mathrm{pr}_{\alpha\alpha} : H^1(\mathbf{Q}_p, V^-) \longrightarrow H^1(\mathbf{Q}_p, V^{-}_{\alpha\alpha})$). The computations carried out in Sections 3.3 and 3.4 of [3] (see in particular Equation (30) of loc. cit. and the discussion preceding it) give the following equality in $H^1(\mathbf{Q}_p, V^{-}_{\alpha\alpha}) \otimes_L \mathscr{I}/\mathscr{I}^2$:

$$2 \cdot \mathrm{pr}_{\alpha\alpha}(\beta^{-}_{\boldsymbol{fgh}}(q_{\alpha\alpha})) = \sum_{\boldsymbol{u}} \log_{\boldsymbol{\xi}} \otimes q_{\alpha\alpha} \cdot (\boldsymbol{u} - u_o),$$

where $(\boldsymbol{\xi}, \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{k}), (\boldsymbol{g}, \boldsymbol{l}), (\boldsymbol{h}, \boldsymbol{m})$. Together with Eq. (20) this implies

$$2\mu_{\boldsymbol{k}} = \mu, \quad 2\mu_{\boldsymbol{l}} \cdot \log_{\boldsymbol{f}} = \mu \cdot \log_{\boldsymbol{g}} \quad \text{and} \quad 2\mu_{\boldsymbol{m}} \cdot \log_{\boldsymbol{f}} = \mu \cdot \log_{\boldsymbol{h}},$$

thus $\mu = \mu_{\boldsymbol{k}} = \mu_{\boldsymbol{l}} = \mu_{\boldsymbol{m}} = 0$ by the assumption on the analytic $\mathscr{L}$-invariants made in the statement. The lemma follows from this and Eq. (19). □

Let $(\boldsymbol{u}, \boldsymbol{\xi})$ denote one of $(\boldsymbol{k}, \boldsymbol{f})$, $(\boldsymbol{l}, \boldsymbol{g})$ and $(\boldsymbol{m}, \boldsymbol{h})$. For each local class x in $H^1(\mathbf{Q}_p, V^-)$, denote by $x_{\beta\beta} = \mathrm{pr}_{\beta\beta}(x)$ in $H^1(\mathbf{Q}_p, V^{-}_{\beta\beta})$ its $\beta\beta$-component and (with the notations introduced in Sect. 2.2) set

$$\ell_{\boldsymbol{u}}(x) = (-1)^{u_o} \cdot \left(x_{\beta\beta}(p^{-1})_f - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{\xi}} \cdot x_{\beta\beta}(e(1))_f\right).$$

For each pair $(\boldsymbol{u}, \boldsymbol{v})$ of distinct elements of $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}\}$ define

$$\tilde{\mathtt{D}}_{\boldsymbol{u},\boldsymbol{u}} = \ell_{\boldsymbol{u}}(\mathfrak{x}_{\boldsymbol{u}}) \quad \text{and} \quad \tilde{\mathtt{D}}_{\boldsymbol{u},\boldsymbol{v}} = \ell_{\boldsymbol{u}}(\mathfrak{x}_{\boldsymbol{v}}) + \ell_{\boldsymbol{v}}(\mathfrak{x}_{\boldsymbol{u}}).$$

Lemma 2.9. *For each pair $(\boldsymbol{u}, \boldsymbol{v})$ of elements of $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}\}$ one has*

$$2\left(1 - p^{-1}\right) \cdot \tilde{D}_{\boldsymbol{u},\boldsymbol{v}}(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) = \tilde{\mathtt{D}}_{\boldsymbol{u},\boldsymbol{v}}.$$

Proof. We give the proof for $(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{k}, \boldsymbol{l})$ and $(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{k}, \boldsymbol{k})$, the other cases being similar. We use the notations introduced in the proof of Lemma 2.8. Section 3 of [3] (see in particular Equations (8) and (30) of loc. cit.) gives the identities

$$2 \cdot \beta^{-}_{\boldsymbol{fgh}}(q_{\beta\beta}) = \sum_{\boldsymbol{u}} (-1)^{u_o} \cdot \log_{\boldsymbol{\xi}} \otimes q_{\beta\beta} \cdot (\boldsymbol{u} - u_o) \quad \text{and} \quad \mathrm{pr}_{\beta\beta}(\beta^{-}_{\boldsymbol{fgh}}(q_{\alpha\alpha})) = 0.$$

Equation (19) (and the definition of derivatives $\tilde{D}_{\boldsymbol{u},\boldsymbol{v}}$) then yields

$$\begin{aligned} &2\left(1 - p^{-1}\right) \cdot \tilde{D}_{\boldsymbol{k},\boldsymbol{l}}(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) - \ell_{\boldsymbol{k}}(\mathfrak{x}_{\boldsymbol{l}}) - \ell_{\boldsymbol{l}}(\mathfrak{x}_{\boldsymbol{k}}) \\ &\quad = \frac{\nu}{2}\left(\ell_{\boldsymbol{k}}(\log_{\boldsymbol{g}} \otimes q_{\beta\beta}) - \ell_{\boldsymbol{l}}(\log_{\boldsymbol{f}} \otimes q_{\beta\beta})\right) = 0 \end{aligned}$$

and

$$2\left(1 - p^{-1}\right) \cdot \tilde{D}_{\boldsymbol{k},\boldsymbol{k}}(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) - \ell_{\boldsymbol{k}}(\mathfrak{x}_{\boldsymbol{k}}) = -\frac{\nu}{2} \cdot \ell_{\boldsymbol{k}}(\log_{\boldsymbol{f}} \otimes q_{\beta\beta}) = 0,$$

quod erat demonstrandum. □

Lemma 2.10. *Assume that either $\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \neq \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}}$ or $\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \neq \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}}$. Then one has*

$$c_p(f) \cdot \langle\langle q_{\alpha\alpha}, \kappa(f, g_\alpha, h_\alpha)\rangle\rangle_{\boldsymbol{fgh}} = -\frac{m_p \cdot \mathrm{ord}_p(q_A)}{\deg(\wp_\infty)} \cdot \sum_{\boldsymbol{u}} \ell_{\boldsymbol{k}}(\mathfrak{x}_{\boldsymbol{u}}) \cdot (\boldsymbol{u} - u_o).$$

Proof. Under the assumption in the statement $\mathfrak{x}_{\boldsymbol{u}}$ belongs to $H^1(\mathbf{Q}_p, V_{\beta\beta}^-)$ by Lemma 2.8. Together with the equality $\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} = \frac{\log_p(q_A)}{\mathrm{ord}_p(q_A)}$ (cf. [6]), this gives

$$\ell_{\boldsymbol{k}}(\mathfrak{x}_{\boldsymbol{u}}) = \mathfrak{x}_{\boldsymbol{u}}(p^{-1})_f - \mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} \cdot \mathfrak{x}_{\boldsymbol{u}}(e(1))_f = -\frac{1}{\mathrm{ord}_p(q_A)} \cdot \mathfrak{x}_{\boldsymbol{u}}(q_A)_f. \tag{21}$$

According to Equation (15) of [3], one has

$$q_{\alpha\alpha}^+ = \frac{m_p}{\deg(\wp_\infty)} \cdot (q_A \hat{\otimes} 1) \otimes q_{\beta\beta}^*,$$

where $q_{\beta\beta}^*$ in $V_{\alpha\alpha}^+$ is the dual basis of $q_{\beta\beta}$ under the perfect pairing $\pi_{fgh}(-1)$. Lemma 2.6, the skew-symmetry of $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{fgh}}$ and local class field theory then give

$$\langle\!\langle q_{\alpha\alpha}, \kappa(f, g_\alpha, h_\alpha) \rangle\!\rangle_{\boldsymbol{fgh}} = -\langle\!\langle \kappa(f, g_\alpha, h_\alpha), q_{\alpha\alpha} \rangle\!\rangle_{\boldsymbol{fgh}} = \frac{m_p}{\deg(\wp_\infty)} \cdot \sum_{\boldsymbol{u}} \mathfrak{x}_{\boldsymbol{u}}^{\beta\beta}(q_A) \cdot (\boldsymbol{u} - u_o),$$

where $\mathfrak{x}_{\boldsymbol{u}}^{\beta\beta}$ in $H^1(\mathbf{Q}_p, \mathbf{Q}_p)$ is defined by $\mathfrak{x}_{\boldsymbol{u}} = \mathfrak{x}_{\boldsymbol{u}}^{\beta\beta} \otimes q_{\beta\beta}$. The lemma follows from the previous equation, Eq. (21) and the identity $\mathfrak{x}_{\boldsymbol{u}}(q_A)_f = \mathfrak{x}_{\boldsymbol{u}}^{\beta\beta}(q_A) \cdot c_p(f)$, □

Lemma 2.11. *Assume that either $\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} \neq \mathfrak{L}_{\boldsymbol{g}}^{\mathrm{an}}$ or $\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} \neq \mathfrak{L}_{\boldsymbol{h}}^{\mathrm{an}}$, so that $\mathfrak{x}_{\boldsymbol{u}}$ belongs to $H^1(\mathbf{Q}_p, V_{\beta\beta}^-)$ for $\boldsymbol{u} = \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}$ by Lemma 2.8. Then*

$$\langle\!\langle \kappa(f, g_\alpha, h_\alpha), \cdot \rangle\!\rangle_{\boldsymbol{fgh}} = \log_{\alpha\alpha}\big(\mathrm{res}_p(\cdot)\big) \cdot \sum_{\boldsymbol{u}} \mathfrak{x}_{\boldsymbol{u}}(e(1))_f \cdot (\boldsymbol{u} - u_o)$$

as $\mathscr{I}/\mathscr{I}^2$-valued L-linear forms on the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V)$.

Proof. Let y be a Selmer class in $\mathrm{Sel}(\mathbf{Q}, V)$, and let $\tilde{y} = \imath_{\mathrm{ur}}(y)$ in $\tilde{H}_f^1(\mathbf{Q}, V)$ be the corresponding class in the extended Selmer group (cf. Section 2.3 of [4]). By construction $\tilde{y}^+$ belongs to the Bloch–Kato finite subspace of $H^1(\mathbf{Q}, V^+)$, and $\mathrm{res}_p(y) = i^+(\tilde{y}^+)$ is its image under the map i^+ induced in cohomology by the inclusion $V^+ \hookrightarrow V$. Define $\tilde{y}_{\alpha\alpha}^+$ in $\mathbf{Z}_p^* \otimes_{\mathbf{Z}_p} L$ by the identity

$$\mathrm{pr}_{\alpha\alpha}(\tilde{y}^+) = \tilde{y}_{\alpha\alpha}^+ \otimes (\eta_f \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha}),$$

in $H_{\mathrm{fin}}^1(\mathbf{Q}_p, V_{\alpha\alpha}^+) = H_{\mathrm{fin}}^1(\mathbf{Q}_p, L(1)) \otimes_L V_{\alpha\alpha}^+(-1)$ (where as usual $H_{\mathrm{fin}}^1(\mathbf{Q}_p, L(1))$ is identified with $\mathbf{Z}_p^* \otimes_{\mathbf{Z}_p} L$ via the local Kummer map). Then one has

$$\log_{\alpha\alpha}(\mathrm{res}_p(y)) = \log_p(\tilde{y}_{\alpha\alpha}^+)$$

where $\log_p$ is the L-linear extension of the p-adic logarithm on $\mathbf{Z}_p^*$. Write similarly

$$\mathfrak{x}_{\boldsymbol{u}} = \mathrm{pr}_{\beta\beta}(\mathfrak{x}_{\boldsymbol{u}}) = \mathfrak{x}_{\boldsymbol{u}}^{\beta\beta} \otimes (\omega_f \otimes \eta_{g_\alpha} \otimes \eta_{h_\alpha})$$

in $H^1(\mathbf{Q}_p, V_{\beta\beta}^-) = H^1(\mathbf{Q}_p, L) \otimes_L V_{\beta\beta}^-$ for some $\mathfrak{x}_{\boldsymbol{u}}^{\beta\beta}$ in $H^1(\mathbf{Q}_p, L)$, so that

$$\langle \mathfrak{x}_{\boldsymbol{u}}, \tilde{y}^+ \rangle_{\mathrm{Tate}} = -\mathfrak{x}_{\mathrm{u}}^{\beta\beta}(\tilde{y}_{\alpha\alpha}^+) = -\log_p(\tilde{y}_{\alpha\alpha}^+) \cdot \mathfrak{x}_{\boldsymbol{u}}^{\beta\beta}(e(1)) = \log_{\alpha\alpha}(\mathrm{res}_p(y)) \cdot \mathfrak{x}_{\boldsymbol{u}}(e(1))_f$$

by local class field theory. The statement then follows from Lemma 2.6. □

We can finally conclude the proof of Proposition 2.4.

Proof of Proposition 2.4. To lighten the notation set $\kappa_{\alpha\alpha} = \kappa(f, g_\alpha, h_\alpha)$. By definition the extended height $\tilde{h}_p^{\alpha\alpha}(\kappa_{\alpha\alpha} \otimes y)$ is equal (up to sign) to

$$\langle\!\langle q_{\alpha\alpha}, q_{\beta\beta} \rangle\!\rangle_{\boldsymbol{fgh}} \cdot \langle\!\langle \kappa_{\alpha\alpha}, y \rangle\!\rangle_{\boldsymbol{fgh}} - \langle\!\langle q_{\alpha\alpha}, \kappa_{\alpha\alpha} \rangle\!\rangle_{\boldsymbol{fgh}} \cdot \langle\!\langle q_{\beta\beta}, y \rangle\!\rangle_{\boldsymbol{fgh}}$$

$$+\langle\langle q_{\beta\beta},\kappa_{\alpha\alpha}\rangle\rangle_{\boldsymbol{fgh}}\cdot\langle\langle q_{\alpha\alpha},y\rangle\rangle_{\boldsymbol{fgh}}$$

for each Selmer class y in $\mathrm{Sel}(\mathbf{Q},V)$. Since $\kappa_{\alpha\alpha}$ is (the specialisation of) a balanced class, one has $\log_{\alpha\alpha}(\mathrm{res}_p(\kappa_{\alpha\alpha}))=0$ (cf. Section 9.1 of [5]), hence $\langle\langle q_{\beta\beta},\kappa_{\alpha\alpha}\rangle\rangle_{\boldsymbol{fgh}}$ is equal to zero by Lemma 2.5. As a consequence

$$\tilde{h}_p^{\alpha\alpha}(\kappa_{\alpha\alpha}\otimes y)=\det\begin{pmatrix}\langle\langle q_{\alpha\alpha},q_{\beta\beta}\rangle\rangle_{\boldsymbol{fgh}} & \langle\langle q_{\alpha\alpha},\kappa_{\alpha\alpha}\rangle\rangle_{\boldsymbol{fgh}}\\ \langle\langle q_{\beta\beta},y\rangle\rangle_{\boldsymbol{fgh}} & \langle\langle \kappa_{\alpha\alpha},y\rangle\rangle_{\boldsymbol{fgh}}\end{pmatrix}. \tag{22}$$

Assume first $\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}}=\mathfrak{L}_{\boldsymbol{g}}^{\mathrm{an}}=\mathfrak{L}_{\boldsymbol{h}}^{\mathrm{an}}$. Then $\langle\langle q_{\alpha\alpha},q_{\beta\beta}\rangle\rangle_{\boldsymbol{fgh}}$ is equal to zero by Lemma 2.5, so that Eq. (22) and Lemmas 2.5 and 2.10 yield the equality (up to sign)

$$\begin{aligned}\tilde{h}_p^{\alpha\alpha}(\kappa_{\alpha\alpha}\otimes y)&=\langle\langle q_{\alpha\alpha},\kappa_{\alpha\alpha}\rangle\rangle_{\boldsymbol{fgh}}\cdot\langle\langle q_{\beta\beta},y\rangle\rangle_{\boldsymbol{fgh}}\\ &=\frac{m_p\cdot\mathrm{ord}_p(q_A)}{2\cdot\deg(\wp_\infty)}\cdot\log_{\alpha\alpha}(\mathrm{res}_p(y))\cdot(\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m})\cdot\sum_{\boldsymbol{u}}\ell_{\boldsymbol{k}}(\mathfrak{x}_{\boldsymbol{u}})\cdot(\boldsymbol{u}-u_o).\end{aligned}$$

Moreover one has (by definition) $\ell_{\boldsymbol{k}}=-\ell_{\boldsymbol{l}}=-\ell_{\boldsymbol{m}}$, hence

$$(\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m})\cdot\sum_{\boldsymbol{u}}\ell_{\boldsymbol{k}}(\mathfrak{x}_{\boldsymbol{u}})\cdot(\boldsymbol{u}-u_o)=\sum_{\boldsymbol{u},\boldsymbol{v}}\tilde{\mathtt{D}}_{\boldsymbol{u},\boldsymbol{v}}\cdot(\boldsymbol{u}-u_o)(\boldsymbol{v}-v_o).$$

Proposition 2.4 follows from the previous two equations and Lemmas 2.3 and 2.9.

Assume from now on that the analytic $\mathscr{L}$-invariants $\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}},\mathfrak{L}_{\boldsymbol{g}}^{\mathrm{an}}$ and $\mathfrak{L}_{\boldsymbol{h}}^{\mathrm{an}}$ are not all equal. Then Eq. (22), Lemmas 2.5, 2.10 and 2.11 yield

$$\tilde{h}_p^{\alpha\alpha}(\kappa_{\alpha\alpha}\otimes y)=\frac{m_p\cdot\mathrm{ord}_p(q_A)}{2\cdot\deg(\wp_\infty)}\cdot\log_{\alpha\alpha}(\mathrm{res}_p(y))\cdot\det(\mathtt{H}) \tag{23}$$

in $\mathscr{I}^2/\mathscr{I}^3$ for each Selmer class y in $\mathrm{Sel}(\mathbf{Q},V)$, where

$$\mathtt{H}=\begin{pmatrix}(\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}}-\mathfrak{L}_{\boldsymbol{g}}^{\mathrm{an}})\cdot(\boldsymbol{l}-1)+(\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}}-\mathfrak{L}_{\boldsymbol{h}}^{\mathrm{an}})\cdot(\boldsymbol{m}-1) & -\sum_{\boldsymbol{u}}\ell_{\boldsymbol{k}}(\mathfrak{x}_{\boldsymbol{u}})\cdot(\boldsymbol{u}-u_o)\\ \boldsymbol{l}+\boldsymbol{m}-\boldsymbol{k} & \sum_{\boldsymbol{u}}\mathfrak{x}_{\boldsymbol{u}}(e(1))_{\boldsymbol{f}}\cdot(\boldsymbol{u}-u_o)\end{pmatrix}.$$

A direct computation gives

$$\det(\mathtt{H})=-\sum_{\boldsymbol{u},\boldsymbol{v}}\tilde{\mathtt{D}}_{\boldsymbol{u},\boldsymbol{v}}\cdot(\boldsymbol{u}-u_o)(\boldsymbol{v}-v_o). \tag{24}$$

Proposition 2.4 follows from Eqs. (23) and (24) and Lemmas 2.3 and 2.9. □

2.4. Heegner Points and Diagonal Classes

Assume from now on

$$\mathrm{ord}_{s=1}L(f\otimes g\otimes h,s)=2 \tag{25}$$

and that Assumption 1.2 (stated in Sect. 1) is satisfied.

For each finite order character $\mu:G_K\longrightarrow\mathbf{Q}(\varrho)^*$, let $\mathrm{Ind}_K^{\mathbf{Q}}\mu$ be the $\mathbf{Q}(\varrho)$-module of functions $c:G_{\mathbf{Q}}\longrightarrow\mathbf{Q}(\varrho)$ satisfying $c(\tau\sigma)=\mu(\tau)\cdot c(\sigma)$ for each τ in G_K and σ in $G_{\mathbf{Q}}$, equipped with the action of $G_{\mathbf{Q}}$ defined by $(\sigma'\cdot c)(\sigma)=c(\sigma\sigma')$

for each σ and σ' in $G_{\mathbf{Q}}$. For $\xi = g, h$, the $\mathbf{Q}(\varrho)[G_{\mathbf{Q}}]$-module $\mathrm{Ind}_K^{\mathbf{Q}}\nu_\xi$ affords the representation ϱ_ξ. With the notations of Sect. 1 we can then take

$$V_\xi = \mathrm{Ind}_K^{\mathbf{Q}}\nu_\xi.$$

One has an isomorphism of $\mathbf{Q}(\varrho)[G_{\mathbf{Q}}]$-modules

$$V_{gh} = V_g \otimes_{\mathbf{Q}(\varrho)} V_h \simeq \mathrm{Ind}_K^{\mathbf{Q}}\varphi \oplus \mathrm{Ind}_K^{\mathbf{Q}}\psi, \tag{26}$$

where $\varphi = \nu_g \cdot \nu_h$ and $\psi = \nu_g \cdot \nu_h^c$ are dihedral characters of K (cf. Sect. 1). The Artin formalism then yields the factorisation

$$L(f \otimes g \otimes h, s) = L(A/K, \varphi, s) \cdot L(A/K, \psi, s), \tag{27}$$

where $L(A/K, \chi, s) = L(f \otimes \vartheta_\chi, s)$ is the Hasse–Weil L-function of the base change of A to K twisted by $\chi = \varphi, \psi$ (viz. the Rankin–Selberg convolution of f and the weight-one theta series ϑ_χ associated with χ).

Let χ denote either φ or ψ, let K_χ be the ring class field of K cut out by χ, and let $A(K_\chi)^\chi$ be the submodule of $A(K_\chi) \otimes_{\mathbf{Z}} \mathbf{Q}(\varrho)$ on which $\mathrm{Gal}(K_\chi/K)$ acts via χ. Fix a primitive Heegner point P in $A(K_\chi)$ and set

$$P_\chi = \sum_{\sigma \in \mathrm{Gal}(K_\chi/K)} \chi(\sigma)^{-1} \cdot \sigma(P) \in A(K_\chi)^\chi.$$

Equations (25) and (27) and Assumption 1.1.(1) imply that $L(A/K, \chi, s)$ has a simple zero at $s = 1$, hence the Gross–Zagier–Kolyvagin–Zhang theorem yields

$$P_\chi \neq 0 \quad \text{and} \quad A(K_\chi)^\chi \otimes_{\mathbf{Q}(\varrho)} L = L \cdot P_\chi = \mathrm{Sel}(K_\chi, V_p(A))^\chi, \tag{28}$$

where $\mathrm{Sel}(K_\chi, V_p(A))$ is the Bloch–Kato Selmer group of the restriction of $V_p(A)$ to G_{K_χ}, one denotes by $\mathrm{Sel}(K_\chi, V_p(A))^\chi$ the submodule of $\mathrm{Sel}(K_\chi, V_p(A)) \otimes_{\mathbf{Q}_p} L$ on which the Galois group of K_χ/K acts via the character χ, and one considers $A(K_\chi)^\chi$ as a submodule of $\mathrm{Sel}(K_\chi, V_p(A))^\chi$ via the K_χ-rational Kummer map.

Let σ_p in $G_{\mathbf{Q}} - G_K$ be an arithmetic Frobenius at p. For $\xi = g, h$ and each pair (a, b) of elements of $\mathbf{Q}(\varrho)$, denote by $[a, b]_\xi$ in V_ξ the $\mathbf{Q}(\varrho)$-valued function on $G_{\mathbf{Q}}$ sending the identity to a and σ_p to b. Then G_K acts on the line $L \cdot [1, 0]_\xi$ via ν_ξ, and on the line $L \cdot [0, 1]_\xi$ via the conjugate ν_ξ^c of ν_ξ by the nontrivial element $c = \sigma_p|_K$ of $\mathrm{Gal}(K/\mathbf{Q})$. Moreover, since $\nu_\xi(\sigma_p^2) = \nu_\xi^{\mathrm{cen}}(p) = \varepsilon_K(p) \cdot \chi_\xi(p) = -\chi_\xi(p) = \alpha_\xi^2$ (cf. Sect. 1), one has $\sigma_p \cdot [a, b]_\xi = [b, \alpha_\xi^2 \cdot a]_\xi$ for each a and b in $\mathbf{Q}(\varrho)$. Set

$$v_{\xi,\alpha} = [1, \alpha_\xi]_\xi \in V_\xi^{\sigma_p = \alpha_\xi} \quad \text{and} \quad v_{\xi,\beta} = [1, -\alpha_\xi]_\xi \in V_\xi^{\sigma_p = \beta_\xi}.$$

(recall that $\beta_\xi = -\alpha_\xi$), and for each pair (i, j) of elements of $\{\alpha, \beta\}$ set

$$v_{ij} = v_{g,i} \otimes v_{h,j} \in V_g^{\sigma_p = i_g} \otimes_{\mathbf{Q}(\varrho)} V_h^{\sigma_p = j_h} \longhookrightarrow V_{gh}^{\sigma_p = i_g \cdot j_h}.$$

A direct computation shows that the vectors

$$v_\varphi = v_{\alpha\alpha} + v_{\alpha\beta} + v_{\beta\alpha} + v_{\beta\beta} \quad \text{and} \quad v_\psi = v_{\alpha\alpha} - v_{\alpha\beta} + v_{\beta\alpha} - v_{\beta\beta}$$

of V_{gh} are qual to $4 \cdot [1, 0]_g \otimes [1, 0]_h$ and $4\alpha_\xi \cdot [1, 0]_g \otimes [0, 1]_h$ respectively, hence G_K acts on them via $\varphi = \nu_g \cdot \nu_h$ and $\psi = \nu_g \cdot \nu_h^c$ respectively. For $\chi = \varphi, \psi$ define

$$P(\chi) = \gamma_{gh}\big(P_\chi \otimes \sigma_p(v_\chi) + \sigma_p(P_\chi) \otimes v_\chi\big)$$

in $\mathrm{Sel}(\mathbf{Q}, V)$ to be image of $P_\chi \otimes \sigma_p(v_\chi) + \sigma_p(P_\chi) \otimes v_\chi$ in $A(K_\varrho)^\varrho$ under the embedding γ_{gh} introduced in Eq. (10), so that (cf. Eqs. (26) and (28))

$$\mathrm{Sel}(\mathbf{Q}, V) = L \cdot P(\varphi) \oplus L \cdot P(\psi). \tag{29}$$

Write $\varepsilon = \alpha_f$ and for χ equal to φ or ψ define

$$P_\chi^\varepsilon = P_\chi + \varepsilon \cdot \sigma_p(P_\chi).$$

The point P_χ^ε is non-zero. This follows from Eq. (28) if χ is not quadratic. When χ is quadratic, one has $\sigma_p(P_\chi) = \chi_1(p) \cdot P_\chi$, hence P_χ^ε is non-zero by Eq. (28) and Assumption 1.2. In order to lighten the notation, set $\kappa_{\alpha\alpha} = \kappa(f, g_\alpha, h_\alpha)$. The main result Theorem A of [2] proves the identity

$$\log_{\beta\beta}(\mathrm{res}_p(\kappa(f, g_\alpha, h_\alpha))) = \log_{\omega_f}(P_\varphi^\varepsilon) \cdot \log_{\omega_f}(P_\psi^\varepsilon) \in L^*/\mathbf{Q}(\varrho)^*. \tag{30}$$

Here $\log_{\omega_f} : A(K_\chi) \otimes_{\mathbf{Z}} L \longrightarrow L \otimes_{\mathbf{Q}_p} K_p$ denotes the L-linear extension of the logarithm $\log_{\omega_f}$ on $A(K_\chi)$ introduced in Sect. 2.1.4. (Note that the right hand side of the previous identity is an element of $L \otimes_{\mathbf{Q}_p} K_p$ fixed by the action of σ_p, id est of L.)

Recall that the roots α_ξ and $\beta_\xi = -\alpha_\xi$ of the pth Hecke polynomial of $\xi = g, h$ are distinct, and that $\alpha_g \cdot \alpha_h = \alpha_f = \beta_g \cdot \beta_h$ (cf. Eq. (1)). We can then replace in the above constructions the Hida family $\boldsymbol{\xi} = \boldsymbol{\xi}_\alpha$ with the one $\boldsymbol{\xi}_\beta$ specialising to the p-stabilisation $\xi_\beta(q) = \xi(q) - \alpha_\xi \cdot \xi(q^p)$ at weight one, for $\xi = g, h$. This produces a diagonal class $\kappa(f, g_\beta, h_\beta)$ in the Selmer group $\mathrm{Sel}(\mathbf{Q}, W)$ of the p-adic representation $W = V(\boldsymbol{f}, \boldsymbol{g}_\beta, \boldsymbol{h}_\beta) \otimes_{w_o} L$. Fix an isomorphism of $L[G_{\mathbf{Q}}]$-modules $\mu : W \simeq V$, and let

$$\kappa_{\beta\beta} = \mu(\kappa(f, g_\beta, h_\beta)) \in \mathrm{Sel}(\mathbf{Q}, V)$$

be the image of $\kappa(f, g_\beta, h_\beta)$ under the isomorphism it induces in cohomology. The analogue of Eq. (30) proves that the $\alpha\alpha$-logarithm of $\kappa_{\beta\beta}$ is non-zero:

$$\log_{\alpha\alpha}(\mathrm{res}_p(\kappa_{\beta\beta})) \in L^*. \tag{31}$$

Since by the definition of the balanced local condition (cf. Sect. 2.1.1) one has

$$\log_{\alpha\alpha}(\mathrm{res}_p(\kappa_{\alpha\alpha})) = \log_{\beta\beta}(\mathrm{res}_p(\kappa_{\beta\beta})) = 0, \tag{32}$$

it follows that the diagonal classes $\kappa_{\alpha\alpha}$ and $\kappa_{\beta\beta}$ are linearly independent, hence

$$\mathrm{Sel}(\mathbf{Q}, V) = L \cdot \kappa_{\alpha\alpha} \oplus L \cdot \kappa_{\beta\beta}. \tag{33}$$

2.4.1. Conclusion of the Proof. Consider the L-basis (cf. Eqs. (6) and (8))

$$q_\flat = \frac{1}{\sqrt{m_p}} \cdot q(f) \otimes v_g^\alpha \otimes v_h^\alpha \quad \text{and} \quad q_\natural = \frac{1}{\sqrt{m_p}} \cdot q(f) \otimes v_g^\beta \otimes v_h^\beta$$

of $H^0(\mathbf{Q}_p, V^-)$, where $v_\xi^\cdot = \gamma_\xi(v_{\xi,\cdot})$ for $\xi = g, h$ and $\cdot = \alpha, \beta$. It is the image of the $\mathbf{Q}(\varrho)$-basis $\{q(A) \otimes v_{g,\alpha} \otimes v_{h,\alpha}, q(A) \otimes v_{g,\beta} \otimes v_{h,\beta}\}$ of $\mathcal{Q}_p(A, \varrho)$ (cf. Eq. (9)) under the isomorphism $\mathcal{Q}_p(A, \varrho)_L \simeq H^0(\mathbf{Q}_p, V)$ arising from the modular parametrisation $\wp_\infty$ fixed in Sect. 2.1.1 and the embeddings γ_g and γ_h fixed in Eq. (8). Define $\mathtt{M}$ and $\mathtt{N}$ in $\mathrm{GL}_2(L)$ by the identities (cf. Eqs. (29) and (33))

$$\begin{pmatrix} \kappa_{\alpha\alpha} \\ \kappa_{\beta\beta} \end{pmatrix} = \mathtt{M} \begin{pmatrix} P(\chi) \\ P(\psi) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q_{\alpha\alpha} \\ q_{\beta\beta} \end{pmatrix} = \mathtt{N} \begin{pmatrix} q_\flat \\ q_\sharp \end{pmatrix}.$$

By the definition of the p-adic regulator $R_p^{\alpha\alpha}(A,\varrho)$ and Proposition 2.4 one has

$$R_p^{\alpha\alpha}(A,\varrho) = \frac{\log^2_{\alpha\alpha}(\mathrm{res}_p(\kappa_{\beta\beta}))}{\det(\mathtt{M})^2\cdot\det(\mathtt{N})^2}\cdot L_p^{\alpha\alpha}(A,\varrho) \pmod{\mathscr{I}^5} \tag{34}$$

in the quotient of $\mathscr{I}^4/\mathscr{I}^5$ by the multiplicative action of $\mathbf{Q}(\varrho)^{*2}$.

Set $\hat{L} = L\otimes_{\mathbf{Q}_p}\hat{\mathbf{Q}}_p^{\mathrm{nr}}$ and for $\xi = g, h$ denote by

$$\hat{\pi}_\xi : V(\xi)\otimes_L V(\xi)\otimes_{\mathbf{Q}_p}\hat{\mathbf{Q}}_p^{\mathrm{nr}} \longrightarrow \hat{L}$$

the $\hat{\mathbf{Q}}_p^{\mathrm{nr}}$-base change of the perfect pairing π_ξ introduced in Sect. 2.1.1. Since

$$\hat{\pi}_g(\eta_{g_\alpha}\otimes\omega_{g_\alpha})\cdot\hat{\pi}_h(\eta_{h_\alpha}\otimes\omega_{h_\alpha}) = G(\chi_g)\cdot G(\chi_h) = 1$$

(cf. Assumption 1.1.(2) and the definitions introduced in Sect. 2.1.2), one has

$$\mathtt{N} = \frac{1}{\sqrt{m_p}}\cdot\begin{pmatrix}\hat{\pi}_g(v_g^\alpha\otimes\eta_{g_\alpha})\cdot\hat{\pi}_h(v_h^\alpha\otimes\eta_{h_\alpha}) & 0\\ 0 & \hat{\pi}_g(v_g^\beta\otimes\omega_{g_\alpha})\cdot\hat{\pi}_h(v_h^\beta\otimes\omega_{h_\alpha})\end{pmatrix}$$

(in $H^0(\sigma_p, \mathrm{GL}_2(\hat{L})) = \mathrm{GL}_2(L)$), hence

$$\det(\mathtt{N}) = m_p^{-1}\cdot\pi_g(v_g^\alpha\otimes v_g^\beta)\cdot\pi_h(v_h^\alpha\otimes v_h^\beta)\in\mathbf{Q}(\varrho)^* \tag{35}$$

by the normalisation imposed on the embeddings γ_g and γ_h (cf. Eq. (8)).

According to Eqs. (30), (31) and (32) one has

$$\mathtt{M}^{-1} = \begin{pmatrix}\frac{\log_{\beta\beta}(P(\varphi))}{\log_{\beta\beta}(\kappa_{\alpha\alpha})} & \frac{\log_{\alpha\alpha}(P(\varphi))}{\log_{\alpha\alpha}(\kappa_{\beta\beta})}\\ \frac{\log_{\beta\beta}(P(\psi))}{\log_{\beta\beta}(\kappa_{\alpha\alpha})} & \frac{\log_{\alpha\alpha}(P(\psi))}{\log_{\alpha\alpha}(\kappa_{\beta\beta})}\end{pmatrix}$$

(where $\log_{ii} : \mathrm{Sel}(\mathbf{Q}, V)\longrightarrow L$, for $i = \alpha, \beta$, is a shorthand for $\log_{ii}\circ\mathrm{res}_p$). After retracing the definitions given in Sect. 2.4, a direct computation yields

$$\log_{\alpha\alpha}(P(\chi)) = \varepsilon\cdot\log_{\omega_f}(P_\chi^\varepsilon)\cdot\hat{\pi}_g(v_g^\alpha\otimes\eta_{g_\alpha})\cdot\hat{\pi}_h(v_h^\alpha\otimes\eta_{h_\alpha})$$

(in $H^0(\sigma_p,\hat{L}) = L$, where as usual χ denotes either φ or ψ) and

$$\log_{\beta\beta}(P(\chi)) = \varepsilon_\chi\cdot\varepsilon\cdot\log_{\omega_f}(P_\chi^\varepsilon)\cdot\hat{\pi}_g(v_g^\beta\otimes\omega_{g_\alpha})\cdot\hat{\pi}_h(v_h^\beta\otimes\omega_{h_\alpha}),$$

where $\varepsilon_\varphi = 1$ and $\varepsilon_\psi = -1$. As a consequence

$$\frac{\log_{\alpha\alpha}(\kappa_{\beta\beta})}{\det(\mathtt{M})} = 2\cdot\frac{\log_{\omega_f}(P_\varphi^\varepsilon)\cdot\log_{\omega_f}(P_\psi^\varepsilon)}{\log_{\beta\beta}(\kappa_{\alpha\alpha})}\cdot\pi_g(v_g^\alpha\otimes v_g^\beta)\cdot\pi_h(v_h^\alpha\otimes v_h^\beta)\in\mathbf{Q}(\varrho)^* \tag{36}$$

by Eqs. (30) and (8).

Equations (34), (35) and (36) give the identity

$$L_p^{\alpha\alpha}(A,\varrho) \pmod{\mathscr{I}^5} = R_p^{\alpha\alpha}(A,\varrho)$$

in the quotient of $\mathscr{I}^4/\mathscr{I}^5$ by the multiplicative action of $\mathbf{Q}(\varrho)^{*2}$. To conclude the proof of the Theorem stated in Sect. 1, it remains to prove that both sides of the previous identity are non-zero. This follows by combining Eq. (30) with [5, Theorem A] and [2, Proposition 2.2], which prove the equality

$$\frac{\partial^2\mathscr{L}_p^{\alpha\alpha}(A,\varrho)}{\partial\boldsymbol{k}^2}(w_o) = c_p(f)\cdot\frac{\deg(\wp_\infty)}{2m_p\mathrm{ord}_p(q_A)}\left(1-\frac{1}{p}\right)^{-1}\cdot\log_{\beta\beta}(\kappa_{\alpha\alpha}).$$

Funding Open access funding provided by Università degli Studi di Milano within the CRUI-CARE Agreement.

References

[1] Bellaïche, J., Dimitrov, M.: On the eigencurve at classical weight 1 points. Duke Math. J. **165**(2), 245–266 (2016)

[2] Bertolini, M., Seveso, M.A., Venerucci, R.: Balanced diagonal classes and rational points on elliptic curves. Astérisque (2021) (**To appear**)

[3] Bertolini, M., Seveso, M.A., Venerucci, R.: On exceptional zeros of Garrett–Hida p-adic L-functions (2021) (**Preprint**)

[4] Bertolini, M., Seveso, M.A., Venerucci, R.: On p-adic analogues of the Birch and Swinnerton–Dyer conjecture for Garrett L-functions (2021) (**Preprint**)

[5] Bertolini, M., Seveso, M.A., Venerucci, R.: Reciprocity laws for balanced diagonal classes. Astérisque (2021) (**To appear**)

[6] Greenberg, R., Stevens, G.: p-adic L-functions and p-adic periods of modular forms. Invent. Math. **111**(2), 407–447 (1993)

[7] Hida, H.: Galois representations into $\mathrm{GL}_2(Z_p[[X]])$ attached to ordinary cusp forms. Invent. Math. **85**(3), 545–613 (1986)

[8] Hsieh, M.-L.: Hida families and p-adic triple product L-functions. Am. J. Math. **143**(2), 411–532 (2021)

[9] Mazur, B., Tate, J., Teitelbaum, J.: On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer. Invent. Math. **84**(1), 1–48 (1986)

[10] Venerucci, R.: p-adic regulators and p-adic families of modular forms. Ph.D. Thesis, Universitá degli Studi di Milano (2013)

[11] Venerucci, R.: Exceptional zero formulae and a conjecture of Perrin–Riou. Invent. Math. **203**(3), 923–972 (2016)

[12] Venerucci, R.: On the p-converse of the Kolyvagin–Gross–Zagier theorem. Comment. Math. Helv. **91**(3), 397–444 (2016)

Massimo Bertolini
Universität Duisburg-Essen
Fakultät für Mathematik
Mathematikcarrée
Thea-Leymann-Straße 9
45127 Essen
Germany
e-mail: massimo.bertolini@uni-due.de

Marco Adamo Seveso and Rodolfo Venerucci
Università degli Studi di Milano
Dipartimento di Matematica Federigo Enriques
Via Cesare Saldini 50
20133 Milano
Italia

Marco Adamo Seveso
e-mail: seveso.marco@gmail.com

Rodolfo Venerucci
e-mail: rodolfo.venerucci@unimi.it

Received: March 8, 2021.
Accepted: April 12, 2021.