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GENERALIZATION AND CONCEPTUALIZATION IN A STEAM TEACHING LEARNING SEQUENCE FOR PRIMARY SCHOOL ABOUT AXIAL SYMMETRY

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In interdisciplinary teaching, students' attitude to generalize mathematical knowledge to new contexts of application is encouraged naturally. Moreover, it fosters the development of creativity and critical thinking. In our research, we focused on integration of Mathematics and Arts in primary school. We designed and tested a Teaching Learning Sequence about axial symmetry, to develop mathematical skills through the execution of artistic techniques and reflections on products and actions carried out. In this paper, we present tasks and results about students' mathematical activity obtained analyzing classroom implementations with children in 4th and 5th grade in Italy. The generalization processes make interesting information about their conceptualization and schemes application and validation emerge.

INTRODUCTION

When students, by themselves or guided by teachers, search for new situations and contexts in which applying and revising their mathematical knowledge, they develop successfully key aspects of mathematical thinking, like problem solving and generalization; design research should "offer teachers an empirically grounded theory on how a certain set of instructional activities can work." (Gravemeijer, 2004). In interdisciplinary tasks, students' attitude to generalize mathematical knowledge to new contexts of application is encouraged naturally. The European Union recently published recommendations (EU Council, 2018) to integrate all areas of the scientific disciplines with their applications in technology and engineering, and with artistic expressions (STEAM). Benefits would be, for example, the positive effects of art in interaction with different disciplines, including mathematics, from the affective and motivational point of view. Moreover, it fosters the development of creativity and critical thinking (ibid., 2018). Among the several possibilities to pursue such goals, we focused on integration of Mathematics and Arts in primary school. We decided to design and test a Teaching Learning Sequences (TLS, Psillos & Kariotoglou, 2016) about axial symmetry, where students were asked to "reinvent" mathematical concepts (Gravemeijer, 2004) and develop mathematical skills through the execution of artistic techniques and reflections on products and actions carried out.

In this work, we present some tasks of this interdisciplinary TLS and some results we obtained analyzing classroom implementations. We focus particularly on the students' mathematical activity. We worked with children in 4th and 5th grade in different schools across Italy. We collected data through video and audio recordings,

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observation and materials produced during the lessons by the students. Results show that encouraging students to generalize make interesting information about their conceptualization emerge. Moreover, we show how linguistic practices and discussions are important to self-realize this generalization and conceptualization mechanism.

LITERATURE REVIEW

In many research, it has been shown that learning axial symmetry is not trivial for primary school students. First of all, the term "symmetry" might be used in different ways (Chesnais, 2012): (a) symmetry as a property of a given figure; (b) axial symmetry as a ternary relationship involving two figures and an axis and/or (c) symmetry as geometrical transformation involving points.

Moreover, axial symmetry is a mathematical concept but also an everyday concept (ibid., 2012). From a mathematical point of view, the geometrical transformation comes before symmetry as a property, being the property a result of the invariance of the figure under the transformation. On the other hand, in the everyday concept, the geometrical transformation could be seen only in the paper folding movement. If not expanded upon, it can lead to the main misconceptions about symmetry, that can be an obstacle to global characterization of the properties of a figure and of the geometrical transformation of the plane (ibid., 2012). It is possible for the teachers not to see these conceptions, since students will continue to produce results as constructing the mirror image of a figure or identifying axes of symmetry on a single simple figure. In general, students are more confident with tasks that require an *intrafigural* perspective (Piaget & Garcia, 1989), where attention is directed towards the internal relationships of figures, than with tasks involving *interfigural* demands requiring attention to the relationships between the figures and objects that are external to them (Healy, 2004). Relying on this review, we decided to orient the students' activity gradually towards the construction of the mathematical concept and an interfigural approach, encouraging them, by means of generalization and verbalization tasks, to reframe the everyday characterization of the axial symmetry.

THEORETICAL FRAMEWORK

In this study, we refer to generalization as *the process of applying a given argument in a broader context* (Harel & Tall, 1991). Generalization is classified as *expansive generalization* when the subject expands the applicability range of an existing scheme without reconstructing it; *reconstructive generalization* when the subject reconstructs a scheme to widen its applicability range (ibid., 1991). A common trait is the need to change the applicability range of a given concept, extending it to a broader concept. In reconstructive generalization, the old scheme is changed and extended, to be embedded in a more general scheme, that still "contains", or is a generalization of, the first schema.

According to Vygotsky (2012), concepts can be spontaneous or scientific, where the former are the result of a generalization process of everyday personal experience. Considering our tasks and our target grade, we refer essentially to the Theory of Conceptual Fields (Vergnaud, 1998) to frame the notions of concept and scheme.

According to Vergnaud (1998; 2013), mathematical knowledge is centered and constructed around a *concept*; a concept results from a process of actions and perceptions. Concept is constituted by three components: the set of situations the concept is rooted in and has meaning on, a set of operational invariants and the set of different linguistic and non-linguistic representations used to represent it.

A scheme (Vergnaud, 2013) is defined as "invariant organization of activity and behavior for a certain class of situations" (p. 47); to tackle new situations extend the scope of application of the scheme. It is made of four categories of components: goals and anticipation, a set of rules of action, operational invariants and possibilities of inferences. Operational invariants, which make the scheme operate and often remain implicit, can be of two kinds: theorems-in-action and concepts-in-action (ibid., 2013). They can be expressed by words and sentences, but their original function is action and the application of schemes is based on them.

METHODOLOGY

We designed the TLS following these principles: a growing challenge level; to foster generalization (in the meaning given by Harel and Tall (1991), to promote linguistic practices that can be meaningful to connect the different activities and to build up to a gradual conceptualization (in the sense of Vergnaud's Theory of Conceptual Fields, 1998; 2013), developing a more precise language and promoting argumentation.

In the first two tasks, students met the first two situations:

Task (1), artistic symmetry: folding the paper with colors, a "similar" figure is obtained (same shape, same, area, same colors).

Task (2), modelling the art: doing "the same things" on the left and on the right, at the same height and the same distance with respect to a line, a figure is obtained that resembles the figure obtained by folding.

We told the students that the line obtained folding and the line drawn in the second situation were both called 'axes of symmetry', that the figure obtained by folding was 'the symmetric figure' with respect to the starting one and that the whole 'figure is symmetric'. Thus, we introduced some terms and the relationships between different elements of a conceptual field named 'symmetry'.

Task (3), TEP: "explain to a younger student how it is possible to build a symmetrical figure with respect to another figure".

Here students are asked to produce a textual *eigenproduction* (TEP, D'Amore & Maier, 2002), i.e. texts produced by students in an autonomous way to describe some mathematical situation. The goal of TEPs is that of better understanding and exploring the true conceptualization of the student. We expected the students to find linguistic and/or not linguistic representations of their concepts and to start making explicit their actions that they should then organize to make them become schemes.

Task (4), square: "find, by folding, the axes of symmetry of a square".

Students are expected to generate a first version of their concept of axis of symmetry including: three situations (1, 2 and 4), an operational invariant (*concept in action*: if, folding, the two parts are overlapping exactly, the fold represents an axis of symmetry) and graphic and linguistic representations of the axis. Meanwhile, since they have to solve a new task, they are also asked to start generalizing their previous actions to a scheme, composed by: one goal (to find axes of symmetry), the rules of action (correct procedure to build a fold that is an axis and a control procedure to check if it is an axis or not), an operational invariant (*concept in action* of axis of symmetry), a set of possibility of inference (conditions to carry out the procedure: possibility to fold the paper, possibility to check if the pieces of the figure have the same features).

Task (5) star: "find, by folding, the axes of symmetry of a regular 5-pointed star".

Students are expected to enrich their previous concept, including another situation and to reinforce the previous scheme. Students are expected (and encouraged) to use their linguistic characterization of the *concept in action*, on which the scheme should be based (task 3), to validate their actions in the different situations (4 and 5).

Task (6), snowflake: "build, as you want, this snowflake" (see Figure 1).

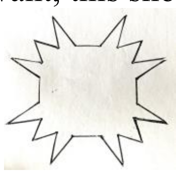

Figure 1: The snowflake to build from a blank sheet of Task 6.

Students are expected to recognize that the figure is symmetric, what are the axes of symmetry, and to decide to exploit this property to build the figure without retracing it, folding a sheet of paper (scheme 1) and/or using the distances from the axes (scheme 2). To do this, the students should: study the situation in terms of possibilities of inference; recognize the same goals of Task 4 and 5 (to find axes of symmetry) even if it is not mentioned in the description of the task; carry out a set of rules to identify the correct folds. Only after the application of the scheme, the students should draw the starting figure, reproducing it symmetrically, to have the most correct result.

Our research questions are:

1. How do the students face spontaneously tasks in which a concept is expected to be applied in a new situation? What kind of information can the observation of a process of generalization give about the students' conceptualization?
2. Whether and how the verbalization tasks and the classroom discussions lead the students to a refinement or a generalization of their personal concepts?

Context and participants

The TLS was implemented in classes of students 8 to 10 years old (two 4th grade and four 5th grade classes of primary school) as part of an in-service teacher training lasting one semester. Class context and formation are variable both geographically through the country and in terms of background of the students. The class teacher acted as main teacher for the TLS; one or more of the authors planned the lesson with the teachers involved, collected data about the students, assisted and helped, intervening occasionally, during all teaching blocks.

Data collection and analysis

The explorative nature of the study led us to use qualitative techniques for data collection towards an interpretative approach. The research data were collected over several sessions at school and consists of (1) audio and video recordings, (2) documents review, (3) researchers' field notes and (4) students' textual productions (TEPs, D'Amore & Meier, 2002).

In particular, (1) videos were analyzed by more researchers and transcripts were finally used as data which we present here. Video analysis (Powell et al., 2003) has been done in more phases: a first review of the videos, cataloguing their content and annotating some particular episodes; a deeper analysis with transcription of some episodes, that were flagged as occurring of generalization; connection of single episodes to consider the overall development of the students' conceptualization. Focus was, as said, on the understanding of the students' conceptualization of axial symmetry, analyzing data inside Vergnaud's Theory of Conceptual Fields (1998; 2013) and with an eye on the generalization processes that took place (Harel & Tall, 1991).

RESULTS

In relation to our first research question, we observed, in the majority cases, in the tasks from Task 3 to Task 6, spontaneous application of previous knowledge to the new situations they are facing. However, is the procedure always correct? Re-applying the spontaneous concepts (in this sense, generalizing; Vygotsky, 2012) can lead the students to different situations. A spontaneous expansive generalization process can be correct but still lead to some non-correct conclusion, due to a concept in action that is either incomplete, and therefore not extendable to other cases without adding other conditions, or valid only in some situations, thus becoming not correct when the related scheme is applied to a new range of situations. Examples can be seen in Table 1.

We can observe that one of the main risks here is that students go on with what they think is a good property (concept in action), and apply it in a range where it will not work without realizing it will not actually be valid. However, without asking students questions that encourage them to apply their schemes in a new situation, these incomplete or situated concepts would not be identified and revised by the students.

From the video analysis, we could pinpoint also different cases in which correct generalization occurs, both expansive and reconstructive. Some students connect the two schemes, performing in this way a sort of reconstructive generalization. Viola and Andrea, for instance, in Task 6, overlapping the drawing with a folding, realize that "sides cannot be longer or shorter, they need to have the same measures!", connecting the two schemes and reconstructing

Scheme (2), which allows them to re-describe the concepts in action of the paper folding Scheme (1) in terms of measures and distances.

Initial concept		*Situation / Concept*	*Examples of students' sentences/indicators*	*What happens when re-applying the concept*
Incomplete concept	**I1**	Folding the paper	"Axes of Symmetry are lines" (also "zigzag" lines) "Symmetry is just folding the paper"	Students identify every fold/line, or every line dividing the figure in two parts with the same area, with an axis of symmetry. The right answers based on this incomplete concept, are true but partial. There is a need for a strengthening of the *concept in action*.
	I2	Two parts with the same area	"a line that divides the paper in two halves with the same area"	
Concept valid in some situation but that becomes not correct if changing the applicability range	**S1**	Axis has to be vertical	"the axis of symmetry is a vertical line dividing the figure in two parts"	Students apply the concept they inferred from a particular example, but it is not working when changing the setting. More difficult to correct, there is a need to revise the *concept in action*, removing some features of the line (S1) or referring the concept to a given figure (S2).
	S2	Axes not related to the figure	when "finding all the axes of symmetry of a figure", students iterate the procedure, with the new figures obtained by folding the first one.	

Table 1. Examples of data analysis

Expansive generalization occurs in many more cases, in all tasks: Task 3 – Task 6, i.e. students keep one scheme they built, always applying the same to a new situation and expanding it, without seeing the connection between folding and overlapping on one hand, lengths and measures on the other hand. This is for example the case of Dora, who generalizes in every situation her scheme about symmetry as folding (1), even when it was easier to use Scheme 2, and never compare the two.

On some occasions, the attempt to generalize the concept will first lead to a non-correct conclusion in a broader situation, but it can also help realize the mistake and therefore adjust the concept and definition the students are trying to identify. For example, as in the transcript below, after an I2 occurring, Elin and then Sara realize there is something not working with their previously discussed definition of axis of symmetry as "a line that divides the paper in two halves with the same area" (Andrea I2 misconception).

Teacher: Why is the diagonal of the square an axis of symmetry?

Andrea: Because it is a line that divides the paper in two halves that are the same, the quantity is the same. […]

Teacher: So, if I do this, *showing a square that is folded in two parts with the same area, but where the fold is not an axis of symmetry*, I fold the square and obtain two pieces with the same area, are they the same? Is this fold representing an axis of symmetry?

Class: Yes! No! Yes!

Teacher: Why is it or why not? Please try to provide some arguments.

Michael: Yes, because there is a line, anyways… [I1 misconception]

Andrea: It works because there is the same half [on both sides – I2 misconception]

Elin: I say no, because…because the figure is rotated. It is the same half on both sides, but one goes up and the other goes down… the same figure is turned one facing up and the other facing down […]

Sara: I say no, because…so, it looks like it is, because it forms a line that divides the sheet into two parts that are equal. But in my opinion, it is not an axis of symmetry because…it should have been like this" indicates the diagonal folding with the hands […]

James: "the angles are not corresponding…"

Sara: Ok, if I try again with the colors experiment and fold the paper it will not work. If I do once more the thing with the thread, it could not work on the other side. The two sides are different [they will not overlap]".

During the discussion, students realize their starting point was correct only if applied to the initial problem of a rectangle divided in two parts, but also that not all lines, even if dividing the figure in two equal parts with the same area, are axes of symmetry for a figure. Therefore, the discussion led to an enrichment of the concept, reconstructed to be adapted to the new situation.

DISCUSSION AND CONCLUSION

We observed that students facing tasks in which a concept is expected to be applied in a new situation re-apply their previous schemes and *concepts in action* to the new situation. While this spontaneous generalization inclination does not surprise, as it seems to be in fact natural in the students, it is interesting to observe the complete process students are undertaking, to get information about their conceptualization. The kind of tasks proposed are revealing students' misconceptions (as in Table 1), which cannot always be observed with standard "textbook exercises" and which cannot be identified by the class teachers themselves, who were surprised by this discovery during the implementations.

While re-applying schemes is a spontaneous process, the same cannot be said of the processes of evaluation of the consistency between the *concept in action* and the linguistic representations and the control of the rules applied in the new situation.

With an appropriate mediation by the teachers and encouraging discussion with peers and argumentation, the lack of a proper control or validation structure for the generalization process can be identified. Properly guided by the teacher, students can understand that their set of rules might not be applicable to every situation and revise their *concept in action* and scheme to adapt them to the new situations. In Task 3 and Task 6 students are encouraged to connect two schemes based on two different *concepts in action* and to carry out a reconstructive generalization by means of a verbalization task and a problem- solving activity. While in the first task this process of generalization never occurs, we observed it in the problem-solving activity, and other students did it during the discussion about their solutions, using one Scheme (2) to check the validity of the procedure carried out with the other Scheme (1).

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