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Supergravity black hole and brane solutions in various dimensions

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PhD thesis of:
Federico Michele Faedo

Director of the school: Prof. Matteo Paris
Supervisor of the thesis: Prof. Silke Klemm

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Final examination commission:

Prof. Silke Klemm, *Università degli Studi di Milano*

Prof. Yolanda Lozano, *Universidad de Oviedo*

Prof. Mario Trigiante, *Politecnico di Torino*

PhD thesis referees:

Prof. Patrick Meessen, *Universidad de Oviedo*

Dr. Lucrezia Ravera, *Politecnico di Torino*

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A mio padre
per il suo costante supporto

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Introduction

> After dinner, the weather being warm, we went into the garden and drank thea under the shade of some appletrees, only he and myself. Amidst other discourse, he told me he was just in the same situation, as when formerly the notion of gravitation came into his mind. “Why should that apple always descend perpendicularly to the ground,” thought he to himself: occasion’d by the fall of an apple, as he sat in a contemplative mood: “why should it not go sideways, or upwards? But constantly to the Earth’s centre? Assuredly, the reason is that the Earth draws it. There must be a drawing power in matter. And the sum of the drawing power in the matter of the Earth must be in the Earth’s center, not in any side of the Earth. Therefore dos this apple fall perpendicularly, or toward the center. If matter thus draws matter; it must be in proportion of its quantity. Therefore the apple draws the Earth, as well as the Earth draws the apple.”

William Stukeley, *Memoirs of Sir Isaac Newton’s life*, 1752

Once upon a time, in Cambridge, there lived a young scientist who was forced, by a deadly plague that was devastating London and the surrounding areas, to return to its birthplace and spend there one year of his life in total isolation. His name was Isaac Newton and in that period, they say the summer of 1666, he formulated the *law of universal gravitation*[1].

General relativity

Many years, or rather centuries, passed before Albert Einstein’s *general relativity (GR)*, a complete breakthrough in the conception of gravity. Einstein’s theory brought a paradigm shift as gravity was not considered anymore as an authentic force, but rather as the result of geometric effects: spacetime is curved by sources of energy (matter, electromagnetic fields, etc.) and all the bodies feel these sources through the bending of the geodesics they move along. Two consequences immediately follow: not only matter can deflect the motion of objects and not only massive bodies can be deflected. So far, GR has been widely tested and a huge number of experiments and observations confirmed its reliability, from the perihelion precession of Mercury’s orbit and the bending of light, to the detection of gravitational waves and the first “photograph” of a black hole (M87*) ever taken. Nevertheless, some issues still affect this well-established theory, mainly its non-renormalizability, the presence of singularities and the obscure interpretation of black hole entropy.

The first problem can be stated as follows: GR coupling constant, i.e. the gravitational constant, is not dimensionless, but rather $[G_{\mathrm{N}}] = \mathrm{E}^{-2}$, hence this theory is non-renormalizable by simple power counting. Indeed, any perturbation expansion would involve higher and higher

[1] . . . even though Robert Hooke made a claim that Newton had obtained the inverse square law from him.

powers of the Riemann curvature tensor and therefore infinitely many counterterms to reabsorb the divergencies appearing in the graviton scattering Feynman diagrams. Despite not being trustworthy above a certain threshold, GR still remains predictive at low energies, or, put in other words, it is an effective theory in search for an UV completion.

A second open question is how to deal with divergencies. In general relativity singularities occur almost everywhere, from the center of a black hole to the very origin of our universe, and like everything is believed to be born from the Big Bang singularity, everything trapped inside a (physical) black hole is believed to die reaching its inner singularity. Their appearance is commonly perceived as a signal of the breakdown of the theory and a hint that a quantum theory of gravity is crucial for its consistent completion. Indeed, in the neighbourhood of these points curvature and density become extremely high, therefore quantum effects start to be more and more important and a theory of *quantum gravity* is needed to take into account these contributions. It is generally expected that such a theory will not feature any singularity.

The last point concerns black holes and their "thermodynamics". In 1972 Jacob Bekenstein conjectured that black holes are characterized by an entropy, a hypothesis made stronger by the subsequent works of Stephen Hawking that assigned also a temperature to these objects based on their surface gravity. The result is the well-known *Bekenstein-Hawking entropy* $S_{\rm BH} = A_{b.h.}/4$, which teaches us that the black hole entropy is proportional to the area of the event horizon. While on the one hand it represents a striking insight, on the other three big issues arise. From a statistical mechanics point of view, a black hole is much like a macroscopic system, it is described by a small set of (macroscopic) variables, therefore its entropy can be interpreted as a measure of the number of possible microscopic arrangements for the given macroscopic state. Only the knowledge of the inner structure of a black hole could tell us what these microscopic arrangements are. Moreover, we are used to systems whose entropy scales as the volume, whereas for black holes it is proportional to the area, as if all the (microscopic) degrees of freedom lived on the horizon. This behaviour suggests the possible existence of an underlying holographic principle, according to which all the information contained in a given region, in this case the black hole, is encoded entirely on its boundary, the event horizon. A different problem arises considering particles falling inside a black hole, since their evolution does not appear to be unitarity, at odds with quantum mechanics. All the information carried by particles seems to be lost once they cross the event horizon and this loss appears permanent after the black hole evaporation due to Hawking radiation; because of it, this phenomenon is called *black hole information paradox.*

The way towards unification

Only a few years had passed since the conception of general relativity when another revolution took place in the world of theoretical physics: it was the 1920s and quantum mechanics was coming into being. In the following years this theory enjoyed remarkable progress, incorporating consistently special relativity and giving birth to *quantum field theory (QFT)*, the foundations on which a completely new formulation of electromagnetism was constructed. Taking a step back in time, between the unification of gravity on Earth with planetary motion performed by Newton and the unification of space with time and mass with energy realized by Einstein's special relativity, the unification of electric and magnetic phenomena together with light was achieved by James C. Maxwell. Combining electromagnetism with quantum mechanics seemed a huge aim when the latter was still at its very first stage, but it was ultimately achieved with the formulation of *quantum electrodynamics (QED).* This effort required the introduction of new tools, like Feynman diagrams and renormalization, that we already mentioned, but served also as a propellant and a template for many of the following quantum theories. In the 1960s QED was unified with weak nuclear force into electroweak interaction by Glashow, Salam and Weinberg, and this, in turn, was combined together with *quantum chromodynamics (QCD)*, formulated to describe the strong interaction between quarks, inside the *Standard Model (SM)* in the 1970s. All these theories have been successfully tested and are now supported by several pieces of evidence, like the extremely accurate value of the anomalous magnetic moment of the electron $g_e - 2$ produced by QED, the prediction of the W and Z bosons provided by the electroweak theory and the existence of top and charm quarks conjectured by the SM. A fundamental piece was added in 2012 with the detection of the Higgs boson at the LHC, the last particle that was still evading observations.

However, the picture is not yet complete. The first issue affecting the SM is a particular example of what is generally called *hierarchy problem.* Due to renormalization effects, the mass of the Higgs boson undergoes quantum corrections of the order of the cut-off energy ($\delta m_{\mathrm{H}}^2 \sim \Lambda_{GUT}^2$), which is at least twelve orders of magnitude higher: in order to have such a small mass coming from such huge contributions we need incredibly fine-tuned cancellations. Secondly, this theory contains many unrelated and totally arbitrary parameters, like particle masses and three coupling constants. Why there are so many parameters and the reason for their values are still subjects of scientific debate, but perhaps some masses could have a common origin and likewise the coupling constants. In the same spirit, we cannot explain why matter is organized in exactly three families. A third big open problem concerns cosmology. Observations spanning the last twenty years—among which the cosmological redshift of supernovae and CMB measurements—concluded that the universe we live in is expanding and that this expansion is accelerated. Within the framework of GR, this phenomenon can be accounted for by a small, positive cosmological constant, to which a new kind of unknown matter contributes, the *dark matter.* The "dark" epithet comes from the fact that it cannot be detected directly and, unfortunately, the SM is unable to explain its nature.

There is a thread running through the whole process that, starting from charges and magnets, leads to the Standard Model: the idea of unification. With three out of four fundamental forces combined into one single theory—although a complete unification, if possible, has do date not been achieved—the following, natural step is the inclusion of gravity inside this picture. Combining GR with the SM means not only merging all the existing fundamental interactions, but also combining the infinitely big with the infinitely small, the motion of planets with the motion of particles, bringing quantization inside gravity and, conversely, bringing spacetime curvature inside the quantum world.

Supersymmetry and supergravity

The goals achieved along the path towards unification not only show how this direction could be the most appropriate in order to construct theories able to explain observations and make predictions, but also display the power of using symmetries as an organizing principle. Indeed, the structure of the Standard Model is dictated by symmetry principles, namely by the group $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$. With this information in hand and remembering that the SM is invariant under the action of the Poincaré group, one could search for a bigger symmetry group containing both the group of internal symmetries, namely G, and the Poincaré group in a non-trivial way, i.e. containing $G \times \mathrm{ISO}(D-1,1)$ as a subgroup, with the two factors not commuting. This idea had to collide with reality in 1967, when Coleman and Mandula proved that it is not possible to construct non-trivial theories with tensorial conserved charges other than the Poincaré generators [1].

As a way out to evade the Coleman-Mandula theorem, the concept of Lie algebra was generalized and the Poincaré group was extended to the so-called *super-Poincaré group* in order to accommodate new spinorial (anticommuting) generators, the *supercharges.* These generators carrying spinorial indices give rise to a totally new kind of symmetry connecting bosons and fermions, known as *supersymmetry (SUSY).* Matter is organized in groups of bosons and fermions, called *supermultiplets*, according to the irrep of the super-Poincaré group they belong to. In each supermultiplet there must be the same amount of bosonic and fermionic degrees of freedom and all these states have the same mass and are related by supersymmetry transformations. An unexpected feature of this mathematical construction is the *R-symmetry*, a subgroup of the internal symmetries acting also as an automorphism on the supercharges.

In the years following its formulation, supersymmetry was applied in the context of particle physics to construct supersymmetric extensions of the SM, like the *Minimal Supersymmetric Standard Model (MSSM)*. The inclusion of supersymmetry made it possible to resolve, or at least stem, some of the problems affecting the SM. For example, supersymmetry protects the Higgs mass from the most severe renormalization corrections, making its value less fine-tuned, and some of the SUSY partners, specifically the so-called lightest superparticles, are candidates for dark matter. As we mentioned, supersymmetry predicts that every particle and its superpartner have the same mass, but so far none of these equal-mass pairs has been observed. Therefore, supersymmetry must be broken in nature, but the mechanism through which SUSY should be broken is still a matter of discussion—although it is strongly believed that a spontaneous symmetry breaking process is involved. And a matter of discussion is also the existence itself of these superparticles, since LHC and other particle accelerators have ruled them out from a wide range of masses up to a threshold.

Another key element in the construction of the SM is the role played by *gauge symmetries*. Put in simple words, we talk about gauge symmetries when we take a (global) symmetry of a theory and make it local by requiring the parameter of the transformation to depend on the spacetime coordinates. Like in the simplest example of QED, whose gauge group is U(1), this process introduces interactions in a natural way. In the case of the Standard Model, the aforementioned $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ group is precisely the gauge group and, in some sense, also general relativity can be seen as a gauge theory with the Poincaré group as gauge group (see, e.g., [2]), although with some caveats.

Given the results delivered by gauge theories, it seems reasonable to try to apply the same argument to supersymmetry. One of the most important properties of supersymmetry is that the supercharges (Q) are closely related to the generators of translations (P), schematically $\{Q, Q\} \sim P$, hence local SUSY transformations imply local translations, i.e. diffeomorphisms. A locally supersymmetric theory requires invariance under diffeomorphisms, which, in turn, requires the GR formalism and gravity. The result is *supergravity*, a theory of general relativity with local SUSY and in which fermions, that account for the vast majority of the matter known at the moment, are naturally and inevitably incorporated. The first example was constructed in 1976 by Freedman, Ferrara and van Nieuwenhuizen and consists of a 4-dimensional gravity theory including one generator of SUSY [3].

Proceeding in the direction established by gauge symmetries, the further step that one could take is to gauge the group of internal symmetries, or just a subgroup of it, giving rise to what is known as *gauged supergravity*. The structure of a gauged supergravity is much richer compared to the ungauged case, and likewise its set of solutions, known or still to be constructed. For a given ungauged theory there are many symmetries or subsymmetries that can be gauged and many ways to gauge them and the scalar potential introduced by the gauging procedure allows for, and enforces, the existence of non-trivial scalar fields. The most studied framework is 4-dimensional gauged supergravity with one, two or eight supercharges and where the gauging involves only part of the R-symmetry, but many examples with different gauging or in five or more dimensions exist as well.

String theory

Approximately in the same years when supergravity was developed, another theory started to attract the attention of a group of theoretical physicists. Born in the 1960s to describe strong interactions, *(bosonic) string theory* is grounded in the idea that observed particles are nothing but the vibrational modes of more fundamental 1-dimensional objects named strings. In order to properly describe fermionic matter, supersymmetry was consistently incorporated, giving rise to the *superstring theory*. Quite remarkably, as it was discovered later on, this allows to solve one of the worst inconsistencies of bosonic strings, the presence in their spectrum of tachyons, i.e. particles with negative squared mass [4].

The scientific community lost interest in string theory in the mid-1970s, when it was replaced by the aforementioned QCD in the description of the strong force, but, nevertheless, this juncture did not mark its end. Indeed, string theory was still considered appealing because in its spectrum was found a massless spin-2 particle, the *graviton*, i.e. the quantum of gravity, or, put in other words, an elementary particle mediator of the gravitational interaction. Furthermore, not only string theory is a quantum theory that includes the graviton, but it also reduces to supergravity in the low-energy limit and is consistent and renormalizable in the opposite regime, all properties that a good theory of quantum gravity should have[2]. Lastly, it was realized that string theory allows for gauge groups big enough to contain the SM gauge group and its interactions, making it a promising candidate as a *theory of everything*.

When trying to make contact with the world we observe, the first obstacle that one encounters are the ten spacetime dimensions in which string theory must be framed in order to avoid the presence of *ghosts*, i.e. states with negative norm. Indeed, going to a weakly-coupled and low-energy regime leads to 10d supergravity, far from the four dimensions we are used to. A way to circumvent this problem had been known from the 1920s and goes under the name of *dimensional reduction*. Compactifying some of the spatial dimensions on tori, Calabi-Yau manifolds or spheres it is possible to obtain consistent lower-dimensional theories and connect string theory with the many, widely investigated gauged and ungauged supergravities.

In the mid-1990s subsequent developments, going under the name of second superstring revolution, brought a deeper understanding in the theory of superstrings. Theoretical physicists realized that all the five existing string theories are related by a web of dualities connecting all of them, possibly in different regimes, and that, in the end, they are different realizations, in the weakly-coupled limit, of one single 11-dimensional theory called *M-theory* [5]. The latter is believed to be the UV completion of the 11d supergravity constructed years before, which is unique too. Moreover, extended dynamical objects were discovered in the spectrum of superstrings [6] and it was shown how specific bound states of these objects may give rise to the black hole solutions known for decades. This fact makes it possible to give a microscopic interpretation to the macroscopic entropy of black holes. Indeed, Strominger and Vafa were the first to match the entropy of a class of 5-dimensional extremal black holes with the degeneracy of the associated bound states [7]. Within this context, *BPS states* play a crucial role thanks to the number of non-renormalization theorems that protect certain quantities, thus allowing to perform some computations perturbatively and to extend the results to the strongly-coupled regime as well. The last breakthrough that happened in string theory is the formulation of the *AdS/CFT correspondence*, a conjectured correlation between the dynamics on AdS spaces and the CFTs that can be constructed on the boundary of the latter [8]. This correlation is a possible realization of the holographic principle as well as an example of gauge/gravity duality

[2] The mild behaviour at high energies is due to the non-local nature of string interactions, caused by the fact that strings are extended objects. In the Feynman diagrams associated to string scatterings there are no point-like vertices, but rather "tubes meeting on smooth surfaces", and this ensures that at each order of the perturbation theory, i.e. at each loop, there are no ultraviolet divergences.

and might arise as a special limit of a more profound, still unknown open/closed strings correspondence. The power of the AdS/CFT correspondence results from its nature of strong/weak duality, providing a non-perturbative formulation of string theory and a powerful toolkit for studying strongly-coupled QFTs. Additionally, it seems to be able to resolve, at least in some special cases, the black hole information paradox.

Outline

This thesis is organized with a first introductory part, inspired by [9], in which we present the main concepts that will be required in order to follow the second, more technical one. Chapter 1 is devoted to string theory, with an overview of the possible five different superstrings and the dualities connecting them, the introduction of some non-perturbative extended objects that populate the string spectrum and a sketch of the aforementioned M-theory. In chapter 2 we move to the weakly-coupled low-energy limit of string and M-theory, reviewing the unique 11d and the different 10d supergravities that arise. Moreover, we give a flavour of the different dimensional reduction techniques that enable to relate 11- and 10-dimensional supergravities to lower-dimensional gauged supergravity theories. In chapter 3 we discuss 4- and 5-dimensional $\mathcal{N} = 2$ gauged supergravities, starting with their matter content and the structure of their moduli space and ending with the gauging procedure and the explicit Lagrangians and equations of motion.

The second part constitutes the core of the dissertation. Chapter 4 is focused on 4d gauged supergravity coupled to vector multiplets, specifically the $\overline{\mathbb{C}\mathrm{P}}^n$ and t^3 models, and the construction of black hole solutions: in both cases BPS solutions are built and in the former also a non-extremal extension is presented. Homogeneous 5d black holes represent the main topic of chapter 5. Here we consider vector-coupled 5d gauged supergravity and backgrounds with Sol-invariant spatial cross-sections, construct a magnetic black hole in pure gauged supergravity and search for BPS and attractor solutions in the general case. Chapters 6 and 7 both deal with 10d supergravity, although tackling very different problems. In chapter 6 we study how defect CFTs can emerge in the context of 11d and type-IIA supergravity. Starting from some specific brane configurations, we analyze the near-horizon backgrounds generated, which turn out to possess an AdS_3 factor; by means of a suitable change of coordinates it is possible to reconstruct higher-dimensional AdS spaces, proving the existence of higher-dimensional CFTs broken to 2d defect CFTs by the presence of some sort of "defect branes". In chapter 7 we discuss the inclusion of α' corrections to heterotic supergravity and their effect on the conserved charges of four- and three-charge black holes. The result is an α'-exact shift to the zeroth-order charges or, alternatively, to the near-horizon charges, which are not modified by higher-curvature terms.

Appendix A includes some additional material on spinors and supergravity theories, appendix B exhibits, as an example, the realization of a 4-dimensional black hole within string theory, while in appendix C we extend the new class of solutions of type-IIA supergravity constructed in subsection 6.2.1.

Chapter 1

A brief introduction to string theory

String theory is rooted in the studies of the late sixties focused on the research of a consistent description of strong nuclear interactions. Despite the failure in this direction, made definitive by the development of QCD, the discovery in the spectrum of string theory of a massless spin-2 particle, that could play the role of mediator of the gravitational interaction, gave new fuel to this line of investigation. The core idea was to represent particles as quantized excited states of vibrating 1-dimensional objects, the *strings*, and the presence of the graviton among these particles naturally brought gravity inside the framework, a clue for a possible quantum description of gravity.

In this chapter we shall give a general idea of the main characteristics and properties of string theory, starting with the bosonic string in section 1.1 and introducing supersymmetry in the following section. Subsequently, in section 1.3, we discuss the problems that arise quantizing the theory and classify the five existing string theories. Section 1.4 is devoted to the methods available to explore the non-perturbative regime of string theory. Specifically, it focuses on string dualities, in subsection 1.4.2, branes and other non-perturbative objects, in 1.4.1, and M-theory, the main topic of subsection 1.4.3. This chapter is based on the books of Green, Schwarz and Witten [10], Becker, Becker and Schwarz [11] and Ortín [12] and on the lectures written by Tong [13], great resources to which we refer for a more thorough discussion.

1.1 The bosonic string

Like for any particle, the motion of a string can be consistently studied introducing its action. In analogy with classical particles moving in a 3-dimensional space along trajectories parametrized by the time coordinate, we describe a string as a $(1+1)$-dimensional surface, the *worldsheet*, embedded in a D-dimensional background spacetime. The worldsheet is parametrized by the coordinates $\sigma^\alpha = (\tau, \sigma)$, with $\alpha = 0, 1$, and the embedding is realized by means of the coordinates $X^\mu(\sigma^\alpha)$, with $\mu = 0, \ldots, D-1$. The first action that was historically introduced is the *Nambu-Goto action*, which is proportional to the area of the worldsheet. The Nambu-Goto action presents many difficulties in its quantization, but luckily these problems were circumvented by the introduction of a classically equivalent action, the *Polyakov action*

$$S = -\frac{T}{2} \int_{\text{w.s.}} d^2\sigma \sqrt{-h}\, h^{\alpha\beta} G_{\mu\nu}(X)\, \partial_\alpha X^\mu \partial_\beta X^\nu \,, \tag{1.1}$$

where $h_{\alpha\beta}$ is the worldsheet metric, $h := \det h_{\alpha\beta}$ and $G_{\mu\nu}$ is the background metric. In this formulation string theory takes the form of a non-linear σ-model of D free scalars living in a 2-dimensional spacetime and spanning a D-dimensional target space. T is the *string tension*, it represents the mass of the string per unit length, thus $[T] = \mathrm{L}^{-2}$, and is related to another fundamental quantity that will appear later on, the *Regge slope* α', through

$$T = \frac{1}{2\pi\alpha'} \,. \tag{1.2}$$

Due to its dimensionality the Regge slope sets the fundamental length of the theory, the *string length* $\ell_s := \sqrt{\alpha'}$.

In what follows we shall restrict to the case of a Minkowskian background, i.e. $G_{\mu\nu} = \eta_{\mu\nu}$. The equations of motion for $h_{\alpha\beta}$ and X^μ can be derived by means of a principle of least action and read

$$T_{\alpha\beta} := -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h^{\alpha\beta}} = 0 \,, \tag{1.3}$$

$$\Box X^\mu = 0 \,. \tag{1.4}$$

Action (1.1) is invariant under the following transformations:

- Poincaré transformations of the background (global)

$$\delta h_{\alpha\beta} = 0 \,, \qquad \delta X^\mu = a^\mu{}_\nu X^\nu + b^\mu \tag{1.5}$$

- reparametrizations of the worldsheet, i.e. the diffeomorphisms $\delta\sigma^\alpha = -\xi^\alpha$ (local)

$$\delta h_{\alpha\beta} = \xi^\gamma \partial_\gamma h_{\alpha\beta} + 2h_{\gamma(\alpha}\partial_{\beta)}\xi^\gamma \,, \qquad \delta X^\mu = \xi^\gamma \partial_\gamma X^\mu \tag{1.6}$$

- Weyl rescalings (local)[1]

$$\delta h_{\alpha\beta} = \lambda\, h_{\alpha\beta} \,, \qquad \delta X^\mu = 0 \tag{1.7}$$

Thanks to the last two transformations it is always possible to "gauge fix" the worldsheet metric and move to the *conformal frame* in which $h_{\alpha\beta} = \eta_{\alpha\beta}$ and the equations for the scalars (1.4) read

$$(\partial_\sigma^2 - \partial_\tau^2) X^\mu = 0 \,. \tag{1.8}$$

In order to have a well defined variational problem, attention must be paid to the boundary terms. Indeed, taking a closer look to the variation of the action, we can observe the presence of the terms

$$\int_{\text{w.s.}} d\tau \left[\partial_\sigma X^\mu \delta X_\mu\right]_{\sigma=0}^{\sigma=\pi} \,, \tag{1.9}$$

where 0 and π are the two endpoints of the string. They vanish imposing the following boundary conditions:

- $\partial_\sigma X^\mu\big|_{\sigma=0,\pi} = 0$ — *open strings* with *Neumann b.c.* (N)
- $\delta X^\mu\big|_{\sigma=0,\pi} = 0 \implies X^\mu\big|_{\sigma=0,\pi} = c^\mu_{0,\pi}$ — *open strings* with *Dirichelet b.c.* (D)
- $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\pi)$ — *closed strings* with *periodic b.c.*

Dirichelet boundary conditions introduce something new in our theory. Imposing these conditions on $(D-p-1)$ coordinates X^μ $(\mu \neq 0)$, the string endpoints are forced to be attached to $(p+1)$-dimensional hypersurfaces; these surfaces can be interpreted as the worldvolume of extended object, the so-called D*p-branes*, whose position is specified by the constants c^μ_0/c^μ_π. The nature of these new players will become clear in subsection 1.4.1.

[1] At least at classical level, this symmetry guarantees the vanishing of the trace of the stress-energy tensor $T_{\alpha\beta}$.

A general solution to equation (1.8) can be easily written in the worldsheet light-cone coordinates $\sigma^{\pm} := \tau \pm \sigma$

$$X^{\mu}(\tau,\sigma) = X^{\mu}_{L}(\sigma^{+}) + X^{\mu}_{R}(\sigma^{-})\,, \tag{1.10}$$

where $X^{\mu}_{L/R}$ are called *left-/right-moving modes.* Provided this decomposition, the equations of motion for $h_{\alpha\beta}$ (1.3) and the boundary conditions for open strings boil down to

$$\left(\partial_{+}X^{\mu}_{L}\right)^{2} = \left(\partial_{-}X^{\mu}_{R}\right)^{2} = 0\,, \tag{1.11}$$
$$\partial_{+}X^{\mu}_{L} = \partial_{-}X^{\mu}_{R} \quad (\text{N}) \qquad \text{or} \qquad \partial_{+}X^{\mu}_{L} = -\partial_{-}X^{\mu}_{R} \quad (\text{D})\,. \tag{1.12}$$

1.2 The superstring

The inclusion of fermionic fields is a key element in the attempt of string theory to describe the world around us. This requirement implies the introduction of *supersymmetry (SUSY)* in order to relate bosons and fermions and the resulting theories are thus called *superstring theories.* This construction can be obtained following two different, yet equivalent, approaches:

- *Ramond-Neveu-Schwarz (RNS) formalism*: supersymmetry is introduced at the level of the worldsheet by means of a new "fermionic coordinate" ψ^{μ};
- *Green-Schwarz (GS) formalism*: supersymmetry is introduced in the target space by means of the superspace formulation.

In what follows we shall give a flavour of the former.

The RNS formalism incorporates the fermions into the bosonic string theory including the additional 2-dimensional Majorana spinors $\psi^{\mu}(\tau,\sigma)$ and $\chi_{\alpha}(\tau,\sigma)$[2], respectively paired to X^{μ} and $h_{\alpha\beta}$. This process leads to the construction of an action with the same symmetries of the Polyakov action, but also invariant under local SUSY transformations and an additional transformation involving χ_{α}. Thanks to these properties we can choose the conformal gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$ and set $\chi_{\alpha} = 0$, getting the gauge-fixed action

$$S_{g.f.} = -\frac{T}{2}\int_{\text{w.s.}} d^{2}\sigma\left(\partial_{\alpha}X^{\mu}\partial^{\alpha}X_{\mu} - i\,\bar{\psi}^{\mu}\rho^{\alpha}\partial_{\alpha}\psi_{\mu}\right), \tag{1.13}$$

where ρ^{α} are the 2-dimensional Dirac matrices, generators of the Clifford algebra $C\ell_{1,1}(\mathbb{R})$ associated to the worldsheet and defined by $\{\rho^{\alpha},\rho^{\beta}\} = -2\eta^{\alpha\beta}$. The gauge-fixed superstring action (1.13) is still invariant under the global SUSY transformation

$$\delta X^{\mu} = \bar{\epsilon}\psi^{\mu}\,, \qquad \delta\psi^{\mu} = -i\rho^{\alpha}(\partial_{\alpha}X^{\mu})\epsilon\,, \tag{1.14}$$

where ϵ is a constant Majorana spinor.

The equations of motion of the complete action comprise the ones for the auxiliary fields that we gauged away, and here we omit[3], and the following wave and Dirac equations

$$\Box X^{\mu} = 0\,, \qquad \rho^{\alpha}\partial_{\alpha}\psi^{\mu} = 0\,. \tag{1.15}$$

Like for the bosonic string, a general solution to (1.15) can be written in worldsheet light-cone coordinates as

$$\psi^{\mu}(\tau,\sigma) = \psi^{\mu}_{L}(\sigma^{+}) + \psi^{\mu}_{R}(\sigma^{-})\,. \tag{1.16}$$

[2] We suppress spinorial indices for clarity.

[3] The equations for $h_{\alpha\beta}$ and χ_{α} are nothing more than the vanishing of the stress-energy tensor and the supercurrent, the conserved currents associated to (rigid) translations and (global) SUSY transformations of the worldsheet. Notice that these currents are conserved by means of the equations of motion for X^{μ} and ψ^{μ}.

In order to be consistent, equations (1.15) need to be supplemented with appropriate boundary conditions. We already presented in the previous section the conditions for X^μ in the case of open or closed strings, so here we shall focus on the ones to be imposed on the spinors[4]:

- $\psi^\mu_L\big|_{\sigma=\pi} = +\,\psi^\mu_R\big|_{\sigma=\pi}$ *open strings* with *Ramond b.c.* (R)
- $\psi^\mu_L\big|_{\sigma=\pi} = -\,\psi^\mu_R\big|_{\sigma=\pi}$ *open strings* with *Neveu-Schwarz b.c.* (NS)
- $\psi^\mu_{L/R}(\tau,\sigma) = +\,\psi^\mu_{L/R}(\tau,\sigma+\pi)$ *closed strings* with *periodic b.c.* (R)
- $\psi^\mu_{L/R}(\tau,\sigma) = -\,\psi^\mu_{L/R}(\tau,\sigma+\pi)$ *closed strings* with *antiperiodic b.c.* (NS)

In the case of closed strings the boundary conditions for the left- and right-moving modes can be chosen independently, giving rise to four distinct combinations, the four closed string sectors R-R, R-NS, NS-R and NS-NS. We shall come back on them at the end of the next section.

1.3 Quantization of the string

Now that we have included fermions, the next step to take is to quantize the theory. In what follows we shall sketch the main passages, starting from the bosonic string.

For closed strings the most general solution to (1.8), a wave equation, can be written as a Fourier expansion of X^μ_L and X^μ_R with coefficients α^μ_n and $\tilde{\alpha}^\mu_n$, with $n \in \mathbb{Z}$ and $n \neq 0$. Analogous is the case of open strings, for which the only coefficients are the α^μ_n because now left- and right-moving modes turn out not to be independent. Subsequently, imposing the canonical Poisson brackets on X^μ and its conjugate variable it is possible to compute the Poisson brackets of α^μ_n (and $\tilde{\alpha}^\mu_n$). The proper quantization proceeds in the usual way promoting the α^μ_n (and $\tilde{\alpha}^\mu_n$) to operators and their Poisson brackets to commutators. Like in the quantum harmonic oscillator, these operators behave like *creation* (raising) and *annihilation* (lowering) *operators* and allow us to define a *vacuum*, or ground state, $|0\rangle$, which is destroyed by the latter and originates all the states acting on it with the former. The oscillation modes of the string can be seen as particles in a quantum field theory, laying the first brick in the interpretation of string theory as a theory of unification.

This quantization procedure is not free of issues, namely the appearance in the spectrum of ghosts and tachyons. A *ghost* is a state of negative norm and its presence is due to the negative eigenvalue of the background metric $\eta_{\mu\nu}$. Three are the main techniques that have been developed to tackle this problem, all of them equivalent and giving the same results:

- *(Old) covariant quantization*: similar to the Gupta–Bleuler formalism applied to QED;
- *BRST quantization*: manifestly covariant, relies on the introduction of Faddeev-Popov ghosts;
- *Light-cone gauge quantization*: breaks the Lorentz covariance to rule out the unwanted states.

All these approaches state that ghosts are absent from the theory if the dimension of the background is $D = 26$[5].

Analyzing the mass levels of the closed string spectrum we discover that the vacuum state is a *tachyon*, i.e. a state of negative mass squared, with mass $M^2 = -4/\alpha'$. Its existence cannot

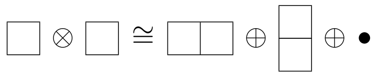
[4]For convention, we fix the overall relative sign for open strings such that $\psi^\mu_L\big|_{\sigma=0} = +\,\psi^\mu_R\big|_{\sigma=0}$.

[5]For example, this value arises in the covariant quantization requiring that the unphysical states decouple from the others, while in the BRST quantization this condition implies the vanishing of the (unwanted) conformal anomaly.

be cured within the bosonic string theory and the introduction of superstrings turns out to be necessary. On the contrary, in $D = 26$ the first excited states are physical, specifically massless, and can be constructed as[6]

$$\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0\rangle \,, \tag{1.17}$$

where α^i_{-1} and $\tilde{\alpha}^i_{-1}$ are the creation operators for transverse polarizations. These states belong to the tensor product of two vector representations of SO(24), the one related to massless vectors in twenty-six dimensions, which can be decomposed in the sum of a symmetric traceless, an antisymmetric and a trace part:



The three irreps of the decomposition are in a one-to-one correspondence with three massless objects, a symmetric $(0, 2)$-tensor, the *graviton* $g_{\mu\nu}$, a 2-form, the *Kalb-Ramond (KR) field* $B_{\mu\nu}$, and a real scalar, the *dilaton* Φ.

The quantization of the superstring proceeds essentially along the same lines as in the bosonic string. Spinors $\psi^\mu_{L/R}$ are expanded in Fourier modes and their coefficients, d^μ_n (and $\tilde{d}^\mu_n$) with $n \in \mathbb{Z}$ in the R sector and b^μ_r (and $\tilde{b}^\mu_r$) with $r \in \mathbb{Z} + \frac{1}{2}$ in the NS one, are promoted to creation and annihilation operators. The theory is free of ghosts in dimensions $D = 10$, whereas tachyons are still present in the NS sector of open strings.

Taking a closer look to the spectrum of open strings we discover that

- **R sector:** the ground state is a massless spinor and the excited states are massive fermions;
- **NS sector:** the ground state is a tachyonic scalar, the first excited state is a massless vector and the other excited states are massive bosons.

This spectrum is clearly problematic. First of all, as we already mentioned, it features tachyons in its NS sector. Secondly, it breaks supersymmetry, which is really bad since, as can be proved, the spectrum contains a massless gravitino, the gauge field of local supersymmetry. This unwanted behaviour emerges considering, e.g., that there are no tachyonic fermions to be paired to the tachyonic bosons[7]. All these issues can be solved applying the *Gliozzi-Scherk-Olive (GSO) projection*, which leaves a supersymmetric spectrum free of tachyons. The basic idea is to partition the whole spectrum according to the number of b and d creators needed to give rise to each state and then to discard all the states with an even number of b's. The result is that the NS sector will contain only states with an odd fermion number F_{NS}, while the R sector will be split into two sets according to the chirality of the vacuum. The ground state of the NS sector will be a massless vector, the vacuum of the R one a (chiral) massless spinor. Both of them belong to a certain representation of SO(8) according to their nature.

The recombination of left- and right-moving modes can be done in various ways and similarly we have the freedom to choose between the two different R subsectors, thus obtaining five inequivalent string theories.

- **Type-II string theories:** for what concerns closed strings, the GSO projection can be performed on ψ_L and ψ_R independently, thus we have two inequivalent ways to combine the NS and $\text{R}_\pm$ sectors, where the $\pm$ is related to the chirality of the ground state. When opposite chiralities for the L and R modes are taken we have *type-IIA string theory*, otherwise *type-IIB string theory*. The different sectors are organized in irreps of SO(8).

[6]Closed string modes always come in couples according to the *matching level condition*.

[7]We remind that a necessary, but not sufficient, condition for supersymmetry is the presence of an equal number of bosonic and fermionic degrees of freedom at each mass level.

- NS-NS sector: it is common to IIA and IIB and features the same field content of the bosonic string (apart from the different dimensionality): the graviton $g_{\mu\nu}$ (35 states), the Kalb-Ramond field $B_{\mu\nu}$ (28 states) and the dilaton Φ (1 state).
- NS-R and R-NS sectors: each of them contains the spin-3/2 fermion *gravitino* Ψ_μ (56 states) and the spin-1/2 *dilatino* λ (8 states). In IIA (NS-R$_\pm \oplus$R$_\mp$-NS) the two gravitinos have opposite chirality, whereas in IIB (NS-R$_\pm \oplus$R$_\pm$-NS) the chirality is the same.
- R-R sector: made up of bosons obtained by tensoring a pair of spinors. In IIA (R$_\pm$-R$_\mp$) one has a 1-form $C^{(1)}_\mu$ (8 states) and a 3-form $C^{(3)}_{\mu\nu\rho}$ (56 states). In IIB (R$_\pm$-R$_\pm$) one has a 0-form $C^{(0)}$ (1 state), a 2-form $C^{(2)}_{\mu\nu}$ (28 states) and a 4-form $C^{(4)}_{\mu\nu\rho\sigma}$ with self-dual field strength (35 states).

These theories preserve 32 real supercharges, thus in ten dimensions they are $\mathcal{N} = 2$ SUSY and maximally supersymmetric (cf. appendices A.1 and A.2 for further details on spinors and SUSY generators in different dimensions).

- **Heterotic string theories:** they contain closed strings constructed combining bosonic left-moving modes with superstring right-moving modes. The 16 extra dimensions are compactified, giving rise to internal gauge symmetries, with gauge group SO(32) or $E_8 \times E_8$. The massless spectrum includes the graviton, the KR field and the dilaton along with a gravitino and a dilatino. Additionally, there are vector fields, which gauge the internal symmetry group, and the related spin-1/2 superpartners, the *gauginos*. These theories preserve 16 real supercharges, hence they are $\mathcal{N} = 1$ SUSY.
- **Type-I string theory:** it is the only theory featuring open strings. For consistency, it requires closed strings as well, which can be obtained by modding out type-II theories with respect to the parity symmetry $\mathbb{Z}_2$ of the worldsheet coordinates. The massless spectrum contains the graviton, the dilaton and a R-R 2-form, along with a gravitino and a dilatino. It is $\mathcal{N} = 1$ SUSY.

1.4 Beyond the perturbative regime

When undertaking calculations of scattering amplitudes between strings, the easiest and most immediate approach is a perturbative one in terms of Feynman diagrams. In doing this, one discovers that the "loop" expansion is governed by the vacuum expectation value (vev) of the dilaton, or, more precisely, is organized in powers of $g_s := \langle e^\Phi \rangle$, which takes the role, and the name, of *string coupling constant*. However, perturbative methods are generally unable to grasp the whole physics behind a quantum theory and non-perturbative effects are inevitably lost. In many QFTs additional techniques have been developed in order to overcome this problem, like path-integral or lattice methods, but unfortunately a complete non-perturbative formulation of string theory is still missing. Nevertheless, the remarkable progresses made in string theory during the mid-1990s provided some tools that allow to partially explore its non-perturbative regime.

1.4.1 Branes and other extended objects

The whole story of string theory does not end with the classification presented at the end of the previous section. Its spectrum is much richer and, apart from a complete tower of massive oscillating string modes, it includes many solitonic states, some of them intimately non-perturbative. Among these are extended objects like *orientifold (Op) planes*, *solitonic branes* or *Kaluza-Klein (KK) monopoles*, but probably the most important example are the Dp-branes we introduced in section 1.1.

We already talked about D-branes as the locus where open string endpoints are attached and their existence is strictly related to the Dirichelet boundary conditions. In particular, the latter represent the position of the branes in the background spacetime. Dirichelet conditions are explicitly non-covariant, thus breaking Poincaré invariance, and for this reason they have been considered problematic for many years, but the situation changed in 1995, when the solitonic nature of D-branes started to be considered and their dynamics to be studied on its own [6]. As we shall see, a Nambu-Goto-like action to describe the evolution of branes can be constructed and relies on the extension of the concept of worldsheet to objects with more than one spatial dimension. Dp-branes extend along p spatial directions, hence they have a $(p+1)$-dimensional *worldvolume* which breaks Lorentz invariance of the background according to the worldvolume and the transverse coordinates. In the example of flat branes with no backreaction on the background we have[8]

$$\mathrm{SO}(9,1) \to \mathrm{SO}(p,1) \times \mathrm{SO}(9-p)\,. \tag{1.18}$$

The analysis of the first excited states in the open string spectrum, which is massless, in presence of Dirichelet boundary conditions enriches the understanding of the physics of Dp-branes:

- **Oscillations along the brane:** they are generated by $(p-1)$ different raising operators transforming under the vector representation of $\mathrm{SO}(p-1)$, thus they give rise to a massless vector in the $(p+1)$-dimensional worldvolume, which can be seen as a $\mathrm{U}(1)$ gauge vector in analogy to the massless photon.

- **Oscillations transverse to the brane:** they are generated by $(9-p)$ different raising operators transforming as scalars under the Lorentz group of the brane and can be thought as $(9-p)$ scalar fields describing the fluctuations of the brane in the transverse directions.

To sum up, every Dp-brane has a massless vector living inside its worldvolume and $(9-p)$ scalars describing its transverse oscillations.

D-branes naturally couple to the R-R potentials included in the bosonic sector of closed strings in type-II theories and, at the same time, play the role of sources. In reality, D-branes are stable objects exactly because they are charged under the R-R fields and this fact establishes which Dp-branes can exist in a given theory. In particular, Dp-branes couple electrically to $C_{(p+1)}$ and magnetically to $C_{(7-p)}$, so, starting from their R-R spectrum, we can derive the D-brane content of type-II string theories:

- **type-IIA:** D0, D2, D4, D6 + D8 (p even)
- **type-IIB:** D(−1), D1, D3, D5, D7 + D9 (p odd)

Some annotations must be made. The branes to the left of the plus are D-branes that couple to the R-R p-forms previously introduced, whereas the other two are branes added "by hand" as open string endpoints. In particular, D8-branes couple to a potential with a 10-form field strength, which is non-dynamical; their relevance will become clear later on, in subsection 2.2.2. D9-branes are spacetime filling branes and for this they lead to Neumann boundary conditions in every dimension. D(−1)-branes arise because they couple to $C_{(0)}$ and by construction are localized in space and time, thus they are a particular type of *instantons*. Finally, D3-branes carry self-dual charges being both electrically and magnetically charged under $C_{(4)}$, whose field strength is self-dual.

[8] Here and in the rest of the section we restrict to $D = 10$.

It is worth noting that D-branes break part of the supersymmetry, halving the number of conserved supercharges. Still they retain half of the SUSY generators, which makes them *BPS states*. This means that D-branes must saturate the *Bogomol'nyi–Prasad–Sommerfield (BPS) bound* which equates the mass of the brane, related to its tension, and the central charge, related to the brane's conserved charges, hence D-branes must be charged.

The dynamics of a D-brane and the fields defined on its worldvolume are in close correlation with the modes of the open strings attached to it. In the limit in which the energy of the former is low compared to the energy of the latter, the dynamics of the D-brane is uniquely determined by the open string massless modes. Given a Dp-brane living in a background generated by the closed string modes $g_{\mu\nu}$, $B_{\mu\nu}$ and Φ, its effective action reads

$$S_{\mathrm{D}p} = -T_{\mathrm{D}p} \int_{\mathrm{w.v.}} d^{p+1}\sigma\, e^{-\Phi} \sqrt{-\det\big(g_{\alpha\beta} + B_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}\big)} + \mu_p \int_{\mathrm{w.v.}} e^{B+2\pi\alpha' F} \wedge \sum_k C_{(k)} \,. \tag{1.19}$$

The first term is the *Dirac-Born-Infeld (DBI) action* and describes the interaction of the brane with the background. The Dp-brane is embedded by means of the coordinates $X(\sigma^\alpha)$ and $T_{\mathrm{D}p}$ is its tension. $g_{\alpha\beta}$ and $B_{\alpha\beta}$ are the pull-back of the background metric and Kalb-Ramond field on the D-brane worldvolume, e.g. $g_{\alpha\beta} = g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$. $F_{\alpha\beta}$ is the 2-form field strength associated to the abelian vector A^α living on the D-brane and implicit in $g_{\alpha\beta}$ is the kinetic term for the scalars describing the D-brane oscillations. The second term is the *Wess-Zumino (WZ) action* and describes the interaction of the brane with the R-R potentials. μ_p is the Dp-brane charge under $C_{(p+1)}$. The tension of the Dp-brane is explicitly given by

$$T_{\mathrm{D}p} = \frac{1}{g_s (\alpha')^{(p+1)/2} (2\pi)^p} \,. \tag{1.20}$$

It is manifest that in a weakly-coupled regime, where $g_s \to 0$, the tension of the brane becomes large, making the brane "heavy". In this way its dynamics decouples from the background and the open and closed strings become completely independent.

A different kind of extended objects arise when considering the NS-NS sector of string theories, indeed it includes the KR 2-form B, which couples electrically to 1-dimensional objects, the *fundamental strings (F1)* (or NS1) and magnetically to 5d branes, the *solitonic 5-branes (NS5)*. These new players are common to both IIA and IIB

- **type-IIA/B:** F1, NS5

and heterotic theories as well.

Lastly, we introduce the *Kaluza-Klein monopoles*. They are purely gravitational solutions and their geometry is described by the product of a Minkowski spacetime and the Euclidean Taub-NUT space. Hence, by definition, a KK-monopole always has a compact direction, the Taub-NUT direction. They are sometimes referred to as KK5-branes, even though, strictly speaking, they (may) become branes only upon T-duality or dimensional reduction.

The great importance of the objects presented in this section is that, once combined in appropriate ways, something surprising may happen: the appearance of black holes. Indeed, these macroscopic objects can arise from intricate bound states of microscopic entities, like branes and other solitonic states, wrapping dimensions compactified to give, e.g., n-spheres or n-tori, and the charges of the former can be given in terms of characteristic quantities of the latter. This fact will be the starting point of chapter 7, in which we shall study black holes constituted of F1-strings, NS5-branes and KK-monopoles, but a probably better-known example is the BPS four-charge black hole originating from a bound state of D2 and D6-branes (see appendix B for the explicit construction).

1.4.2 Dualities

Although, to date, a complete and well-established formulation of string theory that goes beyond the perturbative limit is not known, yet we can learn much about its non-perturbative regime by means of the existence of *string dualities*.

S-duality

S-duality is a non-perturbative (in g_s) equivalence between two different string theories with coupling constants g_s and $1/g_s$ respectively; since it relates strongly-coupled and weakly-coupled theories, it is an example of strong/weak duality. In particular, S-duality maps type-IIB string theory into itself (in different regimes) and type-I to heterotic SO(32) and vice versa. Thanks to its non-perturbative nature, S-duality allows to gain some insight into otherwise inaccessible regions of a theory from the perturbative analysis of an equivalent one. Classically, S-duality is analogous to the electromagnetic duality that allows to interchange the electric and magnetic fields in the homogeneous Maxwell equations. In the case of IIB superstrings, S-duality can be extended to a bigger symmetry described by the discrete group $\mathrm{SL}(2,\mathbb{Z})$.

Let us now review how S-duality affects the extended objects of the type-II spectrum. As we explained, this transformation relates different regimes of type-IIB string theory, thus it maps IIB objects to other IIB objects. In particular it exchanges

$$\text{D1-branes} \longleftrightarrow \text{F1-strings} \qquad \text{and} \qquad \text{D5-branes} \longleftrightarrow \text{NS5-branes}\,,$$

leaves D3-branes unmodified and maps D7-branes to a different kind of 7-branes. Considering the whole $\mathrm{SL}(2,\mathbb{Z})$ transformation, (F1, D1) and (NS5, D5) transform as doublets, D3-branes as a singlet and D7-branes as a triplet. Specifically, we have

$$\begin{aligned} &(\mathrm{F1},\mathrm{D1}) \mapsto (p,q) \text{ strings}\,, && \mathrm{D3} \mapsto \text{D3-branes}\,, \\ &(\mathrm{NS5},\mathrm{D5}) \mapsto (p,q) \text{ 5-branes}\,, && \mathrm{D7} \mapsto (p,q,r) \text{ 7-branes}\,. \end{aligned}$$

The new (p,q) strings (respectively, 5-branes) can be regarded as a bound state of p F1-strings (NS5-branes) and q D1-branes (D5-branes).

We shall come back on S-duality and present its action on the IIB massless spectrum in subsection 2.3.1.

T-duality

A different kind of duality is the *T-duality*, which relates string theories compactified on circles of different radius, namely R and ℓ_s^2/R. From a worldsheet perspective, T-duality acts as

$$X_R^9 \mapsto -X_R^9 \qquad \text{and} \qquad \psi_R^9 \mapsto -\psi_R^9\,, \tag{1.21}$$

where "9" denotes the compact direction. It is possible to show that this mapping exchanges the open strings' Neumann and Dirichelet boundary conditions in the compact direction (cf. (1.12)) and flips the chirality of ψ_R. Whereas the consequences of the former will be clear in the next section, the latter implies that T-duality relates type-IIA and type-IIB superstrings; for what concerns heterotic superstrings, it connects the SO(32) theory to the $\mathrm{E}_8 \times \mathrm{E}_8$ one. Unlike S-duality, T-duality is perturbative and holds order by order in the g_s expansion, but relates theories compactified on huge scales to others compactified on tiny circles.

As we mentioned, T-duality relates type-IIA and type-IIB string theories, thus it maps IIA to IIB objects and vice versa. Starting with Dp-branes and remembering that T-duality exchanges Neumann and Dirichelet conditions, applying it along the brane worldvolume generates a D-brane with one dimension less, a D$(p-1)$-brane; conversely, its action transversely to the brane gives rise to a D$(p+1)$-brane

$$\text{D}p\text{-branes} \mapsto \begin{cases} \text{D}(p-1)\text{-branes} & \text{along} \\ \text{D}(p+1)\text{-branes} & \text{transverse} \end{cases}$$

For what concerns NS5-branes, KK-monopoles may arise[9]

$$\text{NS5-branes} \mapsto \begin{cases} \text{NS5-branes} & \text{along} \\ \text{KK-monopoles} & \text{transverse} \end{cases}$$

KK-monopoles behave differently whether the T-dualization is performed along the Taub-NUT direction or not

$$\text{KK-monopoles} \mapsto \begin{cases} \text{NS5-branes} & \text{along the Taub-NUT direction} \\ \text{KK-monopoles} & \text{along the brane} \end{cases}$$

Again, we shall come back on this topic later on, in subsection 2.3.2, where we shall relate the massless spectra of IIA and IIB superstrings.

Given two theories dual to one another, also their spectra must be dual, therefore IIA and IIB spectra must be T-dual and IIB spectrum must be S-duality invariant. Due to its strong/weak duality nature, S-duality interchanges perturbative with non-perturbative states, hence type-IIB superstring theory must include non-perturbative states dual to the F1-strings already present. T-duality brings non-perturbative states inside type-IIA theory as well and a chain of S- and T-dualities implies that the whole set of non-perturbative states we mentioned in this section must be present in type-II superstrings[10].

1.4.3 M-theory

Let us take a break from string theory and consider a generic $(D+1)$-dimensional theory and imagine to perform a Kaluza-Klein dimensional reduction along the $(D+1)$-th coordinate compactifying it on a circle of radius R. The procedure is well known and it does not come as a surprise the appearance of an infinite tower of massive modes in addition to the massless ones. The mass of the n-th excited state is

$$m_n = \frac{n}{R} \qquad \text{with} \qquad n \in \mathbb{Z}\,, \tag{1.22}$$

and the prescription for a consistent truncation requires to take the limit $R \to 0$ in order to be able to neglect all the modes, but the massless ones. In this way the compactified dimension "disappears" and we are left with a D-dimensional theory.

Going back to string theory, the type-IIA spectrum features the D0-brane among the others, whose mass is given by (1.20)

$$M_{\text{D0}} = T_{\text{D0}} = \frac{1}{g_s \ell_s}\,, \tag{1.23}$$

[9] In this case the Taub-NUT direction of the KK-monopole is the direction of the T-dualization.

[10] From the perspective of boundary conditions, even in a system of Neumann conditions only, T-duality would bring Dirichelet conditions back into the game. Therefore, D-branes must be included in the spectrum in order to be consistent with this transformation.

remembering that $\alpha' = \ell_s^2$. For a stack of n D0-branes the mass reads $n \times M_{\rm D0}$. Comparing it with (1.22) we see that we can interpret this state as a KK excitation and reconstruct the full tower of KK massive modes of an 11-dimensional theory compactified on a circle of radius $R_{10} = g_s \ell_s$. The perturbative regime of type-IIA, $g_s \to 0$, corresponds to the limit $R_{10} \to 0$, in which we have a proper 10-dimensional theory, whereas in the type-IIA strong-coupling limit, $g_s \to \infty$, we observe the decompactification of the circular eleventh dimension. The 11-dimensional theory obtained in this limit is called *M-theory*, but, unfortunately, a Lagrangian formulation is still missing. As we already pointed out, since their tension diverges as $g_s \to 0$, D0-branes have a non-perturbative nature, thus the connection we just established goes beyond the perturbative regime.

What are the extended objects that populate the M-theory spectrum? Guided by the fact that they will necessarily be related to type-IIA branes we can state that the following M-branes exist:

- *M2-brane*: gives rise to an F1-string when wrapped on the compactified direction and to a D2-brane otherwise;
- *M5-brane*: gives rise to a D4-brane when wrapped on the compactified direction and to an NS5-brane otherwise.

Moreover, the dimensionality of the branes allows us to infer the existence of a 3-form potential to which the M2-branes couple electrically and the M5-branes magnetically. For what concerns the M-theory origin of D6-branes, the latter are the magnetic dual of D0-branes, originated by the KK excitations; closing the circle, it is possible to show that D6-branes descend from (11d) KK-monopoles compactified along their Taub-NUT direction. On the other hand, how D8-branes arise from M-theory is still not understood, as will become clear in subsection 2.2.2.

One last remark deserves to be done: contrary to string theories, M-theory is not characterized by any coupling constant, indeed all the parameters of IIA are encoded in geometrical quantities, like R_{10}. This becomes explicit taking a look at the tension of the M-branes, $T_{{\rm M}p} = 2\pi(2\pi\ell_{\rm P})^{-(p+1)}$, with $\ell_{\rm P}$ the Planck length, which depends on fundamental constants solely.

We shall come back on this topic in the next chapter, where 11d and type-IIA supergravities, the low-energy limit of M-theory and IIA superstrings respectively, will be presented and their connection will be made explicit (cf. sections 2.1 and 2.2), shedding more light on these otherwise vague concepts.

In a similar way, M-theory arises in the strongly-coupled regime of heterotic $\rm E_8 \times E_8$ string theory, where the eleventh dimension opens up. We shall not discuss it here, instead we refer to the books cited in the introduction of this chapter and to the seminal paper [14].

By means of these new relations the web of string theories in now complete and all the five theories, type-I, type-II and heterotic, can be accommodated in a unified picture along with M-theory, as displayed in figure 1.1.

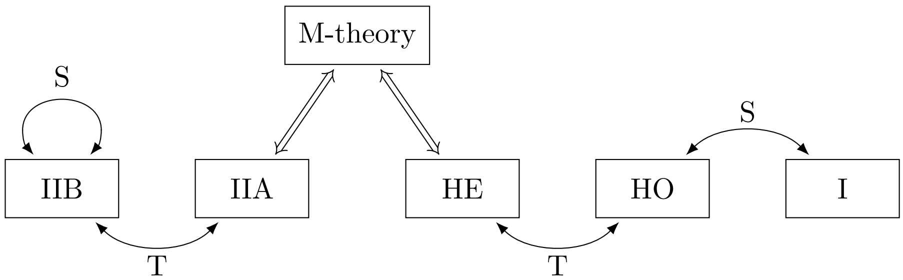


Figure 1.1: *The web of dualities connecting all the five string theories. I, IIA and IIB are type-I/IIA/IIB, HE and HO are heterotic $E_8 \times E_8$ and SO(32). Double lines are dimensional reductions, single lines are S/T-dualities.*

Chapter 2

Higher-dimensional supergravities

String theory is a quantum theory of gravity and, as such, suffers from the presence of non-perturbative effects, which, unfortunately, are fundamental to the understanding of the microscopic behaviour of gravity. The main obstruction to a full control over the theory is our inability to take into account the contributions from the whole string spectrum and the open-closed string interaction, which together contribute to the non-perturbative effects. In this direction, M-theory could prove to be useful since, as we already mentioned, all the non-perturbative effects are encoded into geometrical quantities or in the M-branes and the stringy parameters resolve into them, but a complete formulation is still missing. Luckily, there exist a particular regime of string theories, the weak-coupling and low-energy limit, in which they become tractable and a study can be brought forward. Although not being fundamental theories, they still capture some of the important features of string theories and many crucial information about their non-perturbative states.

Before delving into the low-energy limit of the many string theories we presented, it is first necessary to better understand the two parameters we encountered so far, the Regge slope α' and the string coupling g_s.

- α' encodes the stringy nature of the theory since, we recall, $\alpha' = \ell_s^2$. The $\alpha' \to 0$ limit can be interpreted as the limit in which the string length is ignored and a theory of particles is recovered and an expansion in α' represents an expansion around the point-particle limit. Given that α' is not dimensionless, a meaningful limit is $\alpha'/L^2 \ll 1 \implies L \gg \ell_s$, where L is the system characteristic length, or $\alpha' E^2 \ll 1 \implies E \ll m_s$, where m_s is related to the mass of the string modes and E is the energy scale of the theory. In this sense the $\alpha' \to 0$ limit is a low-energy limit.

- g_s describes the quantum behaviour of string theory, indeed, like the coupling constant in any QFT, it drives the expansion in the number of (string) loops or, in this case, the genus of the string world sheet. A weak string coupling limit, $g_s \to 0$, allows to safely neglect quantum corrections. This can be seen from a different point of view: from (2.12) we have that $g_s \ll 1 \implies \ell_s \gg \ell_{\mathrm{P}}$, hence in this regime the stringy physics, occurring at length scales around ℓ_s, decouples from the strong quantum effects taking place at ℓ_{P}. In what follows we shall work in this regime.

The expression of the tension of a Dp-brane in (1.20) tells us one important thing: not only in the weakly-coupled regime, $g_s \to 0$, but also in the low-energy limit, $\alpha' \to 0$, the tension (and the mass) of the brane becomes large, hence the brane rigid (heavy). It follows that the gauge theory on the brane decouples from the gravity theory of the background or, in other words, the open strings defining the former decouples from the closed strings describing the latter. Moreover, at low energies the massive string modes, whose mass goes like $M^2 \sim 1/\alpha'$, become too heavy to be observed and the massless states, i.e. the ground states, are the only relevant ones. The massless modes belonging to the open string spectrum define a specific supersymmetric QFT on the brane worldvolume. On the other hand, the dynamics of the closed string massless modes can be described by an effective theory in terms of the corresponding massless fields. In the limit of vanishing string coupling this low-energy theory gives rise to a *supergravity theory*, a classical theory of gravity with interacting fields and local supersymmetry, whose "gauge" field is the gravitino itself. Within this picture, branes are defined by a configuration of classical fields, the metric, the KR 2-form, the dilaton etc., solution of the supergravity equations of motion and describing how the background is modified interacting with the brane itself.

The effective supergravities we shall consider in the following sections are the product of a double perturbative expansion, both in α' and in g_s, of the string theories of the previous chapter. But a natural question arises: to what extent can we trust the results obtained in this regime? As a general rule, supergravity solutions are reliable as long as the curvature is small enough, $R \ll \ell_s^{-2}$, and the quantum effects are negligible, $e^{\Phi} \ll 1$. Outside this perimeter things become quite tough: for curvatures of the order of ℓ_s^{-2} stringy contributions invalidate the effective action and its solutions and higher-order corrections must be taken into account, whereas, on the other hand, a finite value of g_s brings quantum effects into play. Luckily, dualities can come to the aid connecting theories at different length and energy scales.

This chapter opens with 11-dimensional supergravity, the low-energy limit of the aforementioned M-theory. The following three sections, namely section 2.2, 2.3 and 2.4, are devoted to type-IIA, type-IIB and heterotic supergravities respectively, focusing on their effective action, their properties and their equations of motion. Finally, in the last section are presented the truncation recipes to obtain 7d minimal and 6d Romans gauged supergravities from 11d and IIA supergravities. These formulae will be very useful in chapter 6, where we shall embed solutions of the 7d and 6d theories in higher-dimensional spacetimes.

2.1 11-dimensional supergravity

We begin our journey among (some of) the higher-dimensional supergravities starting with the effective formulation of M-theory, *11-dimensional supergravity*. This theory was constructed by brute force in 1978 by Cremmer, Julia and Scherk [15], but it found its place only after the second superstring revolution, when M-theory was proposed and 11d SUGRA acquired its natural UV-completion, circumventing its lack of renormalizability.

The field content includes the graviton (44 states), one gravitino (128 states) and a 3-form potential (84 states) (cf. table A.4):

$$\{\hat{g}_{MN}, \hat{\Psi}_M, \hat{A}_{MNP}\}\,, \qquad M = 0, \ldots, 10\,. \tag{2.1}$$

The gravitino is a Majorana spinor with 32 real components (cf. table A.1). The system preserves 32 supercharges, the maximum number allowed in eleven dimensions (cf. appendix A.2), thus 11d SUGRA is a maximal supergravity, in particular $\mathcal{N} = 1$. Requiring invariance under gauge transformations of $\hat{A}$ and supersymmetry uniquely fixes the expression of the action, whose bosonic part is[1]

$$\hat{S}_{11\mathrm{d}} = \frac{1}{16\pi G_{\mathrm{N}}^{(11)}}\Big\{\int d^{11}\hat{x}\sqrt{-\hat{g}}\Big(\hat{R} - \frac{1}{2}|\hat{G}|^2\Big) - \frac{1}{6}\int \hat{A}\wedge\hat{G}\wedge\hat{G}\Big\}\,, \tag{2.2}$$

[1] We restrict to bosonic fields only, the complete action and SUSY variations can be found in the original paper [15] or, e.g., in [12].

where $G_{\mathrm{N}}^{(11)}$ is the 11d Newton's constant given by $16\pi G_{\mathrm{N}}^{(11)} = (2\pi\ell_{\mathrm{P}})^9/(2\pi)$, $\hat{G}$ is the 4-form field strength associated to $\hat{A}$, namely $\hat{G} := d\hat{A}$, and we introduced the notation (ω p-form)

$$|\omega|^2 := \frac{1}{p!}\,\omega_{\mu_1\ldots\mu_p}\,\omega^{\mu_1\ldots\mu_p}\,. \tag{2.3}$$

The SUSY variation of the gravitino reads

$$\delta_\epsilon\hat{\Psi}_M = \left[\hat{\mathcal{D}}_M + \frac{1}{288}\Big(\Gamma_M{}^{NPQR} - 8\delta_M^N\Gamma^{PQR}\Big)\hat{G}_{NPQR}\right]\epsilon\,, \tag{2.4}$$

where $\hat{\mathcal{D}}_M$ is the *Lorentz covariant derivative*

$$\hat{\mathcal{D}}_M := \partial_M + \frac{1}{4}\,\hat{\omega}_M{}^{AB}\Gamma_{AB}\,, \tag{2.5}$$

$\hat{\omega}_M{}^{AB}$ is the *spin connection* related to the *elfbein* $\hat{e}_M^A$, Γ^A are the 11-dimensional Dirac matrices and $\Gamma^{A_1A_2\ldots A_n} := \Gamma^{[A_1}\Gamma^{A_2}\ldots\Gamma^{A_n]}$.

By means of a variational principle we can derive the equations of motion of the 11d supergravity, specifically the Einstein and Maxwell equations

$$\hat{R}_{MN} = \frac{1}{12}\,\hat{G}_{MPQR}\hat{G}_N{}^{PQR} - \frac{1}{144}\,\hat{g}_{MN}\hat{G}_{PQRS}\hat{G}^{PQRS}\,, \tag{2.6a}$$

$$d\star\hat{G} + \frac{1}{2}\,\hat{G}\wedge\hat{G} = 0\,, \tag{2.6b}$$

which can be supplemented with the Bianchi identity

$$d\hat{G} = 0\,. \tag{2.7}$$

2.2 Type-IIA supergravity

The low-energy limit of type-IIA string theory is called *type-IIA supergravity.* Its field content consists of the massless spectrum of IIA strings, which we present here split into the NS-NS, NS-R$\oplus$R-NS and R-R sectors

$$\big\{g_{\mu\nu}, B_{\mu\nu}, \Phi\big\}\cup\big\{\Psi_\mu, \lambda\big\}\cup\big\{C_\mu^{(1)}, C_{\mu\nu\rho}^{(3)}\big\}\,,\qquad \mu = 0,\ldots,9\,. \tag{2.8}$$

All the spinors are Majorana-Weyl (cf. table A.1), but here every pair of left and right MW spinors is presented packed into one single Majorana spinor for convenience. The low-energy limit retains all the supersymmetries of the starting theory, thus also type-IIA SUGRA is maximally supersymmetric, in particular $\mathcal{N} = 2$ in ten dimensions.

The bosonic part of the action, here written in the so-called *string frame*[2], is

$$\begin{aligned} S_{\mathrm{IIA}} = \frac{g_s^2}{16\pi G_{\mathrm{N}}^{(10)}}\bigg\{&\int d^{10}x\sqrt{-g}\left[e^{-2\Phi}\Big(R + 4|\partial\Phi|^2 - \frac{1}{2}|H|^2\Big) - \frac{1}{2}\big(|F_{(2)}|^2 + |F_{(4)}|^2\big)\right] + \\ &- \frac{1}{2}\int B\wedge dC_{(3)}\wedge dC_{(3)}\bigg\}\,, \end{aligned} \tag{2.9}$$

where $G_{\mathrm{N}}^{(10)}$ is the 10d Newton's constant given by $16\pi G_{\mathrm{N}}^{(10)} = g_s^2(2\pi\ell_s)^8/(2\pi)$ and we defined the modified field strengths

$$H := dB\,,\qquad F_{(2)} := dC_{(1)}\,,\qquad F_{(4)} := dC_{(3)} - H\wedge C_{(1)}\,. \tag{2.10}$$

[2] An alternative frame is the *Einstein frame*, in which the Einstein-Hilbert part of the action takes the canonical form $\int d^{10}x\sqrt{-g_{\mathrm{E}}}\,R_{\mathrm{E}}$. The rescaled line element reads $ds_{\mathrm{E}}^2 = e^{-\Phi/2}ds_{s.f.}^2$.

The first part of the action includes gravity and the kinetic terms of the NS-NS sector, which exhibits a peculiar coupling to the dilaton. The second one accommodates the kinetic terms of the R-R sector which, at least in the string frame, does not couple to the dilaton. The third is a Chern-Simons term mixing the NS-NS and R-R sectors.

We postpone the presentation of the equations of motion to subsection 2.2.2 since they can be retrieved from the ones therein simply setting to zero the mass parameter $F_{(0)}$.

2.2.1 M-theory origin of type-IIA supergravity

In subsection 1.4.3 we already stated that M-theory emerges in the strongly-coupled regime of IIA string theory, where the excitations of the latter can be packed to give rise to an additional dimension and to new non-perturbative extended objects. This bound reflects in a correlation between their low-energy limits and we shall now give the explicit realization.

Type-IIA SUGRA can be obtained from 11d SUGRA (or from M-theory, using an abuse of terminology) by means of a Kaluza-Klein dimensional reduction[3]. In order to do so we take the eleventh coordinate, say z, to be compactified on an S^1 with radius R_z, assume that none of the fields depend on z and adopt the following ansatz

$$\begin{aligned} d\hat{s}^2 &= e^{-\frac{2}{3}\Phi}ds^2 + e^{\frac{4}{3}\Phi}\left(dz + C_{(1)}\right)^2 , \\ \hat{A} &= C_{(3)} + B\wedge dz \qquad \Longrightarrow \qquad \hat{G} = F_{(4)} + H\wedge\left(dz + C_{(1)}\right) . \end{aligned} \tag{2.11}$$

We shall not deal explicitly with the reduction of the 11d gravitino, but the idea is the following. Each 32-component Majorana spinor of the gravitino $\hat{\Psi}_M$ gives a pair of 16-component Majorana-Weyl spinors of opposite chirality; the first ten, with $M = 0, \ldots, 9$, will give the two 10d gravitinos Ψ_μ, whereas the eleventh ones will be identified with the two dilatinos.

Comparing the 11d and 10d line elements in the dimensional reduction (2.11) and remembering the nature of g_s as the dilaton vev, we derive a correspondence between the characteristic length scale of the two theories

$$\ell_{\mathrm{P}} = g_s^{1/3}\ell_s \, . \tag{2.12}$$

When we perform the dimensional reduction, the integration of the compactified coordinate appears in front of the action[4], hence $G_{\mathrm{N}}^{(11)} = (2\pi R_z)\, G_{\mathrm{N}}^{(10)}$. It is then straightforward to derive $R_z = g_s\ell_s$, matching exactly the relation of subsection 1.4.3 between the radius of the circle of compactification of M-theory and the inverse mass of the D0-brane in type-IIA string theory.

2.2.2 A massive deformation: Romans IIA supergravity

An interesting feature that characterizes type-IIA supergravity is the possibility to deform it introducing a sort of mass parameter m, giving rise to *Romans IIA supergravity*, named after L.J. Romans, who constructed it in 1985 [17].

This theory naturally accommodates D8-branes, making it a cornerstone for the comprehension of the full non-perturbative type-IIA string theory. Indeed, we can interpret Romans' mass m as a (constant) 0-form, i.e. $m = F_{(0)}$, which, by Hodge duality, can be related to a 10-form field strength $F_{(10)}$, which implies the existence of a 9-form potential $C_{(9)}$. Exactly to this potential D8-branes couple electrically. $C_{(9)}$ does not have dynamical degrees of freedom and this fact is confirmed by its Maxwell's equation $d\star F_{(10)} = 0$, as will be clear in few lines when

[3] In reality, type-IIA action was originally constructed by dimensional reduction from 11d supergravity [16].

[4] Actually, with respect to the expression coming from the dimensional reduction, we rescaled type-IIA action by a factor of g_s^2 in order to reabsorb the contribution coming from the dilaton $e^{-2\Phi}$. This can be obtained by a simple shift of the dilaton, which does not affect any other term. When comparing the Newton's constants we must not consider it.

we shall introduce the democratic formulation. Hence, Romans' theory behaves as a consistent effective theory of type-IIA superstrings in presence of D8-branes charged under $C_{(9)}$, which, to this extent, can be regarded as a "massive" type-IIA string theory [6].

Unlike IIA supergravity, the 11-dimensional origin of Romans supergravity is still not known since a massive deformation of 11d SUGRA does not exist [18]. Similarly, remains mysterious the 11-dimensional origin of D8-branes, although the formulation of a consistent theory including the latter hints at the existence of 9-dimensional extended objects in eleven dimensions, like M9-branes, which, indeed, have been conjectured [19–21]. On this spirit, attempts to relate directly massive IIA string theory to M-theory have been made [22].

Going back to Romans supergravity, the bosonic part of the action is

$$S_{\mathrm{IIA}} = \frac{g_s^2}{16\pi G_{\mathrm{N}}^{(10)}}\left\{ \int d^{10}x\sqrt{-g}\left[e^{-2\Phi}\left(R + 4|\partial\Phi|^2 - \frac{1}{2}|H|^2\right) - \frac{1}{2}\left(|F_{(0)}|^2 + |F_{(2)}|^2 + |F_{(4)}|^2\right)\right] + \\ - \frac{1}{2}\int\left(B\wedge dC_{(3)}\wedge dC_{(3)} + \frac{1}{3}F_{(0)}B\wedge B\wedge B\wedge dC_{(3)} + \frac{1}{20}F_{(0)}^2 B\wedge B\wedge B\wedge B\wedge B\right)\right\}, \tag{2.13}$$

where the R-R field strengths are modified as follows

$$F_{(2)} := dC_{(1)} + F_{(0)}B\,, \qquad F_{(4)} := dC_{(3)} - H\wedge C_{(1)} + \frac{1}{2}F_{(0)}B\wedge B\,. \tag{2.14}$$

By means of the gauge freedom of the R-R fields it is possible to gauge away $C_{(1)}$, giving mass, at the same time, to the Kalb-Ramond field through a Stückelberg mechanism; for this reason Romans theory is also called *massive type-IIA supergravity*. The possibility to give mass to one of the fields, while keeping the others massless, is a clear signal that supersymmetry must be broken, and indeed this can be shown directly.

The equations of motion of the theory can be grouped according to the sector they belong to. For the NS-NS sector we have the Einstein equations, the equation for the Kalb-Ramond field and the one for the dilaton[5]

$$R_{\mu\nu} = -2\nabla_\mu\nabla_\nu\Phi + \frac{1}{2}|H|^2_{\mu\nu} + \frac{1}{2}\,e^{2\Phi}\sum_{n=0,2,4}\left(|F_{(n)}|^2_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}|F_{(n)}|^2\right), \tag{2.15a}$$

$$d\left(e^{-2\Phi}\star H\right) + F_{(0)}\star F_{(2)} + F_{(2)}\wedge\star F_{(4)} + \frac{1}{2}F_{(4)}\wedge F_{(4)} = 0\,, \tag{2.15b}$$

$$\Box\Phi - |\partial\Phi|^2 + \frac{1}{4}R - \frac{1}{8}|H|^2 = 0 \qquad \text{with} \qquad \Box := \nabla_\mu\nabla^\mu\,. \tag{2.15c}$$

The Maxwell equations governing the R-R sector are

$$d\star F_{(2)} + H\wedge\star F_{(4)} = 0\,, \qquad d\star F_{(4)} + H\wedge F_{(4)} = 0\,. \tag{2.16}$$

Again, all of these can be supplemented with a set of Bianchi identities

$$dH = 0\,, \qquad dF_{(0)} = 0\,, \qquad dF_{(2)} = F_{(0)}H\,, \qquad dF_{(4)} = H\wedge F_{(2)}\,. \tag{2.17}$$

2.2.3 The democratic formulation

It is possible to make the notation much more compact introducing the magnetic duals of the R-R potentials and applying the *democratic formulation* [23], whose name comes from the fact

[5] Here and in the next two sections we adopt the notation $|\omega|^2_{\mu\nu} := \frac{1}{(p-1)!}\,\omega_{\mu\mu_2\ldots\mu_p}\,\omega_\nu{}^{\mu_2\ldots\mu_p}$.

that all the potentials and their magnetic counterparts will be put on the same footing. The definition of the dual field strengths is unique, up to signs due to different conventions, and reads

$$F_{(6)} := -\star F_{(4)}\,, \qquad F_{(8)} := \star F_{(2)}\,, \qquad F_{(10)} := -\star F_{(0)}\,. \tag{2.18}$$

In this way, Maxwell's equations and Bianchi identities can be expressed in a single Bianchi-like equation

$$dF_{(p)} - H \wedge F_{(p-2)} = 0 \qquad (p = 0, 2, 4, 6, 8, 10)\,, \tag{2.19}$$

always keeping in mind that now the six field strengths $F_{(p)}$ and not independent, but related by

$$F_{(p)} = (-1)^{[p/2]} \star F_{(10-p)}\,. \tag{2.20}$$

Notice that, given the set of Bianchi identities in (2.19) and the duality relations in (2.20) it is possible to reconstruct all the Maxwell equations. Moreover, also the equation for the KR field (2.15b) simplifies

$$d\big(e^{-2\Phi} \star H\big) + \frac{1}{2} \star \mathcal{F} \wedge \mathcal{F} = 0\,, \qquad \text{with} \qquad \mathcal{F} := \sum_p F_{(p)}\,. \tag{2.21}$$

The power of this formalism lies in the freedom of choosing between an "electric", a "magnetic" or a mixed formulation. Finally, given the R-R field strengths it is possible to derive the potentials $C_{(p)}$ with $p = 1, 3, 5, 7, 9$ from the implicit relation

$$F_{(p)} = dC_{(p-1)} - H \wedge C_{(p-3)} + F_{(0)}\, e^B\Big|_p\,, \tag{2.22}$$

where in the expansion of the exponential only the form of degree p must be taken. In massless type-IIA it is possible to define also a magnetic potential for the 3-form H as follows

$$H = e^{2\Phi} \star \left(dB_{(6)} - \frac{1}{2} \sum_{n=1}^{3} \star F_{(2n+2)} \wedge C_{(2n-1)} \right). \tag{2.23}$$

2.3 Type-IIB supergravity

Moving to the chiral type-II string theory, its low-energy limit is *type-IIB supergravity* [24–26], whose field content consists of

$$\big\{g_{\mu\nu}, B_{\mu\nu}, \Phi\big\} \cup \big\{\Psi^i_\mu, \lambda^i\big\} \cup \big\{C^{(0)}, C^{(2)}_{\mu\nu}, C^{(4)}_{\mu\nu\rho\sigma}\big\}\,, \qquad \mu = 0, \ldots, 9\,. \tag{2.24}$$

All the fermions are pairs of Majorana-Weyl spinors ($i = 1, 2$) with the same chirality, in contrast to type-IIA. Also in this case the effective theory is maximally supersymmetric, thus IIB supergravity is $\mathcal{N} = 2$ SUSY.

In eleven dimensions there is no space for chiral spinors, hence any dimensional reduction would naturally lead to a pair of 10-dimensional Majorana-Weyl spinors of opposite chirality, obstructing any possible derivation of a type-IIB supergravity action from 11d supergravity. Moreover, the self-duality condition that $C_{(4)}$ must satisfy does not allow neither for a covariant, nor for a supersymmetric action[6]. Still, a Lagrangian formulation of the theory could turn out to be very useful, so what is usually done is to derive an action which gives rise to the correct

[6]The self-duality constraint cannot be imposed at the level of the action, thus, without imposing it by hand, the number of degrees of freedom of $C_{(4)}$ exceeds the right amount, breaking the symmetry between the number of bosonic and fermionic states.

equations of motion and to supplement it with the required self-duality condition. In the string frame this action takes the following expression

$$S_{\text{IIB}} = \frac{g_s^2}{16\pi G_{\text{N}}^{(10)}}\left\{\int d^{10}x\sqrt{-g}\Big[e^{-2\Phi}\Big(R+4|\partial\Phi|^2-\frac{1}{2}|H|^2\Big)-\frac{1}{2}\big(|F_{(1)}|^2+|F_{(3)}|^2+\frac{1}{2}|F_{(5)}|^2\big)\Big]+ \right.$$
$$\left. -\frac{1}{2}\int B\wedge dC_{(2)}\wedge dC_{(4)}\right\}, \tag{2.25}$$

where the definitions of $F_{(1)}$, $F_{(3)}$ and $F_{(5)}$ are the same as in (2.22), but in the massless case $F_{(0)} = 0$. The structure of the action is very similar to the one of (massless) type-IIA supergravity (2.9), but with an extra factor of $1/2$ in the kinetic term of $C_{(4)}$ that takes into account that the 4-form potential has twice the right number of degrees of freedom, which are indeed halved by the self-duality condition. As we said, this constraint must be imposed by hand

$$\star F_{(5)} = F_{(5)}\,. \tag{2.26}$$

Like we did in type-IIA supergravity, instead of writing down directly the equations of motion we can go one step further and introduce a democratic formulation. In this way, for what concerns the NS-NS sector, the equation for the Kalb-Ramond field can be written as in (2.21), where now the summation in the definition of $\mathcal{F}$ runs over p odd, whereas the equation for the dilaton remains unmodified with respect to (2.15c). Einstein's equations now read

$$R_{\mu\nu} = -2\nabla_\mu\nabla_\nu\Phi+\frac{1}{2}|H|^2_{\mu\nu}+\frac{1}{2}\,e^{2\Phi}\sum_{n=1,3}\Big(|F_{(n)}|^2_{\mu\nu}-\frac{1}{2}\,g_{\mu\nu}|F_{(n)}|^2\Big)+\frac{1}{4}\,e^{2\Phi}|F_{(5)}|^2_{\mu\nu}\,. \tag{2.27}$$

Once again the equations for the R-R fields, i.e. the Bianchi identities and Maxwell equations, can be written in the compact form (2.19), but with odd values of p. The duality relation (2.20) still holds and, additionally, implies the self-duality condition of $F_{(5)}$. The expression of the magnetic potentials $C_{(6)}$ and $C_{(8)}$ can be deduced by equation (2.22). In analogy to type-IIA, we can define a magnetic potential for the 3-form H also in type-IIB

$$H = e^{2\Phi}\star\Big(dB_{(6)}+\frac{1}{2}\sum_{n=0}^{3}\star F_{(2n+3)}\wedge C_{(2n)}\Big)\,. \tag{2.28}$$

2.3.1 S-duality action

In subsection 1.4.2 we introduced S-duality and its action on type-IIB non-perturbative objects, however, in order to completely describe this duality, we need to know also how it acts on the string massless spectrum.

The first piece is the graviton, or the metric, which, written in the Einstein frame, is not affected by the transformation: $d\tilde{s}^2_{\text{E}} = ds^2_{\text{E}}$. Defined the complex scalar

$$\tau := C_{(0)} + i\,e^{-\Phi}\,, \tag{2.29}$$

the action of S-duality is

$$\tilde{\tau} = -\frac{1}{\tau}\,. \tag{2.30}$$

We immediately see how for $C_{(0)} = 0$ this implies $\tilde{g}_s = 1/g_s$, as expected. The NS-NS and R-R 2 forms transform as

$$\tilde{B} = C_{(2)} \qquad \text{and} \qquad \tilde{C}_{(2)} = -B\,, \tag{2.31}$$

while $F_{(5)}$ remains unmodified.

As we already mentioned, type-IIB S-duality invariance can be extended to a symmetry under a bigger group. In the case of type-IIB supergravity this group is $\mathrm{SL}(2,\mathbb{R})$, which in the corresponding string theory is broken to $\mathrm{SL}(2,\mathbb{Z})$ by quantum effects such as charge quantization. Again, the metric expressed in the Einstein frame is inert, while for the complex scalar τ we have

$$\tilde{\tau} = \frac{a\tau + b}{c\tau + d}\,, \tag{2.32}$$

with $ad - cb = 1$. B and $C_{(2)}$ transform as a doublet

$$\begin{pmatrix} \tilde{C}_{(2)} \\ \tilde{B} \end{pmatrix} = S \begin{pmatrix} C_{(2)} \\ B \end{pmatrix}\,, \qquad \text{with} \qquad S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})\,, \tag{2.33}$$

while the combination $C_{(4)} - \frac{1}{2} B \wedge C_{(2)}$ is left unchanged and thus $F_{(5)}$ as well. With these conventions, S-duality corresponds to the $\mathrm{SL}(2,\mathbb{R})$ transformation with $a = d = 0$, $b = -1$ and $c = 1$. Useful formulae are the transformations of the H and $F_{(3)}$ fields strengths

$$\begin{aligned} \tilde{H} &= (d + c\,C_{(0)})H + c\,F_{(3)}\,, \\ \tilde{F}_{(3)} &= \frac{1}{|c\tau + d|^2}\Big[(d + c\,C_{(0)})F_{(3)} - c\,e^{-2\Phi} H\Big]\,, \end{aligned} \tag{2.34}$$

and of the metric in the string frame

$$d\tilde{s}^2 = |c\tau + d|^2 ds^2\,. \tag{2.35}$$

2.3.2 T-duality action

Like we did in the previous subsection for S-duality, we now study the action of T-duality on the massless spectrum of type-II superstrings.

The mapping of the NS-NS sector goes under the name of *Buscher's rules* [27] and applies in the same manner when going from IIA to IIB and vice versa[7]

$$\begin{aligned} \tilde{g}_{\mu\nu} &= g_{\mu\nu} - \frac{g_{\mu x} g_{\nu x} - B_{\mu x} B_{\nu x}}{g_{xx}}\,, & \tilde{g}_{\mu y} &= \frac{B_{\mu x}}{g_{xx}}\,, & \tilde{g}_{yy} &= \frac{1}{g_{xx}}\,, \\ \tilde{B}_{\mu\nu} &= B_{\mu\nu} - \frac{B_{\mu x} g_{\nu x} - g_{\mu x} B_{\nu x}}{g_{xx}}\,, & \tilde{B}_{\mu y} &= \frac{g_{\mu x}}{g_{xx}}\,, \\ \tilde{\Phi} &= \Phi - \frac{1}{2} \ln g_{xx}\,, \end{aligned} \tag{2.36}$$

where x and y are the coordinates involved in the dualization. Keeping in mind the origin of T-duality, they are the angular coordinates of the compactification circles, hence they must be periodic of 2π. The transformations of the R-R sector in the mapping IIA $\mapsto$ IIB read

$$\begin{aligned} \tilde{C}^{(2n)}_{\mu_1 \ldots \mu_{2n-1} y} &= -C^{(2n-1)}_{\mu_1 \ldots \mu_{2n-1}} + (2n-1)\,\tilde{B}_{[\mu_1|y|} C^{(2n-1)}_{\mu_2 \ldots \mu_{2n-1}]x}\,, \\ \tilde{C}^{(2n)}_{\mu_1 \ldots \mu_{2n}} &= C^{(2n+1)}_{\mu_1 \ldots \mu_{2n} x} - 2n\,B_{[\mu_1|x|} \tilde{C}^{(2n)}_{\mu_2 \ldots \mu_{2n}]y}\,, \end{aligned} \tag{2.37}$$

which lead to

$$\begin{aligned} \tilde{F}^{(2n+1)}_{\mu_1 \ldots \mu_{2n} y} &= -F^{(2n)}_{\mu_1 \ldots \mu_{2n}} - 2n\,\tilde{B}_{[\mu_1|y|} F^{(2n)}_{\mu_2 \ldots \mu_{2n}]x}\,, \\ \tilde{F}^{(2n+1)}_{\mu_1 \ldots \mu_{2n+1}} &= F^{(2n+2)}_{\mu_1 \ldots \mu_{2n+1} x} + (2n+1)\,B_{[\mu_1|x|} \tilde{F}^{(2n+1)}_{\mu_2 \ldots \mu_{2n+1}]y}\,, \end{aligned} \tag{2.38}$$

for the related field strengths. The case IIB $\mapsto$ IIA can be retrieved straightforwardly.

[7] Here we adopt the conventions of [12], but many others exist with a flip in the sign of the B and/or $C_{(p)}$ fields.

2.4 Heterotic supergravities

By analogy with type-II, also heterotic string theories admit an effective formulation. We recall the massless field content of the string theory

$$\{g_{\mu\nu}, B_{\mu\nu}, \Phi\} \cup \{\Psi_\mu, \lambda\} \cup \{A^I_\mu, \chi^I\}\,, \qquad \mu = 0, \dots, 9\,, \tag{2.39}$$

where A^I_μ are the SO(32) or $\mathrm{E}_8\times\mathrm{E}_8$ Yang-Mills gauge fields, χ^I their superpartners, the *gauginos*, and we notice the absence of R-R fields. Since performing the low-energy limit no further supersymmetry is broken, heterotic supergravities are $\mathcal{N}=1$ SUSY.

We already argued that at low energies heterotic string theory is effectively described in terms of a double expansion in α' and g_s, whose zeroth-order corresponds to $\mathcal{N}=1$ supergravity. In chapter 7 α' contributions to the effective theory will be crucial, thus we retain first-order α' terms, keeping ourselves content with the tree-level effective action. For simplicity we do not include the Yang-Mills fields, thus the two heterotic theories reduce to the same form[8].

The effective action of the heterotic superstring at first order in α' is given by [29, 30]

$$S_{\mathrm{H}} = \frac{g_s^2}{16\pi G_{\mathrm{N}}^{(10)}} \int d^{10}x\sqrt{-g}\, e^{-2\Phi}\left(R + 4|\partial\Phi|^2 - \frac{1}{2}|H|^2 - \frac{\alpha'}{8} R_{(-)\mu\nu}{}^a{}_b R_{(-)}{}^{\mu\nu b}{}_a + \dots\right). \tag{2.40}$$

Here, $R_{(-)}{}^a{}_b$ is the curvature 2-form of the torsionful spin connection defined as $\omega_{(-)}{}^a{}_b := \omega^a{}_b - \frac{1}{2}H_\mu{}^a{}_b\, dx^\mu$, namely

$$R_{(-)}{}^a{}_b = d\omega_{(-)}{}^a{}_b + \omega_{(-)}{}^a{}_c \wedge \omega_{(-)}{}^c{}_b\,. \tag{2.41}$$

In heterotic theories the field strength of the Kalb-Ramond 2-form B is modified by the inclusion of a the *Lorentz Chern-Simons term*

$$H := dB + \frac{\alpha'}{4}\,\Omega^{\mathrm{L}}_{(-)}\,, \tag{2.42}$$

where

$$\Omega^{\mathrm{L}}_{(-)} = \omega_{(-)}{}^a{}_b \wedge d\omega_{(-)}{}^b{}_a + \frac{2}{3}\,\omega_{(-)}{}^a{}_b \wedge \omega_{(-)}{}^b{}_c \wedge \omega_{(-)}{}^c{}_a\,. \tag{2.43}$$

The corresponding Bianchi identity now reads

$$dH = \frac{\alpha'}{4} R_{(-)}{}^a{}_b \wedge R_{(-)}{}^b{}_a\,. \tag{2.44}$$

Notice that the action includes a tower of corrections of all powers in α' due to the recursive definition of the KR field strength. Historically, the term of quadratic order in curvature in (2.40) was found imposing supersymmetry of the theory at first order in α' after including the Chern-Simons term [31]. Further corrections of higher power in the curvature $R_{(-)}$ are required to recover supersymmetry order by order. The quartic effective action of heterotic theory, constructed in [30], was also obtained using this criterion. Moreover, additional higher-curvature corrections unrelated to the supersymmetrization of the Kalb-Ramond kinetic term also appear. Not much is known about them, although it has been conjectured that it should be possible to write them in terms of contractions of the curvature $R_{(-)}$ and the metric [32].

The equations of motion of the action (2.40), at first order in α', are

$$R_{\mu\nu} = -2\nabla_\mu\nabla_\nu\Phi + \frac{1}{2}|H|^2_{\mu\nu} + \frac{\alpha'}{4} R_{(-)\mu\lambda}{}^a{}_b R_{(-)\nu}{}^{\lambda b}{}_a\,, \tag{2.45a}$$

$$d\big(e^{-2\Phi} \star H\big) = 0\,, \tag{2.45b}$$

$$\Box\Phi - |\partial\Phi|^2 + \frac{1}{4}R - \frac{1}{8}|H|^2 - \frac{\alpha'}{32} R_{(-)\mu\nu}{}^a{}_b R_{(-)}{}^{\mu\nu b}{}_a = 0\,. \tag{2.45c}$$

[8] An example with non-trivial Yang-Mills fields can be found in [28].

As expected, the zeroth-order supergravity equations of motion can be recovered from these expressions by setting $\alpha' = 0$.

2.5 Down to lower dimensions

In the previous chapter we learnt that any consistent string theory requires a 10-dimensional spacetime in which to be embedded, or even 11-dimensional in the case of M-theory. A natural problem immediately arises since the world we experience has four dimensions and this issue can be solved only providing a mechanism that allows us to bring a higher-dimensional theory to a lower number of dimensions. This result can be achieved by means of one of the most important resources in the toolbox of string theory, the *dimensional reduction*.

2.5.1 Kaluza-Klein reduction

The general strategy of any dimensional reduction is to firstly compactify the spacetime, i.e. to consider the whole D-dimensional spacetime, here $\mathcal{M}_D$, as the fibration of a $(D-n)$-dimensional base space $\mathcal{M}_{D-n}$ with an n-dimensional compact internal space X_n as fibers: $X_n \hookrightarrow \mathcal{M}_D \to \mathcal{M}_{D-n}$[9]. Then, one takes the limit in which the size of the compact manifold is small. In some fortunate cases the degrees of freedom associated to the compact dimensions decouple from the others and the internal space can be "integrated away", leading to the dimensional reduction itself. Even though after the dimensional reduction the internal space becomes invisible, its topological properties will determine (part of) the field content and structure of the lower-dimensional theory. The physics of the higher-dimensional theory will still be captured by the lower-dimensional one.

The best-known example of dimensional reduction is the *Kaluza-Klein (KK) reduction*, which we already saw in action connecting 11d and type-IIA supergravities in subsection 2.2.1 and we will encounter again in subsection 3.5.1. This method was developed in the 1920s and applied for the first time in an unfruitful attempt to unify gravity and electromagnetism [33,34]. Since the internal space is compact, the basic idea is to expand all the D-dimensional fields in eigenfunctions of some kind of mass operator on X_n, e.g. the Laplacian in the case of free scalars, and to keep only the zeroth-order, i.e. the massless, modes on the grounds that the massive ones become too heavy to be detected when the internal space is small enough. Performing this reduction a set of scalars and vector fields appear. Whereas the nature of the latter is clear, they are the gauge vectors associated to the group of the symmetries of X_n, e.g. to the abelian U(1) isometry of $X_1 = S^1$, the former can pose problems. The first we address is that setting a scalar to zero in the reduced action could be very dangerous, indeed we could end up with a set of lower-dimensional equations of motion whose solutions are not a solution of the original, higher-dimensional ones. A truncation is called *consistent* if all the gauge bosons of the isometry group of X_n are retained and if setting the truncated fields to zero is consistent with their own equations of motion [35].

Reduction on tori

Starting from the simplest, original KK reduction, it is possible to move further compactifying on a product of n circles, i.e. an n-torus T^n. Although topologically different from $\mathbb{R}^n$, an n-torus is locally flat and this property ensures that the $(D-n)$-dimensional supergravity obtained will preserve all the original SUSY generators. Moreover, it will be ungauged, hence there will not

[9]We point out that the most common methods consider the higher-dimensional background simply as the product manifold $\mathcal{M}_D = \mathcal{M}_{D-n} \times X_n$, where the product can be a Cartesian product or a warped product over the compact manifold.

be any potential for the scalars originated from the dimensional reduction. Here shows up the second problem that may arise dealing with the scalar fields coming from a KK reduction: in the absence of a potential, the vevs of the massless scalars retained in the compactification, the *moduli*, are unconstrained. The intrinsic issue is that such degrees of freedom are neither observed nor predicted by any theory and that their undetermined vevs do not allow for specific predictions for many physical quantities[10]. What we need is a mechanism that gives the scalars a potential, hence a fixed vev; what we need is a process of *moduli stabilization*, on which we shall come back shortly. A well-known example of reduction on tori is the maximal ($\mathcal{N} = 8$) 4d ungauged SUGRA obtained by reduction of 11d supergravity on T^7 [36].

Reduction on Calabi-Yau manifolds

A more elaborate example of compact manifold involved in KK reductions are the *Calabi-Yau n-folds* (CY_n), compact n-dimensional Kähler manifolds with $\mathrm{SU}(n)$ holonomy[11]. By definition, a Calabi-Yau manifold is Ricci-flat and usually the consistent truncations are constructed as the deformations of the CY, in particular of its complex structure or its Kähler form, preserving this condition. Unlike the tori, Calabi-Yau manifolds do not have isometries and for this reason are extremely useful for constructing supergravities with less preserved supersymmetries. Reducing on a CY may still give an ungauged theory, thus again a mechanism to stabilize the scalar fields into a vacuum could be missing. Examples of reduction of IIA and 11d supergravities on CY_3 to give $\mathcal{N} = 2$ 4d and 5d ungauged supergravity can be found in [37] and [38] respectively.

Reduction on spheres

Another interesting case is given by the reduction over coset spaces G/H, of which homogeneous spaces are a special class; in particular, we shall consider n-spheres S^n, known to be diffeomorphic to $\mathrm{SO}(n+1)/\mathrm{SO}(n)$. This kind of truncation preserves all the initial supersymmetries, but, in contrast to toroidal compactifications, gives rise to gauged lower-dimensional supergravities. For further details, see, e.g., [39]. Notable examples of this reduction applied to 11d supergravity are the maximal ($\mathcal{N} = 8$) 4d gauged SUGRA obtained compactifying on an S^7 [40] and the maximal ($\mathcal{N} = 4$) 7d gauged SUGRA with internal space S^4 [41, 42]. Moreover, maximal ($\mathcal{N} = 8$) 5d gauged SUGRA can be constructed reducing type-IIB supergravity on an S^5 [43].

As mentioned, spherical KK reductions automatically generate a potential able to fix the vevs of the various moduli, but this mechanism can be enforced also to other families of internal spaces, like tori or Calabi-Yau manifolds. In the first case one can apply an alternative type of dimensional reduction, the *twisted reduction*, and take full advantage of the symmetries of the internal space [44]. When compactifying on a CY_n a different approach can be adopted, applying the KK reduction and including in the higher-dimensional theory non-trivial gauge fields "living inside" the fiber space. Their field strengths, called *fluxes*, will wrap p-cycles in the internal manifold X_n and the deformations induced on the reduction ansatz will give rise to a scalar potential in the lower-dimensional theory. KK dimensional reductions in which the compact directions support fluxes go under the name of *flux compactifications* (cf. [45] for a review) and one of the most representative examples is the 4d $\mathcal{N} = 2$ gauged SUGRA obtained from type-II supergravities compactified on a CY_3 [46].

In the following subsections we shall present two different realizations of Kaluza-Klein spherical reduction applied to 11d and type-IIA supergravities.

[10] We recall the example of the dilaton in type-IIA supergravity, originated by KK reduction from eleven dimensions and whose vev gives the string coupling g_s.

[11] Actually, there are many, sometimes inequivalent, definitions of Calabi-Yau manifolds, requiring, e.g., the existence of a holomorphic n-form non-zero everywhere or the vanishing of the first Chern class.

2.5.2 M-theory origin of 7d $\mathcal{N} = 1$ supergravity

In this subsection we introduce the M-theory embedding of minimal $\mathcal{N} = 1$ gauged supergravity in 7d. The theory in question preserves 16 supercharges and only the supergravity multiplet is retained by the truncation from 11d. All the oscillations around the AdS_7 vacuum are thus encoded in the 7d gravitational field, a real scalar X_7, a 3-form gauge potential $\mathcal{B}_{(3)}$ and three SU(2) vector fields $\mathcal{A}_7^i$ [47]. We consider a further truncation of the theory in which all the vector fields are vanishing.

The truncation ansatz we are going to discuss has been worked out in [48]. It is characterized by an 11-dimensional metric of the following form

$$\begin{aligned} ds_{11}^2 &= \Sigma_7^{1/3}\big(ds_7^2 + 2g^{-2}X_7^3\, ds_4^2\big)\,, \\ ds_4^2 &= d\xi^2 + c^2\Sigma_7^{-1}X_7^{-4}\, ds_{S^3}^2\,, \end{aligned} \tag{2.46}$$

where $\Sigma_7 := c^2 X_7 + s^2 X_7^{-4}$, $c := \cos\xi$, $s := \sin\xi$ and g is a free parameter whose nature will become clear in few lines. The 11d 4-flux takes the form

$$\begin{aligned} G_{(4)} = &-\frac{4}{\sqrt{2}}\, g^{-3}c^3\Sigma_7^{-2}W_7\, d\xi\wedge \mathrm{vol}_{S^3} - \frac{20}{\sqrt{2}}\, g^{-3}s\, c^4\Sigma_7^{-2}X_7^{-4}\, dX_7\wedge \mathrm{vol}_{S^3} \\ &+ s\,\mathcal{F}_{(4)} + \sqrt{2}\, g^{-1}c\, X_7^4 \star_7\mathcal{F}_{(4)}\wedge d\xi\,, \end{aligned} \tag{2.47}$$

where $W_7 := s^2X_7^{-8} - 2c^2X_7^2 + 3c^2X_7^{-3} - 4X_7^{-3}$ and $\mathcal{F}_{(4)} = d\mathcal{B}_{(3)}$. As it has been pointed out in [48], in order to describe the right number of degrees of freedom, the 3-form $\mathcal{B}_{(3)}$ has to satisfy an “odd-dimensional self-duality condition”

$$X_7^4 \star_7\mathcal{F}_{(4)} = -2h\,\mathcal{B}_{(3)} \qquad \text{with} \qquad h = \frac{g}{2\sqrt{2}}\,, \tag{2.48}$$

where the last relation is fixed by the truncation. The isometry group of the resulting 7d theory is given by $\mathbb{R}^+\times\mathrm{SO}(3)$ and there are two types of gaugings. One is described by the parameter g and corresponds to the gauging of the R-symmetry $\mathrm{SU}(2)_R$, while the other is a Stückelberg deformation of $\mathcal{B}_{(3)}$ described by h. The general form of the superpotential is given by

$$f_7(h,g,X_7) = \frac{1}{2}\Big(h\,X_7^{-4} + \sqrt{2}\,g\,X_7\Big)\,, \tag{2.49}$$

where the two gauging parameters are linked by the truncation through the algebraic relation $h = \frac{g}{2\sqrt{2}}$. The Lagrangian is given by

$$\mathscr{L}_7 = R_7 - 5\,X_7^{-2}\star_7 dX_7\wedge dX_7 - \frac{1}{2}\,X_7^4\star_7\mathcal{F}_{(4)}\wedge\mathcal{F}_{(4)} - h\,\mathcal{F}_{(4)}\wedge\mathcal{B}_{(3)} - V_7 \tag{2.50}$$

and the scalar potential reads

$$V_7 = \frac{4}{5}\,X_7^2\big(D_X f_7\big)^2 - \frac{24}{5}\,f_7^2\,. \tag{2.51}$$

The theory described by (2.50) has a $\mathcal{N} = 1$ AdS_7 vacuum at $X_7 = 1$ and vanishing gauge potentials; in this case the internal 4d manifold of (2.46) becomes a round 4-sphere. Since the theory we consider can be embedded into the maximally supersymmetric supergravity in seven dimensions, we can relate this 7d vacuum to the $\mathrm{AdS}_7\times S^4$ Freund-Rubin vacuum of M5-branes. In this particular case the 4-flux takes the form

$$G_{(4)} = \frac{12}{\sqrt{2}}\,g^{-3}c^3\, d\xi\wedge\mathrm{vol}_{S^3}\,. \tag{2.52}$$

2.5.3 Massive IIA origin of 6d Romans supergravity

We now move to the consistent truncation of massive IIA string theory to 6d Romans supergravity. In its minimal realization, i.e. when only the supergravity multiplet is retained, this truncation produces a 6d gauged supergravity preserving $\mathcal{N} = (1,1)$ supersymmetry, which is usually called Romans supergravity. Its field content is given by the 6d gravitational field, a real scalar X_6, a 2-form gauge potential $\mathcal{B}_{(2)}$, three SU(2) vectors $\mathcal{A}_6^i$ and one abelian vector $\mathcal{A}_6^0$ [49]. Here we shall restrict to the case of vanishing vector fields.

The truncation from massive IIA supergravity to Romans supergravity was worked out in [50]. The ansatz for the metric is characterized by an internal manifold locally realized as a fibration of a 3-sphere over a line,

$$\begin{aligned} ds_{10}^2 &= s^{-1/3}\Sigma_6^{1/2}X_6^{-1/2}\big(ds_6^2 + 2g^{-2}X_6^2\,ds_4^2\big)\,, \\ ds_4^2 &= d\xi^2 + c^2\Sigma_6^{-1}X_6^{-3}\,ds_{S^3}^2\,, \end{aligned} \tag{2.53}$$

where $\Sigma_6 := c^2X_6 + s^2X_6^{-3}$, $c := \cos\xi$, $s := \sin\xi$ and g is a free parameter. The 10d fluxes are decomposed as [50]

$$\begin{aligned} e^{\Phi} &= s^{-5/6}\Sigma_6^{1/4}X_6^{-5/4}\,, \qquad H_{(3)} = s^{2/3}\mathcal{F}_{(3)} + \sqrt{2}\,g^{-1}m\,s^{-1/3}c\,\mathcal{B}_{(2)}\wedge d\xi\,, \\ F_{(0)} &= m\,, \qquad F_{(2)} = m\,s^{2/3}\mathcal{B}_{(2)}\,, \\ F_{(4)} &= -\frac{4\sqrt{2}}{3}\,g^{-3}s^{1/3}c^3\Sigma_6^{-2}W_6\,d\xi\wedge \mathrm{vol}_{S^3} - 8\sqrt{2}\,g^{-3}s^{4/3}c^4\Sigma_6^{-2}X_6^{-3}\,dX_6\wedge\mathrm{vol}_{S^3} \\ &\quad -\sqrt{2}\,g^{-1}s^{1/3}c\,X_6^4\star_6\mathcal{F}_{(3)}\wedge d\xi - m\,s^{4/3}X_6^{-2}\star_6\mathcal{B}_{(2)}\,, \end{aligned} \tag{2.54}$$

where $W_6 := s^2X_6^{-6} - 3c^2X_6^2 + 4c^2X_6^{-2} - 6X_6^{-2}$ and $\mathcal{F}_{(3)} = d\mathcal{B}_{(2)}$. The 6d theory resulting from this truncation preserves 16 real supercharges and has $\mathbb{R}^+\times \mathrm{SO}(4)$ global isometry group. The two parameters g and m are associated, respectively, to the gauging of the $\mathrm{SU}(2)_R$ R-symmetry group, realized as the diagonal SU(2) within SO(4), and to a mass deformation of the 2-form. In particular, the truncation ansatz (2.53) produces a scalar potential in six dimensions defined by the superpotential

$$f_6(m,g,X_6) = \frac{1}{8}\left(m\,X_6^{-3} + \sqrt{2}\,g\,X_6\right)\,, \tag{2.55}$$

where the two gauging parameters are linked as $m = \frac{\sqrt{2}}{3}g$. The 6d Lagrangian has the form

$$\begin{aligned} \mathscr{L}_6 = R_6 - 4\,X_6^{-2}\star_6 dX_6\wedge dX_6 - \frac{1}{2}\,X_6^4\star_6\mathcal{F}_{(3)}\wedge\mathcal{F}_{(3)} - V_6 \\ - m^2\,X_6^{-2}\star_6\mathcal{B}_{(2)}\wedge\mathcal{B}_{(2)} - \frac{1}{3}\,m^2\mathcal{B}_{(2)}\wedge\mathcal{B}_{(2)}\wedge\mathcal{B}_{(2)}\,, \end{aligned} \tag{2.56}$$

where the scalar potential is given by

$$V_6 = 16\,X_6^2\big(D_X f_6\big)^2 - 80\,f_6^2\,. \tag{2.57}$$

The 6d theory (2.56) has a supersymmetric AdS_6 vacuum at $X_6 = 1$ and vanishing gauge potentials. In this case the 4d internal manifold in (2.53) becomes a round 4-sphere[12] and the only non-zero terms in the fluxes and dilaton are

$$F_{(4)} = \frac{20\sqrt{2}}{3}\,g^{-3}s^{1/3}c^3\,d\xi\wedge\mathrm{vol}_{S^3}\,, \qquad e^{\Phi} = s^{-5/6}\,. \tag{2.58}$$

This vacuum corresponds to the string vacuum of the D4-D8 set-up introduced in [51], indeed these are exactly the fluxes and the dilaton describing the near-horizon limit of the solution constructed therein.

[12]More precisely this is the upper hemisphere of a 4-sphere with boundary at $\xi \to 0$ [50, 51].

Chapter 3

Lower-dimensional $\mathcal{N} = 2$ gauged supergravities

The development of a connection between bosons and fermions and the resulting introduction of supersymmetry laid the foundations for many prominent theories. Among these, is supergravity, a deformation of standard gravity able to incorporate supersymmetry as "gauge" theory. Indeed, as we mentioned in the introduction, gravity arises in a natural way when supersymmetry is made local. The first example appeared in literature is the 4-dimensional supergravity theory constructed by Freedman, van Nieuwenhuizen, and Ferrara in 1976 [3], which includes the smallest number of supersymmetry generators, $\mathcal{N} = 1$, and for this reason is also called *minimal*. On the contrary, theories with more generators are named *extended*.

$\mathcal{N} = 2$ supergravities are the simplest theories of extended supergravity and, in addition to the 10-dimensional type-II theories of the previous chapter, they have been widely studied also in lower dimensions, such as $D = 4, 5$. These are exactly the cases we shall focus on in this chapter. Although, on one hand, the increased number of supersymmetries puts more restrictions on possible matter couplings with respect to the minimal case, on the other, it still leaves great leeway in constructing theories if compared to the $\mathcal{N} > 2$ ones, where supersymmetry imposes many constraints, like in $\mathcal{N} = 4$ SYM. This implies that different matter can be coupled and, in addition, that different theories can be written for the same matter content. In fact, the freedom retained by $\mathcal{N} = 2$ theories allows for many explicit realizations of the special and quaternionic geometries governing the couplings of the vector and hypermultiplets.

An even richer structure is obtained gauging some of the isometries of the theory. In addition to the R-symmetry, matter-coupled supergravity enjoys the global symmetries of the spaces generated by the scalar fields of the various supermultiplets and, as we shall see later, (almost) all of them can be gauged. Many are the ways a theory can be gauged, from the isometries (or subgroups of them) we decide to make local to how we do it and all these choices give rise to different theories. As a result, a gauged supergravity theory is uniquely fixed not only by the Lagrangian, but the matter content, the scalar geometry and the gauging adopted are also required.

Besides the interest raised from the wide range of possible different theories that the gauging can produce, we already stated its importance in stabilizing the scalars when deriving lower-dimensional supergravities from eleven or ten dimensions. As we anticipated in the previous chapter, despite being attractive in themselves, lower-dimensional supergravity theories are much studied also for their connection with 11- and 10-dimensional supergravities and the gauging of the former plays a fundamental role in order to produce a scalar potential. In turn, this potential can give the scalars a fixed vev, a prerequisite for lower-dimensional theories to arise from realistic compactifications.

Maximal lower-dimensional ungauged supergravities can be obtained by means of a Kaluza-Klein reduction of an 11- or 10-dimensional theory on tori, whereas the breaking of supersymmetry requires something more sophisticated. As an example, in order to retain only $\mathcal{N} = 2$ SUSY generators in $D = 4$ or $D = 5$ one can compactify the original theory on a Calabi-Yau, cf. [37, 52–54] and [38]. Gauged supergravities can be the result of different processes. Through flux compactification on a CY 3-fold one can go down to four, see [46] or additionally [45, 55–57], or five dimensions, see [58]. An alternative approach is the reduction on spheres, followed by a consistent truncation to $\mathcal{N}=2$: 11d supergravity on an S^7 gives maximal 4d SUGRA [40], while IIB on an S^5 leads to the 5-dimensional counterpart of the latter [43]. Moreover, a direct reduction to $\mathcal{N}=2$ in both cases, restricted to give the *stu* models, can be found in [59]. Lastly, we mention the possibility of obtaining gauged theories reducing on manifolds equipped with a G-structure, see, e.g., [60, 61].

We open this chapter presenting the matter content of the two theories under analysis, 4- and 5-dimensional $\mathcal{N}=2$ gauged supergravity. In section 3.2 we discuss the geometric properties of their moduli spaces, moving to their isometries and the possible gaugings in section 3.3. In the last two sections we introduce the Lagrangian of our theories and derive the equations of motion, focusing also on the Fayet-Iliopoulos gauging. Moreover, we give an explicit realization of the r-map connecting the 4d and 5d supergravities.

3.1 Matter content

In any SUSY theory matter is organized in supermultiplets, which are reviewed in generality in appendix A.3, where we give a flavour of the existing multiplets in presence of $\mathcal{N}$ supersymmetry generators. We shall now specify to the case of $\mathcal{N}=2$ and include n_V vector multiplets and n_H hypermultiplets. The field content is organized as follows:

- **Supergravity multiplet**
  - $D=4,5$: $(e^a_\mu, \psi^{(A)}_\mu, A_\mu)$ with $A=1,2$ SU(2) index; it contains the graviton (or equivalently the vierbein/fünfbein), an SU(2) doublet of spin-3/2 fermions (*gravitinos*)[1] and one vector field (*graviphoton*).
- **Vector multiplets**
  - $D=4$: $(A^\alpha_\mu, \chi^{\alpha\,(A)}, z^\alpha)$ with $\alpha = 1,\ldots,n_V$; each of them contains a vector field, an SU(2) doublet of spinors (*gauginos*) and a complex scalar.
  - $D=5$: $(A^i_\mu, \chi^{i\,(A)}, \phi^i)$ with $i=1,\ldots,n_V$; each of them contains a vector field, an SU(2) doublet of spinors (*gauginos*) and a real scalar.
- **Hypermultiplets**
  - $D=4,5$: $(q^u, \zeta^{A'})$ with $u=1,\ldots,4n_H$, $A'=1,\ldots,2n_H$; each of them contains four scalars (*hyperscalars*) and two spinors (*hyperinos*). Hyperinos do not transform under SU(2), whereas the entire set of all of them transforms in the fundamental representation of $\mathrm{Sp}(2n_H,\mathbb{R})$.

In $D=4$ both gravitinos and gauginos are (possibly vectors of) couples of Weyl spinors with opposite chirality, for a total of four real independent components, and are indicated as $(\psi^{(A)}_\mu, \psi_{(A)\,\mu})$ and $(\chi^{(A)}, \chi_{(A)})$, where the position of the index is related to the chirality of the spinor. In the same way hyperinos can be decomposed as $(\zeta^{A'}, \zeta_{A'})$. See, e.g., [62] for a more complete analysis. In $D=5$ gravitinos and gauginos are symplectic Majorana spinors, each of them with eight real independent components, but the two spinors of the doublet are related

[1] In 4d the gravitinos (and gauginos) form a doublet under the SU(2) factor of the R-symmetry group $\mathrm{U}(2)_R \simeq \mathrm{SU}(2)_R \rtimes \mathrm{U}(1)_R$, whereas in 5d they form a doublet under the $\mathrm{SU}(2)_R \simeq \mathrm{USp}(2)_R$ R-symmetry group (cf. table A.3 in the appendix).

by the symplectic Majorana condition, which halves the total number of degrees of freedom. In the end, in both cases the total number of real components is eight (see appendix A.1 for the list of irreducible spinors in each spacetime dimension).

For what concerns the vector fields, we shall treat the graviphoton and the other vectors on the same footing, thus they will be labelled as

$$\begin{aligned} A^I_\mu &= (A^0_\mu, A^\alpha_\mu)\,, && I = 0, \dots, n_V && D=4\,, \\ A^I_\mu &= (A^i_\mu, A^{n_V+1}_\mu)\,, && I = 1, \dots, n_V+1 && D=5\,, \end{aligned} \tag{3.1}$$

where A^0_μ and $A^{n_V+1}_\mu$ are the graviphoton respectively in $D=4,5$. The reason, and the power, of this choice will be clear later.

The scalar fields belonging to the *matter multiplets*, a common name for the vector and hypermultiplets, parametrize a peculiar manifold called *moduli space*[2] and their kinetic term is described by the non-linear σ-model

$$\mathscr{L}_{\text{scalar}} = \frac{1}{2}\, g_{XY}(\varphi)\, \partial_\mu \varphi^X \partial^\mu \varphi^Y\,, \tag{3.2}$$

where φ is the collective scalar $\varphi^X = (z^\alpha, q^u)$ or $\varphi^X = (\phi^i, q^u)$ respectively in $D=4,5$ and g_{XY} is the metric of the moduli space. The latter can be decomposed as a product manifold

$$\mathcal{M}_s = \mathcal{M}_v \otimes \mathcal{M}_h\,, \tag{3.3}$$

where $\mathcal{M}_v$ and $\mathcal{M}_h$ are the manifolds parametrized by the scalars belonging respectively to the vector multiplets and the hypermultiplets. The nature of these manifolds is[3]

- $\mathcal{M}_v$ special Kähler manifold [63] $D=4$
  very special (real) manifold [64] $D=5$
- $\mathcal{M}_h$ quaternionic-Kähler manifold [65] $D=4,5$

Often the moduli space enjoys some symmetries, whose information is encoded in a Lie group $G_s = G_v \times G_h$, where G_v and G_h act on $\mathcal{M}_v$ and $\mathcal{M}_h$ respectively and the splitting of the total group descends from the product structure of the moduli space. This group leaves invariant the σ-model Lagrangian (3.2) and, from a spacetime perspective, the symmetries it generates are global. As will be clear later, G_s acts non-trivially not only on the coordinates of the moduli space, i.e. the moduli φ^X, but also on the other fields of the matter multiplets which, together with the associated scalars, will organize themselves in representations of the symmetry group.

We are now going to introduce an important class of moduli spaces that will come in handy in the following sections since they will appear in the models under investigation, the *homogeneous manifolds*. A manifold $\mathcal{M}$ is said to be homogeneous if there exists a group of transformations, the *isometry group* G, connecting any two points of $\mathcal{M}$ [4]. The subgroup $H \subset G$ that leaves invariant a given point is called the *isotropy group* of that point and since all these groups, one in every point, are isomorphic we can talk about the isotropy group of $\mathcal{M}$. It can be shown that a homogeneous manifold is diffeomorphic to the coset G/H, thus it can be identified with it: $\mathcal{M} \simeq G/H$.

[2]In what follows we shall refer to the scalars also as *moduli*.

[3]In the case of global supersymmetry these manifolds are respectively *rigid special Kähler manifold*, *rigid very special (real) manifold* and *hyper-Kähler manifold*.

[4]We refer to section 5.1 for a more formal definition.

3.2 Target spaces

In the previous section we classified the different moduli spaces parametrized by the scalars according to the origin of the latter and the spacetime dimension in which the theory is framed. We are now going to present the key features of these spaces, not delving too deep into their study. The knowledge of complex and Kähler manifolds is a prerequisite for this section and we refer to [12, 66, 67] for a detailed presentation.

3.2.1 Special Kähler manifolds

We already mentioned in the previous section how special Kähler geometry is the structure underlying the coupling between the fields of n_V vector multiplets in $\mathcal{N} = 2$, $D = 4$ supergravity and how the complex scalars z^α parametrize a specific target space called special Kähler manifold. Since, as we shall see, many results for BPS and non-BPS solutions rely heavily on its symplectic formulation, this is the path we are going to follow presenting the topic. The first definition was given in [68], later refined in [69, 70], while additional good references on the topic are the books [67] and [12].

A *special Kähler manifold* $\mathcal{M}$ is an n_V-dimensional Kähler-Hodge[5] manifold with holomorphic coordinates z^α which is the base of an $\mathrm{Sp}(2n_V + 2, \mathbb{R})$-bundle with covariantly holomorphic *symplectic sections*

$$\mathcal{V} = \begin{pmatrix} X^I \\ F_I \end{pmatrix} , \qquad \text{with} \qquad I = 0, \ldots, n_V \, , \tag{3.4}$$

obeying the symplectic constraint

$$\langle \mathcal{V}, \bar{\mathcal{V}} \rangle = X^I \bar{F}_I - F_I \bar{X}^I = i \, , \tag{3.5}$$

where $\langle \, , \, \rangle$ denotes the symplectic inner product. The X^I provide a projective parametrization of the manifold, whereas the physical scalars z^α are expressed as

$$z^\alpha = \frac{X^\alpha}{X^0} \, , \qquad \text{with} \qquad \alpha = 1, \ldots, n_V \, . \tag{3.6}$$

A special Kähler manifold is a specific example of Kähler manifold, hence it is completely characterized by a real function $\mathcal{K}$ called *Kähler potential*, which, for example, defines the metric of the manifold $g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{K}$. For every Kähler-Hodge manifold there exist a line bundle defined on it and associated to the so-called *Kähler transformation*

$$\mathcal{K} \mapsto \mathcal{K} + r(z) + \bar{r}(\bar{z}) \, . \tag{3.7}$$

This map acts on the Kähler potential, yet preserving the metric, and can be thought as a U(1) gauge transformation with connection

$$\mathcal{A}_\mu = -\frac{i}{2} (\partial_\alpha \mathcal{K} \, \partial_\mu z^\alpha - \partial_{\bar{\alpha}} \mathcal{K} \, \partial_\mu \bar{z}^{\bar{\alpha}}) \, . \tag{3.8}$$

The Kähler-covariant derivatives are defined as follows[6]

$$\mathcal{D}_\alpha \mathcal{V} := \partial_\alpha \mathcal{V} + \frac{1}{2} (\partial_\alpha \mathcal{K}) \, \mathcal{V} \, , \qquad \mathcal{D}_{\bar{\alpha}} \mathcal{V} := \partial_{\bar{\alpha}} \mathcal{V} - \frac{1}{2} (\partial_{\bar{\alpha}} \mathcal{K}) \, \mathcal{V} = 0 \, . \tag{3.9}$$

[5]The U(1)-bundle associated to the Hodge condition has the interpretation of the $\mathrm{U}(1)_R$ R-symmetry of the supersymmetry algebra.

[6]Their identification as covariant derivatives becomes clear writing the Kähler connection as the 1-form $\mathcal{A} = -\frac{i}{2} (\partial_\alpha \mathcal{K} \, dz^\alpha - \partial_{\bar{\alpha}} \mathcal{K} \, d\bar{z}^{\bar{\alpha}})$, hence $\mathcal{D}_\alpha \mathcal{V} = \partial_\alpha \mathcal{V} + i \, \mathcal{A}_\alpha \mathcal{V}$ and $\mathcal{D}_{\bar{\alpha}} \mathcal{V} = \partial_{\bar{\alpha}} \mathcal{V} + i \, \mathcal{A}_{\bar{\alpha}} \mathcal{V}$.

The upper components of $\mathcal{V}$ are arbitrary functions, reflecting the freedom of choice of the parametrization of the coordinates z^α. Up to symplectic transformations, there always exists a frame in which the lower components F_I can be derived from a homogeneous function of degree two $F(X)$, referred to as *prepotential*, through

$$F_I = \frac{\partial F}{\partial X^I} \,. \tag{3.10}$$

The choice of this function defines uniquely the manifold and the whole model, indeed, as we shall see, it fixes all the kinetic and interaction terms as well. In this case it can be proved that

$$\langle \mathcal{D}_\alpha \mathcal{V}, \mathcal{V} \rangle = \langle \mathcal{D}_\alpha \mathcal{V}, \bar{\mathcal{V}} \rangle = \langle \mathcal{D}_\alpha \mathcal{V}, \mathcal{D}_\beta \mathcal{V} \rangle = 0 \,, \qquad g_{\alpha\bar{\beta}} = i \, \langle \mathcal{D}_\alpha \mathcal{V}, \mathcal{D}_{\bar{\beta}} \bar{\mathcal{V}} \rangle \,. \tag{3.11}$$

As we already pointed out, condition (3.6) defines a projective structure associated to the symplectic one. Thus, we can introduce the *holomorphic section*[7]

$$v(z) = e^{-\mathcal{K}/2} \mathcal{V} := \begin{pmatrix} Z^I(z) \\ \partial_I F(Z) \end{pmatrix} \,, \tag{3.12}$$

where the Z^I are called *homogeneous coordinates* and $F(X)$ and $F(Z)$ have the same functional forms. Condition (3.5) implies

$$e^{-\mathcal{K}} = -i \, \langle v, \bar{v} \rangle \,, \tag{3.13}$$

giving a recipe to compute the Kähler potential, hence the metric on $\mathcal{M}$, once fixed the prepotential.

A fundamental object that will come into play in what follows is the *period matrix* $\mathcal{N}$, which determines the coupling between the scalars and the vectors and is defined by the relations

$$F_I = \mathcal{N}_{IJ} X^J \,, \qquad \mathcal{D}_{\bar{\alpha}} \bar{F}_I = \mathcal{N}_{IJ} \mathcal{D}_{\bar{\alpha}} \bar{X}_J \,, \tag{3.14}$$

or, equivalently, is given by

$$\mathcal{N}_{IJ} = \begin{pmatrix} \mathcal{D}_{\bar{\alpha}} \bar{F}_I & F_I \end{pmatrix} \begin{pmatrix} \mathcal{D}_{\bar{\alpha}} \bar{X}^J & X^J \end{pmatrix}^{-1} \,. \tag{3.15}$$

In a frame in which a prepotential exists, $\mathcal{N}$ can be obtained from

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + 2i \, \frac{N_{II'} Z^{I'} N_{JJ'} Z^{J'}}{Z^K N_{KL} Z^L} \,, \tag{3.16}$$

where, in this context, $F_{IJ} = \partial_I \partial_J F(Z)$ and $N_{IJ} = \mathrm{Im}(F_{IJ})$. Useful identities of the period matrix can be found in section 20.4.2 of [67], where $\kappa^2 = 1$ in our conventions.

We are going now to present some examples of special Kähler manifolds and, since we shall focus only on spaces generated by the vector multiplets, we shall set $n = n_V$.

The $\overline{\mathbb{C}\mathrm{P}}^n$ model

The first explicit model we study is the $\overline{\mathbb{C}\mathrm{P}}^n$, defined by the quadratic prepotential

$$F = \frac{i}{4} X^I \eta_{IJ} X^J \,, \qquad \eta_{IJ} = \mathrm{diag}(-1, 1, \ldots, 1) \,, \tag{3.17}$$

and characterized by the moduli space

$$\mathcal{M}_v = \overline{\mathbb{C}\mathrm{P}}^n = \frac{\mathrm{SU}(n,1)}{\mathrm{SU}(n) \times \mathrm{U}(1)} \,. \tag{3.18}$$

[7] As the name suggests, $\partial_{\bar{\alpha}} v = 0$.

Both in the prepotential and in the moduli space is evident the splitting between the spatial and temporal parts, e.g. in $\mathrm{SU}(n,1)$ and $\mathrm{SU}(n)\times\mathrm{U}(1)$. If we choose homogeneous coordinates by $Z^0=1$ and $Z^\alpha=z^\alpha$, the Kähler potential and the metric of $\mathcal{M}_v$ read respectively

$$e^{-\mathcal{K}}=1-\sum_{\alpha=1}^{n}|z^\alpha|^2\,,\qquad g_{\alpha\bar\beta}=e^{\mathcal{K}}\delta_{\alpha\beta}+e^{2\mathcal{K}}\bar z^{\bar\alpha}z^\beta\,. \tag{3.19}$$

Here we stress that in order to have a well-defined Kähler potential the scalar fields need to satisfy the condition $\sum_\alpha|z^\alpha|^2<1$.

The cubic model

A second example of realization of special Kähler spaces is the class defined by the cubic prepotential

$$F=-\frac{1}{6}\,C_{\alpha\beta\gamma}\frac{X^\alpha X^\beta X^\gamma}{X^0}\,, \tag{3.20}$$

where $C_{\alpha\beta\gamma}$ is a completely symmetric real tensor. These models play a special role since they emerge naturally from truncations of type-II superstrings and M-theory [52].

The *stu* model

This model is a special case of the generic cubic one previously presented where the matter content is made up of three vector multiplets ($n=3$). The three complex scalars are historically called s, t and u, hence the name of the model. The theory is defined by the prepotential ($C_{123}=1$ and others zero) and moduli space

$$F=-\frac{X^1X^2X^3}{X^0}\,,\qquad \mathcal{M}_v=\left(\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right)^3\,. \tag{3.21}$$

We point out that $\mathcal{M}_v$ is an homogeneous manifold, in contrast to the generic cubic model, which includes also non-homogeneous spaces. Introducing the homogeneous coordinates defined by $Z^0=1$ and $Z^\alpha=z^\alpha$, the Kähler potential and scalar metric are

$$e^{-\mathcal{K}}=8\,y^1y^2y^3\,,\qquad g_{\alpha\bar\beta}=\frac{1}{4}\,\mathrm{diag}\left((y^1)^{-2},(y^2)^{-2},(y^3)^{-2}\right)\,, \tag{3.22}$$

where we decomposed $z^\alpha=x^\alpha+i\,y^\alpha$. Clearly we need $y^1y^2y^3>0$ for consistency of the Kähler potential.

The t^3 model

Identifying the three vector multiplets of the *stu* model we obtain the so-called t^3 model, with just one complex scalar ($n=1$). The prepotential ($C_{111}=6$) and moduli space read

$$F=-\frac{(X^1)^3}{X^0}\,,\qquad \mathcal{M}_v=\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\,. \tag{3.23}$$

In the homogeneous coordinates $Z^0=1$, $Z^1=z$ the Kähler potential and scalar metric read

$$e^{-\mathcal{K}}=8\,(\mathrm{Im}\,z)^3\,,\qquad g_{z\bar z}=\frac{3}{4(\mathrm{Im}\,z)^2}\,, \tag{3.24}$$

which implies $\mathrm{Im}\,z>0$.

3.2.2 Very special (real) manifolds

In section 3.1 we presented the nature of the moduli space generated by n_V vector multiplets in $\mathcal{N} = 2$, $D = 5$ gauged supergravity: it is a very special (real) manifold with coordinates ϕ^i. The original papers on this topic are [71, 72], while a complete exposition is given in book [12].

A *very special (real) manifold* $\mathcal{M}$ is an n_V-dimensional Riemannian manifold with coordinates ϕ^i described as an hypersurface in the (n_V+1)-dimensional space spanned by the auxiliary real functions $h^I(\phi^i)$ with $I = 1, \ldots, n_V + 1$; the hypersurface is parametrized by

$$\mathcal{V} := \frac{1}{6} C_{IJK} h^I h^J h^K = 1\,, \tag{3.25}$$

with C_{IJK} a fully symmetric, constant and real tensor. The choice of this tensor defines uniquely the manifold and the whole model, indeed, as we shall see, it fixes all the kinetic and interaction terms as well. Defined the functions

$$h_I := \frac{1}{6} C_{IJK} h^J h^K \qquad \Longrightarrow \qquad h_I h^I = 1\,, \tag{3.26}$$

we can introduce a sort of metric G_{IJ} (and its inverse G^{IJ}) which, apart from a multiplicative factor placed for later convenience, raises and lowers the indices of the h functions

$$h_I = \frac{2}{3} G_{IJ} h^J\,, \qquad G^{IJ} G_{JK} = \delta^I_K\,. \tag{3.27}$$

It is easy to check that a matrix with these properties is given by

$$G_{IJ} = \frac{9}{2} h_I h_J - \frac{1}{2} C_{IJK} h^K = -\frac{1}{2} \frac{\partial}{\partial h^I} \frac{\partial}{\partial h^J} \ln \mathcal{V}\big|_{\mathcal{V}=1}\,. \tag{3.28}$$

For what concerns the derivatives of the auxiliary functions h^I and h_I w.r.t. ϕ^i we have

$$\partial_i h_I = -\frac{2}{3} G_{IJ} \partial_i h^J\,. \tag{3.29}$$

We can now define the metric of the very special manifold $\mathcal{G}_{ij}$ (and its inverse $\mathcal{G}^{ij}$) as the pullback of the $(n_V + 1)$-dimensional space metric G_{IJ} on the hypersurface $\mathcal{V} = 1$

$$\mathcal{G}_{ij} = \partial_i h^I \partial_j h^J G_{IJ}\,, \qquad \mathcal{G}^{ij} \mathcal{G}_{jk} = \delta^i_k\,. \tag{3.30}$$

Here we list some additional useful relations [71]:

$$\begin{gathered} \mathcal{G}^{ij} \partial_i h^I \partial_j h^J = G^{IJ} - \frac{2}{3} h^I h^J\,, \qquad \mathcal{G}^{ij} \partial_i h_I \partial_j h_J = \frac{4}{9} G_{IJ} - \frac{2}{3} h_I h_J\,, \\ \mathcal{G}^{ij} \partial_i h^I \partial_j h_J = -\frac{2}{3} \delta^I_J + \frac{2}{3} h^I h_J\,. \end{gathered} \tag{3.31}$$

As we shall see explicitly in subsection 3.5.1, very special (real) manifolds are closely related to a subgroup of special Kähler manifolds, called *very special Kähler manifolds*. The connection is made through a specific dimensional reduction, the *r-map*, which allows to move from very special (real) spaces to special Kähler spaces characterized by the cubic prepotential (3.20) and vice versa.

The *stu* model

The most studied example of very special (real) manifold is the *stu* model, parametrized by two real scalars and identified by the tensor with only non-vanishing components $C_{123} = 1$. By means of the r-map it can be put in correlation with the special Kähler *stu* model introduced in the previous section. The most common parametrization is

$$h^1 = e^{-\frac{\phi^1}{\sqrt{6}}-\frac{\phi^2}{\sqrt{2}}}\,, \qquad h^2 = e^{-\frac{\phi^1}{\sqrt{6}}+\frac{\phi^2}{\sqrt{2}}}\,, \qquad h^3 = e^{\frac{2\phi^1}{\sqrt{6}}}\,, \tag{3.32}$$

with scalar metric

$$\mathcal{G}_{ij} = \frac{1}{2}\,\delta_{ij}\,. \tag{3.33}$$

3.2.3 Quaternionic-Kähler manifolds

The last class of manifolds we are going to talk about is the one spanned by the scalar fields belonging to the hypermultiplets, which takes the name of quaternionic-Kähler. As we mentioned, these spaces are common both to four and five dimensions, so we shall make no distinction. Good references are again the books [12,67] and the papers [73,74].

A *quaternionic-Kähler manifold* $\mathcal{M}$ is a $4n_H$-dimensional Riemannian manifold with coordinates q^u and endowed with a metric h_{uv} and a *hypercomplex structure* $\vec{J}$ compatible with it, i.e. a triplet of complex structures $\{(J^x)_u{}^v\}_x$, with $x = 1,2,3$, satisfying the conditions $(J^x)^2 = -\,\mathrm{id}_{T\mathcal{M}}$ and $J^1 \circ J^2 = J^3$ (and cyclic); in components

$$(J^x)_u{}^t (J^y)_t{}^v = -\delta^{xy}\,\delta^v_u + \varepsilon^{xyz}(J^z)_u{}^v\,. \tag{3.34}$$

Compatibility implies that h_{uv} is Hermitian, explicitly $J^x\,h\,(J^x)^t = h$ (no sum over x). Additionally, $\mathcal{M}$ must be the base space of an SU(2)-bundle[8] with connection 1-form $\omega^x = (\omega^x)_u\,dq^u$ and the Levi-Civita connection must preserve the hypercomplex structure up to a rotation:

$$\mathsf{D}_t\vec{J}_u{}^v := \nabla_t\vec{J}_u{}^v + \vec{\omega}_t \times \vec{J}_u{}^v = 0\,. \tag{3.35}$$

From a different perspective, $\vec{J}$ must be constant with respect to the SU(2)-covariant derivative D. The manifold being Hermitian, we can define a Kähler 2-form for every complex structure, getting a triplet called *hyper-Kähler 2-form*

$$\vec{K} := \frac{1}{2}\vec{J}_{uv}\,dq^u \wedge dq^v \qquad \text{with} \qquad \vec{J}_{uv} := h_{vt}\vec{J}_u{}^t\,. \tag{3.36}$$

In terms of $\vec{K}$, relation (3.35) reads

$$dK^x + \varepsilon^{xyz}\omega^y \wedge K^z = 0\,, \tag{3.37}$$

which means that the hyper-Kähler form is covariantly closed with respect to the connection $\vec{\omega}$. This constraint closely resembles the condition for a Hermitian manifold to be Kähler, namely $dK = 0$. From the SU(2) connection it is possible to construct the SU(2) curvature 2-form

$$\Omega^x = d\omega^x + \frac{1}{2}\,\varepsilon^{xyz}\omega^y \wedge \omega^z\,. \tag{3.38}$$

In the case of quaternionic-Kähler manifolds we have

$$\Omega^x = \lambda\,K^x\,, \tag{3.39}$$

where the value of the constant λ will be fixed to -1 in the case of $\mathcal{N}=2$ gauged supergravities in order to have a well-defined kinetic term for the hyperscalars. A remarkable case is $\lambda = 0$,

[8] The SU(2) group corresponds to the $\mathrm{SU}(2)_R$ R-symmetry of the supersymmetry algebra. Within this context, x, y, z are indices of the adjoint representation of su(2) and all the tensors carrying them will be su(2)-valued tensors.

which implies a flat SU(2)-bundle, i.e. $\Omega^x = 0$: the result is a *hyper-Kähler manifold*, a special example of quaternionic-Kähler space.

Quaternionic-Kähler manifolds are Einstein spaces, indeed

$$R_{uv} = \frac{R}{4n_H}\, h_{uv}\,, \tag{3.40}$$

where R_{uv} and R are the Ricci tensor and Ricci scalar of $\mathcal{M}$. Moreover, it can be shown that the coefficient of proportionality in (3.39) is fixed by the scalar curvature of the manifold

$$\lambda = \frac{R}{8n_H(n_H+2)}\,. \tag{3.41}$$

Our last remark concerns the holonomy group of the quaternionic-Kähler manifolds, which is $\mathrm{USp}(2n_H) \times \mathrm{SU}(2)$ [9]. First of all, we notice that it is a subgroup of $\mathrm{SO}(4n_H)$, the general holonomy group of the Levi-Civita connection in a Riemannian $4n_H$-dimensional space. Furthermore, $\mathrm{USp}(2n_H) \simeq \mathrm{U}(n_H, \mathbb{H})$, which tells us that, at least locally, each quadruple q^u can be arranged into a quaternion, hence the name of this family of manifolds. Finally, the SU(2) factor is closely related to the hypercomplex structure and acts mixing the three J^x.

The universal hypermultiplet

A special case of quaternionic-Kähler space is the one realized by the *universal hypermultiplet* $q^u = (\phi, a, \xi^0, \tilde{\xi}_0)$ [52]. These four scalars parametrize the manifold

$$\mathcal{M}_h = \frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2)\times \mathrm{U}(1)} \tag{3.42}$$

with metric

$$ds^2 = d\phi^2 + \frac{1}{4}\, e^{4\phi}\left(da - \frac{1}{2}\langle \xi, d\xi\rangle\right)^2 + \frac{1}{4}\, e^{2\phi}\big[(d\xi^0)^2 + (d\tilde{\xi}_0)^2\big]\,, \tag{3.43}$$

where $\langle \xi, d\xi\rangle := \tilde{\xi}_0 d\xi^0 - \xi^0 d\tilde{\xi}_0$. By means of the formulae presented so far we can obtain the hypercomplex structure and, finally, the SU(2) connection

$$\omega^1 = e^{\phi}\, d\tilde{\xi}_0\,, \qquad \omega^2 = e^{\phi}\, d\xi^0\,, \qquad \omega^3 = \frac{1}{2}\, e^{2\phi}\left(da - \frac{1}{2}\langle \xi, d\xi\rangle\right). \tag{3.44}$$

It is possible to make contact with the formulation adopted by Ferrara and Sabharwal in [53] through the field redefinition

$$S = e^{-2\phi} + i\, a + \frac{1}{4}\big[(\xi^0)^2 + (\tilde{\xi}_0)^2\big]\,, \qquad C = \frac{1}{2}(\tilde{\xi}_0 + i\, \xi^0)\,, \tag{3.45}$$

which allows us to interpret ϕ and a as a dilaton and an axion respectively.

3.3 Isometries and gaugings

The naive inclusion in a *pure supergravity theory*, i.e. a supergravity containing only the supergravity multiplet, of the matter multiplets introduced in section 3.1 leads to an *ungauged* supergravity, where both the R-symmetry group and the isometry group of the moduli space G_s act as a global symmetry group. As we already pointed out, the kinetic terms of the matter fields are completely fixed by the prepotential and the same happens for the couplings, which

[9] Actually, it is $\mathrm{USp}(2n_H)\cdot \mathrm{SU}(2) = \mathrm{USp}(2n_H)\times \mathrm{SU}(2)/\mathbb{Z}_2$, but we shall not deal with the $\mathbb{Z}_2$ factor.

are encoded in the period matrix. Since the latter does not depend on the hyperscalars, these decouple from the rest of the bosonic fields.

We already talked in the introduction of this chapter about the importance of gauging (some of) the isometries enjoyed by the supergravity theory in question; this will be the aim of the remainder of this chapter. Before getting to the heart of the matter, we start introducing the possible isometries of the moduli space, moving, then, to the gauging itself.

3.3.1 The moduli space isometries

As we mentioned at the end of section 3.1, $\mathcal{M}_s$ can enjoy a set of symmetries described by a Lie group G, acting "globally" on the fields. In this section we shall expand on this topic and, in order to do so, we shall begin with a general treatment where φ^X plays the role of z^α, ϕ^i or q^u and the same for the related metrics $g_{\alpha\bar{\beta}}$, $\mathcal{G}_{ij}$ and h_{uv}, here collectively denoted by g_{XY}.

Killing vectors have widely proven to be the best tool to describe symmetries, being the (infinitesimal) generators of the isometries. Indeed, given a set of Killing vectors $\{k_\Lambda\}$, with $\Lambda = 1, \dots, \dim G$, the metric g_{XY} is left invariant by the field redefinition $\delta_\epsilon \varphi^X = \epsilon^\Lambda k_\Lambda^X(\varphi)$, seen as a diffeomorphism from the target space, where ϵ^Λ are constant, infinitesimal parameters. The fact that ϵ^Λ are constant makes this symmetry global. Killing vectors are closely related to the symmetry group G, since they generate its Lie algebra $\mathfrak{g}$,

$$[k_\Lambda, k_\Sigma] = f_{\Lambda\Sigma}{}^\Gamma k_\Gamma \,, \tag{3.46}$$

where $f_{\Lambda\Sigma}{}^\Gamma$ are the *structure constants*.

Depending on the target space, further conditions may arise in order to preserve the additional structures they feature.

Special Kähler

In the case of Kähler manifolds, like special Kähler or quaternionic-Kähler, the Killing vectors must preserve, in addition to the metric, the (hyper)complex structure and the Kähler condition as well, up to an SU(2)-rotation in the quaternionic case. These requirements greatly constraint the expression of the allowed vectors, but on the other hand give us the possibility to introduce a set of functions that characterize completely the Killing vectors, the *momentum maps*, or *Killing prepotentials*.

Starting with special Kähler manifolds, compatibility with the complex structure underlying its Hermitian nature forces the Killing vectors to be holomorphic, i.e. to commute with the J map that squares to minus the identity, hence $k_\Lambda^\alpha = k_\Lambda^\alpha(z)$ and similarly for its complex conjugate. Moreover, requiring the Kähler condition to be preserved implies the existence, for each Killing vector, of a real function $\mathcal{P}_\Lambda$, the aforementioned momentum map, such that

$$k_\Lambda^\alpha(z) = i\, g^{\alpha\bar{\beta}} \partial_{\bar{\beta}} \mathcal{P}_\Lambda(z, \bar{z}) \tag{3.47}$$

and likewise for $k_\Lambda^{\bar{\alpha}}$. This relation makes clear the origin of the name "Killing prepotential", since every Killing vector can be derived from the related $\mathcal{P}_\Lambda$. The holomorphicity of k_Λ^α can be encoded in the condition $\nabla_\alpha \partial_\beta \mathcal{P}_\Lambda = 0$.

Besides the complex structure and the Kähler condition, attention must be paid also to the Kähler structure, which can be totally encoded in the Kähler potential. Under the action of a Killing vector, here realized through the corresponding Lie derivative, the latter has to be invariant up to a Kähler transformation (3.7), specifically

$$\mathcal{L}_{k_\Lambda} \mathcal{K} = (k_\Lambda^\alpha \partial_\alpha + k_\Lambda^{\bar{\alpha}} \partial_{\bar{\alpha}}) \mathcal{K} = r_\Lambda(z) + \bar{r}_\Lambda(\bar{z}) \,. \tag{3.48}$$

This modifies the Kähler potential, but leaves the metric unchanged. By means of this relation we can "integrate" equation (3.47) and provide an expression for the momentum maps once the Killing vectors are known

$$\mathcal{P}_\Lambda = -i\,[k^\alpha_\Lambda \partial_\alpha \mathcal{K} - r_\Lambda(z)] = i\,[k^{\bar\alpha}_\Lambda \partial_{\bar\alpha} \mathcal{K} - \bar r_\Lambda(\bar z)] \tag{3.49}$$

or, alternatively,

$$\mathcal{P}_\Lambda = (k^\alpha_\Lambda \mathcal{A}_\alpha + k^{\bar\alpha}_\Lambda \mathcal{A}_{\bar\alpha}) + \frac{i}{2}\big[r_\Lambda(z) - \bar r_\Lambda(\bar z)\big]\,, \tag{3.50}$$

where $\mathcal{A}$ is the U(1) Kähler connection (3.8).

It is easy to convince oneself that momentum maps are defined up to a (real) constant. In the case of non-abelian symmetry groups, these constants can be chosen such that the Killing prepotentials transform in the adjoint representation of $\mathfrak{g}_v$

$$\mathcal{L}_{k_\Lambda} \mathcal{P}_\Sigma = (k^\alpha_\Lambda \partial_\alpha + k^{\bar\alpha}_\Lambda \partial_{\bar\alpha}) \mathcal{P}_\Sigma = f_{\Lambda\Sigma}{}^\Gamma \mathcal{P}_\Gamma\,, \tag{3.51}$$

where $f_{\Lambda\Sigma}{}^\Gamma$ are the structure constants as in (3.46), but specialized to the isometries of the "vector" moduli space. On the other hand, in the case of abelian groups the freedom of shifting $\mathcal{P}$ remains. Substituting (3.47) we end up with the so-called *equivariance relation*

$$g_{\alpha\bar\beta}(k^\alpha_\Lambda k^{\bar\beta}_\Sigma - k^\alpha_\Sigma k^{\bar\beta}_\Lambda) = -i\, f_{\Lambda\Sigma}{}^\Gamma \mathcal{P}_\Gamma\,. \tag{3.52}$$

Lastly, special Kähler manifolds are characterized by a symplectic structure that, of course, must be retained. On one hand this requirement imposes new constraints on the Killing vectors (and momentum maps), but on the other it provides a very useful symplectic invariant formulation. In the rest of the thesis we shall not deal with the isometries of $\mathcal{M}_v$, hence we leave this topic aside and refer, e.g., to [73] for details.

Very special (real)

For what concerns the isometries of very special (real) manifolds, the main ingredient are still the corresponding Killing vectors $k^i_\Lambda(\phi)$. In this case the are no complex structures to be preserved, but attention must be paid in order to maintain the very special structure related to the auxiliary functions h^I. Again, since the isometries of the vector multiplet moduli space will not be gauged in this dissertation, we shall not dwell on them, referring to [12] for further details.

Quaternionic-Kähler

We now move to a brief analysis of the quaternionic-Kähler isometries, following roughly what has been done with the special Kähler manifolds. In this case the Killing vectors must preserve the hypercomplex structure (3.34), i.e. they must be *triholomorphic*, and the quaternionic-Kähler structure, which implies, again, the existence of triplets of momentum maps $\vec{\mathcal{P}}_\Lambda$, one for each Killing vector, such that

$$\vec{J}_{uv} k^v_\Lambda(q) = \mathsf{D}_u \vec{\mathcal{P}}_\Lambda(q) = \partial_u \vec{\mathcal{P}}_\Lambda + \vec\omega_u \times \vec{\mathcal{P}}_\Lambda\,. \tag{3.53}$$

We immediately notice an analogy, which will be recurring, between this expression and (3.47)[10].

With a bit of computation we can derive an expression for the momentum maps in terms of the Killing vectors

$$\vec{\mathcal{P}}_\Lambda = k^u_\Lambda \vec\omega_u + \vec{W}_\Lambda\,, \tag{3.54}$$

[10]The parallel is even more manifest if we make the complex structure appear by means of the relation $J_{\alpha\bar\beta} = i\, g_{\alpha\bar\beta}$, valid for every Hermitian manifold, and rewrite (3.47) as $J_{\bar\beta\alpha} k^\alpha_\Lambda(z) = \partial_{\bar\beta} \mathcal{P}_\Lambda(z,\bar z)$.

where $\vec{\omega}$ is the quaternionic-Kähler connection and $\vec{W}_\Lambda$ is a compensating field analogous to r_Λ. Recalling relation (3.50), it is worth noting how $\mathcal{A}$ and $\vec{\omega}$, inherited from the Kähler-Hodge and quaternionic-Kähler structures, are closely related to the $\mathrm{U}(1)_R$ and $\mathrm{SU}(2)_R$ factors in the R-symmetry group of 4d supergravity (cf. table A.3 in the appendix).

Finally, an equivariance relation analogous to (3.52) can be obtained fixing conveniently the (optional) additive constants of the momentum maps,

$$\vec{J}_{uv} k^u_\Lambda k^v_\Sigma - \frac{\lambda}{2} \vec{\mathcal{P}}_\Lambda \times \vec{\mathcal{P}}_\Sigma = \frac{1}{2} f_{\Lambda\Sigma}{}^\Gamma \vec{\mathcal{P}}_\Gamma \,, \tag{3.55}$$

where $f_{\Lambda\Sigma}{}^\Gamma$ are the structure constants of the Lie algebra of the isometry group G_h. In fact, even in this case the momentum maps are defined up to a constant shift and this property will play a crucial role in gauging part of the R-symmetry in absence of hypermultiplets.

3.3.2 The gauging procedure

Let us move to the discussion of the gauging itself considering a generic 4- or 5-dimensional $\mathcal{N}=2$ ungauged supergravity. In order to stay generic we shall adopt again the notation used at the beginning of the previous subsection, in which the scalar fields φ^X span a target space $\mathcal{M}$ with metric g_{XY}, invariant under the (global) action of a Lie group G.

In general, we can choose not to consider the full isometry group, but rather to gauge a subgroup $G_0 \subseteq G$ and in order to do so we need to construct a subset of Killing vectors generating the subalgebra $\mathfrak{g}_0$. There exists a rigorous method to pick these Killing vectors, "projecting away" the unneeded ones and embedding $\mathfrak{g}_0$ in $\mathfrak{g}$, that goes under the name of *embedding tensor* formalism [75], but since here we only want to give an idea of the gauging procedure, we shall assume that we are able to directly select the Killing vectors required to close the algebra, leaving the orthodox treatment to other references, e.g. [76]. Moreover, G will not be considered anymore, thus, with an abuse of notation, we shall denote k^X_Λ the generators of $\mathfrak{g}_0$ and, from now on, $f_{\Sigma\Gamma}{}^\Lambda$ will be the structure constants of the latter.

Gauging a symmetry means to make it local, in the sense that the parameter of the transformation starts to depend on the (spacetime) coordinates, i.e. $\epsilon^\Lambda = \epsilon^\Lambda(x)$. Gauge theories have been extensively studied in physics, thus the introduction of a (gauge) *covariant derivative* should not come as a surprise. Its action on a given field is different according to the representation of $\mathfrak{g}_0$ the field belongs to and the case of φ^X is unique, because the derivative acts non-linearly as

$$\hat{\partial}_\mu \varphi^X := \partial_\mu \varphi^X + g\, A^\Lambda_\mu k^X_\Lambda \,, \tag{3.56}$$

where g is the *gauge coupling*. The A^Λ play the role of gauge vectors and must be picked from the only ones we have at our disposal, the graviphoton and the vectors belonging to the vector multiplets. Therefore, we have a constraint on the number of possible Killing vectors, hence on the dimension of the symmetry group: $\dim G_0 \leq n_V + 1$. Furthermore, we notice that the gauging is realized with "electric" vectors, thus it is said to be an *electric gauging*[11]. In order for $\hat{\partial}_\mu$ to be covariant, gauge vectors need to transform as

$$\delta_\epsilon A^\Lambda_\mu = -\frac{1}{g}\big(\partial_\mu \epsilon^\Lambda + g\, A^\Sigma_\mu f_{\Sigma\Gamma}{}^\Lambda \epsilon^\Gamma\big) = -\frac{1}{g}\, \hat{\partial}_\mu \epsilon^\Lambda \,, \tag{3.57}$$

[11]In four dimensions, also magnetic, or even dyonic, gaugings are possible, but in these cases a symplectically covariant formulation is required, which can be better achieved with the aforementioned embedding tensor formalism. Examples can be found in [76, 77] and, to a limited extent, in the second part of subsection 4.1.1. Moreover, a symplectic formulation restores electromagnetic duality, typical of the ungauged theories, otherwise broken by a pure electric gauging.

where, inside the parentheses, we recognize the covariant derivative of an object transforming in the adjoint representation of the symmetry algebra. Indeed, equation (3.57) shows that the A^Λ transform, as expected for a connection, in the adjoint of $\mathfrak{g}_0$; this fact has a profound implication that we shall discuss shortly. For the sake of completeness, we present the field strength 2-forms associated to the gauge 1-forms

$$F^\Lambda_{\mu\nu} = \partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu + g\, f_{\Sigma\Gamma}{}^\Lambda A^\Sigma_\mu A^\Gamma_\nu\,. \tag{3.58}$$

The gauging process does not stop here. In addition to the prescribed covariant derivatives, which may give rise to (scalar-dependent) massive terms for the fermions, new terms in the SUSY transformations appear and their presence requires the introduction of a potential for the scalar fields in order to preserve supersymmetry. The exact expression of all these contributions depends heavily on the dimensionality of the theory and the symmetries we are gauging. An exhaustive survey of the 4- and 5-dimensional theories can be found in [73] and [78] respectively. In any case, a general feature is that, contrary to the ungauged theories, the hyperscalars are now coupled to the vectors through the covariant derivatives and to the scalars of the vector multiplets by virtue of the scalar potential.

As a result of the splitting of the moduli space symmetry group $G_s = G_v \times G_h$, also the Killing vectors are divided in two groups depending on whether they generate the isometries of $\mathcal{M}_v$ or $\mathcal{M}_h$, and these groups can be treated separately as the action of each set is independent of the other. However, hidden links are still present due to the fact that both (the subgroups of) G_v and G_h are gauged by means of the same vectors. First of all, given $G_{h,0}$ a non-abelian gauge group of $\mathcal{M}_h$, we know that the gauge vectors A^I [12] transform in a representation, the adjoint, of this group. As a consequence of supersymmetry, all the fields in a given multiplet must transform in the same way, hence, e.g. in $D = 4$, also the complex scalars z^α, or, more precisely, the sections X^I, must transform under $G_{h,0}$, which has to be a symmetry group for $\mathcal{M}_v$ as well[13]. The abelian case works similarly, but since the adjoint representation of an abelian group is the trivial one, gauging an abelian group implies at, in reality, we are not gauging any isometry of $\mathcal{M}_v$ at all, which, as a consequence, is not required to have any symmetry.

In order to be a bit more schematic, but at the same time concrete, we now take as an example 4d supergravity. In four dimensions the total symmetry group is

$$G = G_v \times G_h \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_R \tag{3.59}$$

and, a priori, the gauging could involve any of these groups or subgroups of them. What turns out is that we have only the following possibilities[14]:

- G_{NA}: a non-abelian subgroup of G_v (see, e.g., [79, 80]). If hypermultiplets are present we must have $G_{\mathrm{NA}} \subseteq G_v$ and $G_{\mathrm{NA}} \subseteq G_h$ simultaneously (see, e.g., [81]);
- G_{A}: an abelian subgroup of G_h, e.g. $\mathrm{U}(1)^{n_V+1}$, in which case the isometries of $\mathcal{M}_v$ are not gauged;
- $\mathrm{SU}(2)_R$: it is referred to as non-abelian, or SU(2), Fayet-Iliopoulos gauging (see, e.g., [82]). A subgroup of G_v with an SU(2) factor must be gauged as well for consistency;
- $\mathrm{U}(1) \subset \mathrm{SU}(2)_R$: the abelian, or U(1), Fayet-Iliopoulos gauging we shall consider in the rest of this thesis when working with $\mathcal{N} = 2$ supergravity.

[12] From now on, we shall assume that all the $n_V + 1$ vector fields of the theory take part to the gauging and, consistently with (3.1), we shall collectively denoted them A^I.

[13] The same argument applies to the real scalars ϕ^i and the auxiliary functions h^I in five dimensions.

[14] We mention that appropriate combinations of these options are still symmetry groups that can be gauged.

A special mention deserve theories with no hypermultiplets, i.e. $n_H = 0$. Despite the absence of the hyperscalars and of the related moduli space, the quaternionic momentum maps can still assume a non-vanishing constant value thanks to their definition up to an additive constant,

$$\mathcal{P}_I^x = \xi_I^x \,, \tag{3.60}$$

where ξ_I^x are called *Fayet-Iliopoulos constants* and must satisfy the equivariance relation (3.55). This fact becomes fundamental when gauging only $\mathrm{SU}(2)_R$ or its $\mathrm{U}(1)$ subgroup, in which case the procedure takes the name of *Fayet-Iliopoulos (FI) gauging*. Adopting the abelian FI-gauging it is possible to "fix a direction" inside the $\mathrm{SU}(2)$ group; one common choice is

$$\xi_I^x = (0, 0, \xi_I) \,. \tag{3.61}$$

In this case the gauging is realized through the vector field $A_\mu := \xi_I A_\mu^I$.

The 5-dimensional case can be studied quite alike, despite the absence of the $\mathrm{U}(1)_R$ term in the symmetry group, which anyway could not be gauged, and the very special (real) structure of the $\mathcal{M}_v$ moduli space; clearly, the expression of the covariant derivatives and the scalar potential will be different. In this case, the abelian FI-gauging sets the momentum maps to $\mathcal{P}_I^x = (0,0,V_I)$ and the gauging of the $\mathrm{U}(1)$ group is achieved by means of $A_\mu := V_I A_\mu^I$.

3.4 The $D = 4$ theory

The first example of a 4-dimensional theory with two generators of supersymmetry dates back to 1976, when Ferrara and van Nieuwenhuizen added a second gravitino and a vector field to the $\mathcal{N} = 1$ supergravity [83]. This theory contains no matter multiplets, is not charged under the vector field and is invariant under two local independent supersymmetry transformations. Here we move further, gauging part of the isometries, yet restricting to abelian gauge groups. As we mentioned, this process requires the adoption of covariant derivatives and the introduction of a scalar potential. For a detailed review, including non-abelian gaugings and the SUSY transformations of the fields, see [12, 67].

The bosonic Lagrangian of the $\mathcal{N} = 2$, $D = 4$ (abelian) gauged supergravity coupled to n_V vector multiplets and n_H hypermultiplets is [62, 73][15]

$$\begin{aligned} e^{-1}\mathscr{L}_{\text{bos}} = {} & \frac{R}{2} + \frac{1}{4}(\mathrm{Im}\,\mathcal{N})_{IJ} F^I_{\mu\nu} F^{J\mu\nu} - \frac{1}{8}(\mathrm{Re}\,\mathcal{N})_{IJ}\frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} F^I_{\mu\nu} F^J_{\rho\sigma} \\ & - g_{\alpha\bar\beta}\partial_\mu z^\alpha \partial^\mu \bar z^{\bar\beta} - h_{uv}\hat\partial_\mu q^u \hat\partial^\mu q^v - V(z,\bar z, q) \,, \end{aligned} \tag{3.62}$$

where $F^I_{\mu\nu}$ are the abelian field strength tensors, $g_{\alpha\bar\beta}$ and h_{uv} are respectively the metric of the special Kähler and quaternionic-Kähler target spaces introduced in subsections 3.2.1 and 3.2.3 and the covariant derivative $\hat\partial$ is given by

$$\hat\partial_\mu q^u = \partial_\mu q^u + g\, A^I_\mu k^u_I \,. \tag{3.63}$$

The scalar potential V reads

$$V = g^2 \left[-\frac{1}{2}\vec{\mathcal{P}}_I \cdot \vec{\mathcal{P}}_J \big((\mathrm{Im}\,\mathcal{N})^{-1|IJ} + 8\bar X^I X^J \big) + 4 h_{uv} k^u_I k^v_J X^I \bar X^J \right] , \tag{3.64}$$

where g denotes the gauge coupling and the expression results from the gauging of the theory.

[15] We recall that $\alpha = 1, \ldots, n_V$, $I = 0, \ldots, n_V$ and $u = 1, \ldots, 4n_H$.

The equations of motion follow from (3.62) taking the variation with respect to the bosonic fields $g_{\mu\nu}$, A^I_μ, z^α and q^u. Einstein's equations, Maxwell's equations and the equations for the (vector and hyper-) scalars are respectively

$$R_{\mu\nu} = -(\mathrm{Im}\,\mathcal{N})_{IJ}\Big(F^I_{\mu\lambda}F^{J\,\lambda}_{\ \nu} - \frac{1}{4}\,g_{\mu\nu}F^I_{\rho\sigma}F^{J\rho\sigma}\Big) + 2g_{\alpha\bar\beta}\partial_{(\mu}z^\alpha\partial_{\nu)}\bar z^{\bar\beta} + 2h_{uv}\hat\partial_\mu q^u\hat\partial_\nu q^v + V g_{\mu\nu}\,, \tag{3.65a}$$

$$\nabla_\mu\Big[(\mathrm{Im}\,\mathcal{N})_{IJ}F^{J\mu\nu} - \frac{1}{2}\,(\mathrm{Re}\,\mathcal{N})_{IJ}\,\frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\,F^J_{\rho\sigma}\Big] = -2g\,h_{uv}k^u_I\,\hat\partial^\nu q^v\,, \tag{3.65b}$$

$$\begin{aligned}&\nabla_\mu\big(g_{\alpha\bar\beta}\partial^\mu\bar z^{\bar\beta}\big) - \partial_\alpha g_{\gamma\bar\beta}\partial_\mu z^\gamma\partial^\mu\bar z^{\bar\beta} + \frac{1}{4}\,\partial_\alpha(\mathrm{Im}\,\mathcal{N})_{IJ}F^I_{\mu\nu}F^{J\mu\nu}\\ &\qquad - \frac{1}{8}\,\partial_\alpha(\mathrm{Re}\,\mathcal{N})_{IJ}\,\frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\,F^I_{\mu\nu}F^J_{\rho\sigma} - \partial_\alpha V = 0\,,\end{aligned} \tag{3.65c}$$

$$\nabla_\mu\big(h_{uv}\hat\partial^\mu q^v\big) - \frac{1}{2}\,\partial_u h_{wv}\hat\partial_\mu q^w\hat\partial^\mu q^v - g\,h_{wv}A^I_\mu\partial_u k^w_I\,\hat\partial^\mu q^v - \frac{1}{2}\,\partial_u V = 0\,, \tag{3.65d}$$

where ∂_α and ∂_u denotes the derivatives w.r.t. z^α and q^u.

In the special case of $n_H = 0$ and U(1) FI-gauging, the q-terms consistently drop out and the scalar potential reduces to

$$V = -2g^2\xi_I\xi_J\big[(\mathrm{Im}\,\mathcal{N})^{-1|IJ} + 8\bar X^I X^J\big]\,, \tag{3.66}$$

with ξ_I the FI-constants. Within this context, we define $g_I := g\,\xi_I$.

3.5 The $D = 5$ theory

Moving ahead, we consider 5-dimensional supergravity, adopting, once again, an abelian gauging and leaving non-abelian examples to other references, like [12].

The bosonic Lagrangian of the $\mathcal{N}=2$, $D=5$ (abelian) gauged supergravity coupled to n_V vector multiplets and n_H hypermultiplets is [72, 78][16]

$$\begin{aligned}e^{-1}\mathscr{L}_{\mathrm{bos}} = {}& \frac{R}{2} - \frac{1}{4}\,G_{IJ}F^I_{\mu\nu}F^{J\mu\nu} + \frac{1}{48}\,C_{IJK}\,\frac{\varepsilon^{\mu\nu\rho\sigma\tau}}{\sqrt{-g}}\,F^I_{\mu\nu}F^J_{\rho\sigma}A^K_\tau\\ &- \frac{1}{2}\,\mathcal{G}_{ij}\partial_\mu\phi^i\partial^\mu\phi^j - h_{uv}\hat\partial_\mu q^u\hat\partial^\mu q^v - V(\phi,q)\,,\end{aligned} \tag{3.67}$$

where $F^I_{\mu\nu}$ are the abelian field strength tensors, $\mathcal{G}_{ij}$ and h_{uv} are respectively the metric of the very special (real) and quaternionic-Kähler target spaces introduced in subsections 3.2.2 and 3.2.3 and the covariant derivative $\hat\partial$ is given by

$$\hat\partial_\mu q^u = \partial_\mu q^u + 3g\,A^I_\mu k^u_I\,. \tag{3.68}$$

The scalar potential V reads

$$V = g^2\left[\vec{\mathcal{P}}_I\cdot\vec{\mathcal{P}}_J\Big(\frac{9}{2}\,\mathcal{G}^{ij}\partial_i h^I\partial_j h^J - 6h^Ih^J\Big) + 9h_{uv}k^u_Ik^v_Jh^Ih^J\right]\,, \tag{3.69}$$

where, again, g is the gauge coupling constant.

[16] We recall that $i = 1,\ldots,n_V$, $I = 1,\ldots,n_V+1$ and $u = 1,\ldots,4n_H$.

The equations of motion follow from (3.67) taking the variation with respect to the bosonic fields $g_{\mu\nu}$, A^I_μ, ϕ^i and q^u. Einstein's equations, Maxwell-Chern-Simons' equations and the equations for the scalars are respectively given by

$$R_{\mu\nu} = G_{IJ}\left(F^I_{\mu\lambda}F^{J\,\lambda}_{\nu} - \frac{1}{6}\,g_{\mu\nu}F^I_{\rho\sigma}F^{J\rho\sigma}\right) + \mathcal{G}_{ij}\partial_\mu\phi^i\partial_\nu\phi^j + 2h_{uv}\hat{\partial}_\mu q^u\hat{\partial}_\nu q^v + \frac{2}{3}\,V g_{\mu\nu}\,, \tag{3.70a}$$

$$\nabla_\lambda\big(G_{IJ}F^{J\lambda\tau}\big) + \frac{1}{16}\,C_{IJK}\,\frac{\varepsilon^{\mu\nu\rho\sigma\tau}}{\sqrt{-g}}\,F^J_{\mu\nu}F^K_{\rho\sigma} = 6g\,h_{uv}k^u_I\,\hat{\partial}^\tau q^v\,, \tag{3.70b}$$

$$\nabla_\mu\big(\mathcal{G}_{ij}\partial^\mu\phi^j\big) - \frac{1}{2}\,\partial_i\mathcal{G}_{kj}\partial_\mu\phi^k\partial^\mu\phi^j - \frac{1}{4}\,\partial_i G_{IJ}F^I_{\mu\nu}F^{J\mu\nu} - \partial_i V = 0\,, \tag{3.70c}$$

$$\nabla_\mu\big(h_{uv}\hat{\partial}^\mu q^v\big) - \frac{1}{2}\,\partial_u h_{wv}\hat{\partial}_\mu q^w\hat{\partial}^\mu q^v - 3g\,h_{wv}A^I_\mu\partial_u k^w_I\,\hat{\partial}^\mu q^v - \frac{1}{2}\,\partial_u V = 0\,, \tag{3.70d}$$

where ∂_i and ∂_u denotes the derivatives w.r.t. ϕ^i and q^u.

Excluding the hypermultiplets and adopting the U(1) FI-gauging, the q-terms disappear and the scalar potential reduces to

$$V = g^2 V_I V_J\left(\frac{9}{2}\,\mathcal{G}^{ij}\partial_i h^I\partial_j h^J - 6h^I h^J\right), \tag{3.71}$$

with V_I the FI-constants. When dealing with this 5-dimensional theory we shall understand $g_I := g\,V_I$.

3.5.1 r-map

As we anticipated in section 3.2.2, 5d supergravity is tightly bound to its 4-dimensional counterpart via the r-map, of which we shall now give an explicit realization. This map across dimensions was first constructed in [84] for the ungauged supergravity and later extended to gauged theories in [85], to which we refer for the proof[17].

The starting point is the 5-dimensional Lagrangian (3.67) with $n_V - 1$ interacting vector multiplets, n_H hypermultiplets and an abelian gauging. In what follows, for the sake of simplification, we shall not consider the hypermultiplets since they are not affected by the dimensional reduction and, in fact, their moduli space is the same in $D=4$ and $D=5$. Nevertheless, it is possible to show that their covariant derivative and scalar potential transform consistently [85]. The dimensional reduction is performed along an S^1 with compact coordinate z, hence on the metric and the vector fields we take the Kaluza-Klein ansatz

$$ds_5^2 = e^{\frac{\phi}{\sqrt{3}}}ds_4^2 + e^{-\frac{2\phi}{\sqrt{3}}}(dz + K_\mu dx^\mu)^2\,, \qquad A^I = B^I(dz + K_\mu dx^\mu) + C^I_\mu dx^\mu\,, \tag{3.72}$$

where all the fields depend only on the 4-dimensional coordinates x^μ with $\mu = 0,\ldots,3$. The 4-dimensional Lagrangian (3.62) with n_V vector multiplets in the case of the cubic prepotential $F = -\frac{1}{6}C_{IJK}\frac{X^I X^J X^K}{X^0}$ can be retrieved via the identifications

$$z^I = -B^I + i\,e^{-\frac{\phi}{\sqrt{3}}}h^I\,, \qquad e^{\mathcal{K}} = \frac{1}{8}\,e^{\sqrt{3}\phi}\,, \qquad F^\Lambda_{\mu\nu} = \frac{1}{\sqrt{2}}\,(dK_{\mu\nu}, dC^I_{\mu\nu})\,, \tag{3.73}$$

where here $I = 1,\ldots,n_V$ and $\Lambda = 0,\ldots,n_V$ play the role respectively of α and I in sections 3.2.1 and 3.4 and the complex scalars z^I are defined through the projection $z^I = X^I/X^0$. Moreover, the following additional constraints must be imposed

$$g_{(4)} = 3\sqrt{2}\,g_{(5)} \qquad \text{and} \qquad \mathcal{P}^x_0 = k^u_0 = 0\,, \tag{3.74}$$

[17] In order to get the r-map with our conventions it is necessary to switch the sign of C_{IJK} and h^I and to take $\varepsilon^{\mu\nu\rho\sigma z}_{(5)} = \varepsilon^{\mu\nu\rho\sigma}_{(4)}$.

where $g_{(4)}$ and $g_{(5)}$ are the gauge couplings in four and five dimensions.

As we can see, in the oxidation process the real part of the complex scalars z^I is "eaten" by the vector fields, whereas their imaginary part parametrize the embedding space of the very special (real) target space, leaving, after imposing the condition $\mathcal{V} = 1$, only $n_V - 1$ independent real scalars. An example of this is the oxidation of the 4-dimensional t^3 model, which leads to a 5-dimensional theory with no scalars ($n_V = 0$) and, indeed, to a theory with only one h "function" whose value is fixed to a constant by $\mathcal{V} = 1$.

Chapter 4

Black holes in $\mathcal{N} = 2$, $D = 4$ gauged supergravity

Gauged supergravity theories and their black hole solutions played in the past decades a central role in addressing fundamental questions of gravity, from the "classical" uniqueness theorems to the "quantum" entropy computations.

As we already mentioned, the gauging of (part of) the R-symmetry gives rise to a potential for the scalars inside the matter multiplets which, in a vacuum configuration, behaves effectively as a cosmological constant. Exactly this cosmological constant allows the existence of non-asymptotically flat solutions, often, but not always, approaching AdS at infinity, solutions that, since the formulation of the AdS/CFT correspondence, found many applications to entropy calculations through the counting of black hole microstates (see [86–88] for examples in $\mathcal{N} = 2$, $D = 4$ FI-gauged supergravity).

Moreover, in four dimensions no-hair and uniqueness theorems can be easily circumvented when, for example, a cosmological constant is taken into account, allowing for black holes with non-spherical horizons, like in [89] where the horizon can be a generic Riemann surface when $\Lambda < 0$. For this reason, gauged supergravity is a terrain much explored for the study of this kind of black holes: the scalar potential guarantees the possibility of various asymptotic behaviours, hence allowing to evade the uniqueness theorems and to display horizons that are flat, hyperbolic or generic Riemann surfaces of genus $g > 1$ [90–95].

The first paper on this subject was [96], where the 4-dimensional $\mathcal{N} = 2$ theory can be obtained as a truncation of $\mathcal{N} = 8$ gauged supergravity and static non-extremal black holes carrying electric or magnetic charges were presented. Ref. [97] derived electrically charged 1/2-BPS solutions for arbitrary prepotential, which unfortunately turned out to be naked singularities for non-vanishing values of the gauge coupling constant. In [94] the first examples of genuine supersymmetric black holes in AdS_4 with non-constant scalar fields were presented for the t^3 and the *stu* model.

The vast majority of the 4-dimensional $\mathcal{N} = 2$ gauged supergravities are classified according to their prepotential and are roughly divided in quadratic and cubic ones, even though theories with no prepotential at all do exist. The most known representatives of the first category are the $\overline{\mathbb{C}\mathrm{P}}^n$ models, of which the $\overline{\mathbb{C}\mathrm{P}}^1$ and the related $-iX^0X^1$ are special cases, whereas the second family includes the cubic models introduced in subsection 3.2.1, alternative formulations[1] and possible deformations. The full list of references on this topic would be rather long, here we shall only give a selected bibliography of papers dealing mainly with quadratic prepotentials [98–101], cubic ones [102–112] or both [95, 113].

In this chapter we shall focus on the $\overline{\mathbb{C}\mathrm{P}}^n$ and t^3 models and construct brand new black hole solutions. A first overview of the theory was already given in section 3.4, to which we refer for

[1]The best example is the so-called *magnetic stu* model with $F = -2i\sqrt{X^0X^1X^2X^3}$. It can be derived from the "classic" *stu* by means of a symplectic rotation which, it is worth stressing, changes the gauging of the theory as well.

the Lagrangian and the equations of motion. We begin with a brief review of different methods to construct solutions in supergravity theories, focusing on supersymmetric ones. In section 4.2 we apply the recipes of the first section to construct rotating extremal BPS black holes in the $\overline{\mathbb{C}\mathrm{P}}^n$ model, which preserve two real supercharges. Moreover, we also obtain BPS black holes with NUT charge in the same model. Section 4.3 contains a generalization of the solutions of the previous section to the non-extremal case along with a discussion of the thermodynamics of these backgrounds. The last section, namely 4.4, is dedicated to the t^3 model, for which we present first a supersymmetric near-horizon solution, which is subsequently extended to a full black hole geometry.

4.1 Constructing solutions

When searching for solutions of a certain theory, dealing directly with its equations of motion is for sure the most immediate way, but it takes little time to understand how hard this task could be. Even in the vacuum, Einstein's equations are quite involved and symmetries must be imposed in order to simplify their expression and make them tractable. In the case of gauged supergravity, and, actually, even in less complex cases like Einstein-Maxwell theories, the second-order Einstein equations become strongly coupled and highly non-linear, thus not trivial at all. In order to circumvent this computational problem, throughout the last decades many different approaches to this problem have been developed and exploited extensively.

One of these methods is the *Hamilton-Jacobi approach*, in which the second-order equations governing the theory are derived from a 1-dimensional effective action. It is then possible to apply the classical Hamilton-Jacobi formalism or to manipulate the Lagrangian and write it as a sum of squares in order to derive a set of first-order differential equations equivalent to the initial equations of motion. We refer to [85, 114–118] for some examples applied to four and five dimensions.

The *attractor mechanism* is a powerful tool in the study of extremal black holes and consists in a dynamical process of stabilization of the scalars at the horizon independently from their asymptotic values. Since we will make use of it later on, we postpone its analysis to section 5.5.

A completely different way to tackle the problem is to look for supersymmetric solutions. The basic idea, that will be explored in detail in section 5.4, is to impose that the SUSY variations of the fermionic fields vanish, so to construct solutions that preserve (part of the) SUSY invariance. These equations lead to constraints on the metric and the fields of the theory, the so-called *BPS equations*, which are of first order and thus, in general, easier to solve rather than the second-order equations of motion.

These equations represent the basis on top of which many other methods have been developed. By means of the BPS equations, in four dimensions all the timelike supersymmetric solutions coupled to an arbitrary number of abelian vector multiplets were classified in [119] and shortly we shall exploit the power of this result. In the 5-dimensional case a similar result was obtained firstly for pure gauged supergravity in [120] and it was later generalized in [121] including abelian vector multiplets for asymptotically AdS black holes.

Moreover, but uniquely in four dimensions, it is possible to take advantage of the properties of the special Kähler moduli space generated by the vector multiplets and of its symplectic structure. In [104, 113, 122] the authors made an extensive use of a real, symplectically covariant, formulation of special geometry and its dimensional reduction.

This real formulation was later applied in combination with the rank-four symplectic covariant tensor I_4 [123–125]. Although this way of repackaging 4-dimensional $\mathcal{N}=2$ supergravity is less popular in the literature, it proves to be in some cases very convenient for solving the BPS equations and was thus adopted in, e.g., [111, 112, 126]

In this section we shall present the BPS equations of the theory under examination and a class of half-supersymmetric near horizon geometries. These two ingredients will turn out to be of great use to construct new solutions.

4.1.1 The BPS equations

The most general timelike supersymmetric background of the $\mathcal{N}=2$, $D=4$ U(1) FI-gauged supergravity was constructed in [119] and is given by

$$ds^2 = -4|b|^2(dt+\sigma)^2 + |b|^{-2}(dz^2 + e^{2\Phi}dw\,d\bar{w})\,, \tag{4.1}$$

where the complex function $b(z,w,\bar{w})$, the real function $\Phi(z,w,\bar{w})$ and the 1-form $\sigma = \sigma_w\,dw + \sigma_{\bar{w}}\,d\bar{w}$, together with the symplectic section (3.4)[2] are determined by the equations

$$\partial_z\Phi = 2i\,g_I\left(\frac{\bar{X}^I}{b} - \frac{X^I}{\bar{b}}\right)\,, \tag{4.2}$$

$$\begin{aligned}&4\partial\bar{\partial}\left(\frac{X^I}{\bar{b}} - \frac{\bar{X}^I}{b}\right) + \partial_z\left[e^{2\Phi}\partial_z\left(\frac{X^I}{\bar{b}} - \frac{\bar{X}^I}{b}\right)\right]\\ &\quad - 2i\,g_J\,\partial_z\left\{e^{2\Phi}\left[|b|^{-2}(\mathrm{Im}\,\mathcal{N})^{-1|IJ} + 2\left(\frac{X^I}{\bar{b}} + \frac{\bar{X}^I}{b}\right)\left(\frac{X^J}{\bar{b}} + \frac{\bar{X}^J}{b}\right)\right]\right\} = 0\,,\end{aligned} \tag{4.3}$$

$$\begin{aligned}&4\partial\bar{\partial}\left(\frac{F_I}{\bar{b}} - \frac{\bar{F}_I}{b}\right) + \partial_z\left[e^{2\Phi}\partial_z\left(\frac{F_I}{\bar{b}} - \frac{\bar{F}_I}{b}\right)\right]\\ &\quad - 2i\,g_J\,\partial_z\left\{e^{2\Phi}\left[|b|^{-2}\,\mathrm{Re}\,\mathcal{N}_{IL}(\mathrm{Im}\,\mathcal{N})^{-1|JL} + 2\left(\frac{F_I}{\bar{b}} + \frac{\bar{F}_I}{b}\right)\left(\frac{X^J}{\bar{b}} + \frac{\bar{X}^J}{b}\right)\right]\right\}\\ &\quad - 8i\,g_I e^{2\Phi}\left[\langle\mathcal{I},\partial_z\mathcal{I}\rangle - |b|^{-2}g_J\left(\frac{X^J}{\bar{b}} + \frac{\bar{X}^J}{b}\right)\right] = 0\,,\end{aligned} \tag{4.4}$$

$$2\partial\bar{\partial}\Phi = e^{2\Phi}\left[i\,g_I\partial_z\left(\frac{X^I}{\bar{b}} - \frac{\bar{X}^I}{b}\right) + 2|b|^{-2}g_Ig_J(\mathrm{Im}\,\mathcal{N})^{-1|IJ} + 4\left(\frac{g_IX^I}{\bar{b}} + \frac{g_I\bar{X}^I}{b}\right)^2\right]\,, \tag{4.5}$$

$$d\sigma + 2\star_3\langle\mathcal{I},d\mathcal{I}\rangle - \frac{i}{|b|^2}\,g_I\left(\frac{\bar{X}^I}{b} + \frac{X^I}{\bar{b}}\right)e^{2\Phi}\,dw\wedge d\bar{w} = 0\,. \tag{4.6}$$

Here $\star_3$ is the Hodge star on the 3-dimensional base with metric[3]

$$ds_3^2 = dz^2 + e^{2\Phi}dw\,d\bar{w}\,, \tag{4.7}$$

and we defined $\partial := \partial_w$, $\bar{\partial} := \partial_{\bar{w}}$, as well as

$$\mathcal{I} := \mathrm{Im}\left(\mathcal{V}/\bar{b}\right)\,,\qquad \mathcal{R} := \mathrm{Re}\left(\mathcal{V}/\bar{b}\right)\,. \tag{4.8}$$

Note that equations (4.2)-(4.5) can be written compactly in the symplectic covariant form

$$\partial_z\Phi = 4\langle\mathcal{I},\mathcal{G}\rangle\,, \tag{4.9}$$

$$\begin{aligned}&\Delta\mathcal{I} + 2e^{-2\Phi}\partial_z\left[e^{2\Phi}\big(\langle\mathcal{R},\mathcal{I}\rangle\Omega\mathcal{M}\mathcal{G} - 4\mathcal{R}\langle\mathcal{R},\mathcal{G}\rangle\big)\right]\\ &\quad - 4\mathcal{G}\big(\langle\mathcal{I},\partial_z\mathcal{I}\rangle + 4\langle\mathcal{R},\mathcal{I}\rangle\langle\mathcal{R},\mathcal{G}\rangle\big) = 0\,,\end{aligned} \tag{4.10}$$

$$\Delta\Phi = -8\langle\mathcal{R},\mathcal{I}\rangle\big(\mathcal{G}^t\mathcal{M}\mathcal{G} + 8|\mathcal{L}|^2\big) = 4\langle\mathcal{R},\mathcal{I}\rangle V\,, \tag{4.11}$$

[2] Note that also σ and $\mathcal{V}$ are independent of t.

[3] Whereas in the ungauged case this base space is flat and thus has trivial holonomy, here we have U(1) holonomy with torsion [119].

where $\mathcal{G} := (g^I, g_I)^t$ represents the symplectic vector of gauge couplings[4], $\mathcal{L} := \langle \mathcal{V}, \mathcal{G}\rangle$, Δ denotes the covariant Laplacian associated to the base space metric (4.7) and V is the scalar potential (3.66). Moreover,

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} \mathrm{Im}\,\mathcal{N} + \mathrm{Re}\,\mathcal{N}\,(\mathrm{Im}\,\mathcal{N})^{-1}\,\mathrm{Re}\,\mathcal{N} & -\mathrm{Re}\,\mathcal{N}\,(\mathrm{Im}\,\mathcal{N})^{-1} \\ -(\mathrm{Im}\,\mathcal{N})^{-1}\,\mathrm{Re}\,\mathcal{N} & (\mathrm{Im}\,\mathcal{N})^{-1} \end{pmatrix}. \tag{4.12}$$

Finally, (4.6) can be rewritten as

$$d\sigma + \star_h\left(d\Sigma - \mathcal{A} + \frac{1}{2}\,\nu\,\Sigma\right) = 0\,, \tag{4.13}$$

where the function Σ and the 1-form ν are respectively given by

$$\Sigma := \frac{i}{2}\ln\frac{\bar{b}}{b}\,, \qquad \nu := \frac{8}{\Sigma}\langle \mathcal{G}, \mathcal{R}\rangle\, dz\,, \tag{4.14}$$

$\mathcal{A}$ is the U(1) gauge field of the special Kähler manifold (3.8) and $\star_h$ denotes the Hodge star on the Weyl-rescaled base space metric

$$h_{ij}\, dx^i dx^j = |b|^{-4}(dz^2 + e^{2\Phi} dw\, d\bar{w})\,. \tag{4.15}$$

(4.13) is the generalized monopole equation [127], or more precisely a Kähler-covariant generalization thereof, due to the presence of the 1-form $\mathcal{A}$. In order to cast (4.6) into the form (4.13), one has to use the special Kähler identities (3.11). Note that (4.13) is invariant under Weyl rescaling, accompanied by a gauge transformation of ν,

$$h_{mn}\, dx^m dx^n \mapsto e^{2\psi} h_{mn}\, dx^m dx^n\,, \quad \Sigma \mapsto e^{-\psi}\Sigma\,, \quad \nu \mapsto \nu + 2d\psi\,, \quad \mathcal{A} \mapsto e^{-\psi}\mathcal{A}\,. \tag{4.16}$$

The integrability condition for (4.13) reads

$$D_i\left[h^{ij}\sqrt{h}\,(D_j - \mathcal{A}_j)\,\Sigma\right] = 0\,, \tag{4.17}$$

with the Weyl-covariant derivative

$$D_i := \partial_i - \frac{m}{2}\,\nu_i\,, \tag{4.18}$$

where m denotes the Weyl weight of the corresponding field[5]. It is straightforward to show that (4.17) is equivalent to

$$\langle \mathcal{I}, \Delta\mathcal{I}\rangle + 4e^{-2\Phi}\partial_z\left(e^{2\Phi}\langle \mathcal{I}, \mathcal{R}\rangle\langle \mathcal{G}, \mathcal{R}\rangle\right) = 0\,, \tag{4.19}$$

which follows from (4.10) by taking the symplectic product with $\mathcal{I}$. To show this, one has to use

$$\frac{1}{2}\,(\mathcal{M} + i\,\Omega) = \Omega\bar{\mathcal{V}}\mathcal{V}\Omega + \Omega\,\mathcal{D}_\alpha\mathcal{V}\, g^{\alpha\bar{\beta}}\mathcal{D}_{\bar{\beta}}\bar{\mathcal{V}}\,\Omega\,, \tag{4.20}$$

as well as (4.9) and (3.11).

Given b, Φ, σ and $\mathcal{V}$, the fluxes read

$$\begin{aligned} F^I &= 2(dt+\sigma)\wedge d(bX^I + \bar{b}\bar{X}^I) + |b|^{-2}dz\wedge d\bar{w}\big[\bar{X}^I(\bar{\partial}\bar{b} + i\mathcal{A}_{\bar{w}}\bar{b}) + (\mathcal{D}_\alpha X^I)\, b\,\bar{\partial} z^\alpha \\ &\quad - X^I(\bar{\partial} b - i\mathcal{A}_{\bar{w}} b) - (\mathcal{D}_{\bar{\alpha}}\bar{X}^I)\,\bar{b}\,\bar{\partial}\bar{z}^{\bar{\alpha}}\big] - |b|^{-2}dz\wedge dw\big[\bar{X}^I(\partial\bar{b} + i\mathcal{A}_w\bar{b}) \\ &\quad + (\mathcal{D}_\alpha X^I)\, b\,\partial z^\alpha - X^I(\partial b - i\mathcal{A}_w b) - (\mathcal{D}_{\bar{\alpha}}\bar{X}^I)\,\bar{b}\,\partial\bar{z}^{\bar{\alpha}}\big] \\ &\quad - \frac{1}{2}\,|b|^{-2}e^{2\Phi}dw\wedge d\bar{w}\big[\bar{X}^I(\partial_z\bar{b} + i\mathcal{A}_z\bar{b}) + (\mathcal{D}_\alpha X^I)\, b\,\partial_z z^\alpha - X^I(\partial_z b - i\mathcal{A}_z b) \\ &\quad - (\mathcal{D}_{\bar{\alpha}}\bar{X}^I)\,\bar{b}\,\partial_z\bar{z}^{\bar{\alpha}} - 2i\, g_J(\mathrm{Im}\,\mathcal{N})^{-1|IJ}\big]\,. \end{aligned} \tag{4.21}$$

[4] In the case considered here with electric gaugings only, one has $g^I = 0$.

[5] A field Γ with Weyl weight m transforms as $\Gamma \mapsto e^{m\psi}\Gamma$ under a Weyl rescaling.

4.1.2 1/2-BPS near-horizon geometries

An interesting class of half-supersymmetric backgrounds was obtained in [128] and includes the near-horizon geometry of extremal rotating black holes. The metric and the fluxes read respectively

$$\begin{aligned} ds^2 &= 4e^{-\xi}\left(-r^2dt^2+\frac{dr^2}{r^2}\right)+4(e^{-\xi}-Ke^{\xi})(d\phi+r\,dt)^2+\frac{4e^{-2\xi}d\xi^2}{Y^2(e^{-\xi}-Ke^{\xi})}\,, \\ F^I &= 16i\sqrt{K}\left(\frac{\bar{X}X^I}{1-iY}-\frac{X\bar{X}^I}{1+iY}\right)dt\wedge dr \\ &\quad+\frac{8\sqrt{K}}{Y}\left[\frac{2\bar{X}X^I}{1-iY}+\frac{2X\bar{X}^I}{1+iY}+(\mathrm{Im}\,\mathcal{N})^{-1|IJ}g_J\right](d\phi+r\,dt)\wedge d\xi\,, \end{aligned} \tag{4.22}$$

where $X:=g_IX^I$, $K>0$ is a real integration constant and Y is defined by

$$Y^2=64\,e^{-\xi}|X|^2-1\,. \tag{4.23}$$

The moduli fields z^α depend on the horizon coordinate ξ only and obey the flow equation[6]

$$\frac{dz^\alpha}{d\xi}=\frac{i}{2\bar{X}Y}\,(1-iY)\,g^{\alpha\bar{\beta}}\mathcal{D}_{\bar{\beta}}\bar{X}\,. \tag{4.24}$$

Metric (4.22) is of the form (3.3) of [129] and describes the near-horizon geometry of extremal rotating black holes[7], with isometry group $\mathrm{SL}(2,\mathbb{R})\times\mathrm{U}(1)$. From (4.24) it is clear that the scalar fields have a non-trivial dependence on the horizon coordinate ξ unless $g_I\mathcal{D}_\alpha X^I=0$. As was shown in [128], the solution with constant scalars is the near-horizon limit of the supersymmetric rotating hyperbolic black holes in minimal gauged supergravity [93].

Using Y in place of ξ as a new variable, (4.24) becomes

$$\frac{dz^\alpha}{dY}=\frac{Xg^{\alpha\bar{\beta}}\mathcal{D}_{\bar{\beta}}\bar{X}}{(Y-i)\left[-\bar{X}X+\mathcal{D}_\gamma Xg^{\gamma\bar{\delta}}\mathcal{D}_{\bar{\delta}}\bar{X}\right]}\,. \tag{4.25}$$

This can also be rewritten in a Kähler-covariant form, as a differential equation for the symplectic section $\mathcal{V}$,

$$D_Y\mathcal{V}=\frac{X\mathcal{D}_\alpha\mathcal{V}\,g^{\alpha\bar{\beta}}\mathcal{D}_{\bar{\beta}}\bar{X}}{(Y-i)\left[-\bar{X}X+\mathcal{D}_\gamma Xg^{\gamma\bar{\delta}}\mathcal{D}_{\bar{\delta}}\bar{X}\right]}\,, \tag{4.26}$$

where

$$D_Y:=\frac{d}{dY}+i\mathcal{A}_Y \tag{4.27}$$

denotes the Kähler-covariant derivative.

4.2 SUSY rotating black holes in the $\overline{\mathbb{C}\mathrm{P}}^n$ model

In this section we shall focus on the $\overline{\mathbb{C}\mathrm{P}}^n$ model and obtain a generalization of the asymptotically AdS black holes found in [98, 99] to include an arbitrary number of vector multiplets n. To do so, we shall use some ansätze which are inspired by those articles. We begin constructing in detail a rotating black hole specified in terms of $n+2$ parameters—see (4.51), (4.54) and (4.56).

[6] Note that this is not a radial flow, but a flow along the horizon.

[7] Metrics of the type (4.22) were discussed for the first time in [130] in the context of the extremal Kerr throat geometry.

Moreover, in subsection 4.2.4 we present a solution with NUT charge and in section 4.3 a non-extremal extension.

We already presented the main features of this model among the examples of special Kähler geometries in section 3.2.1, to which we refer. Here we remind the expression of the prepotential and the lower part of the symplectic section:

$$F = \frac{i}{4} X^I \eta_{IJ} X^J \,, \qquad F_I = \frac{i}{2} \eta_{IJ} X^J \,. \tag{4.28}$$

An additional ingredient we will need is the imaginary part of the period matrix and its inverse

$$\mathrm{Im}\,\mathcal{N}_{IJ} = -\frac{1}{2}\eta_{IJ} + \frac{1}{2}\left(\frac{Z_I Z_J}{Z_K Z^K} + \text{c.c.}\right) , \tag{4.29a}$$

$$(\mathrm{Im}\,\mathcal{N})^{-1|IJ} = 2\left[-\eta^{IJ} + \left(\frac{Z^I \bar{Z}^J}{Z^K \bar{Z}_K} + \text{c.c.}\right)\right] , \tag{4.29b}$$

where we defined $Z_I := \eta_{IJ} Z^J$. Finally, the scalar potential (3.66) reads

$$V = 4g^2 - 8\,\frac{|g_0 + \sum_\alpha g_\alpha z^\alpha|^2}{1 - \sum_\beta |z^\beta|^2} \,, \tag{4.30}$$

with $g^2 := g_I \eta^{IJ} g_J$ from now on. V has an extremum at $z^\alpha = -g_\alpha/g_0$, where $V = 12g^2$. For $z^\alpha = -g_\alpha/g_0$ to lie in the allowed region, the vector of gauge couplings g_I must be timelike, i.e. $g^2 < 0$. This extremum corresponds to a supersymmetric AdS vacuum. In addition, it is easy to see that the potential has flat directions given by $g_0 + \sum_\alpha g_\alpha z^\alpha = 0$, where $V = 4g^2$. For $n = 1$ the flat directions degenerate to the point $z^1 = -g_0/g_1$, which lies in the allowed region for $g^2 > 0$. Thus, in this case one has a critical point corresponding to a SUSY-breaking de Sitter vacuum. If there is more than one vector multiplet, the situation is of course more complicated.

4.2.1 Solving the BPS equations

A natural generalization of the successful ansatz used in [98] for the $\overline{\mathbb{CP}}^1$ model is given by

$$\frac{\bar{X}^I}{b} = \frac{f^I(z) + \eta^I(w,\bar{w})}{\gamma(z)} \,, \qquad e^{2\Phi} = h(z)\,\ell(w,\bar{w}) \,, \tag{4.31}$$

where $f^I(z)$ is a purely imaginary function, while $\gamma(z)$, $\eta^I(w,\bar{w})$, $h(z)$ and $\ell(w,\bar{w})$ are real. With these assumptions, the BPS equations, although remaining non-linear, become separable and can be solved. The first of them, (4.2), boils down to

$$\partial_z \ln h = -\frac{8 g_I\, \mathrm{Im}\, f^I}{\gamma} \,. \tag{4.32}$$

The symplectic constraint (3.5) implies that $X^I \eta_{IJ} \bar{X}^J = -1$, which in turn gives

$$|b|^{-2} = \frac{1}{\gamma^2}\eta_{IJ}\left(f^I f^J - \eta^I \eta^J\right) . \tag{4.33}$$

Using these expressions, equation (4.5) reduces to

$$\frac{\partial\bar{\partial}\ln\ell}{\ell} = h\left[-\frac{1}{4}\partial_z^2 \ln h + \frac{4}{\gamma^2} g_I g_J \Big(\eta^{IJ}\eta_{LK}(\eta^L\eta^K - f^L f^K) + 2(f^I f^J + \eta^I\eta^J)\Big)\right] . \tag{4.34}$$

Now we observe that if we take $h/\gamma^2 = const := c_1 > 0$, this differential equation is separable and one can define a constant c_2 such that

$$-\frac{h}{4}\,\partial_z^2 \ln h + \frac{4h}{\gamma^2}\, g_I g_J \big(-\eta^{IJ} f_K f^K + 2 f^I f^J\big) = c_1 c_2\,, \tag{4.35}$$

$$\frac{\partial\bar{\partial}\ln\ell}{\ell} - 4c_1\big(g^2 \eta_K \eta^K + 2\eta^2\big) = c_1 c_2\,, \tag{4.36}$$

where $\eta := g_I \eta^I$ and capital indices are lowered with η_{IJ}. Equations (4.32) and (4.35) can be solved by means of the polynomial ansatz

$$\gamma = c + az^2\,, \qquad h = c_1(c+az^2)^2\,, \qquad f^I = i(\alpha^I z + \beta^I)\,, \tag{4.37}$$

for some real constants a, c, α^I and β^I, which are constrained by

$$g_I \alpha^I = -\frac{a}{2}\,, \quad g_I \beta^I = 0\,, \quad \alpha^I \eta_{IJ} \beta^J = 0\,, \quad -ac + 4g^2\beta^2 = c_2\,, \quad a^2 = 4g^2\alpha^2\,, \tag{4.38}$$

where $\alpha^2 := \alpha^I \eta_{IJ} \alpha^J$ and $\beta^2 := \beta^I \eta_{IJ} \beta^J$.

The Bianchi identities (4.3) are then easily solved and lead to

$$\alpha^I = -\frac{2\eta^{IJ} g_J \alpha^2}{a}\,. \tag{4.39}$$

It is worth noting that the set of constraints obtained so far completely fixes α^I and c_2 in terms of a, c and β^α, while c_1 remains free. As we will see, some of these degrees of freedom can be eliminated by a coordinate transformation.

After some computation, Maxwell's equations (4.4) reduce to

$$\partial\bar{\partial}\eta_I - 4\ell c_1 \eta g_I \left(\eta_K \eta^K + \beta^2 - \frac{\alpha^2 c}{a}\right) = 0\,. \tag{4.40}$$

Together with (4.36), they define a system of $n+1$ second order, non-linear differential equations, and looking for the general solution might seem a hopeless endeavour. Remarkably, the system can be solved using the ansatz of the type considered in [98],

$$\ell = \frac{1+\delta}{\cosh^4(k\tilde{x})}\,, \quad \eta^I = \hat{\eta}^I \tanh(k\tilde{x})\,, \quad \delta = A\cosh^4(k\tilde{x})\,, \quad \frac{dx}{d\tilde{x}} = \frac{\cosh^2(k\tilde{x})}{1+\delta}\,, \tag{4.41}$$

where $A, k, \hat{\eta}^I$ are some constants and $x := (w+\bar{w})/2$. Defining $\hat{\eta} := g_I \hat{\eta}^I$, equation (4.36) becomes

$$k^2 + c_1 c_2 + \sinh^2(k\tilde{x})\big(-2k^2 + c_1 c_2 + 4c_1 g^2 \hat{\eta}_K \hat{\eta}^K + 8c_1\hat{\eta}^2\big) = 0\,, \tag{4.42}$$

which is solved provided

$$k^2 = -c_1 c_2\,, \qquad 3k^2 = 4c_1 g^2 \hat{\eta}_K \hat{\eta}^K + 8c_1 \hat{\eta}^2\,. \tag{4.43}$$

On the other hand, Maxwell's equations (4.40) simplify to

$$k^2 \hat{\eta}_I + \sinh^2(k\tilde{x}) 4c_1 \hat{\eta} g_I \left(\hat{\eta}_K \hat{\eta}^K + \beta^2 - \frac{\alpha^2 c}{a}\right) + 4c_1 \hat{\eta} g_I \left(\beta^2 - \frac{\alpha^2 c}{a}\right) = 0\,, \tag{4.44}$$

which are satisfied if

$$\hat{\eta}^K \hat{\eta}_K + \beta^2 - \frac{\alpha^2 c}{a} = 0\,, \qquad k^2 \hat{\eta}_I + 4c_1 \hat{\eta} g_I \left(\beta^2 - \frac{\alpha^2 c}{a}\right) = 0\,. \tag{4.45}$$

In summary, we can combine (4.38), (4.43) and (4.45) to find

$$k^2 = 4c_1\hat{\eta}^2\,, \qquad g^2\hat{\eta}_I = \hat{\eta} g_I\,. \tag{4.46}$$

This implies that the only independent parameter in (4.41) is A.

Finally, to completely specify the solution we have to integrate (4.6). To this end we use $(\tilde{x}, y, z)$ as coordinates, where $y := (w-\bar{w})/(2i)$. The relevant Hodge duals on the metric (4.7) are

$$\star_3\, d\tilde{x} = \frac{1+\delta}{\cosh^2(k\tilde{x})}\, dy\wedge dz\,, \qquad \star_3 dz = \frac{e^{2\Phi}\cosh^2(k\tilde{x})}{1+\delta}\, d\tilde{x}\wedge dy\,, \tag{4.47}$$

and thus (4.6) takes the form

$$\begin{aligned}&\partial_{\tilde{x}}\sigma_y\, d\tilde{x}\wedge dy - \partial_z\sigma_y\, dy\wedge dz - \frac{k}{\gamma^2}\left(\alpha^I\hat{\eta}_I z\right)\frac{1+\delta}{\cosh^4(k\tilde{x})}\, dy\wedge dz - \frac{2\hat{\eta}\alpha^2 c_1}{a}\frac{\tanh(k\tilde{x})}{\cosh^2(k\tilde{x})}\, d\tilde{x}\wedge dy\\ &+ 4c_1\hat{\eta}\left(\alpha^2 z^2 + \beta^2 + \hat{\eta}_K\hat{\eta}^K\tanh^2(k\tilde{x})\right)\frac{\tanh(k\tilde{x})}{\gamma\cosh^2(k\tilde{x})}\, d\tilde{x}\wedge dy = 0\,,\end{aligned} \tag{4.48}$$

which can be easily integrated to give

$$\sigma_y = \frac{\hat{\eta}}{4g^2k}\left[\frac{ac_1}{\cosh^2(k\tilde{x})} - \frac{k^2}{\gamma}\left(A + \frac{1}{\cosh^4(k\tilde{x})}\right)\right]\,. \tag{4.49}$$

4.2.2 The fields

The metric (4.1) of the solution obtained here can be simplified by the coordinate transformation

$$\begin{pmatrix} t\\ y\end{pmatrix} \mapsto \sqrt{\frac{\mathsf{E}}{-2A}}\begin{pmatrix} 0 & -\frac{aA\mathsf{E}L^3}{8\hat{\eta}}\\ -\frac{1}{kL} & \frac{\mathsf{E}L}{2k}\end{pmatrix}\begin{pmatrix} t\\ y\end{pmatrix}\,, \qquad p = B\tanh(k\tilde{x})\,, \qquad q = D\,z\,, \tag{4.50}$$

where $B = \sqrt{\frac{\mathsf{E}}{-8g^2}}$, $D = \sqrt{\frac{a^2c_1\mathsf{E}}{-8g^2k^2}}$ and E is a positive constant, so A must be negative. In these coordinates the metric takes the Carter-Plebański [131, 132] form[8]

$$\begin{aligned} ds^2 &= \frac{p^2+q^2-\Delta^2}{P}\, dp^2 + \frac{P}{p^2+q^2-\Delta^2}\left(dt + (q^2-\Delta^2)\, dy\right)^2\\ &+ \frac{p^2+q^2-\Delta^2}{Q}\, dq^2 - \frac{Q}{p^2+q^2-\Delta^2}\left(dt - p^2 dy\right)^2\,,\end{aligned} \tag{4.51}$$

where

$$P := (1+A)\frac{\mathsf{E}^2L^2}{4} - \mathsf{E}p^2 + \frac{p^4}{L^2}\,, \qquad Q := \frac{1}{L^2}\left(q^2 + \frac{\mathsf{E}L^2}{2} - \Delta^2\right)^2\,, \tag{4.52}$$

and L^2 and Δ^2 are two positive constants defined by

$$\Delta^2 := \frac{\mathsf{E}\beta^2}{8\hat{\eta}^2}\,, \qquad L^2 := -\frac{1}{4g^2}\,. \tag{4.53}$$

The scalar fields z^α read

$$z^\alpha = \frac{1}{p+iq_1}\left(-\frac{g_\alpha}{g_0}(p+iq) - i\Delta_1\frac{\beta^\alpha}{\beta^0}\right)\,, \tag{4.54}$$

[8] Notice that the metrics (4.51) and (4.89), though looking very similar to the Carter-Plebański form, are not contained in this family since they are not of Petrov type D. Rather, they belong to the more general class called the Benenti-Francaviglia metric [133], which admits a Killing tensor.

with

$$q_1 := q - \Delta_1\,, \qquad \Delta_1 := \frac{\beta^0}{g_0}\sqrt{\frac{-g^2}{\beta^2}}\,\Delta\,. \tag{4.55}$$

If $\Delta_1 = 0$ (or equivalently $\Delta = 0$), the scalars are constant and they assume the value $-g_\alpha/g_0$ for which the potential (4.30) is extremized.

To complete the solution we need an expression for the gauge potentials, which are found by integrating (4.21). This leads to

$$A^I = 2\eta^{IJ} g_J \mathsf{E} L^2 \sqrt{-A}\frac{p}{p^2+q^2-\Delta^2}\big(dt + (q^2-\Delta^2)dy\big)\,. \tag{4.56}$$

The solution is thus specified by $n+2$ free real parameters, and therefore represents a generalization of the black holes with $n=1$ constructed in [98]. The parameters can be taken to be A, E, Δ and β^α/β^0, subject to the constraint $g_I\beta^I = 0$, cf. (4.38).

A particular, interesting choice is given by

$$\sqrt{-A} = \frac{L^2+j^2}{L^2-j^2}\,, \qquad \mathsf{E} = \frac{j^2}{L^2} - 1\,. \tag{4.57}$$

Then, after the change of coordinates

$$p = j\cosh\theta\,, \qquad y = -\frac{\phi}{j\Xi}\,, \qquad t = \frac{T - j\phi}{\Xi}\,, \qquad \Xi := 1 + \frac{j^2}{L^2}\,, \tag{4.58}$$

and defining the functions

$$\rho^2 := q^2 + j^2\cosh^2\theta\,, \quad \Delta_q := \frac{1}{L^2}\left(q^2 + \frac{j^2-L^2}{2} - \Delta^2\right)^2\,, \quad \Delta_\theta := 1 + \frac{j^2}{L^2}\cosh^2\theta\,, \tag{4.59}$$

the metric (4.51), the scalars (4.54) and the gauge potentials (4.56) become respectively

$$\begin{aligned} ds^2 &= \frac{\rho^2-\Delta^2}{\Delta_q}\,dq^2 + \frac{\rho^2-\Delta^2}{\Delta_\theta}\,d\theta^2 + \frac{\Delta_\theta\sinh^2\theta}{(\rho^2-\Delta^2)\,\Xi^2}\big(j\,dT - (q^2+j^2-\Delta^2)\,d\phi\big)^2 \\ &\quad - \frac{\Delta_q}{(\rho^2-\Delta^2)\,\Xi^2}\big(dT + j\sinh^2\theta\,d\phi\big)^2\,, \end{aligned} \tag{4.60a}$$

$$\begin{aligned} z^\alpha &= \frac{1}{j^2\cosh^2\theta + q_1^2}\Bigg[-i\Delta_1\left(\frac{g_\alpha}{g_0}(j\cosh\theta + iq) + \frac{\beta^\alpha}{\beta^0}(j\cosh\theta - iq)\right) \\ &\quad - \frac{g_\alpha}{g_0}\,\rho^2 + \frac{\beta^\alpha}{\beta^0}\,\Delta_1^2\Bigg]\,, \end{aligned} \tag{4.60b}$$

$$A^I = 2\eta^{IJ}g_J L^2\frac{\cosh\theta}{\rho^2-\Delta^2}\big(j\,dT - (q^2+j^2-\Delta^2)\,d\phi\big). \tag{4.60c}$$

The metric depends only on the two constants Δ and j, that can be interpreted respectively as scalar hair and rotation parameters. Note that for $j=0$ the scalars are real, whereas in the rotating case there is a non-trivial axion.

4.2.3 Near-horizon limit

The metric (4.60a) has an event horizon at $\Delta_q = 0$, i.e. for $q = q_h$ with

$$q_{\mathrm{h}}^2 = \Delta^2 + \frac{1}{2}\,(L^2 - j^2)\,. \tag{4.61}$$

To obtain the near-horizon geometry, we set

$$q = q_{\rm h} + \epsilon q_0\, z\,, \qquad T = \frac{\hat{t} q_0}{\epsilon}\,, \qquad \phi = \hat{\phi} + \Omega\,\frac{\hat{t} q_0}{\epsilon}\,, \tag{4.62}$$

and then zoom in by taking the limit $\epsilon \to 0$. The parameter $\Omega := j/(q_{\rm h}^2 + j^2 - \Delta^2)$ represents the angular velocity of the horizon, while $q_0 := \frac{L^2\Xi}{2\sqrt{2}q_{\rm h}}$. In this limit the metric boils down to

$$\begin{aligned} ds^2 = {} & \frac{\rho_{\rm h}^2 - \Delta^2}{4q_{\rm h}^2 z^2} L^2 dz^2 + \frac{\rho_{\rm h}^2 - \Delta^2}{\Delta_\theta}\, d\theta^2 + \frac{L^4 \Delta_\theta \sinh^2\theta}{4(\rho_{\rm h}^2 - \Delta^2)} \left(d\hat{\phi} + \frac{j}{q_{\rm h}}\, z\, d\hat{t} \right)^2 \\ & - \frac{\rho_{\rm h}^2 - \Delta^2}{4q_{\rm h}^2} L^2 z^2 d\hat{t}^2\,, \end{aligned} \tag{4.63}$$

where $\rho_{\rm h}^2 := q_{\rm h}^2 + j^2 \cosh^2\theta$. The final coordinate transformation

$$e^{-\xi} = L^2 \frac{q_{\rm h}^2 + j^2\cosh^2\theta - \Delta^2}{16 q_{\rm h}^2}\,, \qquad x = \frac{q_{\rm h}}{j}\hat{\phi}\,, \tag{4.64}$$

casts the metric into the form (4.22), namely

$$ds^2 = 4e^{-\xi}\left(-z^2 d\hat{t}^2 + \frac{dz^2}{z^2}\right) + 4\big(e^{-\xi} - Ke^{\xi}\big)\big(dx + z\, d\hat{t}\big)^2 + \frac{4e^{-2\xi} d\xi^2}{Y^2(e^{-\xi} - Ke^{\xi})}\,, \tag{4.65}$$

where $K := L^8 \Xi^2 / 1024 q_{\rm h}^4$. There is thus a supersymmetry enhancement for the near-horizon geometry, which preserves half of the eight supercharges of the theory. Exploiting this fact, there is an alternative way to arrive at this solution that goes as follows. In the $\overline{\mathbb{CP}}^n$ model, the flow equation (4.25) becomes

$$\frac{dz^\alpha}{dY} = \frac{(g_\alpha + z^\alpha g_0)(g_0 + \sum_\beta g_\beta z^\beta)}{g^2(Y - i)}\,, \tag{4.66}$$

which is solved by

$$z^\alpha = \frac{\mu_\alpha g_0 - g_\alpha(Y - i)}{g_0(Y - i - \mu_0)}\,. \tag{4.67}$$

Here, $\mu_I = (\mu_0, \mu_\alpha) \in \mathbb{C}^{n+1}$ is a constant vector orthogonal to the gauge coupling, $\mu_I \eta^{IJ} g_J = 0$. One can now compute $|X|^2$ as a function of Y

$$|X|^2 = -\frac{g^4(Y^2+1)}{g^2(Y^2+1) + g_0^2 \mu\cdot\bar{\mu}}\,, \tag{4.68}$$

where $\mu\cdot\bar{\mu} = \mu_I \eta^{IJ} \bar{\mu}_J$. Plugging this into (4.23) gives then $\xi(Y)$ and the metric (4.22) becomes

$$\begin{aligned} ds^2 = {} & \frac{-g^2(Y^2+1) - g_0^2\mu\cdot\bar{\mu}}{16 g^4}\left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + \frac{P(Y)(d\phi + r\, dt)^2}{16g^4[-g^2(Y^2+1) - g_0^2\mu\cdot\bar{\mu}]} \\ & + \frac{-g^2(Y^2+1) - g_0^2\mu\cdot\bar{\mu}}{4P(Y)}\, dY^2\,, \end{aligned} \tag{4.69}$$

where we defined the quartic polynomial

$$P(Y) = [g^2(Y^2+1) + g_0^2 \mu\cdot\bar{\mu}]^2 - K(64 g^4)^2\,. \tag{4.70}$$

Using $Y=-j\cosh\theta/q_{\rm h}$, one finds that the modulus of the parameters μ_I is related to Δ by

$$\mu\cdot\bar{\mu}=\frac{\Delta^2}{4L^2g_0^2q_{\rm h}^2}\,. \tag{4.71}$$

The expression for the vector μ_I can be found requiring that the scalar fields (4.60b) coincide in the near-horizon limit with the expression (4.67), yielding

$$\mu_0=-i\frac{\Delta_1}{q_{\rm h}}\,,\qquad \mu_\alpha=i\frac{\Delta_1\beta^\alpha}{q_{\rm h}\beta^0}\,. \tag{4.72}$$

Since the static supersymmetric black holes in the $\overline{\mathbb{C}\mathrm{P}}^n$ model constructed in [94] have necessarily hyperbolic horizons, one may ask whether spherical rotating horizons are possible. As was discussed in detail in [134], this question is intimately related to the behaviour of $P(Y)$. Namely, for spherical horizons to be feasible $P(Y)$ must have four distinct roots, and then Y is restricted to the region between the two central roots where $P(Y)$ is positive. The latter condition, together with $-g^2(Y^2+1)-g_0^2\mu\cdot\bar{\mu}>0$, is necessary in order for the metric to have the correct signature[9]. Imposing $P(Y)=0$ yields

$$-g^2(Y^2+1)-g_0^2\mu\cdot\bar{\mu}=64g^4\sqrt{K}\,. \tag{4.73}$$

There are thus only two roots $\pm Y_0$ (with $Y_0>0$), and spherical horizons are therefore excluded in the rotating solution as well. One can show that, in the static limit, the near-horizon geometry of the black holes constructed in [94] is recovered.

4.2.4 NUT-charged black holes

In this section we construct supersymmetric NUT-charged black holes. To do so it is sufficient to mimic what was done in [99], where the theory with only one vector multiplet was considered. Since the BPS equations can be solved following the same steps of that paper, we will just briefly summarize the process here and refer to [99] for further details. We assume that both the scalars and the function b depend on the coordinate z only, and use the ansatz

$$\frac{X^I}{\bar{b}}=\frac{\alpha^Iz+\beta^I}{z^2+iDz+C}\,,\qquad \Phi=\psi(z)+\gamma(w,\bar{w})\,, \tag{4.74}$$

where α^I, β^I, C are complex constants and D is a real constant. The dependence of the solution on the coordinates w and $\bar{w}$ is obtained from (4.5), which reduces to

$$-4\partial\bar{\partial}\gamma=\kappa\,e^{2\gamma}\,, \tag{4.75}$$

where κ is a constant whose value will be fixed later. This is the Liouville equation for the metric $e^{2\gamma}dw\,d\bar{w}$, which consequently has constant curvature κ. We will take as a particular solution

$$e^{2\gamma}=\left(1+\frac{\kappa}{4}w\bar{w}\right)^{-2}. \tag{4.76}$$

From (4.2) one gets

$$\psi(z)=\operatorname{Im}\alpha\left(\ln[z^4+z^2(2\operatorname{Re}C+D^2)+2Dz\operatorname{Im}C+|C|^2]\right)\,, \tag{4.77}$$

provided the following constraints are satisfied

$$\begin{gathered}\operatorname{Im}\beta=D\operatorname{Re}\alpha\,,\qquad -2\operatorname{Im}(\bar{\beta}C)=D\operatorname{Im}\alpha\operatorname{Im}C\,,\\ -2\big[\operatorname{Im}(\bar{\alpha}C)+D\operatorname{Re}\beta\big]=\operatorname{Im}\alpha\big(2\operatorname{Re}C+D^2\big)\,,\end{gathered} \tag{4.78}$$

[9] Note that there is a curvature singularity for $-g^2(Y^2+1)-g_0^2\mu\cdot\bar{\mu}=0$.

where $\alpha := g_I\alpha^I$ and $\beta := g_I\beta^I$.

The expressions displayed so far coincide with those found for the case with one vector multiplet; the explicit form of the prepotential has not been used to solve (4.2) and (4.5). In order to solve the remaining BPS equations we choose $\operatorname{Im}\alpha = 1/2$, since in this way they assume a polynomial form. Then the Bianchi identities fix α^I and κ to

$$\alpha^I = \frac{i}{2g^2}\eta^{IJ}g_J\,, \qquad \kappa = 2\operatorname{Re}C - 8g^2\beta^K\bar\beta_K\,. \tag{4.79}$$

On the other hand, Maxwell's equations are automatically satisfied provided these relations hold. Finally, integration of (4.6) gives

$$\sigma = i\frac{D}{32g^2}\frac{\bar w\,dw - w\,d\bar w}{1+\frac{\kappa}{4}w\bar w}\,, \tag{4.80}$$

from which it is evident that the parameter D is related to the NUT charge of the solution. In this case, the warp factor of the metric is

$$|b|^{-2} = -\frac{z^2+4g^2\beta^K\bar\beta_K}{4g^2|z^2+iDz+C|^2}\,, \tag{4.81}$$

where we recall that in this model $g^2<0$. The solution will have an event horizon at $z=z_\mathrm{h}$ if $b(z_\mathrm{h})$ vanishes, which happens for

$$z_\mathrm{h}^2 = -\operatorname{Re}C\,, \qquad Dz_\mathrm{h} = -\operatorname{Im}C\,. \tag{4.82}$$

This is possible if $(\operatorname{Im}C)^2 = -D^2\operatorname{Re}C$ and $\operatorname{Re}C<0$. There is a curvature singularity at $z^2+4g^2\beta^K\bar\beta_K=0$, which is hidden behind the horizon if

$$\operatorname{Re}C < 4g^2\beta^K\bar\beta_K. \tag{4.83}$$

Then, from (4.79) we see that $\kappa<0$ and therefore the horizon is always hyperbolic.

The solution is in principle specified by $2n+2$ real parameters, which can be taken as β^I, D and $\operatorname{Re}C$ with the constraint $\beta=-D/4$, which follows from (4.78). If (4.83) holds, the metric describes a regular black hole. Notice that we can, without loss of generality, set $\kappa=-1$ by means of the scaling symmetry $(t,z,w,C,D,\beta^I,\kappa)\mapsto(t/\lambda,\lambda z,w/\lambda,\lambda^2C,\lambda D,\lambda\beta^I,\lambda^2\kappa)$, thus reducing the number of independent parameters to $2n+1$.

The fluxes can be computed by plugging the results found so far into (4.21). A long but straightforward calculation yields

$$\begin{aligned} F^I &= 4(dt+\sigma)\wedge dz\,\frac{1}{(z^2+4g^2\beta^K\bar\beta_K)^2}\Big[4g^2\left(2\operatorname{Im}C\operatorname{Im}\beta^I-\operatorname{Re}\beta^I\right)z \\ &\quad -2\eta^{IJ}g_JDz\,(1+2\operatorname{Re}C)+\left(-1-2\operatorname{Re}C+2z^2\right)\left(2g^2D\operatorname{Im}\beta^I+\eta^{IJ}g_J\operatorname{Im}C\right)\Big] \\ &\quad -\frac{1}{2}e^{2\gamma}dw\wedge d\bar w\,\frac{i}{4g^2(z^2+4g^2\beta^K\bar\beta_K)}\Big[\eta^{IJ}g_J\left(-1-2(\operatorname{Re}C+z^2)\right) \\ &\quad +4D\eta^{IJ}g_Jz\,(Dz+\operatorname{Im}C)+8g^2D\left(\operatorname{Re}\beta^Iz^2+D\operatorname{Im}\beta^Iz+\operatorname{Re}(\bar C\beta^I)\right)\Big]\,. \end{aligned} \tag{4.84}$$

The magnetic and electric charges of the solution are given by

$$P^I = \frac{1}{4\pi}\int_{\Sigma_\infty}F^I\,, \qquad Q_I = \frac{1}{4\pi}\int_{\Sigma_\infty}G_I\,, \tag{4.85}$$

where Σ_∞ denotes a surface of constant t and z for $z\to\infty$, and G_I is obtained from the action as $G_I=\delta S/\delta F^I$. This leads to

$$\frac{P^I}{V}=\frac{1-2D^2}{8\pi g^2}\eta^{IJ}g_J-\frac{D\,\mathrm{Re}\,\beta^I}{2\pi}\,,\qquad \frac{Q_I}{V}=-\frac{g_I\,\mathrm{Im}\,C}{8\pi g^2}+\frac{D\,\eta_{IJ}\,\mathrm{Im}\,\beta^J}{4\pi}\,, \tag{4.86}$$

where V is defined by

$$V=\frac{i}{2}\int e^{2\gamma}dw\wedge d\bar{w}\,. \tag{4.87}$$

Finally, the scalars read

$$z^\alpha=\frac{2g^2\beta^\alpha+ig_\alpha z}{2g^2\beta^0-ig_0 z}\,. \tag{4.88}$$

4.3 Non-extremal rotating black holes in the $\overline{\mathbb{C}\mathbf{P}}^n$ model

In this section we shall construct a non-extremal deformation of the 1/4-BPS solution presented in section 4.2. To this end we shall take a Carter-Plebański-type ansatz for the metric similar to (4.51), where $Q(q)$ and $P(p)$ are quartic polynomials in q and p respectively,

$$\begin{aligned} ds^2&=-\frac{Q}{W}\left(dt-p^2dy\right)^2+\frac{P}{W}\left(dt+(q^2-\Delta^2)dy\right)^2+W\left(\frac{dq^2}{Q}+\frac{dp^2}{P}\right)\,,\\ Q&=\sum_{n=0}^{4}a_nq^n\,,\qquad P=\sum_{n=0}^{4}b_np^n\,,\qquad W=p^2+q^2-\Delta^2\,, \end{aligned} \tag{4.89}$$

where a_n, b_n and Δ are real constants. The ansatz for the scalars and the gauge potentials is inspired by (4.54) and (4.56),

$$z^\alpha=\frac{1}{p+i(q-\tilde{\Delta})}\left(-\frac{g_\alpha}{g_0}(p+iq)+ic^\alpha\right)\,, \tag{4.90a}$$

$$A^I=\mathsf{P}^I\frac{p}{W}\left[dt+(q^2-\Delta^2)\,dy\right]\,, \tag{4.90b}$$

with P^I real constants related to the magnetic charges, $\tilde{\Delta}$ real and c^α complex constants. Plugging these expressions into the equations of motion (3.65a)-(3.65c) gives a set of constraints for the constants. It then turns out that at a certain point one has to choose whether Δ vanishes or not. In what follows we shall assume $\Delta\neq0$, while the case $\Delta=0$ is postponed to subsection 4.3.3.

For P^I not proportional to the coupling constants g_I one class of solutions is obtained by taking

$$\begin{aligned} a_0&=b_0+b_2\Delta^2-4g^2\Delta^4-\frac{(g_I\mathsf{P}^I)^2}{2g^2}+\frac{\mathsf{P}^2}{4}\,,\qquad a_1=\frac{(g_I\mathsf{P}^I)\sqrt{(g_I\mathsf{P}^I)^2-g^2\mathsf{P}^2}}{2g^2\Delta}\,,\\ a_2&=-b_2+8g^2\Delta^2\,,\qquad a_3=0\,,\qquad a_4=b_4=-4g^2\,,\qquad b_1=b_3=0\,,\\ \tilde{\Delta}&=\Delta\frac{(g_I\mathsf{P}^I)g_0+g^2\mathsf{P}^0}{g_0\sqrt{(g_I\mathsf{P}^I)^2-g^2\mathsf{P}^2}}\,,\qquad c^\alpha=\Delta\frac{(g_I\mathsf{P}^I)g_\alpha-g^2\mathsf{P}^\alpha}{g_0\sqrt{(g_I\mathsf{P}^I)^2-g^2\mathsf{P}^2}}\,. \end{aligned} \tag{4.91}$$

Here we defined $\mathsf{P}^2:=\mathsf{P}^I\eta_{IJ}\mathsf{P}^J$. Fixing the Fayet–Iliopoulos constants g_I the solution depends on $n+4$ parameters b_0, b_2, Δ and P^I. However, our ansatz is left invariant under the scale transformation

$$\begin{aligned} &p\mapsto\lambda p\,,\qquad q\mapsto\lambda q\,,\qquad t\mapsto t/\lambda\,,\qquad y\mapsto y/\lambda^3\,,\\ &\Delta\mapsto\lambda\Delta\,,\qquad a_n\mapsto\lambda^{4-n}a_n\,,\qquad b_n\mapsto\lambda^{4-n}b_n\,, \end{aligned} \tag{4.92}$$

which reduces the number of independent parameters to $n+3$.

With a few lines of computation it is possible to show that this solution contains the one presented in [100] for the prepotential $F=-i\tilde{X}^0\tilde{X}^1$ (a tilde is introduced in order to distinguish between the two solutions). In order to do so, we must consider the case of just one vector multiplet ($n=1$) and perform a symplectic rotation. In particular, introducing the symplectic vectors

$$\mathcal{G}=\begin{pmatrix}0\\ g_I\end{pmatrix}\,,\qquad \mathcal{Q}=\begin{pmatrix}\mathsf{P}^I\\ 0\end{pmatrix}\,,\tag{4.93}$$

and the symplectic matrix

$$T=\left(\begin{array}{cc|cc}1 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}}\\ 1 & -1 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \frac{1}{2} & \frac{1}{2}\\ & & \frac{1}{2} & -\frac{1}{2}\end{array}\right)\,,\tag{4.94}$$

the solution for the rotated $F=-i\tilde{X}^0\tilde{X}^1$ prepotential can be obtained from the same metric and gauge fields in (4.89) and (4.90b), but with the charges and gauge couplings replaced by their rotated counterparts according to $\mathcal{Q}=T\tilde{\mathcal{Q}}$ and $\mathcal{G}=T\tilde{\mathcal{G}}$. On the other hand, the scalar field is

$$\tilde{\tau}=\frac{\tilde{X}^0}{\tilde{X}^1}=\frac{1-z^1}{1+z^1}\,,\tag{4.95}$$

where $\tilde{X}^I$ belongs to the new symplectic section $\tilde{\mathcal{V}}=T^{-1}\mathcal{V}$.

In the supersymmetric, extremal limit we recover the solution presented in section 4.2. To this end, the charge parameters need to be chosen proportional to the gauge couplings, $\mathsf{P}^I=\lambda\,\eta^{IJ}g_J$, with $\lambda\in\mathbb{R}$, hence the relations presented above simplify to

$$\begin{gathered}a_0=b_0+b_2\Delta^2-4g^2\Delta^4-\frac{\lambda^2g^2}{4}\,,\qquad a_1=0\,,\qquad a_2=-b_2+8g^2\Delta^2\,,\\ a_3=0\,,\qquad a_4=b_4=-4g^2\,,\qquad b_1=b_3=0\,,\end{gathered}\tag{4.96}$$

while on the other hand we are no more able to derive an explicit expression for c^α, but we can only assert that they must satisfy the conditions

$$-g_0+c^\alpha g_\alpha=0\,,\qquad c^\alpha c^\alpha=c^\alpha\bar{c}^\alpha=\tilde{\Delta}^2-\frac{\Delta^2g^2}{g_0^2}\,,\tag{4.97}$$

where summation over α is understood. Then, we see that the BPS solution (4.51), (4.54) and (4.56) is recovered for

$$L^2=-\frac{1}{4g^2}\,,\qquad b_0=(1+A)\frac{\mathsf{E}^2L^2}{4}\,,\qquad b_2=-\mathsf{E}\,,\qquad \lambda=2\,\mathsf{E}L^2\sqrt{-A}\,,\tag{4.98}$$

and by identifying $\tilde{\Delta}=\Delta_1$ and $c^\alpha=-\Delta_1\beta^\alpha/\beta^0$.

4.3.1 Properties of the compact horizon case

Since P is an even polynomial we may assume it has two distinct pairs of roots $\pm p_a$ and $\pm p_b$, where $0<p_a<p_b$. We then consider solutions with p in the range $|p|\leq p_a$ by setting $p=p_a\cos\theta$, where $0\leq\theta\leq\pi$, to obtain black holes with a compact horizon. We now use the scaling symmetry (4.92) to set $p_b=L$ without loss of generality, where $L^{-2}=-4g^2$. Defining the rotation parameter j by $p_a^2=j^2$, this means

$$b_0=j^2\,,\qquad b_2=-1-\frac{j^2}{L^2}\,.\tag{4.99}$$

Then, after the coordinate transformation,

$$t \mapsto t + \frac{j\phi}{\Xi}\,, \qquad y \mapsto \frac{\phi}{j\Xi}\,, \tag{4.100}$$

with $\Xi = 1 - \frac{j^2}{L^2}$, metric (4.89) becomes

$$\begin{aligned} ds^2 = &-\frac{Q}{W}\left(dt + \frac{j\sin^2\theta}{\Xi}\,d\phi\right)^2 + \frac{\Delta_\theta \sin^2\theta}{W}\left(j\,dt + \frac{q^2-\Delta^2+j^2}{\Xi}\,d\phi\right)^2 \\ &+ W\left(\frac{dq^2}{Q} + \frac{d\theta^2}{\Delta_\theta}\right)\,, \end{aligned} \tag{4.101}$$

where

$$W = q^2 - \Delta^2 + j^2\cos^2\theta\,, \qquad \Delta_\theta = 1 - \frac{j^2}{L^2}\cos^2\theta\,. \tag{4.102}$$

We notice that for zero rotation parameter, $j=0$, (4.101) boils down to the static non-extremal black holes with running scalar constructed in [113], after the $n=1$ truncation and the symplectic rotation (4.94) are performed.

Metric (4.101) has an event horizon at $q = q_{\rm h}$, where $q_{\rm h}$ is the largest root of Q. The *Bekenstein-Hawking (BH) entropy* of the black hole is given by

$$S = \frac{\pi}{\Xi G}(q_{\rm h}^2 - \Delta^2 + j^2)\,, \tag{4.103}$$

where G denotes the Newton constant. In order to compute the temperature and angular velocity it is convenient to write the metric in the ADM form

$$ds^2 = -N^2 dt^2 + \sigma(d\phi - \omega\,dt)^2 + W\left(\frac{dq^2}{Q} + \frac{d\theta^2}{\Delta_\theta}\right)\,, \tag{4.104}$$

with

$$\begin{gathered} N^2 = \frac{Q\Delta_\theta W}{\Sigma^2}\,, \qquad \sigma = \frac{\Sigma^2\sin^2\theta}{W\Xi^2}\,, \qquad \omega = \frac{j\Xi}{\Sigma^2}[Q - \Delta_\theta(q^2-\Delta^2+j^2)]\,, \\ \Sigma^2 = \Delta_\theta(q^2-\Delta^2+j^2)^2 - Qj^2\sin^2\theta\,. \end{gathered} \tag{4.105}$$

The angular velocity at the horizon and at infinity are thus

$$\omega_{\rm h} = -\frac{j\Xi}{q_{\rm h}^2 - \Delta^2 + j^2}\,, \qquad \omega_\infty = \frac{j}{L^2}\,. \tag{4.106}$$

The angular momentum may be computed as a Komar integral, which leads to

$$J = \frac{a_1 j}{2\Xi^2 G}\,. \tag{4.107}$$

To get the mass of the solution we use the Ashtekar-Magnon-Das (AMD) formalism [135, 136], applied to the conformally rescaled metric $\bar g_{\mu\nu} = (L/q)^2 g_{\mu\nu}$. This gives

$$M = -\frac{a_1}{2\Xi^2 G}\,. \tag{4.108}$$

Notice that the "ground state" $a_1 = 0$ represents a naked singularity. This can be seen as follows. The curvature singularity at $W=0$[10] is shielded by a horizon if $q_{\rm h}^2 - \Delta^2 + j^2\cos^2\theta > 0$, and thus $q_{\rm h}^2 > \Delta^2$, which is equivalent to

$$a_2^2 - \frac{4a_0}{L^2} > \left(1 + \frac{j^2}{L^2}\right)^2\,, \tag{4.109}$$

[10] Notice also that for $W<0$ the real part of the scalar field becomes negative and, thus, ghost modes appear.

where we used the expression for $q_{\rm h}$. This relation is easily shown to be violated for $a_1=0$ by using (4.91).

The magnetic charges p^I are given by

$$\mathsf{p}^I = \frac{1}{4\pi}\oint_{S^2_\infty} F^I = -\frac{\mathsf{P}^I}{\Xi}\,. \tag{4.110}$$

The product of the horizon areas reads

$$\prod_{\Lambda=1}^{4} A_\Lambda = \frac{(4\pi)^4}{\Xi^4}\prod_{\Lambda=1}^{4}(q^2|_{\rm h_\Lambda} - \Delta^2 + j^2) = (4\pi)^4 L^4\left[\frac{(\mathsf{p}^2)^2}{16} + 4G^2J^2\right]\,, \tag{4.111}$$

where $\mathsf{p}^2 := \mathsf{p}^I\eta_{IJ}\mathsf{p}^J$. In the second step we followed what has been done in [100] and the procedure explained in [103]. The charge-dependent term on the rhs of (4.111) is directly related to the prepotential; a fact that was first noticed in [103] for static black holes.

Now that we have computed the physical quantities of our solution, we see that one may choose the $n+3$ free parameters as P^I, Δ, j, or alternatively p^I, M, J. Our black holes are therefore labelled by the values of $n+1$ independent magnetic charges, the mass and the angular momentum.

4.3.2 Thermodynamics and extremality

Imposing regularity of the Wick-rotated metric it is straightforward to compute the Hawking temperature, with the result

$$T = \frac{Q'_{\rm h}}{4\pi(q_{\rm h}^2 - \Delta^2 + j^2)}\,, \tag{4.112}$$

where $Q'_{\rm h}$ denotes the derivative of Q evaluated at the horizon.

Using the extensive quantities S, M, J and p^I computed above, it is possible to obtain the Christodoulou-Ruffini-type mass formula

$$\begin{aligned} M^2 = {}& \frac{S}{4\pi G} + \frac{\pi J^2}{SG} + \frac{\pi}{4SG^3}\frac{(\mathsf{p}^2)^2}{16} + \left(\frac{L^2}{G^2} + \frac{S}{\pi G}\right)\left((g_I\mathsf{p}^I)^2 + \frac{\mathsf{p}^2}{8L^2}\right) + \\ & + \frac{J^2}{L^2} + \frac{S^2}{2\pi^2L^2} + \frac{S^3G}{4\pi^3L^4}\,. \end{aligned} \tag{4.113}$$

Since S, J and p^I form a complete set of extensive parameters, (4.113) gives the thermodynamic fundamental relation $M = M(S, J, \mathsf{p}^I)$. The intensive quantities conjugate to S, J and p^I are the temperature

$$\begin{aligned} T = \left.\frac{\partial M}{\partial S}\right|_{J,\mathsf{p}^I} = {}& \frac{1}{8\pi GM}\left[1 - \frac{4\pi^2J^2}{S^2} - \frac{\pi^2}{S^2G^2}\frac{(\mathsf{p}^2)^2}{16} + 4\left((g_I\mathsf{p}^I)^2 + \frac{\mathsf{p}^2}{8L^2}\right)\right. \\ & \left. + \frac{4SG}{\pi L^2} + \frac{3S^2G^2}{\pi^2L^4}\right]\,, \end{aligned} \tag{4.114}$$

the angular velocity

$$\Omega = \left.\frac{\partial M}{\partial J}\right|_{S,\mathsf{p}^I} = \frac{\pi J}{MGS}\left[1 + \frac{SG}{\pi L^2}\right]\,, \tag{4.115}$$

and the magnetic potentials

$$\begin{aligned} \Phi_I = \left.\frac{\partial M}{\partial \mathsf{p}^I}\right|_{S,J,\mathsf{p}^{K\neq I}} = {}& \frac{1}{MG}\left[\frac{\pi}{4SG^2}\frac{\mathsf{p}^2}{16}\eta_{IK}\mathsf{p}^K\right. \\ & \left. + \left(\frac{L^2}{G} + \frac{S}{\pi}\right)\left((g_K\mathsf{p}^K)g_I + \frac{1}{16L^2}\eta_{IK}\mathsf{p}^K\right)\right]\,. \end{aligned} \tag{4.116}$$

These quantities satisfy the first law of thermodynamics

$$dM = T\,dS + \Omega\,dJ + \Phi_I\,d\mathsf{p}^I\,. \tag{4.117}$$

It is straightforward to verify that expression (4.114) for the temperature agrees with (4.112), while from (4.115) we observe that

$$\Omega = \omega_{\rm h} - \omega_\infty\,, \tag{4.118}$$

with $\omega_{\rm h}$ and ω_∞ given by (4.106). Thus, what enters the first law is the difference between the angular velocities at the horizon and at infinity.

Extremal black holes have vanishing Hawking temperature (4.112), which happens when $q_{\rm h}$ is at least a double root of Q. The structure function Q can then be written as

$$Q = (q-q_{\rm h})^2\left(\frac{q^2}{L^2} + \frac{2q_{\rm h}}{L^2}q + a_2 + \frac{3q_{\rm h}^2}{L^2}\right)\,, \tag{4.119}$$

so we must have

$$a_0 = a_2 q_{\rm h}^2 + \frac{3q_{\rm h}^4}{L^2}\,, \qquad a_1 = -2a_2 q_{\rm h} - \frac{4q_{\rm h}^3}{L^2}\,. \tag{4.120}$$

It is straightforward to check that these relations are satisfied in the supersymmetric limit $P^I = \lambda\eta^{IJ}g_J$ previously described. On the other hand, it may happen that the free parameters are chosen such that (4.91) is compatible with (4.120) even if the charges are not proportional to the gauge couplings. In that case we would obtain an extremal, non-supersymmetric black hole. To obtain the near-horizon geometry of the extremal black holes, we define new (dimensionless) coordinates z, $\hat{t}$ and $\hat{\phi}$ by

$$q = q_{\rm h} + \epsilon q_0 z\,, \qquad t = \frac{q_0}{\Xi\epsilon}\,\hat{t}\,, \qquad \phi = \hat{\phi} + \frac{\omega_{\rm h} q_0}{\Xi\epsilon}\,\hat{t}\,, \tag{4.121}$$

with

$$q_0^2 := \frac{\Xi(q_{\rm h}^2 - \Delta^2 + j^2)}{C}\,, \qquad C = \frac{6q_{\rm h}^2}{L^2} + a_2\,, \tag{4.122}$$

and take $\epsilon \to 0$ keeping z, $\hat{t}$, $\hat{\phi}$ fixed. This leads to

$$\begin{aligned} ds^2 = {}& \frac{q_{\rm h}^2 - \Delta^2 + j^2\cos^2\theta}{C}\left(-z^2 d\hat{t}^2 + \frac{dz^2}{z^2} + C\,\frac{d\theta^2}{\Delta_\theta}\right) \\ &+ \frac{\Delta_\theta (q_{\rm h}^2 - \Delta^2 + j^2)^2\sin^2\theta}{\Xi^2(q_{\rm h}^2 - \Delta^2 + j^2\cos^2\theta)}\left(d\hat{\phi} + \frac{2q_{\rm h}\omega_{\rm h}}{C}z\,d\hat{t}\right)^2\,, \end{aligned} \tag{4.123}$$

where the constant C is explicitly given by

$$C = \left[\frac{(L^2-\Delta^2)^2}{L^4} + \frac{(j^2-\Delta^2)^2}{L^4} + 14\,\frac{(L^2-\Delta^2)(j^2-\Delta^2)}{L^4} + 24(g_I\mathsf{P}^I)^2 + \frac{3\mathsf{P}^2}{L^2}\right]^{1/2}\,. \tag{4.124}$$

Notice that in the extremal limit it is manifest that the entropy is a function of the charges J and p^I by solving (4.114) (for $T=0$) in terms of S.

4.3.3 Case $\Delta = 0$

Solving the equations of motion with the Carter-Plebański-like ansatz (4.89) and the assumption $\Delta = 0$ leads to the relations

$$\begin{aligned} &a_0 = b_0 - \frac{\mathsf{P}^2}{4}\,, \qquad a_2 = -b_2\,, \qquad a_3 = 0\,, \qquad a_4 = b_4 = -4g^2\,, \\ &b_1 = b_3 = 0\,, \qquad \tilde{\Delta} = \frac{(g_I\mathsf{P}^I)g_0 + g^2\mathsf{P}^0}{2g_0 g^2 a_1}g_I\mathsf{P}^I\,, \qquad c^\alpha = \frac{(g_I\mathsf{P}^I)g_\alpha - g^2\mathsf{P}^\alpha}{2g_0 g^2 a_1}g_I\mathsf{P}^I\,. \end{aligned} \tag{4.125}$$

Notice that in this case a_1 is not fixed by any condition and remains, thus, a free parameter. Moreover, the equations of motion yield an additional condition on the charges,

$$(g_I \mathsf{P}^I)^2 = g^2 \mathsf{P}^2 \,. \tag{4.126}$$

This implies that the charges are proportional to the gauge couplings[11]. Nevertheless, notice that the solution is only supersymmetric if the free parameter a_1 is set to zero and the relations (4.98) hold. If $a_1 \neq 0$, the solution generalizes the Kerr-Newman-AdS black hole with n magnetic charges and constant scalars. In order to show this, one has to take b_0 and b_2 in the form (4.99) and identify $a_1 = -2m$, where m and j are the mass and angular momentum of the Kerr-Newman-AdS solution.

The mass, angular momentum and magnetic charges may be computed as in the case $\Delta \neq 0$, which leads to the same expressions. The Christodoulou-Ruffini formula (4.113) is still valid, but with a simplification due to (4.126),

$$M^2 = \frac{S}{4\pi G} + \frac{\pi J^2}{SG} + \frac{\pi}{4SG^3}\frac{(\mathsf{p}^2)^2}{16} - \left(\frac{L^2}{G^2} + \frac{S}{\pi G}\right)\frac{\mathsf{p}^2}{8L^2} + \frac{J^2}{L^2} + \frac{S^2}{2\pi^2 L^2} + \frac{S^3 G}{4\pi^3 L^4} \,. \tag{4.127}$$

This relation reduces correctly to equation (43) of [137] in the KNAdS case if we identify $\mathsf{p}^2 = -4Q^2$.

4.4 SUSY rotating black holes in the t^3 model

As we already mentioned, cubic models are of special interest due to their close correlation with higher-dimensional theories. In this section we shall focus on the simplest representative of this class, the t^3 model, characterized by the presence of only one complex scalar and defined by the prepotential[12]

$$F = -\frac{(X^1)^3}{X^0} \qquad \Longrightarrow \qquad F_I = X^0\left(\tau^3, -3\tau^2\right)^t \,, \tag{4.128}$$

where we adopted the homogeneous coordinate $\tau := X^1/X^0$. The scalar potential reads

$$V = -\frac{4g_1^2}{3\,\mathrm{Im}\,\tau} \,, \tag{4.129}$$

which has no critical point, so the theory does not admit AdS_4 vacua with constant moduli. Still, we shall be able to construct a non-trivial family of black hole solutions, which of course do not asymptote to AdS_4.

4.4.1 A near-horizon solution

Before starting, we notice that when looking for solutions of the $\overline{\mathbb{C}\mathrm{P}}^n$ model, it proved useful to work with a factorized ansatz for the real and imaginary components of $\bar{X}^I/b$. If a similar decomposition is performed in the case at hand, the equations of motion do not factorize unless we assume that the real or imaginary part of $\bar{X}^0/b$ vanishes. We will only explore here the latter possibility, as in the former we just found trivial solutions. Since in homogeneous coordinates

[11] To see this, choose in $(n+1)$-dimensional Minkowski space with metric η_{IJ} a basis in which the only non-vanishing component of g_I is g_0 (note that g_I is timelike). Then, (4.126) boils down to $\mathsf{P}^\alpha = 0$.

[12] See the last example in section 3.2.1 for additional properties. In this section we shall call the complex scalar τ instead of z in order to distinguish it from the spacetime coordinate.

X^0 is purely real, one can see that this is equivalent to setting $\bar{b}=b$. Thus, we shall adopt the ansatz

$$\frac{\bar{X}^0}{b}=\frac{\eta^0(w,\bar{w})}{\gamma(z)}\,,\qquad \frac{\bar{X}^1}{b}=\frac{f^1(z)+\eta^1(w,\bar{w})}{\gamma(z)}\,,\qquad e^{2\Phi}=h(z)\,\ell(w,\bar{w}) \tag{4.130}$$

in the system of BPS equations (4.2)-(4.6). From (4.2) and (4.5) we get

$$\partial_z\ln h=8i\frac{g_1f^1}{\gamma}\,,\qquad \frac{\partial\bar{\partial}\ln\ell}{\ell}=-\frac{1}{4}\partial_z^2h-\frac{32}{3}\frac{h}{\gamma^2}(g_1f^1)^2\,. \tag{4.131}$$

Using the first equation, we find that the second is separable and boils down to

$$\frac{\partial\bar{\partial}\ln\ell}{\ell}=\frac{C_1}{4}\,,\qquad \partial_z^2h-\frac{2}{3}\frac{(\partial_zh)^2}{h}=-C_1\,, \tag{4.132}$$

for some constant C_1. (4.132) determines the dependence on w, $\bar{w}$ and z of the 3-dimensional base space. For $C_1\neq0$[13] the solution for h reads[14]

$$h(z)=\frac{3}{2}C_1\left(z+\frac{c}{a}\right)^2\,, \tag{4.133}$$

which implies

$$\frac{f^1}{\gamma}=-\frac{i}{4g_1\left(z+\frac{c}{a}\right)}\,. \tag{4.134}$$

The first of (4.132) is just Liouville's equation, and thus the explicit form of $\ell(w,\bar{w})$ depends on the choice of a meromorphic function. In order to make further progress, from now on we shall consider a particular case that has been proven successful for our purpose, i.e.

$$\ell(w,\bar{w})=\frac{2}{C_1\sinh^2\left(\frac{w+\bar{w}}{2}\right)}\,. \tag{4.135}$$

Then, the Bianchi identities (4.3) are automatically solved, so that Maxwell's equations (4.4) represent the last obstacle. Setting, like in the $\overline{\mathbb{CP}}^n$ case, $h(z)/\gamma(z)^2$ to a constant, the latter assume a simple form. The value of this constant is totally arbitrary, but with a redefinition of a, and thus of c in order to keep c/a unchanged, we can always bring it to $\frac{3C_1}{2a^2}$, in which case the Maxwell equations become

$$\partial\bar{\partial}\left(\frac{1}{{\eta^0}^2}-48\frac{g_1^2}{a^2}R^2\right)=0\,,\qquad 2\partial\bar{\partial}R-\frac{R}{\sinh^2\left(\frac{w+\bar{w}}{2}\right)}=0\,, \tag{4.136}$$

where

$$R(w,\bar{w}):=\frac{g_I\eta^I}{g_1\eta^0}\,. \tag{4.137}$$

The second equation of (4.136) can be readily solved,

$$R(x)=\Xi_1\coth x+\Xi_2(x\coth x-1)\,, \tag{4.138}$$

where $\Xi_{1,2}$ are integration constants and $x:=(w+\bar{w})/2$. The first of (4.136) implies

$$\frac{1}{{\eta^0}^2}=48\frac{g_1^2}{a^2}R^2+\mathrm{Re}\,F(w)\,, \tag{4.139}$$

[13] The case $C_1=0$ belongs to a qualitatively different family of solutions to (4.132), which however does not seem to be well-suited for solving the remaining differential equations of the system.

[14] a and c are integration constants. Although h and f^1/γ depend only on the ratio c/a, we prefer to keep them both, for reasons that become clear further below.

for some arbitrary function $F(w)$ that in a first step we shall simply set to 0. The equation (4.6) for the shift vector σ boils down to

$$\partial_z\sigma_w = -\frac{3i\partial R}{4g_1^2\left(z+\frac{c}{a}\right)^2}\,, \qquad \partial_z\sigma_{\bar w} = \frac{3i\bar\partial R}{4g_1^2\left(z+\frac{c}{a}\right)^2}\,, \qquad \partial\sigma_{\bar w} - \bar\partial\sigma_w = -\frac{3i\partial\bar\partial R}{2g_1^2\left(z+\frac{c}{a}\right)}\,,$$

which is solved by

$$\sigma = \frac{3i}{4g_1^2\left(z+\frac{c}{a}\right)}\left(\partial R\, dw - \bar\partial R\, d\bar w\right). \tag{4.140}$$

Defining $y := (w-\bar w)/(2i)$, the metric (4.1) becomes

$$ds^2 = -\frac{8g_1^2}{\sqrt{3}\,R}\left[\left(z+\frac{c}{a}\right)dt - \frac{3}{4g_1^2}\,\partial_x R\, dy\right]^2 + \frac{\sqrt{3}\,R}{2g_1^2}\left[\frac{dz^2}{\left(z+\frac{c}{a}\right)^2} + \frac{3(dx^2+dy^2)}{\sinh^2 x}\right], \tag{4.141}$$

while the scalar field is given by

$$\tau = -\frac{g_0}{g_1} + R(x) + i\sqrt{3}R(x) = -\frac{g_0}{g_1} + 2e^{i\pi/3}R(x)\,. \tag{4.142}$$

For $\Xi_2 = 0$ one can readily identify this solution as belonging to the class of half-supersymmetric near-horizon backgrounds presented in subsection 4.1.2. Performing the change of coordinates

$$e^{-\xi} = \sqrt{K}\coth x\,, \qquad r = z + \frac{c}{a}\,, \qquad \phi = -\sqrt{3}\,y\,, \qquad T = \frac{t}{2\sqrt{K}}\,, \tag{4.143}$$

where $\sqrt{K} = \dfrac{\sqrt{3}\,\Xi_1}{8g_1^2}$, the metric is brought to the form (4.22) with $Y = -1/\sqrt{3}$, namely

$$ds^2 = 4e^{-\xi}\left(-r^2dT^2 + \frac{dr^2}{r^2}\right) + 4\left(e^{-\xi} - Ke^{\xi}\right)(d\phi + r\,dT)^2 + \frac{12e^{-2\xi}d\xi^2}{e^{-\xi} - Ke^{\xi}}\,. \tag{4.144}$$

In the same way, one can check that the scalar (4.142) satisfies the flow equation (4.24).

4.4.2 Black hole extension

We shall now construct a black hole whose near-horizon geometry is given by the solution found in the previous subsection. This is achieved with a slight generalization of the ansatz (4.130). We maintain the factorization form of $e^{2\Phi}$ and $\mathrm{Im}(\bar X^I/b)$, but leave $\mathrm{Re}(\bar X^I/b)$ as arbitrary functions of the three spatial coordinates.

The first steps of subsection 4.4.1 that determine the functions $h(z)$, $\ell(w,\bar w)$ and $\mathrm{Im}(\bar X^I/b)$ remain identical. The difference appears in the first of Maxwell's equations, which now read

$$-\frac{r^2}{\sinh^2\left(\frac{w+\bar w}{2}\right)}\,\partial_r^2\left[\frac{1}{r^2\,\mathrm{Re}^2(\bar X^0/b)}\right] - \frac{4}{3}\,\partial\bar\partial\left[\frac{1}{r^2\,\mathrm{Re}^2(\bar X^0/b)}\right] + 64g_1^2\,\partial\bar\partial\left(R^2\right) = 0\,, \tag{4.145a}$$

$$2\partial\bar\partial R - \frac{R}{\sinh^2\left(\frac{w+\bar w}{2}\right)} = 0\,, \tag{4.145b}$$

where $r = z + c/a$. A simple solution to (4.145a) is

$$\mathrm{Re}\left(\bar X^0/b\right) = \frac{1}{4\sqrt{3}g_1 r\sqrt{\alpha r + \beta + R(w,\bar w)^2}}\,, \tag{4.146}$$

while (4.145b) is solved by (4.138). Here, α and β denote integration constants. Then, the scalar, metric and gauge potentials read respectively

$$\tau = -\frac{g_0}{g_1} + R + i\sqrt{3}\sqrt{\alpha r + \beta + R^2}\,, \tag{4.147}$$

$$\begin{aligned} ds^2 = &-\frac{8g_1^2}{\sqrt{3}\sqrt{\alpha r + \beta + R^2}}\left[r\,dt + \frac{3}{4g_1^2}\,\partial_x R\,dy\right]^2 + \\ &+\frac{\sqrt{3}}{2g_1^2}\sqrt{\alpha r + \beta + R^2}\left[\frac{dr^2}{r^2} + \frac{3(dx^2+dy^2)}{\sinh^2 x}\right]\,, \end{aligned} \tag{4.148}$$

$$\begin{aligned} A^0 &= -\frac{2g_1}{3(\alpha r + \beta + R^2)}\left(r\,dt + \frac{3}{4g_1^2}\,\partial_x R\,dy\right)\,, \\ A^1 &= -\frac{2g_1}{3(\alpha r + \beta + R^2)}\left(R - \frac{g_0}{g_1}\right)\left(r\,dt + \frac{3}{4g_1^2}\,\partial_x R\,dy\right) - \frac{\coth x}{2g_1}\,dy\,. \end{aligned} \tag{4.149}$$

Now the scalar depends on the radial coordinate as well and we recover the near-horizon geometry discussed above by rescaling $r \mapsto \epsilon\, r$, $t \mapsto t/\epsilon$ and taking the limit $\epsilon \to 0$.

As we already mentioned, the asymptotic limit of this solution cannot be AdS_4 since the scalar potential has no critical points. For large values of r, the metric behaves as

$$ds^2 = d\rho^2 + \frac{3}{16}\rho^2\left(-\frac{g_1^8}{108\alpha^2}\rho^4\,dt^2 + \frac{8g_1^2\Xi_1}{3\alpha}\sinh^2\theta\,dtdy + \sinh^2\theta\,dy^2 + d\theta^2\right)\,, \tag{4.150}$$

where we defined ρ and θ by $r := \frac{g_1^4\rho^4}{192\alpha}$, $\coth x := \cosh\theta$, and chose $\Xi_2 = 0$.

Chapter 5

Sol-invariant black holes in $\mathcal{N}=2$, $D=5$ gauged supergravity

Black holes in four dimensions are strongly constrained by a large number of uniqueness and no-hair theorems, among which we find Hawking's topology theorem [138, 139], stating that the event horizon of 4-dimensional asymptotically flat stationary black holes obeying the dominant energy condition is topologically a 2-sphere. Such a restrictive uniqueness theorem does not hold in higher dimensions, where we encounter the most famous counterexample, the 5-dimensional black ring of Emparan and Reall [140] with horizon topology $S^2 \times S^1$, and bound states like black saturns [141], di-rings [142] and bicycling rings [143, 144]. An alternative way to evade Hawking's theorem is to add a negative cosmological constant Λ which, besides violating the dominant energy condition, allows for solutions not asymptotically flat, generically approaching AdS at infinity. Examples of this kind were already mentioned at the beginning of chapter 4.

What we shall do in this chapter is to mix both the possibilities described above, i.e. consider the case $D = 5$ and include a negative cosmological constant. More generally, our model will be the 5-dimensional $\mathcal{N} = 2$ gauged supergravity containing vectors and a set of scalar fields with a potential that admits AdS_5 vacua. A class of uncharged black holes in Einstein-Λ gravity was obtained by Birmingham in [145] for an arbitrary number of dimensions D. Their horizons are $(D-2)$-dimensional Einstein manifolds of positive, zero or negative curvature and in our 3-dimensional case these Einstein spaces have necessarily constant curvature, i.e. are homogeneous and isotropic. One can now try to relax these conditions by dropping the isotropy assumption and getting an horizon which is a homogeneous manifold and, for this reason, belongs to one of the nine Bianchi cosmologies, which are in correspondence with the eight Thurston model geometries (cf. section 5.1 for details). For two of these cases, namely Nil and Sol, the corresponding black holes in 5-dimensional gravity with negative cosmological constant were constructed in [146] for the first time.

Despite making the equations of motion more involved, the addition of charge to a theory can present many advantages. Charged black holes generically have an extremal limit and a subclass of these zero-temperature solutions might preserve some fraction of supersymmetry, which, on the one hand, can help the construction of such solutions and, on the other, can be instrumental in holographic computations of the number of microstates. Moreover, extremal black holes are usually governed by an attractor mechanism, which guides them to a charge-dependent configuration insensitive to the asymptotic values of the moduli; this phenomenon can be employed to search for new solutions. Some attempts have already been made to construct charged homogeneous black holes and probably the most representative are the asymptotically AdS_5 and the Nil and $\widetilde{SL}(2,\mathbb{R})$ near-horizon black holes constructed in [147] by means of the classification of [120]. Other known solutions, but not concerning gauged supergravity, are [148, 149] (no Chern-Simons term), [150] (dilaton-like coupling) and [151, 152] (massive vector field).

The first section of this chapter is devoted to the introduction of homogeneous spaces, their classification and some important properties. In 5.2 we write down the equations of motion for electric and magnetic ansätze, which are explicitly solved in the following section for the case of pure gauged supergravity with magnetic U(1) field strength and Sol horizon. The thermodynamics of the resulting solution is then discussed. Section 5.4 is dedicated to the proof of a no-go theorem stating the non-existence of supersymmetric, static, electrically or magnetically charged solutions with spatial cross-sections modelled on solvegeometry. In the last section we apply the attractor mechanism to construct extremal static non-BPS black holes with solvegeometry horizon along with the related effective potential, proving also that such attractors do not exist for purely electric field strengths. We refer to section 3.5 for the Lagrangian and the equations of motion of the theory considered and to subsection 3.2.2 for the definition and properties of the very special (real) moduli space involved.

5.1 Homogeneous manifolds

Let M be a (pseudo)-Riemannian manifold with isometry group G. M is said to be *homogeneous* if G acts transitively on M, i.e. if $\forall\, p, q \in M$ there exists an isometry $\phi \in G$ such that $\phi(p) = q$. The action of G on M is called simply transitive if the element ϕ is unique or, equivalently, if $\dim M = \dim G$. In this case, M itself is said to be *simply transitive*.

Let us restrict our discussion to a simply transitive manifold. Since $\dim M = \dim G$, the Killing vectors ξ_A ($A = 1, \dots, \dim M$) form a basis of the tangent space. However, it is more convenient [153] to choose a G-invariant basis X_A, i.e. a basis such that

$$\mathcal{L}_{\xi_B} X_A = [\xi_B, X_A] = 0 \qquad \forall\, A, B\,, \tag{5.1}$$

with $\mathcal{L}_{\xi_B} X_A$ the Lie derivative of the vector field X_A along ξ_B. The dual basis θ^A of a G-invariant basis X_A is also G-invariant, $\mathcal{L}_{\xi_B}\theta^A = 0$, and satisfies

$$d\theta^A = \frac{1}{2}\, C^A{}_{BC}\theta^B \wedge \theta^C\,, \tag{5.2}$$

with $C^A{}_{BC}$ the structure constants of the Lie algebra of G. Furthermore, a simply transitive homogeneous manifold can be equipped with a metric

$$ds^2 = g_{AB}\theta^A\theta^B\,, \tag{5.3}$$

where the components g_{AB} are constant on M.

Bianchi showed that in total there are nine 3-dimensional Lie algebras, the so-called *nine Bianchi cosmologies*, labelled from type I to type IX. The name “cosmologies” comes from the fact that these manifolds are used as spatial sections in many spatially homogeneous, but anisotropic cosmological models. The Bianchi cosmologies are divided into two classes, A and B, according to the way the structure constants $C^A{}_{BC}$ can be expanded[1]:

$$C^A{}_{BC} = m^{AD}\varepsilon_{BCD} - 2\,\delta^A_{[B}\, a_{C]}\,, \tag{5.4}$$

where m^{AB} is a symmetric matrix and a_A a constant vector. Class A includes models with $a_A = 0$, while the others fall into class B. In particular, class A spacetimes satisfy

$$\sum_A C^A{}_{AB} = 0\,. \tag{5.5}$$

[1]We refer to table 6.2 of [153] for details.

An important result in geometric topology is the *Thurston conjecture* [154], which states that every 3-dimensional closed and orientable manifold has a geometric structure modelled on one of the eight *model geometries*

$$S^3\,,\quad E^3\,,\quad H^3\,,\quad S^2\times\mathbb{R}\,,\quad H^2\times\mathbb{R}\,,\quad \mathrm{Nil}\,,\quad \mathrm{Sol}\,,\quad \widetilde{\mathrm{SL}}(2,\mathbb{R})\,, \tag{5.6}$$

where $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is the universal covering of $\mathrm{SL}(2,\mathbb{R})$. In [155] it was shown that there exists a correspondence, not necessarily one-to-one, between the nine Bianchi cosmologies and the eight Thurston model geometries, which is summarized in table 5.1[2].

Bianchi	Thurston
I, VII_0	E^3
II	Nil
VI_{-1}	Sol
VIII	$\widetilde{\mathrm{SL}}(2,\mathbb{R})$
IX	S^3

Bianchi	Thurston
III	$H^2\times\mathbb{R}$
V, $\mathrm{VII}_{h\neq 0}$	H^3

Table 5.1: *Class A (left) and B (right) spacetimes and corresponding Thurston geometries.*

In the specific case of solvegeometry/VI_{-1}, the one we shall focus on, the non-vanishing structure constants of the isometry group G, the G-invariant 1-forms θ^A and the metric written in terms of them are[3]

$$C^1{}_{13}=-C^1{}_{31}=1\,,\qquad C^2{}_{23}=-C^2{}_{32}=-1\,, \tag{5.7a}$$

$$\theta^1=e^z dx\,,\qquad \theta^2=e^{-z}dy\,,\qquad \theta^3=-dz\,, \tag{5.7b}$$

$$ds^2=e^{2z}dx^2+e^{-2z}dy^2+dz^2\,. \tag{5.7c}$$

5.2 The electric and magnetic ansätze

In order to solve the equations of motion (3.70a)-(3.70c) we use an ansatz inspired by [146], with homogeneous sections $\Sigma_{t,r}$ of constant t and r. Without loss of generality, we take the line element to be

$$ds^2=-f(r)\,dt^2+\frac{dr^2}{f(r)}+\sum_{A=1}^{3}e^{2T_A(r)}(\theta^A)^2\,, \tag{5.8}$$

where the induced metric on $\Sigma_{t,r}$ is written in terms of G-invariant 1-forms θ^A, which, we remind, satisfy

$$d\theta^A=\frac{1}{2}\,C^A{}_{BC}\theta^B\wedge\theta^C\,. \tag{5.9}$$

Henceforth we shall restrict our discussion to solvegeometry (cf. (5.7a)-(5.7c)). The scalar fields are assumed to depend on the radial coordinate only,

$$\phi^i=\phi^i(r)\,. \tag{5.10}$$

For what concerns the vector fields, we shall consider two different configurations, one purely electric and one purely magnetic; dyonic solutions will not be taken into account.

[2]The Bianchi types IV and $\mathrm{VI}_{h\neq -1}$ are not contained in this correspondence. Moreover, the Thurston geometry $\mathrm{S}^2\times\mathbb{R}$ is missing since it corresponds to the Kantowski-Sachs model, in which G does not act simply transitively or does not possess a subgroup with simply transitive action.

[3]A list of these quantities for the other Bianchi cosmologies can be found in [153].

5.2.1 Electric ansatz

For a purely electric ansatz the vector fields are given by

$$A^I = A^I_t(r)\, dt\,, \tag{5.11}$$

and the Maxwell equations (3.70b) imply

$$F^I_{rt} = \partial_r A^I_t = e^{-\sum_A T_A} G^{IJ} q_J\,, \tag{5.12}$$

where the constants q_I represent essentially the electric charge densities.

Using (5.12) and the Bianchi class A condition (5.5), Einstein's equations (3.70a) boil down to

$$\begin{gathered}
\frac{f''}{2} + \frac{f'}{2}\Big(\sum_A T'_A\Big) = \frac{2}{3}\, e^{-2\sum_A T_A} G^{IJ} q_I q_J - \frac{2}{3} V\,, \\
\sum_A T''_A + \sum_A (T'_A)^2 = -\mathcal{G}_{ij}\phi^{i\prime}\phi^{j\prime}\,, \qquad \sum_B C^B{}_{AB} T'_B = 0\,, \\
-f'\, T'_A - f\, T''_A - f\, T'_A\Big(\sum_B T'_B\Big) + \mathcal{J}_A = \frac{1}{3}\, e^{-2\sum_B T_B} G^{IJ} q_I q_J + \frac{2}{3} V\,,
\end{gathered} \tag{5.13}$$

where we defined

$$\mathcal{J}_A := \sum_{B,C}\left[-\frac{1}{2}\, D^B{}_{AC}\big(D^C{}_{AB} + D^B{}_{AC}\big) + \frac{1}{4}\big(D^A{}_{BC}\big)^2\right]\,, \tag{5.14}$$

with

$$D^A{}_{BC} := e^{T_A - T_B - T_C} C^A{}_{BC}\,. \tag{5.15}$$

The third equation in (5.13) is a constraint, which is trivially satisfied for all the class A Bianchi cosmologies except for solvegeometry; in this case it reduces to

$$T'_1 = T'_2\,. \tag{5.16}$$

Finally, using (5.12), the equations (3.70c) for the scalars become

$$\begin{aligned}
&f\Big(\sum_A T'_A\Big)\mathcal{G}_{ij}\phi^{j\prime} + f\,\frac{d\mathcal{G}_{ij}}{dr}\,\phi^{j\prime} + f'\mathcal{G}_{ij}\phi^{j\prime} + f\,\mathcal{G}_{ij}\phi^{j\prime\prime} - \frac{1}{2}\, f\,\partial_i\mathcal{G}_{jk}\phi^{j\prime}\phi^{k\prime} \\
&\quad - \frac{1}{2}\, e^{-2\sum_A T_A}\,\partial_i G^{IJ} q_I q_J - \partial_i V = 0\,.
\end{aligned} \tag{5.17}$$

5.2.2 Magnetic ansatz

In the magnetically charged case we take for the field strength

$$F^I = p^I \theta^1 \wedge \theta^2\,, \tag{5.18}$$

where the p^I are magnetic charge densities. Note that F^I is closed due to the Bianchi class A condition (5.5), so locally there exists a gauge potential A^I such that $F^I = dA^I$. In the following, we shall consider the case of solvegeometry, for which

$$F^I = p^I dx \wedge dy\,, \qquad A^I = p^I x\, dy\,. \tag{5.19}$$

Using (5.7b) the line element (5.8) becomes

$$ds^2 = -f(r)\, dt^2 + \frac{dr^2}{f(r)} + e^{2(T_1(r)+z)} dx^2 + e^{2(T_2(r)-z)} dy^2 + e^{2T_3(r)} dz^2\,. \tag{5.20}$$

Maxwell's equations (3.70b) are automatically satisfied by (5.19) and (5.20), while the non-trivial Einstein equations (3.70a) read

$$\begin{aligned} \frac{f''}{2}+\frac{f'}{2}(2T_1'+T_3') &= \frac{1}{3}\,e^{-4T_1}G_{IJ}p^Ip^J-\frac{2}{3}\,V\,,\\ 2T_1''+T_3''+2(T_1')^2+(T_3')^2 &= -\mathcal{G}_{ij}\phi^{i\prime}\phi^{j\prime}\,,\\ -f'\,T_1'-f\big(T_1''+T_1'(2T_1'+T_3')\big) &= \frac{2}{3}\,e^{-4T_1}G_{IJ}p^Ip^J+\frac{2}{3}\,V\,,\\ -f'\,T_3'-f\big(T_3''+T_3'(2T_1'+T_3')\big)-2e^{-2T_3} &= -\frac{1}{3}\,e^{-4T_1}G_{IJ}p^Ip^J+\frac{2}{3}\,V\,, \end{aligned} \tag{5.21}$$

where we used the condition $T_1'=T_2'$ and the freedom to rescale y in order to set $T_1=T_2$. The scalar field equations (3.70c) become

$$\begin{aligned} &f(2T_1'+T_3')\mathcal{G}_{ij}\phi^{j\prime}+f\,\frac{d\mathcal{G}_{ij}}{dr}\,\phi^{j\prime}+f'\mathcal{G}_{ij}\phi^{j\prime}+f\,\mathcal{G}_{ij}\phi^{j\prime\prime}-\frac{1}{2}\,f\,\partial_i\mathcal{G}_{jk}\phi^{j\prime}\phi^{k\prime}\\ &\quad-\frac{1}{2}\,e^{-4T_1}\,\partial_iG_{IJ}p^Ip^J-\partial_iV=0\,. \end{aligned} \tag{5.22}$$

5.3 Magnetic black hole in pure gauged supergravity

In order to study the above equations in a simplified setting, we restrict our attention to pure gauged supergravity, i.e. the theory (3.67) without vector multiplets ($n=0$). For a purely electric or magnetic configuration, the Chern-Simons term can be consistently truncated and (3.67) boils down to

$$e^{-1}\mathscr{L}=\frac{R}{2}-\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}-\Lambda\,, \tag{5.23}$$

where $\Lambda=-6g^2<0$, $F_{\mu\nu}=F^1_{\mu\nu}$ and we fixed C_{111} in (3.25) and V_1 in (3.71) such that $G_{11}=1$ and $V=-6g^2$. The field strength (5.19) becomes simply $F_{xy}=p$, while the Einstein equations (5.21) reduce to

$$\begin{aligned} \frac{f''}{2}+\frac{f'}{2}(2T_1'+T_3') &= \frac{1}{3}\,e^{-4T_1}p^2-\frac{2}{3}\,\Lambda\,,\\ 2T_1''+T_3''+2(T_1')^2+(T_3')^2 &= 0\,,\\ -V'T_1'-V\big(T_1''+T_1'(2T_1'+T_3')\big) &= \frac{2}{3}\,e^{-4T_1}p^2+\frac{2}{3}\,\Lambda\,,\\ -V'T_3'-V\big(T_3''+T_3'(2T_1'+T_3')\big)-2e^{-2T_3} &= -\frac{1}{3}\,e^{-4T_1}p^2+\frac{2}{3}\,\Lambda\,. \end{aligned} \tag{5.24}$$

One can easily check that in the uncharged case $p=0$ the above equations are satisfied by the solvegeometry solution constructed in [146].

System (5.24) can be solved by taking T_1 to be constant[4]. With this assumption, a particular black hole solution is given by

$$ds^2=-f(r)\,dt^2+\frac{dr^2}{f(r)}+\sqrt{\frac{p^2}{-\Lambda}}\,\big(e^{2z}dx^2+e^{-2z}dy^2\big)+\frac{r^2}{A}\,dz^2\,, \tag{5.25a}$$

$$F=p\,dx\wedge dy\,, \tag{5.25b}$$

with

$$f(r)=-\frac{\Lambda}{2}\,r^2-2A\ln\left(\frac{r}{B}\right)\,, \tag{5.26}$$

[4] Note that this is not the case for the solution of [146].

where A and B are two positive integration constants. It is worth noting that this solution is singular in the limit $p \to 0$, and it is thus disconnected from the one in [146]. The metric (5.25a) and field strength (5.25b) are invariant under the scale transformations

$$t \mapsto t/\nu\,, \qquad r \mapsto \nu r\,, \qquad z \mapsto z + \ln \alpha\,, \qquad x \mapsto \lambda x\,, \qquad y \mapsto \pm\lambda\alpha^2 y\,, \tag{5.27}$$

accompanied by

$$p \mapsto \pm\frac{p}{\lambda^2\alpha^2}\,, \qquad A \mapsto \nu^2 A\,, \qquad B \mapsto \nu B\,. \tag{5.28}$$

This can be used to set, e.g., $p = B = 1/g$ without loss of generality. B and the magnetic charge density p are, thus, not true parameters of the solution, which is specified completely by choosing A. Notice that the scaling symmetries with $\nu = 1$, $\lambda = 1/\alpha$ belong to the Lie group Sol. If the horizon is compactified (cf. [146] for details on the compactification procedure), the transformations in (5.27) involving α and λ are broken down to a discrete subgroup ($\alpha = \lambda^{-1} = e^{n\,a}$, where a is the constant appearing in (II.22) of [146] and $n \in \mathbb{Z}$), which no more allows to scale p to any value. In this case, p becomes actually a genuine parameter of the black hole.

The background (5.25a) exhibits anisotropic scaling. If the horizon is compactified, the geometry approaches asymptotically for $r \to \infty$ a torus bundle over AdS_3. In $r = 0$ there is a curvature singularity, since the Kretschmann scalar behaves as $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \sim (\ln r)/r^4$ for $r \to 0$. Horizons are determined by the roots of the function $f(r)$, which diverges both for $r \to 0$ and $r \to +\infty$ and has a unique minimum in

$$r = r_{\mathrm{min}} = \sqrt{\frac{2A}{-\Lambda}}\,. \tag{5.29}$$

If $f(r_{\mathrm{min}}) > 0$ the solution represents a naked singularity. For $f(r_{\mathrm{min}}) = 0$, i.e. $A = 3e$, we have an extremal black hole, while for $f(r_{\mathrm{min}}) < 0$ ($A > 3e$) there are an inner and an outer horizon and the solution is non-extremal.

Requiring the absence of conical singularities in the Euclidean section gives the Hawking temperature

$$T = \frac{-\Lambda r_{\mathrm{h}}^2 - 2A}{4\pi r_{\mathrm{h}}}\,, \tag{5.30}$$

where r_{h} denotes the radial coordinate of the horizon. The entropy density can be computed by means of the Bekenstein-Hawking formula and is given by

$$s = \frac{S}{V_{\mathrm{solve}}} = \frac{(\ln(g\,r_{\mathrm{h}}))^{1/2}}{12g^3}\,, \tag{5.31}$$

where we set the Newton constant $G = 1$ and V_{solve} is the volume of the compactified manifold modelled on solvegeometry.

The standard Komar integral for the mass goes like Λr^2 for large r and thus diverges for $r \to +\infty$ due to the presence of the vacuum energy, as was to be expected. Moreover, there is no obvious background to subtract and the conditions for the applicability of the Ashtekar-Magnon-Das formalism [135, 136] are not satisfied. Despite these difficulties, we can associate a mass to the black hole (5.25a) by simply integrating the first law. Since p is not a dynamical parameter of the solution, we do not expect a term containing the variation of the magnetic charge in the first law. Hence, the mass density m satisfies

$$dm = T\,ds\,, \tag{5.32}$$

which gives (up to an integration constant, that can be fixed by requiring, e.g., the extremal solution to have zero energy)

$$m = \frac{r_{\mathrm{h}}}{16\pi g(\ln(g\,r_{\mathrm{h}}))^{1/2}}\,. \tag{5.33}$$

5.4 Existence of static, Sol-invariant BPS solutions

Supersymmetric actions are invariant under local SUSY transformations generated by arbitrary spinor functions $\epsilon(x)$, but, as it happens in the vast majority of theories—classical as well—the solutions of their equations of motion usually break most of the symmetries, if not all. Symmetries preserved by these solutions are generally called *residual symmetries*. Residual symmetries are global symmetries and their generators, the *Killing spinors*[5], can be expressed in terms of a certain number of constant parameters. A supersymmetric solution is called n-BPS, where n is the ratio between the number of parameters characterizing the Killing spinors and the total number of real SUSY generators of the theory.

How do we construct Killing spinors? In a very sketchy fashion, the SUSY transformations of bosonic (B) and fermionic (F) fields are

$$\delta_\epsilon B \sim \bar{\epsilon}\, F\,, \qquad \delta_\epsilon F \sim (\partial\epsilon +)\, B\,\epsilon\,. \tag{5.34}$$

Imposing invariance, i.e. $\delta_\epsilon B = 0$ and $\delta_\epsilon F = 0$, we obtain a set of coupled first-order differential equations involving the fields and the spinors ϵ. Since classical solutions are bosonic, we want to restrict ourselves to configurations with vanishing fermions, which immediately implies that the first set of equations is trivially satisfied and the bosonic fields are always invariant. Solving the remaining second set of equations, which take the name of *Killing spinor equations (KSE)*, for a given background yields the desired Killing spinors.

Killing spinor equations are also a very powerful tool to construct new (SUSY) solutions. Imposing the KSEs without specifying any background leads to a set of equations both for the metric and matter fields and for the Killing spinors, and requiring them to allow for non-trivial Killing spinors gives the *BPS equations*, a set of first-order differential equations for the (bosonic) fields of the theory. To be more concrete, the gravitino KSEs involve the derivative of the SUSY parameter and can be schematically written as $\delta\Psi_\mu := \hat{\mathcal{D}}_\mu\epsilon = 0$; its first integrability conditions read

$$\hat{\mathcal{R}}_{\mu\nu}\epsilon := \big[\hat{\mathcal{D}}_\mu, \hat{\mathcal{D}}_\nu\big]\epsilon = 0\,, \tag{5.35}$$

which is a set of algebraic equations that admit a non-trivial solution ϵ iff $\det(\hat{\mathcal{R}}_{\mu\nu}) = 0$. A similar argument can be brought forward for the gaugino KSEs.

The first-order BPS equations are generically much easier to solve than the full second-order Einstein's and scalar equations and, at least in the case in which the Killing vector constructed as a bilinear from the Killing spinor is timelike, the latter are implied by the former once Maxwell's equations are satisfied (cf. [119] and [121] for examples in $D = 4, 5$ respectively).

The supersymmetry variations for the gravitino Ψ_μ and the gauginos λ_i in a bosonic background are given by (see e.g. [156])

$$\delta\Psi_\mu = \left[\mathcal{D}_\mu + \frac{i}{8}\, h_I\big(\Gamma_\mu{}^{\nu\rho} - 4\delta_\mu{}^\nu\Gamma^\rho\big)F^I_{\nu\rho} + \frac{1}{2}\, g_I h^I\, \Gamma_\mu - \frac{3i}{2}\, g_I A^I_\mu\right]\epsilon\,, \tag{5.36}$$

$$\delta\lambda_i = \left[\frac{3}{8}\,\Gamma^{\mu\nu}F^I_{\mu\nu}\partial_i h_I - \frac{i}{2}\,\mathcal{G}_{ij}\Gamma^\mu\partial_\mu\phi^j + \frac{3i}{2}\, g_I\partial_i h^I\right]\epsilon\,, \tag{5.37}$$

where ϵ is the supersymmetry parameter, $h_I = \frac{1}{6}\, C_{IJK}h^Jh^K$ and $\mathcal{D}_\mu$ denotes the Lorentz-covariant derivative[6]. In what follows, we shall specify to solvegeometry with electric or magnetic

[5] The name "Killing spinors" is due to their close relation with Killing vectors, the generators of bosonic symmetries. This correlation is far from being a simple analogy, indeed given any two Killing spinors ϵ_1 and ϵ_2, the bilinear $(\bar{\epsilon}_1\gamma^\mu\epsilon_2)$ is a Killing vector. These Killing vectors are then used to classify supersymmetric solutions.

[6] We recall that $\mathcal{D}_\mu := \partial_\mu + \frac{1}{4}\,\omega_\mu{}^{ab}\Gamma_{ab}$, $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}$ and $\Gamma^{a_1a_2\ldots a_n} := \Gamma^{[a_1}\Gamma^{a_2}\ldots\Gamma^{a_n]}$.

ansatz. For the metric (5.20) the fünfbein can be chosen as

$$e^0_t = \sqrt{f}\,, \qquad e^1_x = e^{T_1+z}\,, \qquad e^2_y = e^{T_2-z}\,, \qquad e^3_z = e^{T_3}\,, \qquad e^4_r = \frac{1}{\sqrt{f}}\,. \tag{5.38}$$

5.4.1 Electric ansatz

In the case of solvegeometry and electric ansatz, the vanishing of the gravitino variations (5.36) leads to

$$\begin{aligned}
&\left[\partial_t + \frac{f'}{4}\,\Gamma_{04} + \frac{i}{2}\,h_I F^I_{rt}\sqrt{f}\,\Gamma_4 + \frac{1}{2}\,g_I h^I\sqrt{f}\,\Gamma_0 - \frac{3i}{2}\,g_I A^I_t\right]\epsilon = 0\,,\\
&\left[\partial_r + \frac{i}{2}\,h_I F^I_{rt}\frac{1}{\sqrt{f}}\,\Gamma_0 + \frac{1}{2}\,g_I h^I\frac{1}{\sqrt{f}}\,\Gamma_4\right]\epsilon = 0\,,\\
&\left[\partial_x + e^{T_1+z}\left(\frac{1}{2}\sqrt{f}\,T_1'\,\Gamma_{14} + \frac{1}{2}\,e^{-T_3}\,\Gamma_{13} - \frac{i}{4}\,h_I F^I_{rt}\,\Gamma_{014} + \frac{1}{2}\,g_I h^I\,\Gamma_1\right)\right]\epsilon = 0\,,\\
&\left[\partial_y + e^{T_2-z}\left(\frac{1}{2}\sqrt{f}\,T_2'\,\Gamma_{24} - \frac{1}{2}\,e^{-T_3}\,\Gamma_{23} - \frac{i}{4}\,h_I F^I_{rt}\,\Gamma_{024} + \frac{1}{2}\,g_I h^I\,\Gamma_2\right)\right]\epsilon = 0\,,\\
&\left[\partial_z + e^{T_3}\left(\frac{1}{2}\sqrt{f}\,T_3'\,\Gamma_{34} - \frac{i}{4}\,h_I F^I_{rt}\,\Gamma_{034} + \frac{1}{2}\,g_I h^I\,\Gamma_3\right)\right]\epsilon = 0\,.
\end{aligned}\tag{5.39}$$

The integrability conditions (5.35) with (μ,ν) equal to (t,x), (t,y) and (t,z) are, respectively,

$$\begin{aligned}
&\left[\frac{1}{2}\,f'\,T_1' - (g_I h^I)^2 + i\sqrt{f}\,T_1' h_I F^I_{rt}\,\Gamma_0 + i(g_I h^I)(h_J F^J_{rt})\,\Gamma_{04}\right]\epsilon = 0\,,\\
&\left[\frac{1}{2}\,f'\,T_2' - (g_I h^I)^2 + i\sqrt{f}\,T_2' h_I F^I_{rt}\,\Gamma_0 + i(g_I h^I)(h_J F^J_{rt})\,\Gamma_{04}\right]\epsilon = 0\,,\\
&\left[\frac{1}{2}\,f'\,T_3' - (g_I h^I)^2 + i\sqrt{f}\,T_3' h_I F^I_{rt}\,\Gamma_0 + i(g_I h^I)(h_J F^J_{rt})\,\Gamma_{04}\right]\epsilon = 0\,,
\end{aligned}\tag{5.40}$$

while for (x,y), (x,z) and (y,z) we have

$$\begin{aligned}
&\Big[f\,T_1'T_2' - (g_I h^I)^2 + \frac{1}{4}(h_I F^I_{rt})^2 - e^{-2T_3} - \frac{i}{2}\sqrt{f}(T_1'+T_2')(h_I F^I_{rt})\,\Gamma_0\\
&\qquad - i(g_I h^I)(h_J F^J_{rt})\,\Gamma_{04}\Big]\epsilon = 0\,,\\
&\Big[f\,T_1'T_3' - (g_I h^I)^2 + \frac{1}{4}(h_I F^I_{rt})^2 + e^{-2T_3} - \frac{i}{2}\sqrt{f}(T_1'+T_3')(h_I F^I_{rt})\,\Gamma_0\\
&\qquad - i(g_I h^I)(h_J F^J_{rt})\,\Gamma_{04} + \sqrt{f}(T_1'-T_3')e^{-T_3}\,\Gamma_{34}\Big]\epsilon = 0\,,\\
&\Big[f\,T_2'T_3' - (g_I h^I)^2 + \frac{1}{4}(h_I F^I_{rt})^2 + e^{-2T_3} - \frac{i}{2}\sqrt{f}(T_2'+T_3')(h_I F^I_{rt})\,\Gamma_0\\
&\qquad - i(g_I h^I)(h_J F^J_{rt})\,\Gamma_{04} - \sqrt{f}(T_2'-T_3')e^{-T_3}\,\Gamma_{34}\Big]\epsilon = 0\,.
\end{aligned}\tag{5.41}$$

The difference of equations (5.40) taken in (all the three possible) pairs leads to

$$\begin{aligned}
&(T_1'-T_2')\left[\frac{1}{2}\,f' + i\sqrt{f}\,h_I F^I_{rt}\,\Gamma_0\right]\epsilon = 0\,,\\
&(T_1'-T_3')\left[\frac{1}{2}\,f' + i\sqrt{f}\,h_I F^I_{rt}\,\Gamma_0\right]\epsilon = 0\,,\\
&(T_2'-T_3')\left[\frac{1}{2}\,f' + i\sqrt{f}\,h_I F^I_{rt}\,\Gamma_0\right]\epsilon = 0\,,
\end{aligned}\tag{5.42}$$

whereas $(x,y)-(x,z)$ and $(x,y)-(y,z)$ read

$$\begin{aligned}&\left[f\,T_1'(T_2'-T_3')-2e^{-2T_3}-\frac{i}{2}\sqrt{f}(T_2'-T_3')(h_I F^I_{rt})\,\Gamma_0-\sqrt{f}(T_1'-T_3')e^{-T_3}\,\Gamma_{34}\right]\epsilon=0\,,\\&\left[f\,T_2'(T_1'-T_3')-2e^{-2T_3}-\frac{i}{2}\sqrt{f}(T_1'-T_3')(h_I F^I_{rt})\,\Gamma_0+\sqrt{f}(T_2'-T_3')e^{-T_3}\,\Gamma_{34}\right]\epsilon=0\,.\end{aligned}\tag{5.43}$$

We can distinguish between two different cases in which (5.42) hold.

- Case A
$$T_1'=T_2'=T_3'\,.\tag{5.44}$$
In this case (5.43) leads directly to the trivial solution $\epsilon=0$.

- Case B
$$\left[\frac{1}{2}\,f'+i\sqrt{f}\,h_I F^I_{rt}\,\Gamma_0\right]\epsilon=0\,.\tag{5.45}$$
Writing this condition schematically as $\mathcal{M}\epsilon=0$, a necessary condition to have non-trivial solutions is $\det\mathcal{M}=0$, and thus
$$\frac{1}{2}\,f'=\pm\sqrt{f}\,h_I F^I_{rt}\,,\tag{5.46}$$
which, once plugged back into (5.45) gives the projection
$$\Gamma_0\epsilon=\pm i\epsilon\,.\tag{5.47}$$
Using (5.47) in (5.43), we get
$$\begin{aligned}&\left[f\,T_1'(T_2'-T_3')-2e^{-2T_3}\pm\frac{1}{2}\sqrt{f}(T_2'-T_3')(h_I F^I_{rt})-\sqrt{f}(T_1'-T_3')e^{-T_3}\,\Gamma_{34}\right]\epsilon=0\,,\\&\left[f\,T_2'(T_1'-T_3')-2e^{-2T_3}\pm\frac{1}{2}\sqrt{f}(T_1'-T_3')(h_I F^I_{rt})+\sqrt{f}(T_2'-T_3')e^{-T_3}\,\Gamma_{34}\right]\epsilon=0\,.\end{aligned}\tag{5.48}$$
To have non-trivial solutions, the determinants of the two coefficient matrices in these linear systems must vanish, leading to $T_1'=T_3'$ and $T_2'=T_3'$, which brings us back to case A.

We can thus state the following

Proposition 1. *There are no static, Sol-invariant solutions to the Killing spinor equations with solvegeometry spatial cross-sections at fixed r and purely electric field strengths.*

5.4.2 Magnetic ansatz

In this case, the Killing spinor equations become

$$\begin{aligned}&\left[\partial_t+\frac{f'}{4}\,\Gamma_{04}+\frac{i}{4}\,h_I p^I\sqrt{f}\,e^{-T_1-T_2}\,\Gamma_{012}+\frac{1}{2}\,g_I h^I\sqrt{f}\,\Gamma_0\right]\epsilon=0\,,\\&\left[\partial_r+\frac{i}{4}\,h_I p^I\frac{1}{\sqrt{f}}\,e^{-T_1-T_2}\,\Gamma_{124}+\frac{1}{2}\,g_I h^I\frac{1}{\sqrt{f}}\,\Gamma_4\right]\epsilon=0\,,\\&\left[\partial_x+e^{T_1+z}\left(\frac{1}{2}\sqrt{f}\,T_1'\,\Gamma_{14}+\frac{1}{2}\,e^{-T_3}\,\Gamma_{13}-\frac{i}{2}\,h_I p^I e^{-T_1-T_2}\,\Gamma_2+\frac{1}{2}\,g_I h^I\,\Gamma_1\right)\right]\epsilon=0\,,\\&\left[\partial_y+e^{T_2-z}\left(\frac{1}{2}\sqrt{f}\,T_2'\,\Gamma_{24}-\frac{1}{2}\,e^{-T_3}\,\Gamma_{23}+\frac{i}{2}\,h_I p^I e^{-T_1-T_2}\,\Gamma_1+\frac{1}{2}\,g_I h^I\,\Gamma_2\right)-\frac{3i}{2}\,g_I p^I x\right]\epsilon=0\,,\\&\left[\partial_z+e^{T_3}\left(\frac{1}{2}\sqrt{f}\,T_3'\,\Gamma_{34}+\frac{i}{4}\,h_I p^I e^{-T_1-T_2}\,\Gamma_{123}+\frac{1}{2}\,g_I h^I\,\Gamma_3\right)\right]\epsilon=0\,.\end{aligned}\tag{5.49}$$

We have thus the following first integrability conditions:

- (t,x)

$$\begin{aligned}&\bigg[\frac{1}{2}\,f'\,T_1'-(g_Ih^I)^2+\frac{i}{2}\,h_Ip^I\sqrt{f}\,T_1'e^{-T_1-T_2}\,\Gamma_{124}+\frac{i}{2}\,h_Ip^I e^{-T_1-T_2-T_3}\,\Gamma_{123}\\&\qquad+i(g_Ih^I)(h_Jp^J)e^{-T_1-T_2}\,\Gamma_{12}\bigg]\epsilon=0\,,\end{aligned}\tag{5.50}$$

- (t,y)

$$\begin{aligned}&\bigg[\frac{1}{2}\,f'\,T_2'-(g_Ih^I)^2+\frac{i}{2}\,h_Ip^I\sqrt{f}\,T_2'e^{-T_1-T_2}\,\Gamma_{124}-\frac{i}{2}\,h_Ip^I e^{-T_1-T_2-T_3}\,\Gamma_{123}\\&\qquad+i(g_Ih^I)(h_Jp^J)e^{-T_1-T_2}\,\Gamma_{12}\bigg]\epsilon=0\,,\end{aligned}\tag{5.51}$$

- (t,z)

$$\bigg[\frac{1}{2}\,f'\,T_3'-(g_Ih^I)^2-i(g_Ih^I)(h_Jp^J)e^{-T_1-T_2}\,\Gamma_{12}-\frac{1}{4}\,(h_Ip^I)^2e^{-2(T_1+T_2)}\bigg]\epsilon=0\,,\tag{5.52}$$

- (x,y)

$$\begin{aligned}&\bigg[f\,T_1'T_2'-(g_Ih^I)^2+(h_Ip^I)^2e^{-2(T_1+T_2)}-e^{-2T_3}\\&\qquad-i\,h_Ip^I\sqrt{f}(T_1'+T_2')e^{-T_1-T_2}\,\Gamma_{124}-3i\,g_Ip^I e^{-T_1-T_2}\,\Gamma_{12}\bigg]\epsilon=0\,,\end{aligned}\tag{5.53}$$

- (x,z)

$$\begin{aligned}&\bigg[f\,T_1'T_3'-(g_Ih^I)^2+e^{-2T_3}+\sqrt{f}(T_1'-T_3')e^{-T_3}\,\Gamma_{34}-i\,h_Ip^I e^{-T_1-T_2-T_3}\,\Gamma_{123}\\&\qquad+\frac{i}{2}\,h_Ip^I\sqrt{f}\,T_1'e^{-T_1-T_2}\,\Gamma_{124}+i(g_Ih^I)(h_Jp^J)e^{-T_1-T_2}\,\Gamma_{12}\bigg]\epsilon=0\,,\end{aligned}\tag{5.54}$$

- (y,z)

$$\begin{aligned}&\bigg[f\,T_2'T_3'-(g_Ih^I)^2+e^{-2T_3}-\sqrt{f}(T_2'-T_3')e^{-T_3}\,\Gamma_{34}+i\,h_Ip^I e^{-T_1-T_2-T_3}\,\Gamma_{123}\\&\qquad+\frac{i}{2}\,h_Ip^I\sqrt{f}\,T_2'e^{-T_1-T_2}\,\Gamma_{124}+i(g_Ih^I)(h_Jp^J)e^{-T_1-T_2}\,\Gamma_{12}\bigg]\epsilon=0\,,\end{aligned}\tag{5.55}$$

- (r,t)

$$\begin{aligned}&\bigg[\frac{1}{2}\,f''-(g_Ih^I)^2-\frac{1}{4}\,(h_Ip^I)^2e^{-2(T_1+T_2)}-\frac{i}{2}\,h_Ip^I\sqrt{f}(T_1'+T_2')e^{-T_1-T_2}\,\Gamma_{124}\\&\qquad-\frac{i}{2}\,\partial_r(h_Ip^I)\sqrt{f}\,e^{-T_1-T_2}\,\Gamma_{124}+\partial_r(g_Ih^I)\sqrt{f}\,\Gamma_4\\&\qquad-i(g_Ih^I)(h_Jp^J)e^{-T_1-T_2}\,\Gamma_{12}\bigg]\epsilon=0\,,\end{aligned}\tag{5.56}$$

- (r,x)

$$\Big[f\,T_1'' + f(T_1')^2 - i\,\partial_r(h_I p^I)\sqrt{f}\,e^{-T_1-T_2}\,\Gamma_{124} + \partial_r(g_I h^I)\sqrt{f}\,\Gamma_4 \\ - \frac{i}{2}\,h_I p^I\sqrt{f}(T_1' - 2T_2')e^{-T_1-T_2}\,\Gamma_{124}\Big]\epsilon = 0\,, \tag{5.57}$$

- (r,y)

$$\Big[f\,T_2'' + f(T_2')^2 - i\,\partial_r(h_I p^I)\sqrt{f}\,e^{-T_1-T_2}\,\Gamma_{124} + \partial_r(g_I h^I)\sqrt{f}\,\Gamma_4 \\ - \frac{i}{2}\,h_I p^I\sqrt{f}(T_2' - 2T_1')e^{-T_1-T_2}\,\Gamma_{124}\Big]\epsilon = 0\,, \tag{5.58}$$

- (r,z)

$$\Big[f\,T_3'' + f(T_3')^2 + \frac{i}{2}\,\partial_r(h_I p^I)\sqrt{f}\,e^{-T_1-T_2}\,\Gamma_{124} + \partial_r(g_I h^I)\sqrt{f}\,\Gamma_4 \\ - \frac{i}{2}\,h_I p^I\sqrt{f}(T_1' + T_2')e^{-T_1-T_2}\,\Gamma_{124}\Big]\epsilon = 0\,. \tag{5.59}$$

From the vanishing of the gauginos variation (5.37) one gets

$$\left[\frac{1}{3}\,\mathcal{G}_{ij}\sqrt{f}\,\partial_r\phi^j\,\Gamma_4 - \partial_i(g_I h^I) + \frac{i}{2}\,\partial_i(h_I p^I)e^{-T_1-T_2}\,\Gamma_{12}\right]\epsilon = 0\,. \tag{5.60}$$

The combination (5.54)+(5.55)−(5.50)−(5.51) gives

$$\left[(T_1' + T_2')\Big(f\,T_3' - \frac{1}{2}\,f'\Big) + 2e^{-2T_3} + \sqrt{f}(T_1' - T_2')e^{-T_3}\,\Gamma_{34}\right]\epsilon = 0\,. \tag{5.61}$$

The determinant of the coefficient matrix of this linear system vanishes if

$$(T_1' + T_2')\Big(f\,T_3' - \frac{1}{2}\,f'\Big) + 2e^{-2T_3} = 0 \quad \wedge \quad \sqrt{f}(T_1' - T_2')e^{-T_3} = 0\,, \tag{5.62}$$

which implies

$$T_1'\Big(f\,T_3' - \frac{1}{2}\,f'\Big) + e^{-2T_3} = 0\,, \qquad T_1' = T_2'\,. \tag{5.63}$$

From the combination (5.57)−(5.58)+(5.54)−(5.55)+2×((5.50)−(5.51)) we obtain

$$\left[f(T_1'' - T_2'') + (T_1' - T_2')\big(f(T_1' + T_2' + T_3') + f'\big) + \sqrt{f}(T_1' + T_2' - 2T_3')e^{-T_3}\,\Gamma_{34}\right]\epsilon = 0\,. \tag{5.64}$$

Using $T_1' = T_2'$, it turns out that the vanishing of the determinant associated to (5.64) requires $T_3' = T_1'$. (5.50)−(5.51) yields

$$i\,h_I p^I e^{-T_1-T_2-T_3}\,\Gamma_{123}\epsilon = 0\,, \tag{5.65}$$

and thus

$$h_I p^I = 0\,. \tag{5.66}$$

Taking into account the above results and defining $T' := T_1' = T_2' = T_3'$, the first integrability conditions become

- (t,x), (t,y), (t,z)

$$\left[\frac{1}{2}\,f'\,T' - (g_I h^I)^2\right]\epsilon = 0\,, \tag{5.67}$$

- (x,y)

$$\left[f(T')^2 - (g_I h^I)^2 - e^{-2T_3} - 3i\,g_I p^I e^{-T_1-T_2}\,\Gamma_{12}\right]\epsilon = 0\,, \tag{5.68}$$

- (x,z), (y,z)

$$\left[f(T')^2 - (g_I h^I)^2 + e^{-2T_3}\right]\epsilon = 0\,, \tag{5.69}$$

- (r,t)

$$\left[\frac{1}{2}\,f'' - (g_I h^I)^2 + \partial_r(g_I h^I)\sqrt{f}\,\Gamma_4\right]\epsilon = 0\,, \tag{5.70}$$

- (r,x), (r,y), (r,z)

$$\left[f\,T'' + f(T')^2 + \partial_r(g_I h^I)\sqrt{f}\,\Gamma_4\right]\epsilon = 0\,. \tag{5.71}$$

(5.68)$-$(5.69) leads to

$$\left[2e^{-2T_3} + 3i(g_I p^I)e^{-T_1-T_2}\,\Gamma_{12}\right]\epsilon = 0\,, \tag{5.72}$$

which implies the Dirac-type quantization condition

$$g_I p^I = \sigma_1 \frac{2}{3}\,e^{T_1+T_2-2T_3}\,, \tag{5.73}$$

where $\sigma_1 = \pm 1$. Plugging this back into (5.72) gives

$$\Gamma_{12}\epsilon = i\sigma_1\epsilon\,. \tag{5.74}$$

With (5.74), the gauginos equation (5.60) becomes

$$\left[\frac{1}{3}\,\mathcal{G}_{ij}\sqrt{f}\,\partial_r\phi^j\,\Gamma_4 - \partial_i(g_I h^I) - \sigma_1\frac{1}{2}\,\partial_i(h_I p^I)e^{-T_1-T_2}\right]\epsilon = 0\,. \tag{5.75}$$

If the scalar fields were constant, $\partial_r\phi^j = 0\;\forall j$, this would imply

$$\partial_i(g_I h^I) + \sigma_1\frac{1}{2}\,\partial_i(h_I p^I)e^{-T_1-T_2} = 0\,, \tag{5.76}$$

and thus T_1 and T_2 should be constant as well, leading to a contradiction with the first equation of (5.62). Note that this conclusion is valid provided $\partial_i(g_I h^I)$ and $\partial_i(h_I p^I)$ do not both vanish. In the latter case, however, using one of the very special geometry relations, we have

$$0 = \mathcal{G}^{ij}\partial_i(h_I p^I)\,\partial_j(h_J p^J) = \frac{4}{9}\,G_{IJ}p^I p^J - \frac{2}{3}\,h_I p^I h_J p^J = \frac{4}{9}\,G_{IJ}p^I p^J\,, \tag{5.77}$$

where the last step follows from (5.66). Since G_{IJ} is positive definite, (5.77) leads to a contradiction. If $\partial_r\phi^i \neq 0$ for at least one i, one can multiply (5.75) by $\partial_r\phi^i$ and sum over i to get[7]

$$\left[\frac{1}{3}\,\mathcal{G}_{ij}\sqrt{f}\,\partial_r\phi^i\partial_r\phi^j\,\Gamma_4 - \partial_r(g_I h^I)\right]\epsilon = 0\,. \tag{5.78}$$

[7] Notice that $\partial_r(h_I p^I) = 0$.

We see immediately that one needs $\partial_r(g_I h^I) \neq 0$, since otherwise $\mathcal{G}_{ij}\partial_r\phi^i\partial_r\phi^j = 0$, which is impossible because $\mathcal{G}_{ij}$ is a definite matrix.

To proceed, we require the determinants associated to the linear systems (5.70) and (5.71) to vanish, which implies the projection condition $\Gamma_4\epsilon = -\sigma_2\epsilon$ ($\sigma_2 = \pm 1$) as well as

$$\begin{aligned} \sigma_2\, \partial_r(g_I h^I)\sqrt{f} &= \frac{1}{2}\, f'' - (g_I h^I)^2\,, \\ \sigma_2\, \partial_r(g_I h^I)\sqrt{f} &= f\, T'' + f(T')^2\,. \end{aligned} \tag{5.79}$$

Deriving the prefactor of ϵ in (5.67) w.r.t. r, one obtains, using also (5.79) and (5.67),

$$\begin{aligned} 2(g_J h^J)\, \partial_r(g_I h^I) &= \frac{1}{2}\,(f''\, T' + f'\, T'') \\ &= \Big[\sigma_2\, \partial_r(g_I h^I)\sqrt{f} + (g_I h^I)^2\Big] T' + \frac{1}{2}\, f' \Big[\sigma_2\, \partial_r(g_I h^I)\frac{1}{\sqrt{f}} - (T')^2\Big] \\ &= \sigma_2\, \partial_r(g_I h^I)\Big(\sqrt{f}\, T' + \frac{f'}{2\sqrt{f}}\Big)\,. \end{aligned} \tag{5.80}$$

Thus, since $\partial_r(g_I h^I) \neq 0$,

$$g_I h^I = \sigma_2 \frac{1}{2}\Big(\sqrt{f}\, T' + \frac{f'}{2\sqrt{f}}\Big)\,. \tag{5.81}$$

Deriving this w.r.t. r and then subtracting the sum of the two equations in (5.79), divided by two, we get

$$0 = \sigma_2 \frac{1}{2}\Big(\frac{f'\, T'}{\sqrt{f}} - \frac{(f')^2}{4 f^{3/2}} - \sqrt{f}\,(T')^2\Big) = -\sigma_2 \frac{\sqrt{f}}{2}\Big(\frac{f'}{2f} - T'\Big)^2 = -\sigma_2 \frac{1}{2 f^{3/2}(T')^2}\, e^{-4T_3}\,, \tag{5.82}$$

where the last step follows from the first equation of (5.62). Evidently, (5.82) leads to a contradiction, which implies

Proposition 2. *There are no static, Sol-invariant solutions to the Killing spinor equations with solvegeometry spatial cross-sections at fixed r and purely magnetic field strengths.*

In particular, there is no BPS limit of the black hole constructed in section 5.3. Notice, in this context, that rotating supersymmetric Nil and $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ near-horizon geometries were found in [147].

5.5 Attractor mechanism

According to the *attractor mechanism* [157–161], the entropy of an extremal black hole and the scalar fields on the event horizon are insensitive to the asymptotic values of the moduli and depend only on the electric and magnetic charges. This property was first discovered in 4-dimensional ungauged supergravity for BPS black holes [157] and subsequently extended to higher dimensions, non-supersymmetric or rotating solutions, and gauged supergravities, cf. [79, 94, 95, 129, 162–166] for an (incomplete) list of references.

Articulating further, the attractor mechanism features two different, yet strictly related, phenomena. On the one hand, the entropy of extremal black holes depends solely on the electromagnetic charges and possibly on the gauge couplings in case of gauged theory[8]. On the other, the scalar fields flow on the horizon to fixed values independent from their asymptotic

[8] An important consequence is that the Bekenstein-Hawking entropy does not take stringy corrections, although, as we shall see in chapter 7, the entropy formula itself gets corrected.

configuration, but given uniquely by the charges and the gaugings. What connects these two aspects is a particular scalar function called *effective potential* V_{eff}. The scalar configuration on the horizon can be determined by extremizing the effective potential and the Bekenstein-Hawking entropy is given by the value of V_{eff} at its extremum[9].

In this section, we study the attractor mechanism for extremal static black holes with solvgeometry horizon in the theory (3.67). It will turn out that there are no such attractors for purely electric field strengths, while in the magnetic case there are attractor geometries, for which we explicitly determine the effective potential.

5.5.1 Magnetic ansatz

As a first step to extend the black hole solution (5.25a) to the matter-coupled case, we consider the near-horizon limit of the ansatz (5.8). Following closely the argument presented in [166], we are interested in magnetically charged, static and extremal black holes with Sol horizon, but without referring to any particular model of very special geometry. Extremality implies that the near-horizon geometry is the product manifold $\text{AdS}_2 \times \text{Sol}$. Assuming the horizon to be located at $r=0$, we have for $r \to 0$

$$f(r) \sim \left(\frac{r}{r_{\text{AdS}}}\right)^2 , \qquad T_1(r) \sim \frac{1}{4} \ln A \, , \qquad T_3(r) \sim \frac{1}{2} \ln B \, , \qquad \phi^i(r) \sim \phi^i_* \, , \tag{5.83}$$

with r_{AdS} the curvature radius of the AdS_2 part, A and B positive constants and ϕ^i_* the horizon values of the scalar fields. The Einstein equations (5.21) become then algebraic and admit the solution

$$A = -\frac{\Sigma_0}{V_*} \, , \qquad B = -\frac{2}{V_*} \, , \qquad r^2_{\text{AdS}} = -\frac{1}{V_*} \, , \tag{5.84}$$

where $V_* := V(\phi^i_*) < 0$ and $\Sigma_* := G_{IJ}(\phi^i_*)\, p^I p^J$. Using (5.83) and (5.84), the equations (5.22) for the scalars boil down to

$$\partial_i V_{\text{eff}}\big|_{\phi^i_*} = 0 \, , \tag{5.85}$$

where

$$V_{\text{eff}}(\phi^i) = \frac{\sqrt{G_{IJ}(\phi^i) p^I p^J}}{2\sqrt{2}\, |V(\phi^i)|} \tag{5.86}$$

is an effective potential whose normalization has been chosen for later convenience. Thus, the attractor solution reads

$$ds^2 = -|V_*|\, r^2 dt^2 + \frac{dr^2}{|V_*|\, r^2} + \sqrt{\frac{\Sigma_*}{|V_*|}} \left(e^{2z} dx^2 + e^{-2z} dy^2\right) + \frac{2}{|V_*|}\, dz^2 \, , \tag{5.87a}$$

$$F^I = p^I dx \wedge dy \, , \qquad \phi^i(r) = \phi^i_* \, . \tag{5.87b}$$

The horizon values ϕ^i_* of the scalars are computed by extremization of the effective potential (5.86) and, unless V_{eff} has flat directions, are completely fixed by the magnetic charges and the constants g_I, in accordance with the attractor mechanism. Finally, the entropy density is given by

$$s = V_{\text{eff}}(\phi^i_*) \, . \tag{5.88}$$

[9] In case the effective potential had flat directions, some of the scalars would not be stabilized and would retain a dependency from their asymptotic value. Nevertheless, the value of V_{eff} at its extremum would be unique and moduli-independent [167, 168].

Notice that, even if V_{eff} has flat directions, the Bekenstein-Hawking entropy depends only on the magnetic charges p^I and the parameters g_I. As a consequence of the results of section 5.4.2, the attractor geometry (5.87a)-(5.87b) breaks all the supersymmetries.

As an example, we consider the *stu* model already presented in subsection 3.2.2. Without specifying a parametrization we can write

$$\begin{aligned} G_{IJ} &= \frac{1}{2}\,\mathrm{diag}\big((h^1)^{-2},(h^2)^{-2},(h^3)^{-2}\big)\,, \\ V &= -18(g_1g_2\,h^1h^2 + g_2g_3\,h^2h^3 + g_3g_1\,h^3h^1)\,. \end{aligned} \tag{5.89}$$

Choosing the parametrization (3.32) and taking $g_I = v/6\ \forall I$, we get the effective potential

$$V_{\text{eff}}(\phi^1,\phi^2) = \frac{\sqrt{(p^1)^2 e^{\frac{2\phi^1}{\sqrt{6}}+\frac{2\phi^2}{\sqrt{2}}} + (p^2)^2 e^{\frac{2\phi^1}{\sqrt{6}}-\frac{2\phi^2}{\sqrt{2}}} + (p^3)^2 e^{-\frac{4\phi^1}{\sqrt{6}}}}}{2v^2\Big(e^{-\frac{2\phi^1}{\sqrt{6}}} + e^{\frac{\phi^1}{\sqrt{6}}+\frac{\phi^2}{\sqrt{2}}} + e^{\frac{\phi^1}{\sqrt{6}}-\frac{\phi^2}{\sqrt{2}}}\Big)}\,, \tag{5.90}$$

which admits an extremum in

$$\phi^1 = \sqrt{\frac{2}{3}}\,\ln\left|\frac{(p^3)^2}{p^1p^2}\right|\,, \qquad \phi^2 = \sqrt{2}\,\ln\left|\frac{p^2}{p^1}\right|\,. \tag{5.91}$$

In this case, the entropy density (5.88) reads

$$s = \frac{|p^1p^2p^3|}{2v^2\sqrt{(p^1)^2(p^2)^2 + (p^2)^2(p^3)^2 + (p^3)^2(p^1)^2}}\,. \tag{5.92}$$

5.5.2 Electric ansatz

We now consider the case of purely electric field strengths. For a horizon modelled on solvegeometry, by means of the constraint (5.16) and the structure constants (5.7a), the fourth equation of (5.13) reduces to $(A = 1,3)$

$$\begin{aligned} -f'\,T_1' - f\,T_1'' - f\,T_1'(2T_1'+T_3') &= \frac{1}{3}\,e^{-2(2T_1+T_3)}G^{IJ}q_Iq_J + \frac{2}{3}\,V\,, \\ -f'\,T_3' - f\,T_3'' - f\,T_3'(2T_1'+T_3') - 2e^{-2T_3} &= \frac{1}{3}\,e^{-2(2T_1+T_3)}G^{IJ}q_Iq_J + \frac{2}{3}\,V\,, \end{aligned} \tag{5.93}$$

which immediately implies that a configuration with T_1 and T_3 constant is not acceptable[10]. One can try to relax the ansatz on T_1 and T_3 by assuming a generic power dependence like

$$e^{2T_1} \sim k_1 r^{\alpha_1}\,, \qquad e^{2T_3} \sim k_3 r^{\alpha_3}\,, \tag{5.94}$$

with k_A and α_A constants, but consistency of equations (5.93) requires $\alpha_A = 0$ and we fall into the previous contradictory case, hence the following

Proposition 3. *There are no static attractors with Sol horizon and purely electric field strengths.*

[10]For $T_1' = T_3' = 0$, the difference of the two equations leads to $e^{-2T_3} = 0$.

Chapter 6

Defects in 10d and 11d brane bound states

In the first two chapters of this thesis we already discussed the emergence, in the low-energy limit, of a gravity theory due to the massless spectrum of closed string and of a gauge theory originated by open strings ending on a D-brane. Within this picture branes play a double role, indeed they deform the background described by closed strings (in accordance with the equations of motion of the related supergravity theory) and their worldvolume is the setting of the QFT defined by open strings.

The first to grasp a correlation between these two phenomena was Maldacena in a seminal paper of 1997 [8, 169, 170]. He considered a system of N coincident D3-branes in type-IIB string theory, characterized by an $\text{AdS}_5 \times S^5$ near-horizon, and postulated, in the limit of large N, the equivalence between the theory on the AdS_5 factor and the 4-dimensional SCFT on the brane worldvolume, in particular the $\mathcal{N} = 4$ SYM theory. This was the first example of application of the *AdS/CFT correspondence*. The key point of this correspondence is that also other branes or bound state of branes can give rise to AdS factors in the near-horizon and the theory therein can be put in correlation to the CFT living on the boundary of the AdS space, in one dimension less. Unfortunately, although it has been successfully tested in various set-ups, AdS/CFT correspondence remains a conjecture.

Restricting to the low-energy limit, we can focus on two different regimes governed by the string coupling g_s and the number of branes N. As we said, in each of them, non-perturbative states manifest two dual and equivalent descriptions, a gravitational one, coming from the closed strings, and a gauge theory one, due to the open strings. Specifically, in one regime the weakly-coupled gravity picture is given in terms of brane solutions of the supergravity equations of motion and corresponds to a strongly-coupled SCFT on the worldvolume of the branes; in the opposite regime weak and strong coupling are interchanged. The strong implication of the AdS/CFT correspondence is that, in any regime, we can pick the description we prefer and gain some knowledge on the alternative picture as well. In particular, the possibility of working out quantitative results of the physics of branes allows to shed new light on the most interesting and mysterious features of quantum gravity.

Despite the huge amount of literature concerning holography, a satisfying holographic interpretation of lower-dimensional AdS backgrounds is still in need of a deeper understanding. A very interesting approach to the study of AdS spaces in lower dimensions is to resolve their dual CFTs within higher-dimensional field theories. In order to address this issue it can be useful to recall an important ingredient of the AdS/CFT correspondence: the number of dynamical degrees of freedom of a holographic CFT is proportional to the coupling constant of the corresponding AdS solution, related, in turn, to the volume of the internal manifold [171]. Two implications follow immediately. When the AdS geometry is realized as part of a higher-dimensional solution with non-compact internal manifold, it may be signalling the presence of an underlying higher-dimensional field theory. Moreover, the partial breaking of the Lorentz—and, in case, conformal—symmetries of the spacetime where the higher-dimensional field theory lives can be considered as entirely due to the presence of the AdS geometry.

A perfect framework for the implementation of these ideas is given by defect CFTs [172–174]. These usually arise when a brane intersection ends on a bound state which is known to be described by an AdS vacuum in the near-horizon limit. The intersection breaks some of the isometries of the vacuum, producing a lower-dimensional AdS solution characterized by a non-trivial warping between AdS and the internal manifold. On the other side of the correspondence, some of the conformal isometries of the higher-dimensional CFT are broken by a deformation driven by a coupling dependent on the position of the intersection, originating a lower-dimensional CFT. Many are the examples discussed so far in literature and for a non-exhaustive list we refer to [175–187].

A useful approach to these systems comes from their description in lower-dimensional supergravities. Indeed, the parametrization of an AdS string solution often hides the possible presence of a higher-dimensional AdS factor arising in some particular limit, whereas in lower dimensions the defect interpretation could be manifest. More concretely, given an AdS_d vacuum associated to a particular brane system, one can consider a d-dimensional Janus-type background

$$ds_d^2 = e^{2U(\mu)} ds^2_{\text{AdS}_p} + e^{2W(\mu)} ds^2_{d-p-1} + e^{2V(\mu)} d\mu^2 \,, \tag{6.1}$$

with non-compact $\mathcal{M}_{d-p-1} \times I_\mu$ transverse space, admitting an asymptotic locally AdS_d vacuum. These backgrounds can then be consistently uplifted to $D = 10$ or 11, producing a warped geometry of the type $\text{AdS}_p \times \mathcal{M}_{d-p-1} \times I_\mu \times X_{D-d}$ which flows asymptotically to $\text{AdS}_d \times X_{D-d}$, where X_{D-d} is the internal manifold of the truncation. Holographically, this is the supergravity picture of a defect $(p-1)$-dimensional CFT realized within a higher $(d-1)$-dimensional CFT.

In this chapter we shall focus our attention on AdS_3, and briefly on AdS_2, solutions with $\mathcal{N} = (0,4)$ supersymmetry, arising as near-horizons of brane intersections in M-theory and type-IIA string theory[1]. These solutions have received renewed interest recently, having been studied in a series of papers [202–207]. Their significance comes from the fact that they provide explicit holographic duals to 2d $\mathcal{N} = (0,4)$ CFTs [208–210], which, in turn, have been shown to play a central role in the microscopic description of 5d black holes [211–214] and the study of 6d $(1,0)$ CFTs deformed away from the conformal point [215, 216]. A precise duality between AdS_3 solutions and 2d $(0,4)$ quiver CFTs has been described recently in [205, 206].

We begin this chapter considering the 11d class of $\mathcal{N} = (0,4)$ $\text{AdS}_3 \times S^3/\mathbb{Z}_k \times \text{CY}_2 \times I$ backgrounds recently constructed in [207], focusing on a subclass describing the near-horizon regime of a particular set-up of M5'-branes on which M2-M5 bound states end. Besides providing the full 11d brane solution reproducing the AdS_3 background in its near-horizon limit, we derive the parametrization that allows to connect this 11d spacetime with a 7d domain wall described by (6.1), found in [195]. This 7d solution reproduces asymptotically locally in the UV an AdS_7 geometry, while it manifests a singular behaviour in the IR corresponding to the locus where the defect M2-M5 branes intersect the M5'-branes. In section 6.2 we study a specific IIA reduction of this system, which gives rise to a NS5-D6 bound state intersected by D2-D4 branes. The analysis is brought forward in analogy to section 6.1 and the AdS_3 near-horizon solution turns out to belong to a new class presented in appendix C. Again, it is possible to relate this 10d solution to the 7d domain wall found in [195]. The following section considers a subclass of the family of $\mathcal{N} = (0,4)$ $\text{AdS}_3 \times S^2 \times \text{CY}_2 \times I$ solutions to massive IIA supergravity constructed in [203] and provides the associated full brane picture in terms of D2-NS5-D6 branes ending on a D4-D8 bound state. Subsequently, the parametrization relating its AdS_3 near-horizon to a 6d domain wall of the type given by (6.1), found in [198], is obtained. Finally, in section 6.4

[1] For a non-exhaustive list of references regarding solutions and classifications of AdS_3 and AdS_2 backgrounds in M-theory and type-II string theory, see [188–201].

we briefly examine the realization of the AdS_2 solutions to massive IIA supergravity recently constructed in [207] as line defect CFTs within the 5d Sp(N) CFT.

6.1 Surface defects in M-theory

In this section we examine a particular brane set-up in M-theory consisting on M2-M5 branes ending on M5'-branes. We consider the most general case in which the 5-branes are placed on ALE singularities, introduced by KK and KK' monopoles. We construct the explicit supergravity solution and show that it admits a near-horizon regime described by an $AdS_3 \times S^3/\mathbb{Z}_k \times S^3/\mathbb{Z}_{k'} \times \Sigma_2$ background with $\mathcal{N} = (0,4)$ supersymmetry. This geometry extends a particular subclass of the solutions recently studied in [207].

The main aspect to note is that the coordinates in which the near-horizon limit emerges "hide" the presence of an underlying $AdS_7/\mathbb{Z}_k$ vacuum arising in the UV. In order to show this explicitly we link the near-horizon geometry to a 7d domain wall asymptotically locally AdS_7. This 7d solution, first worked out in [195], is a Janus-like flow preserving eight real supercharges, characterized by an AdS_3 slicing. In 11d it is featured by a non-compact internal manifold whose asymptotic behaviour reproduces locally the AdS_7 vacuum of M5-branes on an A-type singularity. In the "domain wall coordinates" the near-horizon geometry of our brane set-up gains a consistent description as a flow interpolating between a local AdS_7 geometry and a singularity. The first regime corresponds to the limit in which we are far from the M2-M5 intersection, while the second is equivalent to "zooming in" on the region where the M2-M5 branes end on the M5'-brane, breaking the isometries of the branes that generate the vacuum.

6.1.1 The brane set-up

We start considering the supergravity picture of an M2-M5 bound state ending on orthogonal M5'-branes, with the 5-branes located at singularities defined by Kaluza-Klein monopoles with charges Q_{KK} and $Q_{KK'}$. This intersection, depicted in table 6.1, preserves an $SO(3) \times SO(3)$ bosonic symmetry and four real supercharges.

branes	t	x^1	r	θ^1	θ^2	χ	z	ρ	φ^1	φ^2	ϕ
KK'	×	×	×	×	×	×	×	–	–	–	ISO
M5'	×	×	×	×	×	×	–	–	–	–	–
M2	×	×	–	–	–	–	×	–	–	–	–
M5	×	×	–	–	–	–	–	×	×	×	×
KK	×	×	–	–	–	ISO	×	×	×	×	×

Table 6.1: *1/8-BPS brane system underlying the intersection of M2-M5 branes ending on M5'-branes with KK monopoles. χ (ϕ) is the Taub-NUT direction of the KK (KK') monopoles.*

We consider the following 11d metric

$$\begin{aligned} ds_{11}^2 = {}& H_{M5'}^{-1/3}\Big[H_{M5}^{-1/3}H_{M2}^{-2/3}\,ds^2_{\mathbb{R}^{1,1}} + H_{M5}^{2/3}H_{M2}^{1/3}\Big(H_{KK}\big(dr^2+r^2ds^2_{S^2}\big)+H_{KK}^{-1}\big(d\chi+Q_{KK}\,\omega\big)^2\Big)\Big] \\ &+ H_{M5'}^{2/3}\Big[H_{M5}^{2/3}H_{M2}^{-2/3}\,dz^2 + H_{M5}^{-1/3}H_{M2}^{1/3}\Big(H_{KK'}\big(d\rho^2+\rho^2ds^2_{\tilde{S}^2}\big)+H_{KK'}^{-1}\big(d\phi+Q_{KK'}\,\eta\big)^2\Big)\Big]\,, \end{aligned} \tag{6.2}$$

where ω and η are defined such that $d\omega = \text{vol}_{S^2}$ and $d\eta = \text{vol}_{\tilde{S}^2}$. We take the M2-M5 branes completely localized in the worldvolume of the M5'-branes, i.e. $H_{M2} = H_{M2}(r)$ and $H_{M5} =$

$H_{\rm M5}(r)$. This particular charge distribution breaks the symmetry under the interchange of the two 2-spheres. This is explicit in the 4-form flux $G_{(4)}$,

$$\begin{aligned} G_{(4)} =& \, \partial_r H_{\rm M2}^{-1} \, \mathrm{vol}_{\mathbb{R}^{1,1}} \wedge dr \wedge dz - \partial_r H_{\rm M5} \, r^2 \, \mathrm{vol}_{S^2} \wedge d\chi \wedge dz \\ &+ H_{\rm KK'} H_{\rm M2} H_{\rm M5}^{-1} \partial_z H_{\rm M5'} \, \rho^2 \, d\rho \wedge \mathrm{vol}_{\tilde{S}^2} \wedge d\phi - \partial_\rho H_{\rm M5'} \, \rho^2 \, dz \wedge \mathrm{vol}_{\tilde{S}^2} \wedge d\phi \, . \end{aligned} \tag{6.3}$$

The equations of motion and Bianchi identities of 11d supergravity are then equivalent to two independent sets of equations, one involving the M2-M5 branes and the KK-monopoles,

$$H_{\rm M2} = H_{\rm M5} \, , \qquad \nabla^2_{\mathbb{R}^3_r} H_{\rm M5} = 0 \qquad \text{with} \qquad H_{\rm KK} = \frac{Q_{\rm KK}}{r} \, , \tag{6.4}$$

and the other describing the dynamics of M5'-branes on the ALE singularity introduced by the KK'-monopoles,

$$\nabla^2_{\mathbb{R}^3_\rho} H_{\rm M5'} + H_{\rm KK'} \partial_z^2 H_{\rm M5'} = 0 \qquad \text{with} \qquad H_{\rm KK'} = \frac{Q_{\rm KK'}}{\rho} \, . \tag{6.5}$$

The second equation in (6.4) can be easily solved for

$$H_{\rm M5}(r) = H_{\rm M2}(r) = 1 + \frac{Q_{\rm M5}}{r} \, , \tag{6.6}$$

where we introduced the M2 and M5 charges $Q_{\rm M2}$ and $Q_{\rm M5}$, that in order to satisfy (6.4) have to be equal. One way to look at our system is then in terms of M5'-KK' branes moving on the 11d background generated by M2-M5-KK branes. The 4d transverse manifold parametrized by the coordinates $(\rho, \varphi^1, \varphi^2, \phi)$ arises as a foliation of the Lens space $\tilde{S}^3/\mathbb{Z}_{k'}$ that is obtained by modding out the $\tilde{S}^3$ with $k' = Q_{\rm KK'}$, through the change of coordinates $\rho \mapsto \rho^2/(4Q_{\rm KK'})$ [188].

It is interesting to consider the limit $r \to 0$. This is equivalent to "zooming in" on the locus where the M2-M5 branes intersect the M5'-branes. In this limit, the worldvolume of the M5'-branes becomes $\mathrm{AdS}_3 \times S^3/\mathbb{Z}_k$, with $k = Q_{\rm KK}$, and the full 11d string background takes the form[2]

$$\begin{aligned} ds^2_{11} =& \, 4\, k \, Q_{\rm M5} H_{\rm M5'}^{-1/3} \big(ds^2_{\mathrm{AdS}_3} + ds^2_{S^3/\mathbb{Z}_k} \big) + H_{\rm M5'}^{2/3} \big(dz^2 + d\rho^2 + \rho^2 \, ds^2_{\tilde{S}^3/\mathbb{Z}_{k'}} \big) \, , \\ G_{(4)} =& \, 8\, k \, Q_{\rm M5} \, \mathrm{vol}_{\mathrm{AdS}_3} \wedge dz + 8 \, k \, Q_{\rm M5} \, \mathrm{vol}_{S^3/\mathbb{Z}_k} \wedge dz \\ &+ \partial_z H_{\rm M5'} \, \rho^3 \, d\rho \wedge \mathrm{vol}_{\tilde{S}^3/\mathbb{Z}_{k'}} - \partial_\rho H_{\rm M5'} \, \rho^3 \, dz \wedge \mathrm{vol}_{\tilde{S}^3/\mathbb{Z}_{k'}} \, . \end{aligned} \tag{6.7}$$

Here the two orbifolded 3-spheres are locally described by the metrics

$$ds^2_{S^3/\mathbb{Z}_k} = \frac{1}{4} \left[ds^2_{S^2} + \left(\frac{d\chi}{k} + \omega \right)^2 \right] \qquad \text{and} \qquad ds^2_{\tilde{S}^3/\mathbb{Z}_{k'}} = \frac{1}{4} \left[ds^2_{\tilde{S}^2} + \left(\frac{d\phi}{k'} + \eta \right)^2 \right] \, . \tag{6.8}$$

It is important to stress the relevance of the $Q_{\rm KK}$ monopole charge dissolved in the worldvolume of the M5'-branes in recovering the near-horizon geometry, given by (6.7), from the general brane solution (6.2). Besides ensuring that the supersymmetries of the M2-M5-M5' brane set-up are broken by a half, the presence of the KK-monopoles crucially determines the emergence of the $\mathrm{AdS}_3 \times S^3/\mathbb{Z}_k$ geometry associated to the smeared M2-M5 branes.

This AdS_3 background corresponds to the $\mathcal{N} = (0, 4)$ $\mathrm{AdS}_3 \times S^3/\mathbb{Z}_k \times \mathrm{CY}_2 \times I$ backgrounds recently studied in [207] and defined by equations (B.1) and (B.2) therein. In particular, we recover the solutions with $\mathrm{CY}_2 = T^4$, $u' = 0$ and $H_2 = 0$. Taking a round $\tilde{S}^3$, i.e. $k' = 1$, and the M2-M5 defects smeared on the $(\rho, \tilde{S}^3)$ directions, one obtains the solutions that were the

[2] We redefined the Minkowski coordinates as $(t, x^1) \mapsto 2 Q_{\rm M5} Q_{\rm KK}^{1/2} (t, x^1)$.

focus of [207], with $ds^2_{T^4} = d\rho^2 + \rho^2 ds^2_{\tilde{S}^3}$. Indeed, we can recast the near-horizon solution (6.7) in the form of [207] by choosing

$$k = h_8 \,, \qquad H_{\text{M5}'} = \frac{2^6 Q^3_{\text{M5}} h_8^2}{u^2}\, h_4 \,, \qquad z = \frac{1}{4\, Q_{\text{M5}}}\,\tilde{\rho} \,, \qquad \rho = \frac{u^{1/2}}{4\, Q_{\text{M5}}\, h_8^{1/2}}\,\tilde{r} \,. \tag{6.9}$$

In the next section we shall see how the extra dependence on the ρ coordinate is crucial in order to reach $\text{AdS}_7/\mathbb{Z}_k$ in a particular limit.

Let us finally make some considerations regarding the supersymmetries preserved by our brane solution. Even if the 11d metric in equation (6.2) is invariant under $\text{SO}(3) \times \text{SO}(3)$, the ansatz taken for our branes, which are smeared on the $\tilde{S}^3$, reduces the global symmetries to just the $\text{SO}(3)$ associated to the S^2 contained in the worldvolume of the M5'-branes[3]. This is manifest in the $G_{(4)}$ 4-form flux given by equation (6.3). The preserved $\text{SO}(3)$ is then the R-symmetry group associated to our solutions, which are, by construction, $\mathcal{N} = (0, 4)$ supersymmetric. Regarding the introduction of the two families of KK-monopoles, one can check by studying the supersymmetry projectors of the brane solution that the introduction of one of the two types is for free, in the sense that it does not reduce further the supersymmetries preserved by the rest of the branes. One can see explicitly that this happens thanks to the presence of the M2-branes in the background.

6.1.2 Surface defects as 7d charged domain walls

We can now show that the AdS_3 background (6.7) admits, in a particular limit, a local description in terms of the $\text{AdS}_7/\mathbb{Z}_k$ vacuum of M-theory. The idea is to relate the near-horizon geometry (6.7) to a charged 7d domain wall characterized by an AdS_3 slicing and an asymptotic behaviour that reproduces locally the AdS_7 vacuum of $\mathcal{N} = 1$ 7d supergravity. The reason the vacuum appears asymptotically locally is that the presence of the M2-M5 defect breaks its isometries (this is most manifest by the non-vanishing 4-form flux), as well as half of its supersymmetries.

We start considering $\mathcal{N} = 1$ minimal gauged supergravity in seven dimensions and its embedding in M-theory. In this case the minimal field content (excluding the presence of vectors) is given by the gravitational field, a real scalar X_7 and a 3-form gauge potential $\mathcal{B}_{(3)}$. The 7d background in which we are interested was introduced in [195] and further studied in [187]. It has the following form

$$\begin{aligned} ds^2_7 &= e^{2U(\mu)}\big(ds^2_{\text{AdS}_3} + ds^2_{S^3}\big) + e^{2V(\mu)} d\mu^2 \,, \\ \mathcal{B}_{(3)} &= b(\mu)\big(\text{vol}_{\text{AdS}_3} + \text{vol}_{S^3}\big) \,, \\ X_7 &= X_7(\mu) \,. \end{aligned} \tag{6.10}$$

The BPS equations were worked out in [195] and are given by

$$U' = \frac{2}{5}\, e^V f_7 \,, \qquad X_7' = -\frac{2}{5}\, e^V X_7^2\, D_X f_7 \,, \qquad b' = -\frac{2\, e^{2U+V}}{X_7^2} \,. \tag{6.11}$$

In these equations f_7 is the superpotential, defined in (2.49). The flow (6.11) preserves eight real supercharges (it is 1/2-BPS in 7d). In order to be consistent, this background has to be endowed by the odd-dimensional self-duality condition (2.48), which takes the form

$$b = -\frac{e^{2U} X_7^2}{h} \,. \tag{6.12}$$

[3] Our construction is thus essentially different from the brane set-up that would give rise to the solutions constructed in [192], in which the branes must be localized in the two 3-spheres.

We can work out an explicit solution by choosing a gauge,

$$e^{-V} = -\frac{2}{5}\, X_7^2\, D_X f_7\,, \tag{6.13}$$

such that system (6.11) can be easily integrated to give [195]

$$\begin{aligned} e^{2U} &= 2^{-1/4} g^{-1/2} \left(\frac{\mu}{1-\mu^5}\right)^{1/2}, \qquad & e^{2V} &= \frac{25}{2g^2}\, \frac{\mu^6}{(1-\mu^5)^2}\,, \\ b &= -2^{5/4} g^{-3/2} \frac{\mu^{5/2}}{(1-\mu^5)^{1/2}}\,, \qquad & X_7 &= \mu\,, \end{aligned} \tag{6.14}$$

with μ running between 0 and 1 and $h = \frac{g}{2\sqrt{2}}$. The behaviour at the boundaries is such that when $\mu \to 1$ the domain wall (6.10) is locally AdS$_7$, since we have

$$R_7 = -\frac{21}{4}\, g^2 + \mathcal{O}(1-\mu)^2\,, \qquad X_7 = 1 + \mathcal{O}(1-\mu)\,, \tag{6.15}$$

where R_7 is the 7d scalar curvature. In turn, when $\mu \to 0$ the 7d spacetime exhibits a singular behaviour. We point out that the background (6.10) can be generalized by quotienting the 3-sphere, locally written as in (6.8), without any further breaking of the supersymmetries, i.e. $ds^2_{S^3} \mapsto ds^2_{S^3/\mathbb{Z}_k}$ and $\text{vol}_{S^3} \mapsto \text{vol}_{S^3/\mathbb{Z}_k}$.

The uplift of the 7d background to M-theory takes place using relations (2.46) and (2.47), summarized in subsection 2.5.2. This gives

$$\begin{aligned} ds^2_{11} &= \Sigma_7^{1/3} e^{2U} \left(ds^2_{\text{AdS}_3} + ds^2_{S^3/\mathbb{Z}_k}\right) + \Sigma_7^{1/3} e^{2V} d\mu^2 \\ &\quad + 2g^{-2} \Sigma_7^{1/3} X_7^3\, d\xi^2 + 2g^{-2} c^2\, \Sigma_7^{-2/3} X_7^{-1}\, ds^2_{\tilde{S}^3}\,, \\ G_{(4)} &= (s\, b'\, d\mu + c\, b\, d\xi) \wedge \text{vol}_{\text{AdS}_3} + (s\, b'\, d\mu + c\, b\, d\xi) \wedge \text{vol}_{S^3/\mathbb{Z}_k} \\ &\quad - \frac{4}{\sqrt{2}}\, g^{-3} c^3\, \Sigma_7^{-2} W_7\, d\xi \wedge \text{vol}_{\tilde{S}^3} - \frac{20}{\sqrt{2}}\, g^{-3} s\, c^4 \Sigma_7^{-2} X_7^{-4} X_7'\, d\mu \wedge \text{vol}_{\tilde{S}^3}\,, \end{aligned} \tag{6.16}$$

where $c := \cos\xi$, $s := \sin\xi$, $\Sigma_7 := c^2 X_7 + s^2 X_7^{-4}$ and W_7 is given by (2.47). We can now relate this solution to the near-horizon geometry given by equation (6.7). We consider for simplicity a round $\tilde{S}^3$. This can be immediately generalized to the case in which KK'-monopoles are included by modding out the $\tilde{S}^3$.

One can see that the near-horizon geometry (6.7) takes the form given in (6.16) if one redefines the (z, ρ) coordinates in terms of the "domain wall coordinates" (μ, ξ) as

$$z = \frac{\sqrt{2}}{4\, g\, k\, Q_{\text{M5}}}\, \sin\xi\, e^{2U} X_7^2\,, \qquad \rho = \frac{\sqrt{2}}{4\, g\, k\, Q_{\text{M5}}}\, \cos\xi\, e^{2U} X_7^{-1/2}\,, \tag{6.17}$$

and requires that

$$H_{\text{M5}'} = \frac{2^6 Q_{\text{M5}}^3 k^3 e^{-6U}}{\Sigma_7}\,. \tag{6.18}$$

In this calculation one needs to crucially use the 7d BPS equations (6.11) and the self-duality condition (6.12). The expression for $H_{\text{M5}'}$ given by equation (6.18) satisfies the condition imposed by equation (6.5). The AdS$_7$ geometry arises through a non-linear change of coordinates that relates the (z, ρ) coordinates of the near-horizon AdS$_3$ geometry to the (μ, ξ) coordinates of the 7d domain wall solution, in which the defect interpretation becomes manifest. When $\mu \to 1$ the domain wall reaches locally the AdS$_7/\mathbb{Z}_k$ vacuum, while entering into the 7d bulk the isometries of the vacuum are broken by the AdS$_3$ slicing and 3-form gauge potential, that capture the effects produced by the M2-M5 brane intersection. This allows us to interpret the singular behaviour appearing in 7d when $\mu \to 0$ in terms of M2-M5 brane sources.

6.2 Surface defects in massless IIA

In this section we study the type-IIA regime of the M-theory set-up introduced in the previous section. From a 10d point of view the KK'-M5'-M2-M5-KK system has two different descriptions, depending on whether the reduction is performed on a circle that lies inside or outside the worldvolume of the M5'-branes. We recall that the 11d background has two compact coordinates. The χ coordinate lies inside the worldvolume of the M5'-branes and is identified as the Taub-NUT direction of the KK-monopoles. In turn, the ϕ coordinate lies outside the worldvolume of the M5'-branes and is identified as the Taub-NUT direction of the KK'-monopoles. The two possible reductions to type-IIA are depicted in figure 6.1.

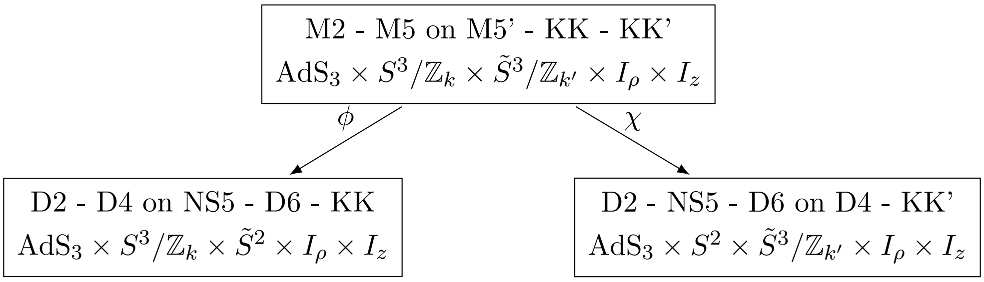


Figure 6.1: *Reductions of the KK'-M5'-M2-M5-KK brane system to type-IIA and their near-horizon limits. Only the reduction along ϕ asymptotes to AdS_7, with the KK-M5'-KK' system becoming KK-NS5-D6.*

In 10d one observes an interesting phenomenon. Both reductions produce a D2-D4-NS5-D6 intersection with Kaluza-Klein monopoles and both of them are described by near-horizon geometries with the same topology and supersymmetries. The charge distributions of the branes are, however, essentially different. In the first reduction the AdS_3 near-horizon geometries constitute a new class of solutions to massless type-IIA, that we shall further explore in this chapter. These solutions enjoy an interesting defect interpretation in terms of KK-NS5-D6 bound states, dual to an AdS_7 geometry, on which D2-D4 branes end. In the second reduction the $\tilde{S}^3/\mathbb{Z}_{k'}$ and I_ρ submanifolds build a $T^4/\mathbb{Z}_{k'}$ manifold, such that the resulting AdS_3 near-horizon geometries become the class I family of solutions to type-IIA recently classified in [203], restricted to the massless case, the $CY_2 = T^4$ and $u' = 0$, with extra KK'-monopoles. We shall see in section 6.3 that these solutions need to be embedded in massive IIA in order to be given a defect interpretation in terms of D4-KK'-D8 branes on which D2-NS5-D6 branes end. Roughly speaking, one could say that in both classes of solutions the D4 and NS5 branes exchange their roles, together with the D6-branes and the KK-monopoles. In fact, it can be shown that these two families of solutions are related upon a chain of T-S-T dualities [217].

The second reduction is the core of the following section 6.3, whereas in the remainder of this one we focus on the first reduction, the one that preserves the AdS_7 asymptotics in the UV. We present the brane picture and show that the resulting near-horizon geometries constitute a new class of AdS_3 solutions to type-IIA supergravity with $\mathcal{N} = (0, 4)$ supersymmetries. The special feature of this class of solutions, as compared to the solutions in [203], is that they asymptote (locally) to the $AdS_7/\mathbb{Z}_k \times S^2 \times I$ solution to massless IIA supergravity and can, thus, be interpreted as surface defect CFTs within the 6d $(1, 0)$ CFT dual to this solution.

6.2.1 New AdS_3 solutions with $\mathcal{N} = (0, 4)$ supersymmetries

In this section we consider the reduction of the 11d background (6.2) along the Taub-NUT coordinate ϕ. The resulting type-IIA configuration, depicted in table 6.2, consists on D2-D4 branes, coming from the smeared M2-M5 brane system appearing in (6.2), ending on a KK-NS5-D6 bound state, that arises upon reduction of the KK-M5'-KK' brane system. As already shown in the literature (see, e.g., [188]) this bound state is described in the near-horizon limit by an $\mathrm{AdS}_7/\mathbb{Z}_k$ vacuum preserving sixteen supercharges and a 3d internal space given by a 2-sphere foliation over a segment. We now add the D2-D4 branes to this system. We introduce firstly

branes	t	x^1	r	θ^1	θ^2	χ	z	ρ	φ^1	φ^2
D6	×	×	×	×	×	×	×	–	–	–
NS5	×	×	×	×	×	×	–	–	–	–
KK	×	×	–	–	–	ISO	×	×	×	×
D2	×	×	–	–	–	–	×	–	–	–
D4	×	×	–	–	–	–	–	×	×	×

Table 6.2: *Brane picture underlying the D2-D4 branes ending on the NS5-D6-KK intersection. The system is 1/8-BPS.*

the 10d metric

$$\begin{aligned} ds_{10}^2 &= H_{\rm D6}^{-1/2}\Big[H_{\rm D4}^{-1/2}H_{\rm D2}^{-1/2}\,ds^2_{\mathbb{R}^{1,1}} + H_{\rm D4}^{1/2}H_{\rm D2}^{1/2}\Big(H_{\rm KK}\big(dr^2+r^2\,ds^2_{S^2}\big)+H_{\rm KK}^{-1}\big(d\chi+Q_{\rm KK}\,\omega\big)^2\Big)\Big] \\ &\quad + H_{\rm D6}^{-1/2}H_{\rm NS5}H_{\rm D4}^{1/2}H_{\rm D2}^{-1/2}\,dz^2 + H_{\rm D6}^{1/2}H_{\rm NS5}H_{\rm D4}^{-1/2}H_{\rm D2}^{1/2}\big(d\rho^2+\rho^2\,ds^2_{\tilde S^2}\big)\,, \end{aligned} \tag{6.19}$$

where we take the D4 and D2 charges completely localized within the worldvolume of the NS5 branes, i.e. $H_{\rm D4} = H_{\rm D4}(r)$ and $H_{\rm D2} = H_{\rm D2}(r)$. Secondly, we consider the following dilaton and gauge potentials,

$$\begin{aligned} e^{\Phi} &= H_{\rm D6}^{-3/4}H_{\rm NS5}^{1/2}H_{\rm D2}^{1/4}H_{\rm D4}^{-1/4}\,, \\ B_{(6)} &= H_{\rm KK}H_{\rm D4}H_{\rm NS5}^{-1}r^2\,\mathrm{vol}_{\mathbb{R}^{1,1}}\wedge dr\wedge \mathrm{vol}_{S^2}\wedge d\chi\,, \\ C_{(3)} &= H_{\rm D2}^{-1}\,\mathrm{vol}_{\mathbb{R}^{1,1}}\wedge dz\,, \\ C_{(5)} &= H_{\rm D6}H_{\rm NS5}H_{\rm D4}^{-1}\rho^2\,\mathrm{vol}_{\mathbb{R}^{1,1}}\wedge d\rho\wedge \mathrm{vol}_{\tilde S^2}\,, \\ C_{(7)} &= H_{\rm KK}H_{\rm D4}H_{\rm D6}^{-1}r^2\,\mathrm{vol}_{\mathbb{R}^{1,1}}\wedge dr\wedge \mathrm{vol}_{S^2}\wedge d\chi\wedge dz\,, \end{aligned} \tag{6.20}$$

where we take the NS5-D6 branes completely localized in their transverse space. From (6.20) one can deduce[4] the fluxes

$$\begin{aligned} H_{(3)} &= -\partial_\rho H_{\rm NS5}\,\rho^2\,dz\wedge \mathrm{vol}_{\tilde S^2} + H_{\rm D2}H_{\rm D4}^{-1}H_{\rm D6}\partial_z H_{\rm NS5}\,\rho^2\,d\rho\wedge\mathrm{vol}_{\tilde S^2}\,, \\ F_{(2)} &= -\partial_\rho H_{\rm D6}\,\rho^2\,\mathrm{vol}_{\tilde S^2}\,, \\ F_{(4)} &= \partial_r H_{\rm D2}^{-1}\,\mathrm{vol}_{\mathbb{R}^{1,1}}\wedge dr\wedge dz - \partial_r H_{\rm D4}\,r^2\,\mathrm{vol}_{S^2}\wedge d\chi\wedge dz\,. \end{aligned} \tag{6.21}$$

As in the 11d picture, the equations of motion and Bianchi identities for the D2-D4-KK branes and the NS5-D6 branes can be solved independently. We have that

$$H_{\rm D2} = H_{\rm D4}\,, \qquad \nabla^2_{\mathbb{R}^3_r}H_{\rm D4} = 0 \qquad \text{with} \qquad H_{\rm KK} = \frac{Q_{\rm KK}}{r}\,, \tag{6.22}$$

and for the NS5-D6 branes,

$$\nabla^2_{\mathbb{R}^3_\rho}H_{\rm NS5} + H_{\rm D6}\,\partial_z^2 H_{\rm NS5} = 0\,, \qquad \nabla^2_{\mathbb{R}^3_\rho}H_{\rm D6} = 0\,. \tag{6.23}$$

[4]We use the conventions for fluxes of [218].

We note that the equations in (6.23) coincide with those found in [218] for the NS5-D6 bound state in the massless limit. The equations in (6.22) can be easily solved for

$$H_{\text{D4}}(r) = H_{\text{D2}}(r) = 1 + \frac{Q_{\text{D4}}}{r}\,, \tag{6.24}$$

where we introduced the D2 and D4 charges Q_{D2} and Q_{D4} that, in order to satisfy (6.22), have to be equal. We point out that uplifting to 11d we get the background (6.2) with $Q_{\text{D2}} = Q_{\text{M2}}$, $Q_{\text{D4}} = Q_{\text{M5}}$, $H_{\text{D6}} = H_{\text{KK}'}/4$ and a rescaling $\rho \mapsto 2\rho$ in the 10d solution.

We now analyze the limit $r \to 0$. As we already saw in the 11d case, the KK-monopole charge $Q_{\text{KK}} = k$ placed on the worldvolume of the NS5-branes realizes the orbifolded 3-sphere $S^3/\mathbb{Z}_k$. The metric (6.19) and the fluxes (6.21) take the form[5]

$$\begin{aligned}
ds^2_{10} &= 4\sqrt{2}k\, Q_{\text{D4}} H_{\text{D6}}^{-1/2}\big(ds^2_{\text{AdS}_3} + ds^2_{S^3/\mathbb{Z}_k}\big) + \sqrt{2}H_{\text{D6}}^{-1/2} H_{\text{NS5}}\, dz^2 + \frac{1}{\sqrt{2}} H_{\text{D6}}^{1/2} H_{\text{NS5}}\big(d\rho^2 + \rho^2 ds^2_{\tilde{S}^2}\big)\,,\\
e^{\Phi} &= 2^{3/4} H_{\text{D6}}^{-3/4} H_{\text{NS5}}^{1/2}\,, \qquad H_{(3)} = -\partial_\rho H_{\text{NS5}}\, \rho^2\, dz \wedge \text{vol}_{\tilde{S}^2} + \frac{1}{2} H_{\text{D6}} \partial_z H_{\text{NS5}}\, \rho^2\, d\rho \wedge \text{vol}_{\tilde{S}^2}\,,\\
F_{(2)} &= \frac{Q_{\text{D6}}}{2}\, \text{vol}_{\tilde{S}^2}\,,\\
F_{(4)} &= 8\, k\, Q_{\text{D4}}\, \text{vol}_{\text{AdS}_3} \wedge dz + 8\, k\, Q_{\text{D4}}\, \text{vol}_{S^3/\mathbb{Z}_k} \wedge dz\,,
\end{aligned} \tag{6.25}$$

with

$$\nabla^2_{\mathbb{R}^3_\rho} H_{\text{NS5}} + \frac{1}{2} H_{\text{D6}} \partial_z^2 H_{\text{NS5}} = 0 \qquad \text{and} \qquad H_{\text{D6}} = \frac{Q_{\text{D6}}}{\rho}\,, \tag{6.26}$$

where the D6-brane charge Q_{D6} equals the KK'-monopole charge of the 11d background (6.7), $Q_{\text{D6}} = k'$.

The AdS_3 backgrounds given by equation (6.25), with H_{NS5} and H_{D6} satisfying (6.26), constitute a new class of 10d backgrounds with $\mathcal{N} = (0, 4)$ supersymmetries. These solutions are of the form $\text{AdS}_3 \times S^3/\mathbb{Z}_k \times S^2$ fibered over two intervals. They preserve the same number of supersymmetries as the $\text{AdS}_3 \times S^2 \times \text{CY}_2 \times I$ solutions constructed in [203] and involve the same types of branes (in the massless limit of the solutions in [203]), plus extra KK-monopoles[6]. As mentioned, the brane intersections are however different.

In appendix C we show that a broader class of $\text{AdS}_3 \times S^3/\mathbb{Z}_k \times S^2$ solutions fibered over two intervals and preserving $\mathcal{N} = (0, 4)$ supersymmetries can in fact be realized from the general class of $\text{AdS}_3 \times S^3/\mathbb{Z}_k \times \text{CY}_2 \times I$ solutions to M-theory recently constructed in [207]. However, in the remainder of the chapter we shall focus our attention on the more restrictive case defined by (6.25). In the next section we shall relate this solution to a domain wall solution that asymptotes locally to $\text{AdS}_7/\mathbb{Z}_k$ and give it an interpretation as dual to D2-D4 surface defects within the corresponding 6d (1, 0) dual CFT.

6.2.2 Surface defects within the NS5-D6-KK brane system

In this section we follow the same strategy of subsection 6.1.2 in order to relate the new $\text{AdS}_3 \times S^3/\mathbb{Z}_k \times S^2$ solutions given by equation (6.25) to an AdS_7 geometry in the UV. In this case we relate the solutions to the uplift of the 7d domain wall discussed in subsection 6.1.2 to massless IIA supergravity. The 10d domain wall solution flows in the UV to the $\text{AdS}_7 \times S^2 \times I$ solution

[5] We redefined the Minkowski coordinates as $(t, x^1) \mapsto 2Q_{\text{D4}}Q_{\text{KK}}^{1/2}(t, x^1)$ and rescaled the function $H_{\text{D6}} \mapsto H_{\text{D6}}/2$.

[6] That can also be introduced in the $\text{AdS}_3 \times S^2 \times \text{CY}_2 \times I$ solutions in [203] without any further breaking of supersymmetries.

to massless IIA supergravity found in [188], modded by $\mathbb{Z}_k$, which arises in the near-horizon limit of a NS5-D6-KK brane intersection. This solution belongs to the general class of solutions to massive IIA supergravity constructed in [219], modded by $\mathbb{Z}_k$, in the massless limit. The solutions to massive IIA supergravity in [219] are the near-horizon geometries of NS5-D6-D8 brane intersections [220] and encode in a natural way the information of the 6d $(1,0)$ dual CFTs that live in their worldvolumes [221]. For this reason, in this section we shall follow the notation of [219, 221].

In the parametrization of [221], the uplift formulae from 7d $\mathcal{N}=1$ supergravity to massive IIA supergravity found in [222] are completely determined by a function $\alpha(y)$ that satisfies the differential equation[7]

$$\dddot{\alpha} = -162\pi^3 F_{(0)}\,. \tag{6.27}$$

Truncating to the massless case, they read

$$\begin{aligned}
ds_{10}^2 &= \frac{16\pi}{g}\left(-\frac{\alpha}{\ddot{\alpha}}\right)^{1/2} X_7^{-1/2} ds_7^2 + \frac{16\pi}{g^3} X_7^{5/2}\left[\left(-\frac{\ddot{\alpha}}{\alpha}\right)^{1/2} dy^2 + \left(-\frac{\alpha}{\ddot{\alpha}}\right)^{1/2} \frac{(-\alpha\ddot{\alpha})}{\dot{\alpha}^2 - 2\alpha\ddot{\alpha}X_7^5}\, ds_{S^2}^2\right],\\
e^{2\Phi} &= \frac{3^8 2^6 \pi^5}{g^3}\frac{X_7^{5/2}}{\dot{\alpha}^2 - 2\alpha\ddot{\alpha}X_7^5}\left(-\frac{\alpha}{\ddot{\alpha}}\right)^{3/2}, \qquad B_{(2)} = \frac{2^3\sqrt{2}\pi}{g^3}\left(-y + \frac{\alpha\dot{\alpha}}{\dot{\alpha}^2 - 2\alpha\ddot{\alpha}X_7^5}\right)\text{vol}_{S^2}\,,\\
F_{(2)} &= -\frac{\ddot{\alpha}}{162\pi^2}\,\text{vol}_{S^2}\,,\\
F_{(4)} &= \frac{2^3}{3^4\pi}\left(\ddot{\alpha}\, dy \wedge \mathcal{B}_{(3)} + \dot{\alpha}\, d\mathcal{B}_{(3)}\right),
\end{aligned} \tag{6.28}$$

where ds_7^2, X_7 and $\mathcal{B}_{(3)}$ are the 7d fields defined in (6.10). As we already mentioned, this solution asymptotes locally to AdS$_7$ when $\mu \to 1$, whereas it exhibits a singular behaviour when $\mu \to 0$. Furthermore, in the $\mu \to 1$ limit we recover the AdS$_7 \times S^2 \times I$ background preserving sixteen supersymmetries constructed in [219], for $g^3 = 2^{7/2}$.

We can now relate the domain wall solution to the AdS$_3 \times S^3/\mathbb{Z}_k \times S^2$ background defined by equation (6.25). The near-horizon geometry (6.25) takes the form given by (6.28) if one redefines the (z,ρ) coordinates in terms of the domain wall coordinates (μ, y) as

$$z = -\frac{1}{3^4\pi k\, Q_{\rm D4}}\,\dot{\alpha}\, b\,, \qquad \rho = \frac{8}{3^4 g^2 k^2 Q_{\rm D4}^2}\,\alpha\, X_7^{-1} e^{4U}\,, \tag{6.29}$$

and requires that

$$Q_{\rm D6} = \frac{(-\ddot{\alpha})}{81\pi^2}\,, \qquad H_{\rm NS5} = 3^8\pi^2 k^3 Q_{\rm D4}^3 \frac{X_7^4 e^{-6U}}{\dot{\alpha}^2 - 2\alpha\ddot{\alpha}X_7^5}\,. \tag{6.30}$$

In this calculation one needs to crucially use the 7d BPS equations (6.11) and the self-duality condition (6.12), and take $h = \frac{g}{2\sqrt{2}}$. Moreover, the S^3 in the 7d background (6.10) must be modded by $\mathbb{Z}_k$. The first condition in (6.30) shows that α_0 is closely related to the number of D6-branes of the solution, while the second one is the 10d version of (6.18). In this case one can see that the constraint on $H_{\rm NS5}$ in (6.23) is satisfied by means of the BPS equations for X_7 and U. The second condition in (6.30) singles out a particular solution in the class defined by (6.25) that asymptotes locally to the AdS$_7/\mathbb{Z}_k \times S^2$ vacuum of massless 10d supergravity. The AdS$_7$ geometry arises through a non-linear change of variables, that relates the (z,ρ) coordinates of the near-horizon AdS$_3$ solution to the (μ, y) coordinates of the uplifted domain wall solution. In the new coordinates the defect interpretation becomes manifest. When $\mu \to 1$ the domain wall

[7] Note that we use y instead of z as in [221] in order to avoid confusion with the notation in the previous sections.

reaches the $AdS_7/\mathbb{Z}_k$ vacuum, while when $\mu \to 0$ a singular behaviour describes D2-D4 brane sources, that create a defect when they intersect the NS5-D6-KK brane system, breaking the isometries of AdS_7 to those of AdS_3.

6.3 Surface defects in massive IIA

The previous section was devoted to the study of the reduction of the 11d AdS_3 solutions and brane set-up along the Taub-NUT direction of the KK'-monopoles, contained in the worldvolume of the M5'-branes. In this section we shall be concerned with the reduction to type-IIA along the Taub-NUT direction χ of the second set of Kaluza-Klein monopoles, the KK-monopoles referred to in table 6.1. As we already pointed out, this reduction destroys the AdS_7 structure in 10d. This appears clear by looking at the near-horizon metric (6.7), where the M-theory circle is taken within the 3-sphere $S^3/\mathbb{Z}_k$, which was part of $AdS_7/\mathbb{Z}_k$. The solutions to type-IIA that arise in this reduction are the $AdS_3 \times S^2 \times CY_2$ solutions recently constructed in [203], with the CY_2 equal to a T^4, further modded out by $\mathbb{Z}_{k'}$. This general class of solutions was constructed as solutions to massive IIA. Upon reduction from M-theory we recover the massless subclass.

In this section we show that these solutions can be given a defect interpretation when embedded in massive IIA. Therefore, we shall be considering the general class of solutions constructed in [203], with $CY_2 = T^4$. We shall see that these solutions can be interpreted as associated to D4-KK'-D8 bound states on which smeared D2-NS5-D6 branes end. The D4-KK'-D8 brane system has as near-horizon geometry the $AdS_6 \times S^4$ background of Brandhuber-Oz [51], further orbifolded by $\mathbb{Z}_{k'}$, i.e. $AdS_6 \times S^4/\mathbb{Z}_{k'}$.

On the same line of subsection 6.2.2, we show that these solutions can be related to a 6d charged domain wall solution characterized by an AdS_3 slicing and a 2-form gauge potential [198]. This domain wall reproduces locally in its asymptotic regime the AdS_6 vacuum in [51] associated to D4-D8 branes, modded out by $\mathbb{Z}_{k'}$. In the opposite limit a singular behaviour describes D2-NS5-D6 brane sources that, intersecting the D4-D8-KK' system, create a defect, breaking the isometries of AdS_6 to those of AdS_3. This behaviour allows us to propose, in analogy to the AdS_7 case, a dual interpretation of the AdS_3 solution as a $\mathcal{N} = (0, 4)$ surface defect CFT within the 5d $Sp(N)$ CFT [223] dual to the Brandhuber-Oz AdS_6 background [51].

6.3.1 The brane set-up

We start considering the well-known D4-D8 brane set-up of massive IIA string theory [51, 218], with D2-NS5-D6 branes ending on it [198]. For the moment we shall ignore the contribution of the KK'-monopoles, since they do not break any further supersymmetry and do not change substantially the properties of the background. We shall include them later by simply replacing the T^4 transverse to the D4-branes by $T^4/\mathbb{Z}_{k'}$, explicitly, substituting $ds^2_{T^4} = d\rho^2 + \rho^2\, ds^2_{\tilde{S}^3}$ with $ds^2_{T^4/\mathbb{Z}_{k'}} = d\rho^2 + \rho^2\, ds^2_{\tilde{S}^3/\mathbb{Z}_{k'}}$.

The D4-D8-D2-NS5-D6 branes set-up depicted in table 6.3 preserves four real supercharges. This is due to the presence of the D8-branes, which relate the charge distributions of the NS5 and D6 branes [218]. As we said, we are interested in a particular realization of branes reproducing locally in the UV the AdS_6 vacuum associated to the D4-D8 brane system. To this end, we consider the following 10d metric

$$\begin{aligned} ds^2_{10} = {} & H_{D4}^{-1/2} H_{D8}^{-1/2} \Big[H_{D6}^{-1/2} H_{D2}^{-1/2}\, ds^2_{\mathbb{R}^{1,1}} + H_{D6}^{1/2} H_{D2}^{1/2} H_{NS5} \big(dr^2 + r^2 ds^2_{S^2}\big) \Big] \\ & + H_{D4}^{1/2} H_{D8}^{1/2} H_{D6}^{-1/2} H_{D2}^{-1/2} H_{NS5}\, dz^2 + H_{D4}^{1/2} H_{D8}^{-1/2} H_{D6}^{-1/2} H_{D2}^{1/2} \big(d\rho^2 + \rho^2 ds^2_{\tilde{S}^3}\big) \,, \end{aligned} \tag{6.31}$$

branes	t	x^1	r	θ^1	θ^2	z	ρ	φ^1	φ^2	ϕ
D8	×	×	×	×	×	–	×	×	×	×
D4	×	×	×	×	×	–	–	–	–	–
D6	×	×	–	–	–	×	×	×	×	×
NS5	×	×	–	–	–	–	×	×	×	×
D2	×	×	–	–	–	×	–	–	–	–

Table 6.3: *Brane picture underlying the intersection of D2-NS5-D6 branes ending on the D4-D8 brane system. The system is 1/8-BPS.*

where we take the D2 and NS5-branes smeared[8] over the space transverse to the D4-branes, i.e. $H_{\rm D2} = H_{\rm D2}(r)$ and $H_{\rm NS5} = H_{\rm NS5}(r)$. Together with metric (6.31), we consider the following set of dilaton and gauge potentials

$$\begin{aligned} e^{\Phi} &= H_{\rm D8}^{-5/4} H_{\rm D4}^{-1/4} H_{\rm D6}^{-3/4} H_{\rm NS5}^{1/2} H_{\rm D2}^{1/4} \,, \\ B_{(6)} &= H_{\rm D8} H_{\rm D4} H_{\rm NS5}^{-1} \rho^3 \,{\rm vol}_{\mathbb{R}^{1,1}} \wedge d\rho \wedge {\rm vol}_{\tilde{S}^3} \,, \\ C_{(3)} &= H_{\rm D8} H_{\rm D2}^{-1} \,{\rm vol}_{\mathbb{R}^{1,1}} \wedge dz \,, \\ C_{(5)} &= H_{\rm D6} H_{\rm NS5} H_{\rm D4}^{-1} r^2 \,{\rm vol}_{\mathbb{R}^{1,1}} \wedge dr \wedge {\rm vol}_{S^2} \,, \\ C_{(7)} &= H_{\rm D4} H_{\rm D6}^{-1} \rho^3 \,{\rm vol}_{\mathbb{R}^{1,1}} \wedge dz \wedge d\rho \wedge {\rm vol}_{\tilde{S}^3} \,, \end{aligned} \tag{6.32}$$

with the $C_{(9)}$ potential for D8-branes defining the Romans mass as $F_{(0)} = m$. One can then derive the fluxes[9]

$$\begin{aligned} H_{(3)} &= \partial_r H_{\rm NS5}\, r^2 \,{\rm vol}_{S^2} \wedge dz \,, \\ F_{(0)} &= m \,, \\ F_{(2)} &= H_{\rm D8} \partial_r H_{\rm D6}\, r^2 \,{\rm vol}_{S^2} \,, \\ F_{(4)} &= H_{\rm D8} \partial_r H_{\rm D2}^{-1} \,{\rm vol}_{\mathbb{R}^{1,1}} \wedge dr \wedge dz + H_{\rm D2} H_{\rm NS5}^{-1} \partial_z H_{\rm D4}\, \rho^3\, d\rho \wedge {\rm vol}_{\tilde{S}^3} - H_{\rm D8} \partial_\rho H_{\rm D4}\, \rho^3\, dz \wedge {\rm vol}_{\tilde{S}^3} \,, \end{aligned} \tag{6.33}$$

for which the Bianchi identities for $F_{(2)}$ and $H_{(3)}$ take the form

$$\partial_z H_{\rm D8} = m \,, \qquad H_{\rm NS5} = H_{\rm D6} = H_{\rm D2} \,, \qquad \nabla^2_{\mathbb{R}^3_r} H_{\rm NS5} = 0 \,. \tag{6.34}$$

Imposing the relations (6.34), the Bianchi identities for $F_{(4)}$ and the equations of motion collapse to the equation describing the D4-D8 system [218],

$$H_{\rm D8}\, \nabla^2_{T^4} H_{\rm D4} + \partial_z^2 H_{\rm D4} = 0 \,. \tag{6.35}$$

We can finally write down a particular solution as

$$H_{\rm NS5}(r) = 1 + \frac{Q_{\rm NS5}}{r} \,, \qquad H_{\rm D6}(r) = 1 + \frac{Q_{\rm D6}}{r} \,, \qquad H_{\rm D2}(r) = 1 + \frac{Q_{\rm D2}}{r} \,, \tag{6.36}$$

where $Q_{\rm D6} = Q_{\rm D2} = Q_{\rm NS5}$ for (6.34) to be satisfied.

Let us consider now the limit $r \to 0$. In this regime the 10d background (6.31) takes the

[8]The existence of this string background has been originally discussed in [198]. Here we provide the explicit solution.

[9]We use the conventions of [218].

form[10]

$$
\begin{aligned}
ds^2_{10} &= Q^2_{\rm NS5} H_{\rm D4}^{-1/2} H_{\rm D8}^{-1/2} \left(4\, ds^2_{\rm AdS_3} + ds^2_{S^2}\right) + H_{\rm D4}^{1/2} H_{\rm D8}^{1/2}\, dz^2 + H_{\rm D4}^{1/2} H_{\rm D8}^{-1/2} \left(d\rho^2 + \rho^2 ds^2_{\tilde{S}^3}\right) ,\\
e^{\Phi} &= H_{\rm D8}^{-5/4} H_{\rm D4}^{-1/4} , \qquad H_{(3)} = -Q_{\rm NS5}\, dz \wedge {\rm vol}_{S^2} ,\\
F_{(0)} &= m ,\\
F_{(2)} &= -Q_{\rm NS5} H_{\rm D8}\, {\rm vol}_{S^2} ,\\
F_{(4)} &= 8\, Q^2_{\rm NS5} H_{\rm D8}\, {\rm vol}_{\rm AdS_3} \wedge dz + \partial_z H_{\rm D4}\, \rho^3\, d\rho \wedge {\rm vol}_{\tilde{S}^3} - H_{\rm D8} \partial_\rho H_{\rm D4}\, \rho^3\, dz \wedge {\rm vol}_{\tilde{S}^3} .
\end{aligned}
\tag{6.37}
$$

In this limit the supergravity solution describes a D4-D8 system wrapping an $\mathrm{AdS}_3 \times S^2$ geometry. As shown in [203], when the D4-D8 brane system is put in this curved background, D2-D4-NS5 branes need to be added in order to preserve supersymmetry. The number of supersymmetries is then reduced to $\mathcal{N} = (0,4)$. The AdS_3 background in (6.37) is indeed included in the classification of $\mathcal{N} = (0,4)$ $\mathrm{AdS}_3 \times S^2 \times \mathrm{CY}_2 \times I$ solutions found in [203]. In particular, it can be reproduced from the class I of AdS_3 solutions written in (3.1) of [203] for the case of $\mathrm{CY}_2 = T^4$ and $u' = 0$, after the redefinitions,

$$
H_{\rm D8} = \frac{h_8}{2Q_{\rm NS5}} , \qquad H_{\rm D4} = \frac{2^5 Q^5_{\rm NS5}}{u^2}\, h_4 \qquad \text{and} \qquad z = \frac{\tilde{\rho}}{2Q_{\rm NS5}} , \qquad \rho = \frac{u^{1/2}}{2^{3/2} Q^{3/2}_{\rm NS5}}\, \tilde{r} . \tag{6.38}
$$

As we mentioned, substituting the $\tilde{S}^3$ with the Lens space $\tilde{S}^3/\mathbb{Z}_{k'}$ in (6.37) one gets the near-horizon regime including KK'-monopoles.

A similar D2-D4-NS5-D6-D8 brane intersection to the one considered in this section was studied in [187]. This brane intersection was obtained as a generalization of the massless solution of [224] to include D8-branes. In these set-ups D2-branes are completely localized in their transverse space and the system finds an interpretation in terms of D2-D4 defect branes ending on NS5-D6-D8 branes. Consistently with this interpretation, it was shown in [187] that the corresponding $\mathcal{N} = (0,4)$ AdS_3 near-horizon geometry asymptotes locally to the AdS_7 vacuum of massive IIA supergravity [219].

6.3.2 Surface defects as 6d charged domain walls

In this section we show that the AdS_3 background (6.37) describes, in a particular limit, the AdS_6 vacuum associated to the D4-D8 system. The idea is to describe the geometry (6.37) in terms of a 6d domain wall characterized by an AdS_3 slicing and an asymptotic behaviour locally reproducing the AdS_6 vacuum. This solution was found in [187] in the context of 6-dimensional $\mathcal{N} = (1,1)$ minimal gauged supergravity (see subsection 2.5.3 for more details on the theory and its embedding in massive IIA).

We consider the following 6d background

$$
\begin{aligned}
ds^2_6 &= e^{2U(\mu)} \left(4\, ds^2_{\rm AdS_3} + ds^2_{S^2}\right) + e^{2V(\mu)} d\mu^2 ,\\
\mathcal{B}_{(2)} &= b(\mu)\, {\rm vol}_{S^2} ,\\
X_6 &= X_6(\mu) .
\end{aligned}
\tag{6.39}
$$

This background is described by the following set of BPS equations [187],

$$
U' = -2\, e^V f_6 , \qquad X_6' = 2\, e^V X_6^2\, D_X f_6 , \qquad b' = \frac{e^{U+V}}{X_6^2} , \tag{6.40}
$$

[10] We redefined the Minkowski coordinates as $(t, x^1) \mapsto 2Q^{3/2}_{\rm NS5}(t, x^1)$.

together with the duality constraint

$$b = -\frac{e^U X_6}{m}\,, \tag{6.41}$$

and the superpotential f_6 written in (2.55). This flow preserves eight real supercharges (1/2-BPS in 6d). In order to obtain an explicit solution of (6.40), a parametrization of the 6d geometry needs to be chosen. The simplest choice is given by

$$e^{-V} = 2\,X_6^2\,D_X f_6\,. \tag{6.42}$$

The system (6.40) can then be integrated out easily [187], to give

$$\begin{aligned} e^{2U} &= 2^{-1/3} g^{-2/3}\left(\frac{\mu}{\mu^4-1}\right)^{2/3}, & e^{2V} &= 8\,g^{-2}\frac{\mu^4}{(\mu^4-1)^2}\,, \\ b &= -2^{-2/3}3\,g^{-4/3}\frac{\mu^{4/3}}{(\mu^4-1)^{1/3}}\,, & X_6 &= \mu\,, \end{aligned} \tag{6.43}$$

with μ running between 0 and 1 and $m = \frac{\sqrt{2}}{3}g$. One can see that for $\mu \to 1$ the 6d background is such that

$$R_6 = -\frac{20}{3}\,g^2 + \mathcal{O}(1-\mu)^{2/3}\,, \qquad X_6 = 1 + \mathcal{O}(1-\mu)\,, \tag{6.44}$$

where R_6 is the scalar curvature. These are the curvature and scalar fields reproducing the AdS_6 vacuum (2.58). In turn, the 2-form gauge potential gives non-zero subleading contributions in this limit, which implies that the asymptotic geometry for $\mu \to 1$ is only locally AdS_6. In the opposite limit $\mu \to 0$, the 6d background is manifestly singular. This is due to the presence of the D2-NS5-D6 brane sources.

Let us consider now the truncation ansatz of massive IIA supergravity (2.53) and (2.54) for the above 6d background,

$$\begin{aligned} ds_{10}^2 &= s^{-1/3}\Sigma_6^{1/2}X_6^{-1/2}e^{2U}\left(4\,ds^2_{\mathrm{AdS}_3} + ds^2_{\tilde{S}^2}\right) + s^{-1/3}\Sigma_6^{1/2}X_6^{-1/2}e^{2V}d\mu^2 \\ &\quad + 2g^{-2}s^{-1/3}\Sigma_6^{1/2}X_6^{3/2}\,d\xi^2 + 2g^{-2}s^{-1/3}c^2\,\Sigma_6^{-1/2}X_6^{-3/2}\,ds^2_{\tilde{S}^3}\,, \\ e^{\Phi} &= s^{-5/6}\,\Sigma_6^{1/4}X_6^{-5/4}\,, \qquad H_{(3)} = s^{2/3}\,b'\,d\mu\wedge \mathrm{vol}_{S^2} + \frac{2}{3}\,s^{-1/3}\,c\,b\,d\xi\wedge\mathrm{vol}_{S^2}\,, \\ F_{(0)} &= m\,, \qquad F_{(2)} = m\,s^{2/3}\,b\,\mathrm{vol}_{S^2}\,, \\ F_{(4)} &= -\frac{4\sqrt{2}}{3}\,g^{-3}s^{1/3}c^3\,\Sigma_6^{-2}\,W_6\,d\xi\wedge\mathrm{vol}_{\tilde{S}^3} - 8\sqrt{2}\,g^{-3}s^{4/3}c^4\,\Sigma_6^{-2}X_6^{-3}X_6'\,d\mu\wedge\mathrm{vol}_{\tilde{S}^3} \\ &\quad - 8\sqrt{2}\,g^{-1}s^{1/3}c\,X_6^4\,b'\,e^{U-V}d\xi\wedge\mathrm{vol}_{\mathrm{AdS}_3} - 8\,m\,s^{4/3}b\,X_6^{-2}e^{U+V}d\mu\wedge\mathrm{vol}_{\mathrm{AdS}_3}\,, \end{aligned} \tag{6.45}$$

with $c := \cos\xi$, $s := \sin\xi$, $\Sigma_6 := c^2X_6 + s^2X_6^{-3}$ and W_6 given by (2.54). It is possible to show that the background (6.45) takes exactly the form of the near-horizon metric (6.37) by performing the change of coordinates

$$z = \frac{3\,s^{2/3}e^U X_6}{\sqrt{2}\,g\,Q_{\mathrm{NS5}}}\,, \qquad \rho = \frac{\sqrt{2}\,c\,e^{3U/2}}{g\,Q_{\mathrm{NS5}}^{3/2}X_6^{1/2}}\,, \tag{6.46}$$

and using the 6d BPS equations (6.40)-(6.41). We can thus express the warp factors describing the D4 and D8 branes in (6.37) in terms of the 6d domain wall realizing the defect as

$$H_{\mathrm{D8}} = \frac{s^{2/3}e^U X_6}{Q_{\mathrm{NS5}}}\,, \qquad H_{\mathrm{D4}} = \frac{Q_{\mathrm{NS5}}^5 e^{-5U}}{\Sigma_6}\,. \tag{6.47}$$

One can check that these expressions satisfy the equations of motion for $H_{\rm D4}$ and $H_{\rm D8}$ written in (6.35).

We have thus shown that the $\rm AdS_3$ background (6.37), describing the near-horizon limit of D2-NS5-D6 branes ending on the D4-D8 brane system, reproduces locally the $\rm AdS_6$ vacuum of [51], for $H_{\rm D4}$ and $H_{\rm D8}$ given by (6.47). This vacuum geometry comes out thanks to a non-linear mixing of the (z,ρ) coordinates that relates the near-horizon geometry to a 6d domain wall admitting $\rm AdS_6$ in its asymptotics. The presence of the 2-form does not allow, however, to globally recover the vacuum in this limit. This is seen explicitly at the level of the uplift (6.45), where one notes that the $F_{(2)}$ and $H_{(3)}$ fluxes break the isometries of the D4-D8 vacuum. This is the manifestation of the D2-NS5-D6 defect, that underlies as well the singular behaviour of the 6d domain wall in its IR regime.

6.4 Line defects in massive IIA

In analogy with our previous analysis, we show in this section that the $\rm AdS_2$ solutions to massive IIA supergravity recently constructed in [207] can be given a line defect CFT interpretation within the Brandhuber-Oz system. The solutions studied in [207] were obtained through double analytical continuation from the $\rm AdS_3 \times S^2 \times CY_2 \times I$ backgrounds constructed in [203]. We showed in the previous section that a subset of these backgrounds, the ones with $\rm CY_2 = T^4$, reproduces locally the $\rm AdS_6$ vacuum of [51], thus allowing for a surface defect interpretation. In this section we show that the solutions in [207] with $\rm CY_2 = T^4$ can be given a similar defect interpretation within the D4-D8 brane system, this time as line defects.

Following the same spirit of the previous sections, a brane solution related to the $\rm AdS_2$ geometries mentioned above was worked out in [199]. This brane solution describes a D0-F1-D4' bound state ending on D4-D8 branes, as depicted in table 6.4. As in the calculation in

branes	t	r	θ^1	θ^2	θ^3	z	ρ	φ^1	φ^2	φ^3
D8	×	×	×	×	×	–	×	×	×	×
D4	×	×	×	×	×	–	–	–	–	–
D0	×	–	–	–	–	–	–	–	–	–
F1	×	–	–	–	–	×	–	–	–	–
D4'	×	–	–	–	–	–	×	×	×	×

Table 6.4: *The brane picture of D0-F1-D4' branes ending on the D4-D8 system [199]. The intersection is 1/8-BPS.*

section 6.3.1, allowing the D4-branes to be completely localized in their transverse space, it is possible to recover a near-horizon geometry describing a D4-D8 system wrapping an $\rm AdS_3 \times S^2$ geometry, to which D0-F1-D4' branes need to be added to preserve supersymmetry [199]. The near-horizon reads

$$ds_{10}^2 = H_{\rm D4}^{-1/2} H_{\rm D8}^{-1/2} \left[Q_1 \left(ds^2_{\rm AdS_2} + 4\, ds^2_{S^3} \right) + H_{\rm D4} H_{\rm D8}\, dz^2 + H_{\rm D4} \left(d\rho^2 + \rho^2\, ds^2_{\tilde{S}^3} \right) \right] , \qquad (6.48)$$

with Q_1 a parameter related to the defect charges of D0-F1-D4' branes. One can check that this background is included in the classification found in (5.1) of [207], for $\rm CY_2 = T^4$ and $u' = 0$, after the redefinitions given by (6.38).

Moreover, the previous brane intersection was related in [199] to a 6d charged domain wall characterized by an $\rm AdS_2$ slicing flowing asymptotically to the $\rm AdS_6$ vacuum of 6d Romans supergravity (see subsection 2.5.3). This domain wall is of the form

$$
\begin{aligned}
ds_6^2 &= e^{2U(\mu)}\left(ds^2_{\mathrm{AdS}_2} + 4\,ds^2_{S^3}\right) + e^{2V(\mu)}d\mu^2\,,\\
\mathcal{B}_{(2)} &= b(\mu)\,\mathrm{vol}_{\mathrm{AdS}_2}\,,\\
X_6 &= X_6(\mu)\,,
\end{aligned}
\tag{6.49}
$$

and, consistently with the whole picture, can be obtained through double analytical continuation from the domain wall solution in (6.39). The BPS equations for this background preserve eight real supercharges and take the same form of (6.40) and (6.41). In analogy with the AdS_3 analysis, the 6d solution (6.49) reproduces locally in the limit $\mu \to 1$ the geometry of the AdS_6 vacuum, together with a singularity in the $\mu \to 0$ limit. Using the uplift formulae to massive IIA given in (2.53), one can check that the resulting domain wall solution in 10d is related to the near-horizon geometry (6.48) through the change of coordinates [199]

$$
z = \frac{3\,s^{2/3}e^{U}X_6}{\sqrt{2}\,g\,Q_1^{1/2}}\,, \qquad \rho = \frac{\sqrt{2}\,c\,e^{3U/2}}{g\,Q_1^{3/4}X_6^{1/2}}\,,
\tag{6.50}
$$

and the requirements for the H_{D4} and H_{D8} functions

$$
H_{\mathrm{D8}} = \frac{s^{2/3}e^{U}X_6}{Q_1^{1/2}}\,, \qquad H_{\mathrm{D4}} = \frac{Q_1^{5/2}e^{-5U}}{\Sigma_6}\,.
\tag{6.51}
$$

These conditions are analogous to (6.46)-(6.47) for AdS_3, which is obviously related to the fact that the AdS_2 solutions and the AdS_3 backgrounds discussed in the previous section are related by double analytical continuation. In this case the solution is interpreted as a D0-F1-D4' line defect within the 5d $\mathrm{Sp}(N)$ fixed point theory.

Chapter 7

Exact charges from heterotic black holes

The fundamental objects of string theory may carry several types of charge. A well-known example is given by a D5-brane in type-IIB theory wrapped on a compact manifold which, besides a unit of D5-brane charge [6][1], carries $-\chi$ units of D1-brane charge as well, where χ is the Euler characteristic of the wrapped space divided by 24 [225]. The somewhat unexpected D1 charge emerges from a quantum correction, which can be read from the three-point function of the R-R 2-form with emission of two gravitons. As originally noted, the shift is necessary for consistency of string duality and the fact that the energy of the left-moving ground state of heterotic string starts at -1. Moreover, the shift must be taken into account for the computation of the degeneracy. If the D5-brane is part of a bound system that can be described as a black hole in the low-energy limit, the D1-brane charge it carries is fundamental to match the microscopic degeneracy with the macroscopic entropy [226]. It is worth emphasizing that the D1 charge is not intrinsic to the D5-brane itself, but depends on the background on which the brane is located. Other examples of similar shifts include [32, 213, 227–230].

Analogous effects appear in the so-called four- and three-charge systems, bound states of an F1-string with winding number w and momentum n, N solitonic 5-branes and, in the first option, a KK-monopole of charge W. For sufficiently large n, w, N, W, when g_s is small but non-vanishing, the gravitational interaction produces the collapse of the systems and these can be described as supersymmetric black holes with four and five non-compact dimensions in terms of classical supergravity fields, at least outside the event horizon. These are arguably the simplest black hole systems that can be considered in string theory (see, e.g., [7, 231–233] and references therein) and, in fact, are the ones we shall study in this chapter. Using type-II/heterotic string duality, it is possible to compare the microscopic degeneracy of the system at vanishing string coupling computed in the former theory with the entropy associated to the horizon of the black hole solution of the latter. Being BPS, the degeneracy of the system is protected under variations of g_s. The precise matching of both quantities constitutes a major achievement of the theory. While the agreement was first revealed for the leading order contribution, subsequent works concluded that it extends to all orders in the α' expansion, see [32, 226, 234] and references therein. In the black hole description, α'-corrections arise in the form of higher-curvature terms added to the effective action, making the analysis more involved.

This complex problem was firstly tackled by means of the attractor mechanism, which was cleverly exploited to decouple the near-horizon region from the rest of the spacetime and to study some of its properties, including the entropy [228, 235–239]. In recent years there has been a renewed interest in exploring this territory [28, 240–242], which allowed to shed some light on relevant aspects that were left aside. The first perturbative corrections beyond the near-horizon region have been obtained for the three- and four-charge systems. Besides the modifications to the fields, which will be subjected to further corrections order by order in the α' expansion, it

[1] By Dp-brane charge we mean the electric charge associated to the R-R $(p+1)$-form.

emerged that the charges (and mass) associated to some of the stringy constituents suffer a shift mediated by the higher-curvature interactions. The interpretation is clear: the α'-corrections modify the equations of motion order by order, introducing delocalized sources with a non-abelian character. Nevertheless, still unknown is whether or not it is possible to derive the value of these shifts when all the α'-corrections have been accounted for.

The structure of the chapter goes as follows. In section 7.1 we briefly review the existing perturbative three- and four-charge black holes solutions of the first-order in α' heterotic theory. Section 7.2 is devoted to the computation of the Wald entropy of both systems and the derivation of the shifts in the black hole charges; the core results are presented in subsection 7.2.3. In section 7.3 we make contact with previous works in the literature that studied the near-horizon regions using lower-dimensional effective actions. Moreover, we consider the Gauss-Bonnet term and discuss why its inclusion is unable to reproduce the relevant properties of the three-charge system [243], while it succeeds for the four-charge black hole [244].

7.1 α'-corrected heterotic black holes

Shifts in the black hole charges are driven by higher-order α'-corrections, which, thus, must be retained to some extent. In this section we shall consider heterotic supergravity up to first order in α', already presented in section 2.4, to which we refer for the action and the equations of motion. Given that we shall deal with black holes of sufficiently large horizon, the supergravity approximation, i.e. small α' and g_s, will be valid.

Since it plays a fundamental role in our discussion, here we stress that any solution to the equations of motion (2.45a)-(2.45c) satisfying ${R_{(-)}}^a{}_b = 0$ is also a solution of the zeroth-order supergravity theory, which can be recovered setting $\alpha' = 0$. Indeed, for the families of supersymmetric black holes that we shall consider ${R_{(-)}}^a{}_b$ vanishes in the near-horizon limit, while it is non-zero in the exterior region of the black hole. Therefore, the higher-curvature corrections do not alter the fields at the horizon, although they introduce modifications in the external region interpolating to asymptotic infinity.

7.1.1 Four-charge black hole

A perturbative solution to first order in α' of the equations of motion of the heterotic supergravity with quadratic curvature terms, namely (2.44) and (2.45a)-(2.45c), was found in [240, 241]. The fields are expressed in terms of four functions $\mathcal{Z}_{\pm,0}$ and $\mathcal{V}$,

$$
\begin{aligned}
ds^2 &= \frac{2}{\mathcal{Z}_-}\, du \left(-dt + \frac{1}{2}\mathcal{Z}_+\, du\right) + \mathcal{Z}_0\, d\sigma_4^2 + d\vec{y}_4^{\,2}\,,\\
e^{-2\Phi} &= g_s^{-2}\frac{\mathcal{Z}_-}{\mathcal{Z}_0}\,,\\
H &= d\mathcal{Z}_-^{-1}\wedge du \wedge dt + \star_4 d\mathcal{Z}_0\,,
\end{aligned}
\tag{7.1}
$$

where the Hodge dual in the last equation is associated to the 4-dimensional metric $d\sigma_4^2$, which is a *Gibbons-Hawking (GH) space*:

$$
d\sigma_4^2 = \mathcal{V}^{-1}(dz+\chi)^2 + \mathcal{V}\, d\vec{x}_3^{\,2} \qquad \text{with} \qquad d\mathcal{V} = \star_3 d\chi\,. \tag{7.2}
$$

It is further assumed that $\mathcal{Z}_{\pm,0}$ and $\mathcal{V}$ only depend on the coordinates $\vec{x}_3$ that parametrize $\mathbb{R}^3$. Before specifying a precise form for these functions, these expressions describe a field configuration preserving four supercharges whose compactification in the u coordinate yields a static spacetime[2]. A spherically symmetric (in $\vec{x}_3$) solution to the equations of motion is given by

$$\begin{aligned}
\mathcal{V} &= 1 + \frac{q_v}{r}\,, \\
\mathcal{Z}_- &= 1 + \frac{q_-}{r}\,, \\
\mathcal{Z}_0 &= 1 + \frac{q_0}{r} - \alpha'\big[F(r;q_0) + F(r;q_v)\big]\,, \\
\mathcal{Z}_+ &= 1 + \frac{q_+}{r} + \frac{\alpha' q_+}{2q_v q_0}\frac{r^2 + r(q_0+q_-+q_v) + q_v q_0 + q_v q_- + q_0 q_-}{(r+q_v)(r+q_0)(r+q_-)}\,,
\end{aligned} \tag{7.3}$$

where

$$F(r;k) := \frac{(r+q_v)(r+2k)+k^2}{4q_v(r+q_v)(r+k)^2}\,. \tag{7.4}$$

Again, one can recover the solution to the zeroth-order supergravity theory simply setting $\alpha' = 0$, obtaining four harmonic functions. The corrections to the harmonic leading terms are in all cases finite and their absolute value is monotonically decreasing. In the near-horizon limit, $r \to 0$, when the corrections take their largest absolute value, their effective contribution is actually zero. The harmonic poles of the zeroth-order solution are responsible for the existence of this well-known decoupling regime. Therefore, the near-horizon solution is unaltered by the correction. Another way to understand this important fact is to study the near-horizon solution in its own, which reads

$$\begin{aligned}
ds^2 &= \frac{2r}{q_-}\,du\left(-dt + \frac{q_+}{2r}\,du\right) + q_0 q_v\left[\frac{dr^2}{r^2} + d\theta^2 + \sin^2\theta\,d\varphi^2 + \left(\frac{dz}{q_v} + \cos\theta\,d\varphi\right)^2\right] + d\vec{y}_4^{\,2}\,, \\
e^{-2\Phi} &= g_s^{-2}\frac{q_-}{q_0}\,, \\
H &= \frac{1}{q_-}\,dr\wedge du\wedge dt + q_0\sin\theta\,d\theta\wedge dz\wedge d\varphi\,.
\end{aligned} \tag{7.5}$$

In the supergravity picture this background represents a 4-dimensional black hole with compact transverse space $T^4 \times S^1 \times \hat{S}^1$. The explicit computation of the curvature of the torsionful spin connection for the near-horizon solution yields $R_{(-)}{}^a{}_b = 0$. Then, as previously stated, (7.5) remains the same in the truncation to the supergravity approximation.

The identification of the q_i parameters in terms of localized, fundamental objects of string theory has been performed in [241]. From the preceding discussion, one sees that such relations can be obtained using the standard techniques on the near-horizon solution of the simpler supergravity theory. The result is

$$q_+ = \frac{\alpha'^2 g_s^2 n}{2R_z R_u^2}\,, \qquad q_- = \frac{\alpha' g_s^2 w}{2R_z}\,, \qquad q_0 = \frac{\alpha' N}{2R_z}\,, \qquad q_v = \frac{W R_z}{2}\,. \tag{7.6}$$

The system describes:

- a string wrapping the circle S^1 parametrized by $u \in (0, 2\pi R_u)$ with winding number w and momentum n,
- a stack of N NS5-branes wrapped on $T^4 \times S^1$,
- a KK-monopole of charge W associated to the circle $\hat{S}^1$ parametrized by $z \in (0, 2\pi R_z)$.

[2] To describe the most general field configuration with these properties, $d\sigma_4^2$ is taken as a generic hyper-Kähler space on which $\mathcal{Z}_{\pm,0}$ vary.

The constituents have four types of charge associated. While w and N behave, respectively, as electric and magnetic localized sources of Kalb-Ramond charge, n and W correspond to momentum carried along the corresponding compact circles. Additionally, the higher-curvature terms induce self-interactions that behave as delocalized charge sources. For the system studied, the non-vanishing terms responsible for this effect occur at the Bianchi identity (2.44) and the (uu) component of the Einstein equation (2.45a), which produce deviations of the functions $\mathcal{Z}_{0,+}$ from the leading harmonic term. They introduce NS5-brane and string momentum charge densities distributed in the exterior of the black hole horizon. The charge contained inside a sphere of radius r^* is $Q_{i,r^*} \sim r^2\partial_r \mathcal{Z}_i|_{r=r^*}$. The total, asymptotic charges are[3]

$$Q_+ = n + \frac{2n}{NW}\,, \qquad Q_- = w\,, \qquad Q_0 = N - \frac{2}{W}\,, \qquad Q_v = W\,. \tag{7.7}$$

The computation of the ADM mass of the black hole yields

$$M = \frac{1}{R_u}\left(n + \frac{2n}{NW}\right) + \frac{R_u}{\ell_s^2}\,w + \frac{R_u}{g_s^2\ell_s^2}\left(N - \frac{2}{W}\right) + \frac{R_z^2 R_u}{g_s^2\ell_s^4}\,W\,. \tag{7.8}$$

Being supersymmetric and, hence, extremal, the mass of the black hole coincides with the sum (up to moduli factors) of the four charges associated to the constituents. This computation reveals that the charge-to-mass ratio of these configurations is not modified by higher-curvature corrections, a behaviour that has been argued to occur in non-supersymmetric extremal black holes [245–247].

In first instance, additional higher-curvature corrections will behave as new delocalized charge sources, modifying the explicit expressions of the functions in (7.3) and, presumably, the asymptotic charges Q_i and ADM mass M. However, it was shown in [242] that the asymptotic NS5-brane charge Q_0 is protected under further corrections. In section 7.2 we review this result and obtain exact relations for the rest of the charges in the α' expansion.

7.1.2 Three-charge black hole

A simpler black hole solution can be described if the KK-monopole is removed from the previous configuration. The field structure in (7.1) is preserved, while the 4-dimensional hyper-Kähler manifold is simply $\mathbb{R}^4$,

$$d\sigma_4^2 = d\rho^2 + \rho^2 ds^2_{S^3}\,. \tag{7.9}$$

This particular case can also be described as a Gibbons-Hawking space with $\mathcal{V} = R_z/(2r)$, introducing a new radial variable $r = \rho^2/(2R_z)$. Then, the near-horizon geometry is identical to that of the four-charge system with $W = 1$. The complete solution reads

$$\begin{aligned}
\mathcal{Z}_- &= 1 + \frac{\tilde{q}_-}{\rho^2}\,,\\
\mathcal{Z}_0 &= 1 + \frac{\tilde{q}_0}{\rho^2} - \alpha'\frac{\rho^2 + 2\tilde{q}_0}{(\rho^2+\tilde{q}_0)^2}\,,\\
\mathcal{Z}_+ &= 1 + \frac{\tilde{q}_+}{\rho^2} + \frac{2\alpha'\tilde{q}_+}{\tilde{q}_0}\frac{\rho^2 + \tilde{q}_0 + \tilde{q}_-}{(\rho^2+\tilde{q}_0)(\rho^2+\tilde{q}_-)}\,,
\end{aligned} \tag{7.10}$$

[3] We normalize the charges such that they are independent of the moduli.

where we introduced $\tilde{q}_i := 2R_z q_i$ for convenience. The near-horizon solution is

$$\begin{aligned} ds^2 &= \frac{2\rho^2}{\tilde{q}_-} du\left(-dt + \frac{\tilde{q}_+}{2\rho^2} du\right) + \tilde{q}_0\left[\frac{d\rho^2}{\rho^2} + \frac{1}{4}\left(d\theta^2 + d\psi^2 + d\varphi^2 + 2\cos\theta\, d\varphi d\psi\right)\right] + d\vec{y}_4^{\,2}\,, \\ e^{-2\Phi} &= g_s^{-2}\frac{\tilde{q}_-}{\tilde{q}_0}\,, \\ H &= \frac{2\rho}{\tilde{q}_-} d\rho\wedge du\wedge dt + \frac{\tilde{q}_0}{4}\sin\theta\, d\theta\wedge d\psi\wedge d\varphi\,, \end{aligned} \tag{7.11}$$

with $\psi = 2z/R_z$. The interpretation of this background as a 5-dimensional black hole with $T^4 \times S^1$ transverse space is direct and similar to the 4-dimensional case goes its description in terms of the stringy constituents, but without KK-monopole. The $\tilde{q}_i$ parameters are

$$\tilde{q}_+ = \frac{\alpha'^2 g_s^2 n}{R_u^2}\,, \qquad \tilde{q}_- = \alpha' g_s^2 w\,, \qquad \tilde{q}_0 = \alpha' N\,, \tag{7.12}$$

in agreement with (7.6). The total, asymptotic charges are

$$Q_+ = n + \frac{2n}{N}\,, \qquad Q_- = w\,, \qquad Q_0 = N - 1\,. \tag{7.13}$$

Likewise, the mass of the solution is of the form of (7.8) after taking into consideration the expressions for the three charges of the solution (7.13).

7.2 Exact entropy and charges in the α' expansion

In this section we compute the Wald entropy of these black holes. As already mentioned, the near-horizon solution is unaltered by the addition of quadratic terms in curvature and, moreover, it is expected to be invariant under further higher-curvature corrections. Moreover, due to the presence of an AdS factor in the near-horizon geometry, the Wald entropy remains unmodified beyond first order in α' [248, 249]. Then, it is possible to compare this result with α'-exact computations of the degeneracy obtained from microscopic counting.

7.2.1 Rewriting of the action

The presence of Chern-Simons terms in the Kalb-Ramond field strength H has been recognized to hamper the direct application of Wald's entropy formula to the action. The reason is that, even if the theory is invariant under anomalous Lorentz gauge transformations, it is difficult to express the functional dependence of H on the Riemann curvature tensor in a manifestly covariant manner. For this reason, following [32, 238] among others, it is convenient to rewrite the action in a classically equivalent manner in terms of the dual of this field strength, whose Bianchi identity is not anomalous. Such transformation involves the addition of total derivative terms which leave the entropy invariant, according to [250], and can therefore be applied for this purpose.

In first place, we perform a (trivial) dimensional reduction of the action to six dimensions by compactifying on T^4 and truncating all the Kaluza-Klein modes. The solutions we consider are of course consistent with this truncation. We obtain

$$S = \frac{g_s^2}{16\pi G_N^{(6)}}\int d^6x\sqrt{-g}\, e^{-2\Phi}\left(R + 4|\partial\Phi|^2 - \frac{1}{2}|H|^2 - \frac{\alpha'}{8}R_{(-)\mu\nu}{}^{a}{}_{b}R_{(-)}{}^{\mu\nu b}{}_{a} + \ldots\right), \tag{7.14}$$

where $G_N^{(6)} = G_N^{(10)}/\mathrm{Vol}(T^4)$. We now introduce the dual 3-form field strength $\tilde{H} = d\tilde{B}$ as $\tilde{H} := e^{-2\Phi} \star H$ and define the equivalent action

$$\widetilde{S} = S + \frac{g_s^2}{16\pi G_N^{(6)}} \int \left(\tilde{H} \wedge H - \frac{\alpha'}{4} \tilde{H} \wedge \Omega^{\mathrm{L}}_{(-)} \right), \tag{7.15}$$

in which $\tilde{B}$ is considered a fundamental field, while H is now an auxiliary field. The equation of motion for $\tilde{B}$ yields

$$d\left(H - \frac{\alpha'}{4} \Omega^{\mathrm{L}}_{(-)} \right) = 0 \,, \tag{7.16}$$

whose general solution is of the form (2.42). On the other hand, the dual Bianchi identity $d\tilde{H} = 0$ is equivalent to (2.45b). It is straightforward to check that the remaining equations of motion obtained taking H as an auxiliary field are identical to those derived from the original action (7.14).

In this form, the modified Lagrangian is manifestly covariant except for the explicit presence of the Chern-Simons 3-form in the last term of (7.15). The next convenient step is to decompose the Chern-Simons 3-form into a standard Chern-Simons 3-form constructed from the Levi-Civita connection and an additional contribution,

$$\Omega^{\mathrm{L}}_{(-)} = \Omega^{\mathrm{L}} + \mathscr{A} \,, \tag{7.17}$$

where Ω^{L} is the standard Lorentz Chern-Simons term, defined as in (2.43), but in terms of the spin connection ${\omega^a}_b$, and

$$\mathscr{A} = \frac{1}{2} d({\omega^a}_b \wedge {H^b}_a) + \frac{1}{4} {H^a}_b \wedge D{H^b}_a - {R^a}_b \wedge {H^b}_a + \frac{1}{12} {H^a}_b \wedge {H^b}_c \wedge {H^c}_a \,, \tag{7.18}$$

where ${H^a}_b = {H_\mu}^a{}_b dx^\mu$ and D is the covariant derivative operator, whose action on ${H^a}_b$ is $D{H^a}_b = d{H^a}_b + {\omega^a}_c \wedge {H^c}_b - {\omega^c}_b \wedge {H^a}_c$. Once plugged in the action, the first term in the above expression becomes a total derivative, so it does not enter the equations of motion or the Wald entropy. Once this term is eliminated, the contribution from $\mathscr{A}$ is manifestly covariant.

Finally, the standard Lorentz Chern-Simons term can also be written in a manifestly covariant form by exploiting the isometries of the spacetimes considered [251]. From (7.25) one sees that, after compactifying on T^4, the 6-dimensional spacetime can be described as the product of two 3-dimensional spaces of the form

$$ds_3^2 = \lambda^2 \left[{}^{(2)}\bar{g}_{mn} dx^m dx^n + \left(dy + \bar{A}_m dx^m \right)^2 \right] , \qquad m, n = 0, 1 \,, \tag{7.19}$$

with (x^0, x^1, y) corresponding to the coordinates (t, r, u) and (θ, φ, z), respectively. The dual 3-form $\tilde{H}$ also factorizes in these two spaces. Hence, the remaining term in the action splits in two portions

$$\tilde{H} \wedge \Omega^{\mathrm{L}} = \tilde{H}_A \wedge \Omega^{\mathrm{L}}_B - \Omega^{\mathrm{L}}_A \wedge \tilde{H}_B \,, \tag{7.20}$$

where the A, B indices refer to the two different 3-dimensional spaces. From this point, we continue the rewriting of the action distinguishing between the two families of solutions that we consider. For the four-charge family, the periodic coordinates u and z parametrize paths of finite length. The Lorentz Chern-Simons 3-form of a space of the form (7.19) can be locally written as [252][4]

$$\Omega^{\mathrm{L}} = \frac{\bar{\varepsilon}^{mn}}{2} \left[{}^{(2)}\bar{R}\, \bar{F}_{mn} + \bar{F}_{mp} \bar{F}^{pq} \bar{F}_{qn} - \partial_m ({}^{(2)}\bar{\omega}_n{}^{ab} \bar{F}_{ab}) \right] dx^0 \wedge dx^1 \wedge dy \,, \tag{7.21}$$

[4]Since there are two different 3-dimensional spaces, there are two copies of each of the elements λ, ${}^{(2)}\bar{g}_{mn}$, $\bar{A}_m$ and so on. In order to simplify the notation we have avoided the introduction of yet another index labelling these copies.

where objects with a bar are associated to the metric ${}^{(2)}\bar{g}_{mn}$, $\bar{F} = d\bar{A}$ and $\bar{\varepsilon}^{01} = +1$.

Once again we observe that, after dropping the last term which contributes as a total derivative, we are left with a manifestly covariant expression to which we can apply Wald's formula. When doing so, the conformal factor in front of the 2-dimensional metric must be taken into account. In particular, the relation between the spacetime and auxiliary metrics, ${}^{(2)}g_{mn} = {}^{(2)}\bar{g}_{mn}\lambda^2$, implies

$$ {}^{(2)}\bar{R} = {}^{(2)}R\,\lambda^2 + 2\nabla^2 \ln\lambda\,. \tag{7.22}$$

The treatment of the three-charge family of solutions is a bit simpler. In this case, the 3-dimensional space parametrized by (θ, φ, z) is a 3-sphere, with the coordinate z parametrizing paths of infinite length at asymptotic spatial infinity. The Lorentz Chern-Simons form of a 3-sphere identically vanishes when evaluated from its definition. Hence, the first term in expression (7.20) is just zero in the three-charge family of solutions. Notice that the decomposition (7.19) becomes singular asymptotically, hence it cannot be used to rewrite this term of the action.

Therefore, we see that topological properties of the asymptotic space make a difference in the explicit expression of the manifestly covariant action. This fact plays a crucial role in the study of these black holes from the near-horizon solution, as described in section 7.3.

7.2.2 Wald entropy

The *Wald entropy* formula for a D-dimensional theory is[5]

$$ \mathbb{S} = -2\pi \int_{\Sigma} d^{D-2}x\sqrt{h}\,\mathcal{E}^{abcd}\epsilon_{ab}\epsilon_{cd}\,, \tag{7.23}$$

where Σ is a cross-section of the horizon, h is the determinant of the metric induced on Σ, ϵ_{ab} is the binormal to Σ with normalization $\epsilon_{ab}\epsilon^{ab} = -2$ and $\mathcal{E}^{abcd}$ is the equation of motion one would obtain for the Riemann tensor R_{abcd} treating it as an independent field of the theory, namely

$$ \mathcal{E}^{abcd} = \frac{g_s^2}{16\pi G_{\mathrm{N}}^{(D)}}\frac{\delta\mathscr{L}}{\delta R_{abcd}}\,, \tag{7.24}$$

where $\mathscr{L}$ is the Lagrangian of the theory.

When first proposed, Wald's entropy formula was meant to be evaluated at the bifurcation surface of the event horizon [253], so it was only defined for non-extremal black holes. In subsequent work [254], it was shown that the expression (7.23) can still be used for any cross-section of the horizon Σ, provided the surface gravity is non-vanishing. One way to understand the origin of this condition is to notice that, in the derivation of the formula, the null Killing vector that generates the horizon ξ^{μ} is normalized to have unit surface gravity. This Killing vector does not appear explicitly in (7.23), whose position is taken by the binormal upon the use of $\mathcal{E}^{abcd}\epsilon_{ab}\epsilon_{cd} = \mathcal{E}^{abcd}\nabla_a\xi_b\nabla_c\xi_d$. When expressed in the form of (7.23), Wald's entropy formula can also be evaluated for extremal black holes.

We can apply this formula to the action of the heterotic theory directly in six dimensions, after performing a trivial compactification on T^4. It is convenient to rewrite the metric as

$$ ds_6^2 = e^{\Phi-\Phi_\infty}\left[\left(\frac{k}{k_\infty}\right)^{-2/3} ds_5^2 + \left(\frac{k}{k_\infty}\right)^2\left(du - \frac{dt}{\mathcal{Z}_+}\right)^2\right]\,, \tag{7.25}$$

[5]Here and in the rest of the chapter we shall denote the entropy with the letter $\mathbb{S}$ in order to distinguish it from the action S.

where the lower-dimensional line elements, the dilaton Φ and the Kaluza-Klein scalars k and ℓ are

$$\begin{aligned} ds_5^2 &= \left(\frac{\ell}{\ell_\infty}\right)^{-1} ds_4^2 + \left(\frac{\ell}{\ell_\infty}\right)^2 (dz+\chi)^2\,, \\ ds_4^2 &= -e^{2U}dt^2 + e^{-2U}\left(dr^2 + r^2 ds_{S^2}^2\right)\,, \\ e^{2\Phi} &= e^{2\Phi_\infty}\frac{\mathcal{Z}_0}{\mathcal{Z}_-}\,, \qquad k = k_\infty \frac{\mathcal{Z}_+^{1/2}}{\mathcal{Z}_0^{1/4}\mathcal{Z}_-^{1/4}}\,, \qquad \ell = \ell_\infty \frac{\mathcal{Z}_0^{1/6}\mathcal{Z}_+^{1/6}\mathcal{Z}_-^{1/6}}{\mathcal{V}^{1/2}}\,, \end{aligned} \tag{7.26}$$

with $e^{\Phi_\infty} = g_s$, the value of the dilaton in the vacuum, and

$$e^{-2U} = \sqrt{\mathcal{Z}_0\,\mathcal{Z}_+\,\mathcal{Z}_-\mathcal{V}}\,. \tag{7.27}$$

For a four-charge configuration, ds_4^2 is the 4-dimensional metric in the Einstein frame, while Φ, k and ℓ provide a parametrization of the three scalars, which are real in the solution considered. The volume form entering Wald's formula is

$$d^4x\sqrt{h} = d\theta\, d\varphi\, dz\, du\, e^{2(\Phi-\Phi_\infty)}\sqrt{\mathcal{Z}_0\,\mathcal{Z}_+\,\mathcal{Z}_-\mathcal{V}}\, r^2\sin\theta\,. \tag{7.28}$$

In order to compute the integrand it is convenient to use flat indices. We define the sechsbein

$$\begin{aligned} e^0 &= e^{\frac{\Phi-\Phi_\infty}{2}}\left(\frac{k}{k_\infty}\right)^{-\frac{1}{3}}\left(\frac{\ell}{\ell_\infty}\right)^{-1/2} e^U dt\,, & e^1 &= e^{\frac{\Phi-\Phi_\infty}{2}}\left(\frac{k}{k_\infty}\right)^{-\frac{1}{3}}\left(\frac{\ell}{\ell_\infty}\right)^{-1/2} e^{-U} dr\,, \\ e^2 &= e^{\frac{\Phi-\Phi_\infty}{2}}\left(\frac{k}{k_\infty}\right)^{-\frac{1}{3}}\left(\frac{\ell}{\ell_\infty}\right)^{-1/2} e^{-U} r\, d\theta\,, & e^3 &= e^{\frac{\Phi-\Phi_\infty}{2}}\left(\frac{k}{k_\infty}\right)^{-\frac{1}{3}}\left(\frac{\ell}{\ell_\infty}\right)^{-1/2} e^{-U} r\sin\theta\, d\varphi\,, \\ e^4 &= e^{\frac{\Phi-\Phi_\infty}{2}}\left(\frac{k}{k_\infty}\right)^{-\frac{1}{3}}\frac{\ell}{\ell_\infty}(dz+\chi)\,, & e^5 &= e^{\frac{\Phi-\Phi_\infty}{2}}\frac{k}{k_\infty}\left(du - \frac{dt}{\mathcal{Z}_+}\right)\,. \end{aligned} \tag{7.29}$$

In this frame, the non-vanishing components of the binormal are $\epsilon_{01} = -\epsilon_{10} = 1$.

The variation of the Lagrangian with respect to the Riemann tensor contains three non-vanishing contributions. The first one comes from the Einstein-Hilbert term in (7.14), which amounts to

$$\mathcal{E}_0^{abcd} = \frac{g_s^2}{16\pi G_N^{(6)}}\frac{\delta}{\delta R_{abcd}}\left(e^{-2\Phi}R\right) = \frac{e^{-2(\Phi-\Phi_\infty)}}{16\pi G_N^{(6)}}\eta^{ac}\eta^{bd}\,. \tag{7.30}$$

This term is responsible for the Bekenstein-Hawking entropy $\mathbb{S}_0 = A_\Sigma/4G_N^{(6)}$, which for large black holes gives the leading contribution to the entropy. The two additional contributions arise from the variation of the Chern-Simons 3-form in the last term of (7.15), each one coming from one of the two factors in the decomposition (7.17). Notice that the last term in (7.14) gives no contribution to the entropy, since it is quadratic in the curvature of the torsionful spin connection, which vanishes at the horizon. By means of the rewriting performed in the previous section, in first place we get

$$\mathcal{E}_1^{abcd} = \frac{g_2^2}{16\pi G_N^{(6)}}\frac{\delta}{\delta R_{abcd}}\left(-\frac{\alpha'}{4(3!)^2}\varepsilon^{efghjk}\tilde{H}_{efg}\mathscr{A}_{hjk}\right) = \frac{e^{-2(\Phi-\Phi_\infty)}}{16\pi G_N^{(6)}}\frac{\alpha'}{8}H^{abf}H_f{}^{cd}\,. \tag{7.31}$$

To obtain the last correction to the entropy, we notice that when $\mathcal{E}^{abcd}$ gets contracted with the binormal, the only relevant values of the flat indices are 0 and 1. Therefore, the remaining non-vanishing contribution to the entropy comes from the second term in the decomposition (7.20) and amounts to

$$\mathcal{E}_2^{abcd} = \frac{g_s^2}{16\pi G_N^{(6)}}\frac{\delta}{\delta R_{abcd}}\left(-\frac{\alpha'\,\varepsilon^{\mu\nu\rho\alpha\beta\gamma}}{4(3!)^2\sqrt{-g}}\tilde{H}_{\mu\nu\rho}\Omega^{\mathrm{L}}_{\alpha\beta\gamma}\right) = \frac{e^{-2(\Phi-\Phi_\infty)}}{16\pi G_N^{(6)}}\frac{\alpha'}{4}H^{tru}\eta^{ac}\eta^{bd}\lambda^2\bar{F}_{tr}\,, \tag{7.32}$$

where t, r, u are curved indices, $\lambda = e^{\frac{\Phi-\Phi_\infty}{2}}(k/k_\infty) = \sqrt{\mathcal{Z}_+/\mathcal{Z}_-}$ and $\bar{A}_t = -\mathcal{Z}_+^{-1}$.

Putting everything together, Wald's entropy is

$$\mathbb{S} = \frac{1}{4G_{\rm N}^{(6)}} \int d\theta\, d\varphi\, dz\, du \sqrt{q_0 q_+ q_- q}\, \sin\theta \left[1 + \frac{\alpha'}{4}\left(-H^{01f} H_f{}^{01} + H^{tru}\lambda^2 \tilde{F}_{tr}\right)\right] . \tag{7.33}$$

The relevant components of the Kalb-Ramond field strength, in flat and curved indices, are

$$H_{015} = -(\mathcal{Z}_0\mathcal{V})^{-1/2}\partial_r \ln \mathcal{Z}_- \,, \qquad H^{tru} = -(\mathcal{Z}_0\mathcal{V})^{-1}\partial_r \mathcal{Z}_- \,. \tag{7.34}$$

Substituting these values in the expression and integrating,

$$\mathbb{S} = \frac{\pi}{G_{\rm N}^{(4)}}\sqrt{q_0 q_+ q_- q}\left(1 + \frac{\alpha'}{2q_0 q}\right) , \tag{7.35}$$

with the 4-dimensional Newton constant given by

$$G_{\rm N}^{(4)} = \frac{G_{\rm N}^{(10)}}{(2\pi R_z)(2\pi R_u)(2\pi\ell_s)^4} = \frac{8\pi^6\alpha'^4 g_s^2}{(2\pi R_z)(2\pi R_u)(2\pi\ell_s)^4} \,. \tag{7.36}$$

Using the relation between the charge parameters q_i and the number of fundamental objects in the system, we finally get

$$\mathbb{S} = 2\pi\sqrt{nwNW}\left(1 + \frac{2}{NW}\right) . \tag{7.37}$$

In recent papers, Wald's formula was applied, but keeping the action written in terms of the KR 2-form H [28, 241]. In these attempts, the authors obtained a correction to the Bekenstein-Hawking term that accounts for half of the total value we derived and interpreted it as the first term of an infinite series expansion of $\sqrt{1 + \frac{2}{NW}}$ for large NW. The entropy in (7.37) shows that such explanation is not correct.

The entropy of the three-charge system is obtained by setting $W = 1$ in this expression since, as we previously noted, the near-horizon solution is identical to that of a four-charge black hole with unit KK-monopole charge.

7.2.3 Corrected charges

We have obtained an expression for the Wald entropy of these families of black holes in terms of the number of fundamental objects of the solution. The result has a clear interpretation: the Chern-Simons term, which is needed for anomaly cancellation, is the sole responsible of the increase in the entropy with respect to the Bekenstein-Hawking term. The near-horizon background remains unperturbed under the curvature corrections of quadratic order, and thus the area of the event horizon is unchanged. This is a consequence of the supersymmetric structure of the theory (and the solutions), which restricts the functional form of the corrections to objects constructed from the curvature of the torsionful spin connection, which vanishes for this background [31]. The non-renormalization of the near-horizon solution has an alternative, yet equivalent, explanation: the central charges of the dual CFT can be computed from the analysis of the anomalies of the theory [249], which are fully described at first order in α'. This phenomenon is also consistent with non-renormalization arguments for black hole entropy in the context of 4-dimensional supergravity [255].

The Wald entropy can be compared with the microscopic degeneracy of the string theory system it represents, whose value is known to all orders in the α' expansion. For the four-charge solution it reads [226]

$$\mathbb{S} = 2\pi\sqrt{Q_- Q_+ (Q_0 Q_v + 4)}\,. \tag{7.38}$$

Here the Q_i are the (asymptotic) charges corresponding to winding (Q_-), momentum (Q_+), NS5-brane (Q_0) and KK-monopole (Q_v). The presumed quantum gravitational consistency of string theory imposes the equality of both the macroscopic and microscopic entropies. This can be used to derive exact relations between the charges and the number of fundamental objects to all orders in α'.

There are, of course, infinitely many alternative expressions for the charge shifts that respect the equality between the macroscopic and microscopic entropies. However, there are a series of arguments that enable us to propose a set of definite relations. We start by recalling the already known exactness of

$$Q_0 = N - \frac{2}{W}\,, \qquad Q_v = W\,. \tag{7.39}$$

The non-renormalization of the KK-monopole charge, $Q_v = W$, follows from the supersymmetry of the solution: any correction to the $\mathcal{V}$ function would make the $d\sigma_4^2$ metric no longer hyper-Kähler. What about the screening in the NS5 charge? It has long been known that an isolated KK-monopole of unit charge, i.e. $W = 1$, carries -1 unit of NS5-brane charge in heterotic theory [227] and the origin of this behaviour can be found in the action of a gravitational instanton, delocalized over the full space, as a negative source of magnetic charge for the Kalb-Ramond field strength[6]. A KK-monopole of charge W, the configuration of interest in four-charge black holes, is characterized by a gravitational instanton number $1/W$ and by this amount contributes negatively to the NS5 charge; the fractional value is a direct consequence of the normalization of the Chern-Simons term entering the field strength. On the other hand, also the presence of torsion in the spin connection has consequences: as described in [240], an additional gravitational instanton, with same charge, is sourced by the stack of NS5 branes. Putting it all together, the total shift in the NS5 charge amounts to $-2/W$ (or simply -1 in the absence of KK-monopole), as was first described in [242], and can be obtained by integrating the Bianchi identity (2.44), whose form is dictated by the anomaly cancellation mechanism.

Taking this information into account, the microscopic entropy is exactly equal to the Wald entropy if the shifts in the charges induced by the higher-curvature corrections satisfy

$$Q_+Q_- = nw\left(1 + \frac{2}{NW}\right)\,. \tag{7.40}$$

Interestingly, this already occurs at first order in α' (cf. equation (7.7)). Then, either the additional higher-curvature corrections do not introduce further charge sources, or they do it in a particular way that preserves the product. Considering that the corrections become less and less relevant order by order and that the F1 charge remains unaltered by the first correction, simplicity suggests that the expressions

$$Q_+ = n\left(1 + \frac{2}{NW}\right)\,, \qquad Q_- = w\,, \tag{7.41}$$

are exact to all orders in α'. While from our analysis we can only assert the validity in that respect of (7.40), we would find natural that the individual relations (7.41) hold. It might be possible to check this guess using dualities.

As we previously mentioned, relation (7.40) is already satisfied when only quadratic corrections in curvature are accounted for, in which case (7.41) hold. This reinforces the idea that the origin of the shift in the charges can be found in the introduction of a Chern-Simons term in the field strength of the Kalb-Ramond 2-form, followed by its corresponding supersymmetrization in the action. Hence, the shifts at first order in α' would be invariant under further corrections

[6] Likewise, given a collection of separated KK-monopoles of unit charge, each of them contributes -1 unit to the NS5 charge [256] and this value is again given by the negative gravitational instanton number.

and this is exactly what happens with the corrections to the Wald entropy [248]. The first set of corrections carries very relevant information, despite being few in an infinite amount.

One last thing to mention about (7.40) is that it shows how the distinction between charges and fundamental objects is crucial in the characterization of a string theory system and to what extent the interpretation of lower-dimensional effective fields within the string world is, therefore, rather subtle[7]. A significant example is represented by black holes with $Q_0 = 0$ and $NW \neq 0$, which were thought to provide a regularization of the singular horizon of small black holes, which do not contain NS5 nor KK, via higher-curvature corrections [257, 258]. As showed in [242], this interpretation was based on a misidentification of the fundamental stringy objects of the solution.

With respect to the three-charge system, the microscopic entropy is [32, 259, 260]

$$\mathbb{S} = 2\pi\sqrt{Q_- Q_+ (Q_0 + 3)}\,. \tag{7.42}$$

The application of the previous arguments gives

$$Q_0 = N - 1\,, \qquad Q_+ Q_- = nw\left(1 + \frac{2}{N}\right)\,, \tag{7.43}$$

which again is satisfied already at first order in α'. The application of the same arguments drives us to assert that the relations valid at first order in α', cf. (7.13),

$$Q_+ = n\left(1 + \frac{2}{N}\right)\,, \qquad Q_- = w\,, \tag{7.44}$$

are again exact to all orders in α'. Moreover, we notice that they are consistent with (7.41) setting $W = 1$.

7.3 Lower-dimensional, near-horizon effective approaches

The study of heterotic black holes and their higher-curvature corrections has been mainly approached in the literature using two different strategies. In the first one, developed around the early-2000s, the target is to find a solution of the form AdS $\times X$, with X some compact manifold, characterized by a given set of charges. It is, then, typically assumed that such solution describes the near-horizon limit of an extremal black hole with the same charges, and its properties are subsequently studied. Several methods have been developed to achieve this purpose, which can be applied in the context of different effective theories of interest. An intriguing result obtained from this line of investigation is that, in some cases, it is possible to reproduce the microscopic entropy by including only a subset of the curvature corrections to the action. The *Gauss-Bonnet (GB) term*, which is known to be one of the corrections to the lower-dimensional effective theory [261], probably provides the most interesting example. The value of the Wald entropy obtained from its inclusion correctly reproduces the microscopic degeneracy of the four-charge system, while it fails to do so for the simpler three-charge system.

The second strategy, which has been recently developed, is the one we followed in previous sections. Starting from a complete black hole solution of the theory of supergravity, the corrections induced by higher-curvature terms are computed using the standard perturbative approach. While conceptually simple, the problem is technically involved and other strategies

[7] A similar behaviour can be observed in section 7.3, where the electric and magnetic charges of the 4-dimensional reduced theory match exactly with the fundamental objects for $\alpha' = 0$, but a shift is introduced by further α'-corrections, cf. (7.58).

were usually preferred. On the other hand, the benefit of this effort is that information about the solution beyond the near-horizon limit becomes available.

At present time there are results that have been obtained using both strategies. It is, therefore, necessary to compare them and see what can be learnt from the analysis. This is the aim of this section.

7.3.1 Compactification of the supergravity theory

From an effective 4-dimensional perspective, the fields relevant for the description of such system are related to the heterotic fields as follows[8]

$$\begin{aligned} g_{\mu\nu} &= \hat{g}_{\mu\nu} - \frac{\hat{g}_{u\mu}\hat{g}_{u\nu}}{\hat{g}_{uu}} - \frac{\hat{g}_{z\mu}\hat{g}_{z\nu}}{\hat{g}_{zz}}\,, \\ s &= e^{-2\Phi}\sqrt{\hat{g}_{uu}\hat{g}_{zz}}\,, \qquad t = \sqrt{\hat{g}_{uu}}\,, \qquad u = \sqrt{\hat{g}_{zz}}\,, \\ A^{(1)}_{\mu} &= \frac{\hat{g}_{u\mu}}{2\hat{g}_{uu}}\,, \qquad A^{(2)}_{\mu} = \frac{\hat{g}_{z\mu}}{2\hat{g}_{zz}}\,, \qquad A^{(3)}_{\mu} = \frac{\tilde{B}_{u\mu}}{2}\,, \qquad A^{(4)}_{\mu} = \frac{\tilde{B}_{z\mu}}{2}\,. \end{aligned} \tag{7.45}$$

Here $(\mu,\nu) \in (t,r,\theta,\varphi)$ and we introduced hats to distinguish the higher-dimensional metric. It is convenient to define $A^{(3,4)}$ in terms of the dual of the Kalb-Ramond 2-form, as in this manner their field strength is closed, $F^{(3,4)} = dA^{(3,4)}$. Using this identification, the zeroth-order supergravity theory compactified to four dimensions reads

$$S = \frac{g_s^2}{16\pi G_{\rm N}^{(4)}}\int d^4x\sqrt{-g}\,s\left[R - a_{ij}\partial_\mu\phi^i\partial^\mu\phi^j - t^2F^2_{(1)} - u^2F^2_{(2)} - \frac{u^2}{s^2}F^2_{(3)} - \frac{t^2}{s^2}F^2_{(4)}\right]\,, \tag{7.46}$$

where we denote the scalars collectively as ϕ^i, with a_{ij} some functions of the scalars[9], and $F^2_{(a)} = F^{(a)}_{\mu\nu}F^{(a)\mu\nu}$.

We are interested in finding solutions to the equations of motion derived from (7.46) describing the near-horizon region of an extremal black hole. The geometry of these is known to be of the form $\mathrm{AdS}_2 \times S^2$. The general field configuration consistent with this isometry and the set of four independent charges we consider in this chapter is

$$\begin{aligned} ds^2 &= v_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_2\,ds^2_{S^2}\,, \\ s &= u_s\,, \qquad t = u_t\,, \qquad u = u_u\,, \\ F^{(1)}_{rt} &= e_1\,, \qquad F^{(2)}_{\theta\varphi} = P_2\sin\theta\,, \qquad F^{(3)}_{rt} = e_3\,, \qquad F^{(4)}_{\theta\varphi} = P_4\sin\theta\,. \end{aligned} \tag{7.47}$$

For the three-charge system we can take the same configuration for the fields, but it is necessary to fix $P_2 = R_z/4$. This is equivalent to the statement that the near-horizon geometry of the three-charge system is identical to that of the four-charge system with unit KK-monopole. The equations of motion in this case imply $u_u = 2\sqrt{v_2}/R_z$[10]. In this manner, there are only three independent vectors and two independent scalars, and the cross-section of the horizon contains a 3-sphere when embedded in the heterotic theory.

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[8] In order to make the comparison with previous literature transparent, the dimensional reduction of this section uses a different parametrization of the scalars than that of subsection 7.2.2.

[9] In the family of near-horizon solutions that we consider the scalars are constant, so the σ-model will play no role.

[10] This substitution should not be made in the function f defined in (7.49), as this would yield incorrect equations of motion.

The well-known attractor mechanism establishes that the parameters of the solution are fully determined in terms of the charges carried by the vectors. The magnetic and electric charges are defined in the standard manner[11],

$$P_a := \frac{1}{4\pi}\int_{S^2} d\theta\, d\varphi\, F^{(a)}_{\theta\varphi}\,, \qquad Q_a := \frac{1}{16\pi}\int_{S^2} d\theta\, d\varphi\, \frac{\delta}{\delta F^{(a)}_{rt}}\left(\sqrt{-g}\,\mathscr{L}\right). \tag{7.48}$$

These integrals can be defined not only for the near-horizon geometry, but also for the full black hole solution. As a consequence of the Bianchi identities $\partial_r F^{(a)}_{\theta\varphi} = 0$ and the Maxwell equations $\partial_r \left[\frac{\delta}{\delta F^{(a)}_{rt}}(\sqrt{-g}\,\mathscr{L})\right] = 0$ of the vectors, the charges are independent of the radius of the sphere on which they are computed. This implies that, from the 4-dimensional effective perspective, the asymptotic and near-horizon charges of the solution coincide, even after the inclusion of higher-curvature corrections. As this behaviour is different from the one displayed by the 10-dimensional fields, one should be very cautious when interpreting lower-dimensional fields in the string theory language. We shall come back to this point later. For the moment, since we do not have higher-derivative terms yet, this distinction is unnecessary.

The relations between the parameters of the near-horizon background and the charges can be determined as follows [262]. One first defines the function

$$f(v_1, v_2, u_i, e_a, P_a) := \int_{S^2} d\theta\, d\varphi \sqrt{-g}\,\mathscr{L}(v_1, v_2, u_i, e_a, P_a)\,, \tag{7.49}$$

where the ansatz (7.47) is used to evaluate the rhs. From (7.48) it follows

$$\frac{1}{16\pi}\frac{\partial f}{\partial e_a} = Q_a\,, \tag{7.50}$$

which can be used to replace e_a by Q_a, if wanted. The solution is obtained by extremizing the function f,

$$\frac{\partial f}{\partial v_1} = 0\,, \qquad \frac{\partial f}{\partial v_2} = 0\,, \qquad \frac{\partial f}{\partial u_i} = 0\,. \tag{7.51}$$

The black hole entropy is proportional to the Legendre transformation of f evaluated on the extremum

$$\mathbb{S} = \frac{g_s^2}{8G_{\rm N}^{(4)}}\left(16\pi e_a Q_a - f\right)\big|_{\rm ext.}\,. \tag{7.52}$$

7.3.2 Near-horizon solutions

It is straightforward to apply this formalism to the compactified zeroth-order heterotic theory (7.46). We obtain

$$\begin{aligned}
&ds^2 = 4P_2Q_3\left(-r^2dt^2 + \frac{dr^2}{r^2} + ds^2_{S^2}\right),\\
&s = \sqrt{\frac{Q_1P_4}{Q_3P_2}}\,, \qquad t = \sqrt{\frac{Q_1}{P_4}}\,, \qquad u = \sqrt{\frac{Q_3}{P_2}}\,,\\
&F^{(1)}_{rt} = \sqrt{\frac{P_2P_4Q_3}{Q_1}}\,, \qquad F^{(2)}_{\theta\varphi} = P_2\sin\theta\,, \qquad F^{(3)}_{rt} = \sqrt{\frac{P_2P_4Q_1}{Q_3}}\,, \qquad F^{(4)}_{\theta\varphi} = P_4\sin\theta\,.
\end{aligned} \tag{7.53}$$

[11] The normalization constants in the definition of the charges have been chosen for later convenience.

We chose to scale the time coordinate such that $v_1 = v_2$ to allow for an easier comparison with previous results in the literature. Using (7.45) it is possible to write the solution for the heterotic fields,

$$\begin{aligned} d\hat{s}^2 &= 4P_2Q_3\left(-r^2dt^2 + \frac{dr^2}{r^2} + ds^2_{S^2}\right) + \frac{Q_1}{P_4}\left(du - 2\sqrt{\frac{P_2P_4Q_3}{Q_1}}\,r\,dt\right)^2 \\ &\quad + \frac{Q_3}{P_2}\left(dz + 2P_2\cos\theta\, d\varphi\right)^2 , \\ e^{-2\Phi} &= \frac{P_4}{Q_3} , \\ H &= 2\sqrt{\frac{P_2Q_1Q_3}{P_4}}\, dr\wedge du\wedge dt + 2Q_3 \sin\theta\, d\theta\wedge dz\wedge d\varphi , \\ \tilde{B} &= 2\sqrt{\frac{P_2P_4Q_1}{Q_3}}\, r\, du\wedge dt - 2P_4\cos\theta\, dz\wedge d\varphi . \end{aligned} \tag{7.54}$$

The expression coincides with the near-horizon limit of our original solutions, after rescaling the time coordinate $t \mapsto t\sqrt{q_0 q_+ q_- q_v}$ in (7.5) and dropping the irrelevant $d\vec{y}_4^2$ term from the metric, with the identifications

$$Q_1 = \frac{q_+}{2g_s^2} , \qquad P_2 = \frac{q_v}{2} , \qquad Q_3 = \frac{q_0}{2} , \qquad P_4 = \frac{q_-}{2g_s^2} . \tag{7.55}$$

It is important to remark that these identifications hold in the zeroth-order solution, but are modified by the α'-corrections. As we shall see shortly, the variables on the lhs correspond to the asymptotic charges, while those on the rhs represent the number of fundamental string theory objects. It is useful to write the 4-dimensional solution in terms of the latter using (7.6),

$$\begin{aligned} ds^2 &= \frac{\alpha' NW}{4}\left(-r^2dt^2 + \frac{dr^2}{r^2} + ds^2_{S^2}\right) , \\ s &= \frac{\alpha'}{R_zR_u}\sqrt{\frac{nw}{NW}} , \qquad t = \frac{\sqrt{\alpha'}}{R_u}\sqrt{\frac{n}{w}} , \qquad u = \frac{\sqrt{\alpha'}}{R_z}\sqrt{\frac{N}{W}} , \\ F^{(1)}_{rt} &= \frac{R_u}{4}\sqrt{\frac{wNW}{n}} , \quad F^{(2)}_{\theta\varphi} = \frac{R_z}{4} W\sin\theta , \quad F^{(3)}_{rt} = \frac{\alpha'}{4R_u}\sqrt{\frac{nwW}{N}} , \quad F^{(4)}_{\theta\varphi} = \frac{\alpha'}{4R_z} w\sin\theta . \end{aligned} \tag{7.56}$$

Likewise, the black hole entropy computed from (7.52) gives

$$\mathbb{S}_0 = 2\pi\sqrt{nwNW} , \tag{7.57}$$

which agrees with the leading order result we obtained in the previous section.

We have obtained these expressions from the zeroth-order supergravity theory. We recall that this field configuration describes both the three- and four-charge systems, with the former being recovered simply setting $W = 1$ or $P_2 = R_z/4$. As we have already stated, the higher-curvature corrections vanish for this background and leave (7.56) invariant. This means that after adding all the relevant higher-curvature terms to the action (7.46) arising from the compactification of (2.40), the form of the function f will change, but it will have an extremum at the same point in this parameter space. On the other hand, if only a subset of the corrections are implemented, the corresponding solution, if exists, will typically take a different expression.

Taking into account this information, it is simple to apply the entropy function formalism to the action that includes all relevant four-derivative terms. In order to do so, it is first necessary to write the action in a manifestly covariant form, see [226], as we did in subsection 7.2.1. After few lines of computation, one can check that (7.56) still gives an extremum for the corrected function f. On the other hand, the charges carried by the 4-dimensional effective fields as defined in (7.48) are now for the four-charge system (for simplicity we set $R_u = R_z = \sqrt{\alpha'} = 4$)

$$Q_1 = n\left(1 + \frac{2}{NW}\right), \qquad P_2 = W\,, \qquad Q_3 = N - \frac{2}{W}, \qquad P_4 = w\,, \tag{7.58}$$

while for the three-charge system these are

$$Q_1 = n\left(1 + \frac{2}{N}\right), \qquad Q_3 = N - 1\,, \qquad P_4 = w\,. \tag{7.59}$$

Hence, we see that the lower-dimensional vector fields carry the asymptotic charges of our original solution of the heterotic theory. It is certainly remarkable how the shift in the charges, which is mediated by the higher-curvature corrections, distinguishes between the four- and three-charge systems, even though their near-horizon background is identical. This is caused by the explicit difference in the expression of the action in both systems when written in a manifestly covariant manner, as described in subsection 7.2.1. The asymptotic structure of the systems is responsible for the effect and, therefore, it is determinant for the analysis of the near-horizon solution.

The Wald entropy is

$$\mathbb{S} = 2\pi\sqrt{nwNW}\left(1 + \frac{2}{NW}\right), \tag{7.60}$$

for the the four-charge system, while the expression for the three-charge system is recovered simply setting $W = 1$. Naturally, the result coincides with (7.37), which provides a consistency check between the two approaches.

In most of the preceding literature, the expressions for the lower-dimensional fields and the Wald entropy are customarily given in terms of the charges carried by the vectors. After a few lines of algebraic computation, we may write for the four-charge α'-corrected solution

$$\begin{aligned}
ds^2 &= 4(P_2Q_3 + 2)\left(-r^2dt^2 + \frac{dr^2}{r^2} + ds^2_{S^2}\right), \\
s &= \sqrt{\frac{P_4Q_1}{P_2Q_3 + 4}}, \qquad t = \sqrt{\frac{Q_1(P_2Q_3 + 2)}{P_4(P_2Q_3 + 4)}}, \qquad u = \sqrt{\frac{Q_3}{P_2}\left(1 + \frac{2}{P_2Q_3}\right)}, \\
F^{(1)}_{rt} &= \sqrt{\frac{P_4(P_2Q_3 + 4)}{Q_1}}, \quad F^{(2)}_{\theta\varphi} = P_2\sin\theta\,, \quad F^{(3)}_{rt} = P_2\sqrt{\frac{P_4Q_1}{P_2Q_3 + 4}}, \quad F^{(4)}_{\theta\varphi} = P_4\sin\theta\,, \\
\mathbb{S} &= 2\pi\sqrt{P_4Q_1(P_2Q_3 + 4)}\,,
\end{aligned} \tag{7.61}$$

while for the three-charge system

$$\begin{aligned}
ds^2 &= 4(Q_3 + 1)\left(-r^2dt^2 + \frac{dr^2}{r^2} + ds^2_{S^2}\right), \\
s &= \sqrt{\frac{P_4Q_1}{Q_3 + 3}}, \qquad t = \sqrt{\frac{Q_1(Q_3 + 1)}{P_4(Q_3 + 3)}}, \qquad u = \sqrt{Q_3 + 1}\,, \\
F^{(1)}_{rt} &= \sqrt{\frac{P_4(Q_3 + 3)}{Q_1}}, \quad F^{(2)}_{\theta\varphi} = \sin\theta\,, \quad F^{(3)}_{rt} = \sqrt{\frac{P_4Q_1}{Q_3 + 3}}, \quad F^{(4)}_{\theta\varphi} = P_4\sin\theta\,, \\
\mathbb{S} &= 2\pi\sqrt{P_4Q_1(Q_3 + 3)}\,.
\end{aligned} \tag{7.62}$$

We find perfect agreement between these expressions and the results of [32, 243], which consider the same action as we do. As far as they can be compared, these solutions are identical to those obtained from 4-dimensional $\mathcal{N} = 2$ supersymmetric theories with corrections of quadratic order in curvature in terms of the Weyl tensor [228, 234–237].

7.3.3 The Gauss-Bonnet correction

A particular higher-derivative correction to the effective tree-level heterotic supergravity theory in four dimensions can be written in terms of the Gauss-Bonnet density [261],

$$\mathscr{L}_{\rm GB} = 2s\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right). \tag{7.63}$$

Even though such term only represents a subset of the relevant corrections at the four-derivative level, it has been noted in the literature that its inclusion leads to the correct value of the Wald entropy in some—but not all—cases. Particularly puzzling is the fact that it seems to give the right answer for the four-charge system, while it fails for the three-charge system. We shall now reanalyze the problem here and find the origin of this behaviour.

Let us begin with the four-charge system. Using the entropy function formalism, it is possible to obtain the near-horizon solution to the GB modified theory,

$$\begin{aligned}
ds^2 &= 4(P_2Q_3+2)\left(-r^2dt^2+\frac{dr^2}{r^2}+ds^2_{S^2}\right),\\
s &= \sqrt{\frac{P_4Q_1}{P_2Q_3+4}}\,, \qquad t=\sqrt{\frac{Q_1}{P_4}}\,, \qquad u=\sqrt{\frac{Q_3}{P_2}}\,,\\
F^{(1)}_{rt} &= \sqrt{\frac{P_4(P_2Q_3+4)}{Q_1}}\,, \quad F^{(2)}_{\theta\varphi}=P_2\sin\theta\,, \quad F^{(3)}_{rt}=P_2\sqrt{\frac{P_4Q_1}{P_2Q_3+4}}\,, \quad F^{(4)}_{\theta\varphi}=P_4\sin\theta\,,\\
\mathbb{S} &= 2\pi\sqrt{P_4Q_1(P_2Q_3+4)}\,.
\end{aligned} \tag{7.64}$$

This solution was first derived in [263]. The action complemented with (7.63) is no longer supersymmetric. It corresponds to an inconsistent truncation of the bosonic sector of the heterotic theory presented in section 2.4, hence one should be cautious when interpreting (7.64) in string theory language. Having this in mind, it seems reasonable to identify the charges of both schemes. Direct comparison with (7.61) reveals that the GB term suffices to capture the corrections to the metric, dilaton, vectors and Wald entropy when written in terms of the charges, while it fails with the scalars t and u. Using (7.58), which in this section can be interpreted as a redefinition of the parameters describing the fields, we get

$$\begin{aligned}
ds^2 &= 4NW\left(-r^2dt^2+\frac{dr^2}{r^2}+ds^2_{S^2}\right),\\
s &= \sqrt{\frac{nw}{NW}}\,, \qquad t=\sqrt{\frac{n}{w}\left(1+\frac{2}{NW}\right)}\,, \qquad u=\sqrt{\frac{1}{W}\left(N-\frac{2}{W}\right)}\,,\\
F^{(1)}_{rt} &= \sqrt{\frac{wNW}{n}}\,, \quad F^{(2)}_{\theta\varphi}=W\sin\theta\,, \quad F^{(3)}_{rt}=\sqrt{\frac{nwW}{N}}\,, \quad F^{(4)}_{\theta\varphi}=w\sin\theta\,,\\
\mathbb{S} &= 2\pi\sqrt{nwNW}\left(1+\frac{2}{NW}\right),
\end{aligned} \tag{7.65}$$

which reproduces the results derived from the heterotic theory, except for the expressions of the scalars t and u. It is useful to write the solution in terms of these variables, as it facilitates making contact with the zeroth-order solution (7.56) (we again set $R_u = R_z = \sqrt{\alpha'} = 4$ for simplicity here).

We now turn our attention to the three-charge system. In preceding subsections, we explained that the corresponding near-horizon solution is obtained setting $W = 1$ in the expressions for the fields and using (7.59) for the shift in the charges. In order to obtain the correct expression for the shift, it was crucial that the higher-curvature corrections to the action are different from those of the four-charge system, as a consequence of the asymptotic structure of the solutions. From this, it is obvious that the Gauss-Bonnet term will not be able to reproduce correctly the properties of the three-charge system. The GB correction has the same impact on the three- and four-charge systems. This means that it gives the right value for the Wald entropy in both cases when expressed in terms of the number of fundamental objects, but it is unable to produce the two different shifts for the charges. Since it gives the shift compatible with the four-charge system, when expressed in terms of the charges the Wald entropy only matches in this case. Therefore, we see that the relevant aspect to understand the puzzling behaviour of the Gauss-Bonnet correction relies on its (in)ability to reproduce the right shift in the charges.

In this sense, the GB term is of course neither unique, nor special. Examples of alternative corrections that produce the exact same effect in the field configuration and its properties are

$$\Delta\mathscr{L} = 2s\big(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu}\big)\,, \qquad \Delta\mathscr{L} = -4sR_{\mu\nu}R^{\mu\nu}\,, \tag{7.66}$$

which correspond to an even lower subset of the corrections than those provided by the GB density. The reason is that the near-horizon background is very symmetric, so the non-vanishing components of the Riemann tensor are proportional to the metric. In flat indices and for a metric of the form (7.47) with $v_1 = v_2$,

$$R_{abcd} = \mp\frac{2}{v_1}\,\eta_{a[c}\eta_{d]b}\,, \tag{7.67}$$

where the $-$ and the $+$ correspond to the AdS_2 and S^2 factors respectively. Hence, any scalar constructed from contractions of two Riemann tensors evaluated in the near-horizon background equals n/v_1^2, for some number n. Once multiplied by $\sqrt{-g}$, such correction is topological, in the sense that it is independent of the metric.

Conclusions

The bizarre character of black holes has attracted the interest of many astrophysicists for years. Supermassive black holes are believed to reside at the center of—probably—every galaxy, it is estimated that millions of stellar black holes crowd the Milky Way, while the hunting for mid-sized ones is still in progress. The existence of these mysterious objects has been observed both indirectly and, recently, directly and their formation is the subject of a large number of studies. On the other hand, black holes are fundamental in quantum theories of gravity as well: the larger the curvature, the larger the energy involved, hence quantum effects are not negligible anymore. Quantum gravity theories investigate the nature of black holes and the microscopic origin of their entropy and within this context supergravity solutions and BPS states play a crucial role. These topics have been addressed throughout the thesis in different set-ups and from different perspectives, thus a recapitulation of the results obtained and of the possible future directions of research seems appropriate.

Chapter 4 focused on 4-dimensional $\mathcal{N} = 2$ FI-gauged supergravity. The introduction of a scalar potential due to the gauging procedure is a possible way to evade the various no-hair and uniqueness theorems, thus allowing for non-constant scalars and non-spherical event horizons, and may lead to solutions with AdS asymptotics, central in the application of the AdS/CFT correspondence. We started examining the $\overline{\mathbb{C}\mathrm{P}}^n$ model and constructed rotating (extremal) BPS black holes and a non-extremal generalization: all of these backgrounds turned out to be asymptotically AdS_4, instrumental in holographic computations. A possible extension could consider the inclusion of hypermultiplets, although in this case the analysis would be much harder. Subsequently, we moved to the t^3 model—tightly bound to 5d gauged supergravity and type-II and 11d supergravities, like many other cubic models—for which we presented a supersymmetric black hole geometry. Further developments could address the generalization of this solution to the *stu* model or the construction of new ones still within the t^3 [264].

The subject of chapter 5 was again $\mathcal{N} = 2$ FI-gauged supergravity, but in five dimensions. All the considerations made for the 4-dimensional theory are still valid, but, furthermore, 3-dimensional event horizons allow for more exotic topologies. We focused on a specific homogeneous, but anisotropic manifold modelled on solvegeometry and constructed a static, magnetically charged solution of the theory with no scalars. Moreover, we proved the non-existence of BPS, static and non-dyonic spacetimes with solvegeometry cross-sections and, lastly, we derived an attractor configuration with magnetic charges. Three are the main directions to take in order to expand these results: apply the same study to configurations with cross-sections modelled on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ [265] or nilgeometry, include matter in the solution with no scalars, e.g. starting with the *stu* model, or extend the attractor configuration outside the horizon to a full black hole (at least for some models).

In chapter 6 we moved to higher-dimensional supergravities and focused on backgrounds including an AdS factor, crucial in the AdS/CFT correspondence. Firstly, we obtained the explicit brane intersections underlying different classes of AdS_3 solutions to 11d and (massive)

IIA supergravity. Moreover, connecting these solutions to domain wall backgrounds admitting asymptotic regions described (locally) by higher-dimensional AdS vacua, 7d and 6d respectively, we were able to provide a surface defect CFT interpretation to the AdS_3 solutions, where the higher-dimensional CFTs are the holographic duals of the higher-dimensional AdS vacua. Lastly, by means of dimensional reduction, we were also able to construct a new solution in type-IIA, along with its brane intersection and surface defect interpretation. This work has been continued in a follow-up paper [217], where the T-duals of the type-IIA brane configurations were studied in the context of IIB supergravity. The counterpart of the massive IIA set-up gives rise to AdS_6 domain wall backgrounds, which belong to a large family of solutions already classified: it could prove to be fruitful to more deeply investigate this class and the possible underlying brane configurations.

Heterotic black holes were the core of chapter 7, where we examined bound states of strings, NS5-branes and, possibly, KK-monopoles in first-order in α' heterotic supergravity. These configurations give rise to black holes in four and five dimensions, of which we computed the entropy exact to all orders in α'. Matching the latter with the microscopic degeneracy of the corresponding systems we were able to infer an α'-exact relation between the black hole (asymptotic) charges and their stringy constituents, proving, furthermore, a shift in the former with respect to their uncorrected ($\alpha' = 0$) values. The analysis was performed for vanishing Yang-Mills fields, thus a first, possible generalization could consider charged black holes.

Appendix A

A collection of useful information

A.1 Irreducible spinors

Spinors in D dimensions can be thought as vectors with $2^{[D/2]}$ complex components transforming under the $2^{[D/2]}$-dimensional representation of the $\mathrm{Spin}(D-1,1)$ group[1]. These are called *Dirac spinors*, and, despite being the most natural, they may not be the most fundamental building blocks to construct a theory. We now survey all the kind of spinors that can be constructed in each spacetime dimension by imposing specific constraints on Dirac spinors. The analysis behind these results is periodic in D and everything is to be intended in dimensions $D \pmod 8$.

- **Weyl spinors (W)**
  In an even number of dimensions the $\mathrm{Spin}(D-1,1)$ representation we mentioned is not irreducible, but is the product of two $(\frac{1}{2}\,2^{[D/2]})$-dimensional irreps. The original Dirac spinor can be decomposed accordingly, giving rise to two vectors with $2^{(D/2)-1}$ complex components called *Weyl spinors*. Weyl spinors are chiral and can be constructed acting on a Dirac spinor with the chiral projectors, which, in fact, exist only in even dimensions.

- **Majorana spinors (M)**
  When dealing with spinors it can prove useful to define the concept of *charge conjugate* of a spinor. This definition allows us to impose some sort of reality constraint requiring that a spinor and its charge conjugate are equal; objects of this kind are called *Majorana spinors* and have half the degrees of freedom of a Dirac spinor. Physically, these spinors describe states in which particles and antiparticles are identical and, in some specific representations of the gamma matrices, they are real, leading to $2^{[D/2]}$ real components. These spinors can be defined in any dimension except for $D = 5, 6, 7$ [2].

- **Symplectic Majorana spinors (S)**
  Although in $D = 5, 6, 7$ Majorana spinors cannot be defined, we can still consider an even number of Dirac spinors and impose "reality constraints" involving the full set. Again, these conditions reduce the number of independent components by a factor of two, but, we stress, can be imposed only taking an even number of spinors.

- **Majorana-Weyl spinors (MW)**
  One might wonder whether, in even dimensions, the Weyl (chirality) and Majorana constraints can be imposed simultaneously. In $D = 2$ these conditions are compatible and the chiral projections of Majorana spinors are still Majorana spinors; the result are spinors

[1] We recall that we are working in Lorentzian signature.

[2] Strictly speaking, in $D = 8, 9$ these spinors are pseudo-Majorana, but since their essential properties are the same, we shall make no distinction with Majorana spinors.

with $2^{(D/2)-1}$ real components that can be seen as the “most fundamental” spinors available in this dimension. In $D = 4, 8$ the two conditions are not compatible, thus we are forced to choose whether to impose one or the other.

- **Symplectic Majorana-Weyl spinors (SW)**
  Like in the MW case, in $D = 6$ the Weyl and symplectic Majorana constraints are compatible and *symplectic Majorana-Weyl spinors* can be defined, still considering an even number of spinors.

Summarizing, Weyl and Majorana conditions, separately, halve the number of degrees of freedom, while imposing the symplectic Majorana condition on an even number of spinors is like trading half of them for the others. In some cases, namely $D = 2 \pmod 4$, Weyl condition can be combined with one of the other two (M or S). A recap can be found in table A.1, where the irreducible spinors in each spacetime dimension are presented along with the number of real degrees of freedom.

$D \pmod 8$	spinor	components
2	MW	1
3	M	2
4	M or W	4
5	S	8
6	SW	8
7	S	16
8	M or W	16
9	M	16
10	MW	16
11	M	32

Table A.1: *Irreducible spinors with related number of (real) components [67]. Changing $D \mapsto D + 8$, the number of components is multiplied by 16.*

A.2 Supergravities across dimensions

In this section we present the various supergravity theories from four to eleven dimensions, organized according to the total number of supersymmetries (table A.2), along with the R-symmetry group allowed by each spacetime dimension (table A.3).

In table A.2 is displayed a survey of supergravity theories, arranged according to the spacetime dimension D and the number of supersymmetries and labelled by the number of SUSY generators $\mathcal{N}$. In each dimension, the latter are spinors of the kind reviewed in section A.1. We point out that the maximum number of supercharges is 32, because higher values would imply the presence of particles with spin $s \geq 5/2$, for which consistent interacting theories are still not known.

Some nomenclature always comes in handy. *Maximal* supergravities have the maximum number of supersymmetries, i.e. 32, thus they populate the first column. On the opposite side are *minimal* supergravities, whose number of supersymmetries depends on the number of components of the specific irreducible spinor, hence on the spacetime dimension. Moreover, in literature some of the entries are known also with different names, like minimal 5d supergravity, which in [12] is referred to as $\mathcal{N} = 1$, $D = 5$. In $D = 6$ alternative names are $(1, 1) \leftrightarrow$ iia, $(2, 0) \leftrightarrow$ iib and $(1, 0) \leftrightarrow$ i, whereas in $D = 10$ we may find IIA $\leftrightarrow (1, 1)$, IIB $\leftrightarrow (2, 0)$ and

	# of supersymmetries				
D	32	24	16	8	4
4	$\mathcal{N}=8$	$\mathcal{N}=6$	$\mathcal{N}=4$	$\mathcal{N}=2$	$\mathcal{N}=1$
5	$\mathcal{N}=8$	$\mathcal{N}=6$	$\mathcal{N}=4$	$\mathcal{N}=2$	
6	$(2,2)$, $(3,1)$, $(4,0)$	$(2,1)$, $(3,0)$	$(1,1)$, $(2,0)$	$(1,0)$	
7	$\mathcal{N}=4$		$\mathcal{N}=2$		
8	$\mathcal{N}=2$		$\mathcal{N}=1$		
9	$\mathcal{N}=2$		$\mathcal{N}=1$		
10	IIA, IIB		I, HE, HO		
11	M				

Table A.2: *A survey of supergravity theories [67]. In addition to what in the table, in $D=4$ there exist $\mathcal{N}=5$ and $\mathcal{N}=3$ theories with, respectively, 20 and 12 (real) supersymmetries.*

I $\leftrightarrow$ $(1,0)$. Finally, in $D=11$ M stands for M-theory, a common abuse of terminology to denote 11d supergravity.

D	spinor	R-symmetry
4	M	$\mathrm{U}(\mathcal{N})$
5	S	$\mathrm{USp}(\mathcal{N})$
6	SW	$\mathrm{USp}(\mathcal{N}_L)\times\mathrm{USp}(\mathcal{N}_R)$
7	S	$\mathrm{USp}(\mathcal{N})$
8	M	$\mathrm{U}(\mathcal{N})$
9	M	$\mathrm{SO}(\mathcal{N})$
10	MW	$\mathrm{SO}(\mathcal{N}_L)\times\mathrm{SO}(\mathcal{N}_R)$
11	M	$\mathrm{SO}(\mathcal{N})$

Table A.3: *R-symmetry groups [67].*

Here are some comments on table A.3. The USp group present in $D=5,6,7$ is defined as $\mathrm{USp}(2N):=\mathrm{Sp}(N,\mathbb{H})\simeq\mathrm{U}(N,\mathbb{H})$, hence it exists only for even values of $\mathcal{N}$. In $D=6,10$ it is possible to impose the Weyl condition independently, thus irreducible spinors are chiral and $\mathcal{N}_{L/R}$ are the number of left-/right-handed SUSY generators. Finally, in type-IIA supergravity $\mathcal{N}_L=\mathcal{N}_R=1$, hence we do not have R-symmetry and likewise in type-I and 11d supergravity, where $\mathcal{N}=1$.

A.3 Supersymmetry multiplets

In theoretical physics particles are firstly organized according to the irreducible representation of the Poincaré group, or better, its little group, they belong to, as prescribed by *Wigner's classification*. Much alike, in a SUSY theory matter is arranged in irreps of the super-Poincaré group, known as *supermultiplets*. All the fields belonging to a certain supermultiplet are related by SUSY transformations and the total number of (on-shell) bosonic and fermionic degrees of freedom must be equal. There exist various supermultiplets and their composition depends on the number of spacetime dimensions D and of SUSY generators $\mathcal{N}$. In this section we shall review the most common massless supermultiplets in broad outline.

Supermultiplets are named according to the highest spin appearing inside, therefore we remind that, as we explained in the previous section, no particles exist with $s\geq 5/2$, hence $s_{\max}=2$. Starting from this value and descending we have:

- **Supergravity multiplets ($s_{\mathbf{max}} = 2$):** they contain the graviton and $\mathcal{N}$ gravitinos, together with other fields depending on D and $\mathcal{N}$.

- **Vector multiplets ($s_{\mathbf{max}} = 1$):** they contain one vector field, which can serve as a gauge field, plus other lower-spin states. They exist only for 16 or less supersymmetries, like for $\mathcal{N} \leq 4$ in $D = 4, 5$.

- **Chiral multiplets/hypermultiplets ($s_{\mathbf{max}} = 1/2$):** they contain only spinors and scalars. Chiral multiplets exist only for $\mathcal{N} = 1$, $D = 4$, while for a total of 8 supersymmetries, e.g. $\mathcal{N} = 2$, $D = 4, 5$, we have hypermultiplets.

- **Tensor multiplets:** they contain antisymmetric tensors $T_{\mu\nu}$. In four and five dimensions these tensors are dual to scalars and vectors respectively, thus these supermultiplets can be omitted. In $D = 6$ things are quite different and tensors multiplets can give non-trivial contributions.

It is worth noting that supermultiplets with $s_{\max} = 3/2$ do not exist because the gravitino necessarily implies the presence of the graviton, thus gravitinos can only appear in the supergravity multiplet.

In order to make easier the computation of the bosonic and fermionic states inside a given supermultiplet, in table A.4 we summarize the number of off-shell and on-shell degrees of freedom of the fields generally encountered.

spin	off-shell	on-shell
0	1	1
1/2	$2^{[D/2]}$	$2^{[D/2]-1}$
1	$D-1$	$D-2$
3/2	$(D-1)\,2^{[D/2]}$	$(D-3)\,2^{[D/2]-1}$
2	$\frac{D(D-1)}{2}$	$\frac{D(D-3)}{2}$
p-form	$\binom{D-1}{p}$	$\binom{D-2}{p}$

Table A.4: *Off-shell and on-shell degrees of freedom for different fields. Spinors are to be intended as Majorana [67].*

Appendix B

Four-charge black hole

In this appendix we shall construct an example of BPS black hole consisting of a bound state of D2-D6 branes in type-IIA string theory [266]. Since the black hole we shall work with has a sufficiently large horizon, we restrict to the weakly-coupled and low-energy regime, i.e. to type-IIA supergravity. Within this context, let us consider the bound state of D2-D6 branes depicted in table B.1. The three sets of D2-branes, labelled by the letters a, b and c, are orthogonal and

branes	t	x^1	x^2	x^3	x^4	x^5	x^6	r	θ	φ
D2$_a$	×	×	×	–	–	–	–	–	–	–
D2$_b$	×	–	–	×	×	–	–	–	–	–
D2$_c$	×	–	–	–	–	×	×	–	–	–
D6	×	×	×	×	×	×	×	–	–	–

Table B.1: *Brane set-up of the four-charge black hole.*

intersect additional D6-branes. In order to solve this system, for the 10d metric we consider the following ansatz

$$\begin{aligned} ds^2 = &-\big(H_a H_b H_c H_{\text{D6}}\big)^{-1/2} dt^2 + \left(\frac{H_b H_c}{H_a H_{\text{D6}}}\right)^{1/2} \big[(dx^1)^2 + (dx^2)^2\big] \\ &+ \left(\frac{H_a H_c}{H_b H_{\text{D6}}}\right)^{1/2} \big[(dx^3)^2 + (dx^4)^2\big] + \left(\frac{H_a H_b}{H_c H_{\text{D6}}}\right)^{1/2} \big[(dx^5)^2 + (dx^6)^2\big] \\ &+ \big(H_a H_b H_c H_{\text{D6}}\big)^{1/2} \big[dr^2 + r^2\big(d\theta^2 + \sin^2\theta\, d\varphi^2\big)\big]\,, \end{aligned} \tag{B.1}$$

where all the branes are localized in the common transverse space and smeared in the other directions, i.e. $H_{a,b,c} = H_{a,b,c}(r)$ and $H_{\text{D6}} = H_{\text{D6}}(r)$. The gauge potentials and dilaton read

$$\begin{aligned} C_{(3)} &= H_a^{-1}\, dt \wedge dx^1 \wedge dx^2 + H_b^{-1}\, dt \wedge dx^3 \wedge dx^4 + H_c^{-1}\, dt \wedge dx^5 \wedge dx^6\,, \\ C_{(7)} &= H_{\text{D6}}^{-1}\, dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6\,, \\ e^{\Phi} &= \big(H_a H_b H_c\big)^{1/4} H_{\text{D6}}^{-3/4}\,, \end{aligned} \tag{B.2}$$

and the other vanishing, and the corresponding field strengths are

$$\begin{aligned} F_{(2)} &= H'_{\text{D6}}\, r^2\, \text{vol}_{S^2}\,, \\ F_{(4)} &= \left(\frac{H'_a}{H_a}\, dt \wedge dx^1 \wedge dx^2 + \frac{H'_b}{H_b}\, dt \wedge dx^3 \wedge dx^4 + \frac{H'_c}{H_c}\, dt \wedge dx^5 \wedge dx^6\right) \wedge dr\,. \end{aligned} \tag{B.3}$$

In general, the equations of motion are satisfied given that

$$\nabla^2_{\mathbb{R}^3} H_a = \nabla^2_{\mathbb{R}^3} H_b = \nabla^2_{\mathbb{R}^3} H_c = \nabla^2_{\mathbb{R}^3} H_{\text{D6}} = 0\,, \tag{B.4}$$

but, from now on, we shall consider the particular solution

$$H_a = 1 + \frac{Q_a}{r}\,, \qquad H_b = 1 + \frac{Q_b}{r}\,, \qquad H_c = 1 + \frac{Q_c}{r}\,, \qquad H_{\mathrm{D6}} = 1 + \frac{Q_{\mathrm{D6}}}{r}\,, \tag{B.5}$$

where, as it is possible to show, the parameters Q_i are closely related to the number of corresponding branes.

The first thing we can do is to "zoom in" on the region close to the branes, $r \to 0$, obtaining

$$\begin{aligned} ds^2 = &-\big(Q_aQ_bQ_cQ_{\mathrm{D6}}\big)^{-1/2} r^2\,dt^2 + \big(Q_aQ_bQ_cQ_{\mathrm{D6}}\big)^{1/2}\left[\frac{dr^2}{r^2} + d\theta^2 + \sin^2\theta\,d\varphi^2\right] \\ &+ \left(\frac{Q_bQ_c}{Q_aQ_{\mathrm{D6}}}\right)^{1/2}\big[(dx^1)^2 + (dx^2)^2\big] + \left(\frac{Q_aQ_c}{Q_bQ_{\mathrm{D6}}}\right)^{1/2}\big[(dx^3)^2 + (dx^4)^2\big] \\ &+ \left(\frac{Q_aQ_b}{Q_cQ_{\mathrm{D6}}}\right)^{1/2}\big[(dx^5)^2 + (dx^6)^2\big]\,. \end{aligned} \tag{B.6}$$

We can compactify the coordinates $x^1 \dots x^6$ and consider them as spanning a 6-torus with (possibly different) constant radii and, subsequently, perform a dimensional reduction to four dimensions, getting the final metric

$$ds^2 = -\frac{r^2}{l^2}\,dt^2 + \frac{l^2}{r^2}\,dr^2 + l^2\big(d\theta^2 + \sin^2\theta\,d\varphi^2\big)\,, \tag{B.7}$$

where we defined $l := (Q_aQ_bQ_cQ_{\mathrm{D6}})^{1/4}$. The result is a product space $\mathrm{AdS}_2 \times S^2$, where the radii of the AdS and the 2-sphere are equal; this is exactly the near-horizon geometry of an extremal Reissner–Nordström black hole with $l = M$. Since our solution is characterized by four independent parameters, it is often called *four-charge black hole*.

In order to better understand solution (B.1) let us take one step back and consider all the charges to be equal. In this case the full solution reduces to

$$ds^2 = -\left(1 + \frac{Q}{r}\right)^{-2} dt^2 + \left(1 + \frac{Q}{r}\right)^{2}\big[dr^2 + r^2\big(d\theta^2 + \sin^2\theta\,d\varphi^2\big)\big] + d\vec{x}_6^{\,2}\,, \tag{B.8}$$

where $d\vec{x}_6^{\,2}$ is the 6-dimensional flat metric and can be removed by means of a dimensional reduction on a T^6. What we obtained is the full geometry of an extremal Reissner–Nordström black hole when the radial coordinate is taken to be $\rho = r + Q$.

Appendix C

An extended class of $\mathrm{AdS}_3 \times S^3/\mathbb{Z}_k \times S^2$ solutions to type-IIA

In this appendix we extend the new class of $\mathrm{AdS}_3 \times S^3/\mathbb{Z}_k \times S^2$ solutions to type-IIA with $\mathcal{N} = (0, 4)$ supersymmetries found in subsection 6.2.1. Our starting point is the general class of $\mathrm{AdS}_3 \times S^3/\mathbb{Z}_k \times \mathrm{CY}_2 \times I$ solutions to M-theory constructed in [207], with $\mathrm{CY}_2 = T^4$. The new solutions are obtained reducing on the Hopf-fibre of the 3-sphere contained in the T^4. This reduction preserves all the supersymmetries and generalizes our solutions given by (6.25). Prior to this reduction we extend the solutions in [207] by modding the 3-sphere contained in T^4 by $\mathbb{Z}_{k'}$. This introduces k' KK-monopoles that give rise to k' D6-branes upon reduction. k' can then be taken to be sufficiently large as in the IIA supergravity limit.

The most general solutions in [207] are $\mathrm{AdS}_3 \times S^2 \times \mathrm{CY}_2$ fibrations over two intervals (see appendix B therein). They take the form

$$\begin{aligned}
ds^2_{11} &= \Delta \left(\frac{u}{\sqrt{h_4 h_8}}\, ds^2_{\mathrm{AdS}_3} + \sqrt{\frac{h_4}{h_8}}\, ds^2_{\mathrm{CY}_2} + \frac{\sqrt{h_4 h_8}}{u}\, dz^2 \right) + \frac{h_8^2}{4\Delta^2} \left(ds^2_{S^2} + \mathrm{D}\tilde{\chi}^2 \right) , \\
G_{(4)} &= -\left(d\left(\frac{u u'}{2h_4} \right) + 2h_8\, dz \right) \wedge \mathrm{vol}_{\mathrm{AdS}_3} - \partial_z h_4\, \mathrm{vol}_{\mathrm{CY}_2} - \frac{u u'}{2(h_4 h_8 + u'^2)} H_2 \wedge \mathrm{vol}_{S^2} \\
&\quad - \frac{h_8}{u} \star_4 d_4 h_4 \wedge dz + \frac{h_8}{2} \left[\frac{1}{2} d \left(-z + \frac{u u'}{4 h_4 h_8 + u'^2} \right) \wedge \mathrm{vol}_{S^2} + \frac{1}{h_8}\, dz \wedge H_2 \right] \wedge \mathrm{D}\tilde{\chi}\, ,
\end{aligned} \tag{C.1}$$

where $H_2 = -d\mathcal{A}$, $\mathrm{D}\tilde{\chi} = d\tilde{\chi} + \tilde{\mathcal{A}} + \omega$, $d\omega = \mathrm{vol}_{S^2}$ and

$$\Delta = \frac{h_8^{1/2} (4 h_4 h_8 + u'^2)^{1/3}}{2^{2/3} h_4^{1/6} u^{1/3}}\, . \tag{C.2}$$

In these solutions h_8 is a constant, h_4 has support on (z, CY_2), u is a function of z and H_2 has support on the CY_2. Note that we have renamed ρ and $\tilde{\psi}$ as in [207] by z and $\tilde{\chi}$, respectively, to connect with our notation in section 6.2 (see below). The quantities $\tilde{\chi}$ and $\tilde{\mathcal{A}}$ are defined as $\tilde{\chi} := \frac{2}{h_8}\chi$ and $\tilde{\mathcal{A}} := \frac{2}{h_8}\mathcal{A}$. In the most general case the connection $\tilde{\mathcal{A}} + \omega$ makes the fibre over the S^2 and the CY_2 non trivial. Here we restrict to the case $\mathcal{A} = 0$ and $\mathrm{CY}_2 = T^4$. In this case the solutions simplify to

$$\begin{aligned}
ds^2_{11} &= \Delta \left(\frac{u}{\sqrt{h_4 h_8}}\, ds^2_{\mathrm{AdS}_3} + \sqrt{\frac{h_4}{h_8}}\, ds^2_{T^4} + \frac{\sqrt{h_4 h_8}}{u}\, dz^2 \right) + \frac{h_8^2}{\Delta^2}\, ds^2_{S^3/\mathbb{Z}_k}\, , \\
G_{(4)} &= -d \left(\frac{u u'}{2h_4} + 2h_8\, z \right) \wedge \mathrm{vol}_{\mathrm{AdS}_3} + 2h_8\, d \left(-z + \frac{u u'}{4 h_4 h_8 + u'^2} \right) \wedge \mathrm{vol}_{S^3/\mathbb{Z}_k} \\
&\quad - \partial_z h_4\, \mathrm{vol}_{T^4} - \frac{h_8}{u} \star_4 d_4 h_4 \wedge dz\, ,
\end{aligned} \tag{C.3}$$

where $k = h_8$ and $ds^2_{S^3/\mathbb{Z}_k}$ is written as in (6.8). Supersymmetry holds when

$$u'' = 0\,, \tag{C.4}$$

while the Bianchi identities of the fluxes impose that

$$\frac{h_8}{u}\nabla^2_{T^4}h_4 + \partial_z^2 h_4 = 0\,. \tag{C.5}$$

The symmetries $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ are realized geometrically on the AdS_3 and the quotiented 3-sphere, respectively.

In order to extend the class of solutions given by (6.25) we mod out the T^4 in equations (C.3) by $\mathbb{Z}_{k'}$, such that $ds^2_{T^4/\mathbb{Z}_{k'}} = d\rho^2 + \rho^2\, ds^2_{\tilde{S}^3/\mathbb{Z}_{k'}}$ with $ds^2_{\tilde{S}^3/\mathbb{Z}_{k'}}$ defined in (6.8). The 4-flux term defined on the CY_2 takes the form $\star_4 d_4 h_4 = \rho^3\partial_\rho h_4\, \mathrm{vol}_{\tilde{S}^3/\mathbb{Z}_{k'}}$ and the equation (C.5) imposed by the Bianchi identities becomes

$$\frac{h_8}{u}\left(\partial_\rho^2 h_4 + \frac{3}{\rho}\,\partial_\rho h_4\right) + \partial_z^2 h_4 = 0\,. \tag{C.6}$$

Reducing now along the $S^1/\mathbb{Z}_{k'} \subset \tilde{S}^3/\mathbb{Z}_{k'}$ we obtain

$$\begin{aligned}
ds^2_{10} &= \rho\left[\tilde{\Delta}\left(\frac{u}{\sqrt{h_4h_8}}\,ds^2_{\mathrm{AdS}_3} + \frac{\sqrt{h_4h_8}}{u}\,dz^2 + \sqrt{\frac{h_4}{h_8}}\,(d\rho^2 + \frac{\rho^2}{4}\,ds^2_{\tilde{S}^2})\right) + \frac{h_8^{3/2}h_4^{1/2}}{\tilde{\Delta}\,k'^2}\,ds^2_{S^3/\mathbb{Z}_k}\right],\\
e^{2\Phi} &= \frac{\tilde{\Delta}}{k'^2}\sqrt{\frac{h_4}{h_8}}\,\rho^3\,, \qquad \tilde{\Delta} = \frac{1}{2k'}\sqrt{\frac{h_8(4h_4h_8+u'^2)}{u}}\,,\\
H_{(3)} &= -\frac{\rho^3}{4k'}\left(\frac{h_8}{u}\partial_\rho h_4 dz - \partial_z h_4 d\rho\right)\wedge \mathrm{vol}_{\tilde{S}^2}\,,\\
F_{(2)} &= \frac{k'}{2}\,\mathrm{vol}_{\tilde{S}^2}\,,\\
F_{(4)} &= -d\left(\frac{uu'}{2h_4} + 2h_8\,z\right)\wedge \mathrm{vol}_{\mathrm{AdS}_3} + 2h_8\,d\left(-z + \frac{uu'}{4h_4h_8+u'^2}\right)\wedge \mathrm{vol}_{S^3/\mathbb{Z}_k}\,.
\end{aligned} \tag{C.7}$$

This extends the new class of 10d backgrounds with $\mathcal{N} = (0,4)$ supersymmetries presented in subsection 6.2.1 to include a new function $u(z)$, satisfying (C.4). Indeed, one can check that the near horizon geometry (6.25) is obtained in the particular case $u' = 0$, with the redefinitions

$$Q_{\mathrm{KK}} = h_8\,, \qquad Q_{\mathrm{D6}} = k'\,, \qquad H_{\mathrm{NS5}} = \frac{2^6 Q^3_{\mathrm{D4}} h_8^2}{u^2}\,h_4 \qquad \text{and} \qquad H_{\mathrm{D6}} = \frac{Q_{\mathrm{D6}}}{\rho}\,, \tag{C.8}$$

where we rescaled the coordinates in (6.25) as

$$z \mapsto \frac{z}{4Q_{\mathrm{D4}}}\,, \qquad \rho \mapsto \frac{u\,\rho^2}{2^5 Q_{\mathrm{D6}}Q^2_{\mathrm{D4}}h_8}\,. \tag{C.9}$$

As previously mentioned, doing the change of coordinates $\rho \mapsto \rho^{1/2}$, one can see that there are k' D6-branes seated at $\rho = 0$,

$$\begin{aligned}
ds^2_{10} &= \rho^{1/2}\left[\tilde{\Delta}\left(\frac{u}{\sqrt{h_4h_8}}\,ds^2_{\mathrm{AdS}_3} + \frac{\sqrt{h_4h_8}}{u}\,dz^2\right) + \frac{h_8^{3/2}h_4^{1/2}}{\tilde{\Delta}\,k'^2}\,ds^2_{S^3/\mathbb{Z}_k}\right] + \frac{\tilde{\Delta}}{4\rho^{1/2}}\sqrt{\frac{h_4}{h_8}}\,(d\rho^2 + \rho^2 ds^2_{S^2}),\\
e^{2\Phi} &= \frac{\tilde{\Delta}}{k'^2}\sqrt{\frac{h_4}{h_8}}\,\rho^{3/2}\,, \qquad F_{(2)} = \frac{k'}{2}\,\mathrm{vol}_{S^2}\,.
\end{aligned} \tag{C.10}$$

List of publications

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