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ON DOMINANT RATIONAL MAPS FROM A VERY GENERAL COMPLETE INTERSECTION SURFACE IN $\mathbb{P}^4$

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Let S be a very general complete intersection surface of multidegree (d_1, d_2) in $\mathbb{P}^4$. The following problem arises: determine the couples (d_1, d_2) such that the surface S does not have any "non-evident" rational map to other surfaces. By non-evident rational map, we mean non-birational dominant map whose target space is not rational. We give a partial solution, presenting a class of multidegrees (d_1, d_2) which satisfy the above condition.

1. Introduction

A classical consequence of the Riemann-Hurwitz formula for curves says that if $\phi\colon C \to C'$ is a noncostant morphism from a very general curve of genus $g > 1$ onto a curve C', then either ϕ is birational or C' is rational. See [1, Cor. 8.32 Chapter XXI]. Lee and Pirola in [6] prove the following theorem which generalizes this result to the case of surfaces in $\mathbb{P}^3$:

Theorem 1.1 ([6, Thm. 1.1]). *Let $X \subset \mathbb{P}^3$ be a very general surface of degree $d > 4$, and let $f\colon X \dashrightarrow Y$ be a dominant rational map from X to another surface Y. Then, either f is birational or Y is rational.*

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Here "very general" means that, if $X = \{F = 0\}$, then the homogeneous polynomial F is very general, *i.e.* it belongs to the complement of a union of countably many proper subvarieties of $\mathbb{P}^{\binom{d+3}{d}}$. In this paper, following the argument of [6] and using results from [4], we try to give an answer to the analogous problem in the case of a very general complete intersection surface:

Question. Let $X \subset \mathbb{P}^4$ be a very general complete intersection surface of multidegree (d_1, d_2) and let $f \colon X \dashrightarrow Y$ be a dominant rational map of degree > 1.

For which degrees $d_2 \geq d_1 > 1$ can we say that Y is a rational surface?

Our main result is the following:

Theorem 1.2. *Let T be the following set of pairs of natural numbers*

$$T := \{(3,t) \mid t \geq 3\} \cup \{(4,t) \mid t \geq 4\} \cup \{(5,t) \mid 5 \leq t \leq 9\} \cup \{(6,6)\}.$$

Then if $(d_1, d_2) \in T$, Y is a rational surface.

This result has an equivalent algebraic formulation:

Theorem 1.3. *Let $(d_1, d_2) \in T$ and let $\mathbb{C}(X)$ be the function field of a complete intersection surface X of multidegree (d_1, d_2). Then every proper subfield $\mathbb{C} \subset K \subset \mathbb{C}(X)$ is a pure transcendental extension of $\mathbb{C}$, if X is very general.*

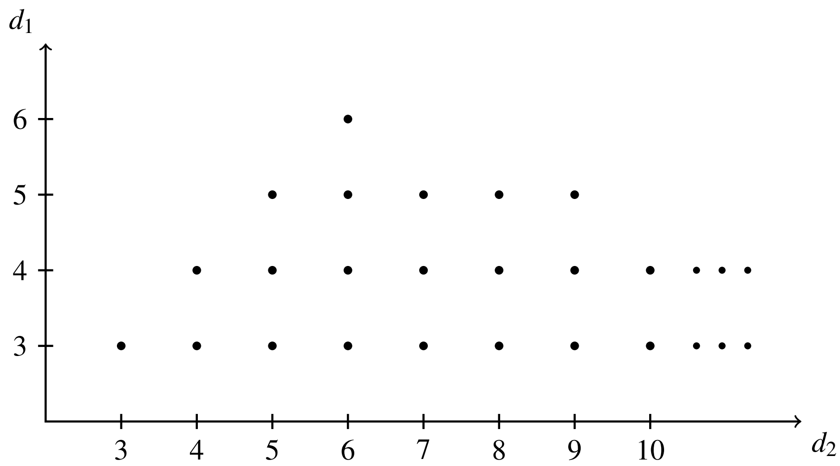

Figure 1: The elements of the set T

The first step to prove Theorem 1.2 uses some results from Hodge theory to find conditions on the surface Y. These conditions are not enough to prove that Y is rational, but they leave us with only two other possibilities. In particular we are able to prove

Lemma 1.4. *If Y is not rational, we can assume that it is a minimal surface which belongs to one of the following classes:*

1. $\mathrm{Kod}(Y) = 1$ *and there exists an elliptic fibration* $Y \to \mathbb{P}^1$ *with exactly two multiple fibers (see [3]);*
2. $\mathrm{Kod}(Y) = 2$, i.e. *Y is a surface of general type.*

In the second step, computations on the moduli dimension of such surfaces prove that the two cases of Lemma 1.4 are not possible if $(d_1, d_2) \in T$; this proves that Y is a rational surface.

2. First restrictions on Y

From now on call F_1, F_2 the homogeneous polynomials of degree respectively d_1, d_2 that define the complete intersection $X \subset \mathbb{P}^4$, *i.e.* $X = \{F_1 = F_2 = 0\}$. Call V_1 and V_2 the hypersurfaces corresponding to F_1 and F_2. Recall that F_1 and F_2 are very general in $\mathbb{P}^{\binom{d_1+4}{d_1}}$ and $\mathbb{P}^{\binom{d_2+4}{d_2}}$, respectively. Up to Veronese embedding of degree d_2, it is often useful to regard X as an hyperplane section on V_1. Recall the definition of fixed and vanishing cohomology on X:

$$H^2(X, \mathbb{Q})_{\text{fixed}} := \mathrm{Im}(i^* \colon H^2(V_1, \mathbb{Q}) \to H^2(X, \mathbb{Q})) \tag{1}$$

and

$$H^2(X, \mathbb{Q})_{\text{van}} := \mathrm{Ker}(i_! \colon H^2(X, \mathbb{Q}) \to H^4(V_1, \mathbb{Q})), \tag{2}$$

where $i \colon X \hookrightarrow V_1$ is the inclusion, i^* is the associated pullback and $i_!$ is the Gysin map. Denote by U the open set parametrising smooth hyperplane sections of V_1. The Noether-Lefschetz locus is the subset of U corresponding to surfaces S such that the restriction map $\mathrm{NS}(V_1) \otimes \mathbb{Q} \to \mathrm{NS}(S) \otimes \mathbb{Q}$ is not surjective. We recall some useful properties of X.

Proposition 2.1. *Let* $X \subset \mathbb{P}^4$ *be a very general complete intersection surface of multidegree* (d_1, d_2) *with* $d_2 \geq d_1 \geq 3$, *and let* $H \subset X$ *be a general hyperplane section. Then,*

(a) *the rational cohomology of X decomposes in the following way*

$$H^2(X, \mathbb{Q}) = \mathbb{Q} \cdot \langle H \rangle \oplus H^2(X, \mathbb{Q})_{\text{van}};$$

in particular the Hodge substructure on the orthogonal complement to the hyperplane section is irreducible;

(b) *the Néron-Severi group* $\mathrm{NS}(X)$ *is generated by* H*;*

(c) $\mathrm{Bir}(X) = \mathrm{Aut}(X) = \mathrm{Lin}\,\mathrm{Aut}(X) = \{\mathrm{id}_X\}$.

Proof. By the Lefschetz hyperplane theorem [8, Thm. 4.25], we have a decomposition of the rational cohomology of X in fixed and vanishing part

$$H^2(X,\mathbb{Q}) = H^2(X,\mathbb{Q})_{\text{fixed}} \oplus H^2(X,\mathbb{Q})_{\text{van}} \tag{3}$$

together with an isomorphism $H^2(X,\mathbb{Q})_{\text{fixed}} \cong H^2(V_1,\mathbb{Q})$. Again by Lefschetz hyperplane theorem, $H^2(V_1,\mathbb{Q}) \cong H^2(\mathbb{P}^4,\mathbb{Q})$, hence we have the decomposition in (a). The irreducibility of the vanishing part, that is the fact that it has no nontrivial Hodge substructures, comes from [8, Cor. 10.23]; see also [9, Sec. 3.2.3].

The Noether-Lefschetz locus is a countable union of proper algebraic subsets of U; [9, Thm. 3.33]. Hence for very general X the map $\mathrm{NS}(V_1) \otimes \mathbb{Q} \to \mathrm{NS}(X) \otimes \mathbb{Q}$ is surjective. Since $\mathrm{NS}(V_1)$ is generated by the hyperplane section, part (b) follows.

To prove part (c), note that $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$ by Lefschetz hyperplane theorem, hence every birational map $X \dashrightarrow X$ leaves H invariant and it comes from an automorphism of $\mathbb{P}^4$ and therefore is linear. A classical result of Matsumura and Monsky [7] states that a general hypersurface of degree ≥ 3 has no nontrivial linear automorphism. It immediately follows that if $3 \leq d_1 < d_2$, any linear automorphism of X should fix V_1 and hence it is the identity. Also in the remaining case $d_1 = d_2 = 3$, one can easily find that such an automorphism fixes a hypersurface given by a polynomial in the ideal (F_1, F_2), hence $\mathrm{Bir}(X) = \mathrm{Aut}(X) = \mathrm{Lin}\,\mathrm{Aut}(X) = \{\mathrm{id}_X\}$ for X very general and $d_2 \geq d_1 \geq 3$. □

Remark 2.2. Actually in [2] the authors prove that

$$\mathrm{Bir}(X) = \mathrm{Aut}(X) = \mathrm{Lin}\,\mathrm{Aut}(X) = \{\mathrm{id}_X\}$$

for X very general complete intersection of degrees $d_2 \geq d_1 \geq 2$. Thanks to this result we could add the line $\{(d_1,d_2) = (2,t) \mid t > 5\}$ to our set T, by the same computation of the following section. Nevertheless in this paper we stick to the case $d_2 \geq d_1 \geq 3$.

Proposition 2.3. *Let X be very general of multidegree (d_1,d_2) such that $d_2 \geq d_1 \geq 3$, and let $f\colon X \dashrightarrow Y$ be a dominant rational map of degree > 1. Then, $p_g(Y) = q(Y) = 0$ and $\pi_1(Y) = 1$.*

Proof.([4, Prop. 3.5.2]). We can define a pullback map $f^*\colon H^2(Y,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ passing through the resolution of indeterminacy of f. More in details consider such a resolution

$$\begin{array}{ccc} & Z & \\ {\scriptstyle\phi}\swarrow & & \searrow{\scriptstyle h} \\ X & \overset{f}{\dashrightarrow} & Y \end{array} \tag{4}$$

and define f^* as the composition of h^* and the Gysin map $\phi_!$. The restriction $f^*\colon H^{2,0}(Y) \to H^{2,0}(X)$ is injective. Let $T_Y \supseteq H^{2,0}(Y)$ (resp. $T_X \supseteq H^{2,0}(X)$) be the Hodge substructure orthogonal to Néron-Severi Hodge substructure of Y (resp. of X). Then,

$$\begin{aligned} H^2(X,\mathbb{C}) &= \mathrm{NS}(X)_{\mathbb{C}} \oplus T_X \\ &= \mathbb{C}\langle H\rangle \oplus H^2(X,\mathbb{C})_{\mathrm{van}}, \end{aligned}$$

By Proposition 2.1(a), T_X is an irreducible Hodge structure. Thus, if $h^{2,0}(Y) \neq 0$ then f^* maps T_Y isomorphically to T_X. In particular, $f^*\colon H^{2,0}(Y) \to H^{2,0}(X)$ is an isomorphism. Then, the canonical map $\varphi_{|K_X|}\colon X \to \mathbb{P}H^0(K_X)$ factors through f. Since $\varphi_{|K_X|}$ is an embedding, f must be a birational map, a contradiction.

For $\pi_1(Y) = 1$ the proof is the same as the one in [4, Prop. 3.5.2] and it uses part (c) of the previous proposition.

Since $\pi_1(Y) = 1$ we immediately deduce that $q(Y) = 0$. □

Assuming that Y is not rational, by classification of algebraic surfaces we may assume that Y is a minimal surface which belongs to one of the following classes:

1. $\mathrm{Kod}(Y) = 1$: there exists an elliptic fibration $Y \to \mathbb{P}^1$ with exactly two multiple fibers (see [3], pp. 133 and 146);

2. $\mathrm{Kod}(Y) = 2$: a surface of general type.

This is exactly Lemma 1.4.

To study the map $f\colon X \dashrightarrow Y$, we restrict to a general hyperplane section C of X, which is a complete intersection curve in $\mathbb{P}^3$.

Lemma 2.4. *Let C be a general hyperplane section of X. Then, the restriction of $f\colon C \to Y$ is birational onto its image.*

Proof. Since C is a general hyperplane section, the images $f(C)$ of such C covers general points of Y. By [8, Cor. 10.23], the general hyperplane section C has a simple Jacobian. The map $f_C\colon C \to f_C(C)$ induces a morphism $C \to D$ where D is the normalization of $f_C(C)$. Then, since $C \to D$ is surjective, $JC \to JD$ is surjective. Assume that f_C is not birational. Then the kernel of $JC \to JD$ is infinite[1], or $C \simeq \mathbb{P}^1$. In the former case it follows that the subabelian variety $\ker^{\circ}(JC \to JD)$ is JC, which means JD is a point. Thus, $D \simeq \mathbb{P}^1$. In the latter the surjectivity of $JC \to JD$ implies again that $D \simeq \mathbb{P}^1$. This is impossible since $f(C)$ covers the general points of Y and Y is not a ruled surface. □

[1] $\sum_{p_i \in f_C^{-1}(q)} n_i p_i$, $\sum n_i = 0$ belongs to the kernel

Since C is a complete intersection curve of multidegree (d_1, d_2) in $\mathbb{P}^3$, if we can prove that a (very) general complete intersection curve in $\mathbb{P}^3$ of multidegree (d_1, d_2) cannot be birationally immersed into any Y as above, then by Lemma 2.4 we can conclude that there is no dominant rational map $f\colon X \dashrightarrow Y$ of degree > 1.

3. Dimension Counting

By contradiction, assume that a very general complete intersection curve C of multidegree (d_1, d_2) has a birational immersion into any Y with $p_g = q = 0$ and $\mathrm{Kod}(Y) \geq 1$. There exists a countable number of families $\{W_i\}_{i\in I}$ for all such birational immersions $\kappa\colon C \to Y$. That is for every birational immersion $\kappa\colon C \to Y$ there is $j \in I$, $u \in W_j$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_j & \xrightarrow{\;K_j\;} & \mathcal{Y}_j \\ {\scriptstyle p_j}\searrow & & \swarrow{\scriptstyle \pi_j} \\ & W_j & \end{array} \tag{5}$$

such that $p_j^{-1}(u) = C$, $\pi_j^{-1}(u) = Y$ and $K_j|_C = \kappa$. To see this, first of all we recall that in general it is possible to find a countable number of families that contain all the algebraic (smooth) projective varieties. This follows from the fact that the Hilbert polynomials are countable and that any Hilbert scheme has a finite number of irreducible components. Any smooth algebraic surface can be embedded in $\mathbb{P}^5$, so the countable union of projective schemes (over $\mathbb{C}$)

$$\bigcup_{i\in I} Hilb^{P_i}, \quad P_i(t) \in \mathbb{Q}[t],\ \deg P_i = 2$$

covers all smooth surfaces, where the $P_i(t)$ are Hilbert polynomials in $\mathbb{P}^5$. Similarly, we can regard the Hilbert scheme $Hilb^Q$ where $Q(t) = d_1d_2t - \frac{1}{2}d_1d_2(d_1 + d_2 - 4)$. This contains the complete intersection curves of multidegree (d_1, d_2) in $\mathbb{P}^3$. For shorthand notation, let $HS_i := Hilb^{P_i}$ and $HC := Hilb^Q$. Let $\mathcal{S}_i$ and $\mathcal{C}$ be the universal families of these Hilbert schemes. We consider the Hilbert scheme of morphisms [1, p.47]

$$\mathbf{H}_i := \mathbf{Hom}_{HC\times HS_i}(\mathcal{C} \times HS_i, HC \times \mathcal{S}_i).$$

The above scheme parametrizes the family of morphisms

$$\mathcal{C}_{[C]} \times \{[Y]\} \to \{[C]\} \times (\mathcal{S}_i)_{[Y]}, \quad ([C],[Y]) \in HC \times HS_i.$$

Since taking graphs gives an embedding of $\mathbf{H}_i$ onto an open subscheme of some Hilbert scheme [1, p.48], $\mathbf{H}_i$ has at most a countable number of irreducible components. Combining all together, we find that there is a countable number of families for all birational immersions $k: C \to Y$. Now, by a Baire's category argument, we may take an irreducible component $W = W_{i_0}$ which dominates the family $\mathcal{M}(d_1,d_2)$ of complete intersection curves of multidegree (d_1,d_2) in $\mathcal{M}_g$, where $g = g(C) = \frac{1}{2}d_1d_2(d_1+d_2-4)+1$ (see [5, Ex. I.7.2 (d)]). It follows that

$$\dim W \geq \dim \mathcal{M}(d_1,d_2).$$

The number $n := \dim W$ is bounded by the number $n'+m'$ where n' is the dimension of the vector space of first order infinitesimal deformations of the morphism $\kappa\colon C \to Y$ with Y fixed, and $m' = h^1(T_Y)$ the dimension of the vector space of first order infinitesimal deformations of Y. If we can prove $n'+m' < \dim \mathcal{M}(d_1,d_2)$ then we get the contradiction

$$\dim \mathcal{M}(d_1,d_2) \leq n \leq n'+m' < \dim \mathcal{M}(d_1,d_2).$$

Proposition 3.1. *We have*

$$\dim \mathcal{M}(d_1,d_2) = \begin{cases} \binom{d_1+3}{3} + \binom{d_2+3}{3} - \binom{d_2-d_1+3}{3} - 17 & \text{if } d_2 > d_1 \\ 2\binom{d+3}{3} - 19 & \text{if } d = d_1 = d_2. \end{cases}$$

Proof. We consider the normal exact sequence

$$0 \to \mathcal{T}_C \to \mathcal{T}_{\mathbb{P}^3}\big|_C \to \mathcal{N}_{C/\mathbb{P}^3} \to 0. \tag{6}$$

The desired dimension can be counted by looking at the dimension of

$$\operatorname{Im}(H^0(\mathcal{N}_{C/\mathbb{P}^3}) \to H^1(\mathcal{T}_C))$$

in the long exact sequence

$$0 \to H^0(\mathcal{T}_C) \to H^0(\mathcal{T}_{\mathbb{P}^3}\big|_C) \to H^0(\mathcal{N}_{C/\mathbb{P}^3}) \to H^1(\mathcal{T}_C) \to \dots.$$

Since $\mathcal{N}_{C/\mathbb{P}^3}$ is isomorphic to $\mathcal{O}_C(d_1) \oplus \mathcal{O}_C(d_2)$, the proof is reduced to a standard cohomological computation using the exact sequence defining C

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-d_1-d_2) \to \mathcal{O}_{\mathbb{P}^3}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^3}(-d_2) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_C \to 0. \tag{7}$$

See also [4, p.303]. □

For a given $\kappa\colon C \to Y$, the dimension of the vector space of first order infinitesimal deformations of κ with fixed target is bounded by $h^0(\mathcal{N}_{C/Y})$. Let $\mathcal{N}'_{C/Y} = \mathcal{N}_{C/Y}/(\mathcal{N}_{C/Y})_{\text{tors}}$ be the torsion-free quotient of $\mathcal{N}_{C/Y}$. We can use

$h^0(\mathcal{N}'_{C/Y})$ instead of $h^0(\mathcal{N}_{C/Y})$ to bound n' since for an actual deformation the corresponding section of the normal bundle cannot vanish on a general point. In the short exact sequence

$$0 \to \mathcal{T}_C \to \kappa^*\mathcal{T}_Y \to \mathcal{N}_{C/Y} \to 0, \tag{8}$$

we get $\deg \mathcal{N}_{C/Y} = \deg \kappa^*\mathcal{T}_Y - \deg \mathcal{T}_C = -\deg \kappa^* K_Y + (2g-2)$. Then, $\deg \mathcal{N}'_{C/Y} \leq 2g-2-\deg \kappa^* K_Y$. The Clifford theorem says $h^0(\mathcal{N}'_{C/Y}) \leq g - \frac{1}{2}\deg \kappa^* K_Y$. Thus, we get $n' \leq g - \frac{1}{2}\deg \kappa^* K_Y$.

Lemma 3.2. *Let $f\colon X \dashrightarrow Y$ be a dominant rational map, and let $\kappa\colon C \to Y$ be the restriction of f to the general hyperplane section. Assume that Y is a minimal surface with $p_g(Y) = q(Y) = 0$, $\pi_1(Y) = 1$, and* $\operatorname{Kod}(Y) \geq 0$. *Then,* $\deg_C \kappa^* K_Y \geq d_1 d_2$.

Proof. Let $\phi : Z \to X$ be the resolution of indeterminacy of f, and let $h\colon Z \to Y$ be the morphism which extends f. Let E be the exceptional divisor of $\phi : Z \to X$, H be the hyperplane divisor of X, and let $H_Z = \phi^* H \in \operatorname{Pic} Z$. By Hurwitz formula,

$$K_Z = h^* K_Y + R,$$

where R is the ramification divisor of h. At the same time, $K_Z = \phi^* K_X + E$, thus

$$h^* K_Y + R = \phi^* K_X + E = (d_1 + d_2 - 5) H_Z + E.$$

Since $\operatorname{Pic} Z$ is generated by H_Z and the irreducible components of E, we may write

$$\begin{aligned} h^* K_Y &= r H_Z - W \\ R &= s H_Z + W + E, \end{aligned}$$

where $W = \sum_i a_i E_i$ and E_i are irreducible components of E, r and s are non-negative integers such that $r + s = d_1 + d_2 - 5$. Then as in [4, Lem. 3.1.1, Rmk. 3.1.3], we have $a_i \geq 0$ for all i and $r > 0$. Since C is a general hyperplane section, we may assume that C does not meet exceptional divisors. Hence, $\deg_C \kappa^* K_Y = (C \,.\, \kappa^* K_Y) = (C \,.\, r H_Z) = d_1 d_2 r \geq d_1 d_2$. □

For the sake of completeness, we reproduce the argument of the section 3.1 of [4]:

Lemma 3.3. *The divisor W is effective and $r > 0$.*

Proof. Write $W = A - B$, where A and B are effective divisors with disjoint irreducible components. By hypothesis $h^*K_Y = rH_Z - A + B$ is a nef divisor, then

$$0 \leq B\,.\,h^*K_Y = -B\,.\,A + B^2 \leq B^2,$$

since $H_Z\,.\,B = 0$ and $A\,.\,B \geq 0$. So $B = 0$, because it is contracted by ϕ, hence W is an effective divisor. Moreover

$$deg(f)K_Y^2 = (h^*K_Y)^2 = r^2H_Z^2 + W^2 = r^2d_1d_2 + W^2 \leq r^2d_1d_2.$$

Now, if $K_Y^2 > 0$ (*i.e.* Y is of general type), we have $r > 0$. Otherwise, by contradiction we suppose $r = 0$, and $K_Y^2 = 0$. Hence $W = 0$ and $h^*K_Y = 0$. This implies that

$$h_*h^*K_Y = deg(f)K_Y = 0.$$

By hypothesis Y is simply connected, therefore $\mathrm{Pic}^0(Y)$ is trivial and so $K_Y = 0$. But this gives that $p_g(Y) = 1$, we get a contradiction. □

Now we want to find the region for (d_1,d_2) satisfying $n' + m' < \dim \mathcal{M}(d_1,d_2)$. The following propositions deal with the two cases of Lemma 1.4.

Proposition 3.4. *Let $T_1 \subset \mathbb{Z}\times\mathbb{Z}$ be the set of pairs (d_1,d_2) defined by*

$$\begin{array}{ll} d_1 = 3, & d_2 \geq 3 \\ d_1 = 4, & d_2 \geq 4 \\ d_1 = 5, & 5 \leq d_2 \leq 13 \\ d_1 = 6, & d_2 = 6,7. \end{array}$$

Let C be a very general hyperplane section of the complete intersection surface X of multidegree $(d_1,d_2) \in T_1$. Then, there is no birational immersion $\kappa\colon C \to Y$ for any surface Y of $\mathrm{Kod}(Y) = 1$, *$p_g(Y) = q(Y) = 0$, and $\pi_1(Y) = 1$.*

Proof. By classification of surfaces, Y admits an elliptic fibration $Y \to \mathbb{P}^1$ with exactly two multiple fibers. By [6, Prop. 2.4], the dimension of the Kuranishi space of deformations of Y has dimension at most 10. Thus, $m' \leq 10$, and $n' \leq g(C) - \frac{1}{2}\deg_C \kappa^*K_Y$. If we can show

$$g(C) - \frac{1}{2}\deg_C \kappa^*K_Y + 10 < \mathcal{M}(d_1,d_2) \qquad (\star)$$

then $n' + m' \leq g(C) - \frac{1}{2}\deg_C \kappa^*K_Y + 10 < \mathcal{M}(d_1,d_2)$, hence we get the desired conclusion. Using Proposition 3.1 and Lemma 3.2 we have $g(C) = \frac{1}{2}d_1d_2(d_1 + d_2 - 4) + 1$ and $\deg_C \kappa^*K_Y \geq d_1d_2$, and we get an inequality on d_1 and d_2 which determines T_1. □

Proposition 3.5. *Let $T_2 \subset \mathbb{Z} \times \mathbb{Z}$ be the set of pairs (d_1, d_2) defined by*

$$\begin{array}{ll} d_1 = 3, & d_2 \geq 3 \\ d_1 = 4, & d_2 \geq 4 \\ d_1 = 5, & 5 \leq d_2 \leq 9 \\ d_1 = 6, & d_2 = 6. \end{array}$$

Let C be a very general hyperplane section of the complete intersection surface X of multidegree $(d_1, d_2) \in T_2$. Then, there is no birational immersion $\kappa \colon C \to Y$ for any surface Y of general type with $p_g(Y) = q(Y) = 0$, and $\pi_1(Y) = 1$.

Proof. By [4, Cor. 2.5.3], $m' \leq 19$. As in Proposition 3.4, it is enough to find the pairs (d_1, d_2) which satisfy the following:

$$g(C) - \frac{1}{2}\deg_C \kappa^* K_Y + 19 < \mathcal{M}(d_1, d_2).$$

It is immediate to see that the set T_2 is exactly the collection of such pairs. □

Combining all the elements, we get our main result, that is Theorem 1.2 of the introduction

Theorem 3.6. *Let $(d_1, d_2) \in T := T_1 \cap T_2 = T_2$. Then, for very general complete intersection surface X of multidegree (d_1, d_2) and any dominant rational map $f \colon X \dashrightarrow Y$, either f is birational or Y is rational.*

Comparing to the result [6] of Lee and Pirola, we expected to find a much wider region for T (for instance, region with bounded complement) by using their method. One of the possible ways to improve T is to find a better bound for $\dim W$ which was bounded by $n' = h^0(\mathcal{N}_{C/Y})$ and $m' = h^1(\mathcal{T}_Y)$. Indeed, $h^0(\mathcal{N}_{C/Y})$ counts the dimension of deformations of the birational immersion $\kappa \colon C \to Y$ with fixed Y, but for us it suffices to look at deformations $\kappa' \colon C' \to Y$ with $C' \in \mathcal{M}(d_1, d_2)$ since we derived contradiction by looking at the restriction to hyperplane sections of the birational maps $X \dashrightarrow Y$. The tangent space $L := T_{[C]}\mathcal{M}(d_1, d_2)$ is identified with the image of the Kodaira-Spencer map $H^0(\mathcal{N}_{C/\mathbb{P}^3}) \to H^1(\mathcal{T}_C)$ along the identification $H^1(\mathcal{T}_C) \simeq T_{[C]}\mathcal{M}_g$. Hence, in the long exact sequence induced by (8), we have

$$H^0(\mathcal{N}_{C/Y}) \xrightarrow{\alpha} H^1(\mathcal{T}_C),$$

and the subgroup $\alpha^{-1}L \subset H^0(\mathcal{N}_{C/Y})$ parametrizes the deformations of the birational immersions $\kappa' \colon C' \to Y$ where C' is lying inside $\mathcal{M}(d_1, d_2)$. If we could find an efficient bound for $\alpha^{-1}L$, we would be able to replace the inequalities in the proofs of Propositions 3.4 and 3.5 to enlarge the area of T. Unfortunately so far, we have no idea how to evaluate the dimension of $\alpha^{-1}L$. Furthermore, even if we have a meaningful difference between dimensions of $\mathcal{M}(d_1,d_2)$ and $\mathcal{M}_g$, we need an upper bound of $\alpha^{-1}L \subset H^0(\mathcal{N}_{C/Y})$ for all possible surfaces Y with $\mathrm{Kod}\,Y \geq 1$, so the global bound for $\alpha^{-1}L$ may not be very useful to enlarge T. In this sense, it might be necessary to find another way to improve the result.

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