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On the linear instability of higher dimensional wormholes supported by self-interacting phantom scalar fields

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To
The Inhabitants of SPACETIME IN GENERAL
And ALL THE TEACHERS of his life IN PARTICULAR
This Work is Dedicated
By a Humble Native of Space
In the Hope that,
Even as he was Initiated into the Mysteries
Of FOUR DIMENSIONS
Having been previously conversant
With ONLY THREE,
So the Citizens of that Celestial Region
May aspire yet higher and higher
To the Secrets of FIVE or EVEN SIX Dimensions
Of their Universe and even of New Ones,
Thereby contributing
To the Enlargement of THE IMAGINATION
And the possible Development
Of that most and excellent Gift of MODESTY
Among the Superior Races
Of FOURDIMENSIONAL HUMANITY.

Agli
Abitanti dello SPAZIOTEMPO IN GENERALE
e a TUTTI GLI INSEGNANTI della sua vita IN PARTICOLARE,
Un Umile Nativo dello Spazio
Dedica questa Opera
Nella Speranza che,
Così come egli fu Iniziato ai Misteri
Delle QUATTRO DIMENSIONI
Dopo aver avuto fino ad allora dimestichezza
Con TRE SOLTANTO,
Così i Cittadini di quella Regione Celeste
Possano aspirare sempre più in alto
Ai Segreti delle CINQUE o PERFINO SEI Dimensioni
Del loro Universo e anche di Nuovi
Contribuendo in tal modo
All'Arricchimento dell'IMMAGINAZIONE
E alla eventuale Diffusione
Del Dono quanto mai raro ed eccelso della MODESTIA
Tra le Razze Superiori
Dell'UMANITÀ QUADRIDIMENSIONALE.

The present dedication is inspired by that of the novel "Flatland" by Edwin A. Abbott.

Contents

Abstract 8
Acknowledgement 10

Introduction **11**
Wormholes in higher dimensional General Relativity and the problem of their stability 11
Organization of the thesis 17
Mathematical and physical background 20

I Static spherically symmetric wormholes supported by self-interacting phantom scalar fields **23**

1 Spacetimes supported by self-interacting scalar fields in arbitrary dimension **24**
1.1 Scalar fields and Klein-Gordon equation 24
1.2 Lagrangian formulation for spacetimes in arbitrary dimension supported by a self-interacting scalar field 26

2 Spherically symmetric wormholes in arbitrary dimension **32**
2.1 Preliminaries 32
2.1.1 Some facts on two-dimensional Lorentzian manifolds . 32
2.1.2 Some facts on Riemannian manifolds with constant curvature 37
2.2 Geometry of spherically symmetric spacetimes 38
2.3 Geometry of (four-dimensional) spherically symmetric wormholes 45
2.4 Necessity of exotic matter: phantom scalar fields 51
2.4.1 Energy conditions 52
2.4.2 The violation of the weak energy condition (WEC) . . 52
2.4.3 An example of exotic matter supporting a wormhole: phantom scalar fields 54

2.5 Field equations for a spherically symmetric spacetime supported by a self-interacting phantom scalar field 56
2.5.1 A constrained field system in the $\beta = 0$ gauge 63

3 Static wormhole solutions 65
3.1 Background equations 65
3.2 EBMT wormhole in higher dimension: the Torii-Shinkai wormhole 66
3.3 Ellis-Bronnikov wormhole 73
3.3.1 Embedding diagram and geodesics of the EBMT and the Ellis-Bronnikov wormholes 74
3.4 A four-dimensional wormhole connecting two AdS universes . 80
3.4.1 The general Bronnikov-Fabris solution 80
3.4.2 The AdS wormhole 81
3.4.3 Embedding diagram and geodesics of the AdS wormhole 83
3.5 A dS wormhole with horizons 87
3.5.1 Another coordinate system for the internal region I . . 89
3.5.2 A first spacetime extension 89
3.5.3 Extending spacetime further: the nonstatic Kruskal type extension beyond the horizons 90

II Linear stability analysis of static wormhole solutions 95

4 Field equations for perturbed static wormhole solutions 96
4.1 Perturbations of the metric and the scalar field 96
4.2 Possible gauge choices 97
4.3 Linearizing the field equations (and the scalar curvature) . . . 101
4.3.1 Linearization of the constrained system and the scalar curvature in the $\delta\beta = 0$ gauge 105

5 A gauge-dependent linear stability analysis of two wormhole solutions 107
5.1 Gauge-dependent linear stability analysis of the Torii-Shinkai wormhole 109
5.1.1 Gauge choice 109
5.1.2 Field equations and the linearization of the scalar curvature 110
5.1.3 Decoupling the field equations 111

5.1.4 Solution of the master equation and linear instability of the Torii-Shinkai wormhole - gauge-dependent formulation . . . 114
5.2 Gauge-dependent linear stability analysis of the AdS wormhole 119
5.2.1 Gauge choice . . . 119
5.2.2 Field equations and the linearization of the scalar curvature . . . 121
5.2.3 Decoupling the field equations . . . 123
5.2.4 Solution of the master equation and linear instability of the AdS wormhole - gauge-dependent formulation . . . 127

6 A gauge-invariant method for the linear stability analysis of wormhole solutions 133
6.1 Study of the constraint equations in the $\delta\beta = 0$ gauge and the $\mathcal{S} = 0$ assumption . . . 134
6.2 The gauge-dependent quantities $\mathcal{A}$, $\mathcal{C}$, $\mathcal{D}$ and the $\delta\Phi = 0$ gauge 138
6.3 The gauge-invariant quantities A, C, E and the linearized field equations for A, C, E in an arbitrary gauge . . . 143
6.3.1 A straightforward deduction of the system (6.54-6.57) and its generalizations to the case $\mathcal{S} \neq 0$ . . . 148
6.4 The gauge-invariant quantities $\mathcal{F}$ and $\mathcal{G}$: the linearized field equations as a constrained wave system . . . 151
6.5 A master equation for the gauge-invariant quantity χ and the solution of the linearized field equations . . . 153
6.5.1 A solution of Eqs. (6.103,6.104) in dependence of χ, and the recovering of the perturbations $\delta\alpha$, $\delta\beta$, $\delta\gamma$, δr, $\delta\Phi$ . . . 157

7 Applications of the gauge-invariant method 160
7.1 A general strategy for obtaining a static solution $(\mathcal{F}_0, \mathcal{G}_0)$ of the constrained wave system (6.74,6.75) . . . 160
7.2 Gauge-invariant stability analysis of the Torii-Shinkai wormhole 163
7.2.1 Solution of the master equation and linear instability of the Torii-Shinkai wormhole - gauge-invariant formulation 165
7.3 Gauge-invariant stability analysis of the Ellis-Bronnikov wormhole . . . 168
7.3.1 Solution of the master equation and linear instability of the Ellis-Bronnikov wormhole . . . 171
7.4 Gauge-invariant stability analysis of the AdS wormhole . . . 174
7.4.1 Solution of the master equation and linear instability of the AdS wormhole - gauge-invariant formulation . . . 176

7.5 Gauge-invariant stability analysis of the static part of the dS wormhole . . . 178
7.5.1 Solution of the master equation and linear instability of the static part of the dS wormhole . . . 178

8 A comparison between the gauge-dependent and the gauge-invariant approaches, and the state of the art 182
8.1 The Ellis-Bronnikov wormhole . . . 182
8.1.1 A comparison between the gauge-invariant approach of Section 7.3 and Refs. [25, 27, 26, 11] . . . 182
8.1.2 Some comments on Ref. [53] . . . 185
8.2 The Torii-Shinkai wormhole . . . 185
8.2.1 A comparison between the gauge-invariant approach of Section (7.2) and Ref. [35] . . . 185
8.2.2 A comparison between the gauge-dependent approach od Section 5.1 and the gauge-invariant approach of Section 7.2 . . . 186
8.3 The AdS wormhole . . . 188
8.3.1 A comparison between the gauge-dependent approach of Section 5.2 and the gauge-invariant approach of Section 7.4 . . . 188

Conclusion and possible future developments 190

A Geodesic motion in a four-dimensional spherically symmetric static spacetime 194

B Supplements on gauge transformations 198
B.1 Gauge transformations of the perturbation functions . . . 198
B.2 On the divergence of linearized quantities . . . 200

C On the spectral decomposition of selfadjoint operators in L^2 202
C.1 (Generalized) orthonormal bases of L^2 . . . 202
C.2 Spectral theorem for selfadjoint operators in L^2 . . . 206
C.2.1 Operators with a purely discrete spectrum . . . 206
C.2.2 Operators with discrete and continuous spectrum . . . 209

D Spectral features of the Schrödinger operators appearing in the master equations 212
D.1 Spectral features of the Schrödinger operator (5.29,5.22) (Torii-Shinkai wormhole) . . . 212

D.1.1 Some general facts on Schrödinger operators with smooth, decaying potential . . . 213
D.1.2 The point spectrum of the Schrödinger operator (5.29,5.22) (Torii-Shinkai wormhole) . . . 215
D.1.3 On the trial function f [Eq. (D.13)] . . . 219
D.1.4 The functions $\chi_0^\pm$ and their zeroes . . . 221
D.1.5 The Kato-Temple inequality, and the derivation of Eq. (D.18) . . . 222
D.2 Spectral features of the Schrödinger operator (7.43,7.39) (Ellis-Bronnikov wormhole) . . . 223
D.3 Spectral features of the Schrödinger operator (5.69,5.60) (AdS wormhole) . . . 225
D.4 Spectral features of the Schrödinger operator (7.64,7.60) (dS wormhole) . . . 229

E Solutions of the master equations 231
E.1 Solution of the master equations of the Torii-Shinkai wormhole 232
E.1.1 Solution of the master equation (7.21) with low regularity 232
E.1.2 Solution of the master equation (5.26) with high regularity . . . 235
E.2 Solution of the master equation of the non reflection symmetric Ellis-Bronnikov wormhole . . . 238
E.2.1 Solution of the master equation (7.40) with low regularity 239
E.3 Solution of the master equations of the AdS wormhole . . . 240
E.3.1 Solution of the master equation (7.54) with low regularity 240
E.3.2 Solution of the master equation (5.66) with high regularity . . . 242
E.4 Solution of the master equations of the dS wormhole . . . 245
E.4.1 Solution of the master equation (7.61) with low regularity 245

Abstract

In this thesis I deal with the linear stability analysis of static, spherically symmetric wormholes supported by phantom self-interacting scalar fields, in the framework of General Relativity with arbitrary spacetime dimension. In the previous literature, a gauge-invariant stability analysis of wormhole configurations often succeeds in decoupling the linearized field equations, yielding a wave-type master equation which, however, is typically singular where the radial coefficient of the metric has a critical point, that is, at the wormhole throat. In order to overcome this problem a regularization method has been proposed in previous works, which transforms the singular wave equation to a regular one; this method is usually referred to as "S-deformation" (and sometimes requires a partly numerical implementation, especially, in the case of scalar fields with nontrivial self-interaction). The first result of my work is the reduction of the linearized field equations to a completely regular, constrained wave system for two suitably defined gauge-invariant functions of the perturbations in the metric coefficients and in the scalar field; the second result is a strategy for decoupling this system, obtaining a single wave-type master equation for another gauge-invariant quantity. No step of this construction causes the appearing of singularities at the wormhole throat or elsewhere (provided that the unperturbed scalar field has no critical points, which occurs in many examples); therefore, it is not necessary to regularize a posteriori the master equation via the S-deformation method. This gauge-invariant and singularity-free formalism, which generalizes to arbitrary spacetime dimensions the approach of my paper [1], is then applied to some known static wormhole solutions (most, but not all of them considered in [1]). The most relevant application is a certain Anti-de Sitter (AdS) wormhole, whose linear stability analysis does not seem to have been performed previously by other authors; by using the present method, it is possible to derive a completely regular master equation describing the perturbations of the AdS wormhole and prove that the latter is actually linearly unstable, after providing a detailed analysis of the spectral properties of the Schrödinger type operator appearing in the master equation. A partial instability result is derived along the same lines for the analogous de Sitter (dS) wormhole, a technically more subtle case due to the presence of horizons. As a further application, I rederive in a singularity-free fashion the master equations for the perturbed Ellis-Bronnikov and Torii-Shinkai wormholes. As a supplement, the linear instability results for the AdS and for the Torii-Shinkai wormholes are also recovered using an alternative, singularity free but gauge-dependent method: in this case a regular master equation is derived for the perturbed radial coordinate, and the gauge-independence of the instability result is

tested a posteriori. This alternative, gauge-dependent approach generalizes that introduced in my paper [2] for the reflection symmetric Ellis-Bronnikov wormhole. Let me also cite [3], from which I report some facts about the previously mentioned wormholes in absence of perturbations.

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Introduction

Wormholes in higher dimensional General Relativity and the problem of their stability

One of the most fascinating features of Einstein's theory of General Relativity (GR) consists in the fact that spacetime may be curved and topologically nontrivial, describing intriguing objects like black holes and wormholes. Black hole spacetimes appear under rather natural conditions in GR, and they are expected to form in Nature, for instance, from the collapse of sufficiently massive stars at the end of their life. Furthermore, there is by now compelling evidence for their existence in our Universe which has recently been reinforced by the observation of gravitational waves from binary black hole mergers [4] and the first image of the shadow of the supermassive black hole in the center of the galaxy M87 [5].

In contrast to this, the occurrence of wormholes [1] is much more speculative, and so far, there is no observational evidence for the existence of such structures. From the theoretical point of view, there are important constraints on their existence, such as the topological censorship theorem [6]. This theorem implies that asymptotically flat, globally hyperbolic wormhole spacetimes (including those whose Cauchy surfaces have topology $\mathbb{R} \times S^2$ and represent a throat connecting two asymptotically flat ends) require the existence of "exotic" matter to support the throat, that is, they require matter whose stress-energy-momentum tensor violates the (averaged) null energy condition. Intuitively, the need for exotic matter can be understood by the fact that a light bundle that traverses a wormhole throat must focus as it approaches the throat, but then must expand again as it moves away from the throat, which is opposite to the focusing effect for light due to ordinary matter [7].

On the other hand, it has also been shown that an infinitesimally small

[1] In this thesis, when talking about wormholes, we always refer to *traversable* Lorentzian wormhole spacetimes in a metric theory of gravity.

quantity of matter violating the averaged null condition is sufficient to support the throat [8]. This leads to the hope that quantum effects may give rise to a semiclassical theory in which wormhole spacetimes are allowed, in a similar way than quantum effects (Hawking radiation) induce black hole evaporation although an area decrease of the event horizon is forbidden in classical GR with matter fields satisfying the null energy condition [7]. Nevertheless, it remains to be seen whether or not such effects are strong enough to give rise to a traversable wormhole throat of macroscopic size [9].

Instead of invoking quantum effects, an alternative way to violate the null energy condition (which has received important motivation from cosmology, see for example Ref. [10]) is the consideration of phantom scalar fields, i.e. scalar fields that have a negative kinetic energy (see for instance Ref. [11] and references therein). Due to this property, such fields may lead to gravitational repulsion, and hence induce interesting effects like the accelerated expansion in the universe, universes with no particle horizon [12] or the ability of supporting a wormhole throat [13, 14]. On the other hand, the presence of unbounded negative kinetic energy might cast doubt on the possibility that any stationary solution found in this theory could ever be stable. (2) Therefore, a pressing question regarding the relevance of static wormhole solutions in such theories (or other GR theories involving exotic matter fields) is their dynamical stability under small perturbations.

The most widely studied wormhole models (including those analyzed in the present thesis) are based on static, spherically symmetric spacetimes in which the world sheet of the throat consists of spheres of minimal area [16]. Within the context of phantom scalar fields, many such solutions have been found; the simplest ones are obtained for a real scalar field and are due to pioneering work by Ellis [13] and by Bronnikov [14]. Since then, these solutions have been generalized to the following supporting fields: a scalar with a self-interaction potential [17, 18], a complex phantom scalar [19], a family of conventional and/or phantom scalars [20, 21, 22], a phantom scalar and an electromagnetic field [23], and, very recently, a k-essence scalar [24]. For the linear stability analysis of many of these solutions, see Refs. [25, 23, 21, 26, 18, 2, 24]; furthermore, all these studies conclude that the static, spherically symmetric wormhole solutions are linearly unstable, with numerical simulations [20, 27, 28] revealing that the throat either collapses to a black hole or expands on timescales comparable to the light-crossing

[2] However, the presence of unbounded negative kinetic energy by itself does not imply that any stationary solution in the theory is *necessarily* unstable. For example, it turns out that the Minkowski spacetime is nonlinearly stable in Einstein theory minimally coupled to a scalar field irrespectively of the sign of the gravitational coupling constant (see the comments and references in appendix B.5 in Ref. [15]).

time of the radius of the throat. Therefore, finding a static, spherically symmetric wormhole solution in GR with exotic matter which can be shown to be linearly stable (or unstable with a large time scale associated with all the unstable modes) remains a challenging open problem. [(3)]

Wormhole spacetimes have been considered even in the context of GR in arbitrary dimension. Higher-dimensional theories have a long story, starting from the work of Kaluza and Klein in the 1920's and marked by the advent of string theory in the 1970's; actually, the interest on wormholes in higher dimensional GR is nothing new: see, for example, the pioneering works of Chodos and Detweiler [31] and of Clément [32], and the generalizations of wormholes in the Einstein-Gauss-Bonnet gravity [33] and in the k-essence scalar theory [34]. However, it was only with [35] that the question of the linear stability of such structures was seriously investigated; indeed, in Ref. [35] the authors introduced a generalization of the reflection symmetric Ellis-Bronnikov wormhole in dimension $(d+1)$, $d \geq 3$ (in the sequel referred to as Torii-Shinkai wormhole) and proved that it is linearly unstable, in any dimension, under spherically symmetric time-dependent perturbations.

In this work, I focus on Einstein gravity in arbitrary dimension minimally coupled to a single, real phantom scalar field Φ with an arbitrary self-interaction potential $V(\Phi)$; the most important result is the development of a new, general, gauge-invariant framework to analyze the linear stability of arbitrary dimensional, static, spherically symmetric wormhole solutions in these theories (generalizing the four-dimensional approach of Ref. [1]). In addition, the latter is tested in specific applications.

In order to clarify which are the novelties of this thesis, it is necessary to sketch the previous state of the art in this area. Linearized perturbations of wormhole solutions of Einstein's equations have been previously discussed, even in a gauge-invariant language. However, most of the previous approaches are based on fixing the radial coordinate and deriving a linearized wave equation for perturbations of the scalar field; due to the fact that the radial coordinate has a critical point at the throat, the effective potential appearing in this wave equation (usually called *master equation*) is necessarily *singular* at the throat. [(4)] As explained in Ref. [25] (see also Ref. [26]) this

[3]See also Ref. [29] for the construction of static, spherically symmetric wormholes in Einstein-Dilaton-Gauss-Bonnet theory, a modified gravity theory, which does not require exotic matter. However, a careful stability analysis has recently revealed that these solutions are linearly unstable as well [30].

[4]Note that these singularities keep affecting the linearized wave equation, even after expressing it in a gauge-invariant fashion, thus no longer assuming the radial coordinate to be fixed (see, for example Ref. [25]).

yields an artificial (mirror-like) boundary condition at the throat which prevents perturbations from traversing the wormhole. This artificial boundary condition effectively restricts the class of physically admissible perturbations, and, as it turns out, the unstable modes associated with wormholes is precluded from this class, leading to the erroneous conclusion that the wormhole is linearly stable.

To overcome these problems, a method for transforming the singular wave equation to a regular one was introduced in Ref. [25] to treat the linearized perturbations of the Ellis-Bronnikov wormhole; this approach was subsequently generalized and referred to as "S-deformation method" in Ref. [26]. Both Refs. [25] and [26] refer to four- dimensional spacetimes. As already mentioned, higher dimensional extensions have been considered in Ref. [35], where the $(d+1)$- dimensional (with $d \geq 3$) Torii-Shinkai wormhole is introduced; here the linear stability analysis of this wormhole was performed, using again the S-deformation method to overcome singularity problems at the throat and eventually showing that the wormhole under consideration is unstable in any dimension.

I am now ready to describe the novelties of the present thesis; this thesis is related to papers [2, 1, 3] which I coauthored or authored during my doctoral studies; however, these articles consider only the case of spacetime dimension $d{+}1$. (5) Here I work in spacetime dimension $d{+}1$ (with $d \geq 3$), in the framework already outlined (a phantom scalar with self-interaction minimally coupled to gravity, the static spherically symmetric wormhole solutions arising from this setting and their linear stability analysis). The first result is the derivation of a coupled, 2×2 linear wave system subject to a constraint, describing the linearized dynamics of time-dependent spherically symmetric perturbations of such static solutions in terms of two gauge-invariant linear combination of the linearized perturbed metric coefficients and scalar field; a key feature of this system is that it is *regular* at the throat, provided the scalar field does not have a critical point there. The second result of my work is that, provided a non-trivial time-independent solution of the coupled 2×2 system is known, it is possible to decouple the system, obtaining a single wave equation (again regular) for an appropriate, gauge-invariant linear combination of the perturbed metric coefficients and scalar field, from which all other perturbations can be reconstructed; in most situations, such a time-independent solution can be found by varying the parameters of the family of static wormhole solutions under consideration. The above two results provide a general frame for spherically symmetric wormholes and their

[5] In particular, the present Introduction is greatly indebted to paper [1].

linear stability analysis which generalize to higher dimensions the approach of Ref. [1] and which represent an alternative to the S-deformation approach of Refs. [25, 26, 35]: no S-deformation of the linearized perturbation equations is necessary in the gauge-invariant approach of this thesis, since there is no singularity to be eliminated.

The method is then applied to the Torii-Shinkai and the Ellis-Bronnikov wormholes, and to a spherically symmetric Anti-de Sitter (AdS)-type wormhole which connects two asymptotic AdS ends (in the sequel referred to as AdS wormhole) (6) Let me recall that such wormholes live in spacetime dimension $d + 1$, with arbitrary $d \geq$ in the Torii-Shinkai case and with $d = 3$ in the Ellis-Bronnikov and AdS cases. In the case of the Ellis-Bronnikov and the AdS wormholes, the derivation reduces exactly to that proposed in Ref. [1]; in particular, for the Ellis-Bronnikov wormhole, I show that the master equation agrees precisely with the one obtained in Ref. [25] by the S-method. Note that the linear instability of the AdS wormhole has been addressed for the first time right in Ref. [1], using the four-dimensional version of the gauge-invariant method presented in this thesis. In the case of the Torii-Shinkai wormhole, I obtain a master equation that coincides with that found by Torii and Shinkai themselves in Ref. [35], after applying the S-method. However, as already mentioned, differently from the deduction of Ref. [35], my inference of the linear instability of the Torii-Shinkai wormhole does not involve the occurrence of any singularity.

The AdS wormhole has a de Sitter (dS) analog which, however, presents horizons; to go beyond the horizons it is necessary to consider a Kruskal-type extension of the dS wormhole spacetime, which, however, is non static and thus it outside the mainstream of the thesis. In any case, following the scheme of Ref. [1], I discuss the above issues and also present a partial result of linear instability, concerning the static part of the wormhole spacetime .

In addition to the just mentioned gauge-invariant method and its applications, in this thesis I provide two examples in which it is possible to decouple the linearized field system by fixing a particular gauge; indeed, by choosing two different coordinate systems for the Torii-Shinkai and the AdS wormholes, I derive (also in this case) completely regular wave-type master equations, describing the temporal behaviour of the radial perturbation. Although this approach does not employ gauge-invariant quantities, the coordinate-independent linear instability of the two wormholes is proved by showing that the perturbed spacetime tends to become singular in the large time limit. This inference of the linear instability of the Torii-Shinkai

[6] This is a special case of a family of static solution of the Einstein-scalar equations derived by Bronnikov and Fabris in Refs. [36, 11].

wormhole generalizes the approach of Ref. [2], in which an analougue proof is provided in the four-dimensional case for the EBMT wormhole. The interest of this deduction lies in the fact that all the equations and the quantities involved are regular and, in contrast to the scheme of Ref. [35] for the Torii-Shinkai wormhole and analogously to that of Ref. [1] for the AdS case, there is no need to introduce any regularization formalism like the S-deformation method.

Finally, in order to complete the stability discussion, in this thesis I also provide a detailed analysis for the behaviour of the solution of the master equations in the Torii-Shinkai, Ellis-Bronnikov, AdS and dS case, based on a rigorous spectral analysis of the Schrödinger operator appearing therein. In general, a negative eigenvalue of the Schrödinger operator gives rise to a pair of modes, one exponentially growing and the other one exponentially decaying with respect to the time variable; a positive eigenvalue gives rise to a pair of oscillating modes, while a positive energy level lying in the continuous spectrum gives rise to a pair of non-normalizable oscillating modes, corresponding to generalized eigenfunctions of the Schrödinger operator; if zero is an eigenvalue it gives rise to a pair of normalizable modes, one of them constant and the other one linearly growing with time. Partially following the results of Ref. [1], I show that in the Ellis-Bronnikov case, the solution can be expanded in terms of an exponentially growing, an exponentially decaying, a constant, a linearly growing mode and a continuum of oscillators associated with non-normalizable modes, while in the Torii-Shinkai and in the EBMT case, the solution can be expanded in terms of an exponentially growing, an exponentially decaying and a continuum of oscillators associated with non-normalizable modes. In contrast to this, in the AdS case the spectrum of the Schrödinger operator is a pure point spectrum, giving rise to an exponentially growing, an exponentially decaying, and to an infinite, discrete set of oscillating normalizable modes: this is due to the Dirichlet-type boundary conditions imposed at the AdS boundary, which give rise to a regular Sturm-Liouville problem.

Although the linear stability analysis is undoubtedly the fundamental issue of this work, some of its sections are devoted to the study of the geometrical properties of static spherically symmetric wormhole configurations, considering in particular their embedding diagrams and geodesic motion. I provide a general method to build the embedding diagrams of the Chaucy surface (at a fixed value of an angular coordinate) of wormholes as two-dimensional "tunnel-shaped" hypersurfaces of suitably chosen ambient spaces; this is applied to the Ellis-Bronnikov, the EBMT and the AdS cases, showing, in particular, that the slices of the EMBT and of the AdS whormholes can be entirely embedded, respectively, into the three-dimensional euclidean space and into a space with constant curvature. In addition, it is proved that the Ellis-Bronnikov and the AdS wormholes' slices can not be embedded in $\mathbb{R}^3$, unless you settle for embedding just a neighbourhood of the throat.

A complete analysis of the null and timelike geodesic motion in the case of the Ellis-Bronnikov and the AdS wormholes is also performed, showing that in both cases it is possible to obtain circular orbits as well as trajectories which cross the wormhole throat; some of these geodesics are plotted in the corresponding embedding diagrams. Admittedly, the Ellis-Bronnikov's diagram and geodesics have already been considered in the recent paper [37]; they have been introduced in the present thesis just for completeness. On the contrary, to my knowledge, the AdS case has not been considered so far. Actually, in the very recent paper [38], which appeared after submission of my work [3], a complete analysis of the geodesic motion near the throats of static, spherically symmetric traversable wormholes is performed, even in the case in which the coupling phantom scalar field has a self-interacting potential; however, the authors of Ref. [38] consider only wormholes which are asymptotically flat, which is not the case of the AdS wormhole.

Organization of the thesis

The present thesis is divided in two parts. The first part is substantially an introduction on static spherically symmetric wormholes supported by self-interacting phantom scalar field in higher dimensional General Relativity. The first part is organized as follows.

Chapter 1 contains some general results on the Lagrangian formulation for $(d+1)$-dimensional ($d \geq 3$) spacetimes supported by a (phantom) scalar field with an arbitrary self-interaction potential, and the derivation of the corresponding field equations, that is, the Einstein's and the Klein-Gordon equations.

In Chapter 2, I specialize this equations to the case of spherically symmetric wormholes: in Sections 2.1 and 2.2, I introduce the most general local representation of a metric describing a spherically symmetric spacetime M_{d+1} diffeomorphic to $M_2 \times S^{d-1}$, where M_2 is a two dimensional Lorentzian manifold and S^{d-1} is the unit $(d-1)$-sphere. In Section 2.3, I derived some conditions that the coefficients of a spherically symmetric metric must fulfill in order to describe a (four-dimensional) static wormhole configuration; in particular, the radial coefficient of the angular part of a wormhole metric has to possess a positive minimum, which represent the throat size. In the same section, fixing a value for the temporal and of an angular coordinate, I propose a general strategy to embed the corresponding two-dimensional wormhole slice as a "tunnel-shaped" hypersurface into a suitably defined ambient space. In the next Section 2.4, I prove that every matter or field configuration supporting a wormhole spacetime, if any, has to violate the weak energy condition; an example of this fact is given by the phantom scalar field supporting the Ellis-Bronnikov-Morris-Thorne (EBMT) wormhole. In the last Section 2.5, I write explicitly the field equations for the spherically-symmetric wormholes and phantom scalar fields.

In Chapter 3, I presented some known static wormhole spacetimes, deriving their metric (and the corresponding scalar field) directly from the static field equation (background equation) given in Section 3.1: Section 3.2 is focused on the $(d+1)$-dimensional Torii-Shinkai wormhole, Section 3.3 on the non reflection symmetric Ellis-Bronnikov wormhole, while Section 3.4 considers a wormhole connecting two Anti-de Sitter (AdS) universes. In particular, for the EBMT, the Ellis-Bronnikov and the AdS wormholes, I build the embedding diagrams, following the general strategy introduced in Chapter 1; in addition, I fully study the qualitative features of timelike and null geodesics in these wormholes, profiting from a general discussion on this topic, contained in Appendix A. In the last Section 3.5, I consider a wormhole connecting two de Sitter (dS) universes; by introducing different coordinations, I built an extension of the dS metric beyond the cosmological horizons of its spacetime.

The second part of the thesis is devoted to the linear stability analysis of static wormhole configurations. Some preliminaries are contained in Chapter 4, where I introduce spherically symmetric perturbations (Section 4.1), I make a few general comments regarding the possibility of transforming their expressions by introducing an infinitesimal gauge transformation (Section 4.2), and I derive the corresponding linearized field equations (Section 4.3). Some of the results of Section 4.2 are explained in detail in Appendix B.

In the next Chapter 5, I describe a gauge-dependent method for studying the linear stability of the Torii-Shinkai and the AdS wormholes; this consists in fixing a particular gauge (which strictly depends on the static solution under consideration) such that the linearized field equations are easily decouplable. I show that, considering a suitably defined coordinate system, it is possible to write two of the perturbations functions in terms on the radial perturbation, which is proved to satisfy a single regular wave-type master equation. This master equation has an associated Schrödinger operator possessing a unique bound state with negative energy, a fact which implies the divergence in the large temporal limit of the radial perturbation. The (gauge-invariant) linear instability of the two considered wormholes is proved by showing that the linearized scalar curvature of the perturbed spacetimes diverge in any coordinate system.

In Chapter 6, which is the core of the thesis, I outline a general gauge-invariant method for decoupling the linearied field equations for a perturbed $(d+1)$-dimensional static spherically symmetric wormhole solution. In particular: I consider gauge-fixed setting (in which the scalar field is held fixed) in order to simplify and partially decouple the linearized field system (Sections 6.1 and 6.2); in Sections 6.3 and 6.4, I introduce a set of combinations of the perturbations which are invariant with respect to infinitesimal coordinate transformations, and the linearized field equations are then cast into a constrained wave system for two of these gauge-invariant quantities; finally, in Section 6.5, I show how to decouple this wave system, provided a static solution of the linearized field equations is available, in which case a single regular master wave equation is obtained.

In the forthcoming Chapter 7, I apply the gauge-invariant method of Chapter 6 to the Torii-Shinkai, the Ellis-Bronnikov and the AdS wormholes, yielding three wave-type master equations. To this purpose, in Section 7.1, a general strategy is given to provide the static solution of the linearized field equations required, by varying the parameters on which the family of wormholes under consideration depend. In Section 7.5, the gauge-invariant method is also applied to the (unextended) static dS wormhole, deriving a master equation concerning perturbations which are confined within the horizons, that is, in the static part of this spacetime.

All the master equations obtained in Chapter 7 contain Schrödinger-type differential operators with a point spectrum consisting in a single, negative eigenvalue, a fact that implies the divergence of the gauge-invariant solutions of the equations, and then the linear instability of the Torii-Shinkai, the Ellis-Bronnikov, the AdS wormholes, and of the static part of the dS wormhole.

The last (non numbered) chapter includes conclusions, limitations and possible future applications (mainly of the gauge-invariant method of Chapter 6).

The technical details concerning the solution of the master equations and the spectral properties of the corresponding Schrödinger-type operators (based on rigorous techniques from functional analysis) are contained in the remaining appendices; in particular, in Appendix C, I recall some general results about the spectral decomposition of selfadjoint operators in L^2 with respect to (generalized) orthonormal bases made up of their proper and improper eigenfunctions; Appendix D is devoted to the analysis of the spectral features of the Schrödinger operators appearing in the master equations for the Torii-Shinkai, the Ellis-Bronnikov, the AdS and the dS wormhole; finally, the spectral decomposition of the solutions of all the corresponding master equations are derived in Appendix E.

Mathematical and physical background

Throughout this thesis I use the the formalism of the higher dimensional General Relativity and the common notations and symbols for the mathematical objects usually involved. In particular, let me stipulate what follows.

(i) A $(d+1)$-*dimensional spacetime* $(M_{d+1}, \mathbf{g})$ (with $d \geq 3$) is a $(d+1)$-dimensional differential manifold M_{d+1} provided with a Lorentzian metric $\mathbf{g}$ with signature $(1, d)$ and the associated Levi-Civita connection, hereafter indicated with ∇.

(ii) A *system of coordinates* $(x^\mu)_\mu \equiv (x^\mu)_{\mu=1,\dots,d+1}$ on M_{d+1} is an homeomorphism onto its image

$$(x^\mu)_\mu : \mathcal{O}_{d+1} \subseteq M_{d+1} \to \mathcal{O}_{d+1} \subseteq \mathbb{R}^{d+1} ,$$

where, $\mathcal{O}_{d+1}$ is an open subset of M_{d+1}; in the sequel, we will use the following abuse of notation: the symbol $(x^\mu)_\mu$ stands for both the above defined mapping, defining the coordinate system, and for a generic point of its image $\mathcal{O}_{d+1}$, which is the range of the coordinate system.

(iii) The *coordinate bases* of the tangent space and the cotangent space in $\mathbf{x} := (x^1, \dots, x^{d+1}) \in \mathcal{O}_{d+1}$ are

$$\left(\frac{\partial}{\partial x^\mu}\bigg|_{\mathbf{x}} \right)_{\mu=1,\dots,d+1} \qquad \text{and} \qquad \left(dx^\mu\Big|_{\mathbf{x}} \right)_{\mu=1,\dots,d+1} .$$

(iv) In terms of the coordinates $(x^\mu)_\mu$, the metric $\mathbf{g}$ can be locally written as [7]

$$\mathbf{g} = g_{\mu\nu}(\mathbf{x}) dx^\mu \otimes dx^\nu , \qquad (\mathbf{x} \in \mathcal{O}_{d+1}) ,$$

where the coefficients

$$g_{\mu\nu} : \mathcal{O} \subseteq \mathbb{R}^{d+1} \to \mathbb{R} \qquad (\mu, \nu = 1, \dots, d+1)$$

are smooth functions such that

$$[g_{\mu\nu}]^T = [g_{\mu\nu}] , \qquad \det[g_{\mu\nu}] < 0 , \qquad [g_{\mu\nu}]^{-1} = [g^{\mu\nu}] ;$$

in the sequel we will write

$$(dx^\mu)^2 := dx^\mu \otimes dx^\mu , \qquad dx^\mu dx^\nu := \frac{1}{2} \left(dx^\mu \otimes dx^\nu + dx^\nu \otimes dx^\mu \right) ,$$

so that

$$\mathbf{g} = g_{\mu\nu}(\mathbf{x}) dx^\mu dx^\nu , \qquad (\mathbf{x} \in \mathcal{O}_{d+1}) .$$

[7] In the sequel, we will use the same symbol for the metric and its local representation.

(v) The *Christoffel coefficients* associated to the Levi-Civita connection ∇ in any coordinate system $(x^\mu)_\mu$ are

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left[\frac{\partial}{\partial x^\mu} g_{\rho\nu} + \frac{\partial}{\partial x^\nu} g_{\rho\mu} - \frac{\partial}{\partial x^\rho} g_{\mu\nu}\right] \qquad (\lambda, \mu, \nu = 1, ..., d+1). \tag{1}$$

(vi) The components of the *Riemann tensor* associated to the Levi-Civita connection ∇ in any coordinate system $(x^\mu)_\mu$ are

$$R^\kappa_{\mu\nu\lambda} = \frac{\partial}{\partial x^\nu}\Gamma^\kappa_{\lambda\mu} - \frac{\partial}{\partial x^\lambda}\Gamma^\kappa_{\nu\mu} + \Gamma^\kappa_{\nu\rho}\Gamma^\rho_{\lambda\mu} - \Gamma^\kappa_{\lambda\rho}\Gamma^\rho_{\nu\mu} \qquad (\kappa, \mu, \nu, \lambda = 1, ..., d+1). \tag{2}$$

(vii) The components of the *Ricci tensor* associated to the Levi-Civita connection ∇ in any coordinate system $(x^\mu)_\mu$ are

$$R_{\mu\nu} = R^\rho_{\mu\rho\nu} \qquad (\mu, \nu = 1, ..., d+1). \tag{3}$$

(viii) The *scalar curvature* associated to the Levi-Civita connection ∇ is

$$R = R^\mu_\mu = g^{\mu\nu} R_{\nu\mu}. \tag{4}$$

(ix) The components of the *Einstein tensor* associated to the Levi-Civita connection ∇ in any coordinate system $(x^\mu)_\mu$ are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \qquad (\mu, \nu = 1, ..., d+1). \tag{5}$$

(x) The *Einstein field equations* (in the sequel usually referred to as *Einstein's equations*) for the spacetime metric $\mathbf{g}$ in presence of energy, matter of fields represented by a *stress-energy tensor field* with components $T_{\mu\nu}$ are

$$G_{\mu\nu} = \kappa T_{\mu\nu} \qquad (\mu, \nu = 1, ..., d+1). \tag{6}$$

where $\kappa = \frac{8\pi G}{c^4}$ is the usual coupling constant.

Remark 1 Throughout this work we choose units in which

$$c = 1\,, \qquad \hbar = 1$$

and the signature convention $(-, +, ...+)$ for the metric $\mathbf{g}$.

Remark 2 Einstein's equations (6) have the equivalent form

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \qquad (\mu, \nu = 1, ..., d+1), \tag{7}$$

where $T := T^{\mu}_{\mu}$.

Remark 3 In the case in which

$$T_{\mu\nu} = -\frac{\Lambda}{\kappa}g_{\mu\nu} \qquad \Lambda \in \mathbb{R} \qquad (\mu, \nu = 1, ..., d+1)\,, \tag{8}$$

Einstein's equations (6) reads

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \qquad (\mu, \nu = 1, ..., d+1)\,. \tag{9}$$

The tensor (8) describes a vacuum state with constant energy density ρ_{vac}, and isotropic pressure p_{vac} defined as

$$\rho_{\text{vac}} = -p_{\text{vac}} = \frac{\Lambda}{\kappa}\,.$$

The equations (9) were introduced by Einstein himself with a somehow different motivation; the constant Λ appearing therein is usually referred to as "cosmological constant".

Part I

Static spherically symmetric wormholes supported by self-interacting phantom scalar fields

Chapter 1

Spacetimes supported by self-interacting scalar fields in arbitrary dimension

In this Chapter, I would like to make a brief overview on scalar fields in the context of General Relativity. In particular, in Section 1.1, I will rederive the Klein-Gordon equation for scalar fields and its generalizations to GR. In Section 1.2, I will recover the field equations for spacetimes of arbitrary dimension, whose gravitational field is minimally coupled to a self-interacting scalar field; these equations are exactly the Einstein's and the Klein-Gordon equations. I will finally prove that the Klein-Gordon equation is actually implied by Einstein's equations. Obviously, all the content of the present chapter can be found in any textbook on Quantum Mechanics and General Relativity (see, e.g. Refs. [39, 40]). However, most of the forthcoming, well-known equations involving scalar fields are rewritten using a parameter ς, which is usually set to 1 for standard fields. In the sequel we will see that the choice $\varsigma = -1$ defines a class of scalar fields, usually referred to as "phantom scalar fields", which is of great importance in studying wormhole configurations in General Relativity.

1.1 Scalar fields and Klein-Gordon equation

In the context of quantum mechanics, it is well known that every system is described by an Hilbert space $\mathcal{H}$ whose vectors $\psi \equiv \psi_t$ represents the possible states of the system at fixed time t; in 1925, Erwin Schrödinger postulated that the evolution of the states of the system was described by the well known equation

$$i\frac{d\psi}{dt} = \hat{H}\psi\,, \tag{1.1}$$

where i is the imaginary unit and $\hat{H}$ is the Hamiltonian operator, namely a linear operator in $\mathcal{H}$ corresponding to the energy of the system (recall that in Remark 1 we have stipulated $\hbar = 1$). Obviously, $\frac{d}{dt}$ denotes the derivative of the vector $\psi \equiv \psi_t$ with respect to the time parameter t. In the case of a quantum particle with mass m and velocity v (i.e. with momentum $p = mv$) moving in the three-dimensional space $\mathbb{R}^3$ and subject to an external potential $U(\mathbf{x}) = \mathbf{U}(\mathbf{x}, \mathbf{u}, \mathbf{z})$, one choose as the Hilbert space $\mathcal{H}$, the functional space $L^2(\mathbb{R}^3)$ made of complex valued, square integrable functions on $\mathbb{R}^3$ and defines the Hamiltonian operator as

$$\hat{H}\psi \equiv \left(\frac{\hat{p}^2}{2m} + \hat{U}\right)\psi := -\frac{1}{2m}\nabla^2\psi + U(\mathbf{x})\psi\,,$$

where $\nabla^2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator of the three-dimensional Euclidean space; with abuse of notation, one can interpret the vectors representing the states of the system as the functions $\psi \equiv \psi_t(x) = \psi(t, x)$ such that $\psi(t, \cdot) \in L^2(\mathbb{R}^3)$ for all $t \in \mathbb{R}$ (actually, the time parameter t should not be considered as a variable of the function). With this abuse of notation, it results that every state function ψ must satisfy the wave-type equation

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi + U(\mathbf{x})\psi\,; \tag{1.2}$$

note that the replacement of symbol for ordinary derivatives $\frac{d}{dt}$ appearing in Eq. (1.1) with that for partial derivatives $\frac{\partial}{\partial t}$ is coherent with the abuse of notation $\psi = \psi(t, x)$.

One of the main limit of the Schrödinger equation (1.2) lies in the fact that it not relativistic invariant and therefore can not describe the motion relativistic particles; in 1929, it was this fact that motivated Oskar Klein and Walter Gordon to look for a new equation which should have been a generalization of the Schrödinger equation. In special relativity, according to an inertial observer, the energy of a particle with rest mass m and three-momentum p is $E = \sqrt{p^2 + m^2}$ (recall that in Remark 1 we have stipulated $c = 1$); the corresponding Hamiltonian operator can be obtained by *quantizing* the energy E, that is

$$\hat{H} := \sqrt{-\nabla^2 + m^2c^4}\,,$$

which, inserted into Eq. (1.1), gives

$$i\frac{d\psi}{dt} = \sqrt{-\nabla^2 + m^2}\psi\,.$$

In order to remove the disadvantages related to the presence of a square root of a differential operator, one can *raise to the square* the equation, getting

$$\left(-\frac{\partial^2}{\partial t^2}+\nabla^2\right)\psi=m^2\psi\,. \tag{1.3}$$

Note that the right hand side of Eq. (1.3) is the d'Alembert operator of the function ψ in the Minkowski space defined by the metric

$$\eta_{\mu\nu}=(-1,1,1,1)\cdot\mathrm{Id}_4$$

(indeed, it is equal to $\eta^{\mu\nu}\partial_\nu\partial_\mu\psi$). Therefore, Eq. (1.3) can be naturally generalized to a higher dimensional curved spacetime with metric $\mathbf{g}\equiv g_{\mu\nu}$ by making the formal substitutions $\eta_{\mu\nu}\mapsto g_{\mu\nu}$ and $\partial_\mu\mapsto\nabla_\nu$; renaming the unknown field ψ with the Greek letter Φ, these replacements leads to the so-called Klein-Gordon equation

$$\nabla^\mu\nabla_\mu\Phi=m^2\Phi\,, \tag{1.4}$$

where $\nabla^\mu\nabla_\mu=g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alambert operator of the metric $\mathbf{g}$.
Therefore, we say that a "scalar field with mass m" in General Relativity is a scalar function defined on a spacetime $(M_{d+1},\mathbf{g})$

$$\Phi\,:\,M_{d+1}\to\mathbb{R} \tag{1.5}$$

which satisfies the Klein-Gordon equation (1.4); for $m=0$ one speaks of a massless scalar field. The most common generalization of the above definition consists in replacing the term $m^2\Phi$ on the right hand side of Eq. (1.4) with an arbitrary function of Φ, which is sometimes (but non necessarily) polynomial in Φ; the scalar field satisfying such a modified nonlinear equation is usually referred to as "self-interacting scalar field". In the next section, we will use the Lagrangian formulation to provide a precise definition of such fields.

1.2 Lagrangian formulation for spacetimes in arbitrary dimension supported by a self-interacting scalar field

Let us consider a $(d+1)$-dimensional spacetime $(M_{d+1},\mathbf{g})$ with $d\geq 3$ and the associated Levi-Civita connection, as already mentioned indicated with ∇; moreover, we assume that the gravitational field $\mathbf{g}$ is minimally coupled to a self-interacting scalar field Φ as in Eq. (1.5). In the sequel we will indicate with the same letter Φ, the local representation of the scalar field Φ according to the coordinate system $(x^\mu)_\mu$, that is, we will assume

$$\Phi \,:\, \mathcal{O}_{d+1} \subseteq \mathbb{R}^4 \to \mathbb{R}\,, \tag{1.6}$$

where $\mathcal{O}_{d+1}$ is the range of the coordinates. The common assumption is that this configuration is defined by the sum of the Hilbert action functional S_H and a functional S_{KG} (to which we will refer to as the Klein-Gordon action functional) defined as

$$\begin{gathered} S[\mathbf{g},\Phi] := S_H[\mathbf{g},\Phi] + S_{KG}[\mathbf{g},\Phi]\,, \\ S_H[\mathbf{g},\Phi] := \int_{M_{d+1}} \frac{R}{2\kappa} dv\,, \quad S_{KG}[\mathbf{g},\Phi] := \int_{M_{d+1}} -\left[\frac{\varsigma}{2}\nabla^\mu\Phi\cdot\nabla_\mu\Phi + V(\Phi)\right] dv\,; \end{gathered} \tag{1.7}$$

where the scalar function

$$V \,:\, \Phi(\mathcal{O}_{d+1}) \subseteq \mathbb{R} \to \mathbb{R}$$

is a potential which describes the self-interaction of the scalar field Φ, while $\kappa = 8\pi G$, R and $dv = \sqrt{|\det[g_{\mu\nu}]|}\prod_{\mu=0}^{d} dx^\mu$ are, respectively, the usual coupling constant, the scalar curvature of the spacetime M_{d+1} and the volume element associated with the metric $\mathbf{g}$. In the case of standard scalar fields, the constant ς is equal to 1. However, we will see that sometimes it can be useful (and acceptable) to consider different values of ς; hence, for the moment, we keep its value undetermined. The evolution of the system is described by the stationary conditions

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0\,, \qquad \frac{\delta S}{\delta \Phi} = 0\,; \tag{1.8}$$

since the functional derivatives of the Hilbert action with respect to $g_{\mu\nu}$ is $\frac{\delta S_H}{\delta g_{\mu\nu}} = -\frac{1}{2\kappa}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)$, [8] while, obviously, $\frac{\delta S_H}{\delta\Phi} = 0$, the stationary conditions (1.8) can be rephrased as

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 2\kappa\frac{\delta S_{KG}}{\delta g_{\mu\nu}}\,, \qquad \frac{\delta S_{KG}}{\delta\Phi} = 0\,. \tag{1.9}$$

The next theorem states that the equations in (1.9) are respectively equivalent to Einstein's equations with a stress-energy tensor field related to the scalar field Φ and to the generalized Klein-Gordon equation with self-interacting potential, hence justifying the assumption (1.7).

[8]See, e.g., Ref. [39] (page 454): here, the functional derivative is computed with respect to the variation $\delta g^{\mu\nu}$ which is defined as the variation of the inverse metric $g^{\mu\nu}$ corresponding to $\delta g_{\mu\nu}$; in the sequel we will call this quantity $\delta[g^{\mu\nu}]$ and prove that it is equal to $-g^{\mu\rho}g^{\nu\lambda}\delta g_{\lambda\rho}$ [see Eq. (1.12)]. Therefore, we have that $\frac{\delta S_{KG}}{g_{\mu\nu}} = -g^{\mu\rho}g^{\nu\lambda}\frac{\delta S_{KG}}{g^{\mu\nu}}$.

Theorem 1 *The stationary conditions* (1.9) *are equivalent to*

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}\,, \qquad \textit{(Einstein's equations with a scalar field)}$$

$$T_{\mu\nu} := \nabla_\mu \Phi \cdot \nabla_\nu \Phi - g_{\mu\nu} \left(\frac{\varsigma}{2} \nabla^\lambda \Phi \cdot \nabla_\lambda \Phi + V(\Phi) \right) \tag{1.10}$$

$$\varsigma \nabla^\mu \nabla_\mu \Phi = V'(\Phi) \qquad \textit{(Klein-Gordon equation)} \tag{1.11}$$

Proof. Let us start computing the functional derivative of the Klein-Gordon action S_{KG} with respect to the metric $\mathbf{g}$. Let be $\delta g_{\mu\nu}$ an infinitesimal variation of the metric tensor $g_{\mu\nu}$ of compact support if seen as a function from the spacetime M_{d+1} to the tensor bundle $T^2_0 M_{d+1}$; hence, denoting with g the absolute value of the metric determinant $\det[g_{\mu\nu}]$ and with $\delta[\cdot]$ the variation of a given quantity corresponding to $\delta g_{\mu\nu}$, one has that

$$\delta[g] = g\, g^{\mu\nu}\, \delta g_{\mu\nu}\,, \qquad \delta\left[\sqrt{g}\right] = \frac{1}{2} \sqrt{g}\, g^{\mu\nu}\, \delta g_{\mu\nu}\,, \qquad \delta\left[g^{\mu\nu}\right] = -g^{\mu\rho}\, g^{\nu\lambda}\, \delta g_{\lambda\rho}\,. \tag{1.12}$$

The first equality of Eq. (1.12) is proven directly observing that

$$\begin{aligned} g + \delta[g] &= \det\left[g_{\mu\nu} + \delta g_{\mu\nu}\right] = \det[g_{\mu\lambda}] \det[\delta^\lambda_\nu + g^{\lambda\rho}\, \delta g_{\rho\nu}] \\ &= g \left(1 + \mathrm{Tr}\left(g^{\lambda\rho}\, \delta g_{\rho\nu} \right) \right) = g\, g^{\lambda\rho}\, \delta g_{\lambda\rho}\,. \end{aligned} \tag{1.13}$$

The second equality of Eq. (1.12) is a consequence of the first one since $\delta\left[\sqrt{g}\right] = \frac{1}{2\sqrt{g}} \delta g$. Finally, the third equality of Eq. (1.12) follows noting that $0 = \delta[\delta^\nu_\rho] = \delta[g_{\rho\lambda}\, g^{\lambda\nu}] = \delta[g_{\rho\lambda}]\, g^{\lambda\nu} + g_{\rho\lambda}\, \delta[g^{\lambda\nu}]$, which implies that $g_{\rho\lambda}\, \delta[g^{\lambda\nu}] = -g^{\lambda\nu}\, \delta[g_{\rho\lambda}]$; the thesis is obtained by multiplying the both the sides of the last equality by $g^{\mu\rho}$.

Setting $d\mathbf{x} := \prod_{\mu=0}^{d} dx^\mu$ and recalling that $dv = \sqrt{g}\, d\mathbf{x}$, we are now ready to compute the variation of S_{KG}:

$$\begin{aligned} \delta S_{KG} &= -\int_{M_{d+1}} \left\{ \left[\frac{\varsigma}{2} \delta[g^{\mu\nu}] \nabla_\nu \Phi \cdot \nabla_\mu \Phi \right] \sqrt{g} + \left[\frac{\varsigma}{2} \nabla^\mu \Phi \cdot \nabla_\mu \Phi + V(\Phi) \right] \delta[\sqrt{g}] \right\} d\mathbf{x} \\ &= -\int_{M_{d+1}} \left\{ - \left[\frac{\varsigma}{2} \nabla^\lambda \Phi \cdot \nabla^\rho \Phi \right] \sqrt{g}\, \delta g_{\lambda\rho} \right. \\ &\quad \left. + \left[\frac{\varsigma}{2} \nabla^\lambda \Phi \cdot \nabla_\lambda \Phi + V(\Phi) \right] \frac{1}{2} \sqrt{g} g^{\mu\nu} \delta g_{\mu\nu} \right\} d\mathbf{x} \\ &= \frac{1}{2} \int_{M_{d+1}} \delta g_{\mu\nu} \left\{ \varsigma \nabla^\mu \Phi \cdot \nabla^\nu \Phi - g^{\mu\nu} \left[\frac{\varsigma}{2} \nabla^\lambda \Phi \cdot \nabla_\lambda \Phi + V(\Phi) \right] \right\} \sqrt{g}\, d\mathbf{x}\,. \end{aligned}$$

This concludes the computation of $\frac{\delta S_{KG}}{\delta g_{\mu\nu}}$ since, by definition, $\delta S_{KG} = \int \delta g_{\mu\nu} \cdot \frac{\delta S_{KG}}{\delta g_{\mu\nu}} dv$; then, Eq. (1.10) is obtained inserting the expression for $\frac{\delta S_{KG}}{\delta g_{\mu\nu}}$ in the first equality of Eq. (1.9) and lowering the indices.
Now, it remains to prove Eq. (1.11). Let be $\delta\Phi$ an infinitesimal variation of the scalar field Φ of compact support; analogously as before, we denote with $\delta[\cdot]$ the variation of a given quantity due to the variation $\delta\Phi$. It is not difficult to prove that

$$\delta[\nabla^\mu \Phi] = \nabla^\mu \delta\Phi \,, \qquad \delta[\nabla_\mu \Phi] = \nabla_\mu \delta\Phi \,, \qquad \delta[V(\Phi)] = V'(\Phi)\delta\Phi \,,$$

so that the variation of S_{KG} corresponding to $\delta\Phi$ is given by

$$\begin{aligned} \delta S_{KG} &= \int_{M_{d+1}} -\left[\frac{\varsigma}{2}\delta[\nabla^\mu]\Phi \cdot \nabla_\mu \Phi + \frac{\varsigma}{2}\nabla^\mu \Phi \cdot \delta[\nabla_\mu \Phi] + \delta[V(\Phi)]\right] dv \\ &= \int_{M_{d+1}} -\left[\frac{\varsigma}{2} g^{\mu\nu} \nabla_\nu \delta\Phi \cdot \nabla_\mu \Phi + \frac{\varsigma}{2} g^{\mu\nu} \nabla_\nu \Phi \cdot \nabla_\mu \delta\Phi + V'(\Phi)\delta\Phi\right] dv \\ &= \int_{M_{d+1}} \delta\Phi \left[\varsigma \nabla^\mu \nabla_\mu \Phi - V'(\Phi)\right] dv \,. \end{aligned}$$

In the last equality we have used the fact that $\nabla^\mu \Phi \nabla_\mu \delta\Phi = \nabla_\mu(\nabla^\mu \Phi \cdot \delta\Phi) - \nabla_\mu \nabla^\mu \Phi \cdot \delta\Phi = \mathrm{div} X - \nabla^\mu \nabla_\mu \Phi \cdot \delta\Phi$, where $X^\mu = \nabla^\mu \Phi \cdot \delta\Phi$, and the fact that $\int \mathrm{div} X \, dv{=}0$ (which is due to Stokes's theorem and to the fact that $\delta\Phi$ has a compact support). The expression for the functional derivative $\frac{\delta S_{KG}}{\delta \Phi}$ is obtained again from the variation δS_{KG} since $\delta S_{KG} = \int \delta\Phi \cdot \frac{\delta S_{KG}}{\delta\Phi} dv$; then, Eq. (1.11) is obtained inserting the expression for $\frac{\delta S_{KG}}{\delta\Phi}$ into the second stationary condition in Eq. (1.9).

□

Actually, the two field equations (1.10,1.11) for the metric **g** and the scalar field Φ are not independent, as stated by the following

Theorem 2 *Einstein's equations* (1.10) *imply the Klein-Gordon equation* (1.11).

Proof. Since the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ has null divergence, every stress-energy tensor satisfying Einstein's equations must have the same property, namely $\nabla_\mu T^\mu_\nu = 0$; indeed this is a condition on the energy and the matter that can be interpreted as a conservation law.
In the case in which the matter is described by a scalar field Φ, the stress-energy tensor $T_{\mu\nu}$ reads as in Eq. (1.10); the computation of its the divergence

is

$$
\begin{aligned}
\nabla_\mu T^\mu_\nu &= \nabla_\mu \left[\varsigma \nabla^\mu \Phi \cdot \nabla_\nu \Phi - \delta^\mu_\nu \left(\frac{\varsigma}{2} \nabla^\lambda \Phi \cdot \nabla_\lambda \Phi + V(\Phi)\right)\right] \\
&= \varsigma(\nabla_\mu \nabla^\mu \Phi) \cdot \nabla_\nu \Phi + \varsigma \nabla^\mu \Phi \cdot (\nabla_{\mu\nu} \Phi) \\
&\quad - \delta^\mu_\nu \left[\frac{\varsigma}{2}(\nabla_\mu \nabla^\lambda \Phi) \cdot \nabla_\lambda \Phi + \frac{\varsigma}{2} \nabla^\lambda \Phi \cdot (\nabla_{\mu\lambda} \Phi) + V'(\Phi) \nabla_\mu \Phi\right] \\
&= \varsigma(\nabla_\mu \nabla^\mu \Phi) \cdot \nabla_\nu \Phi + \varsigma \nabla^\mu \Phi \cdot \nabla_{\mu\nu} \Phi - \frac{\varsigma}{2}(\nabla_\nu \nabla^\lambda \Phi) \cdot \nabla_\lambda \Phi \\
&\quad - \frac{\varsigma}{2} \nabla^\lambda \Phi \cdot \nabla_{\nu\lambda} \Phi - V'(\Phi) \nabla_\nu \Phi \\
&= \varsigma(\nabla_\mu \nabla^\mu \Phi) \cdot \nabla_\nu \Phi + \varsigma \nabla^\mu \Phi \cdot \nabla_{\mu\nu} \Phi - \frac{\varsigma}{2} \nabla^\mu \Phi \cdot \nabla_{\nu\mu} \Phi - \frac{\varsigma}{2} \nabla^\mu \Phi \cdot \nabla_{\nu\mu} \Phi \\
&\quad - V'(\Phi) \nabla_\nu \Phi \\
&= [\varsigma \nabla_\mu \nabla^\mu \Phi - V'(\Phi)] \cdot \nabla_\nu \Phi \,;
\end{aligned}
$$

in the penultimate equality we have use the fact that $(\nabla_\nu \nabla^\lambda \Phi) \cdot \nabla_\lambda \Phi = \nabla_\nu (g^{\mu\lambda} \nabla_\mu \Phi) \cdot \nabla_\lambda \Phi = \nabla^\mu \Phi \cdot \nabla_{\nu\mu} \Phi$, while in the last equality we have used the symmetry of $\nabla_{\mu\nu} \Phi$.
Therefore, the conservation of energy $\nabla_\mu T^\mu_\nu = 0$ is equivalent to

$$
[\varsigma \nabla_\mu \nabla^\mu \Phi - V'(\Phi)] \cdot \nabla_\nu \Phi = 0\,. \tag{1.14}
$$

Let us prove that Eq. (1.14) actually implies the Klein-Gordon equation (1.11). Let be D the subset of M_{d+1} defined as

$$
D := \{m \in M_{d+1} \;:\; \nabla_\nu \Phi(m) \neq 0\}\,;
$$

obviously, on this subset Eq. (1.14) implies the Klein-Gordon equation. Our goal is to show that the Klein-Gordon equation holds even on the complement $M_{d+1} \backslash D$. Evidently, this is nontrivial as long as $M_{d+1} \backslash D \neq \emptyset$; hence we assume that the $D \neq M_{d+1}$, or in other words that the boundary of D is not empty. Since in this case the Klein-Gordon equation has to be satisfied on the boundary of D by continuity, we have that $\varsigma \nabla_\mu \nabla^\mu \Phi(m) = V'(\Phi(m))$ for every m in the closed set $\overline{D}$. We observe now that from the definition of D and (again) by continuity one has that $\nabla_\nu \Phi = 0$ on the complement of $\overline{D}$, that is $\Phi = \text{const}$ on each connected component of $M_{d+1} \backslash \overline{D}$; without loss of generality, from now on we can assume that $M_{d+1} \backslash \overline{D}$ is connected since, if it does not, one can simply make the following considerations for everyone of its connected components. Now, since the Klein-Gordon equation holds on the boundary of D, where, by continuity, $\Phi = \text{const}$ and $\nabla^\nu_\mu \Phi = 0$, one has that $V'(\text{const}) = 0$; but this means that the Klein-Gordon equation holds on $M_{d+1} \backslash D$, since therein $\Phi = \text{const}$ and $\nabla^\nu_\mu \Phi = 0$, and thus the equation is equivalent to $V'(\mathrm{const}) = 0$.

□

Before concluding the present section, for completeness, we give an alternative form to Einstein's equations (1.10).

Proposition 1 *Einstein's equations* (1.10) *are equivalent to*

$$R_{\mu\nu} = \kappa \left[\varsigma \nabla_\mu \Phi \cdot \nabla_\nu \Phi + V(\Phi) g_{\mu\nu} \right] . \tag{1.15}$$

Proof. The trace of the stress-energy tensor of scalar-field Φ in Eq. (1.10) is

$$T = T^\mu_\mu = \varsigma \nabla^\mu \Phi \cdot \nabla_\mu \Phi - 4 \left(\frac{\varsigma}{2} \nabla^\lambda \Phi \cdot \nabla_\lambda \Phi + V(\Phi) \right) = -\varsigma \nabla^\lambda \Phi \cdot \nabla_\lambda \Phi - 4V(\Phi) \, ;$$

the thesis is obtained immediately by inserting the expressions of $T_{\mu\nu}$ and T into the alternative form of Einstein's equation (7).

□

Chapter 2

Spherically symmetric wormholes in arbitrary dimension

2.1 Preliminaries

2.1.1 Some facts on two-dimensional Lorentzian manifolds

Let us consider a two-dimensional Lorentzian manifold $(M_2, \mathbf{g_2})$ and a coordinate system (t,x)

$$(t,x) : \mathcal{O} \subseteq M_2 \to \mathcal{O} \subseteq \mathbb{R}^2$$

such that the metric $\mathbf{g_2}$ can be locally written as

$$\mathbf{g_2} = -\alpha(t,x)^2 dt^2 + \gamma(t,x)^2 \left(dx + \beta(t,x)dt\right)^2 , \tag{2.1}$$

where the coefficients $\alpha, \gamma : \mathcal{O} \subseteq \mathbb{R}^2 \to (0,+\infty)$ and $\beta : \mathcal{O} \subseteq \mathbb{R}^2 \to \mathbb{R}$ are three smooth functions.

Remark 4 Every two-dimensional Lorentzian metric $\mathbf{g_2}$ has the local representation (2.1) in any coordinate system (t,x) with x spacelike.

Proof. We start recalling that, in an arbitrary coordinate system (t,x), the two-dimensional Lorentzian metric $\mathbf{g_2}$ has the general local representation

$$\mathbf{g_2} = g_{00}(t,x)dt^2 + 2g_{10}(t,x)dt\,dx + g_{11}(t,x)dx^2 \tag{2.2}$$

where $g_{00}, g_{10}, g_{11} : \mathcal{O} \subseteq \mathbb{R}^2 \to \mathbb{R}$ are three smooth functions such that

$$\begin{gathered} \det[g_{\mu\nu}](t,x) := g_{00}(t,x)g_{11}(t,x) - g_{01}(t,x)^2 < 0\,, \\ \text{for every } (t,x) \in \mathcal{O} \subseteq \mathbb{R}^2\,. \end{gathered} \tag{2.3}$$

Moreover, if the coordinate x is spacelike, then

$$0 < \mathbf{g_2}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = g_{11}(t,x)$$

for every $(t,x) \in \mathcal{O} \subseteq \mathbb{R}^2$. Hence, we set

$$\gamma(t,x) := \sqrt{g_{11}(t,x)}\,, \qquad \beta(t,x) := \frac{g_{01}(t,x)}{\gamma(t,x)^2}\,;$$

the function γ is smooth and never vanishes, whence β is well defined and smooth. Note that, using the functions γ and β, Eq. (2.3) yields

$$\det[g_{\mu\nu}](t,x) = \gamma^2(t,x)\left[g_{00}(t,x) - \gamma(t,x)^2\beta(t,x)^2\right] < 0\,,$$

from which it turns out that the function

$$\alpha(t,x) := \sqrt{-g_{00}(t,x) + \gamma(t,x)^2\beta(t,x)^2}$$

is well defined, smooth and strictly positive. Finally, in terms of the functions α, γ, β, the expression (2.2) becomes

$$\mathbf{g_2} = (-\alpha(t,x)^2 + \gamma(t,x)^2\beta(t,x)^2)dt^2 + 2\gamma^2(t,x)\beta(t,x)dx\,dt + \gamma(t,x)^2dx^2\,,$$

which is exactly Eq. (2.1).

□

Actually, by making an appropriate choice of the coordinates (t,x) on M_2 it is possible to make the function β disappear. This fact is stated (and proved) in the following

Lemma 1 *Let $(M_2, \mathbf{g_2})$ be a two-dimensional Lorentzian manifold and let (t,x) be an arbitrary coordinate system on M_2 with domain $\mathcal{O} \subseteq M_2$. Suppose that there exists a smooth function $\hat{x} : \mathcal{O} \subseteq M_2 \to \mathbb{R}$ with the property that its gradient grad$(\hat{x})$ is everywhere spacelike, that is,*

$$\mathbf{g_2}\Big(grad(\hat{x}), grad(\hat{x})\Big) > 0\,. \tag{2.4}$$

Then, for every $m \in M_2$ there exists a neighbourhood $\hat{\mathcal{O}} \subseteq \mathcal{O}$ of m and a smooth function $\hat{t} : \hat{\mathcal{O}} \subseteq \mathcal{O} \subseteq M_2 \to \mathbb{R}$ such that:

(i) $(\hat{t}, \hat{x})$ is a coordinate system on M_2 with domain $\hat{\mathcal{O}} \subseteq M_2$ and range $(\hat{t}, \hat{x})(\hat{\mathcal{O}}) = \hat{\mathcal{O}} \subseteq \mathbb{R}^2$;

(ii) $(\hat{t},\hat{x})$ *is an orthogonal coordinate system, that is, in these coordinates, the metric* $\mathbf{g_2}$ *possesses a local diagonal representation*

$$\mathbf{g_2} = -\hat{\alpha}(\hat{t},\hat{x})^2 d\hat{t}^2 + \hat{\gamma}(\hat{t},\hat{x})^2 d\hat{x}^2\,, \tag{2.5}$$

where $\hat{\alpha},\hat{\gamma} \,:\, \hat{\mathcal{O}} \subseteq \mathbb{R}^2 \to (0,+\infty)$ *are two smooth and positive functions.*

Moreover, $\hat{t}$ *has the property that its gradient* $grad(\hat{t})$ *is everywhere timelike, that is,*

$$\mathbf{g_2}\Big(grad(\hat{t}), grad(\hat{t})\Big) < 0\,; \tag{2.6}$$

Proof. We have already recalled that in an arbitrary coordinate system (t,x), the two-dimensional Lorentzian metric $\mathbf{g_2}$ has the general local representation (2.2), where the three smooth coefficients $g_{00}, g_{10}, g_{11} \,:\, \mathcal{O} \subseteq \mathbb{R}^2 \to \mathbb{R}$ satisfy Eq. (2.3); in addition, the gradient of the metric $\mathbf{g_2}$ applied to any smooth function $f \,:\, \mathcal{O} \subseteq M_2 \to \mathbb{R}$ has the local representation

$$\mathrm{grad}(f) = \left[g^{00}(t,x)\frac{\partial f}{\partial t} + g^{01}(t,x)\frac{\partial f}{\partial x}\right]\frac{\partial}{\partial t} + \left[g^{01}(t,x)\frac{\partial f}{\partial t} + g^{11}(t,x)\frac{\partial f}{\partial x}\right]\frac{\partial}{\partial x}\,, \tag{2.7}$$

where $[g^{\mu\nu}]_{\mu,\nu=0,1}$ denotes the inverse matrix of $[g_{\mu\nu}]_{\mu,\nu=0,1}$. Let us observe that the metric (2.2) is diagonal (i.e. $g_{01} = 0$) if and only if

$$\mathbf{g_2}\Big(\mathrm{grad}(t), \mathrm{grad}(x)\Big) = 0\,.$$

To prove this fact, we notice that, for $f = t$ and $f = x$, Eq. (2.7) reads

$$\mathrm{grad}(t) = g^{00}(t,x)\frac{\partial}{\partial t} + g^{01}(t,x)\frac{\partial}{\partial x}\,, \qquad \mathrm{grad}(x) = g^{01}(t,x)\frac{\partial}{\partial t} + g^{11}(t,x)\frac{\partial}{\partial x}\,,$$

which implies that

$$\begin{aligned}
&\mathbf{g_2}\Big(\mathrm{grad}(t), \mathrm{grad}(x)\Big) \\
&\quad = g_{00}(t,x)dt\Big(\mathrm{grad}(t)\Big)dt\Big(\mathrm{grad}(x)\Big) + g_{01}(t,x)dt\Big(\mathrm{grad}(t)\Big)dx\Big(\mathrm{grad}(x)\Big) \\
&\qquad\quad + g_{10}(t,x)dx\Big(\mathrm{grad}(t)\Big)dt\Big(\mathrm{grad}(x)\Big) + g_{11}(t,x)dx\Big(\mathrm{grad}(t)\Big)dx\Big(\mathrm{grad}(x)\Big) \\
&\quad = g_{00}(t,x)g^{00}(t,x)g^{01}(t,x) + g_{01}(t,x)g^{00}(t,x)g^{11}(t,x) \\
&\qquad\quad + g_{10}(t,x)g^{01}(t,x)g^{01}(t,x) + g_{11}(t,x)g^{01}(t,x)g^{11}(t,x) = \\
&\quad = \frac{1}{\det[g_{\mu\nu}]^2}\big[-g_{00}(t,x)g_{01}(t,x)g_{11}(t,x) + g_{00}(t,x)g_{01}(t,x)g_{11}(t,x) \\
&\qquad\quad + g_{01}(t,x)^3 - g_{00}(t,x)g_{01}(t,x)g_{11}(t,x)\big] = -\frac{g_{01}(t,x)}{\det\,[g_{\mu\nu}]^2}\,;
\end{aligned} \tag{2.8}$$

therefore $\mathbf{g_2}((\mathrm{grad}(t), \mathrm{grad}(x)) = 0$ if and only it $g_{01} = 0$.
Note that, in the diagonal case, one has that the coefficient g_{11} is positive if and only if

$$\mathbf{g_2}\Big(\mathrm{grad}(x), \mathrm{grad}(x)\Big) > 0\,;$$

this is easily proved noting that, if $g_{01} = 0$ then

$$\mathbf{g_2}\Big(\mathrm{grad}(x), \mathrm{grad}(x)\Big) = g_{11}(t,x)g^{11}(t,x)g^{11}(t,x) = \frac{g_{11}(t,x)g_{00}(t,x)^2}{\det[g_{\mu\nu}]^2}\,. \quad (2.9)$$

Moreover, if $g_{01} = 0$ and $g_{11} > 0$ then $g_{00} < 0$ as

$$0 > \det[g_{\mu\nu}] = g_{00}(t,x)g_{11}(t,x)\,. \quad (2.10)$$

As a consequence of the previous considerations, we can replace the item (ii) of the statement with the following item:

(ii') $\hat{t}$ has the property that its gradient grad($\hat{t}$) is orthogonal to the gradient of $\hat{x}$, that is,

$$\mathbf{g_2}\Big(\mathrm{grad}(\hat{t}), \mathrm{grad}(\hat{x})\Big) = 0\,. \quad (2.11)$$

Let us prove the equivalence of (ii) and (ii'). If (ii) is verified, that is the metric is diagonal $g_{01} = 0$, then, from Eq. (2.8) we have that (2.11) holds, namely (ii') is verified. Conversely, if (ii') is verified, that is the condition (2.11) holds, then, from Eq. (2.8), the metric is diagonal Eq. (2.8); in addition, since $g_{01} = 0$, the hypothesis (2.4) is equivalent to $g_{11} > 0$ [Eq.(2.9)], which guarantees that $g_{00} < 0$ because of Eq. (2.10).
In addition, it is not difficult to see that (ii') actually implies (i). Hence, we are left to prove only (ii').
In order to prove item (ii'), let us introduce a one-form $\hat{\theta}$ such that

$$\hat{\theta}\Big(\mathrm{grad}(\hat{x})\Big) = 0\,. \quad (2.12)$$

Note that it always possible to define (at least locally) a form $\hat{\theta}$ satisfying (2.12); indeed, in the coordinates (t,x) we have that $\hat{\theta} = \hat{\theta}_0 dt + \hat{\theta}_1 dx$ and $\mathrm{grad}(\hat{x}) = \mathrm{grad}(\hat{x})^0 \frac{\partial}{\partial t} + \mathrm{grad}(\hat{x})^1 \frac{\partial}{\partial x}$, so that the condition (2.12) reads

$$\frac{\hat{\theta}_0}{\hat{\theta}_1} = -\frac{\mathrm{grad}(\hat{x})^1}{\mathrm{grad}(\hat{x})^0}\,,$$

which can always be satisfied.
In addition, for any point $m \in \mathcal{O} \subseteq M_2$, we introduce the following one-dimensional subspaces of the tangent space of M_2 in m

$$\mathcal{V}^1_m := \{v \in T_m M_2 \,:\, \hat{\theta}(v) = 0\} \subseteq T_m M_2\,; \quad (2.13)$$

since the dependence on m is smooth, the union of the sets (2.13) as m varies in $\mathcal{O} \subseteq M_2$ is a one-dimensional distribution on $\mathcal{O}$:

$$\mathcal{V}^1 := \bigcup_{m \in \mathcal{O}} \mathcal{V}^1_m .$$

Using the Frobenius's Theorem one can easily see that every one-dimensional distribution is integrable: this is equivalent to say that for every point $m \in \mathcal{O} \subseteq M_2$ there exists an open neighbourhood $\hat{O} \subseteq \mathcal{O}$ of m and a function $\hat{t} : \hat{\mathcal{O}} \subseteq \mathcal{O} \subseteq M_2 \to \mathbb{R}$ such that

$$\mathcal{V}^1_m = \{v \in T_m M_2 \, : \, d\hat{t}(v) = 0\} . \tag{2.14}$$

By comparing Eq. (2.13) and Eq. (2.14), we deduce that the one-forms $\hat{\theta}$ and $d\hat{t}$ share the same zeroes. Therefore, since $\mathrm{grad}(\hat{x})$ is a zero of $\hat{\theta}$ [Eq. (2.12)], then $\mathrm{grad}(\hat{x})$ is also a zero of $d\hat{t}$; this implies that

$$0 = d\hat{t}\Big(\mathrm{grad}(\hat{x})\Big) = \mathbf{g_2}\Big(\mathrm{grad}(\hat{t}), \mathrm{grad}(\hat{x})\Big) ,$$

as, by definition, $d\hat{t} = \mathbf{g_2}\left(\mathrm{grad}(\hat{t}), \,\cdot\,\right)$. This concludes the proof of item (ii'). Finally, in order to prove Eq. (2.6), let us note that

$$\begin{aligned} \mathbf{g_2}\Big(\mathrm{grad}(\hat{t}), \mathrm{grad}(\hat{t})\Big) &= g_{00}(\hat{t},\hat{x}) g^{00}(\hat{t},\hat{x}) g^{00}(\hat{t},\hat{x}) \\ &= \frac{g_{00}(\hat{t},\hat{x}) g_{11}(\hat{t},\hat{x})^2}{\det[g_{\mu\nu}]^2} = -\frac{\alpha(\hat{t},\hat{x})^2 \gamma(\hat{t},\hat{x})^4}{\det[g_{\mu\nu}]^2} < 0 . \end{aligned}$$

□

Remark 5 It can be proved that there is still one degree of freedom in the choice of the coordinates of M_2; indeed, one can define another orthogonal coordinate system $(\check{t}, \check{x})$ such that $\mathbf{g}$ is conformal to the 2-dimensional Minkowski flat metric, namely the metric has the form (2.5) with $\alpha = \gamma$, that is, $\mathbf{g} = \gamma(\check{t}, \check{x})^2 \left(-d\check{t}^2 + d\check{x}^2\right)$; this gauge is often referred to as "conformally flat gauge".

Remark 6 The static case corresponds to the situation in which the functions α and γ are independent of the variable t, thus we write $\alpha(t,x) = \alpha(x)$ and $\gamma(t,x) = \gamma(x)$; since in this case the metric (3.1) is defined for every $t \in \mathbb{R}$ then the range $\mathcal{O} \subseteq \mathbb{R}^2$ of the coordinates (t,x) is rectangular, that is

$$\mathcal{O} = \mathbb{R} \times x(\mathcal{O}) . \tag{2.15}$$

Remark 7 In the static case, it is possible to use the remaining degree of freedom in the choice of the gauge on M_2 such that the metric $\tilde{g}$ has the form (2.5) with $\alpha\gamma = 1$. This can be easily reached if one defines the new spatial coordinate $\breve{x}$ as

$$\breve{x} = \int_0^{\hat{x}} \alpha(y)\gamma(y)dy + x_0$$

(here x_0 is an integration constant).

2.1.2 Some facts on Riemannian manifolds with constant curvature

We start introducing the concept of sectional curvature for a $(d-1)$-dimensional Riemannian manifold with $d \geq 2$, which is related to the concept of constant curvature manifolds:

Definition 1 *Let $(N, \mathbf{a})$ be an $(d-1)$-dimensional Riemannian manifold with $d \geq 2$; in addition, let be n an element of N and v, w two linearly independent vectors of the tangent space T_nN. The sectional curvature $K(v, w)$ of v and w is defined as the Gaussian curvature of the two-dimensional manifold described by the geodesics which are tangent in n to the plane $\pi :=$ span $\langle v|w\rangle$. Given a real number $K \in \mathbb{R}$, the manifold $(N, \mathbf{a})$ is said to have a constant curvature K if $K(v, w)$ is constant and equal to K for every $v, w \in T_nN$ and for every $n \in N$.*

The next classical theorem provides a characterization for a manifold to have a constant curvature (see, e.g. Corollary 2.2.5 at page 56 of [41]).

Theorem 3 *Let $(N, \mathbf{a})$ be an $(d-1)$-dimensional Riemannian manifold with $d \geq 2$ and let $(x^2, ..., x^d)$ a coordinate system on N. Then, N is a manifold of constant curvature K if and only if the Riemann tensor R^i_{jkl} is defined as:*

$$R^i_{jkl} = K\left(\delta^i_k\, a_{jl} - \delta^i_l\, a_{kj}\right) \qquad (i, j, k, l = 2, ..., d)\,, \tag{2.16}$$

where a_{ij} are the coefficients of the metric $\mathbf{a}$ in the coordinates $(x^2, ..., x^d)$, that is, $\mathbf{a} = a_{ij}dx^i dx^j$. Moreover, if Eq. (2.16) holds then the Ricci tensor R_{ij} and the scalar curvature R of N read, respectively,

$$R_{ij} = (d-2)Ka_{ij} \qquad (i, j = 2, ..., d)\,, \qquad R = (d-1)(d-2)K\,. \tag{2.17}$$

Example 1 In the Euclidean space $\mathbb{R}^d$ $(d \geq 1)$ with orthogonal coordinates $(x^1, ..., x^d)$, we consider the hypersurface $(d-1)$-sphere of radius $\mathfrak{r}_0 > 0$, which is defined as

$$S^{d-1} := \{\mathbf{x} \in \mathbb{R}^d \,:\, ||\mathbf{x}||^2 = \mathfrak{r}_0^2\}\,, \tag{2.18}$$

where $||\mathbf{x}|| := \sqrt{(x^1)^2 + ... + (x^d)^2}$ is the Euclidean norm. Let $(x^2, ..., x^d)$ be the coordinates system induced on S^{d-1}:

$$(x^2, ..., x^d) \,:\, S^{d-1} \to \mathcal{O}_{d-1} \subseteq \mathbb{R}^{d-1}\,,$$

where $\mathcal{O}_{d-1} := (x^i)_{i=2,...,d+1}(S^{d-1})$ is the range of the coordinates; in addition, let $\mathbf{a} = a_{ij} dx^i\, dx^j$ be the induced metric on S^{d-1}, where $a_{ij} : \mathcal{O}_{d-1} \subseteq \mathbb{R}^{d-1} \to \mathbb{R}$ are smooth functions for every $i, j = 2, ..., d$. Then $(S^{d-1}, \mathbf{a})$ is a $(d-1)$-dimensional Riemannian manifold of constant curvature $\frac{1}{\mathfrak{r}_0^2}$; in particular, the Ricci tensor and the scalar curvature of S^{d-1} reads, respectively,

$$R_{ij} = \frac{d-2}{\mathfrak{r}_0^2} a_{ij}\,, \qquad R = \frac{(d-1)(d-2)}{\mathfrak{r}_0^2} \qquad (i,j = 2, ..., d) \tag{2.19}$$

Example 2 Let us focus on the three-dimensional case of the previous example, that is, lets set $d = 3$ and introduce the spherical coordinates on $\mathbb{R}^3$

$$(\mathfrak{r}, \vartheta, \varphi) \,: \mathbb{R}^3 \setminus \{(0,0,0)\} \to (0, +\infty) \times [0, \pi) \times [0, 2\pi)\,.$$

Then, the induced metric on the 2-sphere with radius $\mathfrak{r}_0 > 0$ (defined in Eq. (2.18)) reads

$$\mathbf{a} = \mathfrak{r}_0^2 \big(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\big) =: \mathfrak{r}_0^2\, d\Omega^2\,; \tag{2.20}$$

moreover, the Ricci tensor and the scalar curvature in Eq. (2.19) become

$$[R_{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\vartheta \end{pmatrix}, \qquad R = \frac{2}{\mathfrak{r}_0^2}\,. \tag{2.21}$$

2.2 Geometry of spherically symmetric spacetimes

Throughout the present section we stipulate the forthcoming assumptions:

(i) $(M_{d+1}, \mathbf{g})$ is a $(d+1)$-dimensional spacetime with $d \geq 3$ and the symbol ∇ indicates the Levi-Civita connection associated with the Lorentzian metric $\mathbf{g}$.

(ii) $(M_2, \mathbf{g_2})$ denotes a two-dimensional Lorentzian manifold with coordinates (t, x) such that x is spacelike, so that the metric $\mathbf{g_2}$ can be locally written in the form (2.1); $\mathcal{O} \subseteq \mathbb{R}^2$ is the range of (t, x).

(iii) $(x^2, ..., x^d)$ is the coordinate system of the unit $(d-1)$-sphere S^{d-1} introduced in Example 1 and $\mathbf{a} = a_{ij} dx^i dx^j$ $(i, j = 2, ..., d)$ is the induced metric: in this case, the Ricci tensor and the scalar curvature are as in Eq. (2.19) with $\mathfrak{r}_0 = 1$; $\mathcal{O}_{d-1} \subseteq \mathbb{R}^{d-1}$ is the range of the coordinates $(x^2, ..., x^d)$.

Let us recall a very important class of spacetimes:

Definition 2 *Let $(M_{d+1}, \mathbf{g})$ be an $(d+1)$-dimensional spacetime with $d \geq 2$. M_{d+1} is spherically symmetric if M_{d+1} possesses a subgroup of its isometry group which is isomorphic to the d-dimensional rotation group $SO(d)$, and the orbits of this subgroup (i.e. the points resulting from the action of the subgroup on a given point) are $(d-1)$-spheres.*

In addition to the assumptions (i)(ii)(iii), from now on, we assume that the spacetime $(M_{d+1}, \mathbf{g})$ is spherically symmetric; actually, in order to further simplify the discussion, we assume that

(iv) M_{d+1} is spherically symmetric and is diffeomorphic to the Lorentzian manifold
$$M_{d+1} \simeq M_2 \times S^{d-1}\,, \tag{2.22}$$
so that the metric field $\mathbf{g}$ is defined as
$$\mathbf{g}_{(m,p)} := \mathbf{g_2}_{(m)} + r(m)^2\, \mathbf{a}_{(p)}\,, \quad \text{for every } (m,p) \in M_2 \times S^{d-1}\,,$$
where $r : M_2 \to (0, +\infty)$ is a smooth functions.

Remark 8 Given the coordinates (t, x) on M_2 and the coordinates $(x^2, ..., x^d)$ on S^{d-1} of items (ii-iii), one can easily see that
$$(t, x, x^2, ..., x^d) : \mathcal{O} \times S^{d-1} \to \mathcal{O} \times \mathcal{O}_{d-1} \subseteq \mathbb{R}^2 \times \mathbb{R}^{d-1} \tag{2.23}$$
is actually a coordinate system on M_{d+1} and the metric $\mathbf{g}$ can be locally written as
$$\begin{aligned} \mathbf{g} = &- \alpha(t,x)^2 dt^2 + \gamma(t,x)^2 (dx + \beta(t,x) dt)^2 \\ &+ r(t,x)^2 a_{ij}(x^2, ..., x^d)\, dx^i\, dx^j\,, \end{aligned} \tag{2.24}$$
where the coefficients $\alpha, \gamma, r : \mathcal{O} \subseteq \mathbb{R}^2 \to (0, +\infty)$ and $\beta : \mathcal{O} \subseteq \mathbb{R}^2 \to \mathbb{R}$ are three smooth functions. In the sequel, we will use the Greek letters $\mu, \nu, ... \in \{0, ..., d\}$ as subscript to denotes the components of the tensors with respect to the frame $(t, x, x^2, ..., x^d)$ and the Latin letters $i, j, ... \in \{2, ..., d\}$ as subscript to denotes the components of the tensors with respect to the coordinates $(x^2, ..., x^d)$. Moreover, in the following, a dot and a prime always refer to partial differentiation with respect to t and x, respectively: hence, for every smooth function $f(t, x)$ defined on $\mathcal{O} \subseteq \mathbb{R}^2$ we will write $\dot{f}$ and f' to indicate, respectively, $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial x}$.

Remark 9 In the four-dimensional case $d = 3$, one can choose for the manifold S^2 the coordinate system $(x^2 := \vartheta, x^3 := \varphi)$ as in Example 2; in this case the metric (2.24) reads

$$\mathbf{g} = -\alpha(t,x)^2 dt^2 + \gamma(t,x)^2 (dx + \beta(t,x)dt)^2 + r(t,x)^2 d\Omega^2 \,, \tag{2.25}$$

where (again) $d\Omega^2 := d\vartheta^2 + \sin^2\vartheta d\varphi^2$.

Remark 10 As stated by Lemma 1, one can introduce an orthogonal coordinate system $(\hat{t}, \hat{x})$ on M_2 such that the function β vanishes and the metric (2.24) reduces to

$$\mathbf{g} = -\alpha(\hat{t},\hat{x})^2 d\hat{t}^2 + \gamma(\hat{t},\hat{x})^2 d\hat{x}^2 + r(\hat{t},\hat{x})^2 a_{ij}(x^2, ..., x^d)\, dx^i\, dx^j \,; \tag{2.26}$$

in the sequel, for the sake of simplicity, we will often make use of these orthogonal coordinates; in those cases, in order not to make heavy the notation, we will keep using the symbols (t, x) instead of $(\hat{t}, \hat{x})$. As just mentioned in Remark 9, in the four-dimensional case $d = 3$, one can set $(x^2 := \vartheta, x^3 := \varphi)$, so that the metric (2.26) reduces to

$$\mathbf{g} = -\alpha(t,x)^2 dt^2 + \gamma(t,x)^2 dx^2 + r(t,x)^2 d\Omega^2 \,. \tag{2.27}$$

Remark 11 Generalizing Remark 6,we have that the static case corresponds to the situation in which the functions α, γ, r are independent of the variable t, thus we write $\alpha(t,x) = \alpha(x), \gamma(t,x) = \gamma(x), r(t,x) = r(x)$; since in this case the metric (2.24) is defined for every $t \in \mathbb{R}$, the range of the coordinates (t, x) is rectangular, that is $\mathcal{O} = \mathbb{R} \times x(\mathcal{O})$.

Example 3 The Anti-de Sitter spacetime is a static, spherically symmetric, four-dimensional Lorentzian manifold $(M_{\text{AdS}}, \mathbf{g}_{\text{AdS}})$ with constant negative scalar curvature

$$R_{\text{AdS}} := -12k^2 \qquad (k > 0)\,;$$

note that is possible to introduce a coordinate system $(t, x, \vartheta, \varphi)$ such that the metric $\mathbf{g}_{\text{AdS}}$ has the form (2.25) with

$$\alpha(t,x) = \gamma^{-1}(t,x) = \sqrt{1 + k^2 x^2}\,, \qquad \beta(t,x) = 0\,, \qquad r(t,x) = x\,. \tag{2.28}$$

The coordinates $(t, x, \vartheta, \varphi)$ are an "almost global coordinate system" since the metric (2.28) is singular only at $x = 0$ and therefore describes almost fully the manifold M_{AdS}. Note that the metric (2.28) is a solution to vacuum Einstein's equations with the cosmological constant $\Lambda = -3k^2$ (see Remark 3).

Example 4 The de Sitter spacetime is a static, spherically symmetric, four-dimensional Lorentzian manifold $(M_{\rm dS}, \mathbf{g}_{\rm dS})$ with constant positive scalar curvature

$$R_{\rm dS} := 12k^2 \qquad (k > 0)\,;$$

note that is possible to introduce a coordinate system $(t, x, \vartheta, \varphi)$ such that the metric $\mathbf{g}_{\rm dS}$ has the local form (2.25) with

$$\alpha(t,x) = \gamma^{-1}(t,x) = \sqrt{1-k^2x^2}\,, \qquad \beta(t,x) = 0\,, \qquad r(t,x) = x\,. \quad (2.29)$$

The coordinates $(t, x, \vartheta, \varphi)$ are a coordinate system which is singular at $x = 0$ and for $x = \pm\frac{1}{k}$; it can be proved that the singularities $x = \pm\frac{1}{k}$ are just two cosmological horizons, i.e. it is possible to introduce two new coordinates (T, X) such that the metric (2.29) can be extended continuously beyond the two horizons; the extended metric results in a conformal factor times the line element of the static Einstein universe, that is,

$$\mathbf{g}_{\rm dS} = \frac{1}{k^2\cos^2 T}\Big[-dT^2 + dX^2 + \sin^2 X d\Omega^2\Big]\,. \qquad (2.30)$$

Note that this extended metric is not static. Finally, let us recall that the metric (2.29) is a solution to vacuum Einstein's equations with the cosmological constant $\Lambda = 3k^2$ (see Remark 3).

The Einstein tensor and the scalar curvature of a spherically symmetric spacetime

For a $(d+1)$-dimensional spherically symmetric spacetime (2.22) with metric (2.24), it results that the components of the Einstein tensor $G_{\mu\nu}$ [Eq. (5)] that do not vanish are those corresponding to the indices $(\mu,\nu) \in \{(0,1),(1,0),(\mu,\mu)\ :\ \mu = 0,...,d\}$; however, due to the symmetry in the indices and the spherical symmetry of the metric tensor (2.24), it turns out that $G_{01} = G_{10}$ and that $\frac{G_{ii}}{a_{ii}} = \frac{G_{jj}}{a_{jj}}$ for every $i,j = 2,...,d$. Therefore, setting $G_a := \frac{G_{ii}}{a_{ii}}$ for any $i = 2,...,d$, one has that the non vanishing components of the Einstein tensor are [9]

$$\begin{aligned}
G_{00} &:= \frac{d-1}{2}\left[\frac{(d-2)\alpha^2}{r^2} + \frac{1}{r}\left(\frac{(d-2)\dot{r}^2}{r} + \frac{2\dot{\gamma}\dot{r}}{\gamma}\right)\right. \\
&\quad \left. + \frac{\alpha^2}{\gamma^2 r}\left(\frac{2\gamma' r'}{\gamma} - \frac{(d-2)r'^2}{r} - 2r''\right)\right] + \frac{d-1}{r} G^{\beta}_{00}\,, && (2.31)\\
G_{01} &:= (d-1)\left[\frac{1}{r}\left(\frac{\alpha'\dot{r}}{\alpha} + \frac{\dot{\gamma} r'}{\gamma} - \dot{r}'\right)\right] + \frac{d-1}{r} G^{\beta}_{01}\,, && (2.32)\\
G_{11} &:= \frac{d-1}{2}\left[-\frac{(d-2)\gamma^2}{r^2}\left(1+\frac{\dot{r}^2}{\alpha^2}\right) + \frac{2\gamma^2}{\alpha^2 r}\left(\frac{\dot{\alpha}\dot{r}}{\alpha} - \ddot{r}\right)\right. \\
&\quad \left. + \frac{r'}{r}\left(\frac{2\alpha'}{\alpha} + \frac{(d-2)r'}{r}\right)\right] + \frac{d-1}{r} G^{\beta}_{11}\,, && (2.33)\\
G_{a} &:= \frac{(d-2)(d-3)}{2}\left(\frac{r'^2}{\gamma^2} - \frac{\dot{r}^2}{\alpha^2} - 1\right) \\
&\quad + \frac{(d-2)r}{\alpha^2}\left(\frac{\dot{\alpha}\dot{r}}{\alpha} - \frac{\dot{\gamma}\dot{r}}{\gamma} - \ddot{r}\right) \\
&\quad + \frac{r^2}{\alpha\gamma}\left(\frac{\dot{\alpha}\dot{\gamma}}{\alpha^2} - \frac{\alpha''\gamma'}{\gamma^2} + \frac{\alpha''}{\gamma} - \frac{\ddot{\gamma}}{\alpha}\right) \\
&\quad + \frac{(d-2)r}{\gamma^2}\left(\frac{\alpha' r'}{\alpha} - \frac{\gamma' r'}{\gamma} + r''\right) + \frac{r^2}{\alpha^2} G^{\beta}_{a}\,, && (2.34)
\end{aligned}$$

where

[9] In order to compute the Ricci tensor $R_{\mu\nu}$ (and consequently the scalar curvature R) of the metric $\mathbf{g}$ it is sufficient to observe that, for any $i, j = 1, ..., d$,

$$R_{ij} = \tilde{R}_{ij} + a_{ij}\, r\left[\frac{\ddot{r}}{\alpha^2} - \frac{r''}{\gamma^2} + \left(\frac{\gamma'}{\gamma} - \frac{\alpha'}{\alpha} - (d-2)\frac{r'}{r}\right)\frac{r'}{\gamma^2}\left(\frac{\dot{\gamma}}{\gamma} - \frac{\dot{\alpha}}{\alpha} + (d-2)\frac{\dot{r}}{r}\right)\frac{\dot{r}}{\alpha^2}\right]$$

where $\tilde{R}_{ij}$ is the Ricci tensor of the metric $\mathbf{a}$, which is given by Eq. (2.19) with $\mathfrak{r}_0^2 = 1$.

$$
\begin{aligned}
G_{00}^{\beta} \;:=\;& \left[\left(\beta' + \frac{\dot\gamma}{\gamma}\right) r' + \left(2\frac{\alpha'}{\alpha} - \frac{\gamma'}{\gamma}\right)\dot r - (d-2)\frac{r'\dot r}{r} - 2\dot r'\right]\beta \\
& + \Bigg[-\frac{d-2}{2}\frac{\gamma^2}{r} - r'\frac{(\alpha\gamma)'}{\alpha\gamma} + (d-2)\frac{r'^2}{r} + 2r'' \\
& + \frac{\gamma^2}{\alpha^2}\left(\dot\beta r' + \frac{\dot\alpha \dot r}{\alpha} - \frac{d-2}{2}\frac{\dot r^2}{r} - \ddot r\right)\Bigg]\beta^2 \\
& - \left[\left(\beta' + \frac{\dot\alpha}{\alpha}\right) r' + \left(\frac{\alpha'}{\alpha} - (d-2)\frac{r'}{r}\right)\dot r - 2\dot r'\right]\frac{\gamma^2}{\alpha^2}\beta^3 \\
& + \left[\left(\frac{\alpha'}{\alpha} - \frac{d-2}{2}\frac{r'}{r}\right) r' - r''\right]\frac{\gamma^2}{\alpha^2}\beta^4 - \dot r \beta' , \qquad (2.35)
\end{aligned}
$$

$$
\begin{aligned}
G_{01}^{\beta} \;:=\;& \Bigg[-\frac{d-2}{2}\frac{\gamma^2}{r} + \left(-\frac{\gamma'}{\gamma} + \frac{d-2}{2}\frac{r'}{r}\right) r' + r'' \\
& + \frac{\gamma^2}{\alpha^2}\left(r'\dot\beta + \left(\frac{\dot\alpha}{\alpha} - \frac{d-2}{2}\frac{\dot r}{r}\right)\dot r - \ddot r\right)\Bigg]\beta \\
& - \left[\left(\beta' + \frac{\dot\alpha}{\alpha}\right) r' + \left(\frac{\alpha'}{\alpha} - (d-2)\frac{r'}{r}\right)\dot r - 2\dot r'\right]\frac{\gamma^2}{\alpha^2}\beta^2 \\
& + \left[\left(\frac{\alpha'}{\alpha} - \frac{d-2}{2}\frac{r'}{r}\right) r' - r''\right]\frac{\gamma^2}{\alpha^2}\beta^3 , \qquad (2.36)
\end{aligned}
$$

$$
\begin{aligned}
G_{11}^{\beta} \;:=\;& \left[\left(\beta' + \frac{\dot\alpha}{\alpha}\right) r' + \left(\frac{\alpha'}{\alpha} - (d-2)\frac{r'}{r}\right)\dot r - 2\dot r'\right]\gamma^2\beta \\
& + \left[\left(\frac{\alpha'}{\alpha} - \frac{d-2}{2}\frac{r'}{r}\right) r' - r''\right]\frac{\gamma^2}{\alpha^2}\beta^2 + \frac{\gamma^2}{\alpha^2} r'\dot\beta , \qquad (2.37)
\end{aligned}
$$

$$
\begin{aligned}
G_{a}^{\beta} \;:=\;& \Bigg[\left(\frac{\alpha'}{\alpha} - 2(d-2)\frac{r'}{r} - 3\frac{\gamma'}{\gamma}\right)\beta' - \beta'' \\
& - \left(\frac{\gamma'}{\gamma} + (d-2)\frac{r'}{r}\right)\frac{\dot\alpha}{\alpha} - \left(\frac{\alpha'}{\alpha} - (d-2)\frac{r'}{r}\right)\frac{\dot\gamma}{\gamma} \\
& + (d-2)\left(\frac{\gamma'}{\gamma} - \frac{\alpha'}{\alpha} + (d-3)\frac{r'}{r}\right)\frac{\dot r}{r} + 2\frac{\dot\gamma'}{\gamma} + 2(d-2)\frac{\dot r'}{r}\Bigg]\beta \\
& + \left[\frac{\alpha'}{\alpha}\frac{\gamma'}{\gamma} + (d-2)\left(\frac{\alpha'}{\alpha} - \frac{\gamma'}{\gamma} - \frac{d-3}{2}\frac{r'}{r}\right)\frac{r'}{r} - \frac{\gamma''}{\gamma} - (d-2)\frac{r''}{r}\right]\beta^2 \\
& + \left(\frac{\gamma'}{\gamma} + (d-2)\frac{r'}{r}\right)\dot\beta + \left(2\frac{\dot\gamma}{\gamma} - \frac{\dot\alpha}{\alpha} + (d-2)\frac{\dot r}{r} - \beta'\right)\beta' + \dot\beta' . \qquad (2.38)
\end{aligned}
$$

Here and in the following, the dots and the primes refer to the partial derivatives with respect to t and x, respectively, as mentioned in Remark 8.

In addition, for future convenience, we compute the scalar curvature [Eq. (4)]; this reads

$$
\begin{aligned}
R := & -\frac{2}{\alpha^2}\left(\frac{\dot\alpha}{\alpha}\frac{\dot\gamma}{\gamma} - \frac{\ddot\gamma}{\gamma} - (d-1)\frac{\ddot r}{r}\right) + \frac{2}{\gamma^2}\left(\frac{\alpha'}{\alpha}\frac{\gamma'}{\gamma} - \frac{\alpha''}{\alpha} - (d-1)\frac{r''}{r}\right) \\
& + 2(d-1)\frac{\alpha}{\gamma^3}\frac{r'}{r}\frac{\partial}{\partial x}\left[\frac{\gamma}{\alpha}\right] - 2(d-1)\frac{\gamma}{\alpha^3}\frac{\dot r}{r}\frac{\partial}{\partial t}\left[\frac{\alpha}{\gamma}\right] \\
& + \frac{(d-2)(d-1)}{r^2}\left(1 - \frac{r'^2}{\gamma^2} + \frac{\dot r^2}{\alpha^2}\right) + R^\beta\,,
\end{aligned} \tag{2.39}
$$

where

$$
\begin{aligned}
R^\beta := 2\frac{\beta}{\alpha^2} & \left[\beta'\left(-\frac{\alpha'}{\alpha} + 2(d-1)\frac{r'}{r} + 3\frac{\gamma'}{\gamma}\right) + \frac{\dot\alpha}{\alpha}\left((d-1)\frac{r'}{r} + \frac{\gamma'}{\gamma}\right)\right. \\
& + \frac{\dot\gamma}{\gamma}\left(\frac{\alpha'}{\alpha} - (d-1)\frac{r'}{r}\right) + (d-1)\frac{\dot r'}{r}\left(\frac{\alpha'}{\alpha} - \frac{\gamma'}{\gamma}\right) + \beta'' \\
& \left. - 2(d-1)\frac{\dot r'}{r} - (d-2)(d-1)\frac{\dot r r'}{r^2} - 2\frac{\dot\gamma'}{\gamma}\right] \\
& + 2\frac{\beta^2}{\alpha^2}\left[(d-1)\frac{r'}{r}\left(\frac{\gamma'}{\gamma} - \frac{\alpha'}{\alpha}\right) - \frac{\alpha'}{\alpha}\frac{\gamma'}{\gamma}\right. \\
& \left. + \frac{(d-1)(d-2)}{2}\frac{r'^2}{r^2} + (d-1)\frac{r''}{r} + \frac{\gamma''}{\gamma}\right] \\
& + \frac{2}{\alpha^2}\left[\beta'\left(\beta' + \frac{\dot\alpha}{\alpha} - (d-1)\frac{\dot r}{r} - 2\frac{\dot\gamma}{\gamma}\right) - \dot\beta\left((d-1)\frac{r'}{r} + \frac{\gamma'}{\gamma}\right) - \dot\beta'\right].
\end{aligned} \tag{2.40}
$$

Remark 12 In the orthogonal coordinate system introduced in Lemma 10 (in which the function β vanishes), the expressions for the Einstein tensor $G_{\mu\nu}$ (2.31-2.34) and the scalar curvature (2.39) significantly simplify since the quantities G^β_{00}, G^β_{01}, G^β_{11}, G^β_a and R^β in Eqs. (2.35-2.38,2.40) are zero.

Remark 13 The $(d+1)$-dimensional static, spherically symmetric spacetime defined by

$$\alpha = \gamma = 1\,, \qquad r = x$$

is flat and generalizes the Minkowski spacetime to higher dimension: indeed, in this case, the scalar curvature R is zero everywhere.

2.3 Geometry of (four-dimensional) spherically symmetric wormholes

In this section I try to define the general properties that a metric has to possess in order to describe a spherically symmetric, static wormhole spacetime; for simplicity, we focus on four-dimensional configurations, although our approach can be easily generalized to the $(d+1)$-dimensional case. Therefore, from now on we keep the following assumptions:

(i) $(M, \mathbf{g})$ is a four-dimensional spherically-symmetric static spacetime;

(ii) $(t, x, \vartheta, \varphi)$ is the orthogonal coordiante system for M introduced in Remark 10;

(iii) in the coordinates $(t, x, \vartheta, \varphi)$, the static metric $\mathbf{g}$ reads as in Eq. (2.27), that is

$$\mathbf{g} = -\alpha(x)^2 dt^2 + \gamma(x)^2 dx^2 + r(x)^2 d\Omega^2 \,, \tag{2.41}$$

where the coefficients $\alpha, \gamma, r : x(\mathcal{O}) \to (0, +\infty)$ are three smooth functions (recall that in the static case the range of the coordinates (t, x) is $\mathcal{O} = \mathbb{R} \times x(\mathcal{O})$).

Let us start with few elementary considerations about the metric (2.41):

(a) one can easily see that the spacetime $(M, \mathbf{g})$ is spherically symmetric and static from the fact that the coefficients of the metric $\mathbf{g}$ do not depend on the angular coordinates ϑ and φ nor on the temporal coordinate t; in addition, the coordinates (x, ϑ, φ) define a coordinate system for the "space" of a static observer in this spacetime, while the temporal coordinate t represents its "time";

(b) the first term $-\alpha(x)^2 dt^2$ of $\mathbf{g}$ is the proper time (physical time) measured by someone which is at rest according to the static observer; note that the proper time depends on the spatial variable x and, therefore, the function α can be used to quantify the "gap" between the time lengths signed by two clocks at rest for the static observer depending on their relative positions;

(c) the second term

$$\gamma(x)^2 dx^2 + r(x)^2 d\Omega^2 \tag{2.42}$$

is the metric of each of the three-dimensional spherically symmetric manifolds $\mathbf{t} := \{m \in M \,|\, t(m) = const\} \subset M$ which represent the space seen by the static observer at a fixed time; for this reason, we can say that the functions γ and r determines what we reasonably would call the "shape" of the spacetime $(M, \mathbf{g})$.

A very useful tool for visualizing the geometrical properties of the spatial slice $\mathbf{t}$ is the embedding diagram. In general, a smooth map $\iota : M \to N$ between two differential manifolds M and N is an embedding if both ι and its differential $d\iota$ are everywhere injective and $\iota : M \to \iota(M)$ is a homeomorphism. Of course, it might be impossible to embed the three-dimensional slice $\mathbf{t}$ in a four-dimensional flat space; however, if it is the case, profiting from the spherical symmetry of (2.42), one can build up a picture of the embedded slice $\iota(\mathbf{t})$ fixing the value of an angle. In fact, the embedding ι transforms the spacetime slices $\mathbf{t}_{\vartheta_0} := \{m \in M \,|\, t(m) = const,\, \vartheta(m) = \vartheta_0\} \subset M$ into two-dimensional surfaces in the three-dimensional euclidean space: these representations of the spatial part of a spacetime are called *embedding diagrams*. Note that, as the value of ϑ_0 is immaterial, from now on we fix $\vartheta_0 = \pi/2$.

After these preliminary considerations, we can propose a naive definition of a static spherically symmetric wormhole spacetime metric: the metric (2.41) represents a static spherically symmetric wormhole if each of its $\mathbf{t}_{\frac{\pi}{2}}$ slices (defined by the metric (2.42) with $\vartheta = \pi/2$), once embedded in a three-dimensional flat space, look as a "tunnel-shaped" hypersurface, a form familiar from popular accounts of wormholes. However, this statement is too restrictive since, as we will see later, there are spacetimes whose $\mathbf{t}_{\frac{\pi}{2}}$ slices cannot be embedded in a flat space, but nevertheless, in some sense these slices have still the "shape of a tunnel" if embedded in a different suitable ambient space. Indeed, the distinctive aspect of a wormhole spacetime is the presence of a throat; therefore, one can simply ask that a wormhole spacetime is embeddable as a three-dimensional "tunnel-shaped" hypersurface only in a neighbourhood of the throat. This is the main idea of the following

Definition 3 (Local definition of wormhole metric)
The metric (2.41) describes a spherically symmetric static wormhole if there exist a point x_0 and neighbourhood $\mathcal{U}_{x_0}$ of x_0 such that the following hold:
(i) each of the $\mathbf{t}_{\frac{\pi}{2}}$ slices (defined by the metric (2.42) with $\vartheta = \pi/2$), once intersected with $\mathcal{U}_{x_0}$, can be embedded via an embedding ι into the three-dimensional euclidean flat space $\mathbb{R}^3$;
(ii) introducing the cylindrical coordinates $(z, \rho, \hat{\varphi}) \in \mathbb{R} \times (0, +\infty) \times (0, 2\pi)$ on $\mathbb{R}^3$, the embedding of $\mathbf{t}_{\frac{\pi}{2}} \cap \mathcal{U}_{x_0}$ is a "tunnel-shaped" hypersurface, namely an hypersurface defined as

$$\mathcal{S} := \{(z, \rho, \hat{\varphi}) \,:\, \rho = F(z)\} \subset \mathbb{R}^3\,,$$

where the function $F : \mathrm{dom}(F) \subseteq \mathbb{R} \to (0, +\infty)$ *is smooth and possesses a positive minimum of size* $b > 0$ *at a certain point of its domain* z_0*.*

In order to simplify the forthcoming discussion, we prescribe that z_0 is the only minimum point of F. In short, we require that the function F has the following properties:

$$\begin{aligned} F \in \mathcal{C}^\infty\,, \qquad F(z_0) = b > 0\,, \\ F'(z_0) = 0\,, \qquad F'(z)(z - z_0) > 0 \;\; \text{for all } z \in \mathrm{dom}(F)/\{z_0\}\,. \end{aligned} \tag{2.43}$$

Note that the function F can be effectively regarded as the "profile" of the tunnel; with the assumption (2.43), it is clear that the minimum b of the function F represents the size of the tunnel throat.

The local definition of wormholes has a critical aspect: since the Definition 3 considers only the topological features of wormholes in a neighbourhood of the throat, the profile function F cannot be used to "see" the behaviour of the wormhole in the large x limits. For these reasons, in the following, I provide an alternative (and more general) definition of wormhole spacetimes. In this regard, let us consider a three-dimensional Riemannian manifold $(M_A, \mathbf{g}_A)$ with the cylindrical coordinates $(z, \rho, \hat{\varphi}) \in \mathbb{R} \times (0, +\infty) \times (0, 2\pi)$ and the metric $\mathbf{g}_A$ defined as

$$\mathbf{g}_A = A(\rho)^2 dz^2 + \frac{1}{A(\rho)^2} d\rho^2 + \rho^2 d\hat{\varphi}^2\,, \tag{2.44}$$

where $A : (0, +\infty) \to (0, +\infty)$ is a smooth and positive function; in addition, we assume that $A(0) = 1$. This is the ambient space in which we ask that the embedding of the $\mathbf{t}_{\frac{\pi}{2}}$ slices of the wormhole have the shape of a tunnel.

Definition 4 (Global definition of wormhole metric) [10]
The metric (2.41) describes a spherically symmetric static wormhole if the following hold:
(i) each of the $\mathbf{t}_{\frac{\pi}{2}}$ *slices (defined by the metric (2.42) with* $\vartheta = \pi/2$*) can be embedded via an embedding* ι *into the three-dimensional Riemannian manifold* $(M_A, \mathbf{g}_A)$*, where* $\mathbf{g}_A$ *is defined in Eq. (2.44);*
(ii) in the cylindrical coordinates $(z, \rho, \hat{\varphi}) \in \mathbb{R} \times (0, +\infty) \times (0, 2\pi)$ *on* M_A*, the embedding of* $\mathbf{t}_{\frac{\pi}{2}}$ *is a "tunnel-shaped" hypersurface, namely an hypersurface defined as*

$$\mathcal{S} := \{(z, \rho, \hat{\varphi}) \,:\, \rho = F(z)\} \subset M_A\,, \tag{2.45}$$

[10] The adjective "global" might be confusing, since we are defining whether a *local* representation of a metric describes a wormhole configuration; actually, the globality of this definition refers to the possibility of embed the $\mathbf{t}_{\frac{\pi}{2}}$ slices *entirely* in a suitable ambient space a a "tunnel-shaped" hypersurface.

where the function $F : \mathrm{dom}(F) \subseteq \mathbb{R} \to (0, +\infty)$ *is smooth and possesses a positive minimum of size* $b > 0$ *at* $z = z_0$*.*

Even in this case, the function F can be regarded as the "profile" of the wormhole: indeed, in a neighbourhood of $\rho = 0$, M_A approaches to the flat three-dimensional space and the hyperfurface $\mathcal{S}_{\frac{\pi}{2}}$ [Eq. (2.45)] can be locally interpreted as a "tunnel-shaped" hypersurface in $\mathbb{R}^3$ in a neighbourhood of the throat $z = z_0$. Moreover, the large z limits represent the far ends of two separate hypersurfaces of M_A linked by the tunnel throat; these hypersurfaces are defined, respectively, by $\{(z, F(z), \hat{\varphi}) \,|\, z > z_0\}\}$ and $\{(z, F(z), \hat{\varphi}) \,|\, z < z_0\}\}$.

In the following, we look for some properties that the functions α, γ and r has to possess in such a way that the corresponding metric (2.41) describes a wormhole according to Definition 4.
Let us start observing that, as $\rho = F(z)$ and $d\rho = F'(z)dz$ on the hypersurface $\mathcal{S}$, the induced metric on $\mathcal{S} \subset M_A$ reads

$$d\mathcal{S}^2 = \left[A(F(z))^2 + \frac{F'(z)^2}{A(F(z))^2} \right] dz^2 + F(z)^2 d\hat{\varphi}^2 \,. \tag{2.46}$$

Therefore, we are looking for a metric (2.41) such that, given a "profile function" F, it exists an embedding $\iota : \mathbf{t}_{\frac{\pi}{2}} \subset M \to M_A$ such that the embedded slice $\mathcal{S} := \iota(\mathbf{t}_{\frac{\pi}{2}}) \subset M_A$ has the metric (2.46). Working in coordinates, the embedding ι is specified by three smooth bijections:

$$z = z(x, \varphi)\,, \quad \rho = \rho(x, \varphi)\,, \quad \hat{\varphi} = \hat{\varphi}(x, \varphi)\,;$$

note that, as we are in a spherically symmetric configuration, we can take $\hat{\varphi} = \varphi$ and the functions $z(\cdot)$ and $\rho(\cdot)$ to be angles-independent, i.e we can set

$$z = z(x)\,, \quad \rho = \rho(x)\,, \quad \hat{\varphi} = \varphi\,. \tag{2.47}$$

Provided $\mathrm{Im}(z) \subseteq \mathrm{dom}(F)$, one can insert the embedding functions (2.47) and their differentials into (2.46), obtaining the original form of the $\mathbf{t}_{\frac{\pi}{2}}$ slice metric:

$$d\mathbf{t}^2_{\frac{\pi}{2}} = A(F(z(x)))^2 z'(x)^2 \left[1 + \frac{F'(z(x))^2}{A(F(z(x)))^4} \right] dx^2 + F(z(x))^2 d\hat{\varphi}^2 \,. \tag{2.48}$$

Without loss of generality, from now on we also stipulate that $z(0) = 0$ and $z'(x) > 0$ for all $x \in \mathbb{R}$.
By comparing the two expression (2.42) and (2.48) for the metric of the slice

$\mathbf{t}_{\frac{\pi}{2}}$, we get the expressions of the functions γ and r in dependence of F and $z = z(x)$:

$$r(x) = F(z(x)), \tag{2.49}$$

$$\gamma(x) = A(F(z(x)))z'(x)\sqrt{1 + \frac{F'(z(x))^2}{A(F(z(x)))^4}}. \tag{2.50}$$

Summing up, in this section we have proved the following

Proposition 2 *For all smooth functions* $\alpha : \mathbb{R} \to (0, +\infty)$, $z : \mathbb{R} \to \mathrm{Im}(z)$, $F : \mathrm{dom}(F) \subseteq \mathbb{R} \to [b, +\infty)$, $A : (0, +\infty) \to (0, +\infty)$ *such that*
(i) z *is a bijection with* $z'(x) > 0$ *for all* $x \in \mathbb{R}$,
(ii) $\mathrm{Im}(z) \subseteq \mathrm{dom}(F)$,
(iii) $z(0) = 0 \in \mathrm{dom}(F)$,
(iv) F *satisfies the requirements in Eq. (2.43),*
(v) $A(0) = 1$,
the metric (2.41) with r *and* γ *as in Eqs. (2.49,2.50) represents a static spherically symmetric wormhole spacetime; indeed, the embedded hypersurface* $\mathcal{S} = \iota(\mathbf{t}_{\frac{\pi}{2}}) \subset M_k$ *has a "tunnel-shaped" structure with a throat of size* b *located at* $z = z_0$*. Moreover, the slice* $\mathbf{t}_{\frac{\pi}{2}}$ *of the wormhole spacetime has the throat of size* b *at* $x = x_0 =: z^{-1}(z_0)$ *and links together the two separate universes defined, respectively by* $\{m \in M \,|\, x(m) > x_0\}$ *and* $\{m \in M \,|\, x(m) < x_0\}$.

Remark 14 From equation (2.49), from the fact that $z(x)$ is injective and from the properties (2.43) on the function F we have that the radial coefficient r of a wormhole spacetime metric satisfies

$$r(x) > r(x_0) = b > 0\,, \quad r'(x_0) = 0\,, \quad r'(x)\,(x - x_0) > 0 \quad \text{for all } x \neq x_0\,. \tag{2.51}$$

Now, given a static metric (2.41) describing a wormhole, I propose the following question: when and how is it possible to find a profile function F and an embedding z satisfying the requirements (i)-(iv) of Proposition 2? The answer to this question strongly depends on the choice of the ambient space M_A, i.e. on the function A appearing in Eq. (2.44): this has to be guessed for example by looking the asymptotic behaviour of the wormhole. Hence, let us suppose that we are given with four functions α, γ, r and A. Recalling that $\rho(x) = F(z(x))$ on the embedded hypersurface $\mathcal{S} = \iota(\mathbf{t}_{\frac{\pi}{2}})$, Eqs. (2.49,2.50) becomes

$$r(x) = \rho(x)\,, \tag{2.52}$$

$$\gamma(x) = A(\rho(x))\sqrt{z'(x)^2 + \frac{\rho'(x)^2}{A(\rho(x))^4}}\,, \tag{2.53}$$

$$F(z) := \rho(x(z))\,, \tag{2.54}$$

where $x = x(z)$ is the inverse function of $z = z(x)$. Hence, Eqs. (2.52,2.53) are a system of two differential equations in the unknown $\rho = \rho(x)$, $z = z(x)$ which is trivially solved by setting

$$\rho(x) = r(x)\,, \quad z(x) = \int_0^x \frac{1}{A(r(\tilde{x}))}\sqrt{\gamma(\tilde{x})^2 - \frac{r'(\tilde{x})^2}{A(r(\tilde{x}))^2}}d\tilde{x} \tag{2.55}$$

(note that in solving Eq. (2.53) we have required that $z(0) = 0$). Finally, the profile function F is obtained by composing the function ρ and the inverse function of z, according to Eq. (2.54). Obviously, the function z in Eq. (2.55) is an embedding of the whole slice $\mathbf{t}_{\frac{\pi}{2}}$ only if

$$\gamma(x)^2 \geq \frac{r'(x)^2}{A(r(x))^2} \qquad \text{for every } x \in x(\mathcal{O})\,. \tag{2.56}$$

Moreover, Eq. (2.55) can always be used to build a local embedding of the wormhole spacetime in the three-dimensional Euclidean space, in a neighbourhood $\mathcal{U}_0$ of the throat: this is reached by setting $A(\rho) \equiv 1$ in Eq. (2.55). The neighbourhood $\mathcal{U}_0$ corresponds to the region where the embedding function z is well defined, i.e. where

$$\gamma(x)^2 \geq r'(x)^2\,. \tag{2.57}$$

Remark 15 To my knowledge, there is at least one example in literature of the previous construction in the case of a static spherically symmetric spacetime whose spatial part is embeddable as a “tunnel-shaped” hypersurface in a flat ambient space: in his textbook [42], Hartle describes how to built the embedding of the $\mathbf{t}_{\frac{\pi}{2}}$ slice of a very simple wormhole (therein referred generically as “a Wormhole Spacetime") in the three-dimensional euclidean space $\mathbb{R}^3$. We will return to Hartle’s example in Subsection 3.3.1.

2.4 Necessity of exotic matter: phantom scalar fields

Throughout this section we set:

(i) $(M_{d+1}, \mathbf{g})$ is a $(d+1)$-dimensional spherically-symmetric static spacetime;

(ii) $(t, x, x^2, ..., x^d)$ is the coordiante system for M_{d+1} introduced in Remark 10;

(iiii) in the coordinates $(t, x, x^2, ..., x^d)$, the static metric $\mathbf{g}$ reads (see Eq. (2.27))

$$\mathbf{g} = -\alpha(x)^2 dt^2 + \gamma(x)^2 dx^2 + r(x)^2 a_{ij}(x^2, ..., x^d)\, dx^i\, dx^j\,, \tag{2.58}$$

where the coefficients $\alpha, \gamma, r\ :\ x(\mathcal{O}) \to (0, +\infty)$ and $a_{ij}\ :\ \mathcal{O}_{d-1} \to \mathbb{R}$ are three smooth functions for every $i, j = 2, ..., d$.

Remark 16 Setting $d = 3$, in Remark 14 we have shown that a static spherically symmetric metric (2.58) describes a wormhole configuration according to Definition 3 and Definition 4) if and only if the radial coefficient r is such that Eq. (2.51) holds. For simplicity, let us assume that $x_0 = 0$. In particular, by continuity, it turns out that there exists a neighbourhood $\mathcal{U}_0$ of zero such that $r''(x) > 0$ for every $x \in \mathcal{U}_0 \backslash \{0\}$. Actually, $r''(0)$ might vanish, although this is not the case for all the wormholes presented in this work; for this reason, from now on, along with Eq. (2.51), we assume that for every wormhole with metric (2.58) there exists a neighbourhood $\mathcal{U}_0$ of the throat such that

$$r''(x) > 0 \quad \text{for every } x \in \mathcal{U}_0\,. \tag{2.59}$$

Obviously, all these considerations (and all the considerations of the previous section) can be easily extended to higher dimensions $d \geq 3$.

At this point, one might wonder whether and how it is possible to introduce some matter or fields which could support a spherically symmetric wormhole configuration; in other words, the question is: is it possible to find a physically meaningful stress-energy tensor $T_{\mu\nu}$ which satisfies Einstein's equations [Eq. (6)] for the metric (2.58) with the conditions (2.51,2.59) on the function r?

In order to deal with this problem, in the next subsection I will recall three main prescriptions that, according to GR, every stress-energy tensor field is believed to satisfy when describes some physically reasonable classical matter of fields: the weak, the strong and the dominant energy condition.

2.4.1 Energy conditions

Let $T_{\mu\nu}$ be a stress-energy tensor field associated to a physically reasonable classical configuration of matter or fields; then $T_{\mu\nu}$ is supposed to satisfy at least one of the following three conditions.

Definition 5 *A stress-energy tensor field $T_{\mu\nu}$ is said to satisfy the weak energy condition (WEC) if its energy density is not negative for all observes; since the energy density of $T_{\mu\nu}$ according to an observer with four-velocity ξ^{μ}[11] is represented by the quantity $T_{\mu\nu}\xi^{\mu}\xi^{\nu}$, then the WEC can be written as*

$$T_{\mu\nu}\xi^{\mu}\xi^{\nu} \geq 0 \qquad \textit{for every timelike vector } \xi^{\mu} \qquad \textit{(WEC)}. \tag{2.60}$$

Definition 6 *A stress-energy tensor field $T_{\mu\nu}$ is said to satisfy the strong energy condition (SEC) if the quantity $R_{\mu\nu}\xi^{\mu}\xi^{\nu}$ (here, $R_{\mu\nu}$ is the Ricci tensor) is nonnegative; the alternative form of Einstein's equation [Eq. (7)] implies that this condition is equivalent to*

$$T_{\mu\nu}\xi^{\mu}\xi^{\nu} \geq -\frac{1}{2}T \qquad \textit{for every timelike vector } \xi^{\mu} \qquad \textit{(SEC)}. \tag{2.61}$$

Definition 7 *A stress-energy tensor field $T_{\mu\nu}$ is said to satisfy the dominant energy condition (DEC) if the speed of the four-current density of matter according to any observer is less then the speed of light; since the four-current density according to an observer with four-velocity ξ^{μ} is represented by by the quantity $-T^{\mu\nu}\xi^{\mu}$, then the DEC can be written as*

$$-T^{\mu}_{\nu}\xi^{\mu} \textit{ is timelike or null for every timelike vector } \xi^{\mu} \qquad \textit{(DEC)}. \tag{2.62}$$

Remark 17 For the sake of completeness, let us recall that the only relation among the three energy conditions is (DEC)$\Rightarrow$(WEC). In particular, despite of their names, it can be verified that (SEC)$\not\Rightarrow$(WEC); the strengthen of the (SEC) lays in the fact that the assumption Eq. (2.62) is physically stronger to be assumed when comparing to the assumption (2.60).

2.4.2 The violation of the weak energy condition (WEC)

Let us now return to our question, namely, whether and how it is possible to find a physically meaningful stress-energy tensor $T_{\mu\nu}$ which satisfies Einstein's equations for the static spherically symmetric metric with the conditions

[11] This means that ξ^{μ} is a timelike vector such that $\xi^{\mu}\xi_{\mu} = -1$.

(2.51,2.59) on the function r. In this section we will show that every stress-energy tensor supporting a wormhole configuration, howsoever it is defined, necessarily violates the WEC (2.60); this important fact tells us that the matter or field configurations described by such a $T_{\mu\nu}$, if any, can not be "classical" but rather "exotic".
We start computing the Einstein tensor $G_{\mu\nu}$ for a static spherically symmetric metric, namely we insert the assumption Eq. (2.58) into Eqs. (2.31-2.34), that is

$$\alpha = \alpha(x)\,, \qquad \gamma = \gamma(x)\,, \qquad r = r(x)\,, \qquad \beta = 0\,;$$

it turns out that G_{01} vanishes; therefore, the most general form that $T_{\mu\nu}$ has to possess in order to satisfy Einstein's equations [Eq. (6)] is

$$T_{\mu\nu} = \begin{pmatrix} \rho\,\alpha^2 & 0 & 0 & \cdots & 0 \\ 0 & p_r\,\gamma^2 & 0 & \cdots & 0 \\ 0 & 0 & p_a\,r^2\,a_{1,1} & \cdots & p_a\,r^2\,a_{1,d-1} \\ 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & p_a\,r^2\,a_{d-1,1} & \cdots & p_a\,r^2\,a_{d-1,d-1} \end{pmatrix} \tag{2.63}$$

where ρ, p_r and p_a are three real functions depending on the spatial coordinate x. It is well known that ρ and p_r may be interpreted, respectively, as the rest energy density and the radial pressure of the matter.
I now recall the following lemma which provides a necessary condition required to $T_{\mu\nu}$ in the form (2.63) in order to satisfy the WEC (2.60)

Lemma 2 *Let (M_{d+1}, g) be a $(d+1)$-dimensional static spherically symmetric spacetime with the metric of the form (2.58) supported by a stress-energy tensor field $T_{\mu\nu}$ of the form (2.63); if $T_{\mu\nu}$ satisfies the WEC (2.60) then the following inequalities hold*

$$\rho \geq 0\,, \qquad \rho + p_r \geq 0\,. \tag{2.64}$$

Proof. Let us observe that if $T_{\mu\nu}$ satisfies the WEC (2.60), then the condition $T_{\mu\nu}\xi^\mu\xi^\nu \geq 0$ holds, in particular, for all the normalised, future-directed, timelike vector fields of the form $\xi^\mu = \left(\frac{\sqrt{1+c^2}}{\alpha}, \frac{c}{\gamma}, 0, ..., 0\right)$ with $c \in \mathbb{R}$; in other words, we have that the following inequality holds

$$(1+c^2)\rho + c^2 p_r \geq 0\,, \quad \text{for every } c \in \mathbb{R}. \tag{2.65}$$

The first condition $\rho \geq 0$ in Eq. (2.64) is simply obtained from the inequality (2.65) with the particular choice $c = 0$; the second condition $\rho + p_r \geq 0$ is obtained dividing both sides of (2.65) by c^2 and taking the limit $c \to +\infty$.

□

We are now ready to prove the following

Proposition 3 *Let* (M_{d+1}, g) *be a* $(d+1)$*-dimensional static spherically symmetric spacetime with the metric of the form (2.58) supported by a stress-energy tensor field* $T_{\mu\nu}$ *of the form (2.63); a necessary condition for the spacetime* M_{d+1} *to have a throat (i.e. to have a radial coefficient satisfying (2.51,2.59)) is that* $T_{\mu\nu}$ *violates the WEC (2.60).*

Proof. Since, according to Eq. (2.63), $T_{00} = \rho\,\alpha^2$ and $T_{11} = p_r\,\gamma^2$, the Einstein's equations corresponding to the indices $(0,0)$ and $(1,1)$ imply that

$$\frac{G_{00}}{\alpha^2} + \frac{G_{11}}{\gamma^2} = \rho + p_r\,. \tag{2.66}$$

Moreover, from the static version of Eqs. (2.31,2.33), we have that the right hand side of the previous inequality reads

$$\frac{G_{00}}{\alpha^2} + \frac{G_{11}}{\gamma^2} = \frac{d-1}{\gamma^2 r}\left[\left(\frac{\alpha'}{\alpha} + \frac{\gamma'}{\gamma}\right) r' - r''\right]\,.$$

Thus, since at the wormhole throat $r(0) = b > 0$ and $r'(0) = 0$, then Eq. (2.66) reads

$$-\frac{d-1}{b}\frac{r''(0)}{\gamma(0)^2} = \rho + p_r\,;$$

obviously, in order to have $r''(0) > 0$ one has that $\rho + p_r < 0$, which violates the WEC (2.60).

□

Remark 18 Let me finally highlight that the previous result is well known in the four-dimensional case; e.g., an analogue proof of Proposition 3 with $d = 3$ can be found in Ref. [11].

2.4.3 An example of exotic matter supporting a wormhole: phantom scalar fields

The most simple static four-dimensional spherically symmetric wormhole spacetime that one can guess has the metric (2.41) with

$$\alpha = \gamma^{-1} = 1\,, \qquad r = \sqrt{x^2 + b^2}\,, \tag{2.67}$$

where $b > 0$ is a positive constant with the dimension of a length; indeed, the coefficient r satisfies the conditions (2.51,2.59). For $x \to \pm\infty$, the metric $\mathbf{g}$ approaches the flat Minkowski metric $-dt^2 + dx^2 + x^2 d\Omega^2$. Hence, the region with $x \simeq 0$ represents the wormhole throat, of size b which connects the regions $x \gg 0$, $x \ll 0$, representing two asymptotically flat universes. This spacetime geometry received special attention in the classical 1988 paper by Morris and Thorne [16], considered as the origin of modern investigations on wormholes. Actually, the line element (2.67) had appeared in the literature before [16] (a fact on which Thorne apologized in Ref. [43]); indeed, this spacetime geometry was considered in a 1973 paper by Ellis [13], with the denomination of "drainhole" (and with a somehow different motivation, namely, to model an elementary particle); here the metric (2.67) was derived solving Einstein's equations in presence of a scalar field Φ minimally coupled to gravity, after changing artificially the sign of the action functional for Φ. Again in Ref. [13], the scalar field was found to depend on x with the law

$$\Phi = \sqrt{\frac{2}{\kappa}} \arctan \frac{x}{b} \ . \tag{2.68}$$

Almost simultaneously to Ellis, Bronnikov [14] proposed a family of scalar field solutions of Einstein's equations containing, as a special case, the solution (2.67,2.68). [12] The scalar fields considered by Ellis and Bronnikov, with an anomalous sign in (the kinetic part of) their action functional, have become popular with the denomination of "phantom scalar fields"; their stress-energy violates the usual conditions of positivity of the energy density (that is, the WEC), thus mimicking at the classical level a well known feature of quantum fields in their vacuum states [7, 44]. The action functional describing the system considered by Ellis and Bronnikov is as in Eq. (1.7) with the choices $\varsigma = -1$ (and $V(\Phi) = 0$).
In the rest of this paper I will refer to the names of the previously mentioned authors (Ellis, Bronnikov, Morris and Thorne) using the initials EBMT in the expressions "EBMT wormhole", "EBMT solution" which will be used to indicate the phantom field solution (2.67,2.68) of Einstein's equations.
In Chapter 3, I will recover how the EBMT solution arises directly from the Einstein-scalar equations (1.10,1.11) in the phantom case ($\varsigma = -1$).

Remark 19 The example of the EBMT wormhole directs our attention to the phantom scalar fields as a possible exotic source to support other

[12] We will return to this solution in Section 3.3; for the moment, let me only mention that the family of Bronnikov solutions (that we will call "Ellis-Bronnikov wormhole") depends on a "mass" parameter γ_1, which is zero in the case (2.67,2.68).

wormhole solutions. Hence, from now on we will focus on static spherically symmetric spacetimes minimally coupled to a phantom scalar field Φ that self-interacts according to a potential $V(\Phi)$. As just mentioned, this configuration is described by the action functional Eq. (1.7) with the choice $\varsigma = -1$.

Remark 20 For completeness, let me mention that a phantom scalar field is not the unique source producing the EBMT metric via Einstein's equations. Another source has been considered by Shatskii, Novikov and Kardashev [45]: this consists of a "phantom" fluid (with negative mass-energy density) and of an electromagnetic field. Of course, this alternative source requires a separate analysis for the stability problem. Bronnikov, Lipatova, Novikov and Shatskiy [46] have shown that, assuming a non conventional equation of the state for the fluid, the system is linearly stable under radially symmetric and axial perturbations; the same authors have conjectured the linear stability under arbitrary perturbations.

2.5 Field equations for a spherically symmetric spacetime supported by a self-interacting phantom scalar field

In the present section we are interested in deriving the field equations for a $(d+1)$-dimensional (non-static) spherically symmetric spacetime $(M_{d+1}, \mathbf{g})$ with $d \geq 3$ whose gravitational field $\mathbf{g}$, in the coordinates $(t, x, x^2, ..., x^d)$ introduced in Remark 8, has the local representation (2.24) and which is minimally coupled to a phantom scalar field Φ that self-interacts according to a potential $V(\Phi)$. Obviously, in order to support a (non-static) spherically symmetric configuration, the scalar field has to be independent of the coordinates $(x^2, ..., x^d)$; thus, we suppose that

$$\Phi \,:\, \mathcal{O} \subseteq \mathbb{R}^2 \to \mathbb{R}\,, \qquad V \,:\, \Phi(\mathcal{O}) \subseteq \mathbb{R} \to \mathbb{R} \tag{2.69}$$

are two smooth functions.
The field equations corresponding to this configuration are (1.10,1.11) with $\varsigma = -1$ (phantom case); we write

$$\mathcal{E}_{\mu\nu} = 0\,, \quad \mathcal{E}_{\mu\nu} := G_{\mu\nu} - \kappa\left[\nabla_\mu\Phi\cdot\nabla_\nu\Phi - g_{\mu\nu}\left(-\frac{1}{2}\nabla^\lambda\Phi\cdot\nabla_\lambda\Phi + V(\Phi)\right)\right]\,, \tag{2.70}$$

$$\mathcal{E}_{KG} = 0\,, \quad \mathcal{E}_{KG} := \nabla^\mu\nabla_\mu\Phi + V'(\Phi)\,, \tag{2.71}$$

where $g_{\mu\nu}$ are the coefficients of the metric $\mathbf{g}$ defined by Eq. (2.24) and Φ, V are as in Eq. (2.69). From the symmetry in the indices and the spherical symmetry of the metric and stress-energy tensor fields it turns out that the only non trivial Einstein's equations (2.70) are those corresponding to the indices $(\mu,\nu) \in \{(0,1),(1,0),(\mu,\mu) \,:\, \mu = 0,...,d\}$; moreover, one has that $\mathcal{E}_{01} = \mathcal{E}_{10}$ and that $\frac{\mathcal{E}_{ii}}{a_{ii}} = \frac{\mathcal{E}_{jj}}{a_{jj}}$ for every $i,j = 2,...,d$. Therefore, setting $\mathcal{E}_a := \frac{\mathcal{E}_{ii}}{a_{ii}}$ for any $i = 2,...,d$, and using the expressions for $G_{\mu\nu}$ given in Eqs. (2.31-2.34), the field equations (2.70,2.71) reduce to

$$\mathcal{E}_{00} = 0\,, \quad \mathcal{E}_{01} = 0\,, \quad \mathcal{E}_{11} = 0\,, \quad \mathcal{E}_a = 0\,, \quad \mathcal{E}_{KG} = 0\,, \tag{2.72}$$

where

$$\begin{aligned}
\mathcal{E}_{00} &= \frac{d-1}{2}\Bigg[\frac{(d-2)\alpha^2}{r^2} + \frac{1}{r}\left(\frac{(d-2)\dot{r}^2}{r} + \frac{2\dot{\gamma}\dot{r}}{\gamma}\right) \\
&\quad + \frac{\alpha^2}{\gamma^2 r}\left(\frac{2\gamma' r'}{\gamma} - \frac{(d-2)r'^2}{r} - 2r''\right)\Bigg] \\
&\quad + \frac{\kappa\alpha^2}{2}\left(\frac{\dot{\Phi}^2}{\alpha^2} + \frac{\Phi'^2}{\gamma^2} - 2V(\Phi)\right) + \frac{d-1}{r}\mathcal{E}^{\beta}_{00}\,, \qquad (2.73) \\
\mathcal{E}_{01} &= (d-1)\left[\frac{1}{r}\left(\frac{\alpha'\dot{r}}{\alpha} + \frac{\dot{\gamma}r'}{\gamma} - \dot{r}'\right)\right] + \kappa\dot{\Phi}\Phi' + \frac{d-1}{r}\mathcal{E}^{\beta}_{01}\,, \qquad (2.74) \\
\mathcal{E}_{11} &= \frac{d-1}{2}\Bigg[-\frac{(d-2)\gamma^2}{r^2}\left(1 + \frac{\dot{r}^2}{\alpha^2}\right) + \frac{2\gamma^2}{\alpha^2 r}\left(\frac{\dot{\alpha}\dot{r}}{\alpha} - \ddot{r}\right) \\
&\quad + \frac{r'}{r}\left(\frac{2\alpha'}{\alpha} + \frac{(d-2)r'}{r}\right)\Bigg] + \frac{\kappa\gamma^2}{2}\left(\frac{\dot{\Phi}^2}{\alpha^2} + \frac{\Phi'^2}{\gamma^2} + 2V(\Phi)\right) \\
&\quad + \frac{d-1}{r}\mathcal{E}^{\beta}_{11}\,, \qquad (2.75) \\
\mathcal{E}_a &= \frac{(d-2)(d-3)}{2}\left(\frac{r'^2}{\gamma^2} - \frac{\dot{r}^2}{\alpha^2} - 1\right) + \frac{(d-2)r}{\alpha^2}\left(\frac{\dot{\alpha}\dot{r}}{\alpha} - \frac{\dot{\gamma}\dot{r}}{\gamma} - \ddot{r}\right) \\
&\quad + \frac{r^2}{\alpha\gamma}\left(\frac{\dot{\alpha}\dot{\gamma}}{\alpha^2} - \frac{\alpha''\gamma'}{\gamma^2} + \frac{\alpha''}{\gamma} - \frac{\ddot{\gamma}}{\alpha}\right) + \frac{(d-2)r}{\gamma^2}\left(\frac{\alpha' r'}{\alpha} - \frac{\gamma' r'}{\gamma} + r''\right) \\
&\quad + \frac{\kappa r^2}{2}\left(\frac{\dot{\Phi}^2}{\alpha^2} - \frac{\Phi'^2}{\gamma^2} + 2V(\Phi)\right) + \frac{r^2}{\alpha^2}\mathcal{E}^{\beta}_{a}\,, \qquad (2.76) \\
\mathcal{E}_{KG} &= \frac{\ddot{\Phi}}{\alpha^2} + \frac{\dot{\Phi}}{\alpha^2}\left(\frac{\dot{\alpha}}{\alpha} - \frac{\dot{\gamma}}{\gamma} - (d-1)\frac{\dot{r}}{r}\right) + \frac{\Phi''}{\gamma^2} \\
&\quad + \frac{\Phi'}{\gamma^2}\left(\frac{\alpha'}{\alpha} - \frac{\gamma'}{\gamma} - (d-1)\frac{r'}{r}\right) + V'(\Phi) + \frac{1}{\alpha^2}\mathcal{E}^{\beta}_{KG}\,, \qquad (2.77)
\end{aligned}$$

and

$$
\begin{aligned}
\mathcal{E}^{\beta}_{00} &:= G^{\beta}_{00} + \frac{\kappa}{d-1}\Phi'\dot{\Phi}\beta \\
&\quad + \frac{\kappa}{d-1} r \Big(-\Phi'^2 + (d-2)\gamma^2 r V(\Phi) + \frac{1}{2}\frac{\gamma^2\dot{\Phi}^2}{\alpha^2} \Big)\beta^2 \\
&\quad - \frac{\kappa}{d-1} r \Phi'\dot{\Phi}\frac{\gamma^2}{\alpha^2}\beta^3 + \frac{\kappa}{2(d-1)} r \Phi'^2 \frac{\gamma^2}{\alpha^2}\beta^4 \,, \qquad (2.78) \\
\mathcal{E}^{\beta}_{01} &:= G^{\beta}_{01} + \frac{\kappa}{2(d-1)}\Big(2\gamma^2 V(\Phi) - \Phi'^2 + \frac{\gamma^2}{\alpha^2}\dot{\Phi}^2 \Big) r\beta \\
&\quad - \frac{\kappa}{d-1} r \Phi'\dot{\Phi}\frac{\gamma^2}{\alpha^2}\beta^2 + \frac{\kappa}{2} r \Phi'^2 \frac{\gamma^2}{\alpha^2}\beta^3 \,, \qquad (2.79) \\
\mathcal{E}^{\beta}_{11} &:= G^{\beta}_{11} + \frac{\kappa}{d-1} r \Phi'\dot{\Phi}\gamma^2\beta + \frac{\kappa}{2(d-1)} r \Phi'^2 \frac{\gamma^2}{\alpha^2}\beta^2 \,, \qquad (2.80) \\
\mathcal{E}^{\beta}_{a} &:= G^{\beta}_{a} - \kappa\Phi'\dot{\Phi}\beta + \frac{\kappa}{2}\Phi'^2\beta^2 \,, \qquad (2.81) \\
\mathcal{E}^{\beta}_{KG} &:= \Big[-\Big(\frac{\alpha'}{\alpha} - \frac{\gamma'}{\gamma} - (d-1)\frac{r'}{r} \Big)\dot{\Phi} \\
&\quad - \Big(\frac{\dot{\alpha}}{\alpha} - \frac{\dot{\gamma}}{\gamma} - (d-1)\frac{\dot{r}}{r} + 2\beta' \Big)\Phi' + 2\dot{\Phi}' \Big]\beta \\
&\quad + \Big[\Big(\frac{\alpha'}{\alpha} - \frac{\gamma'}{\gamma} - (d-1)\frac{r'}{r} \Big)\dot{\Phi} - \dot{\Phi}'' \Big]\beta^2 + \Phi'\dot{\beta} + \beta'\dot{\Phi} \,. \qquad (2.82)
\end{aligned}
$$

(G^{β}_{00}, G^{β}_{01}, G^{β}_{11} and G^{β}_{a} appearing in Eqs. (2.78-2.82) are defined as in Eqs. (2.35-2.38.) Clearly, the quantities $\mathcal{E}^{\beta}_{00}$, $\mathcal{E}^{\beta}_{01}$, $\mathcal{E}^{\beta}_{11}$, $\mathcal{E}^{\beta}_{a}$ and $\mathcal{E}^{\beta}_{KG}$ vanish as long as the coefficient β vanishes: this happens, for example, if (t, x) is an orthogonal coordinate system on M_2 (see Remark 10).

Remark 21 Let us remind that Einstein's equations $\mathcal{E}_{\mu\nu} = 0$ and the Klein-Gordon equation $\mathcal{E}_{KG} = 0$ are non independent, since the latter is automatically satisfied if Einstein's equations hold (see Theorem 2); despite of this fact, by considering all the equations in (2.72), one can introduce a useful recombination of the quantities $\mathcal{E}_{00}, ..., \mathcal{E}_{KG}$ such that the field system (2.72) can be rewritten in the following way

$$\begin{aligned}
\mathfrak{E}_1 = 0\,, \quad \mathfrak{E}_1 :=& \frac{\partial}{\partial t}\left[\frac{\dot{\gamma}}{\alpha}\right] - \frac{\partial}{\partial x}\left[\frac{\alpha'}{\gamma}\right] + (d-2)\left(-\frac{\gamma}{\alpha}\frac{\dot{r}^2}{r^2} + \frac{\alpha}{\gamma}\frac{r'^2}{r^2} - \frac{\alpha\gamma}{r^2}\right) \\
&+ \frac{d-3}{2r}\left(\frac{\partial}{\partial t}\left[\frac{\gamma}{\alpha}\dot{r}\right] - \frac{\partial}{\partial x}\left[\frac{\alpha}{\gamma}r'\right]\right) \\
&- \frac{\kappa}{2}\left(\frac{\gamma}{\alpha}\dot{\Phi}^2 - \frac{\alpha}{\gamma}\Phi'^2\right) + \frac{\gamma}{\alpha}\mathfrak{E}_1^{\beta}\,,
\end{aligned} \tag{2.83}$$

$$\begin{aligned}
\mathfrak{E}_2 = 0\,, \qquad \mathfrak{E}_2 :=& \frac{d-1}{2}\left(\frac{\partial}{\partial t}\left[\frac{\gamma}{\alpha}r\dot{r}\right] - \frac{\partial}{\partial x}\left[\frac{\alpha}{\gamma}rr'\right]\right) \\
&+ \frac{(d-1)(d-3)}{2}\left(\frac{\gamma}{\alpha}\dot{r}^2 - \frac{\alpha}{\gamma}r'^2\right) + \frac{d(d-3)}{2}\alpha\gamma \\
&- \alpha\gamma\left(\kappa r^2 V(\Phi) - 1\right) + \frac{d-1}{2}\frac{\gamma r^2}{\alpha}\mathfrak{E}_2^{\beta}\,,
\end{aligned} \tag{2.84}$$

$$\begin{aligned}
\mathfrak{E}_3 = 0\,, \quad \mathfrak{E}_3 :=& \frac{\partial}{\partial t}\left[\frac{\gamma}{\alpha}r^2\dot{\Phi}\right] - \frac{\partial}{\partial x}\left[\frac{\alpha}{\gamma}r^2\Phi'\right] \\
&+ (d-3)\left(\frac{\gamma}{\alpha}\dot{r}\dot{\Phi} - \frac{\alpha}{\gamma}r'\Phi'\right)r - \alpha\gamma r^2 V'(\Phi) + \frac{\gamma r^2}{\alpha}\mathfrak{E}_3^{\beta}\,,
\end{aligned} \tag{2.85}$$

$$\begin{aligned}
\mathcal{H} = 0\,, \qquad \mathcal{H} :=& \frac{d-1}{2}\frac{\alpha}{\gamma}\left[2\frac{r''}{r} + \frac{r'}{r}\left((d-2)\frac{r'}{r} - 2\frac{\gamma'}{\gamma}\right)\right] \\
&- \frac{d-1}{2}\frac{\gamma}{\alpha}\frac{\dot{r}}{r}\left(\frac{\dot{r}}{r} + 2\frac{\dot{\gamma}}{\gamma}\right) - \frac{d(d-3)+2}{2}\frac{\alpha\gamma}{r^2} \\
&- \frac{\kappa}{2}\left(\frac{\gamma}{\alpha}\dot{\Phi}^2 + \frac{\alpha}{\gamma}\Phi'^2\right) + \kappa\alpha\gamma V(\Phi) + (d-1)\frac{\gamma}{\alpha}\mathcal{H}^{\beta}\,,
\end{aligned} \tag{2.86}$$

$$\mathcal{M} = 0\,, \qquad \mathcal{M} := (d-1)\left(\frac{\dot{r}'}{r} - \frac{\dot{r}}{r}\frac{\alpha'}{\alpha} - \frac{r'}{r}\frac{\dot{\gamma}}{\gamma}\right) - \kappa\dot{\Phi}\Phi' + (d-1)\mathcal{M}^{\beta}\,, \tag{2.87}$$

where

$$
\begin{aligned}
\mathfrak{E}_1^\beta \;:=\; & \frac{1}{4}\Bigg[-\kappa\Phi'(t,x)^2+(d-2)(d-1)\frac{r'^2}{r^2}+2(d-1)\Bigg(\frac{r''}{r}-\frac{\alpha'}{\alpha}\frac{r'}{r}\Bigg)\Bigg]\frac{\gamma^2}{\alpha^2}\beta^4\\
&+\frac{1}{2}\Bigg[2\left(-\frac{\alpha'}{\alpha}+3\frac{\gamma'}{\gamma}+(d-3)\frac{r'}{r}\right)\beta'(t,x)+2\beta''(t,x)\\
&\quad+\frac{\dot\alpha}{\alpha}\Bigg(2\frac{\gamma'}{\gamma}+(d-3)\frac{r'}{r}\Bigg)+\left(2\frac{\alpha'}{\alpha}+(5-3d)\frac{r'}{r}\right)\frac{\dot\gamma}{\gamma}\\
&\quad+\left(-(d+1)\frac{\alpha'}{\alpha}-(d-3)\frac{\gamma'}{\gamma}+4(d-2)\frac{r'}{r}\right)\frac{\dot r}{r}-4\frac{\dot\gamma'}{\gamma}+4\frac{\dot r'}{r}\Bigg]\beta\\
&+\frac{(d-1)}{2}\Bigg[\Bigg(\beta'+\frac{\dot\alpha}{\alpha}\Bigg)\frac{r'}{r}+\frac{\kappa}{(d-1)}\Phi'\dot\Phi\\
&\quad+\Bigg(\frac{\alpha'}{\alpha}-(d-2)\frac{r'}{r}\Bigg)\frac{\dot r}{r}-2\frac{\dot r'}{r}\Bigg]\frac{\gamma^2}{\alpha^3 2}\beta^3\\
&+\frac{1}{4}\Bigg[(d-1)\Bigg((d-2)\frac{\dot r^2}{r^2}+2\frac{\ddot r}{r}-\frac{\kappa}{d-1}\dot\Phi^2-2\frac{r'\dot\beta}{r}-2\frac{\dot\alpha\dot r}{\alpha r}\Bigg)\frac{\gamma^2}{\alpha^2}\\
&\quad+(d-2)(d-1)\frac{\gamma^2}{r^2}+\kappa\left(\Phi'^2-2\gamma^2V(\Phi)\right)\\
&\quad+2\Bigg(2\frac{\alpha'}{\alpha}+(3d-5)\frac{\gamma'}{\gamma}-(d-2)(d+3)\frac{r'}{r}\Bigg)\frac{r'}{r}\\
&\quad-4\frac{\alpha'}{\alpha}\frac{\gamma'}{\gamma}-2(d+1)\frac{r''}{r}+4\frac{\gamma''}{\gamma}\Bigg]\beta^2\\
&+\left(\frac{\dot\alpha}{\alpha}-2\frac{\dot\gamma}{\gamma}-\frac{d-3}{2}\frac{\dot r}{r}+\beta'\right)\beta'-\dot\beta'-\left(\frac{\gamma'}{\gamma}+\frac{d-3}{2}\frac{r'}{r}\right)\dot\beta\,,
\end{aligned}
\tag{2.88}
$$

$$
\begin{aligned}
\mathfrak{E}_2^\beta \;:=\;& \Bigg\{\Bigg[\bigg(-\frac{\alpha'}{\alpha}+\frac{1}{2}\frac{\gamma'}{\gamma}+\frac{3(d-2)}{2}\frac{r'}{r}\bigg)\frac{r'}{r} \\
&\quad+\frac{\kappa}{2(d-1)}\bigg(2\alpha^2V(\Phi)+\dot{\Phi}^2-3\Phi'^2\bigg)\Bigg]\frac{\alpha^2}{\gamma^2}-\frac{d-2}{2}\frac{\alpha^2}{r^2} \\
&\quad+\bigg(\frac{\dot{\alpha}}{\alpha}-\frac{d-2}{2}\frac{\dot{r}}{r}+\dot{\beta}\bigg)\frac{\dot{r}}{r}+\bigg(3\frac{r''}{r}-\frac{\ddot{r}}{r}\bigg)\Bigg\}\frac{\gamma^2}{\alpha^2}\beta^2 \\
&+\Bigg[\frac{\dot{\alpha}}{\alpha}\frac{r'}{r}+\frac{\dot{r}}{r}\bigg(\frac{3\alpha'}{\alpha}-\frac{2(d-2)r'}{r}\bigg)+2\beta'\frac{r'}{r} \\
&\quad+\frac{\dot{\gamma}}{\gamma}\frac{r'}{r}-\frac{\gamma'}{\gamma}\frac{\dot{r}}{r}-4\frac{\dot{r}'}{r}+\frac{2\kappa}{d-1}\Phi'\dot{\Phi}\Bigg]\beta \\
&+\Bigg[-\bigg(\beta'+\frac{\dot{\alpha}}{\alpha}\bigg)\frac{r'}{r}+\bigg((d-2)\frac{r'}{r}-\frac{\alpha'}{\alpha}\bigg)\frac{\dot{r}}{r} \\
&\quad+2\frac{\dot{r}'}{r}-\frac{\kappa}{d-1}\Phi'\dot{\Phi}\Bigg]\frac{\gamma^2}{\alpha^2}\beta^3 \\
&+\Bigg[\bigg(\frac{\alpha'}{\alpha}-\frac{d-2}{2}\frac{r'}{r}\bigg)\frac{r'}{r}-\frac{r''}{r}+\frac{\kappa}{2(d-1)}\Phi'^2\Bigg]\frac{\gamma^2}{\alpha^2}\beta^4 \\
&-\dot{\beta}\frac{r'}{r}-\beta'\frac{\dot{r}}{r}\,,
\end{aligned}
\tag{2.89}
$$

$$
\begin{aligned}
\mathfrak{E}_3^\beta \;:=\;& \Bigg[\bigg(\frac{\dot{\alpha}}{\alpha}-(d-1)\frac{\dot{r}}{r}-\frac{\dot{\gamma}}{\gamma}+2\beta'\bigg)\Phi' \\
&\quad+\bigg(\frac{\alpha'}{\alpha}-(d-1)\frac{r'}{r}-\frac{\gamma'}{\gamma}\bigg)\dot{\Phi}-2\dot{\Phi}'\Bigg]\beta \\
&+\Bigg[\bigg(\frac{\gamma'}{\gamma}-\frac{\alpha'}{\alpha}+(d-1)\frac{r'}{r}\bigg)\Phi'+\Phi''\Bigg]\beta^2-\dot{\beta}\Phi'-\beta'\dot{\Phi}\,,
\end{aligned}
\tag{2.90}
$$

$$
\begin{aligned}
\mathcal{H}^\beta \;:=\; & \Bigg\{ \left(\frac{\alpha'}{\alpha} + \frac{\gamma'}{\gamma} - 2(d-2)\frac{r'}{r} \right) \frac{r'}{r} - 2\frac{r''}{r} \\
& \quad + \Bigg[\left(\frac{d-2}{2}\frac{\dot r}{r} - \frac{\dot\alpha}{\alpha} \right) \frac{\dot r}{r} + \frac{d-2}{2}\frac{\alpha^2}{r^2} - \dot\beta \frac{r'}{r} \\
& \quad + \frac{\ddot r}{r} - \frac{\kappa}{2(d-1)} \left(\dot\Phi^2 + 2\alpha^2 V(\Phi) + 2\alpha^2\gamma^2\Phi'^2 \right) \Bigg] \frac{\gamma^2}{\alpha^2} \Bigg\} \beta^2 \\
& + \Bigg[\left(-2\frac{\alpha'}{\alpha} + \frac{\gamma'}{\gamma} + (d-2)\frac{r'}{r} \right) \frac{\dot r}{r} \\
& \quad + \left(\beta' - \frac{\dot\gamma}{\gamma} \right) \frac{r'}{r} + 2\frac{\dot r'}{r} - \frac{\kappa}{d-1}\Phi'\dot\Phi \Bigg] \beta \\
& + \Bigg[\left(\frac{\dot\alpha}{\alpha} + \beta' \right) \frac{r'}{r} + \left(\frac{\alpha'}{\alpha} - (d-2)\frac{r'}{r} \right) \frac{\dot r}{r} \\
& \quad - 2\frac{\dot r'}{r} + \frac{\kappa}{d-1}\Phi'\dot\Phi \Bigg] \frac{\gamma^2}{\alpha^2}\beta^3 \\
& + \Bigg[\left(-\frac{\alpha'}{\alpha} + \frac{d-2}{2}\frac{r'}{r} \right) \frac{r'}{r} + \frac{r''}{r} - \frac{\kappa}{2(d-1)}\Phi'^2 \Bigg] \frac{\gamma^2}{\alpha^2}\beta^4 + \frac{\dot r}{r}\beta' \,,
\end{aligned}
\tag{2.91}
$$

$$
\begin{aligned}
\mathcal{M}^\beta \;:=\; & \Bigg\{ \Bigg[\left(\frac{d-2}{2}\frac{\dot r}{r} - \frac{\dot\alpha}{\alpha} \right) \frac{\dot r}{r} - \dot\beta\frac{r'}{r} + \frac{\ddot r}{r} \Bigg] \frac{\gamma^2}{\alpha^2} \\
& \quad + \left(\frac{\gamma'}{\gamma} - \frac{d-2}{2}\frac{r'}{r} \right) \frac{r'}{r} + \frac{d-2}{2}\frac{\gamma^2}{r^2} - \frac{r''}{r} \\
& \quad + \frac{\kappa}{2} \left(\Phi'^2 - \frac{\gamma^2}{\alpha^2}\dot\Phi^2 - 2\gamma^2 V(\Phi) \right) \Bigg\} \beta \\
& + \Bigg[\left(\frac{\dot\alpha}{\alpha} + \beta' \right) \frac{r'}{r} + \left(\frac{\alpha'}{\alpha} - (d-2)\frac{r'}{r} \right) \frac{\dot r}{r} \\
& \quad - 2\frac{\dot r'}{r} + \frac{\kappa}{d-1}\Phi'\dot\Phi \Bigg] \frac{\gamma^2}{\alpha^2}\beta^2 \\
& + \Bigg[\left(\frac{d-2}{2}\frac{r'}{r} - \frac{\alpha'}{\alpha} \right) \frac{r'}{r} + \frac{r''}{r} - \frac{\kappa}{2(d-1)}\Phi'^2 \Bigg] \frac{\gamma^2}{\alpha^2}\beta^3 \,.
\end{aligned}
\tag{2.92}
$$

Note that the quantities $\mathfrak{E}_1^\beta$, $\mathfrak{E}_2^\beta$, $\mathfrak{E}_3^\beta$, $\mathcal{H}^\beta$ and $\mathcal{M}^\beta$ vanish when $\beta = 0$, that is, when (t, x) is an orthogonal coordinate system for M_2 (see Remark 10).

Proof. In order to prove the statement of the previous remark, we have to show that the systems (2.72) is actually equivalent to the system (2.83-2.87). Indeed, one can see that

$$\mathfrak{E}_1 = -\frac{\gamma}{2\alpha}\mathcal{E}_{00} + \frac{\alpha}{2\gamma}\mathcal{E}_{11} - \frac{\alpha\gamma}{r^2}\mathcal{E}_a\,, \tag{2.93}$$

$$\mathfrak{E}_2 = \frac{\gamma r^2}{2\alpha}\mathcal{E}_{00} - \frac{\alpha r^2}{2\gamma}\mathcal{E}_{11}\,, \tag{2.94}$$

$$\mathfrak{E}_3 = -\alpha\gamma r\mathcal{E}_{KG}\,, \tag{2.95}$$

$$\mathcal{H} = -\frac{\gamma}{\alpha}\mathcal{E}_{00}\,, \tag{2.96}$$

$$\mathcal{M} = -\mathcal{E}_{01}\,; \tag{2.97}$$

clearly, this implies that if Eq. (2.72) holds than the system (2.83-2.87) is trivially satisfied. Conversely, lets suppose that Eqs. (2.83-2.87) hold; then, since α, γ, r vanish nowhere in the coordinates domain, from Eqs. (2.95-2.97) one has that the equations $\mathcal{E}_{KG} = 0$, $\mathcal{E}_{00} = 0$ and $\mathcal{E}_{01} = 0$ are satisfied. Finally, from Eq. (2.94) and Eq. (2.93) and from the fact that $\mathcal{E}_{00}$ vanishes, one infers that even $\mathcal{E}_{11}$ and $\mathcal{E}_a$ vanish too.

□

2.5.1 A constrained field system in the $\beta = 0$ gauge

Let us consider the orthogonal gauge on the manifold M_2 introduced in Remark 10: this is equivalent to set the coefficient $\beta = 0$. In this case the field system (2.83-2.87) can be interpreted as a constrained evolution system in the sense stated by the following

Proposition 4 *In the gauge $\beta = 0$, the equations $\mathcal{H} = 0$ and $\mathcal{M} = 0$ [Eqs. (2.86,2.87)] are two constraints for the second order evolution equations $\mathfrak{E}_1 = 0$, $\mathfrak{E}_2 = 0$, $\mathfrak{E}_3 = 0$ [Eqs. (2.83-2.85)]; this means that if $(\alpha, \gamma, r, \Phi)$ is a (time-dependent) solution of the system (2.83-2.85) satisfying the equations $\mathcal{H} = 0$, $\mathcal{M} = 0$ at time $t = 0$, then $(\alpha, \gamma, r, \Phi)$ satisfies equations $\mathcal{H} = 0$, $\mathcal{M} = 0$ for every time t.*

Proof. Let $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$, $\mathcal{H}$ and $\mathcal{M}$ the quantities defined in Eqs. (2.83-2.87) with $\mathfrak{E}_1^\beta = \mathfrak{E}_2^\beta = \mathfrak{E}_3^\beta = \mathcal{H}^\beta = \mathcal{M}^\beta = 0$ (since we have chosen a gauge in which $\beta = 0$). Then, it is not difficult to prove that the following identities hold

$$\begin{aligned}\dot{\mathcal{H}} &= \left[\frac{\alpha'}{\alpha} + \frac{\alpha}{\gamma}\left(-\frac{\gamma'}{\gamma} + (d-1)\frac{r'}{r}\right)\right]\mathcal{M} + \left[\frac{\dot{\alpha}}{\alpha} - \frac{\dot{\gamma}}{\gamma} - (d-1)\frac{\dot{r}}{r}\right]\mathcal{H} + \frac{\alpha}{\gamma}\mathcal{M}' \\ &\quad - (d-1)\frac{\dot{r}}{r}\mathfrak{E}_1 - \frac{\kappa\dot{\Phi}}{r^2}\mathfrak{E}_2 - \frac{2}{r^2}\left[\frac{\dot{\gamma}}{\gamma} + \frac{d-1}{2}\frac{\dot{r}}{r}\right]\mathfrak{E}_3\,, \qquad (2.98)\\ \dot{\mathcal{M}} &= \left[\frac{\dot{\alpha}}{\alpha} - \frac{\dot{\gamma}}{\gamma} - (d-1)\frac{\dot{r}}{r}\right]\mathcal{M} + \left[\frac{\alpha'}{\alpha} + \frac{\alpha}{\gamma}\left(-\frac{\gamma'}{\gamma} + (d-1)\frac{r'}{r}\right)\right]\mathcal{H} + \frac{\alpha}{\gamma}\mathcal{H}' \\ &\quad - (d-1)\frac{\alpha}{\gamma}\frac{r'}{r}\mathfrak{E}_1 - 2\frac{\alpha}{\gamma r^2}\left[\frac{\gamma'}{\gamma} - \frac{d-5}{2}\frac{r'}{r}\right]\mathfrak{E}_2 + 2\frac{\alpha}{\gamma r^2}\mathfrak{E}_2' - \frac{\alpha}{\gamma}\frac{\kappa\Phi'}{r^2}\mathfrak{E}_3\,. \qquad (2.99)\end{aligned}$$

Now, let $(\alpha, \gamma, r, \Phi)$ be a solution of the evolution equations (2.83-2.85); then $(\alpha, \gamma, r, \Phi)$ satisfies the identities (2.98,2.99) with $\mathfrak{E}_1 = \mathfrak{E}_2 = \mathfrak{E}_2' = \mathfrak{E}_3 = 0$; fixing the spatial coordinate x, these can be seen as a dynamical system in the unknowns $\mathcal{H}(t) \equiv \mathcal{H}(t,x)$ and $\mathcal{M}(t) \equiv \mathcal{M}(t,x)$ with smooth coefficients. Since by hypothesis the solution $(\alpha, \gamma, r, \Phi)$ satisfies Eqs. (2.86,2.87) at time $t = 0$, then the dynamical system (2.98,2.99) has the initial conditions $\mathcal{H}(0) = \mathcal{M}(0) = 0$; obviously, this implies that $\dot{\mathcal{H}}(t) = \dot{\mathcal{M}}(t) = 0$ for every time t, i.e. $\mathcal{H}(t,x) = \mathcal{M}(t,x) = 0$ for every time t and every x.

□

Remark 22 Actually, in the gauge $\delta\beta = 0$, the equations $\mathcal{H} = 0$ and $\mathcal{M} = 0$ [Eqs. (2.86,2.87)] are, respectively, the Hamiltonian and the momentum constraints of pag. 187 of Ref. [47]; indeed, in this case the frame $(t, x, x^2, ..., x^d)$ coincides exactly with the Chauchy adapted frame $(\theta^i)_{i=1,...,d}$ with lapse $N = \alpha(t,x)$. Indeed, coherently with the result of Proposition 4, these two equations can be interpreted as prescriptions on the geometry of the spacetime at "time zero", that is, on the initial data for the evoultion of the spacetime M_{d+1} determined by the system $\mathfrak{E}_1 = 0$, $\mathfrak{E}_2 = 0$, $\mathfrak{E}_3 = 0$ [Eqs. (2.83.2.85)].

Chapter 3

Static wormhole solutions

In this chapter we deal with some examples of static wormhole solutions that have been considered previously in the literature. To this purpose, we consider a $(d+1)$-dimensional spherically symmetric spacetime $(M_{d+1}, \mathbf{g})$ as in items (i-iv) of Section 2.2 where the gravitational field $\mathbf{g}$ minimally couples to a phantom scalar field Φ with a self-interacting potential $V(\Phi)$. In order to simplify the computations, throughout this chapter we assume for the manifold M_2 the orthogonal gauge given in Remark 10 in which the function β vanishes. In addition, since we are interested in static wormhole solutions, we assume that the metric $\mathbf{g}$ (as in Eq. (2.24)) and the scalar field Φ are t-independent, and (generalizing the results of Section 2.3) we assume that the radial coefficient of the metric satisfies the prescriptions (2.51,2.59). Let us recall that in the static case the range of the coordinates (t, x) is $\mathbb{O} = \mathbb{R} \times x(\mathcal{O})$ (see Remark 11). Hence, we set

$$\beta = 0\,, \qquad (\alpha, \gamma, r, \Phi) := (\alpha(x), \gamma(x), r(x), \Phi(x))\,; \tag{3.1}$$

if not otherwise specified, $V(\Phi)$ is an arbitrary real smooth function depending on the scalar field Φ:

$$V\,:\,\Phi(x(\mathbb{O})) \subseteq \mathbb{R} \to \mathbb{R}\,.$$

3.1 Background equations

Let us consider the constrained field system (2.83-2.87). In the static case under consideration (3.1) the momentum constraint $\mathcal{M}$ is identically satisfied; hence, the field system reduces to Eqs. (2.83-2.86). Note that the the latter can be recombined, leading to a new equivalent system:

$$\begin{aligned} \mathfrak{E}_3 &= 0\,, & (3.2)\\ 2\mathfrak{E}_2 + r^2\mathcal{H} &= 0 & (3.3)\\ \mathfrak{E}_2 + r^2\mathcal{H} &= 0\,, & (3.4)\\ (d-1)r^2\mathfrak{E}_1 + (d+1)\mathfrak{E}_2 + (d-1)r^2\mathcal{H} &= 0\,, & (3.5) \end{aligned}$$

Indeed, Eq. (3.2) is exactly Eq. (2.85); moreover, since the couple (3.3,3.4) is clearly equivalent to the couple (2.84,2.86), then the remaining equations (3.3,3.4,3.5) are clearly equivalent to Eqs. (2.83,2.84,2.86).
The interesting fact of the new static system (3.2-3.5) is that one can isolate the the xx derivatives of the static functions α, Φ and r from the first three equations, and the square r'^2 from the remaining one; this yields to

$$\begin{aligned} \Phi'' &= \left(\frac{\gamma'}{\gamma} - \frac{\alpha'}{\alpha} - (d-1)\frac{r'}{r}\right)\Phi' - \gamma^2 V'(\Phi)\,, & (3.6)\\ \alpha'' &= \left(\frac{\gamma'}{\gamma} - (d-1)\frac{r'}{r}\right)\alpha' - \frac{2\kappa}{d-1}\alpha\gamma^2 V(\Phi)\,, & (3.7)\\ r'' &= \left(\frac{\gamma'}{\gamma} + \frac{\alpha'}{\alpha}\right)r' + \frac{\kappa}{d-1} r\Phi'^2\,, & (3.8)\\ r'^2 &= \gamma^2 - \frac{\kappa r^2}{(d-1)(d-2)}\left(2\gamma^2 V(\Phi) + \Phi'^2\right) - \frac{2}{d-2}\frac{\alpha' r'}{\alpha}\,. & (3.9) \end{aligned}$$

In the sequel, we will refer to Eqs. (3.6-3.9) as *background equations* since they fully describe a general static solution (3.1) whose stability features will be discussed in the chapters of the second part of this work.

3.2 EBMT wormhole in higher dimension: the Torii-Shinkai wormhole

In this section we assume a zero potential, that is $V(\Phi) \equiv 0$, in addition to the assumption (3.1). In this case, as already mentioned in Remark 7, one can further adjust the coordinate x so that $\alpha\gamma = 1$. In this case, the static field equations (2.83-2.87) can be reduced to the following four ordinary differential equations [13]

$$
\begin{aligned}
r'^2 + \frac{\kappa r^2 \Phi'^2}{(d-1)(d-2)} - \frac{1}{\alpha^2} &= \frac{4r^2(\alpha\alpha')' + 2(d-3)rr'\alpha\alpha'}{(d+1)(d-2)\alpha^2}\,, && (3.10)\\
r'' - \frac{d-2}{r}\left(\frac{1}{\alpha^2} - r'^2\right) &= -2\frac{\alpha' r'}{\alpha}\,, && (3.11)\\
[\alpha^2 r^{d-1}\Phi']' &= 0\,, && (3.12)\\
\frac{r}{\alpha^2}\left[\alpha^2 r'\right]' + [rr']' + (d-3)r'^2 &= \frac{\kappa}{d-1} r^2\Phi'^2 + \frac{d-2}{\alpha^2}\,. && (3.13)
\end{aligned}
$$

Obviously, Eq. (3.12) gives the relation

$$
\Phi' = \frac{\Phi_0}{\alpha^2 r^{d-1}}\,, \tag{3.14}
$$

where Φ_0 is an integration constant. If we now set $\alpha = 1$ (and hence $\gamma = 1$), the right hand sides of Eqs. (3.10,3.11) vanish. Moreover, since we are interested in static wormhole-type spacetimes, we introduce the prescription (2.51), which guarantees the existence of a throat of size b at $x = 0$. With these assumptions, Eq. (3.10) is satisfied if and only if the following holds

$$
r(0) = b\,, \quad r' = \mathrm{sign}(x)\sqrt{1 - \frac{b^{2(d-2)}}{r^{2(d-2)}}}\,, \quad \Phi_0 = b^{d-2}\sqrt{\frac{(d-1)(d-2)}{\kappa}}\,, \tag{3.15}
$$

where

$$
\mathrm{sign}(x) := \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases} \tag{3.16}
$$

One can easily verify that if Eq. (3.15) holds then the remaining field equations (3.11,3.13) are automatically satisfied. Summing up, we have found the (implicitly defined) static solution

$$
\begin{gathered}
\alpha = \gamma^{-1} = 1\,, \qquad r(0) = b\,, \qquad r' = \mathrm{sign}(x)\sqrt{1 - \frac{b^{2(d-2)}}{r^{2(d-2)}}}\,,\\
\Phi = \sqrt{\frac{(d-1)(d-2)}{\kappa}}\, b^{d-2}\int_{x_0}^{x} \frac{d\tilde{x}}{r(\tilde{x})^{d-1}}\,,
\end{gathered} \tag{3.17}
$$

[13] Lets prove that the equivalence of the field system (2.83-2.87) and the system (3.10-3.13). As we have already observed, that in the static case the momentum $\mathcal{M}$ is identically null: hence we can get rid of Eq. (2.87). Moreover, we observe that Eq. (3.10) arises directly from the recombination $2\mathfrak{E}_1 + \frac{d-3}{d-1}\mathcal{H} = 0$, Eq. (3.11) arises immediately from $\frac{2}{(d-1)\alpha^2 r}\mathfrak{E}_2 = 0$, Eq. (3.12) is equal to $-r^{d-3}\mathfrak{E}_3 = 0$, while Eq. (3.13) is equal to $\frac{2}{d-1}\frac{r^2}{\alpha^2}\mathcal{H} = 0$. Therefore, if $\mathfrak{E}_1 = 0$, $\mathfrak{E}_2 = 0$, $\mathfrak{E}_3 = 0$ and $\mathcal{H} = 0$ hold, then Eqs. (3.10-3.13) are trivially satisfied; conversely, if the system (3.10-3.13) holds then the quantities $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$ and $\mathcal{H}$ must vanish, since the their coefficients in the recombinations which define Eqs. (3.10-3.13) never vanish nor diverge.

where the value of x_0 is immaterial. An elementary qualitative analysis, based on the conservation of energy and on the initial data for r, also gives that r is an even function such that

$$r(x) > b \quad \text{for } x \in \mathbb{R}\backslash\{0\}\,, \qquad r(x) \to \pm\infty \quad \text{for } x \to \pm\infty\,. \tag{3.18}$$

The next theorem provides an explicit solution to the ODE for r appearing in Eq. (3.17) in terms of the hypergeometric function ${}_2F_1$ and the gamma function Γ.

Theorem 4 *The solution of the Cauchy problem*

$$\begin{cases} r' = \mathrm{sign}(x)\sqrt{1 - \frac{b^{2(d-2)}}{r^{2(d-2)}}} \\ r(0) = b \end{cases} \tag{3.19}$$

is

$$r(x) = bF^{-1}\left(\left|\frac{x}{b}\right|\right) \qquad \textit{for every } x \in \mathbb{R}\,, \tag{3.20}$$

where

$$F : [1, +\infty) \to [0, +\infty)\,,$$

$$\varrho \mapsto F(\varrho) := C_d + \varrho\, {}_2F_1\left(\frac{1}{2}, -\frac{1}{2(d-2)}, \frac{2d-5}{2(d-2)}; \frac{1}{\varrho^{2(d-2)}}\right)\,, \tag{3.21}$$

$$C_d := -\frac{\sqrt{\pi}\,\Gamma\left(-\frac{1}{2(d-2)}\right)}{(d-1)\,\Gamma\left(-\frac{d-1}{2(d-2)}\right)}\,; \tag{3.22}$$

moreover, the solution (3.20) has the following asymptotic expansion at infinity

$$r(x) = |x| - b\,C_d + O\left(\frac{1}{|x|^{2d-5}}\right) \qquad \textit{for } x \mapsto \pm\infty \qquad (C_d \textit{ as in Eq. (3.22)})\,. \tag{3.23}$$

Proof. We introduce the function $\rho(\mathtt{x}) := \frac{r(b\mathtt{x})}{b}$ defined for every $\mathtt{x} \in \mathbb{R}$; then the function r satisfies the Cauchy problem (3.19) if and only in the the function ρ satisfies the Cauchy problem

$$\begin{cases} \rho' = \mathrm{sign}(\mathtt{x})\sqrt{1 - \frac{1}{\rho^{2(d-2)}}} \\ \rho(0) = 1 \end{cases} \tag{3.24}$$

Integrating by separation of variables, it turns out that, for every $\mathtt{x} \in \mathbb{R}$

$$|\mathtt{x}| = F(\rho(\mathtt{x}))\,, \tag{3.25}$$

where

$$F(\varrho) := \int_1^\varrho \frac{dr}{\sqrt{1-\frac{1}{r^{2(d-2)}}}}$$
$$= \frac{1}{2(d-2)}\sqrt{1-\frac{1}{\varrho^{2(d-2)}}}\int_0^1 \mathrm{t}^{-\frac{1}{2}}(1-\mathrm{t})^0\left[1-\left(1-\frac{1}{\varrho^{2(d-2)}}\right)\mathrm{t}\right]^{\frac{-2d+3}{2(d-2)}} d\mathrm{t}\,; \tag{3.26}$$

in the last equality I have performed the change of variable $r \mapsto \mathrm{t}$ defined by

$$r = \left[1-\left(1-\frac{1}{\varrho^{2(d-2)}}\right)\mathrm{t}\right]^{-\frac{1}{2(d-2)}} \qquad 0 \le \mathrm{t} \le 1\,.$$

Now, one can use the standard integral representation of the hypergeometric function ${}_2F_1$ (see Ref. [48], Eq. 15.3.1) in order to obtain a first hypergeometric representation for the function F

$$F(\varrho) = \frac{1}{2(d-2)}\sqrt{1-\frac{1}{\varrho^{2(d-2)}}}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)}\,{}_2F_1\left(\frac{2d-3}{2(d-2)},\frac{1}{2},\frac{3}{2};1-\frac{1}{\varrho^{2(d-2)}}\right)$$
$$= \frac{1}{d-2}\sqrt{1-\frac{1}{\varrho^{2(d-2)}}}\,{}_2F_1\left(\frac{1}{2},\frac{2d-3}{2(d-2)},\frac{3}{2};1-\frac{1}{\varrho^{2(d-2)}}\right)\,. \tag{3.27}$$

(In the last equality I have used the symmetry of the hypergeometric function in the first two parameters ${}_2F_1(a,b,c;z) = {}_2F_1(b,a,c;z)$ and the fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$ and $\Gamma\left(1\right) = 1$.) The hypergeometric representation for F given in Eq. (3.21) can obtained from Eq. (3.27) performing a Kummer transformation (see Ref. [48], Eq. 15.3.6):

$$\begin{aligned}F(\varrho) =& \frac{1}{d-2}\sqrt{1-\frac{1}{\varrho^{2(d-2)}}}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(-\frac{1}{2(d-2)}\right)}{\Gamma\left(1\right)\Gamma\left(\frac{d-3}{2(d-2)}\right)}\\ &\times {}_2F_1\left(\frac{1}{2},-\frac{2d-3}{2(d-2)},-\frac{2d-3}{2(d-2)};\frac{1}{\varrho^{2(d-2)}}\right)\\ &+\frac{1}{d-2}\sqrt{1-\frac{1}{\varrho^{2(d-2)}}}\,\varrho\,\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2(d-2)}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2d-3}{2(d-2)}\right)}\\ &\times {}_2F_1\left(1,\frac{d-3}{2(d-2)},\frac{2d-5}{2(d-2)};\frac{1}{\varrho^{2(d-2)}}\right)\\ =& \,C_d + \varrho\,{}_2F_1\left(\frac{1}{2},-\frac{1}{2(d-2)},\frac{2d-5}{2(d-2)};\frac{1}{\varrho^{2(d-2)}}\right)\,,\end{aligned}$$

with C_d defined as in Eq. (3.22). In the last equality of the previous equation, to treat the first hypergeometric term, I have used the identity ${}_2F_1(a,b,b;z) = (1-z)^{-a}$ (see Ref. [48], Eq. 15.1.8)) and the fact that

$$\Gamma\left(\frac{d-3}{2(d-2)}\right) = \Gamma\left(-\frac{d-1}{2(d-2)}+1\right) = -\frac{d-1}{2(d-2)}\Gamma\left(-\frac{d-1}{2(d-2)}\right) ;$$

while, to deal with the second hypergeometric term, I have performed a linear transformation (see Ref. [48], Eq. 15.3.3)) together with the previous mentioned symmetric property of the hypergeometric function and the fact that

$$\Gamma\left(\frac{2d-3}{2(d-2)}\right) = \Gamma\left(\frac{1}{2(d-2)}+1\right) = \frac{1}{2(d-2)}\Gamma\left(\frac{1}{2(d-2)}\right) .$$

In addition, the definition of F as an integral over r of a positive quantity given in Eq. (3.26) makes evident that F is strictly increasing on $[1,+\infty)$, with $F(1) = 0$ and $\lim_{\varrho\to+\infty} F(\varrho) = +\infty$; thus F is one to one between $[1,+\infty)$ and $[0,+\infty)$, so that, for every $\mathtt{x} \in \mathbb{R}$, one has that

$$|\mathtt{x}| = F(\rho(\mathtt{x})) \quad \text{if and only if} \quad \rho(\mathtt{x}) = F^{-1}(|\mathtt{x}|) .$$

This concludes the proof of the first statement, recalling that $r(x) = b\rho\left(\frac{x}{b}\right)$. In order to analyze the large $|\mathtt{x}|$ behaviour of ρ (and, consequently, the large $|x|$ behaviour of r), I consider the hypergeometric representation for F given in Eq. (3.21). Noting that, ${}_2F_1(a,b,c;\epsilon) = 1+O(\epsilon)$ for $\epsilon \to 0$, we readily infer from this representation that $F(\varrho) = C_d + \varrho(1+O(1/\varrho^{2d-4}))$ for $\varrho \to +\infty$, i.e.,

$$F(\varrho) = \varrho + C_d + O\left(\frac{1}{\varrho^{2d-5}}\right) \quad \text{for } \varrho \to +\infty ; \tag{3.28}$$

if we now set $\varrho = \rho(\mathtt{x})$, recalling Eq. (3.18) and that $|\mathtt{x}| = F(\rho(\mathtt{x}))$, we infer the following:

$$\rho(\mathtt{x}) = |\mathtt{x}| - C_d + O\left(\frac{1}{|\mathtt{x}|^{2d-5}}\right) \quad \text{for } \mathtt{x} \to \pm\infty , \tag{3.29}$$

whence Eq. (3.23) follows, recalling (again) that $r(x) = b\rho\left(\frac{x}{b}\right)$.

□

Remark 23 Note that, if $\rho(\mathtt{x})$ is a solution to the Cauchy problem (3.24), then it is not difficult to see that

$$\rho'' = \frac{d-2}{\rho^{2d-3}} ; \tag{3.30}$$

this and Eq. (3.29) imply the following asymptotic expansions for the first and the second derivatives

$$\rho'(\mathtt{x}) = \pm\left(1 - \frac{1}{2|\mathtt{x}|^{2d-4}}\right) + O\left(\frac{1}{|\mathtt{x}|^{2d-3}}\right) \qquad \text{for } \mathtt{x} \to \pm\infty,$$
$$\rho''(\mathtt{x}) = \frac{d-2}{|\mathtt{x}|^{2d-3}} + O\left(\frac{1}{|\mathtt{x}|^{2d-2}}\right) \qquad \text{for } \mathtt{x} \to \pm\infty.$$

Therefore, we have proved that Eq. (3.17) implicitly defines a $(d+1)$-dimensional solution to the Einstein-scalar equations, which has the explicit form

$$\begin{aligned} &\alpha = \gamma^{-1} = 1\,, \qquad r = b\rho\left(\frac{x}{b}\right), \qquad \Phi = \sqrt{\frac{(d-1)(d-2)}{\kappa}}\,\phi\left(\frac{x}{b}\right), \\ &\rho(\mathtt{x}) = F^{-1}(\mathtt{x}) \quad (F \text{ as in Eq. (3.21)})\,, \qquad \phi(\mathtt{x}) = \int_{\mathtt{x}_0}^{\mathtt{x}} \frac{1}{\rho(\tilde{\mathtt{x}})^{d-1}}d\tilde{\mathtt{x}}\,, \end{aligned} \tag{3.31}$$

where $\mathtt{x}_0$ is an immaterial real constant.

Remark 24 Note that, due to the asymptotic expansion (3.23), the metric **g** defined by Eq. (3.31) is asymptotic for $x \to \pm\infty$ to the flat line element $-dt^2 + dx^2 + x^2 a_{ij}dx^i dx^j$ (see Remark 13); moreover, since r possesses a positive minimum at $x = 0$ (see Eq. (3.18)), the metric (3.31) describes a wormhole made up of two asymptotically flat universes connected by a throat of size $r(0) = b$. Let us remark that the fact that the wormhole metric approaches to the flat metric in the large x limit requires that, asymptotically, the role of the scalar field becomes irrelevant; indeed, from the integral expression for Φ given in Eq. (3.31), it turns out that

$$\Phi = \sqrt{\frac{(d-1)(d-2)}{\kappa}}\int_{\mathtt{x}_0}^{\frac{x}{b}} \frac{d\tilde{\mathtt{x}}}{\rho(\tilde{\mathtt{x}})^{d-1}} \;\to\; \pm\sqrt{\frac{(d-1)(d-2)}{\kappa}}\int_{\pm\mathtt{x}_0}^{+\infty} \frac{d\tilde{\mathtt{x}}}{\rho(\tilde{\mathtt{x}})^{d-1}} \quad \text{for } \mathtt{x} \to \pm\infty \tag{3.32}$$

and the above integral converges, since $1/\rho(\tilde{\mathtt{x}})^{d-1} \sim 1/\tilde{\mathtt{x}}^{d-1}$ for $\tilde{\mathtt{x}} \to +\infty$ and $d-1 \geq 2$. This implies that Φ is constant in the large x limit and then it becomes immaterial (this is a consequence of the fact that the scalar field appears in the static field equations only through its x and xx derivatives).

Remark 25 Actually, the wormhole solution given in Eq. (3.31) was obtained by Torii and Shinkai [13] and therefore we will refer to it as "Torii-Shinkai wormhole" or "Torii-Shinkai solution".

For future convenience, we introduce the rescaled coordinates

$$\mathtt{t} := \frac{t}{b}\,, \qquad \mathtt{x} := \frac{x}{b}\,; \tag{3.33}$$

in these coordinates the Torii-Shinkai wormhole (3.31) reads

$$\alpha = \gamma = 1\,, \qquad r = \rho(\mathtt{x})\,, \qquad \Phi = \sqrt{\frac{(d-1)(d-2)}{\kappa}}\,\phi(\mathtt{x})\,,$$
$$\text{with } \rho \text{ and } \phi \text{ as in Eq. (3.31).} \tag{3.34}$$

(The metric (3.31) has been obtained from the metric (2.24,3.31) applying the coordinate transformation (3.33) and dividing the resulting metric by b^2.)

Remark 26 In the four-dimensional case $d = 3$, one can introduce for M_4 the coordinates $(t, x, \vartheta, \varphi)$ defined in Remark 10; it turns out that the equation for r in (3.24) can be solved by means of transcendental functions as well as the integral appearing in the expression for Φ, leading to

$$\alpha = \gamma^{-1} = 1\,, \qquad r = \sqrt{\mathtt{x}^2 + 1^2}\,, \qquad \Phi = \sqrt{\frac{2}{\kappa}}\arctan \mathtt{x}\,. \tag{3.35}$$

Performing the change of variables $(\mathtt{t}, \mathtt{x}) \mapsto (t, x)$ defined in Eq. (3.33), one gets the solution

$$\alpha = \gamma^{-1} = 1\,, \qquad r = \sqrt{x^2 + b^2}\,, \qquad \Phi = \sqrt{\frac{2}{\kappa}}\arctan \frac{x}{b}\,, \tag{3.36}$$

which is exactly the EBMT solution (2.67,2.68) introduced in Subsection 2.4.3. Therefore, the Torii-Shinkai wormhole can be interpreted as a full-fledged $(d+1)$-dimensional generalization of the EBMT wormhole.

Remark 27 We conclude this subsection observing that, in the present case, Eq. (2.39) for the scalar curvature and the asymptotics (3.23) for r give

$$\begin{aligned} R =& -(d-1)(d-2)\frac{1}{\rho^{2(d-1)}(\mathtt{x})} \\ =& -(d-1)(d-2)\frac{1}{|\mathtt{x}|^{2(d-1)}} + O\left(\frac{1}{|\mathtt{x}|^{2d-1}}\right) \quad \text{for } \mathtt{x} \to \pm\infty. \end{aligned} \tag{3.37}$$

3.3 Ellis-Bronnikov wormhole

Let us now focus on the zero potential, static four-dimensional case, that is, we stipulate $V(\Phi) \equiv 0$, Eq. (3.1) and $d = 3$; in this case, it is possible introduce a useful recombination of the equations $\mathfrak{E}_1 = 0$, $\mathfrak{E}_2 = 0$, $\mathfrak{E}_3 = 0$, $\mathcal{H} = 0$ such that the field system (2.83-2.87) can be rewritten as [14] Eq. (3.13) plus

$$[\alpha^2 r^2]'' = 2\,, \qquad [\alpha^2 r r']' = 1\,, \qquad [\alpha^2 r^{d-1}\Phi']' = 0\,. \tag{3.38}$$

The equations (3.38) are obviously satisfied by the functions [15]

$$\alpha = \gamma^{-1} = e^{\gamma_1 \arctan\frac{x}{b}}\,, \qquad r = \sqrt{x^2+b^2}\,\gamma\,, \qquad \Phi = \Phi_1 \arctan\frac{x}{b}\,, \tag{3.39}$$

where, $b > 0$, γ_1 and Φ_1 are integration constants; the remaining equation (3.13) enforces the relation

$$\kappa\Phi_1^2 = 2(1+\gamma_1^2)\,. \tag{3.40}$$

Remark 28 In the literature, the solution (3.39,3.40), firstly found by Bronnikov [14], is often referred to as Ellis-Bronnikov wormhole and describes an asymptotically flat traversable wormhole with a throat of size $\tilde{b} := b\sqrt{1+\gamma_1^2}\, e^{-\gamma_1 \arctan \gamma_1}$ located at $x = \gamma_1 b$. Indeed, it is not difficult to verify that the metric $\mathbf{g}$ defined by Eq. (3.39) is asymptotic for $x \to \pm\infty$ to the flat line element $-dt^2 + dx^2 + x^2 d\Omega^2$ and that $r(\gamma_1 b) = \tilde{b}$ with $r'(\gamma_1 b) = 0$. Let us also mention the recent paper of Yazadjiev [49] for an important uniqueness result of this family (every spherically symmetric (traversable) asymptotically flat wormhole belongs to the family of the Ellis-Bronnikov wormhole).

Remark 29 Note that in the case $\gamma_1 = 0$, the Ellis-Bronnikov metric (3.39) is reflection symmetric with respect to x and coincides with the EBMT metric (3.36).

[14] More explicitly, the three field equations in Eq. (3.38) have been obtained in the following way: the first equation arises form the recombination $2r^2(\mathfrak{E}_1 + \mathcal{H}) + 6\mathfrak{E}_3 = 0$; the second and the third equations are actually equivalent to Eq. (2.84) and Eq. (2.85), respectively; as already stated in Footnote (13), Eq. (3.13) is equal to $\frac{2}{d-1}\frac{r^2}{\alpha^2}\mathcal{H} = 0$. The equivalence of the fields equations (2.83-2.87) and the system (3.38,3.13) is proved easily in the same way as in the Footnote (13).

[15] Actually, the most general solution of Eq. (3.38) is $\alpha = \gamma^{-1} = e^{\gamma_1 \arctan\frac{x}{b} + \gamma_0}$ $(\gamma_0, \gamma_1 \in \mathbb{R})$ and r, Φ as in Eq. (3.39). Obviously, this is exactly the solution Eq. (3.39) up to the constant multiplier $e^{2\gamma_0}$; hence, without loss of generality, we can set $\gamma_0 = 0$.

Remark 30 At this point, one could wonder if it is possible to find a $(d+1)$- dimensional extension of the Ellis-Bronnikov wormhole which extends the Ellis-Bronnikov metric (3.39,3.40) in the same way as the Torii-Shinkai wormhole (3.34) extends the EBMT metric (3.36). The most obvious and naive ansatz that one could consider for this possible extension is

$$r = r_{\rm TS}\gamma \,, \qquad \Phi = \Phi_2 \Phi_{\rm TS} \,, \tag{3.41}$$

where $r_{\rm TS}$ and $\Phi_{\rm TS}$ are, respectively, the radius and the scalar field of the Torii-Shinkai wormhole [Eq. (3.31)], while γ and Φ_2 are, respectively, the dx^2 coefficient of the metric and a constant to be determined. Unfortunately, inserting the ansatz (3.41) into the third field equation (3.12), it results that the latter is satisfied if and only if one of the following two possibilities occur: $\gamma = \alpha^{-1} = const$ or, alternatively, $d = 3$. This means that the only solutions to the Einstein-scalar equations of the form (3.41) are the Ellis-Bronnikov wormhole ($d = 3$) and the Torii-Shinkai wormhole ($d \geq 3, \gamma = \alpha^{-1} = 1$).

3.3.1 Embedding diagram and geodesics of the EBMT and the Ellis-Bronnikov wormholes

Embedding diagram of the EBMT and the Ellis-Bronnikov wormholes

In this subsection we consider all the notations and the results of Section 2.3; therein, we have mentioned that in Hartle's book [42] there is an example of the embedding of a wormhole in $\mathbb{R}^3$. This is actually the EBMT wormhole (3.35) which is recovered, for example, from the Ellis-Bronnikov wormhole (3.39) by setting $\gamma_1 = 0$ (see Remark 29). The construction made by Hartle leads to an embedding diagram for this wormhole, described (in our notation) by the profile and the embedding functions

$$F_{\rm EBMT}(z;b) := b\cosh\frac{z}{b}\,, \qquad z_{\rm EBMT}(x;b) := b\,\mathrm{arcsinh}\,\frac{x}{b}\,; \tag{3.42}$$

the profile function $F_{\rm EBMT}$ and the embedded slice

$$\mathcal{S}_{\rm EBMT}(b) := \{(z,\rho,\varphi)\,:\,\rho = F_{\rm EBMT}(z;b)\} \subset \mathbb{R}^3$$

are represented in Figure 3.1 for $b = 1$.

In the following, I will show that the functions (3.42) can be generalized to the case $\gamma_1 \neq 0$ in order to describe the embedding of the Ellis-Bronnikov wormhole. So, from now on, let us consider the metric (3.39) assuming that $\gamma_1 \neq 0$. Moreover, since for every fixed value of γ_1, the metrics corresponding

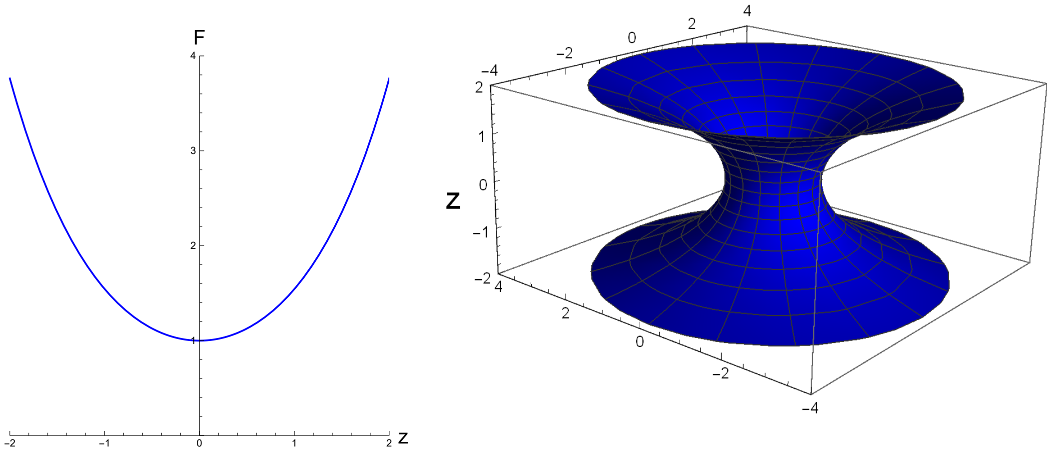

Figure 3.1: *The profile function F_{EBMT} and the embedding diagram $\mathcal{S}_{\mathrm{EBMT}}$ of the EBMT wormhole with $b = 1$.*

to the values $\pm\gamma_1$ are actually the same metric [16], from now on we will focus on the case

$$\gamma_1 > 0\,. \tag{3.43}$$

From the inequality (2.57), it is not difficult to see that the $\mathbf{t}_{\frac{\pi}{2}}$ slices of this metric can be embedded in a three-dimensional flat space only in the interval

$$x \geq \frac{b(\gamma_1^2 - 1)}{2\gamma_1} \qquad (\gamma_1 > 0)\,. \tag{3.44}$$

Let us recall that the minimum of the radial coefficient r of the Ellis-Bronnikov wormhole is located at $x = b\gamma_1$; since $b\gamma_1 > \frac{b(\gamma_1^2-1)}{2\gamma_1}$ for any $b, \gamma_1 > 0$, the interval in Eq. (3.44) does contain the wormhole throat $x = b\gamma_1$.
At this point, one might want to look for a different ambient space $(M_A, \mathbf{g}_A)$ [Eq. (2.44)] in which the $\mathbf{t}_{\frac{\pi}{2}}$ slices of the Ellis-Bronnikov wormhole can be integrally embedded. However, this can be very hard to do; moreover, the interval (3.44) is unbounded and contains even one of the asymptotically flat universes linked by the wormhole. For these reasons, we settle for the local embedding in the euclidean space in the region (3.44): this is defined by the integral expression appearing in Eq. (2.55) with $A(\rho) = 1$, which reads

$$z(x) \equiv z_{\mathrm{EB}}(x; \gamma_1, b) = \sqrt{b} \int_0^x e^{-\gamma_1 \arctan\frac{\tilde{x}}{b}} \sqrt{\frac{2\gamma_1 \tilde{x} + b - b\gamma_1^2}{\tilde{x}^2 + b^2}}\, d\tilde{x}\,. \tag{3.45}$$

[16] Indeed, to see this, it is sufficient to perform the change of coordinates $x \to -x$.

For every fixed values of the constants b, γ_1, the embedding function $z_{\rm EB}$ is integrated numerically and then inverted, allowing to obtain the profile function $F(z) \equiv F_{\rm EB}(z;\gamma_1,b)$ according to Eq. (2.54), that is

$$F(z) \equiv F_{\rm EB}(z;k,b) = \sqrt{x(z)^2+b^2}\, e^{-\gamma_1 \arctan \frac{x(z)}{b}}\,. \tag{3.46}$$

An elementary computation shows that, for all $b > 0$ and $\gamma_1 > 0$, the functions z and F satisfy the conditions (i-iv) of Proposition 2. Note that Eqs. (3.45,3.46) actually generalize Eq. (3.42) to the case $\gamma_1 > 0$ as it is not difficult to see that $z_{\rm EB}(x;0,b) = z_{\rm EBMT}(x;b)$ and $F_{\rm EB}(x;0,b) = F_{\rm EBMT}(x;b)$.
In Figure, 3.2, the embedded surface

$$\mathcal{S}_{\rm EB}(\gamma_1,b) := \{(z,\rho,\varphi)\,:\,\rho = F_{\rm EB}(z;\gamma_1,b)\} \subset \mathbb{R}^3$$

has been plotted in the region (see Eq. (3.44))

$$z \geq z^{-1}\left(\frac{b(\gamma_1^2-1)}{2\gamma_1}\right) \qquad (\gamma_1 > 0)$$

for the particular choice $b = \gamma_1 = 1$.

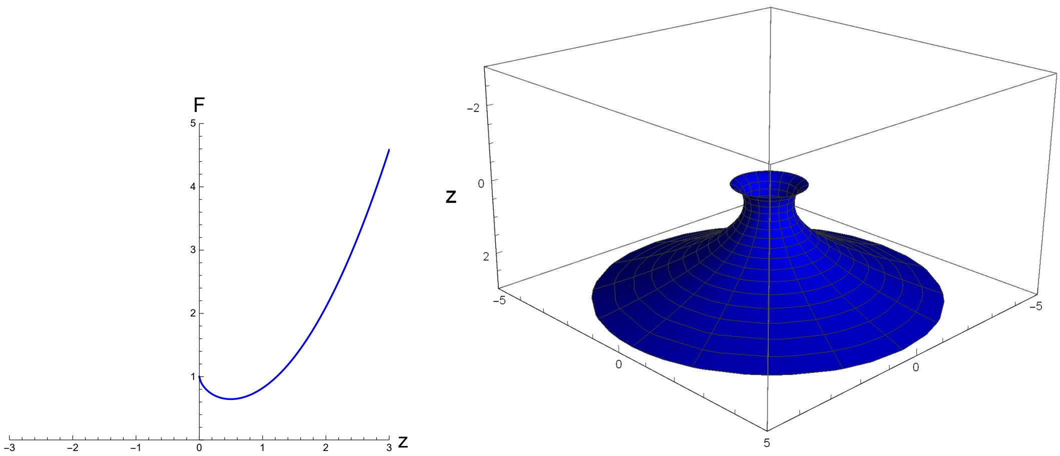

Figure 3.2: *The profile function $F_{\rm EB}$ and the embedding diagram $\mathcal{S}_{\rm EB}$ of the Ellis-Bronnikov wormhole with $b = 1$ and $\gamma_1 = 1$.*

Geodesics of the EBMT and the Ellis-Bronnikov wormholes

Let us now specialize the considerations on the study of the geodesic motion in a four-dimensional spherically symmetric static spacetime of Appendix A to the Ellis-Bronnikov wormhole; in this case we have that the effective potential (A.18) reads

$$V_{\text{eff}}(x) \equiv V_{\text{eff,EB}}(x; b, \gamma_1, \mathtt{k}, L) := \frac{L^2}{2} \frac{e^{4\gamma_1 \arctan \frac{x}{b}}}{x^2 + b^2} + \frac{\mathtt{k}}{2} \left(e^{2\gamma_1 \arctan \frac{x}{b}} - 1 \right) . \quad (3.47)$$

Note that the effective potential for the EBMT wormhole is obtained by setting $\gamma_1 = 0$ in the previous expression.
In the following, a complete analysis of the analytical properties of V_{eff} will be performed by varying the values of the parameters $b, \gamma_1, \mathtt{k}, L$ in their respective ranges; [(17)] in this way we can deduce some information about the geodesic motion in the considered wormholes.

We start from the limit case of the EBMT wormhole corresponding the choice $\gamma_1 = 0$: depending on the value of the angular momentum L, we encounter only two qualitatively different situations: if $L = 0$, the potential V_{eff} is identically null, while, if $L \neq 0$, it possesses an asymptotically null "bell curve" shape with the maximum $L^2/(2b^2)$ located in $x = 0$. This means that in the EBMT wormhole:
(i) there is no difference between timelike and null geodesics;
(ii) the are not stable orbits;
(iii) if $E > L^2/(2b^2)$ both particles and light rays heading towards the center of the wormhole will pass from one universe to the other and never come back (unless they accelerate or are deviated).
Figure 3.3 contains the plots of three possible effective potentials V_{eff} and values of the total energy E for the geodesic motion in the EBMT wormhole; for each possibility the motion of a particular null geodesic $\mathcal{P}(\tau)$, $\tau \in [0, \tau_1]$ has been computed numerically and plotted in the embedding diagram (3.1).

We now focus on the Ellis-Bronnikov case, so suppose $\gamma_1 > 0$. We consider firstly the motion of a light ray ($\mathtt{k} = 0$). In this case we have a situation similar to that of the EBMT wormhole: if $L = 0$, the potential is again identically zero, while if $L \neq 0$, the potential V_{eff} has once more a "bell curve" shape with a vanishing horizontal asymptote. However, in this case the potential is not reflection symmetric in the coordinate x, as the maximum

[17] We recall that the EBMT wormhole corresponds to the choice $\gamma_1 = 0$, while the Ellis-Bronnikov wormhole corresponds to the choice $\gamma_1 > 0$, according to the assumption (3.43).

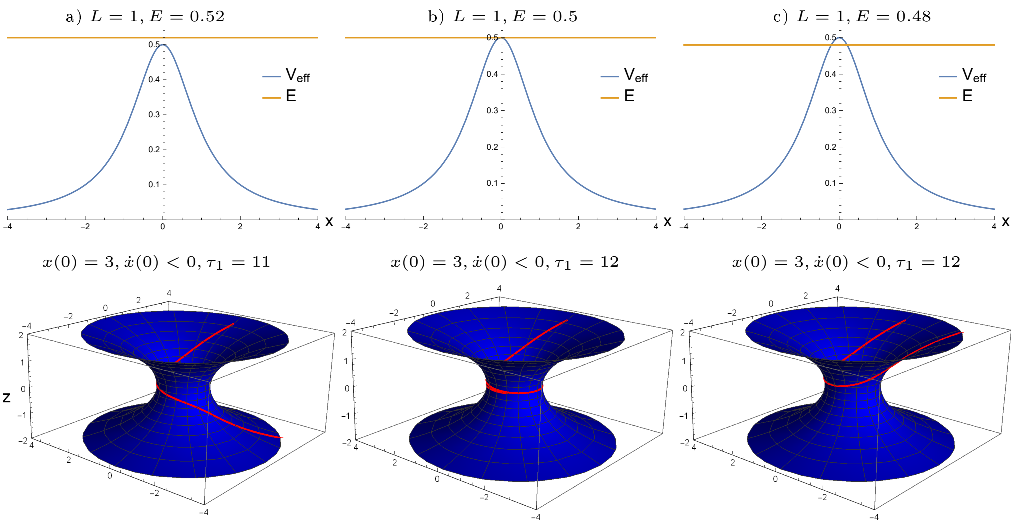

Figure 3.3: *Effective potential and embedding diagram of some geodesics in the EBMT wormhole with $b = 1$.*

of value $L^2 e^{4\gamma_1 \arctan(2\gamma_1)}/(2b^2(1 + 4\gamma_1^2))$ is located at $x = 2b\gamma_1$; note that the point of maximum is greater than the point of minimum $x = b\gamma_1$ of the radial coefficient. In Figure 3.3, the effective potential V_{eff} and the total energy E have been plotted in three different possible occurrences; in each case the motion of a particular (null or timelike) geodesic $\mathcal{P}(\tau)$, $\tau \in [0, \tau_1]$ has been computed numerically and plotted in the embedding diagram of the Ellis-Bronnikov wormhole.

Secondly, we consider the timelike geodesic motion ($\mathtt{k} = 1$). For every values of the angular momentum L, the effective potential is asymptotic to the values $(e^{\pm\pi\gamma_1} - 1)/2$, respectively for $x \to \pm\infty$. Let us now look for any local extrema of the potential; since from the expression

$$V'_{\text{eff}}(x) = \frac{e^{2\gamma_1 \arctan \frac{x}{b}}}{(x^2 + b^2)^2} \left[L^2 e^{2\gamma_1 \arctan \frac{x}{b}} (2b\gamma_1 - x) + b\gamma_1 (x^2 + b^2) \right],$$

we have that $V'_{\text{eff}}(2b\gamma_1) = 0$ only if $\gamma_1 = 0$, then all the possible local extrema of $V_{\text{eff}}(x)$ are contained in the solution of the equation

$$f_1(x) = f_2(x), \qquad f_1(x) := e^{2\gamma_1 \arctan \frac{x}{b}}, \quad f_2(x) := \frac{b\gamma_1 (x^2 + b^2)}{L^2 (x - 2b\gamma_1)}. \tag{3.48}$$

The function f_1 is bounded between $e^{-\pi\gamma_1}$ and $e^{\pi\gamma_1}$, while the function f_2 is unbounded from above for $x \to -\infty$ and $x \to 2b\gamma_1^-$, and is unbounded from below for $x \to 2b\gamma_1^+$ and $x \to +\infty$; moreover, f_2 has one point of minimum

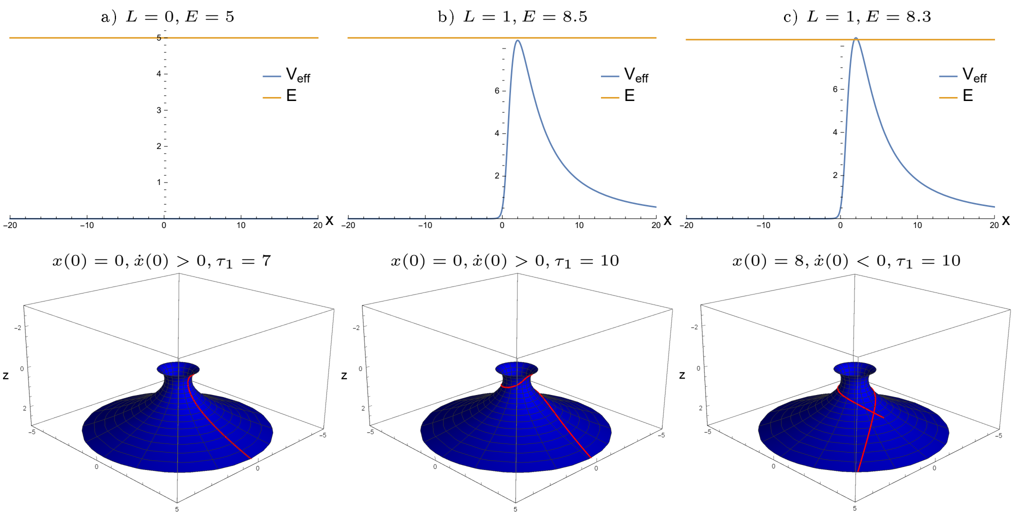

Figure 3.4: *Effective potential and embedding diagram of null geodesics in the Ellis-Bronnikov wormhole with $b = 1$ and $\gamma_1 = 1$.*

and the point of maximum, respectively at $x_m := 2b\gamma_1 + b\sqrt{1+4\gamma_1^2}$ and $x_M := 2b\gamma_1 - b\sqrt{1+4\gamma_1^2}$. Since for $x < 2b\gamma_1$, $f_1(x) > 0$ while $f_2(x) < 0$, the eventual solutions of Eq. (3.48) are contained in the interval $x > 2b\gamma_1$; hence, the graphics of f_1 and f_2 certainly intersect twice if $f_1(x_m) \geq f_2(x_m)$, while they certainly do not if intersect $f_2(x_m) \geq e^{\pi\gamma_1}$. Given the values of b and γ_1, the equation $f_2(x_m) \geq e^{\pi\gamma_1}$ holds if and only if

$$|L| \leq L_1 := \sqrt{2\gamma_1}\, b\, e^{-\gamma_1 \frac{\pi}{2}} \sqrt{2\gamma_1 + \sqrt{1+4\gamma_1^2}}\,, \tag{3.49}$$

while the equation $f_1(x_m) \geq f_2(x_m)$ is satisfied if and only if

$$|L| \geq L_2 := \sqrt{2\gamma_1}\, b\, e^{-\gamma_1 \arctan\left(2\gamma_1 + \sqrt{1+4\gamma_1^2}\right)} \sqrt{2\gamma_1 + \sqrt{1+4\gamma_1^2}}\,. \tag{3.50}$$

(Note that, obviously, $L_1 < L_2$.) Summing up, given the values of b and γ_1, we have proved that:
(i) if $|L| \leq L_1$ with L_1 as in Eq. (3.49), then the effective potential has no local extrema;
(ii) if $|L| \geq L_2$ with L_2 as in Eq. (3.49) then the effective potential has exactly one local maximum and one local minimum;
(iii) there exists a value $L_0 \in (L_1, L_2)$ such that: if $|L| \leq L_0$ the effective potential has no local extrema (and for $|L| = L_0$ has exactly one inflection) and if $|L| > L_0$ the effective potential has exactly one local maximum $V_{\text{eff}}^{\max}$

and one local minimum $V_{\rm eff}^{\rm min}$.
As a consequence, in the Ellis-Bronnikov wormhole the timelike geodesics can be bounded only in the case in which $L > L_0$ and the energy is $E \in (V_{\rm eff}^{\rm min}, V_{\rm eff}^{\rm max})$. In Figure 3.5, I set $b = \gamma_1 = 1$ and for three different values of (E, L) such that $L > L_2 \simeq 0.88$, I plotted the effective potential $V_{\rm eff}$ (with two local extrema) and the total energy E; in each case the motion of a particular timelike geodesic $\mathcal{P}(\tau)$, $\tau \in [0, \tau_1]$ has been computed numerically and plotted. Note that in the first and in the second cases the orbits are not bounded since, respectively, $E > V_{\rm eff}^{\rm max}$ and $\lim\limits_{x \to +\infty} V_{\rm eff}(x) < E < V_{\rm eff}^{\rm max}$.

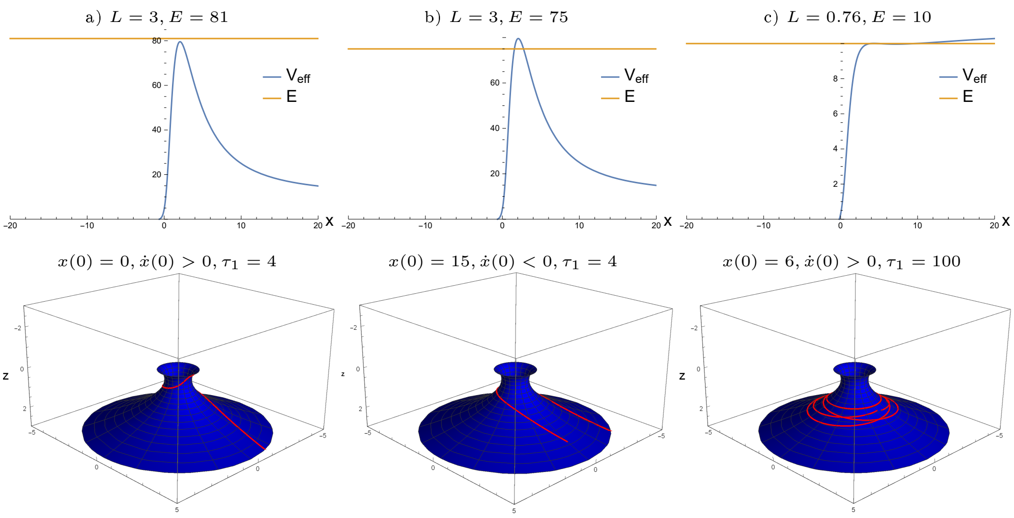

Figure 3.5: *Effective potential and embedding diagram of timelike geodesics in the Ellis-Bronnikov wormhole with $b = 1$ and $\gamma_1 = 1$.*

3.4 A four-dimensional wormhole connecting two AdS universes

3.4.1 The general Bronnikov-Fabris solution

We consider the static four-dimensional case, that is, we stipulate Eq. (3.1) and $d = 3$; moreover, we set the gauge such that $\alpha\gamma = 1$ (see Remark 7). In this section we look for solutions with self-interacting scalar field, therefore we do not assume the potential $V(\Phi)$ to be zero.
Let us show that a family of such solutions can be obtained by putting

$$r = \sqrt{x^2 + b^2}\,, \tag{3.51}$$

where $b > 0$. Indeed, with this choice it is easy to show that the combination of Eq, (2.84) and Eq. (2.86) defined as $\mathfrak{E}_2 + r^2\mathcal{H} = 0$, is satisfied if and only if

$$\Phi = \sqrt{\frac{2}{\kappa}} \arctan\frac{x}{b} + \Phi_0 \tag{3.52}$$

with Φ_0 a constant. With this expression for the scalar field, Eq. (2.83) leads to

$$\alpha = \gamma^{-1} = \sqrt{1 - K\left(b^2 + x^2\right) + M\left(b^2 + x^2\right)\arctan\frac{x}{b} + bMx}\,, \tag{3.53}$$

where K and M are two integration constants. The remaining two equations Eq. (2.84) (or, alternatively, Eq. (2.86)) [18] and Eq. (2.85) can be solved putting

$$V(\Phi(x)) = \frac{K(b^2 + 3x^2) - M\left(b^2 + 3x^2\right)\arctan\frac{x}{b} - 3bMx}{\kappa\left(b^2 + x^2\right)}\,.$$

Choosing, without loss of generality, $\Phi_0 = 0$ and by inverting Eq. (3.52), we obtain for $V(\Phi)$

$$V(\Phi) = \frac{K}{\kappa}\left[3 - 2\cos^2\left(\sqrt{\frac{\kappa}{2}}\Phi\right)\right] - \frac{M}{\kappa}\left\{3\sin\left(\sqrt{\frac{\kappa}{2}}\Phi\right)\cos\left(\sqrt{\frac{\kappa}{2}}\Phi\right) + \sqrt{\frac{\kappa}{2}}\Phi\left[3 - 2\cos^2\left(\sqrt{\frac{\kappa}{2}}\Phi\right)\right]\right\}. \tag{3.54}$$

Remark 31 Actually, the solution (3.51,3.52,3.53,3.54) is exactly the general solution given by Bronnikov and Fabris in Ref. [36] and reconsidered in the recent survey [11] (with some reparametrization of the involved constants). This describes an heterogeneous family of wormholes with throats of size b, since the function r in Eq. (3.51) clearly satisfies the requirements throat.

3.4.2 The AdS wormhole

Let us consider now the Bronnikov-Fabris solution (3.51-3.54) with the choice

$$M = 0\,, \tag{3.55}$$

[18] Indeed, the system made up of the equations $\mathfrak{E}_1 = 0$, $\mathfrak{E}_2 = 0$, $\mathfrak{E}_3 = 0$, $\mathcal{H} = 0$ [Eqs. (2.83-2.86)] is clearly equivalent to the system $\mathfrak{E}_1 = 0$, $\mathfrak{E}_2 = 0$, $\mathfrak{E}_3 = 0$, $\mathfrak{E}_2 + r^2\mathcal{H} = 0$, but is also equivalent to the system $\mathfrak{E}_1 = 0$, $\mathfrak{E}_2 + r^2\mathcal{H} = 0$, $\mathfrak{E}_3 = 0$, $\mathcal{H} = 0$.

which corresponds to a reflection symmetric metric with respect to the throat; moreover, we set

$$K \equiv -k^2\,, \qquad (k>0)\,. \tag{3.56}$$

With the choices (3.55.3.56), the Bronnikov-Fabris solution simplifies to

$$\alpha = \gamma^{-1} = \sqrt{1+k^2(x^2+b^2)}\,, \qquad r = \sqrt{x^2+b^2}\,, \tag{3.57}$$

$$\Phi = \sqrt{\frac{2}{\kappa}}\arctan\frac{x}{b}\,, \qquad V(\Phi) = -\frac{k^2}{\kappa}\left[3-2\cos^2\left(\sqrt{\frac{\kappa}{2}}\Phi\right)\right]\,. \tag{3.58}$$

Remark 32 In the limit case $b \to 0$ the metric (3.57) becomes singular at $x=0$; if we replace its second equality with $r=x$ for $x>0$ and with $r=-x$ for $x<0$, the corresponding metrics describe two AdS universes with cosmological constant $\Lambda = -3k^2$ (see Example 3). Therefore, if we have $b>0$, the metric (3.57) is regular for every

$$x \in (-\infty, +\infty)\,; \tag{3.59}$$

since $r(x) \sim |x|$ and $\alpha = \gamma^{-1} \sim \sqrt{1+k^2x^2}$ for $x \to \pm\infty$, we can interpret the metric in (3.57) as describing a wormhole with a throat of size b connecting two separate asymptotically AdS universes with the same cosmological constant $\Lambda = -3k^2$. For this reason one could call the solution (3.57,3.58) an "AdS-AdS wormhole"; in the sequel this expression will be always shortened into "AdS wormhole" or "AdS solution".

Remark 33 Let us note that, for $k \to 0$, the potential $V(\Phi)$ in Eq. (3.58) vanishes and the AdS wormhole (3.57,3.58) (with b fixed) becomes the reflection symmetric Ellis-Bronnikov wormhole (EBMT wormhole) [Eq. (3.36)].

Remark 34 For further convenience, we introduce the new parameter

$$B := bk\,, \qquad B>0 \tag{3.60}$$

and the change of variables

$$\begin{aligned} t &= \frac{s}{2k\sqrt{1+B^2}}\,, \qquad s \in (-\infty,\infty) \\ x &= \frac{\sqrt{1+B^2}}{k}\tan\frac{u}{2}\,, \qquad u \in (-\pi,\pi) \end{aligned} \tag{3.61}$$

so that in the new coordinate system (s, u) the AdS wormhole (3.57,3.58) is transformed into a solution with a self-interacting scalar field and a metric $\mathbf{g}$ of the form (2.24) with (t, x) replaced by (s, u) such that

$$\begin{aligned} \alpha = \gamma = \frac{1}{2k\cos\frac{u}{2}}\,, \qquad \beta = 0\,, \qquad r = \frac{\sqrt{1+2B^2-\cos u}}{\sqrt{2}k\cos\frac{u}{2}}\,, \\ \Phi = \sqrt{\frac{2}{\kappa}}\arctan\left(\frac{\sqrt{1+B^2}\tan\frac{u}{2}}{B}\right)\,, \qquad V(\Phi) \text{ as in Eq. (3.58)}\,. \end{aligned} \tag{3.62}$$

Let us observe that the limits $x \to \pm\infty$, describing the far ends of the AdS wormhole, are equivalent to the limits $u \to \pm\pi$.
Finally, we note that in the coordinates (s, u) it is no more possible to recover the EBMT wormhole from the AdS wormhole, since in the limit $k \to 0$ the transformation (3.61) becomes singular.

3.4.3 Embedding diagram and geodesics of the AdS wormhole

Embedding diagram of the AdS wormhole

In this subsection we consider all the notations and the results of Section 2.3. Moreover, we consider the AdS metric in the form (3.57) with $k > 0$; note that, in the limit case $k = 0$, we have to recover the results of Subsection 3.3.1 for the EBMT wormhole (see Remark 33).
Not surprisingly, the $\mathbf{t}_{\frac{\pi}{2}}$ slices of the AdS wormhole can not be embedded in a three-dimensional flat space: this is due to the fact that in the large x limit, the $\mathbf{t}_{\frac{\pi}{2}}$ slices of the AdS wormhole approaches to two two-dimensional surfaces with constant negative curvature and such manifolds cannot be globally embedded in $\mathbb{R}^3$. Indeed, the inequality (2.57) has the solution

$$x \in \left[-\frac{\sqrt{B}}{\sqrt{2}k}\sqrt{\sqrt{4+B^2}-B}\,, \frac{\sqrt{B}}{\sqrt{2}k}\sqrt{\sqrt{4+B^2}-B}\right]\,,$$

which is a neighbourhood of the throat $x = 0$ in which the $\mathbf{t}_{\frac{\pi}{2}}$ slices of the AdS wormhole can be embedded in $\mathbb{R}^3$.
However, it is not difficult to see that the the two-dimensional asymptotic surfaces of the wormhole can be trivially embedded in a three-dimensional space with the same asymptotic negative constant curvature $-3k^2$: this ambient space is given by the metric (2.44) with

$$A(\rho) := \sqrt{1 + k^2\rho^2}\,. \tag{3.63}$$

Indeed, with this position, the inequality (2.57) is automatically satisfied and the second integral in Eq. (2.55) can be easily solved giving the following explicit expression for the function $z = z(x)$

$$z(x) \equiv z_{\mathrm{AdS}}(x;k,b) = \frac{b}{\sqrt{1+b^2k^2}} \operatorname{arcsinh}\left(\frac{x}{b\sqrt{1+k^2(x^2+b^2)}}\right) ; \qquad (3.64)$$

this can be inverted so that Eq. (2.54) reads

$$F(z) \equiv F_{\mathrm{AdS}}(z;k,b) = \frac{b\cosh\left(\frac{\sqrt{1+b^2k^2}z}{b}\right)}{\sqrt{1+b^2k^2\left(1-\cosh^2\left(\frac{\sqrt{1+b^2k^2}z}{b}\right)\right)}} . \qquad (3.65)$$

An elementary computation shows that, for all $b > 0$ and $k > 0$, the functions z and F satisfy the conditions (i-iv) of Proposition 2. Note that Eqs. (3.64,3.65) actually generalize Eq. (3.42) to the case $k > 0$ as trivially $z_{\mathrm{AdS}}(x;0,b) = z_{\mathrm{EBMT}}(x;b)$ and $F_{\mathrm{AdS}}(x;0,b) = F_{\mathrm{EBMT}}(x;b)$.

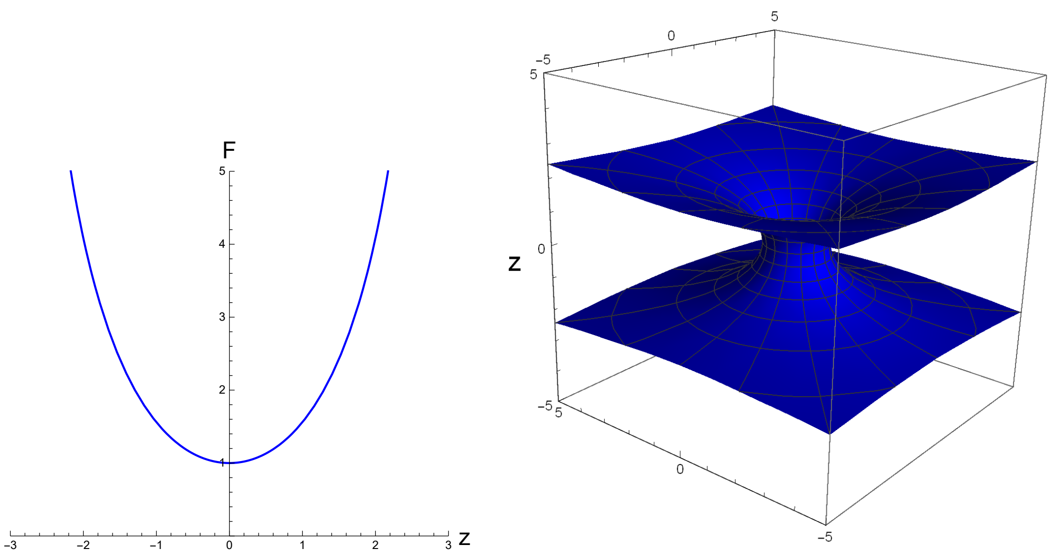

Figure 3.6: *The profile function* F_{AdS} *and the embedding diagram* $\mathcal{S}_{\mathrm{AdS}}$ *of the AdS wormhole with* $b = 1$ *and* $k = 0.1$.

At this point one might want to visualize the embedding of the AdS wormhole, namely a three-dimensional picture of the embedded slice $\iota(\mathbf{t}_{\frac{\pi}{2}})$. Obviously, this is not possible since the ambient space M_A is not flat, unless we settle for an approximation. Locally, for $k \to 0$ or $\rho \to 0$ the metric of the ambient space (2.44) tends to become flat; this means that the bidimensional surface in $\mathbb{R}^3$ defined as

$$\mathcal{S}_{\text{AdS}}(k,b) := \{(z,\rho,\varphi)\,:\,\rho = F_{\text{AdS}}(z;k,b)\} \subset \mathbb{R}^3$$

approaches to $\iota(\mathbf{t}_{\frac{\pi}{2}}) \subset M_A$ in a region "suitably close" to the origin; this region can be very large if k is very small. In Figure, 3.6 one can see the plots of the profile function F_{AdS} and the embedding diagram $\mathcal{S}_{\text{AdS}} \subset \mathbb{R}^3$ for a particular choice of the parameters b and k.

Geodesics of the AdS wormhole

Let us now specialize the considerations on the study of the geodesic motion in a four-dimensional spherically symmetric static spacetime of Appendix A to the AdS wormhole; in this case we have that the effective potential (A.18) reads

$$V_{\text{eff}}(x) \equiv V_{\text{eff,AdS}}(x;b,k,\mathtt{k},L) := \frac{L^2}{2}\left(\frac{1}{x^2+b^2}+k^2\right)+\frac{\mathtt{k}}{2}k^2(x^2+b^2)\,. \quad (3.66)$$

In the following, a complete analysis of the analytical properties of V_{eff} will be performed by varying the values of the parameters $b, k, \mathtt{k}, L$ in their respective ranges; in this way we can deduce some information about the geodesic motion in the AdS wormhole. We will focus on the pure AdS case, so suppose $k > 0$.
We consider firstly the motion of a light ray ($\mathtt{k} = 0$). In this case we have a situation similar to that of the EBMT wormhole (see Subsection 3.3.1): if $L = 0$, the potential is again identically zero, while if $L \neq 0$, the potential V_{eff} has once more a "bell curve" shape with the maximum of value $L^2(k^2+1/b^2)/2$ located in $x = 0$ and a non vanishing horizontal asymptote of value $L^2k^2/2$. In Figure 3.7, the effective potential V_{eff} and the total energy E have been plotted in three different possible occurrences; in each case the motion of a particular null geodesic $\mathcal{P}(\tau)$, $\tau \in [0,\tau_1]$ has been computed numerically and plotted in the embedding diagram of the AdS wormhole.
Secondly, we consider the timelike geodesic motion ($\mathtt{k} = 1$). If $|L| \leq b^2k$, the effective potential is a convex function with the minimum $V_0 := b^2k^2/2 + L^2(k^2+1/b^2)/2$ at $x = 0$; if $|L| > b^2k$, the effective potential has a "Mexican hat" shape, with the local maximum of value V_0 located in $x = 0$, the two local minima in $x = \pm\sqrt{|L|/k - b^2}$ both of value $k|L|(1+k|L|/2)$ and limits $V_{\text{eff}}(x) \to +\infty$ for $x \to \pm\infty$.

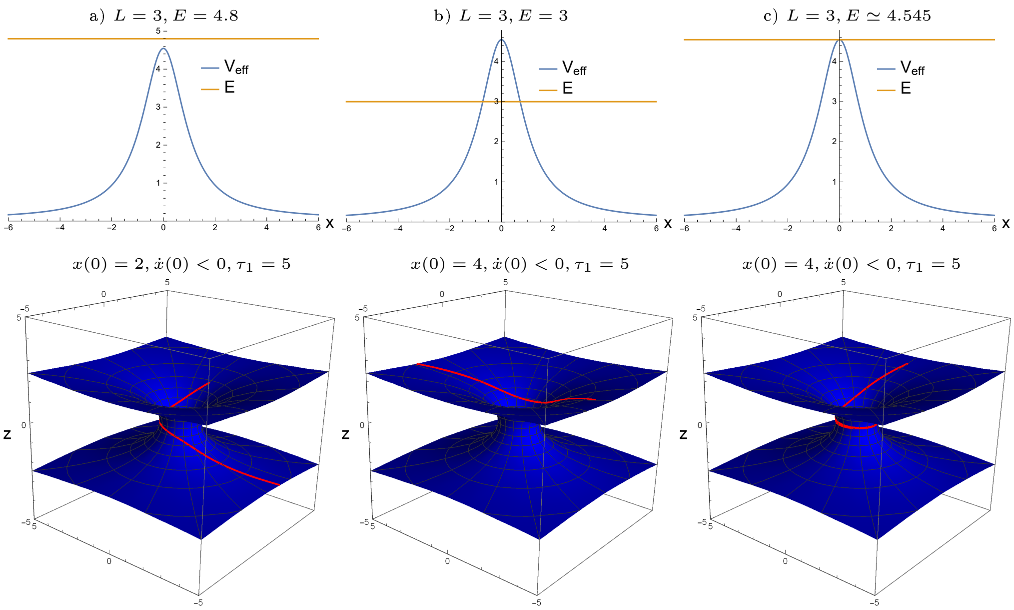

Figure 3.7: *Effective potential and embedding diagram of null geodesics in the AdS wormhole with* $b = 1$ *and* $k = 0.1$.

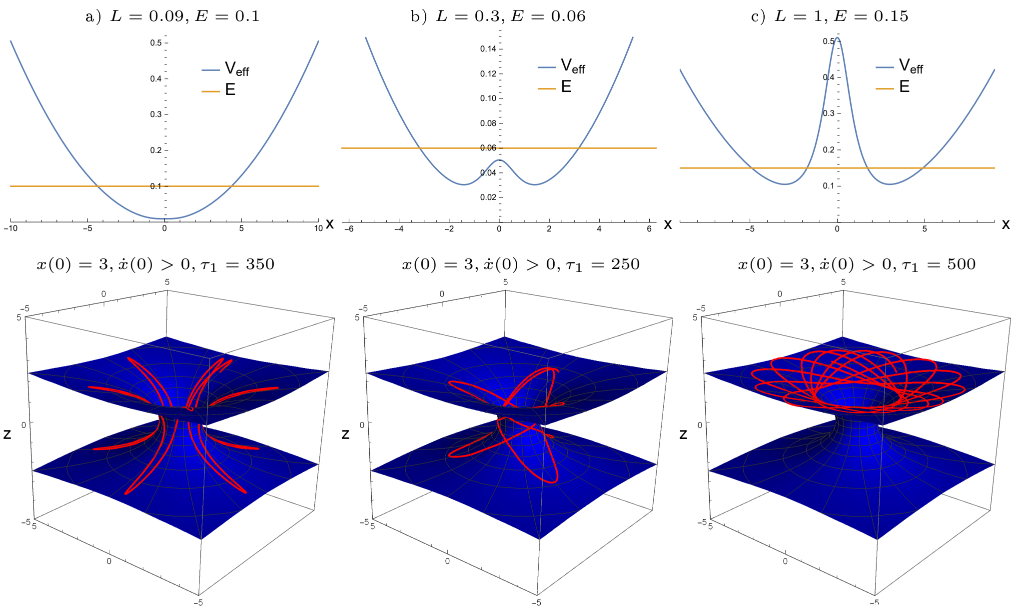

Figure 3.8: *Effective potential and embedding diagram of timelike geodesics in the AdS wormhole with* $b = 1$ *and* $k = 0.1$.

Therefore, in the AdS wormhole the timelike geodesics:
(i) orbit in a bounded region of the spacetime which depends on b, k, E, L and is defined by

$$\left\{ m \,:\, |x(m)| \leq \frac{1}{\sqrt{2}k}\sqrt{2E - k^2(L^2 + 2b^2) + \sqrt{4E^2 + k^2L^2(k^2L^2 - 4(E+1))}} \right\} ;$$

(ii) if $E > V_0$ the particles pass from one universe to the other and keep oscillating between them (unless they accelerate).
Figure 3.8 contains the plot of three possible effective potentials V_{eff} and values of the total energy E for the timelike geodesic motion; as usual, for each possibility the motion of a particular null geodesic $\mathcal{P}(\tau)$, $\tau \in [0, \tau_1]$ has been computed numerically and plotted in the embedding diagram of the AdS wormhole.

3.5 A dS wormhole with horizons

Let us return to the Bronnikov-Fabris wormhole solution (3.51-3.54), depending on the parameters M and K. Keeping the assumption $M = 0$ [Eq. (3.55)] we can as well consider, as an alternative to (3.56), the choice

$$K \equiv k^2 \,, \qquad (k > 0) \,. \tag{3.67}$$

In this way we obtain

$$\alpha = \gamma^{-1} = \sqrt{1 - k^2(x^2 + b^2)} \,, \qquad r = \sqrt{x^2 + b^2} \,, \tag{3.68}$$

$$\Phi = \sqrt{\frac{2}{\kappa}} \arctan \frac{x}{b} \,, \qquad V(\Phi) = \frac{k^2}{\kappa}\left[3 - 2\cos^2\left(\sqrt{\frac{\kappa}{2}}\Phi\right)\right] \,. \tag{3.69}$$

For $b \to 0$ the metric (3.68) becomes singular at $x = 0$, exactly as for the AdS wormhole (see Remark 32); if we replace (again) $\sqrt{x^2}$ with $r = x$ for $x > 0$ and by $r = -x$ for $x < 0$, the corresponding metrics represent the static part of two dS universes with cosmological constant $\Lambda = 3k^2$ (see Example 4). Therefore, by analogy with the terminology of section 3.4, we refer to the solution (3.68,3.69) as a "dS wormhole"; let us note that the expressions for α, γ and r in Eq. (3.68) and for Φ and $V(\Phi)$ in Eq. (3.69) can be obtained formally from the analogous expressions of the AdS case [Eqs. (3.57,3.58)] making the replacement $k \mapsto ik$. However, differently from the AdS case, the metric (3.68) makes sense only if $0 < kb < 1$ (otherwise, the coefficients α and γ are singular for every $x \in \mathbb{R}$); hence, from now on we intend

$$b \in \left(0, \frac{1}{k}\right) \,. \tag{3.70}$$

In the rest of this section we will study in detail the geometry of the spacetime (3.68) for b as in Eq. (3.70).

We set

$$B := b\,k \in (0,1)\,, \qquad \ell := \frac{\sqrt{1-B^2}}{k}\,; \tag{3.71}$$

the limitation of B, which is due to the assumption (3.70), makes the definition of ℓ consistent. The metric (3.68) in the coordinates $(t, x, \vartheta, \varphi)$ (see Remark 10) with the above constants reads

$$\begin{aligned}\mathbf{g} &= -\left[1-k^2(x^2+b^2)\right]dt^2 + \frac{dx^2}{1-k^2(x^2+b^2)} + \left(x^2+b^2\right)d\Omega^2 \\ &= -(1-B^2)\left(1-\frac{x^2}{\ell^2}\right)dt^2 + \frac{dx^2}{(1-B^2)\left(1-\frac{x^2}{\ell^2}\right)} + \left(x^2+b^2\right)d\Omega^2\,. \end{aligned} \tag{3.72}$$

Let us introduce the regions

$$\begin{gathered} I := \{(t,x)\,|\,t\in\mathbb{R}\,,\,x\in(-\ell,\ell)\}\,, \\ E^- := \{(t,x)\,|\,t\in\mathbb{R}\,,\,x\in(-\infty,-\ell)\}\,, \quad E^+ := \{(t,x)\,|\,t\in\mathbb{R}\,,\,x\in(\ell,+\infty)\}\,; \end{gathered} \tag{3.73}$$

then the expressions for α and γ in Eq.(3.68) are well defined in a literal sense over I; more substantially, the metric (3.72) is well defined over $I \times S^2$ and the vector fields ∂_t and ∂_x are, respectively, timelike and spacelike on this domain. However, Eq. (3.72) also gives a Lorentzian metric on each one of the regions E^- and E^+; here ∂_t is spacelike and ∂_x is timelike, so the metric is non static. In the sequel we often refer to I as the internal region and to $E^\pm$ as the exterior regions in (t, x) space. At $x = \pm\ell$ the metric seems to be ill defined but, as explained hereafter, these are just apparent singularities related to the coordinate system: the hypersurfaces $x = \pm\ell$ are indeed cosmological horizons and the metric is non singular across them. Let us note that the $b \to 0$ limit of the previous statement corresponds to well known features of the dS universes, having an horizon at $x = \ell = \frac{1}{k}$ (if $x > 0$). In the rest of the present section, I will follow Ref. [1], where the authors introduce an alternative parametrizations for the dS wormhole, yielding a Kruskal type extension of the metric (3.72) which is regular across $x = \pm\ell$. The extended universe constructed in this way can also be interpreted as a regular black hole with an expanding cosmology beyond the horizons, and is hence referred to as a "black universe" in Ref. [11].

3.5.1 Another coordinate system for the internal region I

In this subsection we consider an alternative coordinatization for the internal region I introducing the analogs of the AdS wormhole coordinates (s,u) (see Eq. (3.61)); let us put

$$t = \frac{\ell}{2\,(1-B^2)}\,s\,, \qquad x = \ell \tanh\frac{u}{2}\,, \qquad (s,u) \in \mathbb{R}^2\,; \tag{3.74}$$

the map $(s,u) \mapsto (t,x)$ is one to one between $\mathbb{R}^2$ and the inner region I with inverse

$$s = \frac{2(1-B^2)}{\ell}\,t\,, \qquad u = 2\,\mathrm{arctanh}\frac{x}{\ell} = \log\frac{\ell+x}{\ell-x}\,.$$

We can regard (s,u) as an alternative coordinate system for I; this does not eliminate the apparent singularities at $x = \pm\ell$ but sends them to infinity since the limits $x \to \pm\ell$ correspond to the limits $u \to \pm\infty$. In the new coordinates the metric (3.72) becomes

$$\mathbf{g} = \frac{1}{4k^2\cosh^2\frac{u}{2}}\left[-ds^2 + du^2 + 2\big(\cosh u - 1 + 2B^2\big)d\Omega^2\right], \tag{3.75}$$

with radial null geodesics given by the straight lines $s = \pm u + const.$

Remark 35 To conclude this subsection, let us remark that the transformation (3.74) and the expression (3.75) for the metric can be obtained from their AdS analogs [Eqs. (3.61,3.62)] making the formal replacements $k \mapsto ik, B \mapsto iB, s \mapsto is, u \mapsto iu$.

3.5.2 A first spacetime extension

We start our construction from the internal region I, that we describe in terms of the coordinates (s,u):

$$I := \{(s,u)\,|\,s,u \in \mathbb{R}\}\,, \tag{3.76}$$

Let us set

$$s = \log\left(-\frac{U}{V}\right), \quad u = -\log(-UV)\,, \quad (U,V) \in (0,+\infty)\times(-\infty,0)\,; \tag{3.77}$$

the transformation $(U,V) \mapsto (s,u)$ is one to one between the sets $(0,+\infty) \times (-\infty,0)$ and $\mathbb{R}^2$. By compositions with (3.74) we obtain the transformation

$$t = \frac{\ell}{2(1-B^2)}\log\left(-\frac{U}{V}\right), \quad x = \ell\,\frac{1+UV}{1-UV}\,, \quad (U,V) \in (0,+\infty)\times(-\infty,0)\,, \tag{3.78}$$

which is a diffeomorphism between $(0,+\infty)\times(-\infty,0)$ and the inner region I. The first cosmological horizon $x=-\ell$ corresponds to $U\to+\infty$ or $V\to-\infty$, while the second cosmological horizon $x=\ell$ coincides with $UV=0$. Now the metric (3.75) reads

$$\mathbf{g}=\frac{1}{k^2(1-UV)^2}\Big[-4dUdV+\big[B^2(1-UV)^2+(1-B^2)(1+UV)^2\big]d\Omega^2\Big]. \tag{3.79}$$

It is evident that this metric is regular on the cone $UV=0$ and can be extend beyond the corresponding horizon to the region $\mathcal{R}\times S^2$, where

$$\mathcal{R}:=\{(U,V)\in\mathbb{R}^2\mid UV<1\}, \tag{3.80}$$

is a region bounded by the two branches of the hyperbola $UV=1$, corresponding to the spacelike infinity $x=+\infty$. The two branches of the hyperbola $UV=-1$ correspond to the throat $x=0$. To go on, let us extend the transformation (3.78) setting

$$t=\frac{\ell}{2(1-B^2)}\log\left|\frac{U}{V}\right|, \quad x=\ell\,\frac{1+UV}{1-UV}, \quad (U,V)\in\mathcal{R}\,|\,V\neq 0. \tag{3.81}$$

The map $(U,V)\mapsto x$ is smooth throughout the region $\mathcal{R}$, while $(U,V)\mapsto t$ is well defined and smooth on the subregion $\{(U,V)\in\mathcal{R}\mid UV\neq 0\}$. The correspondence $(U,V)\mapsto(t,x)$ gives diffeomorphisms between the following pairs of regions:

$$\begin{aligned}(0,+\infty)\times(-\infty,0) &\simeq I,\\ (-\infty,0)\times(0,+\infty) &\simeq I,\\ \{(U,V)\in(0,+\infty)^2\,|\,UV<1\} &\simeq E^+,\\ \{(U,V)\in(-\infty,0)^2\,|\,UV<1\} &\simeq E^+,\end{aligned}$$

where E^+ is the exterior region defined in Eq. (3.73). Under each one of these four diffeomorphisms, the metric of Eq. (3.72) takes the form (3.79). To conclude we note that, writing Φ as in Eq. (3.69) and x as in Eq. (3.81) we obtain a smooth extension of the scalar field Φ to the whole region $\mathcal{R}$.

3.5.3 Extending spacetime further: the nonstatic Kruskal type extension beyond the horizons

We now consider a "compactification" of the extended region $\mathcal{R}$ [Eq. (3.80)] based on the reparametrization

$$U=\tan\mathcal{U}, \qquad V=\tan\mathcal{V}. \tag{3.82}$$

We know that the cone $UV = 0$ and the limits $U \to +\infty$, $V \to -\infty$ and $U \to -\infty$, $V \to +\infty$ correspond to the horizons $x = \pm\ell$ in (3.81); the cone and the indicated limits are associated, according to (3.82) to finite values of $\mathcal{U}$ and $\mathcal{V}$, so the effect of the above transformation is to bring both the horizons at finite distances. One could use $\mathcal{U}$ and $\mathcal{V}$ as an alternative set of coordinates and reexpress the metric (3.79) and so on; but the situation can be described in a simpler way making a further transformation (essentially, a rotation of $\frac{\pi}{4}$ and a translation of the axes)

$$\mathcal{U} = \frac{T}{2} - \frac{X}{2} + \frac{\pi}{4}, \qquad \mathcal{V} = \frac{T}{2} + \frac{X}{2} - \frac{\pi}{4}. \tag{3.83}$$

The composition of Eqs. (3.82,3.83), whenever they make sense, gives

$$U = \tan\left(\frac{T}{2} - \frac{X}{2} + \frac{\pi}{4}\right), \qquad V = \tan\left(\frac{T}{2} + \frac{X}{2} - \frac{\pi}{4}\right); \tag{3.84}$$

the application $(T, X) \mapsto (U, V)$ is a bijection between the following regions:

$$\mathcal{R} \simeq \mathfrak{R}$$

$$\mathfrak{R} \text{ as in Eq. (3.80) and}$$

$$\mathcal{R} := \left\{(T, X) \in \mathbb{R}^2 | -\frac{\pi}{2} < T < \frac{\pi}{2}, -\frac{\pi}{2} < X - T, X + T < \frac{3}{2}\pi\right\}.$$

In the coordinates $(T, X, \vartheta, \varphi)$ the metric (3.79) assumes the form

$$\mathbf{g} = \frac{1}{k^2 \cos^2 T}\Big[-dT^2 + dX^2 + \big[B^2 \cos^2 T + (1 - B^2)\sin^2 X\big]d\Omega^2\Big], \tag{3.85}$$

which clearly admits a further extension to the region $\mathcal{S} \times S^2$, where

$$\mathcal{S} := \left\{(T, X) \in \mathbb{R}^2 \mid -\frac{\pi}{2} < T < \frac{\pi}{2}\right\}. \tag{3.86}$$

Eqs. (3.85,3.86) provide the final form of our dS wormhole spacetime; the strip $\mathcal{S}$ is represented in Fig. 3.9, which also accounts for some facts illustrated hereafter. Note that the metric (3.85) is invariant under the spatial translation, the spatial reflection and the time reflection

$$\begin{gathered}\mathfrak{T} : (T, X) \mapsto (T, X + \pi), \qquad \mathfrak{S} : (T, X) \mapsto (T, \pi - X), \\ \mathfrak{R} : (T, X) \mapsto (-T, X).\end{gathered} \tag{3.87}$$

Let us also remark that, in the limit case $B \to 0$, the expression (3.85) reduces to the non-static extension of the dS metric (2.30). For any $B > 0$, the connection between the spacetime (3.85,3.86) and the original setting (3.72,3.73) is understood expressing the original variables (t, x) in terms of the new variables (T, X). To this purpose we note that the compositions of the transformations (3.81,3.84), whenever they make sense, gives

$$t = \frac{\ell}{2(1-B^2)} \log \left| \frac{\sin T + \cos X}{\sin T - \cos X} \right| , \qquad x = \ell \frac{\sin X}{\cos T} . \tag{3.88}$$

The map of $(T, X) \mapsto x$ is everywhere smooth on $\mathcal{S}$, while the map of $(T, X) \mapsto t$ has singularities at the points of $\mathcal{S}$ where the argument of the logarithm vanishes or diverges; this occurs at points where $\sin T = \mp \cos X$, which are just the points where $x = \mp\ell$. Moreover, we note that $(t, x) \circ \mathfrak{T} = (-t, -x)$, $(t, x) \circ \mathfrak{S} = (-t, x)$ and $(t, x) \circ \mathfrak{R} = (-t, x)$; the behaviour of $\mathbf{g}, t, x$ under $\mathfrak{T}$ implies the invariance of each one of these three objects under the translation $\mathfrak{T}^2 : (T, X) \mapsto (T, X + 2\pi)$.

To go on, let us now introduce the diamond $\mathcal{I}$ and the triangles $\mathcal{E}^\mp$ defined by

$$\begin{aligned} \mathcal{I} &:= \left\{ (T, X) \in \mathbb{R}^2 \,|\, -\frac{\pi}{2} < T - X, T + X < \frac{\pi}{2} \right\}, \\ \mathcal{E}^- &:= \left\{ (T, X) \in \mathbb{R}^2 \,|\, T < \frac{\pi}{2}, \, T - X > \frac{\pi}{2}, \, T + X > -\frac{\pi}{2} \right\}, \\ \mathcal{E}^+ &:= \left\{ (T, X) \in \mathbb{R}^2 \,|\, T < \frac{\pi}{2}, \, T - X > -\frac{\pi}{2}, \, T + X > \frac{\pi}{2} \right\} \end{aligned} \tag{3.89}$$

(see again Fig. 3.9); then the map $(T, X) \to (t, x)$, described by Eq. (3.88), gives isometric diffeomorphisms between the following pairs of regions

$$\mathcal{I} \quad \simeq \quad I \, , \qquad \mathcal{E}^\mp \quad \simeq \quad E^\mp \, ,$$

where I and $E^\mp$ are, respectively, the internal region and the two the exterior regions (3.73) with the metric (3.72). Moreover, we have that $x = \pm\ell$ along the sides of $\mathcal{I}$, $x = -\ell$ and $x = -\infty$ along the sides of $\mathcal{E}^-$ and $x = \ell$ and $x = +\infty$ along the sides of $\mathcal{E}^+$ (see once more Fig. 3.9). It easy to construct infinitely many replicas of the previous statement using the previous information of the behaviour of $\mathbf{g}, t, x$ under the transformations (3.87). For example, using the fact that $\mathbf{g}, t, x$ are invariant under all the iterates $\mathfrak{T}^{2h} : (T, X) \mapsto (T, X + 2h\pi)$ $(h \in \mathbb{Z})$, one readily shows that for each $h \in \mathbb{Z}$, the map (3.88) gives isometric diffeomorphisms between the following pairs of regions

$$\mathfrak{T}^{2h}(\mathcal{I}) \quad \simeq \quad I \, , \qquad \mathfrak{T}^{2h}(\mathcal{E}^\mp) \quad \simeq \quad E^\mp \, .$$

Moreover, applying the time reflection $\mathfrak{R}$ to each one of the translated triangles $\mathfrak{T}^{2h}(\mathcal{E}^{\mp})$ one gets other regions isometrically diffeomorphic to $E^{\mp}$. Finally, let us recall that we have already noted that the points (T, X) where Eq. (3.88) gives singularities for t are just the points at which the same equation gives $x = \pm\ell$; so from the viewpoint of the extended manifold $\mathcal{S} \times S^2$, the apparent singularities at $x = \pm\ell$ of the original metric (3.72) are just due to the singularities of t as a coordinate on $\mathcal{S}$.

Up to now, we have not considered the scalar field Φ; the prescription

$$\Phi = \sqrt{\frac{2}{\kappa}} \arctan \frac{x}{b}\,, \qquad \text{with } x \text{ as in Eq. (3.88)} \tag{3.90}$$

gives a smooth function everywhere on $\mathcal{S}$, with the properties $\Phi \circ \mathfrak{T} = -\Phi$, $\Phi \circ \mathfrak{T}^2 = \Phi$ and so on. The triple $\mathcal{S} \times S^2, \mathbf{g}, \Phi$ in Eqs. (3.86,3.85,3.90) is a solution to the Einstein-scalar equations (with field self-potential $V(\Phi)$ as in (3.69)).

Of course, the extended spacetime $\mathcal{S} \times S^2$ has the topology of $\mathbb{R}^2 \times S^2$. For any fixed $p = 1, 2, 3, ...$ we can take the quotient of the strip $\mathcal{S}$ with respect to the iterated translation $\mathfrak{T}^p$; the quotient $\mathcal{S}/\mathfrak{T}^p$ has the topology of $\mathbb{R} \times S^1$ and the metric (3.85) can be projected on $(\mathcal{S}/\mathfrak{T}^p) \times S^2$, thus getting a new spacetime with the topology $\mathbb{R} \times S^1 \times S^2$. The function Φ of Eq. (3.90) is projectable on this quotient spacetime for p even, since in this case $\Phi \circ \mathfrak{T}^p = \Phi$; on the contrary, Φ is not projectable for p odd because $\Phi \circ \mathfrak{T}^p = -\Phi$. Finally, let us mention that all spacetimes $\mathcal{S} \times S^2$ and $(\mathcal{S}/\mathfrak{T}^p) \times S^2$ $(p = 1, 2, 3...)$ are time orientable: in fact, $\partial/\partial T$ is a smooth timelike vector field, defined everywhere on $\mathcal{S} \times S^2$ and projectable on $(\mathcal{S}/\mathfrak{T}^p) \times S^2$ both for p even and for p odd. One could also consider the quotients $(\mathcal{S}/(\mathfrak{T}^p \circ \mathfrak{R}))$ with $p = 1, 2, 3, \ldots$ involving the time reflection, which yield smooth spacetimes which are, however, not time orientable.

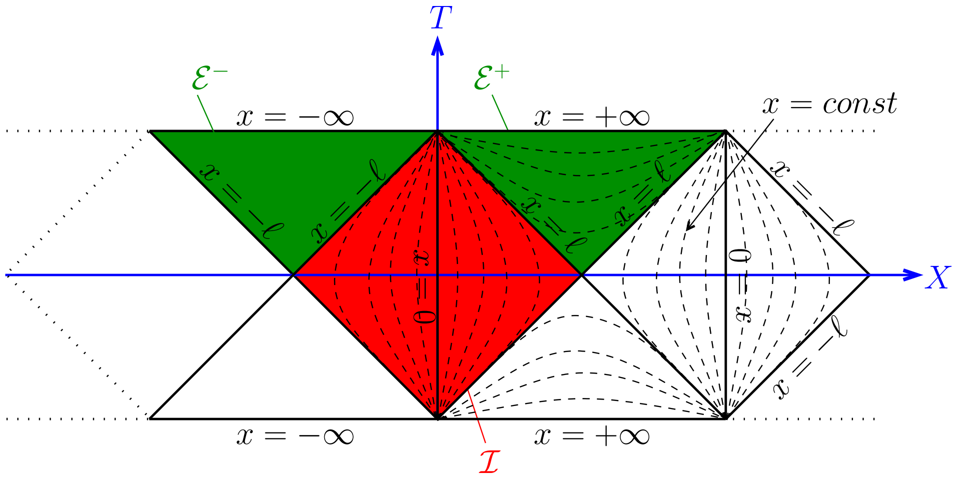

Figure 3.9: *Penrose diagram showing the strip $\mathcal{S}$ in the final extended spacetime of our dS wormhole (Eqs. (3.86,3.85)). The dashed lines are lines with constant x, determined according to Eq. (3.88). Also indicated are the red diamond region $\mathcal{I}$ and the green triangular regions $\mathcal{E}^{\pm}$ of Eq. (3.89) which correspond to the original regions (3.73) in the (t, x) coordinate space; the same can be said of the images of $\mathcal{I}$ and $\mathcal{E}^{\pm}$ under any translation $\mathfrak{T}^{2h} : (T, X) \mapsto (T, X + 2h\pi)$ $(h \in \mathbb{Z})$. Applying the time reflection $\mathfrak{R} : (T, X) \mapsto (-T, X)$ to the triangles $\mathcal{E}^{\mp}$ and to the translated triangles mentioned before, one obtains other regions which are isometric to $E^{\mp}$.*

Part II

Linear stability analysis of static wormhole solutions

Chapter 4

Field equations for perturbed static wormhole solutions

Throughout this chapter, we consider an arbitrary $(d+1)$-dimensional spherically symmetric static solution of the Einstein-scalar equations (2.72) (or, alternatively, (2.83-2.87)) in the gauge $\beta = 0$ (see Remark 10) with self-interacting potential $V(\Phi)$. Hence, we set again

$$\beta = 0\,, \qquad (\alpha, \gamma, r, \Phi) := (\alpha(x), \gamma(x), r(x), \Phi(x))\,. \tag{4.1}$$

We recall that, the in this case, the range of the coordinates (t, x) is as in Eq. (2.15). As proved in Section 3.1, the static solution (4.1) is fully described by the background equations (3.6-3.9).

4.1 Perturbations of the metric and the scalar field

In this chapter, we consider a (non-static) perturbation of the static solution (3.1), i.e., we introduce five smooth real functions

$$\begin{aligned} &\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi \,:\, \mathcal{O} = \mathbb{R} \times x(\mathcal{O}) \subseteq \mathbb{R}^2 \to \mathbb{R} \\ &(t, x) \mapsto \delta\alpha(t, x), \delta\beta(t, x), \delta\gamma(t, x), \delta r(t, x), \delta\Phi(t, x) \end{aligned} \tag{4.2}$$

and a small real parameter ϵ, such that the perturbed metric has the form (2.24) with the coefficients defined as [19]

$$\alpha(t,x) := \alpha(x) + \epsilon\,\delta\gamma(t,x)\,, \qquad \beta(t,x) := \epsilon\,\delta\beta(t,x)\,,$$
$$\gamma(t,x) := \gamma(x) + \epsilon\,\delta\alpha(t,x)\,, \qquad r(t,x) := r(x) + \epsilon\,\delta r(t,x)\,, \tag{4.3}$$

and such that the perturbed field reads

$$\Phi(t,x) := \Phi(x) + \epsilon\,\delta\Phi(t,x)\,. \tag{4.4}$$

Note that a physically reasonable perturbation leaves the function $V(\cdot)$, describing the self-interaction of the potential, unchanged.

In order to fully study the stability of the static configuration (4.1), ideally, one has to insert the perturbed metric and scalar field (4.3,4.4) into the field equations (2.83-2.87) and solve the resulting system of PDEs in the unknown functions $(\delta\alpha, \delta\gamma, \delta r, \delta\Phi)$; then the behaviour of this solution in the large t limit might be used to infer a stability result. However, this can be impractical to do because of the difficulties in solving the exact system (2.83-2.87); for this reason, it is usual to consider linear perturbations and settle for a linear stability result; therefore, from now on we restrict our study to linear perturbations, i.e., to perturbations of the form (4.3,4.4) for which we neglect any power of the parameter ϵ greater then one.

4.2 Possible gauge choices

Let us pass from the coordinate system $(t,x) : \mathcal{O} \subseteq M_2 \to \mathcal{O} \subseteq \mathbb{R}^2$ [20] to a new coordinate system

$$(\tilde{t},\tilde{x}) := \phi_\epsilon \circ (t,x) : \tilde{\mathcal{O}} \subseteq M_2 \to \tilde{\mathcal{O}} \subseteq \mathbb{R}^2\,, \tag{4.5}$$

[19] This notation might seem a little misleading, since, very often, the use of the prefix δ implies that the quantities $\delta\alpha, \ldots$ are small; if one desires to introduce the small parameter ϵ in defining the perturbations as in Eqs. (4.3,4.4), then the letter δ should be replaced, for example, by the letter Δ. Actually, our unusual choice of notation is merely motivated by the coherence with the notation of [1]; hence, throughout this thesis, the symbols $\delta\alpha, \ldots$ represent just smooth functions.

[20] In this section, as elsewhere in the thesis, we use the following abuse of notation: the symbol (t,x) stands for both the mapping $\mathcal{O} \subseteq M_2 \to \mathcal{O} \subseteq \mathbb{R}^2$ defining the coordinate system and for a generic point of its range $\mathcal{O}$. A similar, twofold meaning is given to the notation $(\tilde{t},\tilde{x})$. Other abuses of notation employed in this section will be pointed out in the sequel; this can be useful, due to some subtleties in the subject of the section.

where ϵ must be ultimately sent to zero and ϕ_ϵ is a diffeomorphism ϵ-close to the identity: [(21)]

$$\begin{aligned}&\phi_\epsilon \,:\, \mathcal{O} \subseteq \mathbb{R}^2 \to \tilde{\mathcal{O}} \subseteq \mathbb{R}^2\\ &(t,x) \mapsto (\tilde{t},\tilde{x}) = \phi_\epsilon(t,x) = (t - \epsilon\,\delta t(t,x) + O(\epsilon^2), x - \epsilon\,\delta x(t,x) + O(\epsilon^2))\,.\end{aligned} \tag{4.6}$$

The pair of functions

$$\delta\mathbf{X} := (\delta t, \delta x) : \mathcal{O} \subseteq \mathbb{R}^2 \to \mathbb{R}^2$$

are assumed to be smooth, and define a vector field on $\mathcal{O}$. The transformation ϕ_ϵ can be inverted: the inverse function $\phi_\epsilon^{-1} = \psi_\epsilon$ has the form

$$\begin{aligned}&\psi_\epsilon \,:\, \tilde{\mathcal{O}} \subseteq \mathbb{R}^2 \to \mathcal{O} \subseteq \mathbb{R}^2\\ &(\tilde{t},\tilde{x}) \mapsto (t,x) = \psi_\epsilon(\tilde{t},\tilde{x}) = (\tilde{t} + \epsilon\,\delta t(\tilde{t},\tilde{x}) + O(\epsilon^2), \tilde{x} + \delta x(\tilde{t},\tilde{x}) + O(\epsilon^2))\,.\end{aligned} \tag{4.7}$$

Let us now recall that the spacetime manifold M_{d+1} carries other coordinates $x^2, ..., x^d$, that we can use to define the coordinate systems $(t, x, x^2, ..., x^d)$ and $(\tilde{t}, \tilde{x}, x^2, ..., x^d)$; these are connected by the transformations

$$(t, x, x^2, ..., x^d) \mapsto (\tilde{t}, \tilde{x}, x^2, ..., x^d) = (\phi_\epsilon(t,x), , x^2, ..., x^d), \tag{4.8}$$

$$(\tilde{t}, \tilde{x}, x^2, ..., x^d) \mapsto (t, x, x^2, ..., x^d) = (\psi_\epsilon(\tilde{t},\tilde{x}), , x^2, ..., x^d), \tag{4.9}$$

that in the sequel, by a natural abuse of notation, will be as well indicated with ϕ_ϵ and ψ_ϵ. In the same spirit, we will write $\delta\mathbf{X}$ for the vector field on $\mathbb{R}^{d+1}$ whose first two components are $\delta t(t,x)$, $\delta x(t,x)$, while all other components are zero.

In this paragraph we want to understand how the transformation ϕ_ϵ acts on the perturbed metric (2.24,4.3).

Remark 36 Before going on, let us observe that the perturbed metric (2.24,4.3) can be rewritten in the form

$$\mathbf{g} = \mathbf{g}_0 + \epsilon\,\delta\mathbf{g} + O(\epsilon^2) \tag{4.10}$$

where $\mathbf{g}_0$ is the static unperturbed metric and $\delta\mathbf{g}$ is the perturbation of the metric:

$$\mathbf{g}_0 := -\alpha^2 dt^2 + \gamma^2 dx^2 + r^2\, a_{ij}(x^2, ..., x^d)\, dx^i\, dx^j\,, \tag{4.11}$$

$$\delta\mathbf{g} := -2\alpha\,\delta\alpha\, dt^2 + 2\gamma^2\,\delta\beta\, dt\, dx + 2\gamma\,\delta\gamma\, dx^2 + 2r\,\delta r\, a_{ij}(x^2 ..., x^d)\, dx^i\, dx^j\,. \tag{4.12}$$

[21] The minus sign is introduced in Eq. (4.6) before $\epsilon\delta t$, $\epsilon\delta x$ for coherence with [1].

It is well known that, under the transformation ϕ_ϵ, the perturbed metric in Eq. (4.10) and the perturbed scalar field $\Phi(t,x)$ in Eq. (4.4) are transformed by the pullback of the inverse map $\phi_\epsilon^{-1} = \psi_\epsilon$, that is [22]

$$\mathbf{g} \mapsto \tilde{\mathbf{g}} := \psi_\epsilon^* \mathbf{g} = \mathbf{g} + \epsilon\, \pounds_{\delta\mathbf{X}}\, \mathbf{g} = \mathbf{g}_0 + \epsilon\, \left(\delta\mathbf{g} + \pounds_{\delta\mathbf{X}}\, \mathbf{g}_0\right) + O(\epsilon^2)\,, \quad (4.13)$$

$$\begin{aligned} \Phi(t,x) \mapsto \tilde{\Phi}(\tilde{t},\tilde{x}) &:= \psi_\epsilon^* \Phi(\tilde{t},\tilde{x}) = \Phi(\tilde{t},\tilde{x}) + \epsilon\, \pounds_{\delta\mathbf{X}}\, \Phi(\tilde{t},\tilde{x}) \\ &= \Phi(\tilde{x}) + \epsilon\, \big(\delta\Phi(\tilde{t},\tilde{x}) + \pounds_{\delta\mathbf{X}}\, \Phi(\tilde{x})\big) + O(\epsilon^2)\,; \end{aligned} \quad (4.14)$$

in the first identities of Eqs. (4.13,4.14) we have used the expression for the pullback ψ_ϵ^* of the infinitesimal transformation ψ_ϵ parametrized by the vector field $\epsilon\,\delta\mathbf{X}$ in terms of the Lie derivative $\pounds_{\delta\mathbf{X}}$ with respect to the field $\delta\mathbf{X}$; in the second identities we have used Eqs. (4.10,4.4), neglecting the terms multiplied by ϵ^2. Looking at Eqs. (4.13,4.14), one can assume that in the new coordinates $(\tilde{t},\tilde{x})$ the transformed perturbed metric and field have the same structure as in the coordinates (t,x), namely

$$\tilde{\mathbf{g}} = \tilde{\mathbf{g}}_0 + \epsilon\, \tilde{\delta\mathbf{g}} + O(\epsilon^2) \qquad \tilde{\Phi}(\tilde{t},\tilde{x}) := \tilde{\Phi}(\tilde{x}) + \epsilon\, \tilde{\delta\Phi}(\tilde{t},\tilde{x}) + O(\epsilon^2)\,, \quad (4.15)$$

where the unperturbed solution $(\tilde{\mathbf{g}}_0, \tilde{\Phi}(\tilde{x}))$ remains unchanged, that is

$$\mathbf{g}_0 \mapsto \tilde{\mathbf{g}}_0 := \mathbf{g}_0\,, \qquad \Phi(x) \mapsto \tilde{\Phi}(\tilde{x}) := \Phi(\tilde{x})\,. \quad (4.16)$$

while the perturbation metric $\tilde{\delta\mathbf{g}}$ and the perturbation field $\tilde{\delta\Phi}(\tilde{t},\tilde{x})$ are transformed according to

$$\delta\mathbf{g} \mapsto \tilde{\delta\mathbf{g}} := \delta\mathbf{g} + \pounds_{\delta\mathbf{X}}\, \mathbf{g}_0\,, \quad \delta\Phi(t,x) \mapsto \tilde{\delta\Phi}(\tilde{t},\tilde{x}) := \delta\Phi(\tilde{t},\tilde{x}) + \pounds_{\delta\mathbf{X}}\, \Phi(\tilde{x})\,. \quad (4.17)$$

Let us introduce the transformed perturbation coefficients $(\tilde{\delta\alpha}, \tilde{\delta\beta}, \tilde{\delta\gamma}, \tilde{\delta r})$ of the transformed perturbation metric $\tilde{\delta\mathbf{g}}$ such that

$$\tilde{\delta\mathbf{g}} =: -2\alpha\, \tilde{\delta\alpha}\, d\tilde{t}^2 + 2\gamma^2\, \tilde{\delta\beta}\, d\tilde{t}\, d\tilde{x} + 2\gamma\, \tilde{\delta\gamma}\, d\tilde{x}^2 + 2r\, \tilde{\delta r}\, a_{ij}(x^2,...,x^d)\, dx^i\, dx^j\,; \quad (4.18)$$

[22] Here we are using other notational abuses. We are considering two coordinate systems $(t,x,x^2,...,x^d) : \mathcal{O}_{d+1} \subseteq M_{d+1} \to \mathbb{O}_{d+1} \subseteq \mathbb{R}^{d+1}$ and $(\tilde{t},\tilde{x},x^2,...,x^d) : \mathcal{O}_{d+1} \subseteq M_{d+1} \to \tilde{\mathbb{O}}_{d+1} \subseteq \mathbb{R}^{d+1}$, connected by the transformations (4.8) (4.9) that we have decided to indicate simply with ϕ_ϵ and ψ_ϵ. In principle we should distinguish $\mathbf{g}$ from its local representations $\mathbf{h}$, $\tilde{\mathbf{h}}$, which are the metrics on $\mathbb{O}_{d+1}$, $\tilde{\mathbb{O}}_{d+1}$ such that $\mathbf{g} = (t,x,x^2,...,x^d)^*\mathbf{h} = (\tilde{t},\tilde{x},x^2,...,x^d)^*\tilde{\mathbf{h}}$, with * indicating the pullbacks along the two coordinate systems; it turns out that $\tilde{\mathbf{h}} = \psi_\epsilon^*\mathbf{h} = \mathbf{h} + \epsilon\pounds_{\delta\mathbf{X}}\,\mathbf{h} + \mathbf{O}(\epsilon^2)$. In Eq. (4.13), the notations $\mathbf{g}$, $\tilde{\mathbf{g}}$ stand in fact for the local representations $\mathbf{h}$, $\tilde{\mathbf{h}}$. Eq. (4.14) contains a similar abuse, with Φ and $\tilde{\Phi}$ standing for the local representations of the scalar field in the two coordinate systems. Similar comments could be made for equations appearing elsewhere in this work.

then, by computing the Lie derivatives appearing in Eq. (4.17), it can be proved that the infinitesimal transformation (4.7) transforms the perturbations $(\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi)$ as

$$\delta\alpha \mapsto \tilde{\delta\alpha} := \delta\alpha + \alpha'\delta x + \alpha\,\dot{\delta t}\,, \tag{4.19}$$

$$\delta\beta \mapsto \tilde{\delta\beta} := \delta\beta + \dot{\delta x} - \frac{\alpha^2}{\gamma^2}\delta t'\,, \tag{4.20}$$

$$\delta\gamma \mapsto \tilde{\delta\gamma} := \delta\gamma + (\gamma\delta x)'\,, \tag{4.21}$$

$$\delta r \mapsto \tilde{\delta r} := \delta r + r'\delta x\,, \tag{4.22}$$

$$\delta\Phi \mapsto \tilde{\delta\Phi} := \delta\Phi + \Phi'\delta x\,, \tag{4.23}$$

where the dots and the primes in the above equations stand, respectively, for the partial derivatives with respect to $\tilde{t}$ and $\tilde{x}$. This results has been included in Ref. [1]; for an exhaustive proof see Appendix B.1.

Remark 37 Eqs. (4.19-4.23) make evident that the two degrees of freedom in choosing the transformation functions δt and δx might be used in order to eliminate two of the perturbation functions. For example, using Eq. (4.20), one can eliminate the perturbation $\delta\beta$ (that is, send this perturbation to the perturbation $\tilde{\delta\beta} = 0$) by setting

$$\delta t(\tilde{t}, \tilde{x}) = \int_{\tilde{x}_1}^{\tilde{x}} \frac{\gamma(x)^2}{\alpha(x)^2}\Big(\delta\beta(\tilde{t}, x) + \dot{\delta x}(\tilde{t}, x)\Big)\,dx\,, \qquad \tilde{x}_1 \in \mathbb{R}\,; \tag{4.24}$$

note that this is equivalent to fix the orthogonal gauge on M_2 introduced in Lemma 1 in which the perturbed metric coefficient $\beta(t,x)$ is set to zero (to see this, it is sufficient to look at Eq. (4.3)). In addition, we can prescribe that the transformation ϕ_ϵ sends another perturbation to zero or transforms two perturbations into functions that are proportional; we will see that this choice is crucial in performing the stability analysis. For the moment, we do not impose any prescription on the gauge.

Before concluding this section, let us introduce an important concept that will be widely used throughout the present thesis.

Definition 8 *Consider a functional* [23]

$$F : (\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi) \mapsto F[\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi]\,;$$

this is said to be gauge-invariant if, for every infinitesimal change of coordinates (4.6,4.7), it is

$$F[\tilde{\delta\alpha}, \tilde{\delta\beta}, \tilde{\delta\gamma}, \tilde{\delta r}, \tilde{\delta\Phi}] = F[\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi]$$

where $\tilde{\delta\alpha}, \tilde{\delta\beta}, \tilde{\delta\gamma}, \tilde{\delta r}, \tilde{\delta\Phi}$ are as in Eqs. (4.19-4.23).

[23] F might even depend on the derivatives or on other attributes of the functions $\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi$.

4.3 Linearizing the field equations (and the scalar curvature)

We expand the field equations (2.72) for the perturbed static solution (4.3,4.4) up to the first order in ϵ; neglecting all the powers of ϵ grater or equal then 2, we obtain the linearized equations

$$\begin{aligned} &\mathcal{E}^{(1)}_{ij}(\epsilon) = 0\,, && \mathcal{E}^{(1)}_{ij}(\epsilon) := \mathcal{E}_{ij}\big|_{\epsilon=0} + \epsilon\,\delta\mathcal{E}_{ij} \qquad (i,j=0,1)\,,\\ &\mathcal{E}^{(1)}_{a}(\epsilon) = 0\,, && \mathcal{E}^{(1)}_{a}(\epsilon) := \mathcal{E}_{a}\big|_{\epsilon=0} + \epsilon\,\delta\mathcal{E}_{a}\,, \end{aligned}$$

where

$$\delta\mathcal{E}_{ij} := \left.\frac{d\mathcal{E}_{ij}}{d\epsilon}\right|_{\epsilon=0} \quad (i,j=0,1)\,, \qquad \delta\mathcal{E}_{a} := \left.\frac{d\mathcal{E}_{a}}{d\epsilon}\right|_{\epsilon=0}\,;$$

of course, these equations are satisfied to the zeroth order in ϵ, corresponding to the static solution (4.1), while the first order in ϵ vanishes if and only if the following hold:

$$\begin{aligned} \delta\mathcal{E}_{00} = 0\,,\quad \delta\mathcal{E}_{00} =& -(d-1)\frac{\alpha^2}{\gamma^2}\Bigg[\left(\frac{\alpha'}{\alpha}\frac{r'}{r} + \frac{2\kappa}{d-1}\gamma^2 V(\Phi)\right)\frac{\delta r}{r}\\ &+ \left((d-2)\frac{r'}{r} - \frac{\gamma'}{\gamma}\right)\frac{\delta r'}{r} + \frac{\delta r''}{r}\\ &+ \left(\frac{2\kappa}{d-1}V(\Phi) - \frac{d-2}{r^2}\right)\frac{\delta\gamma}{\gamma}\\ &- \frac{r'}{r}\frac{\delta\gamma'}{\gamma} + \frac{\kappa}{d-1}\left(\gamma^2 V'(\Phi)\delta\Phi - \Phi'\delta\Phi'\right)\Bigg] + \delta\mathcal{E}^{\beta}_{00}\,, \qquad (4.25)\\ \delta\mathcal{E}_{01} = 0\,,\quad \delta\mathcal{E}_{01} =& \frac{\partial}{\partial t}\left[(d-1)\frac{r'}{r}\left(\frac{\delta\gamma}{\gamma} + \frac{\alpha'}{\alpha}\frac{\delta r}{r'} + \frac{\delta r'}{r'}\right) + \kappa\Phi'\delta\Phi\right] + \delta\mathcal{E}^{\beta}_{01}\,, \end{aligned}$$

(4.26)

$$\begin{aligned}
\delta\mathcal{E}_{11} = 0\,, \quad \delta\mathcal{E}_{11} =& (d-1)\Bigg[\gamma\left(\frac{2\kappa}{d-1}V(\Phi) - \frac{d-2}{r^2}\right)\delta\gamma \\
&+ \left(\frac{\alpha'}{\alpha}\frac{r'}{r} + \frac{\kappa}{d-1}\left(2\gamma^2 V(\Phi) - \Phi'^2\right)\right)\frac{\delta r}{r} \\
&+ \left(\frac{\alpha'}{\alpha} + (d-2)\frac{r'}{r}\right)\frac{\delta r'}{r} - \frac{\gamma^2}{\alpha^2}\frac{\delta\ddot{r}}{r} + \frac{r'}{r}\frac{\partial}{\partial x}\left[\frac{\delta\alpha}{\alpha}\right] \\
&+ \frac{\kappa}{d-1}\left(\gamma^2 V'(\Phi)\delta\Phi + \Phi'\delta\Phi'\right)\Bigg] + (d-1)\frac{\gamma^2}{\alpha^2}\delta\mathcal{E}^{\beta}_{11}\,,
\end{aligned} \tag{4.27}$$

$$\begin{aligned}
\delta\mathcal{E}_{a} = 0\,, \quad \delta\mathcal{E}_{a} =& \frac{r^2}{\gamma^2}\Bigg[\left(\frac{\alpha'}{\alpha}\frac{r'}{r} + \frac{2\kappa}{d-1}\gamma^2 V(\Phi)\right)\frac{\delta\alpha}{\alpha} + \left((d-2)\frac{r'}{r} - \frac{\gamma'}{\gamma}\right)\frac{\delta\alpha'}{\alpha} \\
&+ \frac{\delta\alpha''}{\alpha} + \gamma\left(2\kappa V(\Phi) - \frac{(d-2)(d-3)}{r^2}\right)\delta\gamma \\
&- \left(\frac{\alpha'}{\alpha} + (d-2)\frac{r'}{r}\right)\frac{\partial}{\partial x}\left[\frac{\delta\gamma}{\gamma}\right] \\
&+ \left(\frac{\kappa}{d-1}\left(2(d-3)\gamma^2 V(\Phi) - \Phi'^2\right) - 2\frac{\alpha'}{\alpha}\frac{r'}{r}\right)\frac{\delta r}{r} \\
&+ (d-2)\left(\frac{\alpha'}{\alpha} - \frac{\gamma'}{\gamma} + (d-3)\frac{r'}{r}\right)\frac{\delta r'}{r} \\
&+ (d-2)\frac{\delta r''}{r} - \frac{\gamma^2}{\alpha^2}\frac{\partial^2}{\partial t^2}\left[\frac{\delta\gamma}{\gamma} + (d-2)\frac{\delta r}{r}\right] \\
&+ \kappa\left(\gamma^2 V'(\Phi)\delta\Phi - \Phi'\delta\Phi'\right)\Bigg] + \frac{r^2}{\alpha^2}\delta\mathcal{E}^{\beta}_{a}\,,
\end{aligned} \tag{4.28}$$

where [24]

$$\begin{aligned}
&\delta\mathcal{E}^{\beta}_{00} := 0\,, && \delta\mathcal{E}^{\beta}_{01} := 0\,, \\
&\delta\mathcal{E}^{\beta}_{11} := \frac{r'}{r}\delta\dot{\beta}\,, && \delta\mathcal{E}^{\beta}_{a} := \left((d-2)\frac{r'}{r} + \frac{\gamma'}{\gamma}\right)\delta\dot{\beta} + \delta\dot{\beta}'\,.
\end{aligned}$$

[24]Despite the notation, the quantities $\delta\mathcal{E}^{\beta}_{00}$, $\delta\mathcal{E}^{\beta}_{01}$, etc. are not the linearizations of the quantities $\mathcal{E}^{\beta}_{00}$, $\mathcal{E}^{\beta}_{01}$; the same can be said for the quantities $\delta\mathfrak{E}^{\beta}_{1}$, $\delta\mathfrak{E}^{\beta}_{2}$, etc. of Es. (4.34) and for δR^{β} in Eq. (4.38). This choice of notation has been made just to distinguish the parts of the linearized quantities $\delta\mathcal{E}_{00}$, $\delta\mathcal{E}_{01}$, etc. that vanish when $\beta = 0$.

For future convenience, we provide also the linearization of the field system (2.83-2.87) for the perturbed static solution (4.3,4.4), that is

$$\begin{aligned} \mathfrak{E}_i^{(1)}(\epsilon) &= 0\,, & \mathfrak{E}_i^{(1)}(\epsilon) &:= \mathfrak{E}_i\big|_{\epsilon=0} + \epsilon\,\delta\mathfrak{E}_i \qquad (i=1,2,3) \\ \mathcal{H}^{(1)}(\epsilon) &= 0\,, & \mathcal{H}^{(1)}(\epsilon) &:= \mathcal{H}\big|_{\epsilon=0} + \epsilon\,\delta\mathcal{H}\,, \\ \mathcal{M}^{(1)}(\epsilon) &= 0\,, & \mathcal{M}^{(1)}(\epsilon) &:= \mathcal{M}\big|_{\epsilon=0} + \epsilon\,\delta\mathcal{M}\,, \end{aligned}$$

where

$$\delta\mathfrak{E}_i := \frac{d\mathfrak{E}_i}{d\epsilon}\bigg|_{\epsilon=0} \quad (i=1,2,3)\,, \qquad \delta\mathcal{H} := \frac{d\mathcal{H}}{d\epsilon}\bigg|_{\epsilon=0}\,, \qquad \delta\mathcal{M} := \frac{d\mathcal{M}}{d\epsilon}\bigg|_{\epsilon=0}\,;$$

of course, even in this case, the linearized equations are satisfied to the zeroth order in ϵ, as it correspons to the static solution, while the first order in ϵ is satisfied if and only if the following hold: [25]

$$\begin{aligned} \delta\mathfrak{E}_1 = 0\,, \quad \delta\mathfrak{E}_1 =& \frac{\gamma}{\alpha}\frac{\partial^2}{\partial t^2}\left[\frac{\delta\gamma}{\gamma}\right] - \frac{\partial}{\partial x}\left[\frac{\alpha}{\gamma}\frac{\partial}{\partial x}\left[\frac{\delta\alpha}{\alpha}\right]\right] \\ & - \frac{\alpha}{\gamma}\left(\frac{\alpha'}{\alpha} + \frac{d-3}{2}\frac{r'}{r}\right)\frac{\partial}{\partial x}\left[\frac{\delta\alpha}{\alpha} - \frac{\delta\gamma}{\gamma}\right] \\ & + \left[(d-1)\frac{\alpha}{\gamma}\frac{r'}{r} - \frac{d-3}{2}\frac{\partial}{\partial x}\left[\frac{\alpha}{\gamma}\right]\right]\frac{\partial}{\partial x}\left[\frac{\delta r}{r}\right] \\ & - \frac{2(d-2)\alpha\gamma}{r^2}\left(\frac{\delta\gamma}{\gamma} - \frac{\delta r}{r}\right) + \kappa\frac{\alpha}{\gamma}\Phi'\delta\Phi' \\ & + \frac{d-3}{2}\left(\frac{\gamma}{\alpha}\frac{\partial^2}{\partial t^2}\left[\frac{\delta r}{r}\right] - \frac{\alpha}{\gamma}\frac{\partial^2}{\partial t^2}\left[\frac{\delta r}{r}\right]\right) - \frac{\gamma}{\alpha}\delta\mathfrak{E}_1^{\beta}\,, \end{aligned} \qquad (4.29)$$

[25] The expressions in Eqs. (4.31,4.32,4.34) have been simplified using the background equations (3.6,3.8,3.9).

$$\begin{aligned}
\delta\mathfrak{E}_2 = 0\,,\quad \delta\mathfrak{E}_2 =& \left[\frac{d-1}{2}\frac{\gamma}{\alpha}\frac{\partial^2}{\partial t^2}\left[\frac{\delta r}{r}\right] - \frac{d-1}{2}\frac{\partial}{\partial x}\left[\frac{\alpha}{\gamma}\frac{\partial}{\partial x}\left[\frac{\delta r}{r}\right]\right]\right.\\
&- \frac{d-1}{2}\frac{\alpha}{\gamma}\frac{r'}{r}\frac{\partial}{\partial x}\left[\frac{\delta\alpha}{\alpha} - \frac{\delta\gamma}{\gamma} + 2(d-1)\frac{\delta r}{r}\right]\\
&+ \frac{(d-1)(d-2)\alpha\gamma}{r^2}\left(\frac{\delta\gamma}{\gamma} - \frac{\delta r}{r}\right)\\
&\left. - \kappa\alpha\gamma\left[2V(\Phi)\frac{\delta\gamma}{\gamma} + V'(\Phi)\delta\Phi\right]\right] r^2 - \frac{d-1}{2}\frac{\gamma}{\alpha}\delta\mathfrak{E}_2^\beta\,, \qquad (4.30)
\end{aligned}$$

$$\begin{aligned}
\delta\mathfrak{E}_3 = 0\,,\quad \delta\mathfrak{E}_3 =& \frac{\gamma}{\alpha}\delta\ddot{\Phi} - \frac{\partial}{\partial x}\left[\frac{\alpha}{\gamma}\delta\Phi'\right] - (d-1)\frac{\alpha}{\gamma}\frac{r'}{r}\delta\Phi'\\
&- \frac{\alpha}{\gamma}\Phi'\frac{\partial}{\partial x}\left[\frac{\delta\alpha}{\alpha} - \frac{\delta\gamma}{\gamma} + (d-1)\frac{\delta r}{r}\right]\\
&- \alpha\gamma\left[2V'(\Phi)\frac{\delta\gamma}{\gamma} + V''(\Phi)\delta\Phi\right] - \frac{\gamma}{\alpha}\delta\mathfrak{E}_3^\beta\,, \qquad (4.31)
\end{aligned}$$

$$\begin{aligned}
\delta\mathcal{H} = 0\,,\quad \delta\mathcal{H} =& (d-1)(d-2)\frac{\alpha\gamma}{r^2}\left(\frac{\delta r}{r} - \frac{\delta\gamma}{\gamma}\right)\\
&+ \kappa\alpha\gamma\left(2V(\Phi)\frac{\delta\gamma}{\gamma} + V'(\Phi)\delta\Phi\right) + (d-1)\frac{\alpha}{\gamma}\left[\frac{\partial^2}{\partial x^2}\left[\frac{\delta r}{r}\right]\right.\\
&\left. + \left(d\frac{r'}{r} - \frac{\gamma'}{\gamma}\right)\frac{\partial}{\partial x}\left[\frac{\delta r}{r}\right] - \frac{r'}{r}\frac{\partial}{\partial x}\left[\frac{\delta\gamma}{\gamma}\right] - \frac{\kappa}{d-1}\Phi'\delta\Phi'\right] + \delta\mathcal{H}^\beta\,, \qquad (4.32)
\end{aligned}$$

$$\begin{aligned}
\delta\mathcal{M} = 0\,,\quad \delta\mathcal{M} =& (d-1)\frac{\partial}{\partial t}\left[\frac{\partial}{\partial x}\left[\frac{\delta r}{r}\right] - \frac{r}{\alpha}\frac{\partial}{\partial x}\left[\frac{\alpha}{r}\right]\frac{\delta r}{r}\right.\\
&\left. - \frac{r'}{r}\frac{\delta\gamma}{\gamma} - \frac{\kappa}{d-1}\Phi'\delta\Phi\right] + \delta\mathcal{M}^\beta\,, \qquad (4.33)
\end{aligned}$$

where

$$\begin{aligned}
&\delta\mathfrak{E}_1^\beta := \left(\frac{d-3}{2}\frac{r'}{r} + \frac{\gamma'}{\gamma}\right)\delta\dot{\beta} + \delta\dot{\beta}'\,, \qquad \delta\mathfrak{E}_2^\beta := rr'\delta\dot{\beta}\,, \qquad \delta\mathfrak{E}_3^\beta := r^2\Phi'\delta\dot{\beta}\,,\\
&\qquad\qquad \delta\mathcal{H}^\beta := 0\,, \qquad \delta\mathcal{M}^\beta := 0\,. \qquad (4.34)
\end{aligned}$$

For future use, we add the linearization of the scalar curvature (2.39): this is defined as

$$R^{(1)}(\epsilon) := R_0 + \epsilon\, \delta R\,, \tag{4.35}$$

where

$$R_0 := R\Big|_{\epsilon=0}\,, \qquad \delta R := \frac{dR}{d\epsilon}\bigg|_{\epsilon=0}\,;$$

of course the zeroth order in ϵ is exactly the scalar curvature of the static metric (4.1), that is,

$$\begin{aligned} R_0 = \frac{2}{\gamma^2}\left(\frac{\alpha'}{\alpha}\frac{\gamma'}{\gamma} - \frac{\alpha''}{\alpha} - (d-1)\frac{r''}{r}\right) + 2(d-1)\frac{\alpha}{\gamma^3}\frac{r'}{r}\frac{\partial}{\partial x}\left[\frac{\gamma}{\alpha}\right] \\ + \frac{(d-2)(d-1)}{r^2}\left(1 - \frac{r'^2}{\gamma^2}\right), \end{aligned} \tag{4.36}$$

while the first order in ϵ reads [26]

$$\begin{aligned} \delta R = & \frac{2}{d-1}\Bigg\{\Bigg[\left(\frac{1}{d-1}\frac{\alpha'}{\alpha} + \frac{r'}{r}\right)\left(\frac{\delta\gamma}{\gamma}\right)' + \left(\frac{\gamma'}{\gamma} - \frac{\alpha'}{\alpha} - \frac{(d-2)r'}{r}\right)\frac{\delta r'}{r} \\ & + \left(\frac{1}{d-1}\frac{\gamma'}{\gamma} - \frac{r'}{r}\right)\frac{\delta\alpha'}{\alpha} - \frac{1}{d-1}\frac{\delta\alpha''}{\alpha} + \frac{\kappa}{d-1}\Phi'^2\frac{\delta\gamma}{\gamma} - \frac{r''}{r}\Bigg]\frac{1}{\gamma^2} \\ & - \frac{2\kappa}{d-1}\left(\frac{1}{d-1}\frac{\delta\alpha}{\alpha} + \frac{d+1}{d-1}\frac{\delta\gamma}{\gamma} + \frac{\delta r}{r}\right)V(\Phi) \\ & + \left(\frac{1}{d-1}\frac{\delta\ddot{\gamma}}{\gamma} + \frac{\delta\ddot{r}}{r}\right)\frac{1}{\alpha^2} + (d-2)\frac{\delta\gamma}{\gamma}\frac{1}{r^2}\Bigg\} - \frac{2}{\alpha^2}\delta R^\beta\,, \end{aligned} \tag{4.37}$$

where

$$\delta R^\beta := \left((d-1)\frac{r'}{r} + \frac{\gamma'}{\gamma}\right)\dot{\beta} + \delta\dot{\beta}'\,. \tag{4.38}$$

4.3.1 Linearization of the constrained system and the scalar curvature in the $\delta\beta = 0$ gauge

In Section 4.2 (and, in particular, in Remark 37) we have discussed the chance of introducing a coordinate transformation such that in the new gauge the perturbation function $\delta\beta$ vanishes. In this case the linearized field system (4.29-4.33) becomes simpler as the quantities $\mathfrak{E}_1^\beta$, $\mathfrak{E}_2^\beta$, $\mathfrak{E}_3^\beta$, $\mathcal{H}^\beta$ and $\mathcal{M}^\beta$ in Eq.

[26] The expression for δR has been simplified using the background equations (3.6,3.8,3.9).

(4.34) are zero; in addition, it turns out that, analogously to the exact system (2.83-2.87) in the $\beta(t,x) = 0$ gauge (see Subsection 2.5.1), the linearized system (4.29-4.33) is made up of three evolution equations $\delta\mathfrak{E}_1 = 0$, $\delta\mathfrak{E}_2 = 0$, $\delta\mathfrak{E}_3 = 0$ subject to the constraints $\delta\mathcal{H} = 0$ and $\delta\mathcal{M} = 0$. This fact is stated more precisely in the following

Proposition 5 *In the gauge $\delta\beta = 0$, the equations $\delta\mathcal{H} = 0$ and $\delta\mathcal{M} = 0$ [Eqs. (4.32,4.33)] are two constraints for the second order evolution equations $\delta\mathfrak{E}_1 = 0$, $\delta\mathfrak{E}_2 = 0$, $\delta\mathfrak{E}_3 = 0$ [Eqs. (4.29-4.31)]; this means that if $(\delta\alpha, \delta\gamma, \delta r, \delta\Phi)$ is a (time-dependent) solution of the system (4.29-4.31) satisfying equations $\delta\mathcal{H} = 0$, $\delta\mathcal{M} = 0$ at time $t = 0$, then $(\delta\alpha, \delta\gamma, \delta r, \delta\Phi)$ satisfies equations $\delta\mathcal{H} = 0$, $\delta\mathcal{M} = 0$ for every time t.*

Proof. Analogously as in the proof of Proposition 4, we show that if $(\delta\alpha, \delta\gamma, \delta r, \delta\Phi)$ is a solution of the linearized evolution equations (4.29-4.31), then the corresponding quantities $\delta\mathcal{H}$ and $\delta\mathcal{M}$ defined in Eqs. (4.32,4.33) satisfy a first order dynamical system with smooth coefficients. Indeed, the quantities $\delta\mathfrak{E}_1$, $\delta\mathfrak{E}_2$, $\delta\mathfrak{E}_3$, $\delta\mathcal{H}$ and $\delta\mathcal{M}$ defined in Eqs. (4.29-4.33) are related by the following identities [27]

$$\delta\dot{\mathcal{H}} = \left[\left(\frac{\alpha}{\gamma}\right)' + (d-1)\frac{\alpha}{\gamma}\frac{r'}{r}\right]\delta\mathcal{M} + \frac{\alpha}{\gamma}\,\delta\mathcal{M}' \,, \tag{4.39}$$

$$\begin{aligned}\delta\dot{\mathcal{M}} = {} & \left[\left(\frac{\alpha}{\gamma}\right)' + (d-1)\frac{\alpha}{\gamma}\frac{r'}{r}\right]\delta\mathcal{H} + \frac{\alpha}{\gamma}\,\delta\mathcal{H}' \\ & - (d-1)\frac{\alpha}{\gamma}\frac{r'}{r}\delta\mathfrak{E}_1 - 2\frac{\alpha}{\gamma r^2}\left[\frac{\gamma'}{\gamma} - \frac{d-5}{2}\frac{r'}{r}\right]\delta\mathfrak{E}_2 + 2\frac{\alpha}{\gamma r^2}\delta\mathfrak{E}_2' - \frac{\alpha}{\gamma}\frac{\kappa\Phi'}{r^2}\delta\mathfrak{E}_3 \,.\end{aligned} \tag{4.40}$$

Therefore, if the perturbation functions $(\delta\alpha, \delta\gamma, \delta r, \delta\Phi)$ satisfy Eqs. (4.29-4.31) for every t and x, and satisfy Eqs. (4.32,4.33) at $t = 0$, then, for every fixed value of x, the corresponding quantities $\delta\mathcal{H}(t) \equiv \delta\mathcal{H}(t,x)$ and $\delta\mathcal{M}(t) \equiv \delta\mathcal{M}(t,x)$ satisfy the dynamical system (4.39,4.40) with $\delta\mathfrak{E}_1 = \delta\mathfrak{E}_2 = \delta\mathfrak{E}_2' = \delta\mathfrak{E}_3 = 0$ and the initial condition $\delta\mathcal{H}(0) = \delta\mathcal{M}(0) = 0$; obviously, this implies that $\delta\dot{\mathcal{H}}(t) = \delta\dot{\mathcal{M}}(t) = 0$ for every time t, i.e. $\delta\mathcal{H}(t,x) = \delta\mathcal{M}(t,x) = 0$ for every time t and every x.

□

[27] In deriving the system (4.39,4.40) we have used the background equations (3.6-3.9) in order to eliminate the derivatives Φ'', α'', r'' and r'^2 of the static solution.

Chapter 5

A gauge-dependent linear stability analysis of two wormhole solutions

In this chapter we present a first, probably naive but nevertheless instructive, strategy for analyse the stability of some static wormhole configurations of the form (4.1), namely

$$\beta = 0\,, \qquad (\alpha, \gamma, r, \Phi) := (\alpha(x), \gamma(x), r(x), \Phi(x))\,. \tag{5.1}$$

This approach strongly relies on the choice of a particular, suitably defined gauge, which is made in order to "simplify" as much as possible the linearized field equations. Obviously, this choice strictly depends on the expression of the static solution under consideration and therefore can not be generalized in any sense. Having set the coordinate system, with a little luck, one can gradually decouple the system, finally obtaining a single equation involving only one of the unknown perturbations: we refer to this equation as *master equation*. Once the master equation has been solved, the remaining components of the perturbation are derived from its solution. In all the cases that we present, the master equation turns out to be a wave type equation for (a function of) the radial perturbation δr: profiting from the spectral analysis of the Schrödinger-type operator appearing in the master equation, we can infer both qualitative and quantitative features of its solution.

A divergence of δr, as the temporal coordinate goes to infinity, at least for some particular initial data, is a significant hint of the linear instability of the wormhole configuration; actually, this fact is not sufficient to infer the instability, since the expression of δr depends on the gauge chosen. Indeed, in order to prove the linear instability, one has to prove that there exists a solution δr of the master equation such that the corresponding perturbed spacetime becomes singular as the temporal coordinate t approaches infinity. To check this fact, it is sufficient to show that at least one *intrinsic* scalar quantity diverges to infinity, like, for example, the scalar curvature. Of course, since we are considering time dependent linear perturbations of the spacetime, one is led to prove that the *linearization* of the scalar curvature diverges for large value of the coordinate t. Unfortunately, the divergence of the linearization of the scalar curvature for $t \to \pm\infty$ could be an artifact that can be eliminated by an everywhere smooth infinitesimal coordinate change ϕ_ϵ [Eq. (4.6)] (see Appendix B.2 for an example). In conclusion, one has to verify that:

(i) the divergence of the perturbation δr for $t \to +\infty$ corresponds to the divergence of the linearization of the scalar curvature, in the large t limit, at least at a fixed position x;

(ii) the divergence of the linearization of the scalar curvature can not be eliminated by changing the gauge via an infinitesimal coordinate transformation.

In the forthcoming sections we deal with the linear stability problem of two static wormhole solutions presented in Chapter 3: the Torii-Shinkai (3.34) and the Ellis-Bronnikov wormhole (3.39). We follow the approach explained in the previous paragraph; more precisely, for both the solutions we stick to the following general program:

(a) we set a particular gauge;

(b) we write the corresponding linearized Einstein equations and scalar curvature by substituting the static solution under analysis and the ansatz on the perturbations, made in (a), into Eqs. (4.25-4.28) and into Eqs. (4.35,4.36,4.37);

(c) we analyze the system of linear equations for the perturbations arising from (b), and reduce it to a single master equation;

(d) we prove the divergence of the solution of the master equation from a spectral analysis of the differential operator in the master equation;

(e) we infer the linear instability of the wormhole as in items (i-ii), showing in particular that the linearization of the scalar curvature diverges and that this divergence does not depend on the gauge chosen in (a).

5.1 Gauge-dependent linear stability analysis of the Torii-Shinkai wormhole

5.1.1 Gauge choice

We consider a general perturbation of the Torii-Shinkai wormhole solution, that is, the metric (2.24) and the scalar field defined by Eqs. (4.3,4.4) with $(\mathtt{t}, \mathtt{x})$ in place of (t, x), where the static solution (5.1) is given by Eq. (3.34); in this case, it is always possible to introduce a new coordinate system $(\tilde{\mathtt{t}}, \tilde{\mathtt{x}})$ by defining a gauge transformation $(\delta\mathtt{t}, \delta\mathtt{x})$ such that the transformed perturbations $\widetilde{\delta\beta}$ and $\widetilde{\delta\alpha}$ vanishes. Indeed, using Eq. (4.19) with $(\delta\mathtt{t}, \delta\mathtt{x})$ in place of $(\delta t, \delta x)$ and with $(\mathtt{t}, \mathtt{x})$ in place of (t, x) (recalling that $\alpha = 1$, and then $\alpha' = 0$), $\delta\alpha$ is sent to zero if the transformation $\delta\mathtt{t}$ is defined as

$$\delta\mathtt{t}(\tilde{\mathtt{t}}, \tilde{\mathtt{x}}) = \int_{\tilde{\mathtt{t}}_0}^{\tilde{\mathtt{t}}} -\delta\alpha(\mathtt{t}, \tilde{\mathtt{x}})\, d\mathtt{t}\,, \qquad \tilde{\mathtt{t}}_0 \in \mathbb{R}\,; \tag{5.2}$$

moreover, using Eq. (4.20) (again with $(\delta\mathtt{t}, \delta\mathtt{x})$ and $(\mathtt{t}, \mathtt{x})$ in place, respectively, of $(\delta t, \delta x)$ and (t, x)), one can make the transformed perturbation $\widetilde{\delta\beta}$ equal to zero by setting [28]

$$\delta\mathtt{x}(\tilde{\mathtt{t}}, \tilde{\mathtt{x}}) = \int_{\tilde{\mathtt{t}}_1}^{\tilde{\mathtt{t}}} \left(-\delta\beta(\mathtt{t}, \tilde{\mathtt{x}}) + \delta\mathtt{t}'(\mathtt{t}, \tilde{\mathtt{x}})\right) d\mathtt{t}\,, \qquad \tilde{\mathtt{t}}_1 \in \mathbb{R} \tag{5.3}$$

with $\delta\mathtt{t}$ as in Eq. (5.2).
Hence, throughout the present section we consider the coordinates $(\tilde{\mathtt{t}}, \tilde{\mathtt{x}})$ that for the sake of intelligibility, we will denote simply with $(\mathtt{t}, \mathtt{x})$; this is equivalent to assume that

$$\delta\alpha := 0 \qquad \delta\beta := 0\,. \tag{5.4}$$

In addition, just to simplify the subsequent calculations, we introduce three smooth dimensionless functions $\Gamma, \mathcal{R}, \Psi : \mathbb{R}^2 \to \mathbb{R}$ and set

$$\delta\gamma := \Gamma(\mathtt{t}, \mathtt{x})\,, \quad \delta r := \rho^{\frac{5-3d}{2}}(\mathtt{x})\mathcal{R}(\mathtt{t}, \mathtt{x})\,, \quad \delta\Phi := \sqrt{\frac{(d-1)(d-2)}{\kappa}}\,\Psi(\mathtt{t}, \mathtt{x})\,, \tag{5.5}$$

where ρ is the function defined in Eq. (3.31).

[28] The position (5.3) is equivalent to the position (4.24) expressed in the coordinates $(\mathtt{t}, \mathtt{x})$ in place of (t, x) and with $\frac{\gamma^2}{\alpha^2} = 1$.

5.1.2 Field equations and the linearization of the scalar curvature

Let us substitute the expressions (3.34,5.4,5.5) into the linearized Einstein equations (4.25-4.28) with the coordinates (t, x) replaced by $(\mathtt{t}, \mathtt{x})$: whenever we meet the derivatives ρ', ρ'', ϕ' we express them via Eqs. (3.14,3.15). We get

$$\begin{aligned}\delta\mathcal{E}_{00} =0\,, \quad \delta\mathcal{E}_{00} =& \frac{d-1}{\rho^{d-1}}\Bigg\{(d-2)\rho^{d-3}\Gamma + \rho^{d-2}\,\mathrm{sign}\sqrt{1-\frac{1}{\rho^{2(d-2)}}}\Gamma' \\ &- \frac{3d-5}{4\rho^{\frac{5(d-1)}{2}}}\Big[(d+1)\rho^{2(d-2)} - 3(d-3)\Big]\mathcal{R} \\ &+ \frac{2d-3}{\rho^{\frac{d+1}{2}}}\,\mathrm{sign}\sqrt{1-\frac{1}{\rho^{2(d-2)}}}\mathcal{R}' - \frac{\mathcal{R}''}{\rho^{\frac{d-1}{2}}} + (d-2)\Psi'\Bigg\}\,,\end{aligned} \tag{5.6}$$

$$\begin{aligned}\delta\mathcal{E}_{01} =0\,, \quad \delta\mathcal{E}_{01} =& \frac{d-1}{\rho^{d-1}}\Bigg\{\rho^{d-2}\,\mathrm{sign}\sqrt{1-\frac{1}{\rho^{2(d-2)}}}\,\dot{\Gamma} \\ &+ \frac{3d-5}{2\rho^{\frac{d+1}{2}}}\,\mathrm{sign}\sqrt{1-\frac{1}{\rho^{2(d-2)}}}\dot{\mathcal{R}} - \frac{\dot{\mathcal{R}}'}{\rho^{\frac{d-1}{2}}} + (d-2)\dot{\Psi}\Bigg\}\,,\end{aligned} \tag{5.7}$$

$$\begin{aligned}\delta\mathcal{E}_{11} =0\,, \quad \delta\mathcal{E}_{11} =& -\frac{(d-1)(d-2)}{\rho^2}\Bigg\{\Gamma - \frac{d-1}{2\rho^{\frac{7d-11}{2}}}\Big[3 - \frac{3d-5}{d-1}\rho^{2(d-2)}\Big]\mathcal{R} \\ &- \frac{1}{\rho^{\frac{3d-5}{2}}}\,\mathrm{sign}\sqrt{1-\frac{1}{\rho^{2(d-2)}}}\mathcal{R}' + \frac{\ddot{\mathcal{R}}}{(d-2)\rho^{\frac{3d-7}{2}}} - \frac{\Psi'}{\rho^{d-3}}\Bigg\}\,,\end{aligned} \tag{5.8}$$

$$\begin{aligned}\delta\mathcal{E}_{a} =0\,, \qquad \mathcal{E}_{a} =& -(d-2)\Bigg\{(d-3)\Gamma + \rho\,\mathrm{sign}\sqrt{1-\frac{1}{\rho^{2(d-2)}}}\Gamma' + \frac{\rho^2}{d-2}\ddot{\Gamma} \\ &+ \frac{d-3}{4\rho^{\frac{7d-11}{2}}}\Big[9(d-1)^2 - (d+3)(3d-5)\rho^{2(d-2)}\Big]\mathcal{R} \\ &+ \frac{2(d-1)}{\rho^{\frac{3d-5}{2}}}\,\mathrm{sign}\sqrt{1-\frac{1}{\rho^{2(d-2)}}}\mathcal{R}' + \frac{(\ddot{\mathcal{R}} - \mathcal{R}'')}{\rho^{\frac{3d-7}{2}}} + \frac{d-1}{\rho^{d-3}}\Psi'\Bigg\}\,.\end{aligned} \tag{5.9}$$

For future use, we also write down the first order expansion of the scalar curvature (4.35,4.36,4.37), which is as follows (recall that in the gauge (5.4) we have that $\delta R^\beta = 0$):

$$R^{(1)}(\epsilon) = R_0 + \epsilon\, \delta R\,, \tag{5.10}$$

where

$$R_0 = -\frac{(d-1)(d-2)}{\rho^{2(d-1)}}\,, \tag{5.11}$$

$$\begin{aligned}\delta R =& 2(d-1)\Bigg[\frac{d-2}{\rho^{2(d-1)}}\left[1+\rho^{2(d-2)}\right]\Gamma + \frac{1}{\rho}\,\mathrm{sign}\sqrt{1-\frac{1}{\rho^{2(d-2)}}}\Gamma' + \frac{\ddot{\Gamma}}{d-1} \\ &- \frac{3d-5}{4\rho^{\frac{7d-7}{2}}}\left[(d+1)\rho^{2(d-2)} - 3(d-1)\right]\mathcal{R} + \frac{(2d-3)\mathcal{R}_x}{\rho^{\frac{5d-5}{2}}} + \frac{(\ddot{\mathcal{R}}-\mathcal{R}'')}{\rho^{\frac{3(d-1)}{2}}}\Bigg]\,.\end{aligned} \tag{5.12}$$

5.1.3 Decoupling the field equations

Finding the field perturbation Ψ

Integrating with respect to $\mathtt{t}$, we see that Eq. (5.7) holds if and only if

$$\Psi = \frac{\rho^{d-2}}{d-2}\left[-\,\mathrm{sign}\sqrt{1-\frac{1}{\rho^{2(d-2)}}}\left(\Gamma + \frac{3d-5}{2\rho^{3\frac{d-1}{2}}}\mathcal{R}\right) + \frac{\mathcal{R}'}{\rho^{\frac{3d-5}{2}}}\right] + \mathcal{C}(\mathtt{x})$$

where $\mathcal{C} : \mathbb{R} \to \mathbb{R}$ is a smooth function. Inserting this espression for Ψ into Eq. (5.6), we see that the latter holds if and only if $\mathcal{C}$ is constant. Summing up: Eqs. (5.6)(5.7) hold if and only if

$$\begin{aligned}\Psi(\mathtt{t},\mathtt{x}) = \frac{\rho(\mathtt{x})^{d-2}}{d-2}\Bigg[&-\mathrm{sign}(\mathtt{x})\sqrt{1-\frac{1}{\rho(\mathtt{x})^{2(d-2)}}}\left(\Gamma(\mathtt{t},\mathtt{x}) + \frac{3d-5}{2\rho^{3\frac{d-1}{2}}}\mathcal{R}(\mathtt{t},\mathtt{x})\right) \\ &+ \frac{\mathcal{R}'(\mathtt{t},\mathtt{x})}{\rho^{\frac{3d-5}{2}}}\Bigg] + C\end{aligned} \tag{5.13}$$

where $C \in \mathbb{R}$ is a constant. The value of C is immaterial since Ψ appears in the linearized equations (5.6-5.9) only through its derivatives as the same can be said for Φ in the exact equations (2.73-2.76).

The perturbations Γ and Ψ as functions of $\mathcal{R}$

Now we are left with Eqs. (5.8,5.9); $\mathcal{E}_{11}$ and $\mathcal{E}_a$ are reduced, after substituting the expression (5.13) for Ψ, to the expressions

$$
\begin{aligned}
\delta\mathcal{E}_{11} = &-(d-1)\Bigg\{\frac{2(d-2)}{\rho^2}\Gamma + \frac{1}{\rho}\,\mathrm{sign}\sqrt{1-\frac{1}{\rho^{2(d-2)}}}\Gamma' \\
&+ \frac{1}{4\rho^{\frac{7(d-1)}{2}}}\Big[3(d-1)^2 + (3d-5)(d-5)\rho^{2(d-2)}\Big]\mathcal{R} \\
&+ \frac{d-1}{\rho^{\frac{3d-1}{2}}}\,\mathrm{sign}\sqrt{1-\frac{1}{\rho^{2(d-2)}}}\mathcal{R}' - \frac{\mathcal{R}''}{\rho^{\frac{3(d-1)}{2}}} + \frac{\ddot{\mathcal{R}}}{\rho^{\frac{3(d-1)}{2}}}\Bigg\}, \qquad (5.14)\\
\delta\mathcal{E}_a = &-\frac{1}{\rho^2}\Bigg\{-\frac{2(d-2)}{\rho^2}\Gamma - \frac{1}{\rho}\,\mathrm{sign}\sqrt{1-\frac{1}{\rho^{2(d-2)}}}\Gamma' + \ddot{\Gamma} \\
&- \frac{1}{4\rho^{\frac{7(d-1)}{2}}}\Big[3(d-1)^2 + (3d-5)(d-5)\rho^{2(d-2)}\Big]\mathcal{R} \\
&- \frac{d-1}{\rho^{\frac{3d-1}{2}}}\,\mathrm{sign}\sqrt{1-\frac{1}{\rho^{2(d-2)}}}\mathcal{R}' - \frac{\mathcal{R}''}{\rho^{\frac{3(d-1)}{2}}} + (d-2)\frac{\ddot{\mathcal{R}}}{\rho^{\frac{3(d-1)}{2}}}\Bigg\}. \qquad (5.15)
\end{aligned}
$$

Evidently, the system $\delta\mathcal{E}_{11} = 0$, $\delta\mathcal{E}_a = 0$ is equivalent to the system formed by $\delta\mathcal{E}_{11} = 0$ and $-\frac{\delta\mathcal{E}_{11}}{d-1} - \frac{\delta\mathcal{E}_a}{\rho^2} = 0$; the latter combination, which reads

$$\ddot{\Gamma} + \frac{d-1}{\rho^{\frac{3(d-1)}{2}}}\ddot{\mathcal{R}} = 0\,, \qquad (5.16)$$

can be integrated twice in time $\mathtt{t}$, leading to

$$\Gamma(\mathtt{t},\mathtt{x}) = -\frac{d-1}{\rho^{\frac{3(d-1)}{2}}(\mathtt{x})}\mathcal{R}(\mathtt{t},\mathtt{x}) + \mathcal{P}_0(\mathtt{x}) + \mathtt{t}\,\mathcal{P}_1(\mathtt{x}) \qquad (5.17)$$

where $\mathcal{P}_0, \mathcal{P}_1 : \mathbb{R} \to \mathbb{R}$ are smooth integration functions; these are closely related to the set of initial data

$$
\begin{aligned}
\Gamma_0(\mathtt{x}) &:= \Gamma(0,\mathtt{x})\,, \qquad & \Gamma_1(\mathtt{x}) &:= \dot{\Gamma}(0,\mathtt{x})\,,\\
\mathcal{R}_0(\mathtt{x}) &:= \mathcal{R}(0,\mathtt{x})\,, \qquad & \mathcal{R}_1(\mathtt{x}) &:= \dot{\mathcal{R}}(0,\mathtt{x})\,,
\end{aligned} \qquad (5.18)
$$

since (5.17) and its $\mathtt{t}$ derivative imply (once evaluated in $\mathtt{t} = 0$)

$$\mathcal{P}_i(\mathtt{x}) = \Gamma_i(\mathtt{x}) + \frac{d-1}{\rho^{\frac{3(d-1)}{2}}(\mathtt{x})}\mathcal{R}_i(\mathtt{x}) \qquad (i = 0,1)\,. \qquad (5.19)$$

Returning to Eq. (5.13) for Ψ, and substituting therein Eq. (5.17) for Γ, we obtain for the field perturbation the final expression

$$\Psi(\mathtt{t},\mathtt{x}) = \frac{1}{d-2}\Bigg[-\frac{d-3}{2\rho^{\frac{d+1}{2}}(\mathtt{x})}\,\mathrm{sign}(\mathtt{x})\sqrt{1-\frac{1}{\rho^{2(d-2)}(\mathtt{x})}}\mathcal{R}(\mathtt{t},x) + \frac{\mathcal{R}'(\mathtt{t},\mathtt{x})}{\rho^{\frac{d-1}{2}}(\mathtt{x})} -\rho^{d-2}(\mathtt{x})\,\mathrm{sign}(\mathtt{x})\sqrt{1-\frac{1}{\rho^{2(d-2)}(\mathtt{x})}}\Big(\mathcal{P}_0(\mathtt{x})+\mathtt{t}\,\mathcal{P}_1(\mathtt{x})\Big)\Bigg] + \mathcal{C} \quad (5.20)$$

A master equation for the radial perturbation $\mathcal{R}$

We finally substitute the expressions (5.17,5.20) for Γ and Ψ into Eq. (5.14); the equation obtained in this way holds if and only if the following wave equation holds

$$\left[\frac{\partial^2}{\partial \mathtt{t}^2} - \frac{\partial^2}{\partial \mathtt{x}^2} + \mathcal{V}\right]\mathcal{R} = \mathcal{J}_0 + \mathtt{t}\,\mathcal{J}_1\,, \quad (5.21)$$

where

$$\mathcal{V}(\mathtt{x}) \equiv \mathcal{V}_{d,b}(\mathtt{x}) := \frac{1}{4\rho^2(\mathtt{x})}\left[(d-3)(d-5) - \frac{3(d-1)^2}{\rho^{2(d-2)}(\mathtt{x})}\right] \qquad (x\in\mathbb{R})\,, \quad (5.22)$$

$$\mathcal{J}_i(\mathtt{x}) := -\,\rho^{\frac{7-3d}{2}}(\mathtt{x})\left[2(d-2)\,\mathcal{P}_i(\mathtt{x}) + \rho(\mathtt{x})\,\mathrm{sign}(\mathtt{x})\sqrt{1-\frac{1}{\rho^{2(d-2)}(\mathtt{x})}}\,\mathcal{P}_i'(\mathtt{x})\right] \qquad (i=0,1)\,. \quad (5.23)$$

Eq. (5.21) is our master equation for the linear perturbation analysis of the Torii-Shinkai wormhole: this is a wave-type equation for $\mathcal{R}$ with the potential $\mathcal{V}$ and the source term $\mathcal{J}_0(x) + \mathtt{t}\,\mathcal{J}_1(x)$.

Remark 38 The functions $\mathcal{J}_i$ are fully determined by the functions $\mathcal{P}_i$ or, due to Eq. (5.19), by the initial data Γ_i, $\mathcal{R}_i$ $(i=0,1)$:

$$\mathcal{J}_i(\mathtt{x}) = -\,2(d-2)\rho^{\frac{3d-7}{2}}(\mathtt{x})\Gamma_i(\mathtt{x}) - \frac{d-1}{\rho^{2(d-1)}(\mathtt{x})}\left[3(d-1) + (d-5)\rho^{2(2d-1)}(\mathtt{x})\right]\mathcal{R}_i(\mathtt{x}) - \frac{1}{\rho(\mathtt{x})}\,\mathrm{sign}(\mathtt{x})\sqrt{1-\frac{1}{\rho^{2(d-2)}(\mathtt{x})}}\left(\rho^{\frac{3(d-1)}{2}}(\mathtt{x})\Gamma_i'(\mathtt{x}) + (d-1)\mathcal{R}_i'(\mathtt{x})\right). \quad (5.24)$$

Remark 39 For future use, we note that the potential $\mathcal{V}$ in Eq. (5.22) is an even function, with the asymptotics

$$\mathcal{V}(\mathtt{x}) = (d-3)(d-5)\left(\frac{1}{4|\mathtt{x}|^2} + \frac{C_d}{2|\mathtt{x}|^3}\right) + O\left(\frac{1}{|\mathtt{x}|^4}\right) \quad \text{for } \mathtt{x} \to \pm\infty \quad (5.25)$$

(this follows from the fact that ρ is even and from Eq. (3.29) about the large $|\mathtt{x}|$ asymptotics of this function; recall that C_d is defined by Eq.(3.22)).

5.1.4 Solution of the master equation and linear instability of the Torii-Shinkai wormhole - gauge-dependent formulation

For every $d \geq 3$ and every $b > 0$, let us consider the master equation (5.21), which contains the potential (5.22) and the source term $\mathcal{J}_0 + \mathtt{t}\mathcal{J}_1$, where $\mathcal{J}_i$ are defined in Eq. (5.23) for $i = 0, 1$; the master equation can be rewritten as

$$\ddot{\mathcal{R}}(\mathtt{t}) + H\mathcal{R}(\mathtt{t}) = \mathcal{J}_0 + \mathtt{t}\,\mathcal{J}_1 \qquad (\mathtt{t} \in \mathbb{R}), \quad (5.26)$$

where

$$H := -\frac{d^2}{d\mathtt{x}^2} + \mathcal{V} \qquad (\mathcal{V} \equiv \mathcal{V}(\mathtt{x}) \text{ as in Eq. (5.22)}) \quad (5.27)$$

is, formally, a Schrödinger type operator in space dimension 1 with potential $\mathcal{V}$; the unknown of Eq. (5.26) is a function

$$\mathcal{R}(\mathtt{t}) \equiv \mathcal{R}(\mathtt{t}, \cdot) \,:\, \mathtt{x} \mapsto \mathcal{R}(\mathtt{t}, \mathtt{x}) \quad \text{for every } \mathtt{t} \in \mathbb{R}\,.$$

Remark 40 If we want a rigorous functional setting for Eq. (5.26), we are led to consider the Hilbert space

$$\mathfrak{H} := L^2(\mathbb{R}, d\mathtt{x}) \quad (5.28)$$

made of complex valued, square integrable functions on $\mathbb{R}$, for the measure $d\mathtt{x}$ with its inner product $\langle\ |\ \rangle$ and the associated norm $\|\ \|$. [(29)] H can be regarded as a selfadjoint operator in $\mathfrak{H}$, if we give for it the precise definition

$$H := -\frac{d^2}{d\mathtt{x}^2} + \mathcal{V} \,:\, \mathfrak{D} \subset \mathfrak{H} \to \mathfrak{H}\,, \qquad \mathfrak{D} := \{f \in \mathfrak{H} \mid f_{\mathtt{xx}} \in \mathfrak{H}\}\,, \quad (5.29)$$

intending all $\mathtt{x}$-derivatives in the distributional sense. [(30)]
In Appendix D.1 we prove the following facts which are valid for every $d \geq 3$ and every $b > 0$:

[29] For more details, see Remark 84 in Appendix C.
[30] See Footnote 67 in Appendix D.

(i) H possesses a point spectrum consisting of a unique, simple eigenvalue $\mu_1 < 0$;

(ii) H possesses a continuous spectrum which coincides with $[0, +\infty)$.

In addition, in Appendix E.1 we show that it is possible to built a generalized orthonormal basis of the Hilbert space $\mathfrak{H}$, using

(i) a normalized eigenfunction e_1 for the eigenvalue $\mu_1 < 0$, i.e.

$$e_1 \in \mathfrak{D} \quad : \quad \|e_1\| = 1\,, \quad He_1 = \mu_1 e_1$$

(e_1 is proved to be $C^\infty(\mathbb{R})$);

(ii) two suitably chosen and linearly independent "improper eigenfunctions" $e_{i\lambda}$ $(i = 1, 2)$ for each $\lambda \in (0, +\infty)$, i.e.,

$$e_{i\lambda} \in C^\infty(\mathbb{R}) \setminus \mathfrak{D} \quad : \quad He_{i\lambda} = \lambda e_{i\lambda} \qquad (i = 1, 2\,;\ \lambda > 0)\,.$$

Now, we can search the solution $\mathcal{R}(\mathtt{t})$ of the master equation (5.26) with appropriate smoothness properties and with the initial conditions given in Eq. (5.18), that is

$$\mathcal{R}(0) = \mathcal{R}_0\,, \qquad \dot{\mathcal{R}}(0) = \mathcal{R}_1\,, \tag{5.30}$$

where

$$\mathcal{R}_i\,:\,\mathbb{R} \to \mathbb{R}\,, \qquad \mathcal{R}_i : \mathtt{x} \mapsto \mathcal{R}_i(\mathtt{x}) \qquad (i = 0, 1)$$

are two functions with an appropriate regularity.
For all technical details, one can refer to Appendix E.1.2; therein, we introduce the space $\mathcal{E}(\mathbb{R}, \mathbb{R})$ [Eq. (E.30)] and we show that (see Proposition 18), for any initial data such that

$$\mathcal{R}_j, \Gamma_j \in C^\infty(\mathbb{R}, \mathbb{R}) \quad : \quad \mathcal{R}_j, \mathcal{J}_j \in \mathcal{E}(\mathbb{R}, \mathbb{R}) \quad \text{for } j = 0, 1$$

($\mathcal{J}_i$'s are defined by Eqs. (5.19,5.23)), the linearized Einstein equations (5.6-5.9) has a unique solution $(\mathcal{R}(\mathtt{t}, \mathtt{x}), \Gamma(\mathtt{t}, \mathtt{x}), \Phi(\mathtt{t}, \mathtt{x}))$, defined for every $(\mathtt{t}, \mathtt{x}) \in \mathbb{R}^2$, such that:

$$\mathcal{R}(\mathtt{t}, \mathtt{x}), \Gamma(\mathtt{t}, \mathtt{x}), \Phi(\mathtt{t}, \mathtt{x}) \in C^\infty(\mathbb{R}^2, \mathbb{R})\,, \qquad \mathcal{R}(\mathtt{t}, \mathtt{x}) \equiv \mathcal{R}(\mathtt{t}) \in C^\infty(\mathbb{R}, \mathcal{E}(\mathbb{R}, \mathbb{R}))\,;$$

the functions Γ, Φ can be expressed in terms of the function $\mathcal{R}$ via Eqs. (5.17,5.20,5.19), while the function $\mathcal{R}(\mathtt{t})$, for all $\mathtt{t} \in \mathbb{R}$, can be decomposed by means of the previously mentioned generalized orthonormal basis as

$$\begin{aligned}\mathcal{R}(\mathtt{t}) = &\Bigg[\langle e_1|\mathcal{R}_0\rangle \cosh(|\mu_1|^{1/2}\mathtt{t}) + \langle e_1|\mathcal{R}_1\rangle \frac{\sinh(|\mu_1|^{1/2}\mathtt{t})}{|\mu_1|^{1/2}} \\ &+ \langle e_1|\mathcal{J}_0\rangle \frac{\cosh(|\mu_1|^{1/2}\mathtt{t} - 1}{|\mu_1|} + \langle e_1|\mathcal{J}_1\rangle \frac{\sinh(|\mu_1|^{1/2}\mathtt{t}) - |\mu_1|^{1/2}\mathtt{t}}{|\mu_1|^{3/2}}\Bigg] e_1 \\ &+ \sum_{i=1,2} \int_0^{+\infty} \Bigg[\langle e_{i\lambda}|\mathcal{R}_0\rangle \cos(\lambda^{1/2} t) + \langle e_{i\lambda}|\mathcal{R}_1\rangle \frac{\sin(\lambda^{1/2}\mathtt{t})}{\lambda^{1/2}} \\ &+ \langle e_{i\lambda}|\mathcal{J}_0\rangle \frac{1 - \cos(\lambda^{1/2}\mathtt{t})}{\lambda} + \langle e_{i\lambda}|\mathcal{J}_1\rangle \frac{\lambda^{1/2}\mathtt{t} - \sin(\lambda^{1/2}\mathtt{t})}{\lambda^{3/2}}\Bigg] e_{i\lambda}\, d\lambda\,, \end{aligned} \tag{5.31}$$

As explained in Remark 100, the symbols $\langle\cdot|\cdot\rangle$ in the above formula indicate usual inner products in $\mathfrak{H}$, or suitably defined generalizations, while the integrals over λ are understood in a weak sense.

Remark 41 Let us now choose the initial data

$$\mathcal{R}_0(\mathtt{x}) := e_1(\mathtt{x})\,, \quad \mathcal{R}_1(\mathtt{x}) := 0\,, \quad \Gamma_0(\mathtt{x}) = -\frac{(d-1)e_1(\mathtt{x})}{\rho(\mathtt{x})^{\frac{3}{2}(d-1)}}\,, \quad \Gamma_1(\mathtt{x}) := 0\,.$$

Then, from Eqs. (5.19,5.23) we have

$$\mathcal{P}_i = 0 \quad \text{and} \quad \mathcal{J}_i = 0 \qquad (i = 0, 1)\,,$$

while, from the orthonormality of the basis [Eq. (C.4)] we have

$$\langle e_1|\mathcal{R}_0\rangle = 1 \quad \text{and} \quad \langle e_{i\lambda}|\mathcal{R}_0\rangle = 0 \qquad (i = 0, 1)\,.$$

From here and from Eqs. (5.31,5.17,5.20) we have that the solution of the linearized system reads

$$\begin{aligned} \mathcal{R}(\mathtt{t},\mathtt{x}) &= \cosh(|\mu_1|^{1/2}\mathtt{t}) e_1(\mathtt{x})\,, \qquad \Gamma(\mathtt{t},\mathtt{x}) = -\frac{(d-1)\mathcal{R}(\mathtt{t},\mathtt{x})}{\rho(\mathtt{x})^{\frac{3}{2}(d-1)}}\,, \\ \Phi(\mathtt{t},\mathtt{x}) &= -\frac{(d-3)\mathrm{sign}(\mathtt{x})\mathcal{R}(\mathtt{t},\mathtt{x})}{2(d-2)\rho(\mathtt{x})^{\frac{1}{2}(d+1)}} \sqrt{1 - \frac{1}{\rho(\mathtt{x})^{2(d-2)}}} + \frac{\mathcal{R}'(\mathtt{t},\mathtt{x})}{(d-2)\rho(\mathtt{x})^{\frac{1}{2}(d-1)}} + \mathcal{C}\,. \end{aligned} \tag{5.32}$$

Clearly, the solution (5.32) diverges exponentially for $\mathtt{t} \mapsto \pm\infty$; however the functions $(\mathcal{R}(\mathtt{t},\mathtt{x}), \Gamma(\mathtt{t},\mathtt{x}), \Phi(\mathtt{t},\mathtt{x}))$ obviously do depend on the gauge chosen, hence their divergence does not allow to infer a linear instability result for the Torii-Shinkai wormhole. Indeed, as already mentioned at the beginning of this chapter in order to prove the linear instability, one has to prove that the solution (5.32) corresponds to a perturbed spacetime which becomes singular as the timelike coordinate $\mathtt{t}$ approaches infinity. To check this fact, in the rest of the present section we will verify that the perturbed spacetime tends to be singular as $\mathtt{t} \to \pm\infty$ by showing that the linearized scalar curvature of the spacetime blows at the throat, independently on the gauge chosen.

Remark 42 Substituting Eq. (5.32) into Eqs. (5.10-5.12) (and using the relation $He_1 = -\mu_1 e_1$, i.e., $e_1''(\mathtt{x}) = (-\mu_1 + \mathcal{V}(\mathtt{x}))e_1(\mathtt{x})$) we have that the linearization of the scalar curvature reads

$$R^{(1)}(\epsilon) = -\frac{(d-1)(d-2)}{\rho(\mathtt{x})^{2(d-1)}} + \Big[(d-1)(d-2)\mathcal{K}(\mathtt{x})\cosh(|\mu_1|^{1/2}\mathtt{t})\Big]\epsilon\,, \quad (5.33)$$

$$\begin{aligned}\mathcal{K}(\mathtt{x}) :=& \frac{1}{\rho(\mathtt{x})^{\frac{7}{2}(d-1)}}\Bigg[\left(d-1-(d-3)\rho(\mathtt{x})^{2(d-2)} + \frac{2\mu_1\rho(\mathtt{x})^{2(d-1)}}{d-2}\right)e_1(\mathtt{x}) \\ &+ 2\rho(\mathtt{x})^{2d-3}\mathrm{sign}(\mathtt{x})\sqrt{1-\frac{1}{\rho(\mathtt{x})^{2(d-2)}}}\,e_1'(\mathtt{x})\Bigg]\,. \end{aligned} \quad (5.34)$$

For any fixed value of $\epsilon > 0$ the linearization of the scalar curvature (5.33) diverges as $\mathtt{t} \to \pm\infty$; in particular, $R^{(1)}(\epsilon) \to \infty$ in correspondence of the throat $\mathtt{x} = 0$. In order to see this fact it is sufficient to show that the function $\mathcal{K}$ does not vanish at $\mathtt{x} = 0$, that is

$$\mathcal{K}(0) = \frac{2}{d-2}\,(d-2+\mu_1)\,e_1(0) \neq 0\,. \quad (5.35)$$

Let us prove Eq. (5.35). The estimate in Remark 92 of Appendix D.1.2 for the eigenvalue μ_1 gives the numerical evidence that $\mu_1 < 2-d$ for every $d \geq 3$, from which $d-2+\mu_1 \neq 0$. Moreover, let us show that $e_1(0) \neq 0$. To this purpose, we recall that e_1 is an even function (see item (iii)(a) of Appendix D.1.1), whence $e_1'(0) = 0$: if it were also $e_1(0) = 0$, making obvious considerations on the initial value problem for the differential equation $e_1'' = (-\mu_1 + \mathcal{V}(\mathtt{x}))e_1$, we could infer $e_1(\mathtt{x}) = 0$ for all $\mathtt{x} \in \mathbb{R}$, which is clearly impossible.

Remark 43 At the very beginning of the present chapter, we have mentioned that the divergence of the linearization of the scalar curvature $R^1(\epsilon)$ (that is, of the coefficient of ϵ in Eq. (5.33)) could be an artifact that can be eliminated by an everywhere smooth infinitesimal coordinate change ϕ_ϵ [Eq.

(4.6)]. However, in Appendix B.2, we show the following general fact (see Proposition 14): given a coordinate system (t, x) and a linearized function

$$R_0(t, x) + \epsilon\, \delta R(t, x) \tag{5.36}$$

such that:

(a) $R_0(t, x) \equiv R_0(x)$, that is, the term of the zeroth order in ϵ is t-independent;

(b) $\delta R(t, x_0) \to \infty$ as $t \to \infty$ for a given x_0, that is, the linearization (5.36) diverges in the large t limit at $x = x_0$;

(c) $R_0'(x_0) = 0$;

then, the linearization (5.36) written in any coordinate system $(\tilde{t}, \tilde{x})$ ϵ-close to the coordinate system (t, x) still diverges in $\tilde{x} = x_0$ as $\tilde{t} \to \infty$.
In conclusion, we can say that the linearized scalar curvature $R^{(1)}(\epsilon)$ diverges at the spacetime points such that $\mathtt{x} = 0$ if it satisfies items (a-c): items (a) is trivially satisfied and we have already verified that (b) holds for $\mathtt{x} = 0$; in addition, we have that

$$\frac{\partial}{\partial x}\left[-\frac{(d-1)(d-2)}{\rho(\mathtt{x})^{2(d-1)}}\right] = \frac{2(d-1)^2(d-2)\mathrm{sign}(\mathtt{x})}{\rho(\mathtt{x})^{2d-1}}\sqrt{1-\frac{1}{\rho(\mathtt{x})^{2(d-2)}}}$$

which clearly vanishes in $\mathtt{x} = 0$, so that the condition (c) is satisfied.
Hence, we have proved that for $\mathtt{t} \to \pm\infty$ the perturbed spacetime becomes singular *at least* in correspondence of spacetime points such that $\mathtt{x} = 0$, which are clearly reachable by geodesics of the wormhole in finite proper time.

The previous three remarks are the proof of the following

Theorem 5 (Linear instability of the Torii-Shikai wormhole - gauge-dependent version)
For all $d \geq 3$ and for all $b > 0$, the Torii-Shinkai wormhole is linearly unstable under small spherically symmetric perturbations of its metric and the associated scalar field; more precisely, for some special initial data of the perturbation functions, the perturbed spacetime becomes singular as the temporal coordinate $\mathtt{t}$ goes to $\pm\infty$.

Remark 44 Since the previous result is valid for all $d \geq 3$, in the case $d = 3$, it states the linear instability of the EBMT wormhole; moreover, in this case the gauge-dependent approach used in this section reduced exactly to the deduction given in Ref. [2].

Remark 45 Admittedly, the linear instability of the EBMT and the Torii-Shinkai wormholes have firstly been proved, respectively, in Refs. [25] and [35] using a different approach; a closer comparison between the scheme proposed in this section (and in Ref. [2]) and that adopted in Refs. [25, 35] (and in other subsequent papers on the same subject) will be performed in Chapter 8.

5.2 Gauge-dependent linear stability analysis of the AdS wormhole

5.2.1 Gauge choice

We start introducing a general perturbation of the AdS wormhole solution, that is, the metric (2.24) and the scalar field defined by Eqs. (4.3,4.4) with (s,u) in place of (t,x), where the static solution (5.1) is given by Eq. (3.62); we now show that in this case it is always possible to provide a gauge transformation $(\delta s, \delta u)$ such that the transformed perturbations $\tilde{\delta\beta}, \tilde{\delta\alpha}, \tilde{\delta r}$ satisfy

$$\tilde{\delta\beta} = 0\,, \qquad \alpha\,\tilde{\delta\alpha} = \frac{1}{4(1+B^2)}\,r\,\tilde{\delta r}\,. \tag{5.37}$$

From Eqs. (4.19,4.22) with (s,u) and $(\delta s, \delta u)$ in place, respectively, of (t,x) and $(\delta t, \delta x)$, we have that the first condition in Eq. (5.37) can be satisfied by setting [31]

$$\delta u(\tilde{s},\tilde{u}) = \int_{\tilde{s}_1}^{\tilde{s}} \left(-\delta\beta(s,\tilde{u}) + \delta s'(s,\tilde{u})\right) ds\,, \qquad \tilde{s}_1 \in \mathbb{R}\,, \tag{5.38}$$

while the second condition in Eq. (5.37) is obtained as long as the following equation holds

$$\dot{\delta s} = -\frac{\delta\alpha}{\alpha} + \frac{1}{4(1+B^2)}\,\frac{r\,\delta r}{\alpha^2} + \left(\frac{1}{4(1+B^2)}\,r\,r' - \alpha\alpha'\right)\frac{\delta u}{\alpha^2}\,. \tag{5.39}$$

In general, one has to insert the expression for δu given Eq. (5.38) into Eq. (5.39) and then, hopefully, solve the resulting partial differential equation in the unknown $\delta s(\tilde{s},\tilde{u})$; fortunately, in the AdS case of the coefficient of δu in Eq. (5.39) is zero since one can verify that

$$\alpha^2 = \frac{1}{4k^2(1+B^2)} + \frac{1}{4(1+B^2)}r^2\,,$$

[31] The position (5.38) is equivalent to the positions (4.24,5.3) expressed in the coordinates (s,u) in place, respectively of (t,x) and $(\mathtt{t},\mathtt{x})$, and with $\frac{\gamma^2}{\alpha^2}=1$.

which implies that (deriving both sides with respect to x and dividing by 2)

$$\alpha\,\alpha' = \frac{1}{4(1+B^2)}\,r\,r'\,.$$

Hence, Eq. (5.39) can be trivially solved leading to

$$\delta s(\tilde{s},\tilde{u}) = \int_{\tilde{u}_0}^{\tilde{u}} \left(-\frac{\delta\alpha(\tilde{s},u)}{\alpha(u)} + \frac{1}{4(1+B^2)}\,\frac{r(u)\,\delta r(\tilde{s},u)}{\alpha(u)^2} \right) du\,, \quad \tilde{u}_0 \in \mathbb{R}\,. \tag{5.40}$$

In conclusion, the assumptions (5.37) can be reached by setting the transformation δs as in Eq. (5.40) and, consequently, δu as in Eq. (5.38).
Therefore, throughout the present section we assume the coordinates $(\tilde{s},\tilde{u})$ that, for the sake of clearness, we will keep denoting with (s,u); hence we set

$$\delta\beta = 0\,, \qquad \alpha\,\delta\alpha = \frac{1}{4(1+B^2)}\,r\,\delta r\,. \tag{5.41}$$

In addition, in order to simplify the subsequent calculations, we introduce three smooth dimensionless functions $\Gamma, \mathcal{R}, \Psi : \mathbb{R}^2 \to \mathbb{R}$ and set

$$\begin{aligned} \delta\alpha &:= \frac{b\,\mathcal{R}(s,u)}{2\sqrt{2}(1+B^2)\sqrt{1+2B^2-\cos u}}\,, \qquad & \delta\gamma &:= b^2\,\Gamma(s,u)\,, \\ \delta r &:= \frac{b\,\mathcal{R}(s,u)}{1+2B^2-\cos u}\,, & \delta\Phi &:= \sqrt{\frac{2}{\kappa}}\,\Psi(s,u)\,. \end{aligned} \tag{5.42}$$

Note that the second assumption in Eq. (5.41) is obviously satisfied by the perturbations $\delta\alpha$ and δr given in Eq. (5.42).

5.2.2 Field equations and the linearization of the scalar curvature

We now substitute the expressions (3.62,5.42) into the linearized Einstein equations (4.25-4.28) with the coordinates (t, x) replaced by (s, u). It results

$$\begin{aligned}\delta\mathcal{E}_{00} = 0\,, \quad \delta\mathcal{E}_{00} =& \frac{B}{1+2B^2-\cos u}\Bigg[2\sqrt{B^2+1}\left(\Psi' + \tan\frac{u}{2}\Psi\right)\\ &+ \left(B^2+1\right)\left(4\sin\frac{u}{2}\Gamma' - (\cos u - 3)\sec\frac{u}{2}\Gamma\right)\\ &+ \frac{\sqrt{2}\cos\frac{u}{2}}{\sqrt{1+2B^2-\cos u}}\left(\frac{3\sin u}{1+2B^2-\cos u}\mathcal{R}' - 2\mathcal{R}''\right)\\ &+ \Bigg(\frac{24B^4+38B^2+6-(8B^4+16B^2+17)\cos u}{4\sqrt{2}\left(1+2B^2-\cos u\right)^{5/2}}\\ &\quad + \frac{10\left(B^2+1\right)\cos(2u)+\cos(3u)}{4\sqrt{2}\left(1+2B^2-\cos u\right)^{5/2}}\Bigg)\sec\frac{u}{2}\mathcal{R}\Bigg]\,,\end{aligned} \tag{5.43}$$

$$\begin{aligned}\delta\mathcal{E}_{01} = 0\,, \quad \delta\mathcal{E}_{01} =& \frac{B}{1+2B^2-\cos u}\Bigg[2\sqrt{B^2+1}\left(2\sqrt{B^2+1}\sin\frac{u}{2}\dot{\Gamma} + \dot{\Psi}\right)\\ &+ \frac{\sqrt{2}}{\sqrt{1+2B^2-\cos u}}\\ &\quad\times\left(\frac{3+2B^2+\cos u}{1+2B^2-\cos u}\sin\frac{u}{2}\dot{\mathcal{R}} - 2\cos\frac{u}{2}\dot{\mathcal{R}}'\right)\Bigg]\,,\end{aligned} \tag{5.44}$$

$$\begin{aligned}\delta\mathcal{E}_{11} = 0\,, \quad \delta\mathcal{E}_{11} =& \frac{B}{1+2B^2-\cos u}\Bigg[2\sqrt{B^2+1}\left(\Psi' - \tan\frac{u}{2}\Psi\right)\\ &+ 2\left(B^2+1\right)(\cos u - 2)\sec\frac{u}{2}\Gamma\\ &+ \frac{2\sqrt{2}}{\sqrt{1+2B^2-\cos u}}\\ &\quad\times\left(\frac{2+3B^2-\cos u}{1+2B^2-\cos u}\sin\frac{u}{2}\mathcal{R}' - \cos\frac{u}{2}\ddot{\mathcal{R}}\right)\\ &+ \Bigg(-\frac{24B^4+44B^2+24-(24B^4+56B^2+25)\cos u}{4\sqrt{2}\left(1+2B^2-\cos u\right)^{5/2}}\\ &\quad + \frac{4B^2\cos(2u)-\cos(3u)}{4\sqrt{2}\left(1+2B^2-\cos u\right)^{5/2}}\Bigg)\sec\frac{u}{2}\mathcal{R}\Bigg]\,,\end{aligned} \tag{5.45}$$

$$
\begin{aligned}
\delta\mathcal{E}_a = 0\,,\quad \delta\mathcal{E}_a =& B\Bigg[-4\sqrt{B^2+1}\left(\Psi' + \tan\frac{u}{2}\Psi\right) + \\
& \frac{\sqrt{2}\cos\frac{u}{2}}{\sqrt{1+2B^2-\cos u}}\Bigg(-2\sqrt{2}\ddot{\Gamma}\left(1+2B^2-\cos u\right)^{3/2} \\
& \qquad + \frac{3+4B^2-\cos u}{B^2+1}\mathcal{R}'' - 2\ddot{\mathcal{R}}\Bigg) \\
& - 2\left(3+4B^2-\cos u\right)\sin\frac{u}{2}\Gamma' \\
& - \frac{1}{4}\Big(24B^2+17-8\left(B^2+2\right)\cos u - \cos(2u)\Big)\sec\frac{u}{2}\Gamma \\
& - \sqrt{2}\frac{9+10B^2-\cos u}{(B^2+1)\left(1+2B^2-\cos u\right)^{3/2}}\sin\frac{u}{2}\cos^2\frac{u}{2}\mathcal{R}' \\
& - \Bigg(\frac{96B^4+136B^2+26-(32B^4+80B^2+65)\cos u}{8\sqrt{2}\left(1+2B^2-\cos u\right)^{5/2}} \\
& \qquad + \frac{(40B^2+38)\cos(2u)+\cos(3u)}{8\sqrt{2}\left(1+2B^2-\cos u\right)^{5/2}}\Bigg)\sec\frac{u}{2}\mathcal{R}\Bigg]\,. \qquad (5.46)
\end{aligned}
$$

In addition, we consider the first order expansion of the scalar curvature (2.39), which, in the AdS case, reads (note that in the gauge (5.41) $\delta R^\beta = 0$):

$$R^{(1)}(\epsilon) = R_0 + \epsilon\, \delta R\,, \tag{5.47}$$

where

$$\begin{aligned} R_0 =& -2k^2\left(\frac{3\,(6B^4+8B^2+3) - 2\,(3B^4+11B^2+6)\cos u}{(1+2B^2-\cos u)^2}\right.\\ &\left.+\frac{(2B^2+3)\cos(2u)}{(1+2B^2-\cos u)^2}\right)\,, \end{aligned} \tag{5.48}$$

$$\begin{aligned} \delta R =& \frac{B^3\cos\frac{u}{2}}{2\,(1+2B^2-\cos u)^2}\left[8\sqrt{2}\cos^2\frac{u}{2}\sqrt{1+2B^2-\cos u}\right.\\ &\times\left(2\sqrt{2}\,(1+2B^2-\cos u)^{3/2}\,\ddot{\Gamma} - \frac{5+6B^2-\cos u}{B^2+1}\mathcal{R}'' + 4\ddot{\mathcal{R}}\right)\\ &+2\sqrt{2}\sin\frac{u}{2}\cos\frac{u}{2}\left(1+2B^2-\cos u\right)\left(4\sqrt{2}\left(5+6B^2-\cos u\right)\Gamma'\right.\\ &\left.-\frac{24B^4+8B^2-13-4\,(10B^2+9)\cos u+\cos(2u)}{(B^2+1)\,(1+2B^2-\cos u)^{3/2}}\mathcal{R}'\right)\\ &+\Big(240B^4+336B^2+118-\left(48B^4+224B^2+145\right)\cos u\\ &+2\left(8B^2+13\right)\cos(2u)+\cos(3u)\Big)\Gamma\\ &+\left(\frac{192B^4+300B^2+86-(96B^4+224B^2+149)\cos u}{\sqrt{2}\,(1+2B^2-\cos u)^{3/2}}\right.\\ &\left.\left.+\frac{(52B^2+58)\cos(2u)+5\cos(3u)}{\sqrt{2}\,(1+2B^2-\cos u)^{3/2}}\right)\mathcal{R}\right]\,. \end{aligned} \tag{5.49}$$

5.2.3 Decoupling the field equations

Finding the field perturbation Ψ

Eq. (5.44) can be integrated in the temporal variable s, yielding

$$\begin{aligned} \Psi = -2\sqrt{1+B^2}\,\sin\frac{u}{2}\,\Gamma + \frac{\sqrt{2}\cos\frac{u}{2}\,\mathcal{R}'}{\sqrt{1+B^2}\sqrt{1+2B^2-\cos u}} +\\ -\frac{\sin\frac{u}{2}\,(3+2B^2+\cos u)\,\mathcal{R}}{\sqrt{2}\sqrt{1+B^2}\,(1+2B^2-\cos u)^{3/2}} + \mathcal{C}(u) \end{aligned}$$

where $\mathcal{C} : \mathbb{R} \to \mathbb{R}$ is a smooth function. Inserting this espression for Ψ into Eq. (5.43), we see that the latter holds if and only if $\mathcal{C}$ is constant. Summing up: Eqs. (5.43,5.44) hold if and only if

$$\Psi(s,u) = -2\sqrt{1+B^2}\,\sin\frac{u}{2}\,\Gamma(s,u) + \frac{\sqrt{2}\cos\frac{u}{2}\,\mathcal{R}'(s,u)}{\sqrt{1+B^2}\sqrt{1+2B^2-\cos u}} + \\ -\frac{\sin\frac{u}{2}\,(3+2B^2+\cos u)\,\mathcal{R}(s,u)}{\sqrt{2}\sqrt{1+B^2}\,(1+2B^2-\cos u)^{3/2}} + \mathcal{C}\,\cos^2\frac{u}{2} \tag{5.50}$$

where $\mathcal{C} \in \mathbb{R}$ is a constant that depends on the initial data.

The perturbations Γ and Ψ as functions of $\mathcal{R}$

Inserting the expression (5.50) for Ψ into the remaining two linearized Einstein equations (5.45,5.46) one obtains $\delta\mathcal{E}_{11} = 0$ and $\delta\mathcal{E}_a = 0$, where

$$\begin{aligned}
\delta\mathcal{E}_{11} =& -\frac{B\,(1+B^2)\,(\cos u+3)\sec\frac{u}{2}\,\Gamma}{1+2B^2-\cos u} - \frac{4B\,(1+B^2)\sin\frac{u}{2}\,\Gamma'}{1+2B^2-\cos u} \\
&- \frac{B\,(1+B^2)\sec\frac{u}{2}\,(1+8B^2-4\cos u+3\cos(2u))\,\mathcal{R}}{2\sqrt{2}\,(1+2B^2-\cos u)^{7/2}} \\
&- \frac{2\sqrt{2}B\cos\frac{u}{2}\,\ddot{\mathcal{R}}}{(1+2B^2-\cos u)^{3/2}} - \frac{\sqrt{2}B\sin\frac{u}{2}\,(3+2B^2+\cos u)\,\mathcal{R}'}{(1+2B^2-\cos u)^{5/2}} \\
&+ \frac{2\sqrt{2}B\cos\frac{u}{2}\mathcal{R}''}{(1+2B^2-\cos u)^{3/2}} - \frac{2\mathcal{C}B\sqrt{1+B^2}\sin u}{1+2B^2-\cos u}\,,
\end{aligned} \tag{5.51}$$

$$\begin{aligned}
\delta\mathcal{E}_a =& B\,(\cos u+3)\cos\frac{u}{2}\,\Gamma + 4B\sin\frac{u}{2}\cos^2\frac{u}{2}\,\Gamma' \\
&- 4B\cos\frac{u}{2}\left(1+2B^2-\cos u\right)\ddot{\Gamma} \\
&+ \frac{B\cos\frac{u}{2}\,(1+8B^2-4\cos u+3\cos(2u))\,\mathcal{R}}{2\sqrt{2}\,(1+2B^2-\cos u)^{5/2}} - \frac{2\sqrt{2}B\cos\frac{u}{2}\ddot{\mathcal{R}}}{\sqrt{1+2B^2-\cos u}} \\
&+ \frac{\sqrt{2}B\sin\frac{u}{2}\cos^2\frac{u}{2}\,(3+2B^2+\cos u)\,\mathcal{R}'}{(1+B^2)\,(1+2B^2-\cos u)^{3/2}} \\
&- \frac{2\sqrt{2}B\cos^3\left(\frac{u}{2}\right)\mathcal{R}_{uu}}{(1+B^2)\sqrt{1+2B^2-\cos u}}\,.
\end{aligned} \tag{5.52}$$

The system $\delta\mathcal{E}_{11} = 0$ and $\delta\mathcal{E}_a = 0$ is clearly equivalent to the system made up of

$$\delta\mathcal{E}_{11} = 0\,, \qquad \delta\mathcal{E}_{11} + \frac{(1+B^2)\sec^2\frac{u}{2}}{1+2B^2-\cos u}\delta\mathcal{E}_a = 0\,;$$

the latter combination is reduced to

$$-4B\left(1+B^2\right)\sec\frac{u}{2}\ddot{\Gamma}-\frac{\sqrt{2}B\sec\left(\frac{u}{2}\right)\left(3+2B^2+\cos u\right)\ddot{\mathcal{R}}}{\left(1+2B^2-\cos u\right)^{3/2}} \\ -\frac{2\mathcal{C}B\sqrt{1+B^2}\sin u}{1+2B^2-\cos u}=0\,,$$

which can be integrated twice in s, leading to an expression for Γ depending only on $\mathcal{R}$:

$$\Gamma(s,u)=-\frac{\left(3+2B^2+\cos u\right)\mathcal{R}(s,u)}{2\sqrt{2}\left(1+B^2\right)\left(1+2B^2-\cos u\right)^{3/2}} \\ -\frac{\mathcal{C}\sin u\cos\frac{u}{2}\,s^2}{4\sqrt{1+B^2}\left(1+2B^2-\cos u\right)}+\mathcal{P}_0(u)+s\mathcal{P}_1(u)\,, \tag{5.53}$$

where $\mathcal{P}_0,\mathcal{P}_1:\mathbb{R}\to\mathbb{R}$ are smooth integration functions; these are closely related to the set of initial data

$$\begin{aligned}\Gamma_0(u)&:=\Gamma(0,u)\,, &\quad \Gamma_1(u)&:=\dot{\Gamma}(0,u)\,,\\ \mathcal{R}_0(u)&:=\mathcal{R}(0,u)\,, &\quad \mathcal{R}_1(u)&:=\dot{\mathcal{R}}(0,u)\,,\end{aligned} \tag{5.54}$$

since (5.53) and its s derivative imply (once evaluated in $s=0$)

$$\mathcal{P}_i(u)=\Gamma_i(u)+\frac{\left(3+2B^2+\cos x\right)\mathcal{R}_i(u)}{2\sqrt{2}\left(B^2+1\right)\left(1+2B^2-\cos u\right)^{3/2}}\quad(i=0,1)\,. \tag{5.55}$$

Inserting Eq. (5.53) into Eq. (5.50), it is possible to write an expression for the perturbation function Ψ depending only on the function $\mathcal{R}$

$$\Psi(s,u)=\frac{\sqrt{2}\cos\frac{u}{2}\mathcal{R}'}{\sqrt{1+B^2}\sqrt{1+2B^2-\cos u}}+\left[\cos^2\frac{u}{2}+\frac{\sin^2 u\,s^2}{4(1+2B^2-\cos u)}\right]\mathcal{C} \\ -2\sqrt{1+B^2}\sin\frac{u}{2}\big(\mathcal{P}_0(u)+s\,\mathcal{P}_1(u)\big)\,. \tag{5.56}$$

Evaluating the previous expression (5.56) for Ψ in $(s,u)=(0,0)$ and using Eq. (5.54), we have that the constant $\mathcal{C}$ is related to the initial datum $\mathcal{R}_0$ and to the initial datum for the perturbed field

$$\Psi_0:=\Psi(0,0) \tag{5.57}$$

via

$$\mathcal{C}=\Psi_0-\frac{\mathcal{R}_0'(0)}{B\sqrt{1+B^2}}\,. \tag{5.58}$$

A master equation for the radial perturbation $\mathcal{R}$

Inserting the expression (5.53) for the function Γ into equation $\delta\mathcal{E}_{11} = 0$ and dividing both sides by $-\frac{2\sqrt{2}B\cos\frac{u}{2}}{(1+2B^2-\cos u)^{3/2}}$ we get the following wave equation for $\mathcal{R}$

$$\left[\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial u^2} + \mathcal{V}\right]\mathcal{R} = \mathcal{J}_0 + s\,\mathcal{J}_1 + s^2\,\mathcal{J}_2\,, \tag{5.59}$$

where

$$\mathcal{V}(u) \equiv \mathcal{V}_B(u) := -\frac{B^2(2+B^2+\cos u)}{(1+2B^2-\cos u)^2} \qquad (u \in (-\pi,\pi))\,, \tag{5.60}$$

$$\begin{aligned}\mathcal{J}_0(u) :=& -\mathcal{C}\sqrt{2}\sqrt{1+B^2}\sin\frac{u}{2}\sqrt{1+2B^2-\cos u} + \\ & -\frac{1}{2\sqrt{2}}\left(1+B^2\right)\sec^2\frac{u}{2}\sqrt{1+2B^2-\cos u} \\ & \times\left(2\sin u P_0{}'(u) + P_0(u)(\cos u+3)\right)\,,\end{aligned} \tag{5.61}$$

$$\begin{aligned}\mathcal{J}_1(u) :=& -\frac{1}{2\sqrt{2}}\left(1+B^2\right)\sec^2\left(\frac{u}{2}\right)\sqrt{1+2B^2-\cos u} \\ & \times\left(2\sin u P_1{}'(u) + P_1(u)(\cos u+3)\right)\,,\end{aligned} \tag{5.62}$$

$$\mathcal{J}_2(u) := \frac{\mathcal{C}}{4\sqrt{2}}\sqrt{1+B^2}\sin\frac{u}{2}\,\frac{-1+4B^2+2\left(4B^2+1\right)\cos u - \cos(2u)}{\left(1+2B^2-\cos u\right)^{3/2}}\,, \tag{5.63}$$

Eq. (5.59) is our master equation for the linear perturbation analysis of the AdS wormhole: this is a wave-type equation for $\mathcal{R}$ with the potential $\mathcal{V}$ and the source term $\mathcal{J}_0(u) + s\,\mathcal{J}_1(u) + s^2\,\mathcal{J}_2(u)$.

Remark 46 For $i = 0, 1$, the functions $\mathcal{J}_i$ are fully determined by the functions $\mathcal{P}_i$ or, due to Eq. (5.55), by the initial data Γ_i, $\mathcal{R}_i$, that is, [32]

$$
\begin{aligned}
\mathcal{J}_i(u) = &- \frac{1}{2\sqrt{2}} \left(B^2+1\right)(\cos u + 3)\sec^2\frac{u}{2}\sqrt{1+2B^2-\cos u}\,\Gamma_i(u) \\
&- \sqrt{2}\left(B^2+1\right)\tan\frac{u}{2}\sqrt{1+2B^2-\cos u}\,\Gamma_i'(u) \\
&- \left(\frac{1+19B^2+12B^4+4\left(B^4+2B^2-1\right)\cos u}{8\left(1+2B^2-\cos u\right)^2}\right. \\
&\quad \left. - \frac{\left(5B^2+3\right)\cos(2u)}{8\left(1+2B^2-\cos u\right)^2}\right)\sec^2\frac{u}{2}\mathcal{R}_i(u) \\
&- \frac{\tan\frac{u}{2}\left(3+2B^2+\cos u\right)}{2\left(1+2B^2-\cos u\right)}\mathcal{R}_i'(u) \\
&+ (i-1)C\sqrt{2B^2+2}\sin\frac{u}{2}\sqrt{1+2B^2-\cos u}\,.
\end{aligned}
\tag{5.64}
$$

Remark 47 For the following, we assume for the function $\mathcal{R}$, the Dirichlet boundary conditions at the two asymptotic AdS ends, that is,

$$
\mathcal{R}(s, \pm\pi) = 0 \qquad \text{for every } s \in \mathbb{R}. \tag{5.65}
$$

Indeed, from Eq. (5.42) we have that this is equivalent to set (in the gauge (5.41)) that the perturbation function δr vanishes at the far ends $u = \pm\pi$ of the wormhole for every time $s \in \mathbb{R}$, which is a physically reasonable prescription. For general considerations on boundary conditions for field theories on AdS spaces, see Refs. [50, 51, 52].

5.2.4 Solution of the master equation and linear instability of the AdS wormhole - gauge-dependent formulation

Let us consider the master equation (5.59) defined for every $B > 0$: this equation, which contains the potential (5.60) and the source term $\mathcal{J}_0 + s\mathcal{J}_1 + s^2\mathcal{J}_2$, with $\mathcal{J}_i$ defined by Eqs. (5.61-5.63), can be rewritten as

$$
\ddot{\mathcal{R}}(s) + H\mathcal{R}(s) = \mathcal{J}_0 + s\,\mathcal{J}_1 + s^2\,\mathcal{J}_2 \qquad (s \in \mathbb{R}), \tag{5.66}
$$

where

$$
H := -\frac{d^2}{du^2} + \mathcal{V} \qquad (\mathcal{V} \equiv \mathcal{V}(u) \text{ as in Eq. (5.60)}) \tag{5.67}
$$

[32] The letter i in the last line of Eq. (5.64) does not stand for the imaginary unit, but for the parameter $i = 0, 1$ specifying the function $\mathcal{J}_i$.

is, formally, a one-dimensional Schrödinger type operator with potential $\mathcal{V}$. The unknown of Eq. (5.66) is a function

$$\mathcal{R}(s) \equiv \mathcal{R}(s, \cdot) \ : \ u \mapsto \mathcal{R}(s, u) \quad \text{for every } s \in \mathbb{R} \, .$$

Remark 48 In order to provide a technical precise functional framework for discussing Eq. (5.66) with boundary conditions (5.65), we have to introduce the Hilbert space

$$\mathfrak{H} := L^2((-\pi, \pi), du) \tag{5.68}$$

made of complex valued, square integrable functions on $(-\pi, \pi)$, for the measure du with its inner product $\langle \ | \ \rangle$ and the associated norm $\| \ \|$. (33) In order to regard H as a selfadjoint operator in $\mathfrak{H}$, we have to define it as

$$H := -\frac{d^2}{du^2} + \mathcal{V} \ : \ \mathfrak{D} \subset \mathfrak{H} \to \mathfrak{H} \, , \qquad \mathfrak{D} := \{ f \in \mathfrak{H} \mid f_{uu} \in \mathfrak{H} \ , f(\pm\pi) = 0 \} \, ; \tag{5.69}$$

clearly, the u-derivatives have to be intended in the distributional sense. (34) Moreover, in Appendix D.3 we prove that for every $B > 0$ the operator H has a purely discrete $\{\mu_n\}_{n \in \mathbb{N}}$ with a single negative eigenvalue μ_1 and an increasing sequence of positive eigenvalues $\mu_2 < \mu_3 < \dots$.
In addition, in Appendix E.3 we see that one can built an orthonormal basis of the Hilbert space $\mathfrak{H}$, which is made up of the normalized eigenfunctions $\{e_n\}_{n \in \mathbb{N}}$ of H, i.e.

$$e_n \in \mathfrak{D} \quad : \quad \|e_n\| = 1 \, , \quad H e_n = \mu_n e_n \qquad (n \in \mathbb{N})$$

(e_n is proved to be $C^\infty(\mathbb{R})$ for all $n \in \mathbb{N}$).

Let us look for the solution $\mathcal{R}(s)$ of the master equation (5.66) with appropriate smoothness properties and with the initial conditions given in Eq. (5.54), that is

$$\mathcal{R}(0) = \mathcal{R}_0 \, , \qquad \dot{\mathcal{R}}(0) = \mathcal{R}_1 \, , \tag{5.70}$$

where

$$\mathcal{R}_i \ : \ \mathbb{R} \to \mathbb{R} \, , \qquad \mathcal{R}_i : \mathtt{x} \mapsto \mathcal{R}_i(\mathtt{x}) \qquad (i = 0, 1)$$

are two functions with an appropriate regularity.
For all technical details, one we refer to Appendix E.3.2; one introduced the space $\mathcal{E}$ [Eq. (E.30)] and one can show that (see Proposition 21), for any initial data such that

$$\Psi_0 \in \mathbb{R} \, , \quad \mathcal{R}_j, \Gamma_j \in C^\infty((-\pi, \pi), \mathbb{R}) \ : \ \mathcal{R}_j, \mathcal{J}_j \in \mathcal{E}((-\pi, \pi), \mathbb{R}) \quad \text{for } j = 0, 1$$

[33] For more details, see Remark 84 in Appendix C.

[34] See Footnote 69 in Appendix D.

($\mathcal{J}_i$'s are defined by Eqs. (5.55,5.61-5.62)), the linearized Einstein equations (5.43-5.46) has a unique solution $(\mathcal{R}(s,u),\Gamma(s,u),\Phi(s,u))$, defined for every $(s,u)\in\mathbb{R}\times(-\pi,\pi)$, such that:

$$\mathcal{R}(s,u),\Gamma(s,u),\Phi(s,u)\in C^\infty(\mathbb{R}\times(-\pi,\pi),\mathbb{R})\,,$$
$$\mathcal{R}(s,u)\equiv\mathcal{R}(s)\in C^\infty(\mathbb{R},\mathcal{E}((-\pi,\pi),\mathbb{R}))\,;$$

in addition, the functions Γ,Φ can be expressed in terms of the function $\mathcal{R}$ via Eqs. (5.53,5.56,5.55,5.58), while the function $\mathcal{R}(s)$, for all $s\in\mathbb{R}$, can be decomposed by means of the previously mentioned orthonormal basis as

$$\begin{aligned}\mathcal{R}(s)=&\Bigg[\langle e_1|\mathcal{R}_0\rangle\cosh(|\mu_1|^{1/2}s)+\langle e_1|\mathcal{R}_1\rangle\frac{\sinh(|\mu_1|^{1/2}s)}{|\mu_1|^{1/2}}\\&+\langle e_1\,|\,\mathcal{J}_0\rangle\,\frac{\cosh(|\mu_1|^{1/2}s)-1}{|\mu_1|}+\langle e_1\,|\,\mathcal{J}_1\rangle\,\frac{\sinh(|\mu_1|^{1/2}s)-|\mu_1|^{1/2}s}{|\mu_1|^{3/2}}\\&+\langle e_1\,|\,\mathcal{J}_2\rangle\,\frac{2\cosh(|\mu_1|^{1/2}s)-|\mu_1|s^2-2}{|\mu_1|^2}\Bigg]e_1\\&+\sum_{n=2}^{+\infty}\Bigg[\langle e_n|q\rangle\cos(\mu_n^{1/2}s)+\langle e_n|p\rangle\frac{\sin(\mu_n^{1/2}s)}{\mu_n^{1/2}}\\&+\langle e_n\,|\,\mathcal{J}_0\rangle\,\frac{1-\cos(\mu_n^{1/2}s)}{\mu_n}+\langle e_n\,|\,\mathcal{J}_1\rangle\,\frac{\mu_n^{1/2}s-\sin(\mu_n^{1/2}s)}{\mu_n^{3/2}}\\&+\langle e_n\,|\,\mathcal{J}_2\rangle\,\frac{2\cos(\mu_n^{1/2}s)+\mu_n s^2-2}{\mu_n^2}\Bigg]e_n\,.\end{aligned}\tag{5.71}$$

Remark 49 Let us now choose the initial data

$$\Psi_0:=\frac{e_1'(0)}{B\sqrt{1+B^2}}\,,\qquad\mathcal{R}_0(u):=e_1(u)\,,\qquad\mathcal{R}_1(u):=0\,,$$
$$\Gamma_0(u)=-\frac{3+2B^2+\cos u}{2\sqrt{2}(1+B^2)(1+2B^2-\cos x)^{3/2}}e_1(\mathtt{x})\,,\qquad\Gamma_1(\mathtt{x}):=0\,.$$

Then, from Eqs. (5.55,5.58,5.61-5.63) we have

$$\mathcal{C}=0\,,\quad\mathcal{P}_0=\mathcal{P}_1=0\quad\text{and}\quad\mathcal{J}_0=\mathcal{J}_1=\mathcal{J}_2=0\,,$$

while, from the orthonormality of the basis [Eq. (C.1)] we have

$$\langle e_1|\mathcal{R}_0\rangle=1\quad\text{and}\quad\langle e_n|\mathcal{R}_0\rangle=0\qquad(n=2,3,\ldots)\,.$$

From here and from Eqs. (5.71,5.53,5.56) we have that the solution of the linearized system is

$$\mathcal{R}(s,u) = \cosh(|\mu_1|^{1/2} s) e_1(u)\,, \quad \Gamma(s,u) = -\frac{(3+2B^2+\cos u)\mathcal{R}(s,u)}{2\sqrt{2}(1+B^2)(1+2B^2-\cos u)^{3/2}}\,,$$
$$\Phi(s,u) = \frac{\sqrt{2}\cos\frac{u}{2}\mathcal{R}(s,u)}{\sqrt{1+B^2}\sqrt{1+2B^2-\cos u}}\,. \tag{5.72}$$

Clearly, the solution (5.72) diverges exponentially for $s \mapsto \pm\infty$; as already mentioned in the Torii-Shinkai case (see Remark 41) the functions $\mathcal{R}(s,u)$, $\Gamma(s,u)$, $\Phi(s,u)$ obviously do depend on the gauge chosen, hence their divergence does not allow to infer a linear instability result for the AdS wormhole. In the rest of the present section we will show that the solution (5.72) corresponds to a perturbed spacetime which becomes singular as $s \to \pm\infty$.

Remark 50 Substituting Eq. (5.72) into Eqs. (5.48-5.49) (and using the relation $He_1 = -\mu_1 e_1$, i.e., $e_1''(\mathtt{x}) = (-\mu_1 + \mathcal{V}(\mathtt{x}))e_1(\mathtt{x})$) we have that the linearization of the scalar curvature reads

$$\begin{aligned} R^{(1)}(\epsilon) =& \frac{2k^2}{(1+2B^2-\cos u)^2}\Big[-3\left(3+8B^2+6B^4\right) \\ &+ 2\left(6+11B^2+3B^4\right)\cos u - (3+2B^2)\cos(2u)\Big] \\ &+ \left[\frac{8\sqrt{2}\,k^2\,B\,\cos^2\frac{u}{2}\,\mathcal{K}(u)\,\cosh(|\mu_1|^{1/2}s)}{(1+2B^2-\cos u)^{7/2}}\right]\epsilon\,, \end{aligned} \tag{5.73}$$

$$\begin{aligned} \mathcal{K}(u) :=& \cos\frac{u}{2}\Big[B^2(1+\cos u) + 2(1+2B^2-\cos u)^2\mu_1\Big]e_1(u) \\ &+ \sin\frac{u}{2}\Big[2+5B^2+6B^4-(3+7B^2)\cos u + \cos(2u)\Big]e_1'(u)\,. \end{aligned} \tag{5.74}$$

We note that, even in this case, for any fixed value of $\epsilon > 0$ the linearization of the scalar curvature (5.73) diverges as $s \to \pm\infty$; indeed, for example, $R^{(1)}(\epsilon) \to \infty$ at the throat $u = 0$. One can verify this fact noting that the function $\mathcal{K}$ does not vanish at $u = 0$, that is

$$\mathcal{K}(0) = 8B^4\left(\frac{1}{4B^2} + \mu_1\right)e_1(0) \neq 0\,. \tag{5.75}$$

We now prove the latter equation. From the estimate in Eq. D.47 (Appendix D.3), we have that $\mu_1 \equiv \mu_1(B) \leq \varepsilon(B)$ for each $B < 0$, where the function $\varepsilon(B)$ is defined in Eq. (D.46). It is not difficult to see that $\varepsilon(B) < -\frac{1}{4B^2}$ for every $B > \frac{\sqrt{7-\sqrt{17}}}{4} =: B^*$ ($B^* \simeq 0.424$), whence $\mu_1 < -\frac{1}{4B^2}$ for every $B > B^*$.

Unfortunately, for $0 < B \leq B^*$, we have that $-\frac{3}{4B^2} < -\frac{1}{4B^2} \leq \varepsilon(B)$, so that it can be possible that $\mu_1 \equiv \mu_1(B) = -\frac{1}{4B^2}$ (see Eq. (D.51)). However, some numerical considerations lead to the conclusion that $\mu_1 \equiv \mu_1(B) \neq -\frac{1}{4B^2}$ even for $0 < B \leq B^*$. (35) In addition, it turns out that $e_1(0) \neq 0$; indeed, since the potential $\mathcal{V} \equiv \mathcal{V}_B$ is an even function which trivially satisfies the assumptions (D.4) of Appendix D.1.1, then e_1 must be an even function (see item (iii)(a) of the same Appendix), whence $e_1'(0) = 0$. As already stated at the end of Remark 42, this fact implies that $e_1(0) \neq 0$, since the initial conditions $e_1(0) = e_1'(0) = 0$ for the differential equation $e_1'' = (-\mu_1 + \mathcal{V}(u))e_1$ gives $e_1(u) = 0$ for all $u \in (-\pi, \pi)$, which is clearly impossible.

Remark 51 As already mentioned in Remark 43, one has to check that the divergence of the linearization of the scalar curvature $R^1(\epsilon)$ is not a mirage due to a bad (or good) gauge choice. Fortunately, even in this case the linearized scalar curvature satisfies items (a-b) of Remark 43 at $u = 0$ (with (t, x) replaced by (s, u)); in addition, one can see that the derivative

$$\frac{\partial}{\partial u}\left[\frac{2k^2}{(1+2B^2-\cos u)^2}\Big[-3\left(3+8B^2+6B^4\right) + 2\left(6+11B^2+3B^4\right)\cos u - (3+2B^2)\cos(2u)\Big]\right] = \frac{4\,k^2\,B^2\,(1+B^2)\,(1+6B^2-5\cos u)\,\sin u}{(1+2B^2-\cos u)^3} \tag{5.76}$$

vanishes at $u = 0$, so that even item (c) is satisfied. Therefore, this is sufficient to infer that for $s \to \pm\infty$ the perturbed AdS wormhole becomes singular *at least* in correspondence of spacetime points such that $u = 0$, as therein the linearized scalar curvature diverges and these points are clearly reachable by geodesics of the wormhole in finite proper time.

The results of the previous three remarks are contained in the following

Theorem 6 (Linear instability of the AdS wormhole - gauge-dependant version)
For all $B > 0$ and for all $k > 0$ (or, equivalently, for all $b > 0$ and for all $k > 0$), the AdS wormhole is linearly unstable under small spherically symmetric

[35] The eigenvalue equation $-e_1'' + \mathcal{V}_B(u)e_1 = -\frac{1}{4B^2}e_1$ with initial data $e_1(-\pi) = 0, e_1'(-\pi) = 1$ can be solved numerically using any package for ODEs; since one can see that for any $B \in (0, B^*]$, the solutions $e_1(u; B)$ do not vanish in $u = \pi$, it turns out that $e_1(u; B)$ is not an element of $\mathfrak{D}$, whence $-\frac{1}{4B^2}$ is not an eigenvalue of the operator$H$ for every $B \in (0, B^*]$.

perturbations of its metric and the associated scalar field; more precisely, for some special initial data of the perturbation functions, the perturbed spacetime becomes singular as the temporal coordinate s *goes to* $\pm\infty$.

Remark 52 The deduction of the linear instability of the AdS wormhole presented in this section is alternative end equivalent to the gauge-invariant deduction contained in Ref. [1], which will be reproposed in Section 7.4 of this thesis. The present deduction has been included to provide a second example of the gauge-dependent approach that has been adopted to firstly prove the linear instability of the EBMT wormhole without encountering any singularity [2]. For a closer comparison between the approaches of the present deduction and that of [1] see Chapter 8 and, in particular Section 8.3.

Chapter 6

A gauge-invariant method for the linear stability analysis of wormhole solutions

In Ref. [1], a gauge-invariant method has been introduced for decoupling the linearized field equations (4.29,4.33) in the four dimensional case; in this chapter, I provide a generalization of this method to higher dimension. Analogously to [1], I consider a general $(d+1)$-dimensional static solution of the form (5.1), i.e.,

$$\beta = 0\,, \qquad (\alpha, \gamma, r, \Phi) := (\alpha(x), \gamma(x), r(x), \Phi(x))\,; \tag{6.1}$$

differently from the gauge-dependent approach of the previous chapter, the only hypothesis that we make on the static solution (6.1) is that it satisfies the background equations (3.6-3.9). Then, I will prove that the corresponding linearized Einstein-scalar equations can be rewritten, in any gauge, as a 2×2 constrained wave system involving two gauge-invariant quantities; in addition, I will prove that, provided a static solution of the system is known, the latter can be decoupled, leading to a single wave type-equation (which will be referred to as *master equation*) for a suitably defined combination of the above mentioned gauge-invariant quantities. Usually, the static solution of the wave system can be obtained on a case-by-case basis, following a general method which consists in varying the free parameters of the static wormhole under consideration.

Before delving into details, let me underline the following fact; the 2×2 constrained wave system will be obtained initially by simplifying the linearized field equations by stetting a gauge in which $\delta\beta = \delta\Phi = 0$. It might seem counter-intuitive to the reader that the *gauge-invariant* construction of this chapter (and of the construction in Ref. [1]) relies on the possibility of suitable combining of the linearized field equations in a *particular* gauge; actually, we will show that the system obtained with this choice of coordinates can be easily generalized to a system which is valid in any gauge. Therefore, the gauge assumptions that we will make in this chapter do not affect the gauge-invariance of the method, hence they are substantially different from those made in Chapter 5 (see Eq. (5.4) and Eq. (5.41)), in which the approach strongly depended on the coordinates chosen.

6.1 Study of the constraint equations in the $\delta\beta = 0$ gauge and the $\mathcal{S} = 0$ assumption

From now on we assume the gauge introduced in Remark 37 (that for the sake of simplicity we denote again with (t, x)) in which

$$\delta\beta = 0\,. \tag{6.2}$$

As explained in Subsection 4.3.1, in this case the linearized field equations (4.29-4.33) can be interpreted as a constrained evolution system in the unknowns $(\delta\alpha, \delta\gamma, \delta r, \delta\Phi)$. In the present section, we begin the general study of this constrained system from a complete analysis the constraints (4.32,4.33). Let us start integrating the second constraint $\delta\mathcal{M} = 0$ once in t; hence, we get

$$\delta r' = \frac{\alpha'}{\alpha}\delta r + r'\frac{\delta\gamma}{\gamma} + \frac{\kappa}{d-1} r\Phi'\delta\Phi + \frac{2}{d-1} r\Sigma(x)\,, \tag{6.3}$$

where $\Sigma : x(\mathcal{O}) \to \mathbb{R}$ is a smooth function depending on the initial data for the perturbation functions. Eq. (6.3) and its derivatives enable to eliminate $\delta r'$, $\delta r''$ and $\delta\dot{r}'$ from the first constraint $\delta\mathcal{H} = 0$, yielding [(36)]

$$2\,\frac{\alpha}{\gamma}\left[\Sigma'(x) + \left(\frac{\alpha'}{\alpha} - \frac{\gamma'}{\gamma} + (d-1)\frac{r'}{r}\right)\Sigma(x)\right] = 0. \tag{6.4}$$

This differential equation can be solved in the unknown Σ, leading to

$$\Sigma(x) = \sigma\frac{\gamma}{\alpha\, r^{d-1}}\,, \tag{6.5}$$

where σ is an integration constant. Summing up, we have shown that if the two constraints $\delta\mathcal{H} = 0$ and $\delta\mathcal{M} = 0$ are satisfied then Eq. (6.3) holds with

[36] In order to obtain Eq. (6.4) I have used the background equations (3.6-3.9).

the function Σ defined by Eq. (6.5); in other words, the two constraints (4.32,4.33) imply the following equation

$$\mathcal{S} = \sigma \in \mathbb{R}\,, \qquad \mathcal{S} := \frac{d-1}{2}\frac{\alpha\, r^{d-2}}{\gamma}\left(\delta r' - \frac{\alpha'}{\alpha}\delta r - r'\frac{\delta\gamma}{\gamma} - \frac{\kappa}{d-1} r\Phi'\delta\Phi\right). \tag{6.6}$$

Conversely, if $\mathcal{S} = \sigma \in \mathbb{R}$ is satisfied (with $\mathcal{S}$ defined by Eq. (6.6)), then $\delta\mathcal{H} = 0$ and $\delta\mathcal{M} = 0$ are satisfied too, by noting that the following identities hold [37]

$$\delta\mathcal{H} = 2\frac{1}{r^{d-1}}\mathcal{S}'\,, \qquad \delta\mathcal{M} = 2\frac{\gamma}{\alpha r^{d-1}}\dot{\mathcal{S}}\,. \tag{6.7}$$

In conclusion, we have proved the equivalence of the constraint equations (4.32,4.33) with Eq. (6.6); moreover, in some sense, the latter is still a constraint of the evolution field system (4.29-4.31), as stated by the following

Proposition 6 *For every $\sigma \in \mathbb{R}$, the equation $\mathcal{S} = \sigma$ [Eq. (6.6)] is a constraint for the second order evolution equations $\delta\mathfrak{E}_1 = 0$, $\delta\mathfrak{E}_2 = 0$, $\delta\mathfrak{E}_3 = 0$ [Eqs. (4.29-4.31)] in the following sense: if $(\delta\alpha, \delta\gamma, \delta r, \delta\Phi)$ is a (time-dependent) solution of the system (4.29-4.31) satisfying the initial conditions* [38]

$$\mathcal{S}(0,x) = \sigma\,, \qquad \dot{\mathcal{S}}(0,x) = 0\,, \qquad \textit{for every } x \in x(\mathcal{O})\,, \tag{6.8}$$

then $(\delta\alpha, \delta\gamma, \delta r, \delta\Phi)$ satisfies the equation $\mathcal{S} = \sigma$ for every time t and every x.

Proof. The proof is trivial. Indeed, from the identities (6.7) we see that the initial conditions (6.8) (see the equivalent form contained in the footnote (38)) are trivially equivalent to the initial conditions $\delta\mathcal{H}(0,x) = 0$ and $\delta\mathcal{M}(0,x) = 0$; since $\delta\mathcal{H}$ and $\delta\mathcal{M}$ are two constraints of the system $\delta\mathfrak{E}_1 = 0$, $\delta\mathfrak{E}_2 = 0$, $\delta\mathfrak{E}_3 = 0$, then every solution $(\delta\alpha, \delta\gamma, \delta r, \delta\Phi)$ of Eqs. (4.29-4.31) satisfy $\delta\mathcal{H} = 0$ and $\delta\mathcal{M} = 0$ for every x and t. As already proved, this is equivalent to say that $\mathcal{S} = \tilde{\sigma}$ for some $\tilde{\sigma} \in \mathbb{R}$ for every x and t; however, since $\mathcal{S} = \sigma$ at $t = 0$, by continuity we have that $\tilde{\sigma} = \sigma$.

□

[37] In order to prove the first identity of Eq. (6.7), the derivatives Φ'', α'', r'' and r'^2 of the static solution have been removed using the background equations (3.6-3.9).

[38] Obviously, these are equivalent to the conditions $\mathcal{S}(0,0) = \sigma$, $\mathcal{S}'(0,x) = 0$, $\dot{\mathcal{S}}(0,x) = 0$ for every $x \in x(\mathcal{O})$.

Remark 53 For future convenience, in the following lines I provide an alternative proof of the previous proposition. Let us consider the identities (4.39-4.40); replacing the quantities $\delta\mathcal{H}$ and $\delta\mathcal{M}$ with the partial derivatives of $\mathcal{S}$, according to the formulas in Eq. (6.7), one can easily see that the first equation (4.39) is automatically satisfied, while the second equation (4.40) becomes

$$2\frac{\gamma}{\alpha r^{d-1}}\left[\frac{\partial^2}{\partial t^2}-\left(\frac{\alpha}{\gamma}\frac{\partial}{\partial x}\right)^2\right]\mathcal{S}=-(d-1)\frac{\alpha}{\gamma}\frac{r'}{r}\delta\mathfrak{E}_1 - 2\frac{\alpha}{\gamma r^2}\left[\frac{\gamma'}{\gamma}-\frac{d-5}{2}\frac{r'}{r}\right]\delta\mathfrak{E}_2+2\frac{\alpha}{\gamma r^2}\delta\mathfrak{E}_2'-\frac{\alpha}{\gamma}\frac{\kappa\Phi'}{r^2}\delta\mathfrak{E}_3\,. \quad (6.9)$$

Therefore, every perturbation $(\delta\alpha,\delta\gamma,\delta r,\delta\Phi)$ which satisfies Eqs. (4.29-4.31), satisfies the identity (6.9) with the right hand side vanishing; in the new coordinate

$$\rho=\rho(x):=\int_0^x\frac{\gamma(y)}{\alpha(y)}\,dy\,, \quad (6.10)$$

this becomes the one dimensional wave equation

$$\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial\rho^2}\right]\tilde{\mathcal{S}}=0\,, \quad (6.11)$$

where $\tilde{\mathcal{S}}(t,\rho):=\mathcal{S}(t,x(\rho))$, where $x(\rho)$ is the inverse map of the transformation (6.10). It is well known that the solution of the wave equation (6.11) has the general form $\tilde{\mathcal{S}}(t,\rho)=F(\rho-t)+G(\rho+t)$, where F and G are two smooth functions depending on the initial data on $\tilde{\mathcal{S}}$ in the following way: if $\tilde{\mathcal{S}}(0,0)=\tilde{\mathcal{S}}_0$, $\tilde{\mathcal{S}}'(0,\rho)=f(\rho)$ and $\dot{\tilde{\mathcal{S}}}(0,\rho)=g(\rho)$, then $F(\rho-t)=\frac{1}{2}\int_0^{\rho-t}(f(y)-g(y))\,dy+\tilde{\mathcal{S}}_0$ and $G(\rho+t)=\frac{1}{2}\int_0^{\rho+t}(f(y)+g(y))\,dy$. In our case, since $\rho(0)=0=x(0)$ and $\frac{\alpha}{\gamma}\frac{\partial}{\partial x}=\frac{\partial}{\partial\rho}$, the initial conditions (6.8) (in the equivalent form contained in the footnote (38)) read

$$\tilde{\mathcal{S}}(0,0)=\sigma\,,\qquad \tilde{\mathcal{S}}'(0,\rho)=0\,,\qquad \dot{\tilde{\mathcal{S}}}(0,\rho)=0\qquad \text{for every } \rho\in\rho(x(\mathcal{O}));$$

from the general solution with $f(\rho)=g(\rho)=0$ and $\tilde{\mathcal{S}}_0=\sigma$, one has that the solution of the wave equation (6.11) is $\tilde{\mathcal{S}}(t,\rho)=\sigma$ for every $(t,\rho)\in\mathbb{R}\times\rho(x(\mathcal{O}))$. Finally, one notes that this is equivalent to $\mathcal{S}(t,x)=\sigma$ for every $(t,x)\in\mathbb{R}\times x(\mathcal{O})$.

At this point one might want to investigate the meaning of the constant σ appearing in Eq. (6.6): in the rest of this section I will make some considerations on this integration constant. Let us start with a trivial proposition:

Proposition 7 *Let $(\delta\alpha_1, \delta\gamma_1, \delta r_1, \delta\Phi_1)$ and $(\delta\alpha_2, \delta\gamma_2, \delta r_2, \delta\Phi_2)$ be two (time-dependent) solutions of the linearized constrained system (4.29,4.30,4.31,6.6) which satisfy the constraint equation (6.6), respectively, with constants $\sigma = \sigma_1$ and $\sigma = \sigma_2$. Then the solution $(\delta\alpha, \delta\gamma, \delta r, \delta\Phi)$ of the linearized system defined as a linear combination of the solutions $(\delta\alpha_1, \delta\gamma_1, \delta r_1, \delta\Phi_1)$ and $(\delta\alpha_2, \delta\gamma_2, \delta r_2, \delta\Phi_2)$, namely*

$$\begin{aligned} \delta\alpha &:= a_1\delta\alpha_1 + a_2\delta\alpha_2\,, & \delta\gamma &:= a_1\delta\gamma_1 + a_2\delta\gamma_2\,, \\ \delta r &:= a_1\delta r_1 + a_2\delta r_2\,, & \delta\Phi &:= a_1\delta\Phi_1 + a_2\delta\Phi_2\,, \qquad a_1, a_2 \in \mathbb{R}\,, \end{aligned}$$

satisfies the constraint equation (6.6) with the constant $\sigma = a_1\sigma_1 + a_2\sigma_2$.

Remark 54 Suppose now that there exists a static solution

$$(\delta\alpha_0, \delta\gamma_0, \delta r_0, \delta\Phi_0) = (\delta\alpha_0(x), \delta\gamma_0(x), \delta r_0(x), \delta\Phi_0(x)) \tag{6.12}$$

of the linearized system (4.29,4.30,4.31,6.6) that satisfies the constraint (6.6) with a constant $\sigma = \sigma_0 \neq 0$; then the previous proposition assures that for every (time-dependent) solution $(\delta\alpha, \delta\gamma, \delta r, \delta\Phi)$ of the linearized system such that the constraint (6.6) holds with a constant $\sigma \in \mathbb{R}$, then the (time-dependent) solution $(\tilde{\delta\alpha}, \tilde{\delta\gamma}, \tilde{\delta r}, \tilde{\delta\Phi})$ defined as

$$\begin{aligned} \tilde{\delta\alpha} &:= \delta\alpha - \frac{\sigma}{\sigma_0}\delta\alpha_0\,, & \tilde{\delta\gamma} &:= \delta\gamma - \frac{\sigma}{\sigma_0}\delta\gamma_0\,, \\ \tilde{\delta r} &:= \delta r - \frac{\sigma}{\sigma_0}\delta r_0\,, & \tilde{\delta\Phi} &:= \delta\Phi - \frac{\sigma}{\sigma_0}\delta\Phi_0 \end{aligned}$$

is a solution of the linearized system such that the constraint (6.6) holds with the constant $\sigma = 0$. This means that, for every (time-dependent) solution $(\delta\alpha, \delta\gamma, \delta r, \delta\Phi)$ of the linearized system (4.29,4.30,4.31,6.6), it is possible to find another (time-dependent) solution $(\tilde{\delta\alpha}, \tilde{\delta\gamma}, \tilde{\delta r}, \tilde{\delta\Phi})$ of Eqs. (4.29,4.30,4.31) and $\mathcal{S} = 0$, and the latter is such that the difference between the two solutions is static, a fact which implies that their temporal behaviour is qualitatively similar.
One can find a concrete example of this fact in Section 3.1 of Ref. [25], where the authors show directly that the parameters b and γ_1 of the Ellis-Bronnikov wormhole label the static solutions of the linearized field equations.

The considerations of the previous remark suggest that one can substantially simplify the linear stability analysis of a static wormhole solution by considering only the perturbations $(\delta\alpha, \delta\gamma, \delta r, \delta\Phi)$ for which the constraint equation (6.6) holds with $\sigma = 0$. Note that, it can be possible that a static

solution such that $\mathcal{S} = \sigma_0 \neq 0$ can not be provided; in such case we are getting rid of plenty of time-dependent perturbations of the system. However, since we expect that most of known wormhole solutions are actually linearly unstable, the loss of some solutions does not affect the possibility to infer an instability result: indeed, to do this, it is sufficient to find *at least* one perturbation (gauge-invariantly) diverging in time.
Hence, from now on, we assume that Eq. (6.6) holds with $\sigma = 0$, that is, Eq. (6.6) is replaced by

$$\mathcal{S} = 0\,, \qquad \mathcal{S} := \frac{d-1}{2}\frac{\alpha\, r^{d-2}}{\gamma}\left(\delta r' - \frac{\alpha'}{\alpha}\delta r - r'\frac{\delta\gamma}{\gamma} - \frac{\kappa}{d-1} r\Phi'\delta\Phi\right)\,. \tag{6.13}$$

6.2 The gauge-dependent quantities $\mathcal{A}$, $\mathcal{C}$, $\mathcal{D}$ and the $\delta\Phi = 0$ gauge

For future use, it is advantageous to introduce the quantities [39]

$$\mathcal{D} := \frac{\delta\alpha}{\alpha}\,, \qquad \mathcal{A} := \frac{\delta\gamma}{\gamma}\,, \qquad \mathcal{C} := \frac{\delta r}{r}\,. \tag{6.14}$$

Then the system of Eqs. (4.29,4.30,4.31,6.13) (namely, the linearized field system (4.29,4.30,4.31,6.6) with the condition $\sigma = 0$) becomes

$$\begin{aligned} \delta\mathfrak{E}_1 = 0\,, \quad \delta\mathfrak{E}_1 = {} & \frac{\gamma}{\alpha}\ddot{\mathcal{A}} - \frac{\partial}{\partial x}\left[\frac{\alpha}{\gamma}\mathcal{D}'\right] - \frac{\alpha}{\gamma}\left(\frac{\alpha'}{\alpha} + \frac{d-3}{2}\frac{r'}{r}\right)(\mathcal{D}-\mathcal{A})' + \\ & \left[(d-1)\frac{\alpha}{\gamma}\frac{r'}{r} - \frac{d-3}{2}\left(\frac{\alpha}{\gamma}\right)'\right]\mathcal{C}' - \frac{2(d-2)\alpha\gamma}{r^2}(\mathcal{A}-\mathcal{C}) \\ & + \kappa\frac{\alpha}{\gamma}\Phi'\delta\Phi' + \frac{d-3}{2}\left(\frac{\gamma}{\alpha}\ddot{\mathcal{C}} - \frac{\alpha}{\gamma}\mathcal{C}''\right)\,, \end{aligned} \tag{6.15}$$

[39] This choice of notation is somehow awkward; however the reason for it is to maintain compatibility with the notation used in Refs. [53, 1].

$$\begin{aligned}\delta\mathfrak{E}_2 = 0\,, \quad \delta\mathfrak{E}_2 = \Bigg[& \frac{d-1}{2}\frac{\gamma}{\alpha}\ddot{\mathcal{C}} - \frac{d-1}{2}\frac{\partial}{\partial x}\left[\frac{\alpha}{\gamma}\mathcal{C}'\right] \\ & - \frac{d-1}{2}\frac{\alpha}{\gamma}\frac{r'}{r}(\mathcal{D} - \mathcal{A} + 2(d-1)\mathcal{C})' \\ & + \frac{(d-1)(d-2)\alpha\gamma}{r^2}(\mathcal{A} - \mathcal{C}) \\ & - \kappa\alpha\gamma\left[2V(\Phi)\mathcal{A} + V'(\Phi)\delta\Phi\right]\Bigg] r^2\,, \end{aligned} \tag{6.16}$$

$$\begin{aligned}\delta\mathfrak{E}_3 = 0\,, \quad \delta\mathfrak{E}_3 = & \frac{\gamma}{\alpha}\delta\ddot{\Phi} - \frac{\partial}{\partial x}\left[\frac{\alpha}{\gamma}\delta\Phi'\right] - (d-1)\frac{\alpha}{\gamma}\frac{r'}{r}\delta\Phi' \\ & - \frac{\alpha}{\gamma}\Phi'(\mathcal{D} - \mathcal{A} + (d-1)\mathcal{C})' - \alpha\gamma\left[2V'(\Phi)\mathcal{A} + V''(\Phi)\delta\Phi\right]\,, \end{aligned} \tag{6.17}$$

$$\mathcal{S} = 0\,, \qquad \mathcal{S} = \frac{d-1}{2}\frac{\alpha\, r^{d-1}}{\gamma}\left[\mathcal{C}' - \left(\frac{\alpha'}{\alpha} - \frac{r'}{r}\right)\mathcal{C} - \frac{r'}{r}\mathcal{A} - \frac{\kappa}{d-1}\Phi'\delta\Phi\right]. \tag{6.18}$$

Remark 55 Eqs. (6.15-6.18) only assume the gauge $\delta\beta = 0$; as already mentioned, at the linearized level there is still liberty in the choice of the coordinates, which is related to the choice of the transformation functions $(\delta t, \delta x)$ (see Section 4.2 and, in particular, Remark 37). One possible choice that has been considered in several work (see, e.g., Refs. [25, 11]) is keeping fix the radial coefficient r of the static solution, that is, setting $\delta r = 0$; with this prescription, $\mathcal{C} = 0$ and Eqs. (6.18,6.16) allow to express the functions $\mathcal{A}$ and $\mathcal{D}'$ in terms of $\delta\Phi$. Inserting these expressions into Eq. (6.17), one obtains a master equation for the linearized scalar field $\delta\Phi$. [40] Unfortunately, due to this choice of the gauge, it happens that the potential of this wave-type master equation has a singularity in correspondence of the wormhole throat, where $r' = 0$ (see Refs. [25, 11] for more details). This, from a physical point of view, is related to the fact that the assumption $\delta r = 0$ amounts in forcing the perturbations to vanish at the throat, which is much too restrictive for wormhole configurations. On the other hand, at a mathematical level, the occurrence of this singularity can be easily deduced from Eq. (4.22). Indeed, in order to send δr (simultaneously with $\delta\beta$) to zero, the function δt has to be defined as in Eq. (4.24) with $\delta x = -\frac{\delta r}{r'}$; of course, the latter occurs to be singular whenever the derivative of the coefficient r of the static metric

[40] Note that, once Eqs. (6.16-6.18) have been solved, the remaining equation (6.15) is automatically satisfied thanks to the relation (6.9).

vanishes, which is exactly the case for wormhole spacetimes. However, while $r' = 0$ at the wormhole throat, in all the static wormhole solutions introduced in Chapter 3, the spatial derivative of the scalar field Φ' never vanishes: as a consequence, we may choose, instead,

$$\begin{aligned} \delta x(\tilde{t}, \tilde{x}) &= -\frac{\delta\Phi(\tilde{t}, \tilde{x})}{\Phi'(\tilde{x})} \qquad \text{and} \\ \delta t(\tilde{t}, \tilde{x}) &= \int_{\tilde{x}_1}^{\tilde{x}} \frac{\gamma(x)^2}{\alpha(x)^2}\left(\delta\beta(\tilde{t}, x) - \frac{\delta\dot{\Phi}(\tilde{t}, x)}{\Phi'(x)}\right) dx\,, \quad \tilde{x}_1 \in \mathbb{R} \end{aligned} \tag{6.19}$$

and send $\delta\Phi$ (simultaneously with $\delta\beta$) to zero (see Eqs. (4.20,4.23,4.24)). Thus, form now till the end of this section, we fix a gauge in which $\delta\beta = 0$ (as before) and

$$\delta\Phi = 0\,; \tag{6.20}$$

again, we keep denoting the new coordinates simply with (t, x).

In the new gauge (6.2,6.20), the linearized system (6.15-6.18) reduce to

$$\begin{aligned} \delta\mathfrak{E}_1 = 0\,, \quad \delta\mathfrak{E}_1 &:= \frac{\gamma}{\alpha}\ddot{\mathcal{A}} - \frac{\partial}{\partial x}\left[\frac{\alpha}{\gamma}\mathcal{D}'\right] - \frac{\alpha}{\gamma}\left(\frac{\alpha'}{\alpha} + \frac{d-3}{2}\frac{r'}{r}\right)(\mathcal{D} - \mathcal{A})' \\ &+ \left[(d-1)\frac{\alpha}{\gamma}\frac{r'}{r} - \frac{d-3}{2}\left(\frac{\alpha}{\gamma}\right)'\right]\mathcal{C}' \\ &- \frac{2(d-2)\alpha\gamma}{r^2}(\mathcal{A} - \mathcal{C}) + \frac{d-3}{2}\left(\frac{\gamma}{\alpha}\ddot{\mathcal{C}} - \frac{\alpha}{\gamma}\mathcal{C}''\right)\,, \end{aligned} \tag{6.21}$$

$$\begin{aligned} \delta\mathfrak{E}_2 = 0\,, \quad \delta\mathfrak{E}_2 &:= \left[\frac{d-1}{2}\frac{\gamma}{\alpha}\ddot{\mathcal{C}} - \frac{d-1}{2}\frac{\partial}{\partial x}\left[\frac{\alpha}{\gamma}\mathcal{C}'\right]\right. \\ &- \frac{d-1}{2}\frac{\alpha}{\gamma}\frac{r'}{r}(\mathcal{D} - \mathcal{A} + 2(d-1)\mathcal{C})' \\ &\left.+ \frac{(d-1)(d-2)\alpha\gamma}{r^2}(\mathcal{A} - \mathcal{C}) - 2\kappa\alpha\gamma V(\Phi)\mathcal{A}\right] r^2\,, \end{aligned} \tag{6.22}$$

$$\delta\mathfrak{E}_3 = 0\,, \quad \delta\mathfrak{E}_3 := -\frac{\alpha}{\gamma}\Phi'(\mathcal{D} - \mathcal{A} + (d-1)\mathcal{C})' - 2\alpha\gamma V'(\Phi)\mathcal{A}\,, \tag{6.23}$$

$$\mathcal{S} = 0\,, \quad \mathcal{S} := \frac{d-1}{2}\frac{\alpha\, r^{d-1}}{\gamma}\left[\mathcal{C}' - \left(\frac{\alpha'}{\alpha} - \frac{r'}{r}\right)\mathcal{C} - \frac{r'}{r}\mathcal{A}\right]\,. \tag{6.24}$$

Remark 56 Note that, using $\delta\mathfrak{E}_3 = 0$ one can isolate $\mathcal{D}'$ and substitute its expression into Eqs. (6.21,6.22); obviously, this can be done as long as the derivative Φ' of the scalar field does not vanish anywhere, but this is exactly the assumption that we have made for considering the $\delta\Phi = 0$ gauge. Moreover, a suitable recombination of $\delta\mathfrak{E}_1 = 0$ and $\delta\mathfrak{E}_2 = 0$ allows to eliminate the second derivative $\ddot{\mathcal{C}}$ from Eq. (6.21): in this way, making use again of the background equation (3.6), we can rewrite Eqs. (6.21-6.24) as follows

$$\begin{aligned}\mathfrak{E}_{\ddot{\mathcal{A}}} = 0\,, \quad \mathfrak{E}_{\ddot{\mathcal{A}}} := {}& \frac{\gamma}{\alpha}\ddot{\mathcal{A}} - \frac{\partial}{\partial x}\left[\frac{\alpha}{\gamma}(\mathcal{A}' - (d-1)\mathcal{C}')\right] \\ &+ (d-1)\frac{\alpha}{\gamma}\left(\frac{\alpha'}{\alpha} + (d-2)\frac{r'}{r}\right)\mathcal{C}' - (d-1)(d-2)\frac{\alpha\gamma}{r^2}(\mathcal{A} - \mathcal{C}) \\ &+ \frac{2\alpha\gamma V'(\Phi)}{\Phi'}\mathcal{A}' + 2\alpha\gamma\left[\frac{(d-3)\kappa}{d-1}V(\Phi) + \gamma^2\frac{V'(\Phi)^2}{\Phi'^2}\right. \\ &\left. + \left(3\frac{\alpha'}{\alpha} + (d-1)\frac{r'}{r}\right)\frac{V'(\Phi)}{\Phi'} + V''(\Phi)\right]\mathcal{A}\,, \end{aligned} \tag{6.25}$$

$$\begin{aligned}\mathfrak{E}_{\ddot{\mathcal{C}}} = 0\,, \quad \mathfrak{E}_{\ddot{\mathcal{C}}} := {}& \frac{\gamma}{\alpha}\ddot{\mathcal{C}} - \frac{\partial}{\partial x}\left[\frac{\alpha}{\gamma}\mathcal{C}'\right] - (d-1)\frac{\alpha}{\gamma}\frac{r'}{r}\mathcal{C}' + 2(d-2)\frac{\alpha\gamma}{r^2}(\mathcal{A} - \mathcal{C}) \\ &+ 2\alpha\gamma\left[\frac{r'}{r}\frac{V'(\Phi)}{\Phi'} - \frac{2\kappa}{d-1}V(\Phi)\right]\mathcal{A}\,, \end{aligned} \tag{6.26}$$

$$\mathfrak{E}_{\mathcal{D}'} = 0\,, \quad \mathfrak{E}_{\mathcal{D}'} := \mathcal{D}' - \mathcal{A}' + (d-1)\mathcal{C}' + 2\gamma^2\frac{V'(\Phi)}{\Phi'}\mathcal{A}\,, \tag{6.27}$$

$$\mathfrak{E}_{\mathcal{C}'} = 0\,, \quad \mathfrak{E}_{\mathcal{C}'} := \mathcal{C}' - \left(\frac{\alpha'}{\alpha} - \frac{r'}{r}\right)\mathcal{C} - \frac{r'}{r}\mathcal{A}\,. \tag{6.28}$$

The equivalence of the systems (6.21,6.24) and (6.25-6.28) is easily proved observing that

$$\begin{aligned}\mathfrak{E}_{\ddot{\mathcal{A}}} &= \delta\mathfrak{E}_1 + \frac{d-3}{(d-1)r^2}\delta\mathfrak{E}_2 - \frac{1}{r^2\Phi'}\left(2\frac{\alpha'}{\alpha} - \frac{\gamma'}{\gamma} + (d-3)\frac{r'}{r} + \gamma^2\frac{V'(\Phi)}{\Phi'}\right)\delta\mathfrak{E}_3 \\ &\quad + \frac{1}{r^2\Phi'}\delta\mathfrak{E}_3'\,, \end{aligned} \tag{6.29}$$

$$\mathfrak{E}_{\ddot{\mathcal{C}}} = \frac{1}{r^2}\left(\frac{2}{(d-1)}\delta\mathfrak{E}_2 - \frac{r'}{r\Phi'}\delta\mathfrak{E}_3\right)\,, \tag{6.30}$$

$$\mathfrak{E}_{\mathcal{D}'} = -\frac{\gamma}{\alpha r^2\Phi'}\delta\mathfrak{E}_3\,, \tag{6.31}$$

$$\mathfrak{E}_{\mathcal{C}'} = \frac{2}{d-1}\frac{\gamma}{\alpha r^{d-1}}\mathcal{S}\,; \tag{6.32}$$

indeed, one implication of the equivalence is trivial, since $\delta\mathfrak{E}_1 = \delta\mathfrak{E}_2 = \delta\mathfrak{E}_3 = \mathcal{S} = 0$ immediately implies $\mathfrak{E}_{\ddot{\mathcal{A}}} = \mathfrak{E}_{\ddot{\mathcal{C}}} = \mathfrak{E}_{\mathcal{D}'} = \mathfrak{E}_{\mathcal{C}'} = 0$. Conversely, if $\mathfrak{E}_{\mathcal{D}'} = \mathfrak{E}_{\mathcal{C}'} = \mathfrak{E}_{\ddot{\mathcal{C}}} = 0$ then $\delta\mathfrak{E}_3 = \mathcal{S} = \delta\mathfrak{E}_2 = 0$ since the coefficients of

$\delta\mathfrak{E}_3$, $\mathcal{S}$ and $\delta\mathfrak{E}_2$, respectively, in Eq. (6.31), Eq. (6.32) and Eq. (6.30) are smooth and never vanish; finally, Eq. (6.29) combined with $\delta\mathfrak{E}_3 = \delta\mathfrak{E}_2 = 0$ and $\mathfrak{E}_{\ddot{\mathcal{A}}} = 0$ implies immediately that $\delta\mathfrak{E}_1 = 0$.
Note that, obviously, Eq. (6.28) is still a constraint for the system (6.25-6.27): indeed, the constraint $\mathcal{S} = 0$ is equivalent to $\mathfrak{E}_{\mathcal{C}'} = 0$, while the evolution system $\delta\mathfrak{E}_1 = 0$, $\delta\mathfrak{E}_2 = 0$, $\delta\mathfrak{E}_3 = 0$ is equivalent to the system $\mathfrak{E}_{\ddot{\mathcal{A}}} = 0$, $\mathfrak{E}_{\ddot{\mathcal{C}}} = 0$, $\mathfrak{E}_{\mathcal{D}'} = 0$.
In addition, since the recombinations (6.29-6.32) have been introduced in order to make the function $\mathcal{D}$ disappear from Eqs. (6.25,6.26,6.28), then the linearized system (6.25-6.28) can be regarded as an evolution system only for $\mathcal{A}$ and $\mathcal{C}$ (6.25,6.26) subject to the constraint (6.28); from the remaining equation (6.27) one can isolate $\mathcal{D}'$ and then find an expression for the function $\mathcal{D}$ in dependence of $\mathcal{A}$ and $\mathcal{C}$:

$$\mathcal{D}(t,x) = \mathcal{A}(t,x) - (d-1)\mathcal{C}(t,x) + 2\int_{x_0}^{x} \frac{\gamma(y)^2 V'(\Phi(y))}{\Phi'(y)} \mathcal{A}(t,y)dy\,, \quad (6.33)$$

where x_0 is an arbitrary point depending on the initial data of $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{D}$.

All the considerations contained in the previous remark are reassumed in the following

Proposition 8 *In the gauge $\delta\beta = \delta\Phi = 0$, the linearized field equations (4.29-4.33), together with the condition $\mathcal{S} = 0$ [Eq. (6.13)], are equivalent to the system (6.25-6.28), where the functions $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{D}$ are defined in Eq. (6.14). In addition, Eq. (6.27) gives the expression (6.33) for the function $\mathcal{D}$ in terms of the functions $\mathcal{A}$ and $\mathcal{C}$, while the remaining three equations (6.25,6.26,6.28) can be regarded as a constrained evolution system in the following sense: if $(\mathcal{A},\mathcal{C})$ is a (time-dependent) solution of the system $\mathfrak{E}_{\ddot{\mathcal{A}}} = 0$, $\mathfrak{E}_{\ddot{\mathcal{C}}} = 0$ [Eqs. (6.25,6.26)] satisfying the initial conditions*

$$\mathfrak{E}_{\mathcal{C}'}(0,x) = 0\,, \qquad \dot{\mathfrak{E}}_{\mathcal{C}'}(0,x) = 0\,, \qquad \textit{for every } x \in x(\mathcal{O})\,, \qquad (6.34)$$

then $(\mathcal{A},\mathcal{C})$ satisfies the equation $\mathfrak{E}_{\mathcal{C}'} = 0$ [Eq. 6.28] for every time t and every x.

Proof. Basically, the proposition has already been proved; we have just to verify that the initial conditions for $\mathfrak{E}_{\mathcal{C}'}$ in Eq. (6.34) are equivalent to the initial conditions for $\mathcal{S}$ in Eq. (6.8). Actually, in the case $\sigma = 0$, the initial conditions (6.8) reduce to $\mathcal{S}(0,x) = 0$ and $\dot{\mathcal{S}}(0,x) = 0$ for every $x \in x(\mathcal{O})$; these are exactly the conditions (6.34) as the coefficient $\frac{2}{d-1}\frac{\gamma}{\alpha r^{d-1}}$ appearing in Eq. (6.32) is time independent and never vanishes.

□

6.3 The gauge-invariant quantities A, C, E and the linearized field equations for A, C, E in an arbitrary gauge

Let us return for a moment to the gauge transformation ϕ_ϵ of Eq. (4.6) and to the corresponding transformations (4.19-4.23) of the perturbation functions $\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi$; then the following proposition is trivially proved.

Proposition 9 *The following quantities are invariant with respect to gauge transformations ϕ_ϵ [Eq. (4.6)]:*

$$A := \frac{\delta\gamma}{\gamma} - \frac{1}{\gamma}\left(\gamma\frac{\delta\Phi}{\Phi'}\right)', \tag{6.35}$$

$$C := \frac{\delta r}{r} - \frac{r'}{r}\frac{\delta\Phi}{\Phi'}, \tag{6.36}$$

$$E := \left(\frac{\delta\alpha}{\alpha}\right)' - \left(\frac{\alpha'}{\alpha}\frac{\delta\Phi}{\Phi'}\right)' + \frac{\gamma^2}{\alpha^2}\left(\delta\dot{\beta} - \frac{\delta\ddot{\Phi}}{\Phi'}\right). \tag{6.37}$$

and $\mathcal{S}$ defined in Eq. (6.6).

Proof. The gauge transformations (4.6) transform the quantities A, C, E and $\mathcal{S}$ into the quantities $\tilde{A}$, $\tilde{C}$, $\tilde{E}$ and $\tilde{\mathcal{S}}$ defined as A, C, E and $\mathcal{S}$ with $\delta\alpha$, $\delta\beta$, $\delta\gamma$, δr, $\delta\Phi$ replaced by $\tilde{\delta\alpha}$, $\tilde{\delta\beta}$, $\tilde{\delta\gamma}$, $\tilde{\delta r}$, $\tilde{\delta\Phi}$; to prove the invariance, it is sufficient to see that $\tilde{A} = A$, $\tilde{C} = C$, $\tilde{E} = E$ and $\tilde{\mathcal{S}} = \mathcal{S}$ with $\tilde{\delta\alpha}$, $\tilde{\delta\beta}$, $\tilde{\delta\gamma}$, $\tilde{\delta r}$, $\tilde{\delta\Phi}$ defined as in Eqs. (4.19-4.23) (see Definition 8).

□

Remark 57 Given any static solution (6.1) such that Φ' never vanishes, we introduce the following equivalence relation for the perturbation functions $\delta\alpha$,$\delta\beta$,$\delta\gamma$,δr,$\delta\Phi$

$$(\tilde{\delta\alpha}, \tilde{\delta\beta}, \tilde{\delta\gamma}, \tilde{\delta r}, \tilde{\delta\Phi}) \sim (\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi) \quad \Leftrightarrow \quad \exists\phi_\epsilon, \psi_\epsilon \text{ as in Eqs. (4.6,4.7) :}$$
$$\tilde{\delta\alpha} = \delta\alpha + \alpha'\delta x + \alpha\,\delta\dot{t}, \qquad \tilde{\delta\beta} = \delta\beta + \delta\dot{x} - \frac{\alpha^2}{\gamma^2}\delta t',$$
$$\tilde{\delta\gamma} = \delta\gamma + (\gamma\delta x)', \qquad \tilde{\delta r} = \delta r + r'\delta x, \qquad \tilde{\delta\Phi} = \delta\Phi + \Phi'\delta x, \tag{6.38}$$

the corresponding equivalence class

$$\big[(\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi)\big]_\sim := \Big\{(\tilde{\delta\alpha}, \tilde{\delta\beta}, \tilde{\delta\gamma}, \tilde{\delta r}, \tilde{\delta\Phi}) \mid (\tilde{\delta\alpha}, \tilde{\delta\beta}, \tilde{\delta\gamma}, \tilde{\delta r}, \tilde{\delta\Phi}) \sim (\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi)\Big\} \tag{6.39}$$

and the corresponding quotient

$$[C^\infty(\mathcal{O},\mathbb{R})]^5/_\sim := \Big\{ \big[(\delta\alpha,\delta\beta,\delta\gamma,\delta r,\delta\Phi)\big]_\sim \mid \delta\alpha,\delta\beta,\delta\gamma,\delta r,\delta\Phi \in C^\infty(\mathcal{O},\mathbb{R}) \Big\} . \quad (6.40)$$

Note that, every infinitesimal coordinate transformation of the coordinates (t, x) as in Eqs. (4.6,4.7) is class invariant under the equivalence relation $\sim$ (confront Eqs. (4.19-4.23) and Eq. (6.38)), that is, it sends perturbations $(\delta\alpha,\delta\beta,\delta\gamma,\delta r,\delta\Phi)$ to perturbations $(\tilde{\delta\alpha},\tilde{\delta\beta},\tilde{\delta\gamma},\tilde{\delta r},\tilde{\delta\Phi})$ that belong to the same equivalence class of the perturbations $(\delta\alpha,\delta\beta,\delta\gamma,\delta r,\delta\Phi)$.
For future use, let us remark that the equivalence class $\big[(\delta\alpha,\delta\beta,\delta\gamma,\delta r,\delta\Phi)\big]_\sim$ coincides exactly with the the equivalence class $\big[(\delta\alpha,\tilde{\delta\beta},\delta\gamma,\delta r,\tilde{\delta\Phi})\big]_\sim$ for every smooth functions $\tilde{\delta\beta}$ and $\tilde{\delta\Phi}$; in particular, it coincides with the class $\big[(\delta\alpha,0,\delta\gamma,\delta r,0)\big]_\sim$, where 0 stands for the zero function (i.e. the function which vanishes for every t and every x). To see this fact, it is sufficient to prove that, for every smooth functions $\delta\beta,\delta\Phi,\tilde{\delta\beta},\tilde{\delta\Phi}$ there exist two smooth functions δt, δx such that

$$\tilde{\delta\beta} = \delta\beta + \dot{\delta x} - \frac{\alpha^2}{\gamma^2}\delta t' , \qquad \tilde{\delta\Phi} = \delta\Phi + \Phi'\delta x ,$$

but this is reached by setting

$$\delta x = \frac{\tilde{\delta\Phi} - \delta\Phi}{\Phi'} \qquad \text{and} \qquad \delta t := \int \frac{\gamma^2}{\alpha^2}\left(\delta\beta - \tilde{\delta\beta} + \frac{\dot{\tilde{\delta\Phi}} - \dot{\delta\Phi}}{\Phi'}\right) dx \, ; \quad (6.41)$$

indeed, δx and δt are well defined and smooth since $\Phi', \alpha \neq 0$ for every x.
The equivalence relation $\sim$ of the present remark has been introduced in order to prove the forthcoming proposition.

Proposition 10 *The expressions (6.35-6.37) define a one-to-one relation*

$$\begin{aligned} [C^\infty(\mathcal{O},\mathbb{R})]^5/_\sim &\to [C^\infty(\mathcal{O},\mathbb{R})]^3 \\ \big[(\delta\alpha,\delta\beta,\delta\gamma,\delta r,\delta\Phi)\big]_\sim &\mapsto (A, C, E) \end{aligned} \quad (6.42)$$

Proof. For every equivalence class $\big[(\delta\alpha,\delta\beta,\delta\gamma,\delta r,\delta\Phi)\big]_\sim$, one can define the corresponding gauge-invariant quantities A, C, E using the expressions (6.35-6.37), where $(\delta\alpha,\delta\beta,\delta\gamma,\delta r,\delta\Phi)$ is any representative of the class. Let us prove that the correspondence is well defined. By definition, any other representative $(\tilde{\delta\alpha},\tilde{\delta\beta},\tilde{\delta\gamma},\tilde{\delta r},\tilde{\delta\Phi})$ of the class, is related to the representative $(\delta\alpha,\delta\beta,\delta\gamma,\delta r,\delta\Phi)$ as in Eq. (6.38) for an infinitesimal coordinate transformation ψ_ϵ; looking at Eqs. (4.19-4.23), we have that $(\tilde{\delta\alpha},\tilde{\delta\beta},\tilde{\delta\gamma},\tilde{\delta r},\tilde{\delta\Phi})$ are exactly the transformed perturbations of $(\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi)$ via the gauge transformation ψ_ϵ. Hence, the application is well defined since the quantities A, C, E are gauge-invariant. Thus, we write the application (6.42) as

$$A := \frac{[\delta\gamma]_\sim}{\gamma} - \frac{1}{\gamma}\left(\gamma\frac{[\delta\Phi]_\sim}{\Phi'}\right)', \tag{6.43}$$

$$C := \frac{[\delta r]_\sim}{r} - \frac{r'}{r}\frac{[\delta\Phi]_\sim}{\Phi'}, \tag{6.44}$$

$$E := \left(\frac{[\delta\alpha]_\sim}{\alpha}\right)' - \left(\frac{\alpha'}{\alpha}\frac{[\delta\Phi]_\sim}{\Phi'}\right)' + \frac{\gamma^2}{\alpha^2}\left([\delta\dot\beta]_\sim - \frac{[\delta\ddot\Phi]_\sim}{\Phi'}\right), \tag{6.45}$$

where we have introduced the abuse of notation

$$([\delta\alpha]_\sim, [\delta\beta]_\sim, [\delta\gamma]_\sim, [\delta r]_\sim, [\delta\Phi]_\sim) \equiv \big[(\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi)\big]_\sim .$$

Let us show that the application (6.43-6.45) can be inverted in the following way: to any three smooth functions (A, C, E) we associate the equivalence class $\big[(\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi)\big]_\sim$ with representative

$$\delta\alpha := \alpha\int E\,dx\,, \quad \delta\beta := 0\,, \quad \delta\gamma := \gamma A\,, \quad \delta r := rC\,, \quad \delta\Phi := 0\,. \tag{6.46}$$

Let us prove that the the latter is actually the inverse function of Eqs. (6.43-6.45). Firstly, consider (A, C, E), define the equivalence class associated to the perturbations (6.46) and then the quantities $(\tilde A, \tilde C, \tilde E)$ as in Eqs. (6.436.45) with $(\tilde A, \tilde C, \tilde E)$ in place of (A, C, E); hence, we have trivially that $(\tilde A, \tilde C, \tilde E) = (A, C, E)$. Conversely, take an arbitrary equivalence class $\big[(\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi)\big]_\sim$ and fix any representative $(\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi)$, define the quantities (A, C, E) using Eqs. (6.356.37) and then write the perturbations $(\tilde{\delta\alpha}, \tilde{\delta\beta}, \tilde{\delta\gamma}, \tilde{\delta r}, \tilde{\delta\Phi})$ using Eq. (6.46) with $(\tilde{\delta\alpha}, \tilde{\delta\beta}, \tilde{\delta\gamma}, \tilde{\delta r}, \tilde{\delta\Phi})$ in place of $(\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi)$; therefore, we have that $\tilde{\delta\beta} = \tilde{\delta\Phi} = 0$ and

$$\frac{\delta\gamma}{\gamma} - \frac{1}{\gamma}\left(\gamma\frac{\delta\Phi}{\Phi'}\right)' = A = \frac{\tilde{\delta\gamma}}{\gamma} \tag{6.47}$$

$$\frac{\delta r}{r} - \frac{r'}{r}\frac{\delta\Phi}{\Phi'} = C = \frac{\tilde{\delta r}}{r} \tag{6.48}$$

$$\left(\frac{\delta\alpha}{\alpha}\right)' - \left(\frac{\alpha'}{\alpha}\frac{\delta\Phi}{\Phi'}\right)' + \frac{\gamma^2}{\alpha^2}\left(\delta\dot\beta - \frac{\delta\ddot\Phi}{\Phi'}\right) = E = \left(\frac{\tilde{\delta\alpha}}{\alpha}\right)', \tag{6.49}$$

from which

$$\tilde{\delta\alpha} = \delta\alpha - \alpha' - \frac{\delta\Phi}{\Phi'} + \alpha \int \frac{\gamma^2}{\alpha^2}\left(\dot{\delta\beta} - \frac{\ddot{\delta\Phi}}{\Phi'}\right) dx\,, \quad (6.50)$$

$$\tilde{\delta\gamma} = \delta\gamma - \left(\gamma\frac{\delta\Phi}{\Phi'}\right)'\,, \quad (6.51)$$

$$\tilde{\delta r} = \delta r - r'\frac{\delta\Phi}{\Phi'}\,. \quad (6.52)$$

Let us prove that $\big[(\delta\alpha,\delta\beta,\delta\gamma,\delta r,\delta\Phi)\big]_\sim = \big[(\tilde{\delta\alpha},0,\tilde{\delta\gamma},\tilde{\delta r},0)\big]_\sim$. As already proved, $[\delta\beta]_\sim = [0]_\sim$ and $[\delta\Phi]_\sim = [0]_\sim$; this can be verified by defining the transformations δt and δx as in Eq. (6.41) with $\tilde{\delta\beta} = \tilde{\delta\Phi} = 0$, namely

$$\delta x = -\frac{\delta\Phi}{\Phi'} \qquad \text{and} \qquad \delta t := \int \frac{\gamma^2}{\alpha^2}\left(\delta\beta - \frac{\dot{\delta\Phi}}{\Phi'}\right) dx\,; \quad (6.53)$$

hence, we have that $[\delta\alpha]_\sim = [\tilde{\delta\alpha}]_\sim$, $[\delta\gamma]_\sim = [\tilde{\delta\gamma}]_\sim$, $[\delta r]_\sim = [\tilde{\delta r}]_\sim$, whence $(\delta\alpha,\delta\beta,\delta\gamma,\delta r,\delta\Phi) \sim (\tilde{\delta\alpha},0,\tilde{\delta\gamma},\tilde{\delta r},0)$, since Eqs. (6.50-6.52) are exactly the relations in Eq. (6.38) with δt and δx as in Eq. (6.53).

□

Remark 58 It is well known that the (linearized) Einstein-scalar equations are gauge-invariant: this means that, if Eq. (4.29-4.33) are satisfied in a particular gauge, then they are satisfied in every gauge. Now, since the constraints equations (4.32,4.33) are equivalent to the constraint equation (6.6), then, $\mathcal{S} = \sigma$ in a fixed gauge implies that for every gauge there exists a constant $\tilde{\sigma} \in \mathbb{R}$ such that $\mathcal{S} = \tilde{\sigma}$. However, in Proposition 9 we have proved that $\mathcal{S}$ is gauge-invariant: therefore, if $\mathcal{S} = \sigma$ in a fixed gauge, then $\mathcal{S} = \sigma$ in every coordinate system.

Let us now return to the gauge-invariant quantities A, C, E defined in Eqs. (6.35-6.37): these are well defined as long as $\Phi' \neq 0$ and, as already noted, this request is satisfied by all the static wormhole solutions of Chapter 3 and in most of other known wormhole solutions. (41) Moreover, we note that in the particular gauge used in the previous section, for which $\delta\beta = \delta\Phi = 0$, it turns out that $A = \mathcal{A}$, $C = \mathcal{C}$ and $E = \mathcal{D}'$, where $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{D}$ are defined in Eq. (6.14). Therefore, since the linearized field equations are gauge-invariant

[41] As already mentioned in the Introduction, in a very recent paper [22], Carvente et al. have introduced a new class of wormholes with scalar fields that have zero derivative in correspondence of the wormhole throats.

even for the particular choice $\mathcal{S} = 0$ (see Remark 58), the equivalent system (6.25-6.28) obtained in the gauge $\delta\beta = \delta\Phi = 0$ is transformed by any gauge transformation (4.6,4.7) into the same system in which the quantities $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{D}'$ are replaced by A, C and E; this system reads

$$\begin{aligned}\mathfrak{E}_{\ddot{A}} = 0\,, \quad \mathfrak{E}_{\ddot{A}} := {} & \frac{\gamma}{\alpha}\ddot{A} - \frac{\partial}{\partial x}\left[\frac{\alpha}{\gamma}(A' - (d-1)C')\right] \\ & + (d-1)\frac{\alpha}{\gamma}\left(\frac{\alpha'}{\alpha} + (d-2)\frac{r'}{r}\right)C' - (d-1)(d-2)\frac{\alpha\gamma}{r^2}(A - C) \\ & + \frac{2\alpha\gamma V'(\Phi)}{\Phi'}A' + 2\alpha\gamma\left[\frac{(d-3)\kappa}{d-1}V(\Phi) + \gamma^2\frac{V'(\Phi)^2}{\Phi'^2}\right. \\ & \left. + \left(3\frac{\alpha'}{\alpha} + (d-1)\frac{r'}{r}\right)\frac{V'(\Phi)}{\Phi'} + V''(\Phi)\right]A\,, \qquad (6.54)\end{aligned}$$

$$\begin{aligned}\mathfrak{E}_{\ddot{C}} = 0\,, \quad \mathfrak{E}_{\ddot{C}} := {} & \frac{\gamma}{\alpha}\ddot{C} - \frac{\partial}{\partial x}\left[\frac{\alpha}{\gamma}C'\right] - (d-1)\frac{\alpha}{\gamma}\frac{r'}{r}C' + 2(d-2)\frac{\alpha\gamma}{r^2}(A - C) \\ & + 2\alpha\gamma\left[\frac{r'}{r}\frac{V'(\Phi)}{\Phi'} - \frac{2\kappa}{d-1}V(\Phi)\right]A\,, \qquad (6.55)\end{aligned}$$

$$\mathfrak{E}_{E} = 0\,, \quad \mathfrak{E}_{E} := E - A' + (d-1)C' + 2\gamma^2\frac{V'(\Phi)}{\Phi'}A\,, \qquad (6.56)$$

$$\mathfrak{E}_{C'} = 0\,, \quad \mathfrak{E}_{C'} := C' - \left(\frac{\alpha'}{\alpha} - \frac{r'}{r}\right)C - \frac{r'}{r}A\,. \qquad (6.57)$$

Remark 59 We remark again that the system (6.54-6.57) actually generalizes the linearized system (6.25-6.28) to an arbitrary gauge (thus, not assuming that $\delta\beta = \delta\Phi = 0$) and the unknown of this system, A, C, E are gauge-invariant; once that the system has been solved, form the expressions of A, C, E and by using the definitions (6.35-6.37), one can recover the gauge-dependent perturbation functions $\delta\alpha$, $\delta\beta$, $\delta\gamma$, δr, $\delta\Phi$ (of course, in doing this, there are two degrees of freedom, due to the gauge choice - see Proposition 10).

All the previous considerations are summed up in the following

Proposition 11 *In any gauge, the linearized field equations (4.29-4.33), together with the conditions $\Phi' \neq 0$ and $\mathcal{S} = 0$ [Eq. (6.13)], are equivalent to the system (6.54-6.57), where the functions A, C, E are the three gauge-invariant quantities defined in Eqs. (6.35-6.37). Moreover, from Eq. (6.56) the function E can be expressed in terms of the functions A, C*

$$E(t,x) = A'(t,x) - (d-1)C'(t,x) - 2\gamma^2(x)\frac{V'(\Phi(x))}{\Phi'(x)}A(t,x)\,, \qquad (6.58)$$

while the remaining three equations (6.54,6.55,6.57) can be regarded as a constrained evolution system in the following sense: if (A, C) is a (time-dependent) solution of the system $\mathfrak{E}_{\ddot{A}} = 0$, $\mathfrak{E}_{\ddot{C}} = 0$ [Eq. (6.54,6.55)] satisfying the initial conditions

$$\mathfrak{E}_{C'}(0, x) = 0\,, \qquad \dot{\mathfrak{E}_{C'}}(0, x) = 0\,, \qquad \textit{for every } x \in x(\mathcal{O})\,, \tag{6.59}$$

then (A, C) satisfies the equation $\mathfrak{E}_{C'} = 0$ [Eq. (6.57)] for every time t and every x.

Remark 60 The hypothesis $\Phi' \neq 0$ of the previous proposition is no more connected to the choice of the gauge $\delta\Phi = 0$ (see Remark 55), but it is due to the definitions of the gauge-invariant quantities A, C, E in Eqs. (6.35-6.37). Note, however, that the definitions (6.35-6.37) have been clearly inspired by the form of the system (6.25-6.28), which have been obtained by setting $\delta\Phi$ (and $\delta\beta$) equal to zero.

Example 5 As a simple example, let us consider the EBMT wormhole [Eq. (3.36)] for which $\alpha = \gamma = 1$ and $r = \sqrt{x^2 + b^2}$. In this case, the difference between Eq. (6.54) and Eq. (6.55), along with Eq. (6.57) (used to eliminate the derivative C'), gives

$$\ddot{\chi} - \chi'' - \frac{3b^2}{(x^2 + b^2)^2}\chi = 0\,, \qquad \chi := \frac{A - C}{r}\,, \tag{6.60}$$

which coincides with Eq. (15) in Ref. [53]. The interest of this equation is that it involves only one unknown gauge-invariant function $\chi(t, x)$ and reduces the linear stability analysis of the EBMT wormhole to the spectral analysis of the Schrödinger operator $-\frac{d^2}{dx^2} - \frac{3b^2}{(x^2+b^2)^2}$. Since this has one negative eigenvalue (see Refs. [53, 25]), one concludes that the EBMT wormhole is unstable. We will return to the possibility of decoupling the linearized field equations in Section 6.5 and, in particular, to the master equation (6.60) for the EBMT wormhole in Chapter 7.

6.3.1 A straightforward deduction of the system (6.54-6.57) and its generalizations to the case $\mathcal{S} \neq 0$

In Proposition 11 we have stated that the linearized field equations (4.29-4.33) can be rewritten as a system involving only gauge-invariant quantities and the static solution; this equivalent system has been obtained after a long discussion, introducing the assumption $\Phi' \neq 0$ on the static solution and the condition $\mathcal{S} = 0$ on the perturbation functions. We now show that the system (6.54-6.57) can be generalized to a system which is equivalent to the linearized Einstein-scalar equations even for perturbations such that $\mathcal{S} \neq 0$.

Let us consider the quantities $\delta\mathfrak{E}_1$, $\delta\mathfrak{E}_2$, $\delta\mathfrak{E}_3$, $\delta\mathcal{H}$ and $\delta\mathcal{M}$ defined in Eqs. (4.29-4.33) in a general gauge (thus, not assuming that $\delta\beta = 0$ nor $\delta\Phi = 0$), and define the quantities $\widetilde{\mathfrak{E}_{\ddot{A}}}$, $\widetilde{\mathfrak{E}_{\ddot{C}}}$, $\widetilde{\mathfrak{E}_E}$, $\mathfrak{E}_{\delta\mathcal{H}}$ and $\mathfrak{E}_{\delta\mathcal{M}}$ as follows [42]

$$\widetilde{\mathfrak{E}_{\ddot{A}}} := \delta\mathfrak{E}_1 + \frac{d-3}{(d-1)r^2}\delta\mathfrak{E}_2 - \frac{1}{r^2\Phi'}\left(2\frac{\alpha'}{\alpha} - \frac{\gamma'}{\gamma} + (d-3)\frac{r'}{r} + \gamma^2\frac{V'(\Phi)}{\Phi'}\right)\delta\mathfrak{E}_3 + \frac{1}{r^2\Phi'}\delta\mathfrak{E}_3' \,, \tag{6.61}$$

$$\widetilde{\mathfrak{E}_{\ddot{C}}} := \frac{1}{r^2}\left(\frac{2}{(d-1)}\delta\mathfrak{E}_2 - \frac{r'}{r\Phi'}\delta\mathfrak{E}_3\right) \,, \tag{6.62}$$

$$\widetilde{\mathfrak{E}_E} := -\frac{\gamma}{\alpha r^2\Phi'}\delta\mathfrak{E}_3 \,, \tag{6.63}$$

$$\mathfrak{E}_{\delta\mathcal{H}} := \frac{r^{d-1}}{2}\delta\mathcal{H} \,, \tag{6.64}$$

$$\mathfrak{E}_{\delta\mathcal{M}} := \frac{\alpha}{\gamma}\frac{r^{d-1}}{2}\delta\mathcal{M} \,; \tag{6.65}$$

clearly, the system $\widetilde{\mathfrak{E}_{\ddot{A}}} = 0$, $\widetilde{\mathfrak{E}_{\ddot{C}}} = 0$, $\widetilde{\mathfrak{E}_E} = 0$, $\mathfrak{E}_{\delta\mathcal{H}} = 0$, $\mathfrak{E}_{\delta\mathcal{M}} = 0$ is equivalent to the linearized field system.
In addition, we introduce the gauge-invariant quantities A, C, E, defined as in Eqs. (6.35-6.37); isolating the functions $\delta\gamma$, δr and $\delta\dot{\beta}$, we get

$$\delta\gamma = \gamma A + \left(\gamma\frac{\delta\Phi}{\Phi'}\right)' \,, \tag{6.66}$$

$$\delta r = rC + r'\frac{\delta\Phi}{\Phi'} \,, \tag{6.67}$$

$$\delta\dot{\beta} = \left[E + \left(\frac{\delta\alpha}{\alpha}\right)' - \left(\frac{\alpha'}{\alpha}\frac{\delta\Phi}{\Phi'}\right)'\right]\frac{\alpha^2}{\gamma^2} + \frac{\delta\ddot{\Phi}}{\Phi'} \,; \tag{6.68}$$

with the substitutions (6.66-6.68) one can verify that

$$\widetilde{\mathfrak{E}_{\ddot{A}}} = \mathfrak{E}_{\ddot{A}} \,, \qquad \widetilde{\mathfrak{E}_{\ddot{C}}} = \mathfrak{E}_{\ddot{C}} \,, \qquad \widetilde{\mathfrak{E}_E} = \mathfrak{E}_E \,, \tag{6.69}$$

$$\mathfrak{E}_{\delta\mathcal{H}} = \frac{d-1}{2}\frac{\partial}{\partial x}\left[\frac{\alpha r^{d-1}}{\gamma}\mathfrak{E}_{C'}\right] \,, \qquad \mathfrak{E}_{\delta\mathcal{M}} = \frac{d-1}{2}\frac{\partial}{\partial t}\left[\frac{\alpha r^{d-1}}{\gamma}\mathfrak{E}_{C'}\right] \,, \tag{6.70}$$

where $\mathfrak{E}_{\ddot{A}}$, $\mathfrak{E}_{\ddot{C}}$, $\mathfrak{E}_E$, $\mathfrak{E}_{C'}$ are the quantities defined in Eqs. (6.54-6.57). We have already mentioned that $\delta\mathcal{H} = 0$, $\delta\mathcal{M} = 0$ are equivalent to $\mathfrak{E}_{\delta\mathcal{H}} = 0$,

[42] The right hand sides of Eqs. (6.61-6.63) are exactly the right hand sides of Eqs. (6.29-6.31) without the assumption $\delta\beta = 0$.

$\mathfrak{E}_{\delta\mathcal{M}} = 0$; from Eq. (6.70) it results that these are equivalent to

$$\frac{d-1}{2}\frac{\partial}{\partial x}\left[\frac{\alpha r^{d-1}}{\gamma}\mathfrak{E}_{C'}\right] = 0\,, \qquad \frac{d-1}{2}\frac{\partial}{\partial t}\left[\frac{\alpha r^{d-1}}{\gamma}\mathfrak{E}_{C'}\right] = 0\,.$$

Integrating the previous equations in x and t, respectively, it turns out that

$$\mathfrak{E}_{C'} = \sigma\frac{2}{d-1}\frac{\gamma}{\alpha r^{d-1}}\,, \quad \sigma \in \mathbb{R}\,, \qquad \mathfrak{E}_{C'} := C' - \left(\frac{\alpha'}{\alpha} - \frac{r'}{r}\right)C - \frac{r'}{r}A\,. \tag{6.71}$$

Note that Eq. (6.71) actually generalizes Eq. (6.57) to perturbations such that $\mathcal{S} = \sigma \in \mathbb{R}$ (and, obviously, the two equations coincides for $\sigma = 0$): indeed, one can see that [43]

$$\mathfrak{E}_{C'} = \frac{2}{d-1}\frac{\gamma}{\alpha r^{d-1}}\mathcal{S}\,, \tag{6.72}$$

with $\mathcal{S}$ defined in Eq. (6.6) and $\mathfrak{E}_{C'}$ defined in Eq. (6.71), where A and C are as in Eqs. (6.35,6.36.

The results of the present subsection are summed up in the next proposition, which generalizes Proposition 11.

Proposition 12 *In any gauge, the linearized field equations (4.29-4.33), together with the condition $\Phi' \neq 0$, are equivalent to the system (6.54,6.55,6.56, 6.71), where the functions A, C, E are the three gauge-invariant quantities defined in Eqs. (6.35-6.37). Eq. (6.56) gives the expression for the function E given in Eq. (6.58). The arbitrary parameter σ in Eq. (6.71) is related to the perturbation functions as follows: if (A, C, E) is a solution of the system (6.54,6.55,6.56,6.71) for a fixed constant σ, then the corresponding perturbations $(\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi)$ are such that the quantity $\mathcal{S}$ in Eq (6.13) is equal to σ.* [44] *For perturbations such that $\mathcal{S} = 0$, Eq. (6.71) reduces to Eq. (6.57); in this case the three equations (6.54,6.55,6.57) can be regarded as a constrained evolution system in the sense explained in Proposition 11.*

[43] This is exactly Eq. (6.32), expressed in an arbitrary gauge.

[44] Clearly, the perturbations $(\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi)$ are not uniquely determined, since there are two degrees of freedom due to the gauge choice (see Proposition 10); note that, however, the quantity $\mathcal{S}$ is gauge-invariant.

6.4 The gauge-invariant quantities $\mathcal{F}$ and $\mathcal{G}$: the linearized field equations as a constrained wave system

Let us recall Example 5, where we have shown that that, in the EBMT case, the constrained field system (6.54,6.55,6.57) can be decoupled, obtaining a single equation involving only one unknown gauge-invariant function [Eq. (6.60)]. In the rest of this chapter we illustrate a general method to reduce the system (6.54,6.55,6.57) to one master equation (thus, generalizing the considerations in Example 5). To this purpose we start with the following

Proposition 13 *Let (A, C) be a solution of the constrained evolution system Eqs. (6.54,6.55,6.57), then the gauge-invariant quantities*

$$\mathcal{F} := \frac{A - C}{r}\,, \qquad \mathcal{G} := \frac{C}{r} \tag{6.73}$$

satisfy the hyperbolic system of wave equations and the first order ODE

$$\begin{aligned}
&\mathfrak{E}_{\ddot{\mathcal{F}}} = 0\,, \qquad \begin{pmatrix} \mathfrak{E}_{\ddot{\mathcal{F}}} \\ \mathfrak{E}_{\ddot{\mathcal{G}}} \end{pmatrix} := \mathfrak{D} \begin{pmatrix} \mathcal{F} \\ \mathcal{G} \end{pmatrix}, \\
&\mathfrak{E}_{\ddot{\mathcal{G}}} = 0\,, \\
&\mathfrak{D} := \left[\frac{\partial^2}{\partial t^2} - \left(\frac{\alpha}{\gamma} \frac{\partial}{\partial x} \right)^2 + \begin{pmatrix} Y_0 & Y_0 \\ 0 & 0 \end{pmatrix} \frac{\alpha}{\gamma} \frac{\partial}{\partial x} + \frac{\alpha^2}{\gamma^2} \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \right],
\end{aligned} \tag{6.74}$$

$$\mathfrak{E}_{\mathcal{G}'} = 0\,, \qquad \mathfrak{E}_{\mathcal{G}'} := \mathcal{G}' - \left(\frac{\alpha'}{\alpha} - \frac{r'}{r} \right) \mathcal{G} - \frac{r'}{r} \mathcal{F}\,; \tag{6.75}$$

here, the functions Y_0 and W_{ij} are given in terms of the static solution (6.1) as follows:

$$Y_0 \;:=\; 2\alpha\gamma \frac{V'(\Phi)}{\Phi'} + (d-3) \frac{\alpha'}{\gamma} \frac{r'}{r}\,, \tag{6.76}$$

$$W_{11} \;:=\; \frac{r'}{r} \left(2(d-1) \frac{\alpha'}{\alpha} + d \frac{r'}{r} \right) - 3(d-2) \frac{\gamma^2}{r^2} + Z_{11}\,, \tag{6.77}$$

$$W_{12} \;:=\; 2(d-1) \frac{\alpha'^2}{\alpha^2} + (d-3) \frac{r'}{r} \left(\frac{r'}{r} - \frac{\alpha'}{\alpha} \right) + Z_{12}\,, \tag{6.78}$$

$$\begin{aligned} W_{21} \;:=\; & -4 \frac{r'^2}{r^2} + (d-1) \frac{\gamma^2}{r^2} \\ & + (d-3) \left(\frac{2}{d-2} \frac{\alpha'}{\alpha} \frac{r'}{r} + \frac{\kappa}{(d-1)(d-a)} \Phi'^2 \right) + Z_{21}\,, \end{aligned} \tag{6.79}$$

$$W_{22} \;:=\; \frac{r'}{r} \left(- \frac{d^2 + d - 8}{d-2} \frac{\alpha'}{\alpha} + 3 \frac{r'}{r} \right) - \frac{\gamma^2}{r^2} - \frac{(d-3)\kappa}{(d-1)(d-2)} \Phi'^2 + Z_{22}\,, \tag{6.80}$$

where

$$
\begin{aligned}
Z_{11} &:= Z_{12} + \frac{2\kappa}{d-1}\gamma^2 V(\Phi)\,, && (6.81)\\
Z_{12} &:= 2\gamma^2\left[\gamma^2\frac{V'(\Phi)^2}{\Phi'^2} + \left(3\frac{\alpha'}{\alpha} + (d-1)\frac{r'}{r}\right)\frac{V'(\Phi)}{\Phi'} + V''(\Phi)\right]\,, && (6.82)\\
Z_{21} &:= 2\gamma^2\left[-\frac{\kappa}{d-2}V(\Phi) + \frac{r'}{r}\frac{V'(\Phi)}{\Phi'}\right]\,, && (6.83)\\
Z_{22} &:= Z_{21} - \frac{2(d-4)\kappa}{(d-1)(d-2)}\gamma^2 V(\Phi)\,. && (6.84)
\end{aligned}
$$

Conversely, if $(\mathcal{F}, \mathcal{G})$ *is a solution of Eqs.* (6.74,6.75)*, then the quantities* $A := r(\mathcal{F} + \mathcal{G})$ *and* $C := r\mathcal{G}$ *satisfy the constrained system* (6.54,6.55,6.57)*.*

Proof. The equivalence of the systems (6.54,6.55,6.57) and (6.74,6.75) is proved immediately by observing that the quantities $\mathfrak{E}_{\ddot{\mathcal{F}}}$, $\mathfrak{E}_{\ddot{\mathcal{G}}}$ and $\mathfrak{E}_{\mathcal{G}'}$ (with $\mathcal{F}$ and $\mathcal{G}$ defined as in Eq. (6.73)) can be written as combinations of the quantities $\mathfrak{E}_{\ddot{A}}$, $\mathfrak{E}_{\ddot{C}}$, $\mathfrak{E}_{C'}$ and $\mathfrak{E}'_{C'}$ as follows:

$$
\begin{aligned}
\mathfrak{E}_{\ddot{\mathcal{F}}} = \frac{\alpha}{\gamma r}\Bigg[&\mathfrak{E}_{\ddot{A}} - \mathfrak{E}_{\ddot{C}} + (d-1)\frac{\alpha}{\gamma}\left(\frac{\gamma'}{\gamma} - 3\frac{\alpha'}{\alpha} - \frac{d^2-3d+4}{d-1}\frac{r'}{r}\right)\mathfrak{E}_{C'}\\
&- (d-1)\frac{\alpha}{\gamma}\mathfrak{E}'_{C'}\Bigg]\,, \qquad (6.85)\\
\mathfrak{E}_{\ddot{\mathcal{G}}} = \frac{\alpha}{\gamma r}\Bigg[&\mathfrak{E}_{\ddot{C}} + (d+1)\frac{\alpha}{\gamma}\frac{r'}{r}\mathfrak{E}_{C'}\Bigg]\,, \qquad (6.86)\\
\mathfrak{E}_{\mathcal{G}'} = \frac{1}{r}&\mathfrak{E}_{C'}\,. \qquad (6.87)
\end{aligned}
$$

□

Remark 61 Let us underline the following fact: although from Eq. (6.87) it is evident that the constraint equation $\mathfrak{E}_{C'} = 0$ is equivalent to the equation $\mathfrak{E}_{\mathcal{G}'} = 0$, it is not necessary true that the latter is still a constraint for the wave system (6.74), if with the term "constraint" we indicate an equation that acts as a restriction in the choice of the initial data of an evolution system; indeed, the recombinations (6.85-6.86) make evident that the wave system (6.74) has been obtained from the evolution system (6.54,6.55) *using also* the constraint $\mathfrak{E}_{\mathcal{G}'} = 0$. This inevitably modifies the relation among Eqs. (6.74,6.75) which now reads

$$
\ddot{\mathfrak{E}}_{\mathcal{G}'} - \frac{\alpha^2}{\gamma^2}\mathfrak{E}''_{\mathcal{G}'} - 3\frac{\alpha}{\gamma}\left(\frac{\alpha}{\gamma}\right)'\mathfrak{E}'_{\mathcal{G}'} + J_0\mathfrak{E}_{\mathcal{G}'} + \frac{r'}{r}\mathfrak{E}_{\ddot{\mathcal{F}}} + \frac{r}{\alpha}\left(\frac{\alpha}{r}\right)'\mathfrak{E}_{\ddot{\mathcal{G}}} - \mathfrak{E}'_{\ddot{\mathcal{G}}} = 0\,, \quad (6.88)
$$

where

$$J_0 := \frac{\alpha^2}{\gamma^2}\left[\frac{\gamma''}{\gamma} - 3\frac{\gamma'^2}{\gamma^2} - \frac{\alpha'^2}{\alpha^2} + \frac{\alpha'}{\alpha}\left(3\frac{\gamma'}{\gamma} + 2\frac{d^2-2d-1}{d-2}\frac{r'}{r}\right) + \frac{(2d-5)\kappa}{(d-1)(d-2)}\left(2\gamma^2 V(\Phi) + \Phi'^2\right)\right] - (d-3)\frac{\alpha^2}{r^2}\,.$$

The identity (6.88) can be verified by hand using the definitions given in Eqs. (6.74,6.75) and the background equations (3.6-3.9); alternatively, one can see that this identity is exactly Eq. (6.9) reformulated in an arbitrary gauge and with the quantities $\delta\mathfrak{E}_1$, $\delta\mathfrak{E}_2$, $\delta\mathfrak{E}_3$, $\mathcal{S}$ expressed in terms of the new quantities $\mathfrak{E}_{\ddot{\mathcal{F}}}$, $\mathfrak{E}_{\ddot{\mathcal{G}}}$, $\mathfrak{E}_E$, $\mathfrak{E}_{\mathcal{G}'}$ using Eqs. (6.61-6.63,6.69,6.72) and then Eqs. (6.85-6.87) (this can be done since the quantities $\delta\mathfrak{E}_1$, $\delta\mathfrak{E}_2$, $\delta\mathfrak{E}_3$, $\mathcal{S}$ are gauge-invariant). Note that, in doing this, $\mathfrak{E}_E$ disappears from the identity.
In spite of this fact, in Ref. [1], Eq. (6.75) (therein indicated with the number (43)) is regarded as a constraint for the wave system (6.74) (therein indicated with the number (42)); in this case the epithet "constraint" was simply used to highlight that Eq. (6.75) does not contain second temporal derivatives. After this clarification, in order to maintain compatibility with the terminology used in Ref. [1], in the sequel we usually refer to Eqs. (6.74,6.75) as "constrained wave system".

Example 6 Let us observe that in the case of the EBMT wormhole [Eq. (3.36)], it follows that $Y_0 = W_{12} = 0$, so that the equation $\mathfrak{E}_{\ddot{\mathcal{F}}} = 0$ for $\mathcal{F}$ in the system (6.74) decouples trivially from the remaining ones; note that this equation actually coincides with Eq. (6.60) as, in this case, $\mathcal{F}$ and χ are the the same function.

6.5 A master equation for the gauge-invariant quantity χ and the solution of the linearized field equations

In the previous section we have rewritten the linearized equations (6.54,6.57) as the wave system (6.74 subject to the constraint (6.75); in the EBMT case this system is already decoupled (see Example 6). Inspired by this possibility, in the following theorem, we describe a general trick which allows to decouple the constrained wave system (6.74,6.75); this requires that a static solution

$$(\mathcal{F}_0, \mathcal{G}_0) := (\mathcal{F}_0(x), \mathcal{G}_0(x))$$

of Eqs. (6.74,6.75) is known, such that $\mathcal{G}_0(x) \neq 0$ for every $x \in x(\mathcal{O})$.

Theorem 7 *Let* $(\mathcal{F}, \mathcal{G})$ *and* $(\mathcal{F}_0, \mathcal{G}_0)$ *be, respectively, a (time-dependent) solution and a static solution of the constrained wave system (6.74,6.75), such that* $\mathcal{G}_0(x) \neq 0$ *for every* $x \in x(\mathcal{O})$*, then the gauge-invariant quantity*

$$\tilde{\chi} := \mathcal{F} - \frac{\mathcal{F}_0}{\mathcal{G}_0}\mathcal{G} \tag{6.89}$$

satisfies the decoupled wave equation

$$\mathfrak{E}_{\tilde{\chi}} = 0\,, \qquad \mathfrak{E}_{\tilde{\chi}} := \left[\frac{\partial^2}{\partial t^2} - \left(\frac{\alpha}{\gamma}\frac{\partial}{\partial x}\right)^2 + Y_0\frac{\alpha}{\gamma}\frac{\partial}{\partial x} + \frac{\alpha^2}{\gamma^2}\tilde{\mathcal{V}}\right]\tilde{\chi} \tag{6.90}$$

with the potential

$$\tilde{\mathcal{V}} = W_{11} - \frac{\mathcal{F}_0}{\mathcal{G}_0}W_{21} - 2\frac{r'}{r}\left(\frac{\mathcal{F}_0}{\mathcal{G}_0}\right)' + \frac{\gamma}{\alpha}\frac{r'}{r}Y_0\left(\frac{\mathcal{F}_0}{\mathcal{G}_0} + 1\right)\,; \tag{6.91}$$

moreover, the couple $(\mathcal{G}, \tilde{\chi})$ *satisfies the equations*

$$\mathfrak{E}_{\ddot{\mathcal{G}},\tilde{\chi}} = 0\,, \quad \mathfrak{E}_{\ddot{\mathcal{G}},\tilde{\chi}} := \left[\frac{\partial^2}{\partial t^2} - \left(\frac{\alpha}{\gamma}\frac{\partial}{\partial x}\right)^2 + \frac{\alpha^2}{\gamma^2}\left(\frac{\mathcal{F}_0}{\mathcal{G}_0}W_{21} + W_{22}\right)\right]\mathcal{G} + \frac{\alpha^2}{\gamma^2}W_{21}\tilde{\chi}\,, \tag{6.92}$$

$$\mathfrak{E}_{\mathcal{G}',\tilde{\chi}} = 0\,, \quad \mathfrak{E}_{\mathcal{G}',\tilde{\chi}} := \mathcal{G}_0\frac{\partial}{\partial x}\left[\frac{\mathcal{G}}{\mathcal{G}_0}\right] - \frac{r'}{r}\tilde{\chi}\,. \tag{6.93}$$

Conversely, if $(\mathcal{G}, \tilde{\chi})$ *satisfies Eqs. (6.90,6.92,6.93), then the quantities* $\mathcal{F} := \tilde{\chi} + \frac{\mathcal{F}_0}{\mathcal{G}_0}\mathcal{G}$ *and* $\mathcal{G}$ *satisfy the constrained wave system (6.74,6.75).*

Proof. In order to show that the system (6.74,6.75) is equivalent to the system (6.90,6.92,6.93) it is sufficient to note that the following identities hold (with $\tilde{\chi}$ defined as in Eq. (6.89))

$$\mathfrak{E}_{\tilde{\chi}} = \mathfrak{E}_{\ddot{\mathcal{F}}} - \frac{\mathcal{F}_0}{\mathcal{G}_0}\mathfrak{E}_{\ddot{\mathcal{G}}} + K_0\mathfrak{E}_{\mathcal{G}'}\,, \tag{6.94}$$

$$\mathfrak{E}_{\ddot{\mathcal{G}},\tilde{\chi}} = \mathfrak{E}_{\ddot{\mathcal{G}}}\,, \tag{6.95}$$

$$\mathfrak{E}_{\mathcal{G}',\tilde{\chi}} = \mathfrak{E}_{\mathcal{G}'}\,, \tag{6.96}$$

where

$$K_0 := \left\{\frac{\alpha^2}{\gamma^2}\frac{2}{\mathcal{G}_0}\left[\mathcal{F}_0\frac{r'}{r}\left(1 - \frac{\mathcal{F}_0}{\mathcal{G}_0}\right) + \alpha\left(\frac{\mathcal{F}_0}{\alpha}\right)'\right] - W_0\left(1 + \frac{\mathcal{F}_0}{\mathcal{G}_0}\right)\right\}.$$

Note that, in verifying Eqs. (6.94,6.96), one has to use the fact that $(\mathcal{F}_0, \mathcal{G}_0)$ is a solution of the static version of Eqs. (6.74,6.75) in order to get rid of the derivatives $\mathcal{F}_0''$, $\mathcal{G}_0''$ and $\mathcal{G}_0'$. As an example, the proof of Eq. (6.96) reads

$$\begin{aligned}\mathfrak{E}_{\mathcal{G}',\tilde{\chi}} := \mathcal{G}' - \frac{\mathcal{G}_0'}{\mathcal{G}_0}\mathcal{G} - \frac{r'}{r}\tilde{\chi} = \mathcal{G}' - \left[\left(\frac{\alpha'}{\alpha} - \frac{r'}{r}\right)\mathcal{G}_0 + \frac{r'}{r}\mathcal{F}_0\right]\frac{\mathcal{G}}{\mathcal{G}_0} - \frac{r'}{r}\left(\mathcal{F} - \frac{\mathcal{F}_0}{\mathcal{G}_0}\mathcal{G}\right) \\ = \mathcal{G}' - \left(\frac{\alpha'}{\alpha} - \frac{r'}{r}\right)\mathcal{G} - \frac{r'}{r}\mathcal{F} =: \mathfrak{E}_{\mathcal{G}'} .\end{aligned}$$

□

Remark 62 Let us observe that it is always possible to eliminate the first spatial derivative from Eq. (6.90); indeed, if one defines the gauge-invariant quantity

$$\chi := \frac{\tilde{\chi}}{a}, \qquad a(x) = a_0 e^{\int_{x_0}^{x} \frac{Y_0(y)\gamma(y)}{2\alpha(y)} dy}, \tag{6.97}$$

where a_0 and x_0 are two constants, then $\tilde{\chi}$ satisfies Eq. (6.90) if and only if χ satisfies the wave equation [45]

$$\mathfrak{E}_\chi = 0, \qquad \mathfrak{E}_\chi := \left[\frac{\partial^2}{\partial t^2} - \left(\frac{\alpha}{\gamma}\frac{\partial}{\partial x}\right)^2 + \frac{\alpha^2}{\gamma^2}\mathcal{V}\right]\chi \tag{6.98}$$

with the potential

$$\mathcal{V} = \tilde{\mathcal{V}} + \frac{1}{4}\frac{\gamma^2}{\alpha^2}Y_0^2 - \frac{1}{2}\frac{\gamma}{\alpha}Y_0' . \tag{6.99}$$

As a consequence of the previous remark, we can provide a new version of Theorem 7:

Theorem 8 *Let $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}_0, \mathcal{G}_0)$ be, respectively, a (time-dependent) solution and a static solution of the constrained wave system (6.74,6.75), then the gauge-invariant quantity*

$$\chi := \frac{1}{a}\left(\mathcal{F} - \frac{\mathcal{F}_0}{\mathcal{G}_0}\mathcal{G}\right), \qquad a(x) = a_0 e^{\int_{x_0}^{x} \frac{Y_0(y)\gamma(y)}{2\alpha(y)} dy}, \qquad a_0, x_0 \in \mathbb{R} \tag{6.100}$$

[45] This can be easily proved noting that a satisfies

$$a' = \frac{Y_0\gamma}{2\alpha}a, \qquad a'' = \left[\left(\frac{Y_0\gamma}{2\alpha}\right)' + \left(\frac{Y_0\gamma}{2\alpha}\right)^2\right]a .$$

satisfies the decoupled wave equation

$$\mathfrak{E}_\chi = 0\,, \qquad \mathfrak{E}_\chi := \left[\frac{\partial^2}{\partial t^2} - \left(\frac{\alpha}{\gamma}\frac{\partial}{\partial x}\right)^2 + \frac{\alpha^2}{\gamma^2}\mathcal{V}\right]\chi \tag{6.101}$$

with the potential

$$\begin{aligned}\mathcal{V} :=\ & \frac{d-1}{2(d-2)}\left[(d-5)\frac{\alpha'}{\alpha}\frac{r'}{r} - \frac{3\kappa}{2}\Phi'^2\right] + \frac{(d-3)(d-5)}{4}\frac{\gamma^2}{r^2} - 2\frac{r'}{r}\left(\frac{\mathcal{F}_0}{\mathcal{G}_0}\right)' \\ & + \gamma^2\left[2\gamma^2\frac{V'(\Phi)^2}{\Phi'^2} + 4\left(\frac{\alpha'}{\alpha} + \frac{d-1}{2}\frac{r'}{r}\right)\frac{V'(\Phi)}{\Phi'} + V''(\Phi) - \frac{d+1}{d-2}\frac{\kappa}{2}V(\Phi)\right] \\ & + \frac{2}{d-2}\left[(d-2)\frac{\gamma^2}{r^2} - 2(d-1)\frac{\alpha'}{\alpha}\frac{r'}{r} - \kappa\Phi'^2 - \frac{2\kappa}{d-1}\gamma^2 V(\Phi)\right]\frac{\mathcal{F}_0}{\mathcal{G}_0}\,;\end{aligned} \tag{6.102}$$

moreover, the couple $(\mathcal{G}, \tilde{\chi})$ satisfies the equations

$$\mathfrak{E}_{\ddot{\mathcal{G}},\chi} = 0\,, \quad \mathfrak{E}_{\ddot{\mathcal{G}},\chi} := \left[\frac{\partial^2}{\partial t^2} - \left(\frac{\alpha}{\gamma}\frac{\partial}{\partial x}\right)^2 + \frac{\alpha^2}{\gamma^2}\left(\frac{\mathcal{F}_0}{\mathcal{G}_0}W_{21} + W_{22}\right)\right]\mathcal{G} + a\frac{\alpha^2}{\gamma^2}W_{21}\chi\,, \tag{6.103}$$

$$\mathfrak{E}_{\mathcal{G}',\chi} = 0\,, \quad \mathfrak{E}_{\mathcal{G}',\chi} := \mathcal{G}_0\frac{\partial}{\partial x}\left[\frac{\mathcal{G}}{\mathcal{G}_0}\right] - a\frac{r'}{r}\chi\,. \tag{6.104}$$

Conversely, if $(\mathcal{G}, \chi)$ satisfies Eqs. (6.101,6.103,6.104), then the quantities $\mathcal{F} := a\chi + \frac{\mathcal{F}_0}{\mathcal{G}_0}\mathcal{G}$ and $\mathcal{G}$ satisfy the constrained wave system (6.74,6.75).

Proof. Substantially, the theorem has already been proved. Indeed, the function χ in Eq. (6.100) is exactly the function defined in Remark 62 by Eq. (6.100) (recalling the definition of $\tilde{\chi}$ in Eq. (6.89)); analogously, Eq. (6.101) is exactly Eq. (6.98). The new expression for the potential $\mathcal{V}$ in Eq. (6.102) has been computed from the expression in Eq. (6.99) by using the expression for $\tilde{\mathcal{V}}$ in Eq. (6.91) and by using Eqs. (6.76,6.77,6.79) in order to make explicit the dependence of Y_0, W_{11} and W_{21} on the static solution (6.1); the background equations (3.6,3.8,3.9) have also been employed in order to remove the derivatives Φ'', r'' and r'^2. Finally Eqs. (6.103,6.104) are precisely Eqs. (6.92,6.93) in which the function $\tilde{\chi}$ has been replaced by $a\chi$ (see Eq. (6.97)).

□

In the sequel, we always refer to Eq. (6.101) as the *master equation* since this reduces the linear stability analysis of the solution (6.1) to the spectral analysis of the linear, Schrödinger-type operator arising from Eq. (6.101)

$$H := -\left(\frac{\alpha}{\gamma}\frac{d}{dx}\right)^2 + \frac{\alpha^2}{\gamma^2}\mathcal{V}\,.$$

Finally, let me repeat that the above approach for decoupling the linearized field equations (yielding the master equation) requires the knowledge of a static solution $(\mathcal{F}_0(x), \mathcal{G}_0(x))$ of Eqs. (6.74,6.75): in the next chapter I will show that for static wormholes depending on parameters, this static solution can be obtained following a general strategy.

6.5.1 A solution of Eqs. (6.103,6.104) in dependence of χ, and the recovering of the perturbations $\delta\alpha$, $\delta\beta$, $\delta\gamma$, δr, $\delta\Phi$

Once the master equation (6.101) has been solved for the function $\chi(t,x)$, one has still to solve Eqs. (6.103,6.104) in order to find $\mathcal{G}(t,x)$. For the moment, let us suppose that this system is solvable and let $\mathcal{G}$ be its solution; then from χ and $\mathcal{G}$ it is possible to reconstruct the gauge-invariant quantity $\mathcal{F}$ recalling that

$$\mathcal{F}(t,x) = a(x)\chi(t,x) + \frac{\mathcal{F}_0(x)}{\mathcal{G}_0(x)}\mathcal{G}(t,x)\,. \tag{6.105}$$

The quantities A, C, E are immediately obtained by inverting the definitions (6.73) and using Eq. (6.58):

$$A(t,x) = r(x)\left[a(x)\chi(t,x) + \left(1 + \frac{\mathcal{F}_0(x)}{\mathcal{G}_0(x)}\right)\mathcal{G}(t,x)\right]\,, \tag{6.106}$$

$$C(t,x) = r(x)\mathcal{G}(t,x)\,, \tag{6.107}$$

$$E(t,x) = -2\gamma^2 r\frac{V(\Phi)}{\Phi'}\left(1 + \frac{\mathcal{F}_0}{\mathcal{G}_0}\right)\mathcal{G} - \frac{\partial}{\partial x}\left[r\left(d - 2 - \frac{\mathcal{F}_0}{\mathcal{G}_0}\right)\mathcal{G}\right]$$
$$+ a\,r\left[\left(\frac{r'}{r} + \frac{a'}{a} - 2\gamma^2\frac{V'(\Phi)}{\Phi'}\right)\chi + \chi'\right]\,. \tag{6.108}$$

Finally, as already mentioned in Section 6.3, the perturbation functions $\delta\alpha$, $\delta\beta$, $\delta\gamma$, δr, $\delta\Phi$ can be recovered (after fixing a gauge - see Proposition 10) by inverting the definitions of A, C and E in Eqs. (6.35-6.37) and using their expressions in Eqs. (6.106-6.108).

Let us return to the system (6.103,6.104) in the unknown $\mathcal{G}$ (the function χ is the solution of the master equation $\mathfrak{E}_\chi = 0$ and is supposed to be given); in general, there is no hope to find a function satisfying a system of two independent differential equations. Fortunately, Eqs. (6.103,6.104) are not independent: indeed, one can rewrite the identity (6.88) with the quantities $\mathfrak{E}_{\mathcal{G}'}$, $\mathfrak{E}_{\ddot{\mathcal{F}}}$, $\mathfrak{E}_{\ddot{\mathcal{G}}}$ replaced by $\mathfrak{E}_{\mathcal{G}',\chi}$, $\mathfrak{E}_\chi$, $\mathfrak{E}_{\ddot{\mathcal{G}},\chi}$ by using Eqs. (6.94-6.96) and the fact that $\mathfrak{E}_{\tilde{\chi}} = \mathfrak{E}_\chi$, $\mathfrak{E}_{\ddot{\mathcal{G}},\tilde{\chi}} = \mathfrak{E}_{\ddot{\mathcal{G}},\chi}$, $\mathfrak{E}_{\mathcal{G}',\tilde{\chi}} = \mathfrak{E}_{\mathcal{G}',\chi}$ (if $\chi = a\tilde{\chi}$). In this way Eq. (6.88) becomes

$$\ddot{\mathfrak{E}}_{\mathcal{G}',\chi} - \frac{\alpha^2}{\gamma^2}\mathfrak{E}''_{\mathcal{G}',\chi} - 3\frac{\alpha}{\gamma}\left(\frac{\alpha}{\gamma}\right)'\mathfrak{E}'_{\mathcal{G}',\chi} + \left[J_0 - \frac{r'}{r}K_0\right]\mathfrak{E}_{\mathcal{G}',\chi} + \frac{r'}{r}\mathfrak{E}_\chi - \mathfrak{E}'_{\ddot{\mathcal{G}},\chi} + \left[\frac{\alpha'}{\alpha} - \left(1 - \frac{\mathcal{F}_0}{\mathcal{G}_0}\right)\frac{r'}{r}\right]\mathfrak{E}_{\ddot{\mathcal{G}},\chi} = 0\,. \quad (6.109)$$

Suppose now that $\mathcal{G}$ satisfies the equation $\mathfrak{E}_{\mathcal{G}',\chi} = 0$ [Eq. (6.104)] (and, as already mentioned, that χ is a solution of the master equation $\mathfrak{E}_\chi = 0$); hence, the identity (6.109) gives

$$\mathfrak{E}'_{\ddot{\mathcal{G}},\chi} = \left[\frac{\alpha'}{\alpha} - \left(1 - \frac{\mathcal{F}_0}{\mathcal{G}_0}\right)\frac{r'}{r}\right]\mathfrak{E}_{\ddot{\mathcal{G}},\chi}\,. \quad (6.110)$$

Now, Eq. (6.110) can be interpreted as an ordinary differential equation of the first order in the unknown $\mathfrak{E}_{\ddot{\mathcal{G}},\chi}(x) \equiv \mathfrak{E}_{\ddot{\mathcal{G}},\chi}(t,x)$, whose solutions depend on the initial condition $\mathfrak{E}_{\ddot{\mathcal{G}},\chi}(0) \equiv \mathfrak{E}_{\ddot{\mathcal{G}},\chi}(t,0)$. In particular, if $\mathfrak{E}_{\ddot{\mathcal{G}},\chi}(t,0) = 0$ for every time t, then the solution of Eq. (6.110) is $\mathfrak{E}_{\ddot{\mathcal{G}},\chi}(t,x) \equiv 0$ for every t and x, namely, Eq. (6.103) holds. Therefore, in order to find a solution of the system (6.103,6.104), it is sufficient to find (for any fixed solution χ of the master equation $\mathfrak{E}_\chi = 0$) a solution $\mathcal{G}$ of the equation $\mathfrak{E}_{\mathcal{G}',\chi} = 0$ [Eq. (6.104)] such that

$$\mathfrak{E}_{\ddot{\mathcal{G}},\chi}(t,0) = 0 \quad \text{for every time } t, \quad (6.111)$$

where $\mathfrak{E}_{\ddot{\mathcal{G}},\chi}(t,x) \equiv \mathfrak{E}_{\ddot{\mathcal{G}},\chi}$ is defined by Eq. (6.103).
Luckily, Eq. (6.104) can be easily integrated, leading to

$$\mathcal{G}(t,x) = \mathcal{G}_0(x)\int_{x_0}^{x}\frac{r'(y)}{r(y)}\frac{a(y)}{\mathcal{G}_0(y)}\chi(t,y)dy + \mathcal{G}_0(x)P(t)\,, \quad (6.112)$$

where x_0 and $P(t)$ are, respectively, an integration constant and an integration function: we can use these two degrees of freedom in determining the function $\mathcal{G}$ in order to impose the condition (6.111). Indeed, inserting the expression (6.112) into Eq. (6.111) and set $x_0 = 0$, it turns out that the function $P(t)$ has to solve the second order ODE

$$P'' + p_1 P = p_2\, \chi(t,0) \tag{6.113}$$

where p_1 and p_2 are two real constants defined as

$$p_1 := \frac{\alpha(0)^2}{\gamma(0)^2}\left[W_{22}(0) + W_{21}(0)\frac{\mathcal{F}_0(0)}{\mathcal{G}_0(0)}\right], \qquad p_2 := -\frac{\alpha(0)^2}{\gamma(0)^2}\frac{W_{21}(0)}{\mathcal{G}_0(0)}. \tag{6.114}$$

Eq. (6.113) can be solved by means of the variation of constants method, yielding [46]

$$P(t) = \frac{p_2}{2\sqrt{-p_1}}\left[e^{\sqrt{-p_1}t}\int_{t_0}^{t}\chi(s,0)e^{-\sqrt{-p_1}s}ds - e^{-\sqrt{-p_1}t}\int_{t_0}^{t}\chi(s,0)e^{\sqrt{-p_1}s}ds\right]$$
$$\text{with } p_1, p_2 \text{ as in (6.114).} \tag{6.115}$$

Thus, the following theorem is proved

Theorem 9 *In any gauge, the linearized field equations (4.29-4.33), together with the conditions $\Phi' \neq 0$ and $\mathcal{S} = 0$ [Eq. (6.13)], are equivalent to the system (6.101,6.108,6.112,6.115), where the functions χ, $\mathcal{G}$, E are three gauge-invariant quantities defined, respectively, by Eqs. (6.100,6.73,6.35,6.36), by Eqs. (6.73,6.36) and by Eq. (6.37), where $(\mathcal{F}_0(x), \mathcal{G}_0(x))$ is a static solution of the system (6.74,6.75).*
Moreover, the system (6.101,6.108,6.112,6.115) is decoupled since Eq. (6.101) is a wave equation in the only unknown χ and, once this equation is solved, Eq. (6.108) and Eqs. (6.112,6.115) define, respectively, the functions E and $\mathcal{G}$ in terms of χ.

[46] Let us recall again that the present gauge-invariant decoupling method actually generalizes to arbitrary dimension the four-dimensional method introduced in Ref. [1]; here, the constrained system (6.74,6.75) and the master equation (6.101) are derived for $d = 3$. In addition, an attempt find the general solution $(\mathcal{F}, \mathcal{G})$ of (6.74,6.75) in terms of the solution χ of the master equation was made; unfortunately, the authors obtain Eqs. (6.105,6.112) with $P(t) = 0$ for every t, which in general is not true (see Eqs. (61,62) of Ref. [1]).

Chapter 7

Applications of the gauge-invariant method

7.1 A general strategy for obtaining a static solution $(\mathcal{F}_0, \mathcal{G}_0)$ of the constrained wave system (6.74,6.75)

Let us suppose to have a static wormhole solution of the form (6.1) depending on certain parameters $\boldsymbol{\lambda} = \{\lambda_1, ..., \lambda_n\} \in \mathcal{L} \subseteq \mathbb{R}^n$ with $\mathcal{L}$ an open subset of $\mathbb{R}^n$ and $n \geq 1$; we write

$$\beta = 0\,, \qquad (\alpha, \gamma, r, \Phi) := \big(\alpha(\boldsymbol{\lambda}; x), \gamma(\boldsymbol{\lambda}; x), r(\boldsymbol{\lambda}; x), \Phi(\boldsymbol{\lambda}; x)\big)\,. \tag{7.1}$$

For example, one can think of the Torii-Shinkai wormhole (3.31) and the EBMT wormhole (3.36), depending on the parameter $b > 0$, the Ellis-Bronnikov wormhole (3.39), depending on the parameters $b > 0$, $\gamma_1 \in \mathbb{R}$, or the AdS wormhole (3.62) depending on the parameters $k > 0$, $B > 0$. For every fixed value of the parameters $\boldsymbol{\lambda} \in \mathcal{L}$, we introduce a "perturbation" of the static solution (7.1) near $\boldsymbol{\lambda}$, that is, we introduce n real constants $\delta\lambda_1, ..., \delta\lambda_n \in \mathbb{R}$ ($\delta\boldsymbol{\lambda} := \{\delta\lambda_1, ..., \delta\lambda_n\} \in \mathbb{R}^n$) and a small real parameter ϵ, such that the "perturbed" metric has the form (2.24) with the coefficients defined as

$$\begin{aligned} \alpha(t,x) &:= \alpha(\boldsymbol{\lambda} + \epsilon\,\delta\boldsymbol{\lambda}; x)\,, & \beta(t,x) &:= 0 \\ \gamma(t,x) &:= \gamma(\boldsymbol{\lambda} + \epsilon\,\delta\boldsymbol{\lambda}; x)\,, & r(t,x) &:= r(\boldsymbol{\lambda} + \epsilon\,\delta\boldsymbol{\lambda}; x)\,, \end{aligned} \tag{7.2}$$

and such that the "perturbed" field reads

$$\Phi(t,x) := \Phi(\boldsymbol{\lambda} + \epsilon\,\delta\boldsymbol{\lambda}; x)\,. \tag{7.3}$$

The expansions up to the first order in ϵ of the "perturbed" static solution read

$$\begin{aligned}
\alpha(\boldsymbol{\lambda}+\epsilon\delta\boldsymbol{\lambda};x) &= \alpha(\boldsymbol{\lambda};x) + \epsilon\left(\frac{\partial\alpha}{\partial\lambda_1}(\boldsymbol{\lambda};x),...,\frac{\partial\alpha}{\partial\lambda_n}(\boldsymbol{\lambda};x)\right)\cdot\delta\boldsymbol{\lambda} + O(\epsilon^2)\,,\\
\gamma(\boldsymbol{\lambda}+\epsilon\,\delta\boldsymbol{\lambda};x) &= \gamma(\boldsymbol{\lambda};x) + \epsilon\left(\frac{\partial\gamma}{\partial\lambda_1}(\boldsymbol{\lambda};x),...,\frac{\partial\gamma}{\partial\lambda_n}(\boldsymbol{\lambda};x)\right)\cdot\delta\boldsymbol{\lambda} + O(\epsilon^2)\,,\\
r(\boldsymbol{\lambda}+\epsilon\delta\boldsymbol{\lambda};x) &= r(\boldsymbol{\lambda};x) + \epsilon\left(\frac{\partial r}{\partial\lambda_1}(\boldsymbol{\lambda};x),...,\frac{\partial r}{\partial\lambda_n}(\boldsymbol{\lambda};x)\right)\cdot\delta\boldsymbol{\lambda} + O(\epsilon^2)\,,\\
\Phi(\boldsymbol{\lambda}+\epsilon\delta\boldsymbol{\lambda};x) &= \Phi(\boldsymbol{\lambda};x) + \epsilon\left(\frac{\partial\Phi}{\partial\lambda_1}(\boldsymbol{\lambda};x),...,\frac{\partial\Phi}{\partial\lambda_n}(\boldsymbol{\lambda};x)\right)\cdot\delta\boldsymbol{\lambda} + O(\epsilon^2)\,.
\end{aligned} \tag{7.4}$$

where the symbol $\cdot$ denotes the scalar product in $\mathbb{R}^n$. Hence, a small modification of the parameters $\boldsymbol{\lambda}$ can be indeed interpreted as a static perturbation of the static solution (7.1) in the sense of Eqs. (4.3,4.4), where the perturbation functions are defined as

$$\begin{aligned}
\delta\alpha &:= \left(\frac{\partial\alpha}{\partial\lambda_1}(\boldsymbol{\lambda};x),...,\frac{\partial\alpha}{\partial\lambda_n}(\boldsymbol{\lambda};x)\right)\cdot\delta\boldsymbol{\lambda}\,,\\
\delta\gamma &:= \left(\frac{\partial\gamma}{\partial\lambda_1}(\boldsymbol{\lambda};x),...,\frac{\partial\gamma}{\partial\lambda_n}(\boldsymbol{\lambda};x)\right)\cdot\delta\boldsymbol{\lambda}\,,\\
\delta r &:= \left(\frac{\partial r}{\partial\lambda_1}(\boldsymbol{\lambda};x),...,\frac{\partial r}{\partial\lambda_n}(\boldsymbol{\lambda};x)\right)\cdot\delta\boldsymbol{\lambda}\,,\\
\delta\Phi &:= \left(\frac{\partial\Phi}{\partial\lambda_1}(\boldsymbol{\lambda};x),...,\frac{\partial\Phi}{\partial\lambda_n}(\boldsymbol{\lambda};x)\right)\cdot\delta\boldsymbol{\lambda}\,,\\
\delta\beta &:= 0
\end{aligned} \tag{7.5}$$

(to see this, it is sufficient to compare Eqs. (4.3,4.4) and Eqs. (7.2,7.3,7.4, recalling that, at the linearized level, we neglect all the powers of ϵ greater or equal then 2).

Remark 63 The static perturbations (7.5) satisfy the linearized field equations (4.25-4.28) (or, equivalently, (4.29-4.33)) for every perturbations $\delta\boldsymbol{\lambda}$ of the parameters $\boldsymbol{\lambda}$. Although this fact is quite obvious, we give an example for $n = 1$ to make it clearer.
Let us consider the differential equation

$$\mathfrak{E}\left(f(\lambda;x),\frac{\partial f}{\partial x}(\lambda;x),\frac{\partial^2 f}{\partial x^2}(\lambda;x)\right) = 0\,, \tag{7.6}$$

where $\mathfrak{E}$ is a smooth real function defined on a subset of $\mathbb{R}^3$ and λ is a real parameter. Let us introduce a solution $f_0(\lambda, x)$ of Eq. (7.6), defined for any

$\lambda \in \mathcal{L} \subseteq \mathbb{R}$ with $\mathcal{L}$ an open subset of $\mathbb{R}^n$, and a (static) perturbation of this solution defined as

$$f(\lambda; x) = f_0(\lambda; x) + \epsilon\, \delta f(\lambda; x)\,, \tag{7.7}$$

where ϵ is a small real parameter and δf is a smooth function to be determined. The linearization of Eq. (7.6) evaluated in Eq. (7.7) with respect to ϵ reads

$$\begin{aligned}\delta\mathfrak{E} = 0\,, \qquad \delta\mathfrak{E} :=& \epsilon\left(\delta f(\lambda; x), \frac{\partial \delta f}{\partial x}(\lambda; x), \frac{\partial^2 \delta f}{\partial x^2}(\lambda; x)\right)\\ &\cdot \nabla^{\mathbb{R}^3}\mathfrak{E}\left(f_0(\lambda; x), \frac{\partial f_0}{\partial x}(\lambda; x), \frac{\partial^2 f_0}{\partial x^2}(\lambda; x)\right)\,;\end{aligned} \tag{7.8}$$

here and in the following $\nabla^{\mathbb{R}^3}$ and $\cdot$ denote, respectively, the three-dimensional Euclidean gradient and the scalar product in $\mathbb{R}^3$.
Let us now define a (static) perturbation of the solution $f_0(\lambda; x)$ obtained from a perturbation of the parameter λ (analogously to what we have done in Eq. (7.5)):

$$f_0(\lambda + \epsilon\, \delta\lambda; x) = f_0(\lambda; x) + \epsilon\, \frac{\partial f_0}{\partial \lambda}(\lambda; x)\delta\lambda + O(\epsilon^2)\,, \tag{7.9}$$

where, ϵ is a small real parameter and $\delta\lambda$ is a real constant; we now prove that the static perturbation

$$\delta f(\lambda; x) := \frac{\partial f_0}{\partial \lambda}(\lambda; x)\delta\lambda \tag{7.10}$$

is actually a solution of the linearized equation (7.8) for every value of $\delta\lambda \in \mathbb{R}$. We start observing that, since for ϵ small enough $\lambda + \epsilon\,\delta\lambda \in \mathcal{L}$, then $f_0(\lambda + \epsilon\,\delta\lambda; x)$ is still an exact solution of the differential equation (7.6); hence

$$\mathfrak{E}\left(f_0(\lambda + \epsilon\,\delta\lambda; x), \frac{\partial f_0}{\partial x}(\lambda + \epsilon\,\delta\lambda; x), \frac{\partial^2 f_0}{\partial x^2}(\lambda + \epsilon\,\delta\lambda; x)\right) \equiv 0\,. \tag{7.11}$$

Hence, the linearization with respect to ϵ of the previous identity gives

$$\begin{aligned}\delta\mathfrak{E} =& \mathfrak{E}\left(f_0(\lambda; x), \frac{\partial f_0}{\partial x}(\lambda; x), \frac{\partial^2 f_0}{\partial x^2}(\lambda; x)\right)\\ &+ \epsilon\left(\frac{\partial f_0}{\partial \lambda}(\lambda; x)\,\delta\lambda, \frac{\partial}{\partial x}\frac{\partial f_0}{\partial \lambda}(\lambda; x)\,\delta\lambda, \frac{\partial^2}{\partial x^2}\frac{\partial f_0}{\partial \lambda}(\lambda; x)\,\delta\lambda\right)\\ &\quad\cdot \nabla^{\mathbb{R}^3}\mathfrak{E}\left(f_0(\lambda; x), \frac{\partial f_0}{\partial x}(\lambda; x), \frac{\partial^2 f_0}{\partial x^2}(\lambda; x)\right) \equiv 0\,,\end{aligned} \tag{7.12}$$

which implies that the two addends are identically zero; the first term is zero since $f_0(\lambda; x)$ is a solution of Eq. (7.6), while the cancelling of the second term tells us that the (static) perturbation (7.10) is actually a solution of the linearized equation (7.8).

We now return to the method outlined in Section 6.5 for decoupling the linearized field equations, which assumes the knowledge of a static solution $(\mathcal{F}_0(x), \mathcal{G}_0(x))$ of the system (6.74,6.75). The previous remark and the considerations at the beginning of this section provide a general strategy to obtain a solution $(\mathcal{F}_0(x), \mathcal{G}_0(x))$ for static wormholes depending on one or more parameters. Indeed, we have proved that the static perturbations (7.5), obtained by varying the parameter(s) of the considered wormholes, actually satisfy the linearized system (2.83–2.87); as a consequence, the corresponding static gauge-invariant fields $A \equiv A_0(x)$ and $C \equiv C_0(x)$ (defined by Eqs. (6.35,6.36)) fulfill the system (6.54,6.55,6.57), provided the vanishing condition $\mathcal{S} = 0$ [Eq. (6.13)] holds. The latter additional condition is indeed a condition on the variation(s) $\delta\boldsymbol{\lambda} = (\delta\lambda_1, ..., \delta\lambda_n)$ in Eq. (7.5); then, under this prescription, the gauge-invariant fields $\mathcal{F} \equiv \mathcal{F}_0(x)$ and $\mathcal{G} \equiv \mathcal{G}_0(x)$, associated with $A \equiv A_0(x)$ and $C \equiv C_0(x)$ according to Eq. (6.73), are a static solution of the constrained wave system (6.74,6.75). In this chapter we will apply this general strategy to the cases of the Torii-Sinkai, Ellis-Bronnikov and AdS wormholes. In the last section, the problem of the linear stability of the dS wormhole is partially treated in the same fashions.

7.2 Gauge-invariant stability analysis of the Torii-Shinkai wormhole

We firstly deal with the $(d+1)$-dimensional Torii-Shinkai wormhole in the coordinate system $(\mathtt{t}, \mathtt{x}, x^2, ..., x^d)$, defined by Eq. (3.34); we would like to apply the general method presented in Chapter 6 with $(\mathtt{t}, \mathtt{x})$ in place of (t, x). However, the metric (3.34) does not depend on any parameter, a fact that seems to prevent from applying the method of the variation of the parameters which allows to find the static solution $(\mathcal{F}_0, \mathcal{G}_0)$, introduced in the first section of this chapter. [47] This problem can be easily overcome by considering for

[47] Of course it is possible to use the Torii-Shinkai metric in the coordinates (t, x), in which the metric is as in Eq. (3.31), thus depending on the parameter $b > 0$; however, in this coordinate system the forthcoming equations would become less elegant.

any $b > 0$ the equivalent solution

$$\alpha = \gamma = b\,, \qquad r = b\,\rho(\mathtt{x})\,, \qquad \Phi = \sqrt{\frac{(d-1)(d-2)}{\kappa}}\phi(\mathtt{x})\,,$$
$$\text{with } \rho \text{ and } \phi \text{ as in Eq. (3.31).} \tag{7.13}$$

Note that the parameter b in Eq. (7.13) does not represent any more the size of the wormhole throat (which is 1) and its introduction is purely technical. Therefore, the static perturbation (7.5) (with $n = 1$ and $\lambda_1 := b$) reads

$$\delta\alpha = \delta\gamma = \delta b \qquad \delta\beta = \delta\Phi = 0\,, \qquad \delta r = \rho(\mathtt{x})\,\delta b\,; \tag{7.14}$$

in the first section of this chapter we have shown that the functions in Eq. (7.14) satisfy the linearized field system (4.25-4.28) (or, equivalently, (4.29-4.33)) for every variation δb. If now we introduce Eq. (7.14) into Eqs. (6.35,6.36) we get following expressions for the gauge-invariant quantities A and C:

$$A \equiv A_0(\mathtt{x}) = \frac{\delta b}{b}\,, \qquad C \equiv C_0(\mathtt{x}) = \frac{\delta b}{b}\,; \tag{7.15}$$

these are a static solution even of the system (6.54,6.55,6.57) only if the variation δb satisfies the condition $\mathcal{S} = 0$. Inserting Eq. (7.14) into the very definition of $\mathcal{S}$ in Eq. (6.13), we have that $\mathcal{S} \equiv 0$ for every choice of δb; hence, we can set $\delta b := 1$. If we now insert Eq. (7.15) in the definition (6.73) of $\mathcal{F}$ and $\mathcal{G}$, we get a static solution of the system (6.74,6.75):

$$\begin{pmatrix} \mathcal{F} \\ \mathcal{G} \end{pmatrix} \equiv \begin{pmatrix} \mathcal{F}_0(\mathtt{x}) \\ \mathcal{G}_0(\mathtt{x}) \end{pmatrix} := \begin{pmatrix} 0 \\ \frac{1}{b^2\,\rho(x)} \end{pmatrix}. \tag{7.16}$$

Clearly, $\mathcal{G}_0$ is strictly positive. Using the non-trivial solution (7.16) we can now obtain the master equation for the perturbed Torii-Shinkai wormhole; indeed, since $\frac{\mathcal{F}_0}{\mathcal{G}_0} = 0$, the function $\tilde{\chi}$ in Eq. (6.89) coincides with $\mathcal{F}$. Moreover, since in this case

$$Y_0 = \frac{(d-3)\,\mathrm{sign}(\mathtt{x})}{\rho(\mathtt{x})}\sqrt{1 - \frac{1}{\rho(\mathtt{x})^{2(d-2)}}}\,,$$

we have to distinguish between the four-dimensional case, in which $d = 3$ and $Y_0 \equiv 0$, and the higher dimension case, in which $d > 3$ and $Y_0 \not\equiv 0$.

Let us consider firstly the four-dimensional case $d = 3$; since, in this case $Y_0 = 0$, we can set $a = 1$ in Eq. (6.100), so that $\mathcal{V} = \tilde{\mathcal{V}}$ and $\chi = \tilde{\chi}$. Therefore, $\chi = \mathcal{F}$ satisfies the master equation

$$\left[\frac{\partial^2}{\partial \mathtt{t}^2} - \frac{\partial^2}{\partial \mathtt{x}^2} - \frac{3}{(\mathtt{x}^2+1)^2}\right]\chi = 0\,. \tag{7.17}$$

We consider now the case $d > 3$ and introduce the functions a and χ defined as in Eq. (6.100). It turns out that, setting $x_0 = 0$ and $a_0 = 1$,

$$a(\mathtt{x}) = \rho(\mathtt{x})^{\frac{d-3}{2}}\,;$$

hence, the function

$$\chi := \frac{\mathcal{F}}{\rho(\mathtt{x})^{\frac{d-3}{2}}} \tag{7.18}$$

satisfies the master equation (6.101) which reads (recalling that $\alpha/\gamma = 1$ for the Torii-Shinkai wormhole)

$$\left[\frac{\partial^2}{\partial \mathtt{t}^2} - \frac{\partial^2}{\partial \mathtt{x}^2} + \mathcal{V}\right]\chi = 0\,, \tag{7.19}$$

with the potential

$$\mathcal{V}(\mathtt{x}) \equiv \mathcal{V}_{d,b}(\mathtt{x}) := \frac{1}{4\rho^2(\mathtt{x})}\left[(d-3)(d-5) - \frac{3(d-1)^2}{\rho^{2(d-2)}(\mathtt{x})}\right] \qquad \mathtt{x} \in \mathbb{R}\,. \tag{7.20}$$

Remark 64 As already mentioned, in the four-dimensional case, the Torii-Shinkai wormhole reduces to the EBMT wormhole in the coordinates $(\mathtt{t}, \mathtt{x})$ [Eq. (3.35)]. Note that the potential (7.20) is well defined also for $d = 3$ and is equal to $\mathcal{V}(\mathtt{x}) = -\frac{3}{(\mathtt{x}^2+1)^2}$; hence, the master equation (7.19) reduces to Eq. (7.17); note that the latter coincides exactly with the master equation for the EBMT wormhole (6.60), after performing the coordinate change $t = b\,\mathtt{t}$, $x = b\,\mathtt{x}$ [Eq. (3.33)].

7.2.1 Solution of the master equation and linear instability of the Torii-Shinkai wormhole - gauge-invariant formulation

For every $d \geq 3$ and every $b > 0$, the master equation (7.19), containing the potential (7.20) can be written as

$$\ddot{\chi}(\mathtt{t}) + H\chi(\mathtt{t}) = 0 \qquad (\mathtt{t} \in \mathbb{R}), \tag{7.21}$$

where

$$H := -\frac{d^2}{d\mathtt{x}^2} + \mathcal{V} \qquad (\mathcal{V} \equiv \mathcal{V}(\mathtt{x}) \text{ as in Eq. (7.20)}) \tag{7.22}$$

and the unknown is a function

$$\chi(\mathtt{t}) \equiv \chi(\mathtt{t}, \cdot) \,:\, \mathtt{x} \mapsto \chi(\mathtt{t}, \mathtt{x}) \quad \text{for every } \mathtt{t} \in \mathbb{R}\,.$$

Remark 65 In order to rigorously studying the solutions of the master equation (7.21), we have to consider again the Hilbert space (5.28) and the selfadjoint operator H defined in Eq. (5.29); now, we can profit from the discussion in Appendices D.1 and E.1 on the spectra of this operator and the possibility of introducing a generalized orthonormal basis consisting in its proper and improper eigenfunctions. The results therein contained are reassumed in Remark 40.

Then, profiting from the considerations contained in the previous remark, one can search the solution $\chi(\mathtt{t})$ of the master equation (7.21) with appropriate smoothness properties and with the initial conditions

$$\chi(0) = q\,, \qquad \dot{\chi}(0) = p\,, \tag{7.23}$$

where

$$q \,:\, \mathtt{x} \mapsto q(\mathtt{x})\,, \qquad p : \mathtt{x} \mapsto p(\mathtt{x})$$

are sufficiently regular functions.
For all technical details, we refer to Appendix E.1.1; here we introduce two Hilbertian structures on the domains $\mathfrak{D}$ and $\mathfrak{D}^{1/2}$ of the operators H and $|H|^{1/2}$ and we show that, for any

$$q \in \mathfrak{D}\,, \qquad p \in \mathfrak{D}^{1/2}$$

Eqs. (7.21,7.23) have a unique solution $\chi(\mathtt{t})$ defined for every $\mathtt{t} \in \mathbb{R}$ such that

$$\chi \in C^2(\mathbb{R}, \mathfrak{H}) \cap C^1(\mathbb{R}, \mathfrak{D}^{1/2}) \cap C(\mathbb{R}, \mathfrak{D})\,; \tag{7.24}$$

this solution, for all $\mathtt{t} \in \mathbb{R}$, can be decomposed by means of the previously mentioned generalized orthonormal basis as follows

$$\begin{aligned}\chi(\mathtt{t}) = &\left[\langle e_1|q\rangle \cosh(|\mu_1|^{1/2}\mathtt{t}) + \langle e_1|p\rangle \frac{\sinh(|\mu_1|^{1/2}\mathtt{t})}{|\mu_1|^{1/2}}\right] e_1 \\ &+ \sum_{i=1}^{2} \int_0^{+\infty} \left[\langle e_{i\lambda}|q\rangle \cos(\lambda^{1/2}\mathtt{t}) + \langle e_{i\lambda}|p\rangle \frac{\sin(\lambda^{1/2}\mathtt{t})}{\lambda^{1/2}}\right] e_{i\lambda}\, d\lambda\,. \end{aligned}\tag{7.25}$$

As explained in Remark 100, the symbols $\langle\cdot|\cdot\rangle$ in the above formula indicate usual inner products in $\mathfrak{H}$, or suitably defined generalizations, while the integrals over λ are understood in a weak sense. Of course we are interested in the case in which $\chi(\mathtt{t})$ is *real valued* for each $\mathtt{t}$, a fact that occurs if and only if the data q, p are real valued functions.

Remark 66 From the expression (7.25) for $\chi(\mathtt{t})$, one can see that the coefficient of e_1 diverges exponentially both for $\mathtt{t} \to -\infty$ and for $\mathtt{t} \to +\infty$ (except for very special choices of $\langle e_1|q\rangle$ and $\langle e_1|p\rangle$); [48] this suffices to infer the (linear) instability of the Torii-Shinkai wormhole [35]. Indeed, we have proved that for some special initial data, there exists a combination χ of the linearized perturbation functions $\delta\alpha, \delta\beta, \delta\gamma, \delta r, \delta\Phi$ which diverges as the temporal coordinate $\mathtt{t}$ goes to infinity; moreover, even after introducing an infinitesimal coordinate transformation (4.6,4.7), the transformed "perturbation" $\tilde{\chi}$ still diverges for $\tilde{\mathtt{t}} \to \pm\infty$ as the quantity χ is gauge-invariant, that is $\tilde{\chi}(\tilde{\mathtt{t}}, \tilde{\mathtt{x}}) = \chi(\tilde{\mathtt{t}}, \tilde{\mathtt{x}})$.
Finally, let us remark that the integrals over λ in Eq. (7.25) are superpositions of "non normalizable" oscillatory modes, living outside the space $\mathfrak{H} = L^2(\mathbb{R}, d\mathtt{x})$ like the improper eigenfunctions $e_{i\lambda}$.

Hence, we have shown the following final result.

Theorem 10 (Linear instability of the Torii-Shikai wormhole - gauge-invariant formulation)
For all $d \geq 3$ and for all $b > 0$, the Torii-Shinkai wormhole is linearly unstable under small spherically symmetric perturbations of its metric and the associated scalar field; more precisely, there exists a gauge-invariant quantity χ (depending on the perturbations) which, for some special initial data, diverges as the temporal coordinate $\mathtt{t}$ goes to $\pm\infty$.

Remark 67 Since the previous result is valid for all $d \geq 3$, in the case $d = 3$, it reduces to the statement of the linear instability of the EBMT wormhole (see Example 5). Note that the present approach is the same of that in Ref. [1].

Remark 68 As already mentioned in Remark 45, the linear instability of the EBMT wormhole has been proved firstly in Ref. [25] (and then reproposed in Refs. [53, 2]), while the first deduction of the same feature for the Torii-Shinkai wormhole can be found in Ref. [35]. All these papers use a different approach from that of the present section; in particular, in Ref. [2] the linear instability of the EBMT wormhole is derived using the "gauge-dependent" approach explained in Section 5.1 in the four-dimensional case $d = 3$. To see a closer comparison among the schemes of the previous mentioned cited works, see Chapter 8.

[48] For $\langle e_1|q\rangle = \langle e_1|p\rangle = 0$, the coefficient of e_1 in Eq. (7.25) vanishes. For $\langle e_1|q\rangle = \xi\langle e_1|p\rangle/|\mu_1|^{1/2} \neq 0$, with $\xi = \pm 1$, the coefficient of e_1 diverges for $t \to \xi(+\infty)$ and vanishes for $t \to \xi(-\infty)$.

7.3 Gauge-invariant stability analysis of the Ellis-Bronnikov wormhole

We consider the Ellis-Bronnikov solution given in Eqs. (3.39,3.40); hence in this section we set $d = 3$. The Ellis-Bronnikov solution depends on two parameters $b > 0$ and $\gamma_1 \in \mathbb{R}$; therefore, one can apply the general strategy introduced at the beginning of the present chapter in order to find the static solution $(\mathcal{F}_0, \mathcal{G}_0)$ of the system (6.74,6.75). The knowledge of this static solution is sufficient to decouple the latter system, thanks to the gauge-invariant method explained in Chapter 6.
Let us consider the static perturbations (7.5) obtained from the linearization of the solution (3.39,3.40) with respect to small variations of b and γ_1:

$$\begin{aligned}
\delta\alpha &= \left(\arctan\frac{x}{b}\right)\alpha\,\delta\gamma_1 - \frac{\gamma_1 x}{x^2+b^2}\,\alpha\,\delta b\,,\\
\delta\beta &= 0\,,\\
\delta\gamma &= -\left(\arctan\frac{x}{b}\right)\gamma\,\delta\gamma_1 + \frac{\gamma_1 x}{x^2+b^2}\,\gamma\,\delta b\,,\\
\delta r &= -\left(\arctan\frac{x}{b}\right) r\,\delta\gamma_1 + \frac{b+\gamma_1 x}{x^2+b^2}\,r\,\delta b\,,\\
\delta\Phi &= \frac{\gamma_1}{1+\gamma_1^2}(x^2+b^2)\left(\arctan\frac{x}{b}\right)\Phi'\,\frac{\delta\gamma_1}{b} - x\,\Phi'\,\frac{\delta b}{b}\,.
\end{aligned} \tag{7.26}$$

The previous functions satisfy the linearized field system (4.25-4.28) (or, equivalently, (4.29-4.33)) for every choice of the variation $\delta b, \delta\gamma_1$; introducing into Eqs. (6.35,6.36), they give rise to rise to the gauge-invariant quantities

$$A \equiv A_0(x) = -\frac{1}{1+\gamma_1^2}\left[\gamma_1 + \left(1+2\gamma_1\frac{x}{b}\right)\arctan\frac{x}{b}\right]\delta\gamma_1 + \frac{\delta b}{b}\,, \tag{7.27}$$

$$C \equiv C_0(x) = -\frac{1+\gamma_1\frac{x}{b}}{1+\gamma_1^2}\left(\arctan\frac{x}{b}\right)\delta\gamma_1 + \frac{\delta b}{b}\,. \tag{7.28}$$

As explained previously, the functions A_0 and C_0 automatically satisfy the system of equations (6.54,6.55,6.57) for all the variations $\delta b, \delta\gamma_1$ such that $\mathcal{S} = 0$. In this case the definition of $\mathcal{S}$ in Eq. (6.13) gives $\mathcal{S} = -\delta b\,\gamma_1 - b\,\delta\gamma_1 = \delta(b\,\gamma_1)$, so that the condition $\mathcal{S} = 0$ holds if and only if

$$\delta\gamma_1 = -\frac{\gamma_1}{b}\delta b\,. \tag{7.29}$$

Inserting Eqs. (7.27,7.28,7.29) into the definition (6.73) of $\mathcal{F}$ and $\mathcal{G}$ (and choosing for simplicity $\delta b = b$), one obtains the following time-independent solution of the constrained wave system (6.74,6.75): [49]

$$\begin{pmatrix} \mathcal{F} \\ \mathcal{G} \end{pmatrix} \equiv \begin{pmatrix} \mathcal{F}_0(x) \\ \mathcal{G}_0(x) \end{pmatrix} := \frac{1}{r} \begin{pmatrix} \frac{\gamma_1^2}{1+\gamma_1^2}\left[1 + \frac{x}{b}\arctan\frac{x}{b}\right] \\ F(x) \end{pmatrix}, \qquad F(x) := 1 + \frac{\gamma_1}{1+\gamma_1^2}\left(1 + \gamma_1 \frac{x}{b}\right)\arctan\frac{x}{b}\,. \tag{7.30}$$

Note that the function $F : \mathbb{R} \to \mathbb{R}$ is smooth and strictly positive [50] and that

$$\frac{\mathcal{F}_0}{\mathcal{G}_0} = \gamma_1^2 \frac{1 + \frac{x}{b}\arctan\frac{x}{b}}{1 + \gamma_1^2 + \gamma_1\left(1 + \gamma_1\frac{x}{b}\right)\arctan\frac{x}{b}}\,;$$

therefore, we can now apply the general method for decoupling the wave system (6.74,6.75), choosing the static solution $(\mathcal{F}_0(x), \mathcal{G}_0(x))$ as in Eq. (7.30). In this case, for every γ_1 and every $b > 0$, we have $Y_0 = 0$ so that we can choose $a = 1$ in Eq. (6.100), which implies that $\mathcal{V} = \tilde{\mathcal{V}}$ and $\chi = \tilde{\chi}$. Hence, the function χ defined as

$$\chi := \mathcal{F} + \gamma_1^2 \frac{1 + \frac{x}{b}\arctan\frac{x}{b}}{1 + \gamma_1^2 + \gamma_1\left(1 + \gamma_1\frac{x}{b}\right)\arctan\frac{x}{b}}\,\mathcal{G}$$

satisfies the master equation (6.101) for the Ellis-Bronnikov wormhole, which reads

$$\left[\frac{\partial^2}{\partial t^2} - \left(\frac{\alpha}{\gamma}\frac{\partial}{\partial x}\right)^2 + \frac{\alpha^2}{\gamma^2}\mathcal{V}\right]\chi = 0 \tag{7.31}$$

where

$$\mathcal{V}(x) \equiv \mathcal{V}_{b,\gamma_1}(x) = \frac{\gamma^2}{\alpha^2}\frac{1}{b^2} W\left(\frac{x}{b}\right) \qquad x \in \mathbb{R}\,, \tag{7.32}$$

$$W(x) := e^{4\gamma_1 \arctan\frac{x}{b}}\left[-\frac{3}{x^2+1} + 3\frac{(x-\gamma_1)^2}{(x^2+1)^2} + 2\left(\frac{F'}{F}\right)^2 - 4\frac{\gamma_1}{x^2+1}\frac{F'}{F} + 4\frac{\gamma_1}{\gamma_1^2+1}\frac{x-\gamma_1}{(x^2+1)^2}\frac{1}{F}\right] \qquad (F \text{ as in Eq. (7.30)})\,; \tag{7.33}$$

---

[49] The use of the letter F to denote the static solution $\mathcal{G}_0$ can seem somehow confusing; this choice has been made to maintain the notation of Ref. [25].

[50] For $\gamma_1 = 0$, $F = 1$ and the statement is trivial. When $\gamma_1 \neq 0$ one has $F(x) \to +\infty$ for $x \to \pm\infty$, thus F has a global minimum at some $x = x_0$, where $0 = (1 + \gamma_1^2)bF'(x_0) = \gamma_1\left[\gamma_1 \arctan(x_0/b) + (b + \gamma_1 x_0)b/(x_0^2 + b^2)\right]$. Eliminating the arctan term one obtains from this $(1+\gamma_1^2)F(x_0) = (x_0 - b\gamma_1)^2/(x_0^2+b^2)$. However, this minimum value must be strictly positive since otherwise $x_0 = b\gamma_1$ which would imply that $(1+\gamma_1^2)bF'(x_0) = \gamma_1(\gamma_1 \arctan\gamma_1 + 1)$ which cannot be zero since $\gamma_1 \neq 0$.

one can see that the function W defined in Eq. (7.33) coincides exactly with the potential defined in Eq. (32) of Ref. [25]. In other word, we have recovered the master equation found in the just mentioned paper where the whole potential $\frac{\alpha^2}{\gamma^2}\mathcal{V}$ of the master equation Eq. (7.31) agrees up to a rescaling with the potential W, namely

$$\left(\frac{\alpha^2}{\gamma^2}\mathcal{V}\right)(x) = \frac{1}{b^2}W\left(\frac{x}{b}\right) . \tag{7.34}$$

Remark 69 In the reflection symmetric case $\gamma_1 = 0$, the Ellis-Bronnikov solution reduces to the EBMT wormhole (see Remark 29); in this case $\frac{\alpha^2}{\gamma^2} = 1$, the function F in Eq. (7.30) is equal to 1 and

$$\mathcal{V}(x) = -\frac{3b^2}{(x^2+b^2)^2} \tag{7.35}$$

so that the master equation (7.31) becomes

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{3b^2}{(x^2+b^2)^2}\right]\chi = 0\,. \tag{7.36}$$

Note that this is exactly the master equation (6.60); moreover, as already stated in Remark 64, it coincides with the four-dimensional Torii-Shinkai master equation (7.17) up to the rescaling $t = b\,\mathtt{t}$, $x = b\,\mathtt{x}$ [Eq. (3.33)].

Remark 70 As noted in Ref. [25], in the non reflection symmetric case $\gamma_1 \neq 0$, the analysis of the master equation (7.31) can be simplified by introducing the new coordinate

$$\rho = \rho(x) := \int_0^x \frac{\gamma(y)}{\alpha(y)}\, dy\,; \tag{7.37}$$

note that the mapping $x \mapsto \rho(x)$ is a diffeomorphism of $\mathbb{R}$ to itself, and $\rho(x) \sim e^{\mp\pi\gamma_1}x$ for $x \to \pm\infty$. By construction $\frac{\alpha}{\gamma}\frac{\partial}{\partial x} = \frac{\partial}{\partial \rho}$; so, writing (as an abbreviation)

$$\chi(t,\rho) \equiv \chi(t, x(\rho))\,,$$

where $x(\rho)$ is the inverse map of Eq. (7.37), we can rephrase the master equation (7.31) as

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \rho^2} + \mathcal{U}\right]\chi = 0\,, \tag{7.38}$$

with the potential

$$\mathcal{U}(\rho) := \left(\frac{\alpha^2}{\gamma^2}\mathcal{V}\right)(x(\rho)) \quad (\mathcal{V}\text{ as in Eq. (7.32)}) \qquad \rho \in \mathbb{R}\,. \tag{7.39}$$

7.3.1 Solution of the master equation and linear instability of the Ellis-Bronnikov wormhole

The linear instability of the reflection symmetric Ellis-Bronnikov wormhole (EBMT wormhole) given by $\gamma_1 = 0$ has already been proved in Subsection 7.2.1 (see, in particular, Remark 67); for this reason, we now focus on the non reflection symmetric case, so we set

$$\gamma_1 \neq 0\,.$$

For every $b > 0$, the master equation (7.38), with the potential (7.39) can be written as

$$\ddot{\chi}(t) + H\chi(t) = 0 \qquad (t \in \mathbb{R}), \tag{7.40}$$

where

$$H := -\frac{d^2}{d\rho^2} + \mathcal{U} \qquad (\mathcal{U} \equiv \mathcal{U}(\rho) \text{ as in Eq. (7.39)}) \tag{7.41}$$

and the unknown is a function

$$\chi(t) \equiv \chi(t, \cdot) \,:\, \rho \mapsto \chi(t, \rho) \quad \text{for every } t \in \mathbb{R}\,.$$

Remark 71 A rigorous functional setting for Eq. (7.40) is obtained by considering the Hilbert space

$$\mathfrak{H} := L^2(\mathbb{R}, d\rho) \tag{7.42}$$

made of complex valued, square integrable functions on $\mathbb{R}$, for the measure $d\rho$ with its inner product $\langle\ |\ \rangle$ and the associated norm $\|\ \|$. [51] Hence, the operator H can be considered as a selfadjoint operator in $\mathfrak{H}$, if we provide the precise definition

$$H := -\frac{d^2}{d\rho^2} + \mathcal{U} \,:\, \mathfrak{D} \subset \mathfrak{H} \to \mathfrak{H}\,, \qquad \mathfrak{D} := \{f \in \mathfrak{H} \mid f_{\rho\rho} \in \mathfrak{H}\}\,, \tag{7.43}$$

where the ρ-derivatives have to be intended as usual in the distributional sense. [52]

In Appendix D.2 we show the following facts which are valid for every $b > 0$ and every $\gamma_1 \neq 0$:

(i) H possesses a point spectrum consisting of two, simple eigenvalues $\mu_1 < 0$ and $\mu_2 = 0$;

[51] For more details, see Remark 84 in Appendix C.

[52] See Footnote 67 with ρ in place of x.

(ii) H possesses a continuous spectrum which coincides with $(0, +\infty)$.

Moreover, in Appendix E.2 we show that it is possible to built a generalized orthonormal basis of the Hilbert space $\mathfrak{H}$, consisting in

(i) two normalized eigenfunctions e_1, e_2 for the eigenvalues $\mu_1 < 0$ and $\mu_2 = 0$, i.e.

$$e_1, e_2 \in \mathfrak{D} \quad : \quad He_1 = \mu_1 e_1\,, \; He_2 = 0\,, \quad \|e_1\| = \|e_2\| = 1\,;$$

(e_1, e_2 are proved to be $C^\infty(\mathbb{R})$);

(ii) two suitably chosen and linearly independent "improper eigenfunctions" $e_{i\lambda}$ $(i = 1, 2)$ for each $\lambda \in (0, +\infty)$, i.e.,

$$e_{i\lambda} \in C^\infty(\mathbb{R}) \setminus \mathfrak{D} \quad : \quad He_{i\lambda} = \lambda e_{i\lambda} \qquad (i = 1, 2\,;\; \lambda > 0)\,.$$

In this way, profiting from the considerations contained in the previous remark, it is possible to look for the solution $\chi(t)$ of the master equation (7.40) with appropriate smoothness properties and with the initial conditions

$$\chi(0) = q\,, \qquad \dot{\chi}(0) = p\,, \tag{7.44}$$

where

$$q \,:\, \rho \mapsto q(\rho)\,, \qquad p : \rho \mapsto p(\rho)$$

are sufficiently regular functions.
For more technical details, we refer to Appendix E.2, where we introduce two Hilbertian structures on the domains $\mathfrak{D}$ and $\mathfrak{D}^{1/2}$ of the operators H and $|H|^{1/2}$ and we show that, setting

$$q \in \mathfrak{D}\,, \qquad p \in \mathfrak{D}^{1/2}\,,$$

then Eqs. (7.40,7.44) have a unique solution $\chi(t)$ defined for every $t \in \mathbb{R}$ such that

$$\chi \in C^2(\mathbb{R}, \mathfrak{H}) \cap C^1(\mathbb{R}, \mathfrak{D}^{1/2}) \cap C(\mathbb{R}, \mathfrak{D})\,; \tag{7.45}$$

for every $t \in \mathbb{R}$, one can decompose this solution by means of the previously mentioned generalized orthonormal basis as follows

$$\chi(t) = \Big[\langle e_1|q\rangle \cosh(|\mu_1|^{1/2}t) + \langle e_1|p\rangle \frac{\sinh(|\mu_1|^{1/2}t)}{|\mu_1|^{1/2}}\Big]\, e_1 + \Big[\langle e_2|q\rangle + \langle e_2|p\rangle t\Big]\, e_2$$
$$+ \sum_{i=1}^{2} \int_0^{+\infty} \Big[\langle e_{i\lambda}|q\rangle \cos(\lambda^{1/2}t) + \langle e_{i\lambda}|p\rangle \frac{\sin(\lambda^{1/2}t)}{\lambda^{1/2}}\Big]\, e_{i\lambda}\, d\lambda\,. \tag{7.46}$$

As explained in Subsection E.1.2, the symbols $\langle\cdot|\cdot\rangle$ in the above formula indicate usual inner products in $\mathfrak{H}$, or suitably defined generalizations, while the integrals over λ are understood in a weak sense. Since in our setting $\chi(t)$ has to be *real valued* for each t, one has to choose q, p as real valued functions.

Remark 72 Analogously to the reflection symmetric case (EMBT wormhole), the function $\chi(t)$ in Eq. (7.46) has the coefficient of e_1 which is exponentially divergent for $t \to \pm\infty$ (except for very special choices of $\langle e_1|q\rangle$ and $\langle e_1|p\rangle$); [53] in addition, is contains a term diverging linearly for $t \to \pm\infty$ (if $\langle e_2|p\rangle \neq 0$). In any case the wormhole is linearly unstable (see comments in Remark 66 for inference of the linear instability of a wormhole having verified that χ diverges).
Let us note that, as in Eq. (7.25), the present expression for $\chi(t)$ contains an integral over λ of non normalizable oscillatory modes, proportional to the improper eigenfunctions $e_{i\lambda}$ which live outside $\mathfrak{H}$.

Hence, we have proved the following final result.

Theorem 11 (Linear instability of the Ellis-Bronnikov wormhole) *For all $\gamma_1 \in \mathbb{R}$ and for all $b > 0$, the Ellis-Bronnikov wormhole is linearly unstable under small spherically symmetric perturbations of its metric and the associated scalar field; more precisely, there exists a gauge-invariant quantity χ (depending on the perturbations) which, for some special initial data, diverges as the temporal coordinate t goes to $\pm\infty$.*

Remark 73 The proof of the linear instability of the Ellis-Bronnikov wormhole presented in this section is exactly that proposed in the recent paper [1]; however, as a matter of fact, the authors of Ref. [25] arrived to the same conclusion in 2009 following a pretty different approach. In Chapter 8 (and in particular in Section 8.1), we propose a closer comparison between the method used in the two just cited papers.

[53] See the Footnote 48 in the discussion after Eq. (7.25), which is readily adapted to the present framework.

7.4 Gauge-invariant stability analysis of the AdS wormhole

We now analyze the AdS wormhole in the coordinate system (s, u), as described by Eq. (3.62) for arbitrary parameters $k, B > 0$, and apply the general framework presented in Chapter 6 with (s, u) in place of (t, x) and in the case $d = 3$. In this respect, we consider the method of the variation of the parameters introduced at the beginning of the present chapter in order to find the static solution $(\mathcal{F}_0, \mathcal{G}_0)$ of the wave system (6.74,6.75); note that, although the AdS solution formally depends on two parameters B and k, it is important to note that k also appears in the potential function $V(\Phi)$ (see Eq.(3.58)): indeed, since we regard the potential to be fixed in our perturbation analysis, we should exclude the possibility of varying k. In contrast to k, the parameter B is free, and variation of the solution (3.62) with respect to it (see Eq. (7.5)) gives

$$\delta\alpha = \delta\beta = \delta\gamma = 0\,,$$
$$\delta r = \frac{2B}{1 + 2B^2 - \cos u}\, r\, \delta B\,, \qquad \delta\Phi = -\frac{\sin u}{B(1 + B^2)}\, \Phi'\, \delta B\,; \tag{7.47}$$

as already mentioned, the functions in Eq. (7.47) satisfy the linearized field system (4.25-4.28) (or, equivalently, (4.29-4.33)) for every choice of the variation δB. Eq. (7.47), introduced into Eqs. (6.35,6.36), yields the following expressions for the gauge-invariant quantities A and C:

$$A \equiv A_0(u) = \frac{1 + \cos u}{2B(1 + B^2)}\delta B\,, \qquad C \equiv C_0(u) = \frac{\delta B}{B}\,; \tag{7.48}$$

these are a static solution of Eqs. (6.54,6.55,6.57), as long as the variation δB satisfies the condition $\mathcal{S} = 0$. From Eq. (7.47) and from the definition of $\mathcal{S}$ in Eq. (6.13), we see that $\mathcal{S} \equiv 0$, as required, for every choice of the perturbation δB. Inserting Eq. (7.48) in the definition (6.73) of $\mathcal{F}$ and $\mathcal{G}$ (and choosing for simplicity $\delta B = 1$) one obtains a static solution of the wave system (6.74,6.75):

$$\begin{pmatrix} \mathcal{F} \\ \mathcal{G} \end{pmatrix} \equiv \begin{pmatrix} \mathcal{F}_0(u) \\ \mathcal{G}_0(u) \end{pmatrix} := \frac{\sqrt{2}k}{B} \cos\left(\frac{u}{2}\right) \times \begin{pmatrix} -\frac{\sqrt{1+2B^2-\cos u}}{2(1+B^2)} \\ \frac{1}{\sqrt{1+2B^2-\cos u}} \end{pmatrix}. \tag{7.49}$$

Note that $\mathcal{G}_0$ is a strictly positive function of $u \in (-\pi, \pi)$, and that

$$\frac{\mathcal{F}_0}{\mathcal{G}_0} = -\frac{1 + 2B^2 - \cos u}{2(1 + B^2)}.$$

Having found the non-trivial solution (7.49), we can now obtain the master equation governing the spherical symmetric linearized perturbations of the AdS wormhole, following the general method explained in the last chapter. We observe that in the AdS case $Y_0 = -2\tan\frac{u}{2}$; hence, we have to introduce the functions a and χ defined as in Eq. (6.100). Setting $x_0 = 0$ and $a_0 = 4k^2$, it turns out that

$$a(u) = \frac{1}{\alpha^2(u)}\,;$$

hence, the function

$$\chi := \left(\mathcal{F} + \frac{1+2B^2-\cos u}{2(1+B^2)}\,\mathcal{G}\right)\alpha^2 \tag{7.50}$$

satisfies the master equation (6.101) which reads (recalling that $\alpha/\gamma = 1$ for the AdS wormhole in coordinates (s,u))

$$\left[\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial u^2} + \mathcal{V}\right]\chi = 0\,, \tag{7.51}$$

with the potential

$$\mathcal{V}(u) \equiv \mathcal{V}_B(u) := -\frac{B^2\,(2+B^2+\cos u)}{(1+2B^2-\cos u)^2} \qquad u \in (-\pi,\pi)\,. \tag{7.52}$$

Remark 74 For the following, we assume for the gauge-invariant quantity χ, the Dirichlet boundary conditions at the two asymptotic AdS ends, that is,

$$\chi(s,\pm\pi) = 0 \qquad \text{for every } s \in \mathbb{R}. \tag{7.53}$$

Since one has that

$$\chi(s,u) = \frac{1}{\sqrt{2}\sqrt{1+2B^2-\cos u}}\delta\gamma(s,u) - \frac{1+\cos u}{4(1+B^2)(1+2B^2-\cos u)}\delta r(s,u)$$
$$-\frac{\sqrt{\kappa}\sec\frac{u}{2}}{4Bk}\left(\frac{\sqrt{1+B^2}\tan\frac{u}{2}}{\sqrt{1+2B^2-\cos u}}\delta\Phi(s,u) + \frac{\sqrt{1+2B^2-\cos u}}{\sqrt{1+B^2}}\delta\Phi'(s,u)\right),$$

a sufficient condition for Eq. (7.53) to hold is that, in any gauge, the perturbation functions δr, $\delta\gamma$ and $\delta\Phi$ vanish at the far ends $u = \pm\pi$ of the wormhole for every time $s \in \mathbb{R}$, along with the derivative $\delta\Phi'$, which is a physically reasonable prescription, as already mentioned in Remark 47.

7.4.1 Solution of the master equation and linear instability of the AdS wormhole - gauge-invariant formulation

For every $B > 0$, the master equation (7.51), containing the potential (7.52) can be written as

$$\ddot{\chi}(s) + H\chi(s) = 0 \qquad (s \in \mathbb{R}), \tag{7.54}$$

where

$$H := -\frac{d^2}{du^2} + \mathcal{V} \qquad (\mathcal{V} \equiv \mathcal{V}(u) \text{ as in Eq. (7.52)}) \tag{7.55}$$

and the unknown is a function

$$\chi(s) \equiv \chi(s, \cdot) \, : \, u \mapsto \chi(s, u) \quad \text{for every } s \in \mathbb{R}\,.$$

Remark 75 In order to provide a rigorous setting for studying the solutions of the master equation (7.54) with boundary conditions (7.53), one is led to consider again the Hilbert space (5.68) and introduce the selfadjoint operator H defined in Eq. (5.69); in this way, we can take advantage of the discussion in Appendices D.3 and E.3 on the spectrum of this operator and the possibility of introducing an orthonormal basis made up of its normalized eigenfunctions. The results obtained in the previously mentioned appendices have been already reassumed in Remark 48.

Using the results mentioned in the previous remark, we can search the solution $\chi(s)$ of the master equation (7.51) with appropriate smoothness properties and with the initial conditions

$$\chi(0) = q\,, \qquad \dot{\chi}(0) = p\,, \tag{7.56}$$

where

$$q \, : \, \mathtt{x} \mapsto q(\mathtt{x})\,, \qquad p : \mathtt{x} \mapsto p(\mathtt{x})$$

are functions with a regularity to be specified.
All the technical details of the forthcoming statements can be found in Appendix E.3. If one introduce two Hilbertian structures on the domains $\mathfrak{D}$ and $\mathfrak{D}^{1/2}$ of the operators H and $|H|^{1/2}$, it is possible to show that, choosing the initial data as

$$q \in \mathfrak{D}\,, \qquad p \in \mathfrak{D}^{1/2}\,,$$

then Eqs. (7.51,7.56) possess a unique solution $\chi(s)$ defined for every $s \in \mathbb{R}$ which is such that

$$\chi \in C^2(\mathbb{R}, \mathfrak{H}) \cap C^1(\mathbb{R}, \mathfrak{D}^{1/2}) \cap C(\mathbb{R}, \mathfrak{D})\,; \tag{7.57}$$

in addition, for all $s \in \mathbb{R}$, this solution can be decomposed by means of the previously mentioned orthonormal basis as follows

$$\chi(s) = \left[\langle e_1|q\rangle \cosh(|\mu_1|^{1/2}s) + \langle e_1|p\rangle \frac{\sinh(|\mu_1|^{1/2}s)}{|\mu_1|^{1/2}}\right] e_1 + \sum_{n=2}^{+\infty} \left[\langle e_n|q\rangle \cos(\mu_n^{1/2}s) + \langle e_n|p\rangle \frac{\sin(\mu_n^{1/2}s)}{\mu_n^{1/2}}\right] e_n \,. \quad (7.58)$$

Note that we are interested in the case where $\chi(s)$ is *real valued* for each s and this occurs if and only if the data q, p are real valued functions.

Remark 76 The coefficient of e_1 in the expression Eq. (7.58) for χ diverges exponentially both for $s \to -\infty$, and for $s \to +\infty$ (except for very special choices of $\langle e_1|q\rangle$ and $\langle e_1|p\rangle$); (54) so, the AdS wormhole is linearly unstable (see comments in Remark 66 for inference of the linear instability of a wormhole having verified that χ diverges).
In this case, for each $n \geq 2$, the n-th term in Eq. (7.58) represents a "normalizable" oscillatory mode, living like e_n inside the Hilbert space $\mathfrak{H}$ (indeed, inside the subspace $\mathfrak{D} \subset \mathfrak{H}$). This is a relevant difference with respect to the "non normalizable" oscillatory modes that we have found for the perturbed Torii-Shinkai wormhole and Ellis-Bronnikov wormhole, associated with the continuous spectrum and living outside the Hilbert space of the system (see Remarks 66 and 72).

Hence, we have verified the validity of the following final result.

Theorem 12 (Linear instability of the AdS wormhole - gauge-invariant formulation)
For all $B > 0$ and for all $k > 0$ (or, equivalently, for all $b > 0$ and for all $k > 0$), the AdS wormhole is linearly unstable under small spherically symmetric perturbations of its metric and the associated scalar field; more precisely, there exists a gauge-invariant quantity χ (depending on the perturbations) which, for some special initial data, diverges as the temporal coordinate s goes to $\pm\infty$.

Remark 77 The linear instability of the AdS wormhole has been firstly inferred very recently in Ref. [1]: the deduction of this paper is exactly that presented in this section.

[54] See the Footnote 48 in the discussion after Eq. (7.25), which is readily adapted to the present framework.

Remark 78 In Section 5.2 of the present thesis we have proposed an alternative, gauge-dependent deduction of the linear instability of the AdS wormhole; in Chapter 8 (and in particular in Section 8.3), we provide a closer comparison between the present approach and that of Section 5.2.

7.5 Gauge-invariant stability analysis of the static part of the dS wormhole

If one confines the attention to the restriction of the dS wormhole defined as $(I \times S^2, \mathbf{g}, \Phi)$, where I is the inner region (3.73) in (t, x) space and $\mathbf{g}$ and Φ are, respectively, as in Eq. (3.72) and in Eq. (3.69), the analysis of linearized perturbations for the Einstein-scalar equations is rather simple in the framework of Chapter 6.

First of all, one replaces the coordinates (t, x) with the coordinates $(s, u) \in \mathbb{R}^2$ defined by Eq. (3.74). After this, one should in principle apply the general scheme of Chapter 6 (in the coordinates (s, u)) to the linearized perturbations of this solution, ultimately yielding a master equation. As a matter of fact, it is not even necessary to carry on this construction and it suffices to use the following trick: since the dS wormhole under analysis is connected to the AdS wormhole through the formal replacement rules $(k, B, s, u) \mapsto (ik, iB, is, iu)$ (see Remark 35), the master equation for the perturbed dS wormhole can be obtained making formally the same replacements in Eqs. (7.51,7.52) of the AdS case. In conclusion, the master equation governing linear perturbations of the inner region of the dS wormhole, in the unknown gauge-invariant function $\chi(s, u)$, reads

$$\left[\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial u^2} + \mathcal{V}\right]\chi = 0\,, \tag{7.59}$$

and involves the potential

$$\mathcal{V}(u) \equiv \mathcal{V}_B(u) := -\frac{B^2\,(2 - B^2 + \cosh u)}{(-1 + 2B^2 + \cosh u)^2} \qquad (u \in \mathbb{R})\,. \tag{7.60}$$

7.5.1 Solution of the master equation and linear instability of the static part of the dS wormhole

As usual, for every $B > 0$, the master equation (7.59), with the potential (7.60) can be written as

$$\ddot{\chi}(s) + H\chi(s) = 0 \qquad (s \in \mathbb{R}), \tag{7.61}$$

where

$$H := -\frac{d^2}{du^2} + \mathcal{V} \qquad (\mathcal{V} \equiv \mathcal{V}(u) \text{ as in Eq. (7.60)}) \tag{7.62}$$

and the unknown is a function

$$\chi(s) \equiv \chi(s, \cdot) \;:\; u \mapsto \chi(s, u) \quad \text{for every } s \in \mathbb{R}\,.$$

Remark 79 As usual, one has to consider the Hilbert space

$$\mathfrak{H} := L^2(\mathbb{R}, du) \tag{7.63}$$

made of complex valued, square integrable functions on $\mathbb{R}$, for the measure $d\rho$ with its inner product $\langle\ |\ \rangle$ and the associated norm $\|\ \|$. [55] in order to built a rigorous functional setting for studying Eq. (7.61). Therefore, the operator H can be regarded as a selfadjoint operator in $\mathfrak{H}$, if one provides the precise definition

$$H := -\frac{d^2}{du^2} + \mathcal{V} \;:\; \mathfrak{D} \subset \mathfrak{H} \to \mathfrak{H}\,, \qquad \mathfrak{D} := \{f \in \mathfrak{H} \mid f_{uu} \in \mathfrak{H}\}\,, \tag{7.64}$$

where the u-derivatives have to be intended as usual in the distributional sense. [56]
In Appendix D.4 we show the following facts which are true for every $B > 0$:

(i) H possesses a point spectrum consisting of a unique, simple eigenvalues $\mu_1 < 0$;

(ii) H possesses a continuous spectrum which coincides with $[0, +\infty)$.

Moreover, in Appendix E.4 we show that it is possible to built a generalized orthonormal basis of the Hilbert space $\mathfrak{H}$, consisting in

(i) a normalized eigenfunctions e_1 for the eigenvalue $\mu_1 < 0$ i.e.

$$e_1 \in \mathfrak{D} \quad : \quad He_1 = \mu_1 e_1\,, \quad \|e_1\| = 1\,;$$

(e_1 is proved to be $C^\infty(\mathbb{R})$);

(ii) two suitably chosen and linearly independent "improper eigenfunctions" $e_{i\lambda}$ $(i = 1, 2)$ for each $\lambda \in (0, +\infty)$, i.e.,

$$e_{i\lambda} \in C^\infty(\mathbb{R}) \setminus \mathfrak{D} \quad : \quad He_{i\lambda} = \lambda e_{i\lambda} \qquad (i = 1, 2\,;\ \lambda > 0)\,.$$

[55] For more details, see Remark 84 in Appendix C.
[56] See, again, Footnote 67 in Appendix D.

Hence, one can look for a solution $\chi(t)$ of the master equation (7.61) with appropriate smoothness properties and with the initial conditions

$$\chi(0) = q\,, \qquad \dot{\chi}(0) = p\,, \tag{7.65}$$

where

$$q\,:\,u \mapsto q(u)\,, \qquad p : u \mapsto p(u)$$

are sufficiently regular functions.
For more technical details, we refer to Appendix E.4, where we introduce two Hilbertian structures on the domains $\mathfrak{D}$ and $\mathfrak{D}^{1/2}$ of the operators H and $|H|^{1/2}$ and we show that, setting

$$q \in \mathfrak{D}\,, \qquad p \in \mathfrak{D}^{1/2}\,,$$

then Eqs. (7.61,7.65) have a unique solution $\chi(s)$ defined for every $s \in \mathbb{R}$ such that

$$\chi \in C^2(\mathbb{R}, \mathfrak{H}) \cap C^1(\mathbb{R}, \mathfrak{D}^{1/2}) \cap C(\mathbb{R}, \mathfrak{D})\,; \tag{7.66}$$

one can decompose the solution $\chi(s)$ for every $s \in \mathbb{R}$ by means of the previously mentioned generalized orthonormal basis as follows

$$\begin{aligned} \chi(s) = &\left[\langle e_1|q\rangle \cosh(|\mu_1|^{1/2}s) + \langle e_1|p\rangle \frac{\sinh(|\mu_1|^{1/2}s)}{|\mu_1|^{1/2}}\right] e_1 \\ &+ \sum_{i=1}^{2} \int_0^{+\infty} \left[\langle e_{i\lambda}|q\rangle \cos(\lambda^{1/2}s) + \langle e_{i\lambda}|p\rangle \frac{\sin(\lambda^{1/2}s)}{\lambda^{1/2}}\right] e_{i\lambda}\, d\lambda\,. \end{aligned} \tag{7.67}$$

As explained in Remark 100, the symbols $\langle\cdot|\cdot\rangle$ in the above formula indicate usual inner products in $\mathfrak{H}$, or suitably defined generalizations, while the integrals over λ are understood in a weak sense. Of course, in our setting $\chi(s)$ has to be *real valued* for each s, and this is true as long as q, p as real valued functions.
The situation is similar to that of the Torii-Shinkai wormhole: the coefficient of e_1 in the expression (7.67) for $\chi(s)$ exponentially diverges for $s \to \pm\infty$ (except for very special choices of $\langle e_1|q\rangle$ and $\langle e_1|p\rangle$); [(57)] Again as in the Torii-Shinkai case, the present expression for $\chi(s)$ contains an integral over λ of non normalizable oscillatory modes, proportional to the improper eigenfunctions $e_{i\lambda}$ which live outside $\mathfrak{H}$.

Hence, we have proved the following final result.

[57] See the footnote 48 in the discussion after Eq. (7.25), which is readily adapted to the present framework.

Theorem 13 (Linear instability of the static part of the dS wormhole)
For all $k > 0$ and for all $b > 0$ (or, equivalently, for all $k > 0$ and for all $B < 0$), the dS wormhole is linearly unstable under small spherically symmetric perturbations of its metric and the associated scalar field confined in the static part of the spacetime; more precisely, there exists a gauge-invariant quantity χ (depending on perturbations defined in a suitable small neighbourhood of the throat) which, for some special initial data, diverges as the temporal coordinate s goes to $\pm\infty$.

Remark 80 Exactly as for the AdS case, the linear instability of the static part dS wormhole has been inferred for the first time in 2020 in Ref. [1]: in this paper, the Ellis-Bronnikov, the AdS and the dS wormholes are introduced as three possible applications of a four-dimensional gauge-invariant method for studying the linear instability of some wormhole spacetimes. This method has been extended to higher dimension and proposed in Chapter 6 of the present thesis.

Remark 81 For a full understanding of the linear instability of the dS wormhole under discussion, linearized perturbations of the Einstein-scalar equations should be treated on the extended spacetime $\mathcal{S} \times S^2$ of subsection 3.5.3 (or on the quotients $(\mathcal{S}/\mathfrak{T}^p) \times S^2$), possibly in a gauge-invariant fashion. The discussion of this problem would bring us outside the scope of the present thesis, since the extended spacetime $\mathcal{S} \times S^2$ is not static. One can reasonably expect that the instability result about the inner region will eventually produce a precise statement of linear instability for the extended dS wormhole. However, we prefer to postpone these matters to future works; let us also mention that the notion of linear instability is not so obvious if one perturbs a non static spacetime, and requires in our opinion a general discussion before reconsidering the specific case of the extended dS wormhole.

Chapter 8

A comparison between the gauge-dependent and the gauge-invariant approaches, and the state of the art

8.1 The Ellis-Bronnikov wormhole

8.1.1 A comparison between the gauge-invariant approach of Section 7.3 and Refs. [25, 27, 26, 11]

In Ref. [25] the linear instability of the Ellis-Bronnikov solution (and, in particular, of the EBMT wormhole) is derived via a two-steps construction, that I now describe briefly. The first step is the reduction of the linearized Einstein equations and the linearized Klein-Gordon equation to a scalar master equation where the unknown is a suitable gauge-invariant recombination of the perturbation components, here indicated with χ_{sing}; in our notation, the master equation and the function χ_{sing} read

$$\left[\frac{\partial^2}{\partial t^2} - \left(\frac{\alpha}{\gamma}\frac{\partial}{\partial x}\right)^2 + \frac{\alpha^2}{\gamma^2}\mathcal{V}_{\text{sing}}\right]\chi_{\text{sing}} = Q(x)\,, \tag{8.1}$$

$$\chi_{\text{sing}} := r\left(\delta\Phi - \Phi'\frac{\delta r}{r'}\right) = -\frac{r^2\Phi'}{r'}C\,, \tag{8.2}$$

$$\mathcal{V}_{\text{sing}} := 1 - \frac{r'^2}{r^2} + \kappa\frac{\Phi'^2 r^2}{r'^2}\,, \qquad Q(x) := -2\sigma\frac{\Phi'}{r'^2}\frac{\alpha\gamma}{r} \tag{8.3}$$

with C as in Eq. (6.36) (see Eq. (17) of Ref. [25]). The potential $\mathcal{V}_{\text{sing}}$ in this master equation occurs to be singular at the wormhole throat as well as the source term $Q(x)$ and, obviously, the very definition of the recombination $\chi_{\rm sing}$. Note that the constant σ appearing in the definition of $Q(x)$ is exactly the value of the gauge-invariant quantity $\mathcal{S}$ defined in Eq. (6.6), which can be set to zero; so the authors of [25] assume that $Q(x) \equiv 0$. The second step in the construction of Ref. [25] removes the singularities by a clever strategy: the idea is to apply to $\chi_{\rm sing}$ a suitable first order differential operator, so as to obtain a function χ fulfilling a regular master equation. This is in fact possible if one knows a static solution of the singular master equation $\chi_{\rm sing 0}$; the transformation relating $\chi_{\rm sing}$ and χ reads, in our notations

$$\chi = D_+ \chi_{\rm sing}\,, \qquad \text{where} \quad D_+ := \frac{\alpha}{\gamma}\frac{\partial}{\partial x} - \frac{1}{\chi_{\rm sing 0}}\frac{\alpha}{\gamma}\frac{\partial \chi_{\rm sing 0}}{\partial x} \tag{8.4}$$

and the latter is proved to satisfy a master equation which is completely regular (in the case of the Ellis-Bronnikov wormhole). [58] In our notations, this equation coincides exactly with Eq. (7.31), with the nowhere singular potential (7.34) and $b = 1$ (see Eqs. (30,32) of [25]). Note that, in the special EBMT case ($\gamma_1 = 0$), the regular potential coincides with the potential in Eq. (7.35) and the master equation reduces to Eq. (7.36) (again with $b = 1$). In comparison with Ref. [25], the novelty of the gauge-invariant deduction of the linear instability of the Ellis-Bronnikov wormhole (and, in particular, of the EBMT wormhole), firstly presented in Ref. [1] and reproposed in this

[58] Let us give some more details on this fact. In addition to the differential operator D_+ defined in Eq. (8.4), the authors of [25] introduce the operator

$$D_- := -\frac{\alpha}{\gamma}\frac{\partial}{\partial x} - \frac{1}{\chi_{\rm sing 0}}\frac{\alpha}{\gamma}\frac{\partial \chi_{\rm sing 0}}{\partial x}\,;$$

recalling that, by definition, $\left(\frac{\alpha}{\gamma}\frac{\partial}{\partial x}\right)^2 \chi_{\rm sing 0} = \frac{\alpha^2}{\gamma^2}\mathcal{V}_{\rm sing}\chi_{\rm sing 0}$, one can easily prove that

$$D_- D_+ = -\left(\frac{\alpha}{\gamma}\frac{\partial}{\partial x}\right)^2 + \frac{\alpha^2}{\gamma^2}\mathcal{V}_{\rm sing}\,, \quad D_+ D_- = -\left(\frac{\alpha}{\gamma}\frac{\partial}{\partial x}\right)^2 - \frac{\alpha^2}{\gamma^2}\mathcal{V}_{\rm sing} + 2\frac{\alpha^2}{\gamma^2}\left(\frac{1}{\chi_{\rm sing 0}}\frac{\partial \chi_{\rm sing 0}}{\partial x}\right)^2.$$

Hence, in the case $\sigma = 0$, the master equation can be rewritten as

$$\left[\frac{\partial^2}{\partial t^2} + \mathcal{D}_- D_+\right]\chi_{\rm sing} = 0\,;$$

applying to both sides the operator D_+ and recalling the definition (8.4) of χ, one has that the latter satisfies the equation

$$\left[\frac{\partial^2}{\partial t^2} - \left(\frac{\alpha}{\gamma}\frac{\partial}{\partial x}\right)^2 + \frac{\alpha^2}{\gamma^2}\mathcal{V}\right]\chi = 0\,, \qquad \mathcal{V} := -\mathcal{V}_{\rm sing} + 2\left(\frac{1}{\chi_{\rm sing 0}}\frac{\partial \chi_{\rm sing 0}}{\partial x}\right)^2.$$

thesis in Section 7.3, is that we manage to derive a regular master equation in a direct way, with no need to use the previous two-steps construction. Substantially, this result has been achieved by choosing a different gauge at the very beginning of the respective discussions: while the authors of Ref. [25] choose a gauge such that $\delta\beta = \delta r = 0$, in Ref. [1] we set the coordinates (t, x) so that $\delta\beta = \delta\Phi = 0$ (see Remark 55). However, one might wonder how a gauge-invariant treatment such that of Refs. [25, 1] can be affected by a *coordinate* choice? In both papers, the strategy to obtain the master equation for a gauge-invariant quantity, relies on the possibility of decoupling the linearized field equations; in order to do so, the linearized field equations are simplified by a suitable gauge choice. Then, the gauge-invariant quantities χ_{sing} as in Eq. (8.2) and χ as in Eq. (6.100) satisfying, respectively, the master equations (6.101) and (8.1), are defined *a posteriori.* [59] Typically, these gauge-invariant quantities occurs to be singular *exactly* where the gauge transformation $(\delta t, \delta x)$ which realizes the initial gauge choice, is not defined. In particular, the transformation $(\delta t, \delta x)$ which gives $\delta\beta = \delta r = 0$ is singular where $r' = 0$ exactly as χ_{sing} [Eq. (8.2)], while the transformation $(\delta t, \delta x)$ which gives $\delta\beta = \delta r = 0$ is singular where $\Phi' = 0$ exactly as χ [Eq. (6.100)]. Paper [25] has a companion work by the same authors [27] where the exact, nonlinear Einstein equations for the perturbed Ellis-Bronnikov solution are treated numerically, providing evidence that the initial perturbation produces a rapid growth of the wormhole's throat or a collapse to a black hole. (A numerical analysis of the exact, perturbed Einstein equations is also given in the second half of Ref. [53] for the special EBMT case). Admittedly, this issue is beyond the aims of the present thesis.

Returning to the linear stability analysis, let us point out that the two-steps approach (a singular master equation and its subsequent regularization) has been extended by Bronnikov, Fabris and Zhidenko [26] to the whole class of static, radially symmetric scalar field solutions of Einstein's equations with throats (including cases with an external potential for the scalar field) and referred to as "S-deformation method".

Let us also mention a very recent paper of Bronnikov [11], an excellent review about wormholes and black holes supported by scalar fields that considers, amongst else, the two-steps approach to linear stability problems.

[59] For example, see the paragraph just before Eq. (6.54), noting that the definition of χ strictly depends on the choice of A, C.

8.1.2 Some comments on Ref. [53]

In Ref. [53] a first attempt was made to provide a gauge-invariant formulation for the linear stability analysis of the EBMT wormhole which would not require the two step approach of Ref. [25], that is, which would not present the arise of any singularity. The conclusion of the cited article is that a suitable gauge-invariant recombination χ of two gauge-invariant quantities A and C, which are in turn recombinations of the perturbation functions, fulfills exactly the master equation (7.17); however, while the two gauge-invariant quantities A and C were correctly defined, the quantity D defined in Eq. (8) of Ref. [53] is only invariant with respect to the restricted set of gauge transformations for which $\dot{\delta t} = 0$. Unfortunately, in general, this restricted set is not sufficient to achieve both conditions $\delta\Phi = 0$ and $\delta\beta = 0$ simultaneously, on which the derivation in Ref. [53] was based (see Remark 55 and, in particular Eq. (6.19)).

8.2 The Torii-Shinkai wormhole

8.2.1 A comparison between the gauge-invariant approach of Section (7.2) and Ref. [35]

In Ref. [35] the authors, in dealing with the linear stability analysis of the Torii-Shinkai wormhole solution (3.34), derive from linearized Einstein equations (and from the linearized Klein-Gordon equation for the perturbed scalar field) a master equation for a gauge-invariant quantity χ_{sing} analogue to that introduced in Ref. [25](see Eq. (8.2)); actually, Ref. [35] assumes from the very beginning a gauge such that $\delta\beta = 0$ and such that the other components of the perturbation have a sinusoidal time dependence , that is (in our notation)

$$\delta\gamma(\mathtt{t},\mathtt{x}) = \hat{\delta\gamma}(\mathtt{x})\cos(\omega\mathtt{t})\,, \qquad \delta r(\mathtt{t},\mathtt{x}) = \hat{\delta r}(\mathtt{x})\cos(\omega\mathtt{t})\,,$$
$$\delta\Phi(\mathtt{t},\mathtt{x}) = \hat{\delta\Phi}(\mathtt{x})\cos(\omega\mathtt{t})\,,$$

where $\omega > 0$ is a frequency, to be determined like the functions $\hat{\delta\gamma}, \hat{\delta r}, \hat{\delta\Phi}$. The function χ_{sing} is defined as

$$\chi_{\text{sing}}(\mathtt{t},\mathtt{x}) := \hat{\chi}_{\text{sing}}(\mathtt{x})\cos(\omega\mathtt{t})\,, \qquad \hat{\chi}_{\text{sing}}(\mathtt{x}) := r\left(\hat{\delta\Phi} - \Phi'\frac{\hat{\delta r}}{r'}\right)\,. \tag{8.5}$$

However, the coefficients in the linear combination defining χ_{sing} and in the master equation are singular at the wormhole throat $\mathtt{x} = 0$; to remove the singularities, the authors of Ref. [35] generalize the scheme developed by [25] to spacetime dimension $d+1$ with $d \geq 3$ [60] introducing a new gauge-invariant function χ related to χ_{sing} by an ad hoc built, nontrivial differential transformation:

$$\chi = D\,\chi_{\text{sing}}\,, \qquad \text{where} \quad D := \frac{\partial}{\partial \mathtt{x}} - \frac{{\chi_{\text{sing}0}}'}{\chi_{\text{sing}0}}\,;$$

here $\chi_{\text{sing}0}$ is a static solution of the previously mentioned, singular master equation. Then, the gauge-invariant quantity χ is shown to fulfill a regularized master equation, with no singularities at $\mathtt{x} = 0$; in our notation, this reads exactly (7.19) with the nowhere singular potential (7.20). [61]
Differently from the approach of Ref. [35], our computations *never* meet singularities at $\mathtt{x} = 0$ (nor elsewhere) and our master equation for χ has been obtained in direct way, with no need to introduce regularizing transformations. One of the features of our strategy marking a difference with respect to Ref. [35], is that our approach relies on a different initial gauge choice and, consequently, on a differently defined gauge-invariant quantity χ (see Subsection 8.1.1).

8.2.2 A comparison between the gauge-dependent approach od Section 5.1 and the gauge-invariant approach of Section 7.2

In both the gauge-invariant and the gauge-dependent deductions of the linear instability of the Torii-Shinkai wormhole of Sections (5.1,7.2) the linearized field equations are decoupled obtaining a wave-type master equation which is everywhere regular. However, while in the first case we introduce a gauge-invariant quantity satisfying the master equation, in the second case the unknown of the equation is substantially the perturbation coefficient δr (multiplied by a suitable defined time-independent function), which clearly depends on the coordinates chosen. Indeed, using this second approach, in order to infer a coordinate-independent linear instability result, we had to study the behaviour of the linearized scalar curvature of the perturbed spacetime, showing that it diverges as $\mathtt{t}$ approach $\pm\infty$ (and that it is true even after an infinitesimal gauge change). This inconvenient (i.e., the gauge-dependence of the unknown of the master equation) is indeed removed using

[60] See Subsection 8.1.1, for more details on the four-dimensional case.

[61] Actually, due to the ansatz on the temporal dependence of the perturbation functions, Torii and Shinkai found the eigenvalue equation of the operator H in Eq. (7.20) with unknown eigenfunctions $\hat{\chi}_{\text{sing}}$ and unknown eigenvalues ω^2.

the gauge-invariant strategy of Section 7.2.
Actually, there is a deep reason why the the two master equations for the gauge-invariant quantity χ and for the gauge-dependent quantity $\mathcal{R}$ are substantially the same equation (indeed, they are exactly the same except for the presence of a source term in the equation for $\mathcal{R}$). Using the very definition of the gauge-invariant quantities C and E in Eqs. (6.36,6.37) for the static Torii-Shinkai solution (3.34), in the gauge (5.4) for which $\delta\beta = \delta\alpha = 0$ it turns out that

$$\ddot{\delta r} = \rho(\mathtt{x})\ddot{C} - \rho'(\mathtt{x})E\,;$$

then, applying the linearized field equations (6.55,6.57) and the background equation (3.8), we get

$$\ddot{\mathcal{R}} = -(d-2)\frac{A-C}{\rho(\mathtt{x})}\frac{1}{\rho(\mathtt{x})^{\frac{d-3}{2}}} = -(d-2)\frac{\mathcal{F}}{\rho(\mathtt{x})^{\frac{d-3}{2}}} = -(d-2)\chi\,, \tag{8.6}$$

where we have set $\mathcal{R}(\mathtt{t},\mathtt{x}) = \rho(\mathtt{x})^{\frac{3d-5}{2}}\delta r$ as in Eq. (5.5), $\mathcal{F}$ as in Eq. (6.73) and χ as in Eq. (7.18). Let me recall that the master equation (5.21) for $\mathcal{R}$ (in the gauge $\delta\beta = \delta\alpha = 0$) is

$$\left[\frac{\partial^2}{\partial \mathtt{t}^2} - \frac{\partial^2}{\partial \mathtt{x}^2} + \mathcal{V}\right]\mathcal{R} = \mathcal{J}_0(\mathtt{x}) + \mathcal{J}_1(\mathtt{x})\mathtt{t} \tag{8.7}$$

with $\mathcal{V}$ as in Eq. (5.22); deriving both sides twice in $\mathtt{t}$ and recall Eq. (8.6), we get

$$\left[\frac{\partial^2}{\partial \mathtt{t}^2} - \frac{\partial^2}{\partial \mathtt{x}^2} + \mathcal{V}\right]\chi = 0\,, \tag{8.8}$$

which is exactly the master equation (7.19). Note that, since the linearized field equations and the quantity χ are gauge-invariant, Eq. (8.8) is valid in any gauge.
Let us underline that this relation is not so obvious, since the master equation (7.19) is valid only for perturbations such that $\mathcal{S} = 0$, with $\mathcal{S}$ as in Eq. (6.6); however, in this case, introducing the expressions (5.5,5.17,5.20) for the perturbations $\delta r, \delta\gamma, \delta\Phi$ into Eq. (6.6), we get

$$\mathcal{S} := \frac{(d-1)(d-2)}{2}\,\mathcal{C}\,,$$

which vanishes if one set $\mathcal{C} = 0$ (note that this is possible since the constant $\mathcal{C}$ in Eq. (5.20) is immaterial).

8.3 The AdS wormhole

8.3.1 A comparison between the gauge-dependent approach of Section 5.2 and the gauge-invariant approach of Section 7.4

In Section 7.4, we have presented a deduction of the linear instability of the AdS wormhole, which relies in studying the temporal behaviour of the radial perturbation of the AdS metric (which is of course gauge-dependent) and then in using this result in order to show that the perturbed spacetime becomes singular with the increasing of the temporal coordinate. As in the case of the Torii-Shinkai wormhole, there is a reason why the master equation arising from this gauge-dependent deduction is very similar to the master equation obtained following the gauge-invariant approach. In the gauge (5.41), for which $\delta\beta = 0$ and $\alpha\,\delta\alpha = \frac{1}{4(1+B^2)}\, r\,\delta r$, from the very definition of the gauge-invariant quantities C, E in Eqs. (6.36,6.37) for the static AdS solution (3.62) one gets that

$$\ddot{\delta r} = \frac{\sqrt{2}\csc u\, \sin^3\frac{u}{2}\sqrt{1+2B^2-\cos u}}{k}\left[\frac{C - 2(1+B^2)\csc u\, E}{1+2B^2-\cos u} + \csc u\, C' + \csc^2\frac{u}{2}\,\ddot{C}\right];$$

then, applying the linearized field equations (6.55,6.57) and the background equation (3.8), we get

$$\ddot{\mathcal{R}} = -\frac{B\sec\frac{u}{2}\Big(2(1+B^2)A - (1+\cos u)C\Big)}{2\sqrt{2}\sqrt{1+2B^2-\cos u}} = -2Bk(1+B^2)\chi\,, \qquad (8.9)$$

where we have set $\mathcal{R}(s,u) = \frac{1+2B^2-\cos u}{b}\delta r$ as in Eq. (5.42) and χ as in Eq. (7.50) (recalling the definitions (6.73) of $\mathcal{F}$ and $\mathcal{G}$).
Let me rewrite the master equation (5.59) for $\mathcal{R}$ (which is valid in the gauge (5.41))

$$\left[\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial u^2} + \mathcal{V}\right]\mathcal{R} = \mathcal{J}_0(u) + \mathcal{J}_1(u)s + \mathcal{J}_2(u)s^2 \qquad (8.10)$$

with $\mathcal{V}$ as in Eq. (5.60). Deriving both sides of the previous equation twice in `t`, recalling Eq. (8.9), we get

$$\left[\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial u^2} + \mathcal{V}\right]\chi = 2\,\mathcal{J}_2(u)\,. \qquad (8.11)$$

This is not the master equation (7.51) for χ because of the presence of the source term $2\mathcal{J}_2(u)$; indeed, we have just mentioned that the master equation (7.51) is not valid for every perturbations, but only for perturbations such that the gauge-invariant quantity $\mathcal{S}$ vanishes. In the AdS case, if we introduce the expressions (5.42,5.53,5.56) for the perturbations $\delta r, \delta\gamma, \delta\Phi$ into Eq. (6.6), we obtain

$$\mathcal{S} = -\frac{b\sqrt{1+B^2}}{2k}\,\mathcal{C}\,;$$

note that in this case the constant $\mathcal{C}$ in Eq. (5.56) is *not* immaterial, hence we can not set $\mathcal{C} = 0$. However, if we restrict ourselves to field perturbations such that $\mathcal{C} = 0$, then the conditions $\mathcal{S} = 0$ is satisfied; indeed, in this case the coefficient $\mathcal{J}_2$ in Eq. (5.63) vanishes and Eq. (8.11) actually reduces exactly to the master equation (7.51) (in any gauge).

Conclusion and possible future developments

In this thesis I have analyzed the linear stability of a class of $(d+1)$-dimensional (with $d \geq 3$) static, spherically symmetric wormhole solutions in higher dimensional GR with a self-interacting phantom scalar field minimally coupled to gravity. To this purpose, in the first part of the thesis, I have introduced some known wormhole solutions, recovering directly their metrics (and the corresponding scalar field and self-interacting potential) from the $(d+1)$-dimensional Einstein-scalar equations: these are the $(d+1)$-dimensional Torii-Shinkai wormhole [35], the four-dimensional Ellis-Bronnikov wormhole [14] and a four-dimensional AdS wormholes [36], for which the scalar field is subject to a non-trivial self-interaction term. All of them are generalization of the celebrated the Ellis-Bronnikov-Morris-Thorne (EBMT) wormhole, which is included as a special case. Again in the fist part, I have also construct (and plotted) the embedding diagrams of the Ellis-Bronnikov and AdS wormholes; for the same wormholes, I have performed a complete study of the timelike and of the null geodesic motions, plotting some of their trajectories on the previously mentioned embedding diagrams. In the last section of the first part, I have considered a dS wormhole with horizons, showing how it is possible to extend its metric beyond the horizons, obtaining a non static extended spacetime [1].
The second part of the thesis is devoted to the linear stability analysis of the wormholes introduced in the first part. To this purpose, I have considered a general, coordinate-dependent perturbation of an arbitrary $(d+1)$-dimensional static, spherically symmetric scalar-solution, deriving the corresponding linearized field equations and studying their transformation under infinitesimal coordinate changes.
In Chapter 5, I have focused on the Torii-Shinkai and AdS wormholes; by introducing a suitably defined gauge, I have shown that in both cases it is possible to decoupled the linearized field equations, yielding a completely regular master equation for the radial perturbation, which has a source term

depending on the initial data of the perturbation. I have obtained an instability result of the two wormholes (which has been firstly inferred, respectively in Refs. [35] and [1]) by showing that the solutions of the master equations possess a unique, exponentially in time growing mode associated with a bound state of negative energy of the Schrödinger operator arising in the master equations themselves. Indeed, I have proved that these solutions generates perturbed spacetimes that become singular in the large temporal limit, independently of the chosen gauge.

In Chapter 6, I have described a gauge-invariant perturbation formalism that describes the dynamics of linearized, spherically symmetric, time-dependent perturbations of a wormhole metric and of the associated scalar field, resulting in a coupled 2×2 linear wave system subject to a constraint (see Eqs. (6.74,6.75)). Provided that a nontrivial, time-independent solution is known (as is usually the case when a family of static solutions is available), I have shown that this system can be decoupled yielding a master wave equation for a gauge-invariant quantity which is regular at the throat. This construction, which generalizes to arbitrary dimension that introduced in Ref. [1], relies on a basic requirement: the derivative Φ' of the (background) scalar field should vanish nowhere. The relevance of this condition in this approach is indicated by the almost ubiquitous presence of the reciprocal $1/\Phi'$ in the equations of Chapter 6.

Based on the gauge-invariant formalism of Chapter 6, in Chapter 7 I have rederived the regular master equations first obtained in Refs. [35] and [25] for the linear spherical perturbations of the Torii-Shinkai and the Ellis-Bronnikov wormholes. My approach never produces singularities at the throat, differently from Refs. [35, 25], where the intermediate steps in the construction contain singularities, which are eliminated a posteriori by means of the S-method. In addition, I have applied the gauge-invariant approach to the AdS and dS wormhole, recovering the master equations and the linear instability results presented in Ref. [1]; however, in the dS case, the instability result refers only to the static spacetime region within the horizons.

In Appendices CDE, I have performed a spectral analysis of the Schrödinger operators appearing in the master wave equations, providing a detailed and rigorous discussion for the mode decomposition of their solutions in all the aforementioned examples, which revealed that besides the exponentially in time growing modes (whose existence is sufficient to infer a linear instability result), there might be linearly growing modes, while all the remaining modes are oscillatory. In particular, the AdS wormhole has infinitely many normalizable, oscillatory modes in addition to the pair of exponential growing and decaying modes associated with the unique bound state of negative energy of the Schrödinger operator.

Let me conclude with some remarks on the possible future developments of the present work. I have already mentioned that the linear stability theory for non static wormhole solutions, and its application to the (extended) dS wormhole, deserves further work in my opinion. Sticking to the case of static wormholes and of their linearized perturbations, I think that the forthcoming issues (i-iii) are worthy of future investigations.

(i) The intent to avoid the arising of singularities in the analysis of wormholes stability is not only motivated by "aesthetic" instances, because the S-deformation method used to regularize the potential in singular master equations requires to solve a Riccati-type equation, which can often be done only numerically: this fact prevents to obtain an explicit expression for the regularized potential. An example among others of the occurrence of this situation is the linear stability analysis of the M-AdS wormhole of Ref. [18]. In the future, it could be interesting to rephrase the stability analysis of the M-AdS wormhole (and of other similar wormhole configurations) using the gauge-invariant method proposed in this thesis.

(ii) A basic requirement of the gauge-invariant approach presented in this thesis, recalled above, is the condition that Φ has no critical points. Removing this requirement would be interesting since, recently, a large class of new wormhole solutions of the Einstein-scalar equations has been found [22], generalizing previous work [19], in which the scalar field Φ has an extremum at the throat. Since r has a global minimum at the throat (like every wormhole spacetime) and r' converges to zero as fast or faster than Φ', it turns out the gauge-invariant quantity C defined in Eq. (6.36) is still well-defined; unfortunately, it is unclear if a decoupled equation for C can be obtained which is regular at the throat. In connection with this problem, one could try to recover the S-deformation method of Refs. [25, 26] (the formulation of this method in Ref. [26] indeed considers the gauge-invariant quantity C). However, as it happens for the M-AdS wormhole, when the potential $V(\Phi)$ is non-zero, this method seems to require the numerical integration of a Riccati-type equation to find the regularized potential, and further one still needs to justify *a posteriori* the validity of the transformed equation at the throat. An alternative possibility consists in applying a variation of the approach discussed in this article, in which Φ' is absent from all denominators, thanks to the use of new gauge-invariant quantities in place of the functions A, C, E of Eqs. (6.35-6.37); at present, it is not clear whether this will be possible.

(iii) Let me propose the following question: is there a *deep* geometrical reason for which the approaches presented in this thesis succeed, in certain cases, in decoupling the perturbation equations and reducing them to a single, scalar master equation? Typically, the possibility of reducing to a simpler form a PDE or of a system of PDEs is due to the presence of a Lie group of symmetries; an interpretation of this kind could perhaps be given for the decoupling methods of this thesis. As already recalled, the gauge-invariant approach of Chapter 6 uses a static solution of Eqs. (6.74-6.75), arising from variations with respect to the parameters of a *family* of static wormhole solutions. The availability of such parametric families could perhaps be interpreted in terms of a Lie group of symmetries, acting on the static solutions of the Einstein-scalar system; if so, it would be interesting to understand the interplay of these symmetries with the linearized perturbation equations.

Appendix A

Geodesic motion in a four-dimensional spherically symmetric static spacetime

Let us firstly recall that the trajectory described by a free-falling particle or by a light ray in a four-dimensional spacetime $(M, \mathbf{g})$ is represented by a geodesic of M, i.e. a world line $\tau \mapsto \mathcal{P}(\tau)$ such that [54, 39]

$$\frac{\nabla}{d\tau}\frac{d\mathcal{P}}{d\tau} = 0 \tag{A.1}$$

where ∇ is the covariant derivative defined by the Levi-Civita connection of the metric $\mathbf{g}$ and $\tau \in \mathrm{dom}(\mathcal{P}) \subseteq \mathbb{R}$ is the temporal parameter of the geodesic. Moreover, it is always possible to redefine the parameter τ in such a way that, for all τ

$$\mathbf{g}_{\mathcal{P}(\tau)}\left(\frac{d\mathcal{P}}{d\tau}, \frac{d\mathcal{P}}{d\tau}\right) = -\mathtt{k}\,,\ \mathtt{k} = \begin{cases} 1 \text{ for timelike geodesics (free-falling particles)} \\ 0 \text{ for null geodesics (light rays)} \end{cases} \tag{A.2}$$

In this appendix we will study the geodesic motion in a four-dimensional spherically symmetric spacetime defined in the coordinates $(x^\mu) := (t, x, \theta, \varphi)$ by the metric

$$\mathbf{g} = -\alpha(x)^2 dt^2 + \frac{1}{\alpha(x)^2} dx^2 + r(x)^2 d\Omega^2\,, \tag{A.3}$$

where the coefficients $\alpha, r\ :\ x(\mathcal{O}) \to (0, +\infty)$ are two smooth functions; each geodesic $\mathcal{P}$ in this spacetime is described locally by four functions of the parameter τ

$$(x^\mu(\mathcal{P}(\tau))) =: (x^\mu(\tau)) =: (t(\tau), x(\tau), \theta(\tau), \varphi(\tau)) \tag{A.4}$$

satisfying the geodesic equation (A.1). This equation locally reads

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\lambda\nu}\frac{dx^\lambda}{d\tau}\frac{dx^\nu}{d\tau} = 0\,, \qquad \mu = 1, ..., 4\,, \tag{A.5}$$

where $\Gamma^\mu_{\lambda\nu}$ are the Christoffel symbols of the Levi-Civita connection of $\mathbf{g}$. Moreover, it can be proved [55] that the geodesics equations (A.5) are equivalent to the Euler–Lagrange equations $\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} - \frac{\partial\mathcal{L}}{\partial x^\mu} = 0$, $\mu = 1, ..., 4$, for the Lagrangian

$$\mathcal{L}(x^\mu, \dot{x}^\mu) := \frac{1}{2}\mathbf{g}_{\lambda\nu}(x^\mu)\dot{x}^\lambda\dot{x}^\nu = -\frac{\alpha(x)^2}{2}\dot{t}^2 + \frac{1}{2\alpha(x)^2}\dot{x}^2 + \frac{r^2}{2}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right)\,; \tag{A.6}$$

these are satisfied if and only if the following system of four ordinary differential equation holds

$$\begin{aligned}
\frac{d}{d\tau}\left(\alpha(x)^2\dot{t}\right) &= 0\,, && \text{(A.7)}\\
\frac{d}{d\tau}\left(\frac{1}{\alpha(x)^2}\dot{x}\right) &= \alpha(x)r(x)r'(x)\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right) - \alpha(x)^2\alpha'(x)\dot{t}^2\,, && \text{(A.8)}\\
\frac{d}{d\tau}\left(r(x)^2\dot{\theta}\right) &= r(x)^2\sin\theta\cos\theta\,\dot{\varphi}^2\,, && \text{(A.9)}\\
\frac{d}{d\tau}\left(r(x)^2\dot{\varphi}\right) &= \frac{d}{d\tau}\left(r(x)^2\dot{\varphi}\cos^2\theta\right)\,. && \text{(A.10)}
\end{aligned}$$

Before starting with the study of the system (A.7-A.10), let me summarize some general and useful results about Lagrangian systems:

(i) in a time-independent n-dimensional Lagrangian system ($\mathcal{L}(q^i, \dot{q}^i)$, $q^i = q^i(t)$, $i = 1, ..., n$), the total energy function defined by $\mathcal{E} := \frac{\partial\mathcal{L}}{\partial\dot{q}^i}\dot{q}^i - \mathcal{L}$ is conserved;

(ii) in the the hypothesis of (i), the system of the n Euler-Lagrange equations for the Lagrangian $\mathcal{L}$ is equivalent to the system made up of $n-1$ Euler-Lagrange equations and the conservation law $\mathcal{E} = const$;

(iii) in the the hypothesis of (i) and if the Lagrangian $\mathcal{L}$ is a quadratic function in the generalised velocities $\dot{q}^i$, it follows that $\mathcal{E} = \mathcal{L}$ and the conservation law reads $\mathcal{L} = const$.

These results immediately apply to the Lagrangian (A.6); moreover, since for all τ

$$\mathcal{L}(x^\mu(\tau), \dot{x}^\mu(\tau)) = \frac{1}{2}\mathbf{g}_{\lambda\nu}(x^\mu(\tau))\dot{x}^\lambda(\tau)\dot{x}^\nu(\tau) = \frac{1}{2}\mathbf{g}_{\mathcal{P}(\tau)}\left(\frac{d\mathcal{P}}{d\tau}, \frac{d\mathcal{P}}{d\tau}\right)\,,$$

and recalling the position (A.2), we have that the conservation law becomes $\mathcal{L} = -\mathtt{k}/2$.
We are now ready to study of the system (A.7-A.10), starting from the third equation (A.9); obviously, this equation and the initial conditions

$$\tau_0 = 0\,, \quad \theta(\tau_0) = \frac{\pi}{2}\,, \quad \dot{\theta}(\tau_0) = 0 \tag{A.11}$$

imply that $\theta(\tau) = \frac{\pi}{2}$ for every τ. Since it is always possible to redefine the coordinates θ and φ and the parameter τ so that the previous conditions on θ are true, from now on we assume (A.11); in this way the four-dimensional system (A.7-A.10) reduces to the three-dimensional system:

$$\frac{d}{d\tau}\left(\alpha(x)^2\dot{t}\right) = 0\,, \tag{A.12}$$

$$\frac{d}{d\tau}\left(r(x)^2\dot{\varphi}\right) = 0\,, \tag{A.13}$$

$$-\frac{\alpha(x)^2}{2}\dot{t}^2 + \frac{1}{2\alpha(x)^2}\dot{x}^2 + \frac{r(x)^2}{2}\dot{\varphi}^2 = -\frac{\mathtt{k}}{2}\,, \tag{A.14}$$

where we have substituted the second equation (A.8) with the conservation law $\mathcal{L} = -\mathtt{k}/2$, thanks to (i)-(iii) after Eq. (A.10) and the forthcoming remark.
Let us start with Eqs. (A.12,A.13); note that, hopefully performing the parameter change $\tau \mapsto -\tau$, the vector $d\mathcal{P}/d\tau$ can be regarded as future-oriented, so that one can always suppose that $\dot{t} > 0$. Thanks to this remark, it is clear that Eqs. (A.12,A.13) holds if and only if there exists two constants $E \geq -\mathtt{k}/2$ and $L \in \mathbb{R}$ [39] such that

$$\alpha(x)^2\dot{t} = \sqrt{\mathtt{k} + 2E}\,, \qquad r(x)^2\dot{\varphi} = L\,; \tag{A.15}$$

these two equations can be easily solved, leading to

$$t(\tau) = \int_0^\tau \frac{\sqrt{\mathtt{k} + 2E}}{\alpha(x(\tilde{\tau}))^2} d\tilde{\tau} + t(0)\,, \qquad \varphi(\tau) = \int_0^\tau \frac{L}{r(x(\tilde{\tau}))^2} d\tilde{\tau} + \varphi(0)\,. \tag{A.16}$$

Note that it results from Eq. (A.15) that the two constants E and L are fully determined by the initial data:

$$E := \frac{\dot{t}(0)^2 h(x(0))^4 - \mathtt{k}}{2}\,, \quad L = \dot{\varphi}(0) r(x(0))^2\,.$$

It is easy to see that in the limit case of a particle moving slowly in a weak gravitational potential (i.e. $\alpha(x) \simeq 1$) E and L approach, respectively, to

the classical total energy and the angular momentum per unit rest mass of the particle [39]; therefore, in the timelike case, one can interpret L and E as a relativistic generalisation of the total energy and the angular momentum per unit rest mass of a free-falling particle and, in the null case, $\hbar L$ and $\hbar E$ as the angular momentum and the total energy of a photon (recall that in Remark 1 we have stipulated $\hbar = 1$).
Inserting Eq. (A.16) into the conservation law (A.14), we have that the reduced Lagrangian system (A.12,A.13,A.14) is equivalent to the dynamical system made up of Eq. (A.16) and

$$\frac{1}{2}\dot{x}^2 + V_{\text{eff}}(x) = E \tag{A.17}$$

where we have defined the effective potential

$$V_{\text{eff}}(x) := \frac{L^2}{2}\frac{\alpha(x)^2}{r(x)^2} + \frac{\mathtt{k}}{2}\left(\alpha(x)^2 - 1\right) . \tag{A.18}$$

Summing up, provided a suitable change of coordinates, the problem of finding the qualitative behaviour of a timelike ($\mathtt{k} = 1$) or a null ($\mathtt{k} = 0$) geodesic in a spacetime with a metric of the form (A.3) is reduced to studying its radial motion, which satisfies Eq. (A.17) with the potential (A.18); since this raidal motion is the same as the motion of a unit mass particle of energy E in ordinary one-dimensional, nonrelativistic mechanics moving in the effective potential V_{eff}, in order to understand of the qualitative features of the geodesic motion one has to investigate the analytical properties of V_{eff} in dependence of the values of the parameters appearing in its definition (A.18).

Appendix B

Supplements on gauge transformations

B.1 Gauge transformations of the perturbation functions

Let us keep the notation of Section 4.2; in particular, let us introduce the static metric $\mathbf{g}$, the perturbed metric $\delta\mathbf{g}$, and the transformed perturbation metric $\tilde{\delta\mathbf{g}}$ as in Eqs. (4.11,4.12,4.18) these are related according to Eq. (4.17), i.e.

$$\tilde{\delta\mathbf{g}} = \delta\mathbf{g} + \pounds_{\delta\mathbf{X}}\, \mathbf{g}_0\,. \tag{B.1}$$

Note that the previous equation has to be evaluated in the new coordinates $(\tilde{t}, \tilde{x})$; however, for the sake of simplicity in the sequel, we omit the tildes, as there is no possibility of misunderstanding. In Eq. (B.1) the symbol $\pounds_{\delta\mathbf{X}}$ stands for the Lie derivative with respect to the vector field $\delta\mathbf{X}$ on $\mathbb{R}^{d+1}$ with the first two smooth components

$$(\delta t, \delta x) : \mathcal{O} \subseteq \mathbb{R}^2 \to \mathbb{R}^2 \tag{B.2}$$

and all other vanishing; moreover, we write the quantities appearing in Eq. (B.1) as

$$\mathbf{g}_0 := -\alpha^2 dt^2 + \gamma^2 dx^2 + r^2\, a_{ij}(x^2, ..., x^d)\, dx^i\, dx^j\,, \tag{B.3}$$

$$\delta\mathbf{g} := -2\alpha\,\delta\alpha\, dt^2 + 2\gamma^2\,\delta\beta\, dt\, dx + 2\gamma\,\delta\gamma\, dx^2 + 2r\,\delta r\, a_{ij}(x^2, ..., x^d)\, dx^i\, dx^j\,, \tag{B.4}$$

$$\tilde{\delta\mathbf{g}} := -2\alpha\,\tilde{\delta\alpha}\, dt^2 + 2\gamma^2\,\tilde{\delta\beta}\, dt\, dx + 2\gamma\,\tilde{\delta\gamma}\, dx^2 + 2r\,\tilde{\delta r}\, a_{ij}(x^2, ..., x^d)\, dx^i\, dx^j\,. \tag{B.5}$$

Our aim is to show that Eq. (B.1) is satisfied if and only if the transformed perturbation coefficients $(\tilde{\delta\alpha}, \tilde{\delta\beta}, \tilde{\delta\gamma}, \tilde{\delta r})$ are related to the perturbation coefficients $(\delta\alpha, \delta\beta, \delta\gamma, \delta r)$ according to Eqs. (4.19-4.23).
We start computing the Lie derivative appearing in Eq. (B.1); with this respect, let us recall the following

Remark 82 For every vector field $\delta\mathbf{X}$ on $\mathbb{R}^{d+1}$, for every integers $0 \leq \mu, \nu \leq d+1$ and for every smooth function $f : x(\mathcal{O}) \to \mathbb{R}$, the following holds

$$\begin{aligned}
&\pounds_{\delta\mathbf{X}} \left[f(x) dx^\mu \otimes x^\nu \right] \\
&\quad = (\pounds_{\delta\mathbf{X}} f(x)) dx^\mu \otimes x^\nu + f(x)(\pounds_{\delta\mathbf{X}} dx^\mu) \otimes dx^\nu + f(x) dx^\mu \otimes (\pounds_{\delta\mathbf{X}} dx^\nu)\,;
\end{aligned}$$

in addition, if the first two components of the vector field $\delta\mathbf{X}$ are defined as in Eq. (B.2) while the others are zero, then, for every smooth function $f : \mathcal{O} \to \mathbb{R}$

$$\pounds_{\delta\mathbf{X}}\, dt = \dot{\delta t}\, dt + \delta t'\, dx\,, \qquad \pounds_{\delta\mathbf{X}}\, dx = \dot{\delta x}\, dt + \delta x'\, dx\,,$$
$$\pounds_{\delta\mathbf{X}}\, dx^i = 0 \quad (2 \leq i \leq d+1)\,,$$
$$\pounds_{\delta\mathbf{X}}\, f(t, x, x^2, ..., x^d) = \dot{f}(t, x, x^2, ..., x^d)\delta t + f'(t, x, x^2, ..., x^d)\delta x\,.$$

In particular, for every smooth functions $f : x(\mathcal{O}) \to \mathbb{R}$ and $a : \mathcal{O}_{d-1} \to \mathbb{R}$, one has that

$$\pounds_{\delta\mathbf{X}}\, f(x) = f'(x)\delta x\,, \qquad \pounds_{\delta\mathbf{X}}\, a(x^2, ..., x^d) = 0\,.$$

Therefore, from the previous remark it follows that

$$\begin{aligned}
\pounds_{\delta\mathbf{X}} \left[\alpha^2 dt^2\right] &= \pounds_{\delta\mathbf{X}} \left[\alpha^2 dt \otimes dt\right] \\
&= (2\alpha\alpha'\delta x + 2\alpha^2\, \dot{\delta t})\, dt \otimes dt + \alpha^2\, \delta t' (dt \otimes dx + dx \otimes dt) \\
&= (2\alpha\alpha'\delta x + 2\alpha^2\, \dot{\delta t})\, dt^2 + 2\alpha^2\, \delta t'\, dt\, dx\,, \\
\pounds_{\delta\mathbf{X}} \left[\gamma^2 dt^2\right] &= \pounds_{\delta\mathbf{X}} \left[\gamma^2 dt \otimes dt\right] \\
&= (2\gamma\gamma'\delta x + 2\gamma^2\, \delta x')\, dx \otimes dx + \alpha^2\, \dot{\delta x} (dt \otimes dx + dx \otimes dt) \\
&= (2\gamma\gamma'\delta x + 2\alpha^2\, \delta x')\, dx^2 + 2\gamma^2\, \dot{\delta x}\, dt\, dx\,,
\end{aligned}$$

$$\begin{aligned}
&\pounds_{\delta\mathbf{X}} \left[r^2\, a_{ij}(x^2, ..., x^d)\, dx^i\, dx^j\right] \\
&= \frac{1}{2} \pounds_{\delta\mathbf{X}} \left[r^2\, a_{ij}(x^2, ..., x^d)\, dx^i \otimes dx^j\right] + \frac{1}{2} \pounds_{\delta\mathbf{X}} \left[r^2\, a_{ij}(x^2, ..., x^d)\, dx^j \otimes dx^i\right] \\
&= r\, \delta x (dx^i \otimes dx^j + dx^j \otimes dx^i) = 2r\, \delta x\, dx^i\, dx^j\,;
\end{aligned}$$

hence,

$$\begin{aligned}
\pounds_{\delta\mathbf{X}}\, \mathbf{g}_0 = &- \pounds_{\delta\mathbf{X}} \left[\alpha^2 dt^2\right] + \pounds_{\delta\mathbf{X}} \left[\gamma^2 dx^2\right] + \pounds_{\delta\mathbf{X}} \left[r^2\, a_{ij}(x^2, ..., x^d)\, dx^i\, dx^j\right] \\
= &- (2\alpha\alpha'\delta x + 2\alpha^2\, \dot{\delta t})\, dt^2 + (2\gamma\gamma'\delta x + 2\alpha^2\, \delta x')\, dx^2 \\
&+ (2\gamma^2\, \dot{\delta x} - 2\alpha^2\, \delta t')\, dt\, dx + 2r\, \delta x\, dx^i\, dx^j\,. \qquad \text{(B.6)}
\end{aligned}$$

Inserting Eqs. (B.4,B.5,B.6) into Eq. (B.1), we have that the latter is identically satisfied if and only if

$$\tilde{\delta\alpha} = \delta\alpha + \alpha'\delta x + \alpha\,\dot{\delta t}\,, \qquad \tilde{\delta\beta} = \delta\beta + \delta\dot{x} - \frac{\alpha^2}{\gamma^2}\delta t'\,,$$
$$\tilde{\delta\gamma} = \delta\gamma + (\gamma\delta x)'\,, \qquad \tilde{\delta r} = \delta r + r'\delta x\,, \qquad \tilde{\delta\Phi} = \delta\Phi + \Phi'\delta x\,,$$

a fact which justifies Eqs. (4.19-4.23).

B.2 On the divergence of linearized quantities

Let us consider on a two-dimensional Lorentzian manifold M_2 a smooth scalar function of the form

$$R^{(1)} = R_0 + \epsilon\delta R$$

where $R_0, \delta R$ are as well smooth scalars on M_2 and ϵ is a small parameter. In addition, let us consider for M_2 a general coordinate system (t, x) and introduce the local representation of the above scalars in these coordinates, that we indicate for simplicity with the same symbols:

$$R^{(1)}(t,x) = R_0(t,x) + \epsilon\,\delta R(t,x)\,. \tag{B.7}$$

Remark 83 Let us now introduce an arbitrary infinitesimal gauge transformation ϕ_ϵ as in Eq. (4.6); then the quantity (B.7) is transformed by the pullback of the inverse map $\phi_\epsilon^{-1} = \psi_\epsilon$ [Eq. (4.7)], namely

$$\begin{aligned} R^{(1)}(t,x) \mapsto \tilde{R}^{(1)}(\tilde{t},\tilde{x}) &:= \psi_\epsilon^* R^{(1)}(\tilde{t},\tilde{x}) = R^{(1)}(\tilde{t},\tilde{x}) + \epsilon\,\pounds_{\delta\mathbf{X}}\,R^{(1)}(\tilde{t},\tilde{x}) \\ &= R_0(\tilde{t},\tilde{x}) + \epsilon\,\left(\delta R(\tilde{t},\tilde{x}) + \pounds_{\delta\mathbf{X}}\,R_0(\tilde{t},\tilde{x})\right) + O(\epsilon^2) \\ &= R_0(\tilde{t},\tilde{x}) + \epsilon\,\Big(\delta R(\tilde{t},\tilde{x}) + \dot{R}_0(\tilde{t},\tilde{x})\delta t(\tilde{t},\tilde{x}) + R_0'(\tilde{t},\tilde{x})\delta x(\tilde{t},\tilde{x})\Big)\;; \end{aligned} \tag{B.8}$$

in the first identity we have used the expression for the pullback ψ_ϵ^* of the infinitesimal transformation ψ_ϵ parametrized by the vector field $\epsilon\,\delta\mathbf{X}$ in terms of the Lie derivative $\pounds_{\delta\mathbf{X}}$ with respect to the field $\delta\mathbf{X}$; in the second identity we have used Eq. (B.7); in the third identity we have neglected all the powers of ϵ greater or equal then 2 and we have used the fact that for every smooth function $f : \mathcal{O} \subseteq \mathbb{R}^2 \to \mathbb{R}$

$$\pounds_{\delta\mathbf{X}}\,f(t,x) = \dot{f}(\tilde{t},\tilde{x})\delta t + f'(\tilde{t},\tilde{x})\delta x\,.$$

Example 7 Suppose that $\mathcal{O} := \mathbb{R}^2$ and define

$$R^{(1)}(t,x) = (1+x^2)(-e^{-t^2} + 2\epsilon\, t^2) \qquad ((t,x) \in \mathbb{R}^2)\,; \tag{B.9}$$

then $R^{(1)}(t,x) \to +\infty$ for $t \to \pm\infty$ and for every fixed $x \in \mathbb{R}$. Let us show that it is possible to find an infinitesimal gauge transformation ϕ_ϵ as in Eq. (4.6), with inverse ψ_ϵ as in Eq. (4.7) such that the transformed linearized quantity is such that

$$\tilde{R}^{(1)}(\tilde{t},\tilde{x}) \to 0 \quad \text{as } \tilde{t} \to \pm\infty$$

for every fixed $\tilde{x} \in \mathbb{R}$. Indeed, by setting

$$\psi_\epsilon(\tilde{t},\tilde{x}) := (\, t := \tilde{t} + \epsilon\,(1 - e^{\tilde{t}^2}\tilde{t})\,,\; x := \tilde{x}\,) \qquad ((\tilde{t},\tilde{x}) \in \mathbb{R}^2)$$

that is, by setting

$$\delta t(\tilde{t},\tilde{x}) = 1 - e^{\tilde{t}^2}\tilde{t}\,, \qquad \delta x(\tilde{t},\tilde{x}) = 0 \qquad ((\tilde{t},\tilde{x}) \in \mathbb{R}^2)$$

and by using Eq. (B.8), we have that the transformed quantity (B.9) reads

$$\tilde{R}^{(1)}(\tilde{t},\tilde{x}) = (1+\tilde{x}^2)(-e^{-\tilde{t}^2} + 2\epsilon\,\tilde{t}e^{-\tilde{t}^2})\,,$$

which clearly vanishes for $t \to \pm\infty$ and for every fixed $\tilde{x} \in \mathbb{R}$.

In the following proposition we provide some sufficient requirements that the linearized quantity (B.7) has to possess in order to diverge in every ϵ-close coordinate systems.

Proposition 14 *Let us suppose that the linearized quantity* (B.7) *satisfies the following hypotheses:*

(i) $R_0(t,x) \equiv R_0(x)$*;*

(ii) $\delta R(t,x_0) \to \infty$ *as* $t \to \infty$ *for a given* x_0*;*

(iii) $R_0'(x_0) = 0$*;*

then, the linearization (B.7)*, in any coordinate system* $(\tilde{t},\tilde{x})$ ϵ*-close to* (t,x)*, still diverges in* $\tilde{x} = x_0$ *as* $\tilde{t} \to \infty$*.*

Proof. Let us consider an arbitrary infinitesimal gauge transformation ϕ_ϵ as in Eq. (4.6), with inverse ψ_ϵ as in Eq. (4.7). Then, from Eq. (B.8) and using (i) we have that the transformed linearized quantity (B.7) reads

$$\tilde{R}^{(1)}(\tilde{t},\tilde{x}) = R_0(\tilde{x}) + \epsilon\Big(\delta R(\tilde{t},\tilde{x}) + \delta x(\tilde{t},\tilde{x})R_0'(\tilde{x})\Big)\,;$$

therefore, using (ii-iii) we have that

$$\tilde{R}^{(1)}(\tilde{t},x_0) = R_0(x_0) + \epsilon\,\delta R(\tilde{t},x_0) \to \infty \quad \text{as } \tilde{t} \to \pm\infty\,.$$

□

Appendix C

On the spectral decomposition of selfadjoint operators in L^2

Remark 84 Throughout this appendix, we consider the Hilbert space [(62)]

$$\mathfrak{H} := L^2((a,b),dx)\,, \qquad -\infty \le a < b \le +\infty\,,$$

made of complex valued, square integrable functions on $\mathbb{R}$, for the measure dx with its inner product $\langle\ |\ \rangle$ and the associated norm $\|\ \|$ defined by

$$\langle f|g\rangle := \int_a^b \overline{f(x)}g(x)dx\,, \qquad \|f\|^2 = \langle f|f\rangle\,, \qquad \text{for all } f,g\in\mathfrak{H}\,.$$

C.1 (Generalized) orthonormal bases of L^2

Let us start with the following

Definition 9 (Orthonormal basis)
A system of functions $\{e_n\}_{n\in\mathbb{N}}$ such that $e_n \in \mathfrak{H}$ for all $n\in\mathbb{N}$ is called orthonormal basis of $\mathfrak{H}$ if the following conditions (a-b) hold:

(a) For all $f\in\mathfrak{H}$ the following element is trivially well defined

$$\hat{f}_n := \langle e_n\,|\,f\rangle \in \mathbb{C} \qquad (n\in\mathbb{N})\,;$$

then the sequence $\{\hat{f}_n\}_{n\in\mathbb{N}}$ is in the space of the complex sequences $\mathfrak{l}^2$, that is

$$\sum_{n=1}^{+\infty}|\hat{f}_n|^2 = \sum_{n=1}^{+\infty}|\,\langle e_n\,|\,f\rangle\,|^2 < +\infty\,.$$

[62] Throughout the thesis, the expression "Hilbert space" is an abbreviation for "complex, separable Hilbert space".

(b) The linear map

$$\mathfrak{H} \to \mathfrak{l}^2\,, \qquad f \mapsto \{\hat{f}_n\}_{n\in\mathbb{N}} = \big\{\, \langle e_n \,|\, f\rangle \,\big\}_{n\in\mathbb{N}}$$

is unitary, that is, it is one-to-one and preserves the inner products:

$$\langle f \,|\, g\rangle = \sum_{n=1}^{+\infty} \overline{\hat{f}_n}\hat{g}_n = \sum_{n=1}^{+\infty} \overline{\langle e_n \,|\, f\rangle}\,\langle e_n \,|\, g\rangle \qquad \text{for all } f, g \in \mathfrak{H}\,.$$

As a consequence of the previous definition we have the following

Proposition 15 (Decomposition for orthonormal bases)
Let us consider an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $\mathfrak{H}$; then each $f \in \mathfrak{H}$ can be expanded in terms of this basis as follows

$$f = \sum_{n=1}^{+\infty} \hat{f}_n e_n = \sum_{n=1}^{+\infty} \langle e_n \,|\, f\rangle\, e_n\,.$$

Remark 85 The orthonormality of the previously defined basis $\{e_n\}_{n\in\mathbb{N}}$ can be easily recovered from the previous decomposition for the elements of the basis, hence for $f = e_n$, $n \in \mathbb{N}$; indeed, this gives rise to

$$\langle e_n, e_m\rangle = \delta_{nm} := \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \tag{C.1}$$

The concept of orthonormal bases can be generalized as follows

Definition 10 (Generalized orthonormal basis)
A system of functions $\big\{\{e_n\}_{n=1,\dots,N}, \{e_{i\lambda} \,|\, \lambda \in \Lambda\}_{i=1,\dots,I}\big\}$ with $N, I \in \mathbb{N}$ and $\Lambda \subseteq \mathbb{R}$ such that

$$e_n \in \mathfrak{H} \qquad (n = 1, ..., N)\,, \qquad e_{i\lambda} \in C^\infty((a,b),\mathbb{C}) \qquad (i = 1, ..., I; \lambda \in \Lambda)$$

is called generalized orthonormal basis of $\mathfrak{H}$ if the following conditions (a-c) hold [56]:

(a) For all $f \in \mathfrak{C} := C_c((a,b),\mathbb{C}) \subset \mathfrak{H}$ the following element is well defined [63]

$$\hat{f}_i(\lambda) := \langle e_{i\lambda} \,|\, f\rangle \in \mathbb{C} \qquad (i = 1, ..., I; \lambda \in \Lambda)\,;$$

then the function $\lambda \to \hat{f}_i(\lambda)$ is in the space $L^2(\Lambda, d\lambda)$, that is

$$\int_\Lambda |\hat{f}_i(\lambda)|^2 d\lambda = \int_\Lambda |\,\langle e_{i\lambda} \,|\, f\rangle\,|^2 d\lambda < +\infty\,.$$

[63] This is due to the fact that $\overline{e_{i\lambda}} f \in \mathfrak{C} \subseteq \mathfrak{H}$.

(b) For all $i = 1, ..., I$ linear maps

$$\mathfrak{C} \subset \mathfrak{H} \to L^2(\Lambda, d\lambda)\,, \qquad f \mapsto \hat{f}_i(\lambda) = \langle e_{i\lambda} \,|\, f \rangle$$

are continuous with respect to the norms of the Hilbert spaces $\mathfrak{H}$ and $L^2(\Lambda, d\lambda)$; thus, by density of $\mathfrak{C}$ in $\mathfrak{H}$, all these maps have a unique continuous (and linear) extension to $\mathfrak{H}$, that we write has

$$\mathfrak{H} \to L^2(\Lambda, d\lambda)\,, \qquad f \mapsto \hat{f}_i(\lambda) =: \langle e_{i\lambda} \,|\, f \rangle\ .$$

The previous extension gives for each $f \in \mathfrak{H}$ the map $\lambda \to \hat{f}_i(\lambda) = \langle e_{i\lambda} \,|\, f \rangle$ which is said to define the "generalized inner product" between the $e_{i\lambda}$'s and f (for a fixed λ the product $\langle e_{i\lambda} \,|\, f \rangle$ does not make sense, since the generalized inner product is a function in $L^2(\Lambda, d\lambda)$).

(c) Consider the direct sum Hilbert space $\hat{\mathfrak{H}} := \left(\overset{N}{\underset{n=1}{\oplus}} \mathbb{C} \right) \oplus \left(\overset{I}{\underset{n=1}{\oplus}} L^2(\Lambda, d\lambda) \right)$; then the linear map

$$\begin{aligned} \mathfrak{H} \to \hat{\mathfrak{H}}\,, \quad f \mapsto & \left(\hat{f}_1, ..., \hat{f}_N, \hat{f}_1(\lambda), ..., \hat{f}_I(\lambda) \right) \\ & = \Big(\langle e_1 \,|\, f \rangle\,, ..., \langle e_N \,|\, f \rangle\,, \langle e_{1\lambda} \,|\, f \rangle\,, ..., \langle e_{I\lambda} \,|\, f \rangle \Big) \qquad \text{(C.2)} \end{aligned}$$

is unitary, that is, it is one-to-one and preserves the inner products:

$$\begin{aligned} \langle f \,|\, g \rangle &= \sum_{n=1}^{N} \overline{\hat{f}_n} \hat{g}_n + \sum_{i=1}^{I} \int_\Lambda \overline{\hat{f}_{i\lambda}} \hat{g}_{i\lambda} d\lambda \\ &= \sum_{n=1}^{N} \overline{\langle e_n \,|\, f \rangle} \langle e_n \,|\, g \rangle + \sum_{i=1}^{I} \int_\Lambda \overline{\langle e_{i\lambda} \,|\, f \rangle} \langle e_{i\lambda} \,|\, g \rangle \quad \textit{for all } f, g \in \mathfrak{H}\,. \end{aligned}$$

Remark 86 The forthcoming items (i-ii) describe some consequences of the previous definition:

(i) Fix $i \in \{1, ..., I\}$ and consider $F \in L^2(\Lambda, d\lambda)$. Since the map $g \to \hat{g}_i(\lambda) = \langle e_{i\lambda} \,|\, g \rangle$ is linear and continuous from $\mathfrak{H}$ and $L^2(\Lambda, d\lambda)$, then the application

$$\mathfrak{H} \to \mathbb{C}\,, \qquad g \to \int_\Lambda \overline{F(\lambda)} \langle e_{i\lambda} \,|\, g \rangle \, d\lambda$$

is a continuous linear functional of the space $\mathfrak{H}$; therefore, the Riesz representation theorem ensures that there exists a function $f_F \in \mathfrak{H}$ such that

$$\langle f_F \,|\, g \rangle = \int_\Lambda \overline{F(\lambda)} \langle e_{i\lambda} \,|\, g \rangle \, d\lambda\,;$$

the element f_F is often indicated with $\int_\Lambda F(\lambda)e_{i\lambda}d\lambda$, so that the previous identity reads

$$\left\langle \int_\Lambda F(\lambda)e_{i\lambda}d\lambda \,\middle|\, g \right\rangle = \int_\Lambda \overline{F(\lambda)} \left\langle e_{i\lambda} \,|\, g \right\rangle d\lambda \,.$$

The element $\int_\Lambda F(\lambda)e_{i\lambda}d\lambda \in \mathfrak{H}$ is called weak integral of the function $\lambda \mapsto F(\lambda)e_{i\lambda}$.

(ii) The inverse of the unitary map (C.2) can be expressed in terms of weak integrals as

$$\hat{\mathfrak{H}} \to \mathfrak{H} \,, \quad \big(z_1, ..., z_N, F_1(\lambda), ..., F_I(\lambda)\big) \mapsto \sum_{n=1}^{N} z_n e_n + \sum_{i=1}^{I} \int_\Lambda F_i(\lambda)e_{i\lambda}d\lambda \,. \tag{C.3}$$

The following proposition is a direct consequence of the previous remark.

Proposition 16 (Decomposition for generalized orthonormal bases) *Let us consider a generalized orthonormal basis* $\big\{\{e_n\}_{n=1,...,N}, \{e_{i\lambda} \,|\, \lambda \in \Lambda\}_{i=1,...,I}\big\}$ *with* $N, I \in \mathbb{N}$ *and* $\Lambda \subseteq \mathbb{R}$*; then each* $f \in \mathfrak{H}$ *can be expanded in terms of this basis as follows*

$$f = \sum_{n=1}^{N} \hat{f}_n e_n + \sum_{i=1}^{I} \int_\Lambda \hat{f}_{i\lambda} e_{i\lambda} d\lambda = \sum_{n=1}^{N} \left\langle e_n \,|\, f \right\rangle e_n + \sum_{i=1}^{I} \int_\Lambda \left\langle e_{i\lambda} \,|\, f \right\rangle e_{i\lambda} d\lambda \,.$$

Remark 87 The "orthonormality" (in a generalized sense) of the previously defined basis $\big\{\{e_n\}_{n=1,...,N}, \{e_{i\lambda} \,|\, \lambda \in \Lambda\}_{i=1,...,I}\big\}$ can be easily recovered from the previous decomposition for the elements of the basis, hence for $f = e_n$, $n = 1, ..., N$ and for $f = e_{i\lambda}$, $i = 1, ..., I$ and $\lambda \in \Lambda$; indeed, this gives rise to (64)

$$\langle e_n, e_m \rangle = \delta_{nm} := \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \,,$$
$$\langle e_{i\lambda_1}, e_{j\lambda_2} \rangle = \delta_{ij}\, \delta(\lambda_1 - \lambda_2) = \begin{cases} \delta(\lambda_1 - \lambda_2) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \,, \qquad \langle e_{i\lambda}, e_n \rangle = 0 \,. \tag{C.4}$$

[64] With the expression $\delta(\lambda)$ we intend the distribution *Dirac delta function*; since we do not make use of this distribution, we do not recall its definition.

C.2 Spectral theorem for selfadjoint operators in L^2

In this section we consider a selfadjoint operator H in $\mathfrak{H}$ with domain $\mathfrak{D}$, that is, a linear operator

$$H\ :\ \mathfrak{D} \subseteq \mathfrak{H} \to \mathbb{C}$$

such that the domain $\mathfrak{D}$ is dense in $\mathfrak{H}$ and $H = H^\dagger$, where $H^\dagger$ is the adjoint operator of H.

C.2.1 Operators with a purely discrete spectrum

We firstly focus on the case in which the operator H has a purely discrete spectrum

$$\sigma(H) = \sigma_p(H) = \{\mu_n\}_{n\in\mathbb{N}}$$

such that each eigenvalue $\mu_n \in \sigma_p(H)$ is simple, i.e., the corresponding eigenspace is one-dimensional. In this case the spectral theorem for self-adjoint operators in L^2 becomes:

Theorem 14 (Spectral theorem for selfadjoint operators with a purely discrete spectrum)
For each eigenvalue $\mu_n \in \sigma_p(H)$ there exists an eigenfunction e_n, i.e.

$$e_n \in \mathfrak{D} \quad : \quad He_n = \mu_n e_n \qquad n \in \mathbb{N}\,,$$

such that $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathfrak{H}$. Moreover, for every $f \in \mathfrak{D}$, the following representation for the operator H holds

$$\begin{aligned}
\widehat{Hf}_n =& \mu_n \hat{f}_n \qquad (n\in\mathbb{N})\\
&\Updownarrow\\
\langle e_n|Hf\rangle =& \mu_n\langle e_n|f\rangle \qquad (n\in\mathbb{N})\\
&\Updownarrow\\
Hf = \sum_{n=1}^{+\infty}\mu_n\hat{f}_n e_n =& \sum_{n=1}^{+\infty}\mu_n\langle e_n|f\rangle e_n\,;
\end{aligned} \tag{C.5}$$

in addition, the domain of H can be characterized as

$$\mathfrak{D} = \left\{ f\in\mathfrak{H} \mid \left\{\mu_n\hat{f}_n\right\}_{n\in\mathbb{N}} = \left\{\mu_n\left\langle e_n\,|\,f\right\rangle\right\}_{n\in\mathbb{N}} \in \mathfrak{l}^2 \right\}. \tag{C.6}$$

Remark 88 It is well known that the domain $\mathfrak{D}$ of the operator H has the structure of Hilbert space if provided with the (complete) inner product $\langle\ |\ \rangle_{\mathfrak{D}} : \mathfrak{D} \times \mathfrak{D} \to \mathbb{C}$ defined as [65]

$$\langle f|g\rangle_{\mathfrak{D}} := \langle f|g\rangle + \langle Hf|Hg\rangle \qquad \text{for all } f, g \in \mathfrak{D}\,.$$

The decomposition for orthonormal bases and the spectral theorems [Proposition 15 and Theorem 14] give an expression of the inner product in $\mathfrak{D}$ in terms of the eigenfunctions e_n and the eigenvalues μ_n, that is

$$\langle f|g\rangle_{\mathfrak{D}} = \sum_{n=1}^{+\infty}(1+\mu_n^2)\overline{\hat{f}_n}\hat{g}_n = \sum_{n=1}^{+\infty}(1+\mu_n^2)\overline{\langle e_n\,|\,f\rangle}\,\langle e_n\,|\,g\rangle \qquad \text{for all } f, g \in \mathfrak{D}\,. \tag{C.7}$$

Remark 89 As well known, a functional calculus exists for selfadjoint Hilbert space operators (see, e.g., Ref. [57]): for each (Borel-) measurable function defined on the spectrum of the operator H

$$\mathcal{F} : \sigma(H) \to \mathbb{C} \tag{C.8}$$

one can define the operator

$$\mathcal{F}(H) : \mathfrak{D}^{\mathcal{F}} \subset \mathfrak{H} \to \mathfrak{H} \tag{C.9}$$

where $\mathfrak{D}^{\mathcal{F}}$ is a suitable domain, determined by H and $\mathcal{F}$; the operator $\mathcal{F}(H)$ is selfadjoint if $\mathcal{F}$ is real valued. Making reference to the previously mentioned orthonormal basis of eigenfunctions of H, one can prove that for every $f \in \mathfrak{D}^{\mathcal{F}}$, the following representation for the operator $\mathcal{F}(H)$ holds

$$\begin{aligned}
\widehat{\mathcal{F}(H)f}_n &= \mathcal{F}(\mu_n)\hat{f}_n \qquad (n \in \mathbb{N})\\
&\Updownarrow\\
\langle e_n|\mathcal{F}(H)f\rangle &= \mathcal{F}(\mu_n)\langle e_n|f\rangle \qquad (n \in \mathbb{N})\\
&\Updownarrow\\
\mathcal{F}(H)f = \sum_{n=1}^{+\infty}\mathcal{F}(\mu_n)\hat{f}_n e_n &= \sum_{n=1}^{+\infty}\mathcal{F}(\mu_n)\langle e_n|f\rangle e_n\,.
\end{aligned} \tag{C.10}$$

[65] One could as well consider the alternative inner product $\langle\ |\ \rangle'_{\mathfrak{D}} : \mathfrak{D} \times \mathfrak{D} \to \mathbb{C}$ defined setting

$$\langle f|g\rangle'_{\mathfrak{D}} := \langle Hf|Hg\rangle\,;$$

this has a structure similar to those of the inner product $\langle\ |\ \rangle_{\mathfrak{D}}$. However, in the present situation the two inner products are equivalent.

moreover, the following facts and characterization of the domain of $\mathcal{F}(H)$ hold

$$e_n \in \mathfrak{D}^{\mathcal{F}} \quad \text{and} \quad \mathcal{F}(H)e_n = \mathcal{F}(\mu_n)e_n \qquad (n \in \mathbb{N})\,; \tag{C.11}$$

$$\mathfrak{D}^{\mathcal{F}} = \Big\{ f \in \mathfrak{H} \mid \big\{\mathcal{F}(\mu_n)\hat{f}_n\big\}_{n\in\mathbb{N}} = \big\{\mathcal{F}(\mu_n)\,\langle e_n \,|\, f\rangle\,\big\}_{n\in\mathbb{N}} \in \mathfrak{l}^2 \Big\}\,. \tag{C.12}$$

In addition, one can see that the domain $\mathfrak{D}^{\mathcal{F}}$ of the operator $\mathcal{F}(H)$ has the structure of Hilbert space if provided with the (complete) inner product $\langle\ |\ \rangle_{\mathfrak{D}^{\mathcal{F}}} : \mathfrak{D}^{\mathcal{F}} \times \mathfrak{D}^{\mathcal{F}} \to \mathbb{C}$ defined as

$$\langle f|g\rangle_{\mathfrak{D}^{\mathcal{F}}} := \langle f|g\rangle + \langle \mathcal{F}(H)f|\mathcal{F}(H)g\rangle \qquad \text{for all } f,g \in \mathfrak{D}^{\mathcal{F}}\,. \tag{C.13}$$

For our purposes it is important to consider the choice $\mathcal{F}(\gamma) := |\gamma|^{1/2}$ for all $\gamma \in \sigma(H)$, producing a selfadjoint operator that we indicate with

$$|H|^{1/2} : \mathfrak{D}^{1/2} \subset \mathfrak{H} \to \mathfrak{H} \tag{C.14}$$

and that behaves on every $f \in \mathfrak{D}^{1/2}$ as follows in relation to our orthonormal basis:

$$\widehat{|H|^{1/2}f}_n = |\mu_n|^{1/2}\hat{f}_n \qquad (n \in \mathbb{N})$$

$$\Updownarrow$$

$$\langle e_n \big| |H|^{1/2}f\rangle = |\mu_n|^{1/2}\langle e_n|f\rangle \qquad (n \in \mathbb{N})$$

$$\Updownarrow$$

$$|H|^{1/2}f = \sum_{n=1}^{+\infty} |\mu_n|^{1/2}\hat{f}_n e_n = \sum_{n=1}^{+\infty} |\mu_n|^{1/2}\langle e_n|f\rangle e_n\,; \tag{C.15}$$

in addition, Eqs. (C.11,C.12) become

$$e_n \in \mathfrak{D}^{1/2} \quad \text{and} \quad |H|^{1/2}e_n = |\mu_n|^{1/2}e_n \qquad (n \in \mathbb{N})\,; \tag{C.16}$$

$$\mathfrak{D}^{1/2} = \Big\{ f \in \mathfrak{H} \mid \big\{|\mu_n|^{1/2}\hat{f}_n\big\}_{n\in\mathbb{N}} = \big\{|\mu_n|^{1/2}\,\langle e_n \,|\, f\rangle\,\big\}_{n\in\mathbb{N}} \in \mathfrak{l}^2 \Big\}\,. \tag{C.17}$$

Finally, let us write down explicitly in the expression of the inner product (C.13) with which the domain $\mathfrak{D}^{1/2}$ becomes an Hilbert space:

$$\begin{aligned} \langle f|g\rangle_{\mathfrak{D}^{1/2}} = \langle f|g\rangle + \langle |H|^{1/2}f \big| |H|^{1/2}g\rangle &= \sum_{n=1}^{+\infty}(1+|\mu_n|)\overline{\hat{f}_n}\hat{g}_n \\ &= \sum_{n=1}^{+\infty}(1+|\mu_n|)\overline{\langle e_n \,|\, f\rangle}\,\langle e_n \,|\, g\rangle \qquad \text{for all } f,g \in \mathfrak{D}\,. \end{aligned} \tag{C.18}$$

C.2.2 Operators with discrete and continuous spectrum

We now focus on the case in which the operator H has the spectrum $\sigma(H)$ made up of a finite discrete spectrum

$$\sigma_p(H) = \{\mu_n\}_{n=1,...,N} \qquad N \in \mathbb{N}$$

and a continuous spectrum

$$\sigma_c(H) \quad : \quad \mathring{\sigma}_c(H) := \Lambda \subseteq \mathbb{R}$$

$(\sigma(H) = \sigma_p(H) \cup \sigma_c(H))$, such that each eigenvalue $\mu_n \in \sigma_p(H)$ is simple and for each improper eigenvalue $\lambda \in \sigma_c(H)$ there are $I \in \mathbb{N}$ linearly independent improper eigenfunctions. In this case the spectral theorem of selfadjoint operators in L^2 becomes:

Theorem 15 (Spectral theorem for selfadjoint operators with a discrete and continuous spectrum)
For each eigenvalue $\mu_n \in \sigma_p(H)$ there exists an eigenfunction e_n, i.e.

$$e_n \in \mathfrak{D} \quad : \quad He_n = \mu_n e_n \qquad (n = 1, ..., N)\,,$$

and for each improper eigenvalue $\lambda \in \sigma_c(H)$ there exist $I \in \mathbb{N}$ improper eigenfunctions $\{e_{i\lambda}\}_{i=1,...,I}$, i.e. [66]

$$e_{i\lambda} \in C^\infty((a,b)) \setminus \mathfrak{D} \quad : \quad He_{i\lambda} = \lambda e_{i\lambda} \qquad (i = 1, ..., I\,; \lambda \in \Lambda)\,,$$

such that $\big\{\{e_n\}_{n=1,...,N}, \{e_{i\lambda}\,, \lambda \in \Lambda\}_{i=1,...,I}\big\}$ is a generalized orthonormal basis of $\mathfrak{H}$. Moreover, for every $f \in \mathfrak{D}$, the following representation for the operator H holds

$$\begin{aligned}
\widehat{Hf}_n =& \mu_n \hat{f}_n \qquad (n = 1, ..., N)\,,\\
\widehat{Hf}_i(\lambda) =& \lambda \hat{f}_i(\lambda) \qquad (i = 1, ..., I\,; \lambda \in \Lambda)\,,\\
&\Updownarrow\\
\langle e_n|Hf\rangle =& \mu_n \langle e_n|f\rangle \qquad (n = 1, ..., N)\,,\\
\langle e_{i\lambda}|Hf\rangle =& \lambda \langle e_{i\lambda}|f\rangle \qquad (i = 1, ..., I\,; \lambda \in \Lambda)\,,\\
&\Updownarrow
\end{aligned}$$

$$\begin{aligned}
Hf =& \sum_{n=1}^{N} \mu_n \hat{f}_n e_n + \sum_{i=1}^{I} \int_\Lambda \lambda \hat{f}_i(\lambda) e_{i\lambda} d\lambda\\
=& \sum_{n=1}^{N} \mu_n \langle e_n|f\rangle e_n + \sum_{i=1}^{I} \int_\Lambda \lambda \,\langle e_{i\lambda}\,|\,f\rangle\, e_{i\lambda} d\lambda\,; \qquad \text{(C.19)}
\end{aligned}$$

[66] Although $e_{i\lambda}$ does not belong to $\mathfrak{D}$ the expression $He_{i\lambda} = -\frac{\partial^2 e_{i\lambda}}{\partial x^2} + \mathcal{V} e_{i\lambda}$ makes sense as $e_{i\lambda} \in C^\infty((a,b))$.

in addition, the domain of H *can be characterized as*

$$\mathfrak{D} = \left\{ f \in \mathfrak{H} \mid \lambda \hat{f}_i(\lambda) = \lambda \langle e_{i\lambda} \mid f \rangle \in L^2(\Lambda, d\lambda) \quad \textit{for } i = 1, ..., I \right\}. \quad \text{(C.20)}$$

Remark 90 In Remark 88 we have recalled that it is possible to provide the domain $\mathfrak{D}$ of the operator H with the structure of Hilbert space by introducing the (complete) inner product $\langle \mid \rangle_{\mathfrak{D}} : \mathfrak{D} \times \mathfrak{D} \to \mathbb{C}$ defined in Eq. (C.7). The decomposition for generalized orthonormal bases and the spectral theorems [Proposition 16 and Theorem 15] give an expression of the inner product in $\mathfrak{D}$ in terms of the eigenfunctions e_n, $e_{i\lambda}$ and the eigenvalues μ_n, λ, that is

$$\begin{aligned} \langle f|g\rangle_{\mathfrak{D}} &= \sum_{n=1}^{N} (1+\mu_n^2)\overline{\hat{f}_n}\hat{g}_n + \sum_{i=1}^{I} \int_{\Lambda} (1+\lambda^2)\overline{\hat{f}_i(\lambda)}\hat{g}_i(\lambda) d\lambda \\ &= \sum_{n=1}^{N} (1+\mu_n^2)\overline{\langle e_n \mid f\rangle} \langle e_n \mid g\rangle + \sum_{i=1}^{I} \int_{\Lambda} (1+\lambda^2)\overline{\langle e_{i\lambda} \mid f\rangle} \langle e_{i\lambda} \mid g\rangle \, d\lambda \\ &\qquad \text{for all } f, g \in \mathfrak{D}\,. \end{aligned}$$

Remark 91 In Remark 89 we have introduced for for each (Borel-) measurable function (C.8) the operator $\mathcal{F}(H)$ (C.9). Making reference to the previously mentioned generalized orthonormal basis of proper and improper eigenfunctions of H, one can prove that for every $f \in \mathfrak{D}^{\mathcal{F}}$, the following representation for the operator $\mathcal{F}(H)$ holds

$$\begin{aligned} \widehat{\mathcal{F}(H)f}_n &= \mathcal{F}(\mu_n)\hat{f}_n \qquad (n = 1, ..., N)\,, \\ \widehat{\mathcal{F}(H)f}_i(\lambda) &= \mathcal{F}(\lambda)\hat{f}_i(\lambda) \qquad (i = 1, ..., I\,; \lambda \in \Lambda)\,, \\ &\Updownarrow \\ \langle e_n|\mathcal{F}(H)f\rangle &= \mathcal{F}(\mu_n)\langle e_n|f\rangle \qquad (n = 1, ..., N)\,, \\ \langle e_{i\lambda}|\mathcal{F}(H)f\rangle &= \mathcal{F}(\lambda)\langle e_{i\lambda}|f\rangle \qquad (i = 1, ..., I\,; \lambda \in \Lambda)\,, \\ &\Updownarrow \\ \mathcal{F}(H)f &= \sum_{n=1}^{N} \mathcal{F}(\mu_n)\hat{f}_n e_n + \sum_{i=1}^{I} \int_{\Lambda} \mathcal{F}(\lambda)\hat{f}_i(\lambda) e_{i\lambda} d\lambda \\ &= \sum_{n=1}^{N} \mathcal{F}(\mu_n)\langle e_n|f\rangle e_n + \sum_{i=1}^{I} \int_{\Lambda} \mathcal{F}(\lambda)\, \langle e_{i\lambda} \mid f\rangle\, e_{i\lambda} d\lambda\,; \end{aligned} \quad \text{(C.21)}$$

moreover, the following facts and characterization of the domain of $\mathcal{F}(H)$

hold

$$e_n \in \mathfrak{D}^{\mathcal{F}} \quad \text{and} \quad \mathcal{F}(H)e_n = \mathcal{F}(\mu_n)e_n \qquad (n = 1, ..., N)\,; \tag{C.22}$$

$$\mathfrak{D}^{\mathcal{F}} = \Big\{ f \in \mathfrak{H} \mid \mathcal{F}(\lambda)\hat{f}_i(\lambda) = \mathcal{F}(\lambda)\,\langle e_{i\lambda} \,|\, f\rangle \in L^2(\Lambda, d\lambda) \Big\}\,. \tag{C.23}$$

As already mentioned in Remark 89, we are interested in the case $\mathcal{F}(\gamma) := |\gamma|^{1/2}$ for all $\gamma \in \sigma(H)$, which produces the selfadjoint operator (C.14); for every $f \in \mathfrak{D}^{\mathcal{F}}$, this has the following representation:

$$\begin{aligned}
\widehat{|H|^{1/2} f}_n = & |\mu_n|^{1/2} \hat{f}_n \qquad (n = 1, ..., N)\,,\\
\widehat{|H|^{1/2} f}_i(\lambda) = & |\lambda|^{1/2} \hat{f}_i(\lambda) \qquad (i = 1, ..., I; \lambda \in \Lambda)\,,\\
& \Updownarrow\\
\langle e_n \big| |H|^{1/2} f\rangle = & |\mu_n|^{1/2} \langle e_n | f\rangle \qquad (n = 1, ..., N)\,,\\
\langle e_{i\lambda} \big| |H|^{1/2} f\rangle = & |\lambda|^{1/2} \langle e_{i\lambda} | f\rangle \qquad (i = 1, ..., I; \lambda \in \Lambda)\,,\\
& \Updownarrow
\end{aligned}$$

$$\begin{aligned}
|H|^{1/2} f &= \sum_{n=1}^{N} |\mu_n|^{1/2} \hat{f}_n e_n + \sum_{i=1}^{I} \int_\Lambda |\lambda|^{1/2} \hat{f}_i(\lambda) e_{i\lambda} d\lambda\\
&= \sum_{n=1}^{N} |\mu_n|^{1/2} \langle e_n | f\rangle e_n + \sum_{i=1}^{I} \int_\Lambda |\lambda|^{1/2}\, \langle e_{i\lambda} \,|\, f\rangle\, e_{i\lambda} d\lambda\,;
\end{aligned} \tag{C.24}$$

moreover, in this case Eqs. (C.22,C.23) become

$$e_n \in \mathfrak{D}^{1/2} \quad \text{and} \quad |H|^{1/2} e_n = |\mu_n|^{1/2} e_n \qquad (n = 1, ..., N)\,; \tag{C.25}$$

$$\mathfrak{D}^{1/2} = \Big\{ f \in \mathfrak{H} \mid |\lambda|^{1/2}\hat{f}_i(\lambda) = |\lambda|^{1/2}\,\langle e_{i\lambda} \,|\, f\rangle \in L^2(\Lambda, d\lambda) \Big\}\,. \tag{C.26}$$

Finally, let us write down explicitly in the expression of the inner product (C.13) with which the domain $\mathfrak{D}^{1/2}$ becomes an Hilbert space:

$$\begin{aligned}
\langle f | g\rangle_{\mathfrak{D}^{1/2}} &= \langle f | g\rangle + \langle |H|^{1/2} f \big| |H|^{1/2)} g\rangle\\
&= \sum_{n=1}^{N} (1 + |\mu_n|) \overline{\hat{f}_n} \hat{g}_n + \sum_{i=1}^{I} \int_\Lambda (1 + \lambda^2) \overline{\hat{f}_i(\lambda)} \hat{g}_i(\lambda) d\lambda\\
&= \sum_{n=1}^{N} (1 + |\mu_n|) \overline{\langle e_n \,|\, f\rangle}\, \langle e_n \,|\, g\rangle + \sum_{i=1}^{I} \int_\Lambda (1 + \lambda^2) \overline{\langle e_{i\lambda} \,|\, f\rangle}\, \langle e_{i\lambda} \,|\, g\rangle\, d\lambda\\
&\qquad \text{for all } f, g \in \mathfrak{D}\,.
\end{aligned}$$

Appendix D

Spectral features of the Schrödinger operators appearing in the master equations

D.1 Spectral features of the Schrödinger operator (5.29,5.22) (Torii-Shinkai wormhole)

In this section we deal with the spectral features of the operator

$$\begin{gathered} H := -\frac{d^2}{d\mathrm{x}^2} + \mathcal{V} \;:\; \mathfrak{D} \subset \mathfrak{H} \to \mathfrak{H}\,, \qquad \mathfrak{D} := \{f \in \mathfrak{H} \mid f_{\mathrm{xx}} \in \mathfrak{H}\}\,, \\ \mathcal{V}(\mathrm{x}) := \frac{1}{4\rho^2(\mathrm{x})}\left[(d-3)(d-5) - \frac{3(d-1)^2}{\rho^{2(d-2)}(\mathrm{x})}\right] \quad (\rho(\mathrm{x}) \text{ as in Eq. (3.31)}) \end{gathered} \tag{D.1}$$

in the Hilbert space

$$\mathfrak{H} := L^2(\mathbb{R}, d\mathrm{x})\,. \tag{D.2}$$

Note that we will intend all x-derivatives in the distributional sense.[(67)] Let us recall that in Eq. (5.25) we have proved that the potential $\mathcal{V}$ in Eq. (D.1) has the asymptotics

$$\begin{gathered} \mathcal{V}(\mathrm{x}) = (d-3)(d-5)\left(\frac{1}{4|\mathrm{x}|^2} + \frac{C_d}{2|\mathrm{x}|^3}\right) + O\left(\frac{1}{|\mathrm{x}|^4}\right) \quad \text{for } \mathrm{x} \to \pm\infty \\ (C_d \text{ as in Eq. (3.22)}). \end{gathered} \tag{D.3}$$

[67]The conditions $f \in \mathfrak{H}$ and $f_{\mathrm{xx}} \in \mathfrak{H}$ imply $f_{\mathrm{x}} \in \mathfrak{H}$, due to the Gagliardo-Nirenberg interpolation inequality (see e.g. Ref. [58]); $\mathfrak{D}$ is just the usual Sobolev space $W^{2,2}(\mathbb{R}) \equiv H^2(\mathbb{R})$. Let us also remark that, for $f \in \mathfrak{H}$, one has automatically $\mathcal{V}f \in \mathfrak{H}$ due to the boundedness of $\mathcal{V}$.

D.1.1 Some general facts on Schrödinger operators with smooth, decaying potential

In this subsection we deal with the spectral features the Schrödinger type operators of the general form

$$H := -\frac{d^2}{d\mathrm{x}^2} + \mathcal{V}$$
$$\mathcal{V} \in C^\infty(\mathbb{R}, \mathbb{R}) \cap L^1(\mathbb{R}, \mathbb{R})\,, \quad : \quad \mathcal{V}, \mathcal{V}_{\mathrm{x}}, \mathcal{V}_{\mathrm{xx}}, \text{etc.} \in L^\infty(\mathbb{R}, \mathbb{R})\,. \tag{D.4}$$

Here, we are not assuming that $\mathcal{V}$ has any special form. As usual, the x-derivatives appearing in Eq. (D.4) will be intended in the sense of the Schwartz distributions theory [59]. In addition, we consider the Hilbert space

$$\mathfrak{H} := L^2(\mathbb{R}, d\mathrm{x}) \tag{D.5}$$

of complex valued, square integrable functions on $\mathbb{R}$, for the measure dx with its inner product $\langle\ |\ \rangle$ and the associated norm $\|\ \|$.
The forthcoming statements are extracted from Refs. [56, 57, 60], or are simple consequences of statements proved therein, or follow from other references cited hereafter.

(i) Consider the operator H restricted to the domain $\mathfrak{D}$ defied as

$$\mathfrak{D} := \{f \in \mathfrak{H} \mid Hf \in \mathfrak{H}\}\,;$$

then, H is a selfadjoint operator in $\mathfrak{H}$.

(ii) The spectrum $\sigma(H)$ is the union of the point and continuous spectra $\sigma_p(H)$, $\sigma_c(H)$, i.e.

$$\sigma(H) = \sigma_p(H) \cup \sigma_c(H)\,.$$

(iii) The point spectrum $\sigma_p(H)$, the set of the eigenvalues, is a finite (possibly empty), or countable subset of $\mathbb{R}$ such that

$$\sigma_p(H) \subseteq (-\infty, 0)\,.$$

Moreover, each eigenvalue is simple, i.e. has an associate space of square integrable eigenfunctions of dimension 1; the eigenfunctions of H ere proved to be C^∞.
In this section we will set

$$N := \text{number of elements of } \sigma_p(H)\,, \quad N \in \{0, 1, 2, ..., +\infty\}\,.$$

It is known that [68]

$$N \leq 1 + \int_{\mathbb{R}} (1 + |\mathrm{x}|)|\mathcal{V}_-(\mathrm{x})| d\mathrm{x} \qquad \mathcal{V}_-(\mathrm{x}) := \min\{\mathcal{V}(\mathrm{x}), 0\}\,; \tag{D.6}$$

$$\limsup_{\mathrm{x}\to\pm\infty} \mathrm{x}^2\, \mathcal{V}(\mathrm{x}) > -\frac{1}{4} \quad \Rightarrow \quad N < +\infty\,. \tag{D.7}$$

The eigenvalues of H, if any, can be arranged in increasing order, thus we write

$$\sigma_p(H) = \{\mu_n\}_{n\geq 1} \qquad \mu_1 < \mu_2 < \ldots < 0\,.$$

For each $n \geq 1$ the (nonzero) eigenfunctions corresponding to the n-th eigenvalue μ_n have exactly n zeroes.
Let us suppose that $\mathcal{V}$ is an even function; then the following hold:

(a) the eigenfunctions corresponding to μ_n have the same parity as $n + 1$;

(b) let $\chi_0^\pm \in C^\infty(\mathbb{R}, \mathbb{R})$ denote the solutions of the zero energy Schrödinger equations defined, respectively, by the Cauchy problems

$$\begin{cases} H\chi_0^+ = 0\,, \\ \chi_0^+(0) = 1\,, \\ {\chi_0^+}'(0) = 0\,, \end{cases} \qquad \begin{cases} H\chi_0^- = 0\,, \\ \chi_0^-(0) = 0\,, \\ {\chi_0^-}'(0) = 1\,; \end{cases} \tag{D.8}$$

then χ_0^+ and ${\chi_0}^-$ are, respectively, an even and an odd function;

(c) if $N^\pm$ is the number of zeroes of $\chi_0^\pm$, then

$$N^+ \in \{0, 2, 4, ..., +\infty\}\,, \qquad N^- \in \{1, 3, 5, ..., +\infty\}$$

and the number of eigenvalues of H is

$$N = \frac{1}{2} N^+ + \frac{1}{2}(N^- - 1)\,. \tag{D.9}$$

(iv) The continuous spectrum is

$$\sigma_c(H) = [0, +\infty)\,.$$

Every point $\lambda \in (0, +\infty)$ has an associated, two-dimensional space of "generalized" eigenfunctions: these are C^∞ functions f which fulfill the equation $Hf = \lambda f$ but do not belong to $\mathfrak{D}$.

[68] For a proof of Eq. (D.7), see Ref. [61].

(v) According to the Rayleigh-Ritz variational characterization (see e.g. Ref. [57], pages 265-266) one has that

$$\inf \sigma(H) = \inf_{f \in \mathfrak{D} \setminus \{0\}} \mu(f) \,, \qquad \mu(f) := \frac{\langle f \,|\, Hf \rangle}{\|f\|^2} \qquad \text{for every } f \in \mathfrak{D} \setminus \{0\} \tag{D.10}$$

($\mu(f)$ is the familiar mean value of H over f). We note that

$$\begin{aligned} \langle f|Hf \rangle &= \int_{\mathbb{R}} \left(-\overline{f} f'' + \mathcal{V}|f|^2\right) d\mathrm{x} \\ &= \int_{\mathbb{R}} \left(|f'|^2 + \mathcal{V}|f|^2\right) d\mathrm{x} \geq \int_{\mathbb{R}} \mathcal{V}|f|^2 d\mathrm{x} \geq \inf_{\mathrm{x} \in \mathbb{R}} \mathcal{V}(\mathrm{x}) \, \|f\|^2 \end{aligned} \tag{D.11}$$

(the second of these equalities follows from integration by parts). The inequalities (D.11) and Eq. (D.10) imply that

$$\mu(f) \geq \inf_{\mathrm{x} \in \mathbb{R}} \mathcal{V}(\mathrm{x}) \quad \text{for all } f \in \mathfrak{D} \setminus \{0\} \,, \qquad \inf \sigma(H) \geq \inf_{\mathrm{x} \in \mathbb{R}} \mathcal{V}(\mathrm{x}) \,.$$

Finally, on account of previous information on the point and continuous spectrum, we have that

$$\begin{aligned} &\inf_{f \in \mathfrak{D} \setminus \{0\}} \mu(f) \leq 0 \,, \\ &\inf_{f \in \mathfrak{D} \setminus \{0\}} \mu(f) = 0 \;\Leftrightarrow\; \sigma_p(H) = \emptyset \,, \\ &\inf_{f \in \mathfrak{D} \setminus \{0\}} \mu(f) < 0 \;\Leftrightarrow\; \sigma_p(H) \neq \emptyset \,, \\ &\inf_{f \in \mathfrak{D} \setminus \{0\}} \mu(f) < 0 \;\Rightarrow\; \inf_{f \in \mathfrak{D} \setminus \{0\}} \mu(f) = \mu_1 . \end{aligned} \tag{D.12}$$

D.1.2 The point spectrum of the Schrödinger operator (5.29,5.22) (Torii-Shinkai wormhole)

Let us consider the Hilbert space (D.2) and the operator (D.1), with the specific potential $\mathcal{V}$ appearing therein (which depends on $d \geq 3$). From the asymptotics given on Eq. (D.3), we have that the potential $\mathcal{V}$ has all features required in Eq. (D.4), so we can apply to it all the general results of the previous subsection. Hereafter we fix the attention on the point spectrum $\sigma_p(H) \subset (-\infty, 0)$, keeping the notations N, μ_1 of the previous subsection to indicate the total number of eigenvalues and, if $N \geq 1$, the minimum eigenvalue.

Remark 92 On the grounds of numerical tests, the authors of Ref. [35] presume for the operator H the existence in any dimension d of at least one eigenvalue, for which they report the numerical values when $d = 3, 4, ..., 9, 10, 19, 49, 99$. Hereafter we will prove analytically that H actually possesses eigenvalues, and we will derive an upper bound for the minimum eigenvalue μ_1. Our approach relies on the trial function

$$f(\mathtt{x}) := \frac{1}{\rho(\mathtt{x})^{d/2}}, \qquad -\infty < \mathtt{x} < +\infty, \tag{D.13}$$

which is C^∞ and belongs to $\mathfrak{D}$. In the forthcoming Subsection D.1.3 we will compute the mean value

$$\mu(f) := \frac{\langle f|Hf\rangle}{\|f\|^2} \tag{D.14}$$

and find that

$$\mu(f) = -\frac{(d-2)(16d-9)\Gamma\left(\frac{d+1}{2(d-2)}\right)\Gamma\left(\frac{2d-3}{2(d-2)}\right)}{4(2d-1)\Gamma\left(\frac{d-1}{2(d-2)}\right)\Gamma\left(\frac{2d-1}{2(d-2)}\right)} = -2d + O(1) \text{ for } d \to +\infty. \tag{D.15}$$

From the above expression it is evident that $\mu(f) < 0$ for any dimension $d \geq 3$, which implies that

$$\inf_{f\in\mathfrak{D}\setminus\{0\}} \mu(f) \leq \mu(f) < 0\,,$$

these fact and Eq. (D.12) give

$$\sigma_p(E) \neq \emptyset\,, \qquad \mu_1 \leq \mu(f)\,. \tag{D.16}$$

In the subsequent Table 1 the second column reports, for the dimensions $d = 3, 4, ..., 9, 10, 19, 49, 99$, the numerical eigenvalue determined by Ref. [35], that I presume should be identified with the minimum eigenvalue μ_1; the third column of the Table reports $\mu(f)$ which appears to be rather close to μ_1 for any d, and very close to it for large d.

Table 1

d	μ_1 (numer.)	$\mu(f)$
3	-1.39705	-1.30000
4	-2.98496	-2.86581
5	-4.68662	-4.57395
6	-6.46258	-6.36474
7	-8.28976	-8.20752
8	-10.1536	-10.0850
9	-12.0443	-11.9870
10	-13.9552	-13.9068
19	-31.5751	-31.5451
49	-91.3458	-91.2796
99	-191.283	-191.191

Remark 93 The considerations in Remark 92 prove that H has $N \geq 1$ eigenvalues. Recalling the $\mathtt{x} \to \pm\infty$ asymtoptics (D.3) for $\mathcal{V}(\mathtt{x})$, we have that

$$\lim_{\mathtt{x}\to\pm\infty} \mathtt{x}^2 \mathcal{V}(\mathtt{x}) = \frac{1}{4}(d-3)(d-5)\,;$$

applying the criterion (D.7) we infer that $N < +\infty$ for each $d \neq 4$; in the case $d = 4$, we have that $1/4(d-3)(d-5) = -1/4$ and (D.7) cannot be applied.

In any dimension d, the value of N can be determined using Eq. (D.9) which contains the number of zeroes $N^{\pm}$ of the solutions $\chi_0^{\pm}$ of the systems (D.8). We have computed numerically $\chi_0^{\pm}$ for many values of d between 3 and 499 (see Subsection D.1.4 for some detail): these computations give numerical evidence that, for any dimension $d \geq 3$, it is $N^+ = 2$, $N^- = 1$ so that

$$N = 1\ . \tag{D.17}$$

In other terms, there is numerical evidence that H has just one eigenvalue μ_1 for any $d \geq 3$.

Remark 94 Let us accept Eq. (D.17) in any dimension $d \geq 3$. Then, for any d, μ_1 is the unique point of the spectrum of H, which is contained in $(-\infty, 0)$. We will show in Subsection D.1.5 that from here and from the Kato-Temple inequality it follows that

$$\mu(f)\left(1 + \frac{\Delta(f)^2}{\mu(f)^2}\right) \leq \mu_1 \leq \mu(f) \tag{D.18}$$

where $\mu(f)$ is the mean of the previously mentioned trial function f [Eq. (D.13)] defined by Eq. (D.14), while $\varDelta(f)$ is the dispersion of f, which is defined as

$$\varDelta(f) := \frac{\|Hf - \mu(f)f\|}{\|f\|} . \tag{D.19}$$

In Subsection D.1.3 we will show that

$$\varDelta(f) = \left[\frac{3(d-2)(44d-3)\,\Gamma\left(\frac{2d-3}{2(d-2)}\right)\Gamma\left(\frac{d+3}{2(d-2)}\right)}{16(2d+1)\Gamma\left(\frac{2d+1}{2(d-2)}\right)\Gamma\left(\frac{d-1}{2(d-2)}\right)} - \frac{(d-2)^2(16d-9)^2\,\Gamma\left(\frac{2d-3}{2(d-2)}\right)^2\Gamma\left(\frac{d+1}{2(d-2)}\right)^2}{16(2d-1)^2\Gamma\left(\frac{2d-1}{2(d-2)}\right)^2\Gamma\left(\frac{d-1}{2(d-2)}\right)^2}\right]^{\frac{1}{2}} = \frac{d}{2\sqrt{2}} + O(1) \text{ for } d \to +\infty. \tag{D.20}$$

From the explicit expressions of the quantities $\mu(f)$ and $\varDelta(f)$ given by Eqs. (D.15,D.20), one finds that

$$\frac{\varDelta(f)^2}{\mu(f)^2} < 0.1 \qquad \text{for all } d \geq 3 .$$

Moreover, the asymoptotics for $\mu(f), \epsilon(f)$ in Eqs. (D.15,D.20) imply

$$\frac{\varDelta(f)^2}{\mu(f)^2} = \frac{1}{32} + O\left(\frac{1}{d}\right) = 0.03125 + O\left(\frac{1}{d}\right) \qquad \text{for all } d \to +\infty .$$

Summing up the results of the present subsection: the Schrödinger operator (D.1) defined for any dimension $d \geq 3$ in the Hilbert space (D.2) possesses the following features:

(i) H with the domain $\mathfrak{D}$ is a selfadjoint operator in $\mathfrak{H}$;

(ii) the point spectrum of H consists of a unique, simple eigenvalue $\mu_1 < 0$, thus we write

$$\sigma_p(H) = \{\mu_1\}$$

(this results has been deduced by accepting a numerical evidence);

(iii) the continuous spectrum of H is

$$\sigma_c(H) = [0, +\infty) .$$

D.1.3 On the trial function f [Eq. (D.13)]

Let H, $\mathcal{V}$ and f be as in Eqs. (D.1,D.13); like ρ [Eq. (3.31)], these functions are in $C^\infty(\mathbb{R},\mathbb{R})$ and even. The second derivative f'' can be computed using Eqs. (3.24,3.30) for ρ' and ρ''; recalling Eqs. (3.29,D.3) on the large x behaviour of ρ and $\mathcal{V}$ one concludes that

$$f(\mathrm{x}) = \frac{1}{\rho(\mathrm{x})^{d/2}} = O\left(\frac{1}{|\mathrm{x}|^{d/2}}\right) \qquad \text{for } \mathrm{x} \to \mp\infty\,, \tag{D.21}$$

$$-f''(\mathrm{x}) + \mathcal{V}(\mathrm{x})f(\mathrm{x}) = -\frac{5(2d-3)}{4\rho(\mathrm{x})^{\frac{d}{2}+2}} + \frac{4d-3}{4\rho(\mathrm{x})^{\frac{5d}{2}-2}} = O\left(\frac{1}{|\mathrm{x}|^{\frac{d}{2}+2}}\right) \text{ for } \mathrm{x} \to \mp\infty\,. \tag{D.22}$$

It is clear form Eqs. (3.29,D.21,D.22) that $f, -f'' + \mathcal{V}f \in \mathfrak{H}$; thus, f is in the domain of the Schrödinger operator H (D.1). In the sequel we will denote with μ any real number. We have the following:

$$\begin{aligned}
\|f\|^2 &= \int_{-\infty}^{+\infty} f(\mathrm{x})^2 d\mathrm{x} = 2\int_0^{+\infty} \frac{1}{\rho(\mathrm{x})^d} d\mathrm{x}\,,\\
\langle f|Hf\rangle &= \int_{-\infty}^{+\infty} f(\mathrm{x})\Big[-f''(\mathrm{x}) + \mathcal{V}(\mathrm{x})f(\mathrm{x})\Big]d\mathrm{x}\\
&= 2\int_0^{+\infty}\left[-\frac{5(2d-3)}{4\rho(\mathrm{x})^{d+2}} + \frac{4d-3}{4\rho(\mathrm{x})^{3d-2}}\right]d\mathrm{x}\,,\\
\|Hf - \mu f\|^2 &= \int_{-\infty}^{+\infty}\Big[-f''(\mathrm{x}) + \mathcal{V}(\mathrm{x})f(\mathrm{x}) - \mu f(\mathrm{x})\Big]^2 d\mathrm{x}\\
&= 2\int_0^{+\infty}\left[-\frac{5(2d-3)}{4\rho(\mathrm{x})^{\frac{d}{2}+2}} + \frac{4d-3}{4\rho(\mathrm{x})^{5\frac{d}{2}-2}} - \frac{\mu}{\rho(\mathrm{x})^d}\right]^2 d\mathrm{x}\,.
\end{aligned}$$

To compute the above integrals it is convenient to perform the transformation

$$(0,+\infty) \in \mathrm{x} \mapsto w = \rho(\mathrm{x})^{-2d+4} \in (0,1)\,,$$

so that

$$\rho(\mathrm{x}) = w^{-\frac{1}{2(d-2)}}\,,$$

while the measures $dw, d\mathrm{x}$ are related by

$$\begin{aligned}
dw &= \frac{d}{d\mathrm{x}}\left[\rho(\mathrm{x})^{-2d+4}\right]d\mathrm{x} = -2(d-2)\rho(\mathrm{x})^{-2d+3}\rho'(\mathrm{x})d\mathrm{x}\\
&= -2(d-2)\rho(\mathrm{x})^{-2d+3}\sqrt{1-\rho(\mathrm{x})^{-2(d-2)}}d\mathrm{x} = -2(d-2)w^{\frac{2d-3}{2(d-2)}}\sqrt{1-w}\,d\mathrm{x}\,.
\end{aligned}$$

Hence, we have

$$\|f\|^2 = \frac{1}{d-2}\int_0^1 \frac{w^{-\frac{d-3}{2(d-2)}}}{\sqrt{1-w}} dw\,, \tag{D.23}$$

$$\langle f|Hf\rangle = \frac{1}{4(d-2)}\int_0^1 \frac{-5(2d-3)w^{-\frac{d-5}{2(d-2)}} + 5(4d-3)w^{\frac{d+1}{2(d-2)}}}{\sqrt{1-w}} dw\,, \tag{D.24}$$

$$\|Hf-\mu f\|^2 = \frac{1}{d-2}\int_0^1 \frac{w^{-\frac{2d-3}{2(d-2)}}}{\sqrt{1-w}}\Bigg[-\frac{5(2d-3)}{4}w^{\frac{d+4}{4(d-2)}} + \frac{4d-3}{4}w^{\frac{5d-4}{4(d-2)}} - \mu w^{d/(2d-4)}\Bigg]^2 dw\,. \tag{D.25}$$

Each one of the integrals in Eqs. (D.23-D.25) (even the last one, after expanding the term $[...]^2$ therein) can be written as a linear combination of integrals of the form

$$\int_0^1 dw \frac{w^\alpha}{\sqrt{1-w}} = \frac{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{3}{2}\right)} \tag{D.26}$$

for suitable values of $\alpha \in (-1,+\infty)$. Repeated applications of Eq. (D.26) gives explicit expressions for $\|f\|^2, \langle f|Hf\rangle, \|Hf-\mu f\|^2$ containing ratios of Gamma terms of the form $\Gamma\left(\frac{ad+b}{2(d-2)}\right)$, for suitable integers $a > 0$ and b. These expressions can be significantly simplified using the identity $\Gamma(z) = (z-1)\Gamma(z-1)$ until each Gamma term is reduced to a form with $a = 1$ or $a = 2$; the final results are

$$\|f\|^2 = \frac{\sqrt{\pi}\,\Gamma\left(\frac{d-1}{2(d-2)}\right)}{(d-2)\Gamma\left(\frac{2d-3}{2(d-2)}\right)}\,, \tag{D.27}$$

$$\langle f|Hf\rangle = -\frac{(16d-9)\sqrt{\pi}\,\Gamma\left(\frac{d+1}{2(d-2)}\right)}{4(2d-1)\Gamma\left(\frac{2d-1}{2(d-2)}\right)}\,, \tag{D.28}$$

$$\|Hf-\mu f\|^2 = \frac{3(d-2)(44d-3)\sqrt{\pi}\,\Gamma\left(\frac{d+3}{2(d-2)}\right)}{16(2d+1)\Gamma\left(\frac{2d+1}{2(d-2)}\right)} + \frac{(16d-9)\mu\sqrt{\pi}\,\Gamma\left(\frac{d+1}{2(d-2)}\right)}{2(2d-1)\Gamma\left(\frac{2d-1}{2(d-2)}\right)} + \frac{\mu^2\sqrt{\pi}\,\Gamma\left(\frac{d+1}{2(d-2)}\right)}{(d-2)\Gamma\left(\frac{2d-3}{2(d-2)}\right)} \tag{D.29}$$

Eqs. (D.27,D.28) yield the expression (D.15) for the mean value [Eq. (D.14)]; in addition, Eq. (D.27), Eq. (D.29) with $\mu = \mu(f)$ and Eq. (D.15) for $\mu(f)$ yield an expression for

$$\varDelta(f)^2 := \frac{\|Hf - \mu(f)f\|^2}{\|f\|^2}$$

and taking the square root we obtain for the dispersion $\varDelta(f)$ [Eq. (D.19)] the representation in Eq. (D.20). The $d \to +\infty$ asymptotics for $\mu(f)$ and $\varDelta(f)$ in Eqs. (D.15,D.20) are obtained via elementary manipulations, noting the following for any $h, k \in \mathbb{R}$:

$$\Gamma\left(\frac{d+h}{2d+k}\right) = \Gamma\left(\frac{1}{2} + O\left(\frac{1}{d}\right)\right) = \sqrt{\pi} + O\left(\frac{1}{d}\right) \quad \text{for } d \to +\infty\,,$$
$$\Gamma\left(\frac{2d+h}{2d+k}\right) = \Gamma\left(1 + O\left(\frac{1}{d}\right)\right) = 1 + O\left(\frac{1}{d}\right) \quad \text{for } d \to +\infty\,.$$

D.1.4 The functions $\chi_0^\pm$ and their zeroes

Let us consider the operator H and the potential $\mathcal{V}$ of Eq. (D.1) and the functions $\chi_0^\pm \in C^\infty(\mathbb{R}, \mathbb{R})$ mentioned in item (iii) of Subsection D.1.1 and in Remark 93; these are the solutions of the systems D.8.
Let us consider, for example, the function χ_0^+; explicitating $\mathcal{V}$, we obtain

$$-\chi_0^{+\prime\prime} + \frac{1}{4\rho^2}\left[(d-3)(d-5) - \frac{3(d-1)^2}{\rho^{2(d-2)}}\right]\chi_0^+ = 0\,, \quad \chi_0^+(0) = 1\,, \quad \chi_0^{+\prime}(0) = 0\,. \tag{D.30}$$

The function ρ is defined in a rather implicit way as in Eq. (3.31); for our purposes, it is convenient to characterize it as the unique function in $C^\infty(\mathbb{R}, (0, +\infty))$ such that

$$\rho'' = \frac{d-2}{\rho^{2d-3}}\,, \quad \rho(0) = 1\,, \quad \rho'(0) = 0 \tag{D.31}$$

(recall Eqs. (3.24,3.30). We can regard the pair (D.30,D.31) as a Cauchy problem for the unknowns $(\chi_0^+(\mathtt{x}), \rho(\mathtt{x}))$; this can be solved numerically using any package for ODEs (e.g., the ODE routines of `Mathematica`). Similarly, we can write and solve numerically a Cauchy problem for the pair $(\chi_0^-(\mathtt{x}), \rho(\mathtt{x}))$ formed by Eq. (D.31) and by the analogous of Eq. (D.30) for χ_0^-, prescribing the initial data $\chi_0^-(0) = 0$, $\chi_0^{-\prime}(0) = 1$.
Following this strategy, we have determined numerically $\chi_0^\pm$ (and ρ) for many values of d between $d = 3$ and $d = 500$; for all the tested values of d, we found that χ_0^+ has just two zeros and χ_0^- has just one zero (the point $\mathtt{x} = 0$). The forthcoming figures report the graphs of $\chi_0^\pm$ for $d = 4$.

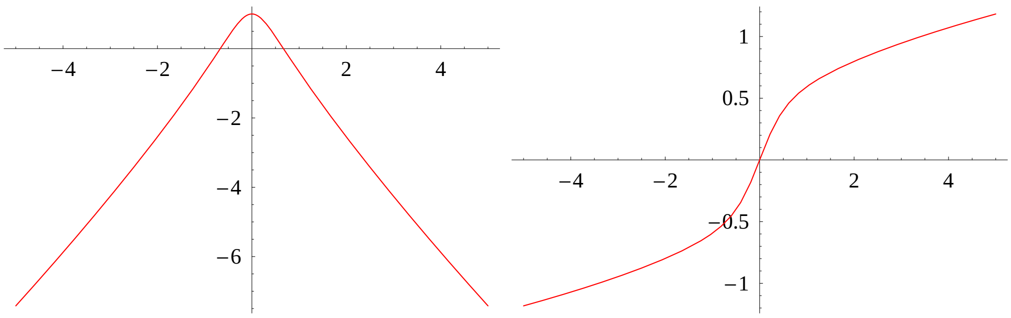

Figure D.1: *Graphs of* χ_0^+ *and* χ_0^+ *in the case* $d = 4$.

D.1.5 The Kato-Temple inequality, and the derivation of Eq. (D.18)

Let us consider a general complex, separable Hilbert space $\mathfrak{H}$ with inner product $\langle \, | \, \rangle$ and norm $\| \, \|$, and a selfadjoint operator

$$H : \mathfrak{D} \subset \mathfrak{H} \to \mathfrak{H}$$

with spectrum $\sigma(H)$. Let us assume the following items (a-c):

(a) α, β are two extended real numbers such that $-\infty \leq \alpha < \beta \leq +\infty$, and $(\alpha, \beta) \cap \sigma(H)$ contains at most a simple proper eigenvalue of H;

(b) we are given a "trial vector" $f \in \mathfrak{D} \setminus \{0\}$; $\mu(f), \Delta(f)$ are the mean value and the dispersion of H over this vector, i.e.,

$$\mu(f) := \frac{\langle f | Hf \rangle}{\|f\|^2}\,, \qquad \Delta(f) := \frac{\|Hf - \mu(f)f\|}{\|f\|}\,;$$

(c) α, β are such that

$$\alpha < \mu(f) < \beta\,, \qquad \Delta(f)^2 < (\mu(f) - \alpha)(\beta - \mu(f))\,.$$

The *Kato-Temple theorem* [62, 63, 64] states that, under the conditions (a-c), the set $(\alpha, \beta) \cap \sigma(H)$ actually consists of a simple eigenvalue $\mu_{\alpha\beta}$ such that

$$\mu(f) - \frac{\Delta(f)^2}{\beta - \mu(f)} \leq \mu_{\alpha\beta} \leq \mu(f) + \frac{\Delta(f)^2}{\mu(f) - \alpha}\,; \qquad \text{(D.32)}$$

the above relation is usually called the *Kato-Temple inequality*.
Let us apply this theorem to the Schrödinger operator H in Eq. (D.1), acting

on the Hilbert space (D.2). For this operator, we accept the conclusions of Remark 93, that is, for any $d \geq 3$, H has only one eigenvalue μ_1. From general facts on Schröedinger operators of the form (D.4) (see Subsection D.1.1, item (iii)), μ_1 is simple and is the unique point of $\sigma(H)$ contained in $(-\infty, 0)$; so, the assumption (a) of the Kato-Temple theorem is fulfilled with $\alpha := -\infty$ and $\beta := 0$. We now consider the trial function f of Eq. (D.13); the corresponding mean and dispersion $\mu(f), \Delta(f)$ are as in Eqs. (D.15,D.20). With our choices for α, β the inequalities of item (c) take the from $-\infty < \mu(f) < 0$, $\Delta(f)^2 < +\infty$ and are obviously satisfied. Thus we have the inequality (D.32) for $\mu_{\alpha\beta} = \mu_1$; due again to the choice of α, β this takes the form

$$\mu(f) + \frac{\Delta(f)^2}{\mu(f)} \leq \mu_1 \leq \mu(f)\,, \tag{D.33}$$

which is clearly equivalent to Eq. (D.18).

D.2 Spectral features of the Schrödinger operator (7.43,7.39) (Ellis-Bronnikov wormhole)

Let us consider the Hilbert space

$$\mathfrak{H} := L^2(\mathbb{R}, d\rho) \tag{D.34}$$

and the operator

$$\begin{gathered} H := -\frac{d^2}{d\rho^2} + \mathcal{U} \;:\; \mathfrak{D} \subset \mathfrak{H} \to \mathfrak{H}\,, \qquad \mathfrak{D} := \{f \in \mathfrak{H} \mid f_{\rho\rho} \in \mathfrak{H}\}\,, \\ \mathcal{U}(\rho) := \left(\frac{\alpha^2}{\gamma^2}\mathcal{V}\right)(x(\rho)) \quad (\mathcal{V} \text{ as in Eq. (7.32)}) \end{gathered} \tag{D.35}$$

with the ρ-derivatives intended in the distributional sense (see Footnote 67 with ρ in place of x). Even in this case we profit from the general results on Schrödinger operators [56] and from the analysis performed in Ref. [25]; this leads to the following statement:

(i) in the case $\gamma_1 \neq 0$, H with the domain $\mathfrak{D}$ is a selfadjoint operator in $\mathfrak{H}$;

(ii) in the case $\gamma_1 \neq 0$, the point spectrum of H consists of two simple eigenvalues $\mu_1 < 0$ and $\mu_2 := 0$, thus we write

$$\sigma_p(H) = \{\mu_1, 0\}\,; \tag{D.36}$$

(iii) in the case $\gamma_1 \neq 0$, the continuous spectrum of H is

$$\sigma_c(H) = (0, +\infty)\,.$$

Remark 95 In the reflection symmetric case $\gamma_1 = 0$, $\rho = x$ and the operator (D.35) reduces to the four-dimensional operator (D.1) (with $d = 3$), after performing the coordinate change $t = b\,\mathtt{t}$, $x = b\,\mathtt{x}$ [Eq. (3.33)]. Therefore, quite obviously, the spectral properties of (D.35) in the case $\gamma_1 = 0$ can be inferred from the results of Section D.1 setting $d = 3$; in particular one sees that

(i) in the case $\gamma_1 = 0$, H with the domain $\mathfrak{D}$ is a selfadjoint operator in $\mathfrak{H}$;

(ii) in the case $\gamma_1 = 0$, the point spectrum of H consists of a unique, simple eigenvalue $\mu_1 < 0$, thus we write

$$\sigma_p(H) = \{\mu_1\}\,; \tag{D.37}$$

(iii) in the case $\gamma_1 = 0$, the continuous spectrum of H is

$$\sigma_c(H) = [0, +\infty)\,.$$

For the sake of completeness, let us sketch a possible proof of the existence of exactly two eigenvalues $\mu_1 < 0$ and $\mu_2 = 0$ [Eq. (D.36)] in the non reflection symmetric case $\gamma_1 \neq 0$ (and of the existence of exactly one eigenvalue $\mu_1 < 0$ [Eq. (D.37)] in the reflection symmetric case $\gamma_1 = 0$).
We start noting that, since $d\rho = \frac{\gamma}{\alpha} dx$, working with the operator (7.32) in the Hilbert (D.34) is equivalent to working directly with the operator

$$-\left(\frac{\alpha}{\gamma}\frac{d}{dx}\right)^2 + \frac{\alpha^2}{\gamma^2}\,\mathcal{V}$$

in the Hilbert space

$$L^2\left(\mathbb{R}, \frac{\gamma}{\alpha} dx\right)\,. \tag{D.38}$$

As shown in Ref. [25], the "zero energy" equation

$$\left[-\left(\frac{\alpha}{\gamma}\frac{d}{dx}\right)^2 + \frac{\alpha^2}{\gamma^2}\,\mathcal{V}\right]\chi_0 = 0$$

has a solution

$$\chi_0(x) = \frac{x - b\gamma_1}{r(x)F(x)}\,, \tag{D.39}$$

which has precisely one zero in the interval $(-\infty, +\infty)$. According to the Sturm oscillation theorem (see for instance Refs. [65, 66] and references therein) it follows that for each γ_1 (including $\gamma_1 = 0$), the Schrödinger operator in the master equation possesses a single bound state with negative energy. Note that for $\gamma_1 \neq 0$ the function χ_0 decays as $1/|x|$ for large $|x|$, so that it describes an element of the Hilbert space (D.38); moreover, it can be proved that in this case χ_0 belongs to the domain of selfadjointness of H, which implies that χ_0 is an eigenfunction of H corresponding to the eigenvalue $\mu_2 = 0$. On the contrary, for $\gamma_1 = 0$ Eq. (D.39) reduces to $\chi_0(x) = x/\sqrt{1+x^2}$ which is not an element of the Hilbert space (D.38) and hence can not be considered as an eigenfunction of H; thus, for $\gamma_1 = 0$, zero is not an eigenvalue of the operator H.

D.3 Spectral features of the Schrödinger operator (5.69,5.60) (AdS wormhole)

In this section we study the spectral features of the operator

$$H := -\frac{d^2}{du^2} + \mathcal{V} \;:\; \mathfrak{D} \subset \mathfrak{H} \to \mathfrak{H}\,, \qquad \mathfrak{D} := \{f \in \mathfrak{H} \mid f_{uu} \in \mathfrak{H}\,, f(\pm\pi) = 0\}\,,$$
$$\mathcal{V}(u) \equiv \mathcal{V}_B(u) = -\frac{B^2\,(2 + B^2 + \cos u)}{(1 + 2B^2 - \cos u)^2}\,, \tag{D.40}$$

in the Hilbert space

$$\mathfrak{H} := L^2((-\pi,\pi), du)\,. \tag{D.41}$$

Here and in the sequel, the u-derivatives like f_{uu} are understood distributionally; a function $f \in \mathfrak{H}$ with $f_{uu} \in \mathfrak{H}$ is in fact in $C^1([-\pi,\pi])$, so it can be evaluated at $u = \pm\pi$ [(69)].

Remark 96 As an operator in the Hilbert space $\mathfrak{H}$, the operator H in Eq. (D.40) has the following properties: [(70)]

(i) it is selfadjoint;

[69] The conditions $f \in \mathfrak{H}$, $f_{uu} \in \mathfrak{H}$ imply $f_u \in \mathfrak{H}$, due to the already mentioned Gagliardo-Nirenberg interpolation inequality [58]. The space $\{f \in \mathfrak{H} \mid f_{uu} \in \mathfrak{H}\}$ coincides with the standard Sobolev space $W^{2,2}(-\pi,\pi) \equiv H^2(-\pi,\pi)$, which is contained in $C^1([-\pi,\pi])$ by the Sobolev embedding theorem (see again Ref. [58]). Let us also remark that, due to the boundedness of the function $\mathcal{V}$, for each $f \in \mathfrak{H}$ one has automatically $\mathcal{V}f \in \mathfrak{H}$.

[70] For some general facts about Hilbert space operators with properties (i-iii), see e.g. Ref. [57] (especially, pages 37, 178 and 265-67).

(ii) it is bounded from below;

(iii) it has a purely discrete spectrum.

As known in general for Hilbert space operators fulfilling (i-iii), it is possible to represent the eigenvalues of H as an increasing sequence $\mu_1 < \mu_2 < \dots$; thus, we write

$$\sigma(H) = \sigma_p(H) = \{\mu_n\}_{n\in\mathbb{N}} \qquad \mu_1 < \mu_2 < \dots . \tag{D.42}$$

In addition, H has the following properties:

(iv) any of its eigenfunctions is in the space $C^\infty([-\pi,\pi])$;

(v) each one of its eigenvalues is simple.

For completeness, let us give more information on the above issues (i-v) for the operator H. In this regard, let us recall that the operator

$$H^0 := -\frac{d^2}{du^2} \ : \ \mathfrak{D} \subset \mathfrak{H} \to \mathfrak{H}$$

(with the domain $\mathfrak{D}$ as in Eq. (D.40)) also has the properties (i-v); in this case the eigenvalues are

$$\mu_n^0 := \frac{n^2}{4} \qquad n \in \mathbb{N}$$

and for each eigenvalue μ_n^0 the corresponding normalized eigenfunction is

$$f_n^0(u) := \frac{1}{\sqrt{\pi}} \sin\left[\frac{n}{2}(u+\pi)\right] \qquad n \in \mathbb{N} .$$

The properties (i-v) of H^0 and the expressions given above for its eigenvalues and eigenfunctions are checked "by hand" (71). Now consider any function $\mathcal{V} \in C^\infty([-\pi,\pi],\mathbb{R})$; then, due to the boundedness of this function, the multiplication operator by $\mathcal{V}$ is a bounded selfadjoint operator on $\mathfrak{H}$. As well known the properties (i), or (i-ii), or (i-iii) of an operator in an abstract Hilbert space are preserved by the addition of a bounded selfadjoint perturbation (see again Ref. [57]); therefore the operator $H^\mathcal{V} := H^0 + \mathcal{V} = -\frac{d^2}{du^2} + \mathcal{V}$ with domain $\mathfrak{D}$ fulfills (i-iii). Moreover, the operator $H^\mathcal{V}$ is proved to have also the properties (iv-v). (72) All the previous statements apply, in particular, with $\mathcal{V}$ as in Eq. (D.40) (hence for $H^\mathcal{V} = H$).

[71] The eigenfunctions of H^0 are proved to be smooth due to the following regularity result: if f is a distribution on an open interval $\Omega \subset \mathbb{R}$ (with derivatives $f^{(i)}$, $i = 0, 1, \dots$) and f fulfills a homogeneous linear ODE $f^{(k)} + \sum_{i=0}^{k-1} a_i f^{(i)} = 0$ of any order $k \in \{1, 2, \dots\}$ with C^∞ coefficients $a_i : \Omega \to \mathbb{C}$, then f is a C^∞ function on Ω: this follows from Theorem IX in Ref. [67], page 130.

[72] For the proof of (iv) for $H^\mathcal{V}$ one can use again the cited regularity result for distributional, homogeneous linear ODEs (see Footnote 71); a derivation of (v) for $H^\mathcal{V}$ can be found e.g. in Ref. [68], page 30.

Remark 97 Using (again) the Sturm oscillation theorem (see Theorem 3.4 in Ref. [66]), one can prove that the operator H has exactly one negative eigenvalue, while all other eigenvalues are positive, namely, one can see that

$$\mu_1 < 0 < \mu_2 < \mu_3 < \dots . \tag{D.43}$$

Indeed, the zero energy Schrödinger equation

$$\left[-\frac{d^2}{du^2} + \mathcal{V}\right] \chi_0 = 0$$

admits for each fixed $B > 0$ the general solution

$$\chi_0(u) = C_1 \frac{\sin\frac{u}{2}}{\sqrt{1+2B^2-\cos u}} + C_2 \frac{-2u\sin\frac{u}{2} + 4B^2\cos\frac{u}{2}}{\sqrt{1+2B^2-\cos u}}\,, \qquad -\pi < u < \pi\,, \tag{D.44}$$

with constants $C_1, C_2 \in \mathbb{C}$. The Dirichlet boundary conditions $\chi_0(\pm\pi) = 0$ are satisfied only in the trivial case $C_1 = C_2 = 0$, which shows that none of these solutions is an eigenfunction of our Schrödinger operator (and thus showing that zero is not an eigenvalue of H). For $C_1 = -2\pi C_2 \neq 0$ the zero energy solution satisfies the left boundary condition, i.e. $\chi_0(-\pi) = 0$, and since this solution has precisely one zero in the interval $(-\pi, \pi)$, [73] it follows from the Sturm oscillation theorem (see Theorem 3.4 in Ref. [66]) that our Schrödinger operator (with Dirichlet boundary conditions) has a single negative eigenvalue $\mu_1 < 0$.

In the remainder of this section, the notations $\mathfrak{H}$, $\mathcal{V}$, H, $\mathfrak{D}$, $\{\mu_n\}_{n\in\mathbb{N}}$ will always indicate, respectively, the Hilbert space in Eq. (D.41), the potential, the operator and its domain in Eq. (D.40) and its eigenvalues in increasing order [Eq. (D.42)]. In addition, it will be useful to emphasize that the potential $\mathcal{V}$ depends on the parameter $B \in (0, +\infty)$, thus originating in a similar dependence for the corresponding operator, and its eigenvalues: $\mathcal{V} \equiv \mathcal{V}_B$, $H \equiv H_B$, $\mu_n \equiv \mu_n(B)$ $(n \in \mathbb{N})$.

In what follows, we provide estimates for the eigenvalues of $\mu_n(B)$.
We start with an upper bound for the ground state energy $\mu_1 \equiv \mu_1(B)$. We

[73] Let us justify this statement on the number of zeroes of χ_0 for the special choice $C_1 = -2\pi C_2 \neq 0$. In this case we can write $\chi_0(u) = \left(-2C_2\cos\frac{u}{2}\right) w(u)/\sqrt{1+2B^2-\cos u}$ where $w : (-\pi,\pi) \to \mathbb{R}$, $u \mapsto w(u) := (u+\pi)\tan\frac{u}{2} - 2B^2$. The zeroes of χ_0 in $(-\pi,\pi)$ coincide with the zeroes of the function w. To find the zeroes of w, it is useful to note that this function has derivative $w'(u) = \left(\frac{1}{2}\sec^2\frac{u}{2}\right)(u + \sin u + \pi) > 0$ for all $u \in (-\pi,\pi)$; from $w' > 0$ it follows that w is a strictly monotonic bijection of $(-\pi,\pi)$ to $(-2B^2-2, +\infty)$, and thus possesses a unique zero.

have already recalled that, for a spectrum of type (D.43), the Rayleigh-Ritz variational characterization ensures that (see Eq. (D.12))

$$\mu_1(B) = \inf_{f\in\mathfrak{D}\setminus\{0\}} \frac{\langle f \mid H_B f\rangle}{||f||^2}\,. \tag{D.45}$$

Choosing in $\mathfrak{D}$ the function

$$f(u) := \cos\frac{u}{2}\,, \qquad -\pi < u < \pi\,,$$

we get

$$\frac{\langle f \mid H_B f\rangle}{||f||^2} = \frac{1}{4} - B^2 + \frac{\sqrt{1+B^2}\,(4B^2-3)}{4B} =: \varepsilon(B)\,, \tag{D.46}$$

which, together with Eq. (D.45), yields the estimate

$$\mu_1(B) \le \varepsilon(B) \qquad \text{for each } B > 0\,. \tag{D.47}$$

It can be checked that $B \mapsto \varepsilon(B)$ is a negative, monotonously increasing function on $(0, +\infty)$ with the properties

$$\lim_{B\to 0^+} \varepsilon(B) = -\infty\,, \qquad \lim_{B\to+\infty} \varepsilon(B) = 0^-\,.$$

Therefore, we obtain the upper bound for the ground state energy,

$$\mu_1(B) \le \varepsilon(B) < 0 \tag{D.48}$$

which provides an independent proof for the fact that it is negative (see Remark 97).
Next, we provide two-sided bounds on the eigenvalues $\mu_n \equiv \mu_n(B)$ for arbitrary n. In order to achieve this, we check that for any fixed $B > 0$, one has

$$\begin{aligned} \min_{u\in[-\pi,\pi]} \mathcal{V}_B(u) &= \mathcal{V}_B(0) = -\frac{1}{4} - \frac{3}{4B^2}\,, \\ \max_{u\in[-\pi,\pi]} \mathcal{V}_B(u) &= \mathcal{V}_B(\pm\pi) = -\frac{1}{4} + \frac{1}{4(1+B^2)}\,. \end{aligned} \tag{D.49}$$

In the Hilbert space $\mathfrak{H}$, let us consider the operator H and the operators

$$H^- := -\frac{d^2}{du^2} - \frac{1}{4} - \frac{3}{4B^2}\,, \qquad H^+ := -\frac{d^2}{du^2} - \frac{1}{4} + \frac{1}{4(1+B^2)}$$

all of them with the same domain $\mathfrak{D}$ as defined in Eq. (D.40); note that all of these operator satisfy the properties (i-v) of Remark (96). Due to Eq. (D.49) we have

$$\left\langle f \,\middle|\, H^- f\right\rangle \le \langle f \,|\, Hf\rangle \le \left\langle f \,\middle|\, H^+ f\right\rangle \quad \text{for all } f \in \mathfrak{D}\,,$$

and this implies (see e.g. Ref. [57], pages 230 and 267)

$$\mu_n^- \le \mu_n \le \mu_n^+ \quad n \in \mathbb{N}$$

where $\mu_1^\mp < \mu_2^\mp < \ldots$ are the eigenvalues of $H^\mp$. On the other hand, the eigenvalues of $H^\mp$ are obtained shifting those of $H^0 = -\frac{d^2}{du^2}$, i.e.,

$$\mu_n^- = \frac{n^2}{4} - \frac{1}{4} - \frac{3}{4B^2}\,, \qquad \mu_n^+ = \frac{n^2}{4} - \frac{1}{4} + \frac{1}{4(1+B^2)}\,.$$

In conclusion, the eigenvalues of H satisfy the two-side bounds

$$\frac{n^2-1}{4} - \frac{3}{4B^2} \le \mu_n(B) \le \frac{n^2-1}{4} + \frac{1}{4(1+B^2)} \qquad n \in \mathbb{N}\,. \tag{D.50}$$

Combining this result with Eq. (D.48) one obtains the following two-side bound for the ground state energy:

$$-\frac{3}{4B^2} \le \mu_1(B) \le \varepsilon(B) = -\frac{1}{2B^2} + \mathcal{O}\left(\frac{1}{B^4}\right)\,. \tag{D.51}$$

D.4 Spectral features of the Schrödinger operator (7.64,7.60) (dS wormhole)

Let us consider the Hilbert space

$$\mathfrak{H} := L^2\left(\mathbb{R}, du\right) \tag{D.52}$$

and the operator

$$\begin{gathered} H := -\frac{d^2}{du^2} + \mathcal{V} \;:\; \mathfrak{D} \subset \mathfrak{H} \to \mathfrak{H}\,, \qquad \mathfrak{D} := \{f \in \mathfrak{H} \mid f_{uu} \in \mathfrak{H}\}\,, \\ \mathcal{V}(u) := -\frac{B^2\left(2 - B^2 + \cosh u\right)}{\left(-1 + 2B^2 + \cosh u\right)^2} \end{gathered} \tag{D.53}$$

(the ρ-derivatives have to be intended in the distributional sense as in Footnote 67). Note that $\mathcal{V}(u)$ is everywhere negative and vanishes like $-\frac{1}{\cosh u}$ for $u \to \pm\infty$; hence the situation of this operator is rather similar to that of the reflection symmetric Ellis- Bronnikov wormhole (see Remark 95):

(i) H with the domain $\mathfrak{D}$ is a selfadjoint operator in $\mathfrak{H}$;

(ii) the point spectrum of H consists of a unique, simple eigenvalue $\mu_1 < 0$, thus we write
$$\sigma_p(H) = \{\mu_1\}\,; \tag{D.54}$$

(iii) the continuous spectrum of H is
$$\sigma_c(H) = [0, +\infty)\,.$$

For the sake of completeness, let us proof the previous items (ii-iii). Consider the zero energy Schrödinger equation
$$\left[-\frac{d^2}{du^2} + \mathcal{V}\right] \chi_0 = 0\,;$$
the general solution of this equation is obtained from the analogous solution (D.44) for the AdS case with the formal replacements $(u, B) \mapsto (iu, iB)$ and reads
$$\chi_0(u) = C_1 \frac{\sinh\frac{u}{2}}{\sqrt{-1 + 2B^2 + \cosh u}} + C_2 \frac{2u \sinh\frac{u}{2} - 4B^2 \cosh\frac{u}{2}}{\sqrt{-1 + 2B^2 + \cosh u}}\,,$$
with constants $C_1, C_2 \in \mathbb{C}$. One has $\chi_0 \in L^2(\mathbb{R}, du)$ if and only if $C_1 = C_2 = 0$, thus zero is not an eigenvalue of H. If $C_1 \in \mathbb{R} \setminus \{0\}$ and $C_2 = 0$ it is evident that χ_0 has a unique zero in $\mathbb{R}$ (namely, $u = 0$). If $C_2 \in \mathbb{R} \setminus \{0\}$ and $C_1 \in \mathbb{R}$, one can show that χ_0 possesses two zeroes in $\mathbb{R}$ (via an analysis rather similar to that given for the function χ_0 of the AdS case after E. (D.44). Summing up, the *minimal* number of zeroes of the real, non identically vanishing solutions χ_0 of the zero energy equation is *one*. The Sturm oscillation theorem (Theorem 14.8 of Ref. [65]) states that such a minimal number of zeroes is the number of negative eigenvalues of H. So, H has a unique negative eigenvalue; in addition, due to general facts on Schrödinger operators (and to the previous remark that 0 is not an eigenvalue), H has continuous spectrum $[0, +\infty)$.

Appendix E

Solutions of the master equations

We start this appendix with a trivial remark on the general solution of a second order ODE with a polynomial source term.

Remark 98 The solution of the second order ordinary Cauchy problem in the unknown $y = y(t)$

$$\begin{cases} \ddot{y} + Ey = J_0 + J_1 s + J_2 s^2 & (E < 0,\ J_0, J_1, J_2 \in \mathbb{R})\,, \\ y(0) = y_0 & (y_0 \in \mathbb{R})\,, \\ \dot{y}(0) = y_1 & (y_1 \in \mathbb{R})\,, \end{cases} \tag{E.1}$$

is

$$y(t) = y_0 \cosh(|E|^{1/2} t) + y_1 \frac{\sinh(|E|^{1/2} t)}{|E|^{1/2}} + \frac{\cosh(|E|^{1/2} t) - 1}{|E|} J_0 + \frac{\sinh(|E|^{1/2} t) - |E|^{1/2} t}{|E|^{3/2}} J_1 + \frac{2\cosh(|E|^{1/2} t) - |E| t^2 - 2}{|E|^2} J_2\,. \tag{E.2}$$

The solution of the second order ordinary Cauchy problem in the unknown $y = y(t)$

$$\begin{cases} \ddot{y} + Ey = J_0 + J_1 s + J_2 s^2 & (E > 0,\ J_0, J_1, J_2 \in \mathbb{R})\,, \\ y(0) = y_0 & (y_0 \in \mathbb{R})\,, \\ \dot{y}(0) = y_1 & (y_1 \in \mathbb{R})\,, \end{cases} \tag{E.3}$$

is

$$y(t) = y_0 \cos(E^{1/2} t) + y_1 \frac{\sin(E^{1/2} t)}{E^{1/2}} + \frac{1 - \cos(E^{1/2} t)}{E} J_0 + \frac{E^{1/2} t - \sin(E^{1/2} t)}{E^{3/2}} J_1 + \frac{2\cos(E^{1/2} t) + E t^2 - 2}{E^2} J_2\,. \tag{E.4}$$

E.1 Solution of the master equations of the Torii-Shinkai wormhole

In this section we deal with the master equations (5.26,7.21); we will show how the resolution of these equations can be reduced to the spectral analysis of the operator (D.1) in the Hilbert space (D.2).

Remark 99 In Section D.1 (and in particular in Subsection D.1.2) we have stated that there is numerical evidence that the spectrum $\sigma(H)$ the operator in Eq. (D.1) is made up of a simple proper eigenvalue $\mu_1 < 0$ and a continuous spectrum $\sigma_c(H) = [0, +\infty)$; from now on we accept this result as true. Therefore, the spectral theorem 15 ensures that one can construct a generalized orthonormal basis of the Hilbert space $\mathfrak{H}$, in the sense of Definition 10 with $N = 1$ and $I = 2$ which is made up of:

(i) a normalized eigenfunction e_1 for the eigenvalue $\mu_1 < 0$, i.e.

$$e_1 \in \mathfrak{D} \quad : \quad \|e_1\| = 1\,, \quad He_1 = \mu_1 e_1$$

(e_1 is proved to be $C^\infty(\mathbb{R})$);

(ii) two suitably chosen and linearly independent "improper eigenfunctions" $e_{i\lambda}$ $(i = 1, 2)$ for each $\lambda \in \mathring{\sigma}_c(H) = (0, +\infty)$, i.e.,

$$e_{i\lambda} \in C^\infty(\mathbb{R}) \setminus \mathfrak{D} \quad : \quad He_{i\lambda} = \lambda e_{i\lambda} \qquad (i = 1, 2\,; \lambda > 0)\,.$$

E.1.1 Solution of the master equation (7.21) with low regularity

Let us start with the master equation (7.21) with the initial conditions (7.23) in the suitable Hilbertian framework introduced in Remark 40, i.e.,

$$\begin{aligned} \ddot{\chi}(\mathtt{t}) + H\chi(\mathtt{t}) = 0 \qquad & H \text{ as in Eq. (D.1)}\,, \\ \chi(0) = q\,, \qquad & \dot{\chi}(0) = p\,, \end{aligned} \tag{E.5}$$

where the unknown is a function

$$\mathbb{R} \ni \mathtt{t} \mapsto \chi(\mathtt{t}) \in \mathfrak{D}\,.$$

We first proceed formally, assuming that the initial data q, p are in suitable spaces to be specified later. Applying $\langle e_1|\ \rangle$ and $\langle e_{i\lambda}|\ \rangle$ to the differential equation in Eq. (E.5), we obtain

$$0 = \left\langle e_1 \,\middle|\, \frac{d^2}{d\mathtt{t}^2}\chi(\mathtt{t}) + H\chi(\mathtt{t}) \right\rangle = \frac{d^2}{d\mathtt{t}^2}\langle e_1\,|\,\chi(\mathtt{t})\rangle + \langle e_1\,|\,H\chi(\mathtt{t})\rangle$$
$$= \left(\frac{d^2}{d\mathtt{t}^2} + \mu_1\right)\langle e_1|\chi(\mathtt{t})\rangle \quad \text{(E.6)}$$

and

$$0 = \left\langle e_{i\lambda} \,\middle|\, \frac{d^2}{d\mathtt{t}^2}\chi(\mathtt{t}) + H\chi(\mathtt{t}) \right\rangle = \frac{d^2}{d\mathtt{t}^2}\langle e_{i\lambda}\,|\,\chi(\mathtt{t})\rangle + \langle e_{i\lambda}\,|\,H\chi(\mathtt{t})\rangle$$
$$= \left(\frac{d^2}{d\mathtt{t}^2} + \lambda\right)\langle e_{i\lambda}|\chi(\mathtt{t})\rangle \qquad (i = 1, 2\,; \lambda > 0)\,. \quad \text{(E.7)}$$

These are two wave equations respectively in the unknowns $y_1(\mathtt{t}) := \langle e_1|\chi(\mathtt{t})\rangle$ and $y_{i\lambda}(\mathtt{t}) := \langle e_{i\lambda}|\chi(\mathtt{t})\rangle$ with the initial conditions given by Eq. (E.5), i.e.

$$\begin{cases} y_1(0) = \langle e_1|\chi(0)\rangle = \langle e_1|q\rangle\,, \\ \dot{y}_1(0) = \langle e_1|\dot{\chi}(0)\rangle = \langle e_1|p\rangle\,, \end{cases} \quad \text{(E.8)}$$

$$\begin{cases} y_{i\lambda}(0) = \langle e_{i\lambda}|\chi(0)\rangle = \langle e_{i\lambda}|q\rangle \qquad (i = 1, 2\,; \lambda > 0)\,, \\ \dot{y}_{i\lambda}(0) = \langle e_{i\lambda}|\dot{\chi}(0)\rangle = \langle e_{i\lambda}|p\rangle \qquad (i = 1, 2\,; \lambda > 0)\,. \end{cases} \quad \text{(E.9)}$$

On account of the previous conditions, from Eq. (E.2) with $E = \mu_1 < 0$ and Eq. (E.4) with $E = \lambda > 0$ with $J_i = 0$ for $i = 0, 1, 2$, we have that the wave equations (E.6,E.7) have solutions

$$\langle e_1|\chi(\mathtt{t})\rangle = \langle e_1|q\rangle \cosh(|\mu_1|^{1/2}\mathtt{t}) + \langle e_1|p\rangle \frac{\sinh(|\mu_1|^{1/2}\mathtt{t})}{|\mu_1|^{1/2}}\,, \quad \text{(E.10)}$$

$$\langle e_{i\lambda}|\chi(\mathtt{t})\rangle = \langle e_{i\lambda}|q\rangle \cos(\lambda^{1/2}\mathtt{t}) + \langle e_{i\lambda}|p\rangle \frac{\sin(\lambda^{1/2}\mathtt{t})}{\lambda^{1/2}}\,, \qquad (i = 1, 2\,; \lambda > 0)\,; \quad \text{(E.11)}$$

the spectral decompositions of the function $\chi(\mathtt{t}) \in \mathfrak{H}$ [Proposition 16] implies that the system (E.5) has the solution

$$\chi(\mathtt{t}) = \left[\langle e_1|q\rangle \cosh(|\mu_1|^{1/2}\mathtt{t}) + \langle e_1|p\rangle \frac{\sinh(|\mu_1|^{1/2}\mathtt{t})}{|\mu_1|^{1/2}}\right] e_1$$
$$+ \sum_{i=1}^{2} \int_0^{+\infty} \left[\langle e_{i\lambda}|q\rangle \cos(\lambda^{1/2}\mathtt{t}) + \langle e_{i\lambda}|p\rangle \frac{\sin(\lambda^{1/2}\mathtt{t})}{\lambda^{1/2}}\right] e_{i\lambda}\, d\lambda\,, \quad \text{(E.12)}$$

thus providing a formal justification for Eq. (7.25).

Remark 100 Let us stress the fact that, as explained in Remark 86, the symbols $\langle\cdot|\cdot\rangle$ in the above formula indicate the usual inner product in $\mathfrak{H}$, or its suitably defined generalization, while the integrals over λ are understood in a weak sense.

Remark 101 It can be checked a posteriori that all the previous manipulations make sense if one assumes that

$$q \in \mathfrak{D}\,, \quad p \in \mathfrak{D}^{1/2}\,, \tag{E.13}$$

where $\mathfrak{D}^{1/2}$ is the domain of the operator $|H|^{1/2}$ defined in Remark 91 (see, in particular Eq. (C.26)). With the assumptions (E.13), Eq. (E.12) describes the unique solution $\chi : \mathbb{R} \ni \mathtt{t} \mapsto \chi(\mathtt{t})$ of the system (E.5) such that

$$\chi \in C^2(\mathbb{R}, \mathfrak{H}) \cap C^1(\mathbb{R}, \mathfrak{D}^{1/2}) \cap C(\mathbb{R}, \mathfrak{D})\,; \tag{E.14}$$

in particular, $\chi(\mathtt{t}) \in \mathfrak{H} \cap \mathfrak{D} = \mathfrak{D}$ for all $t \in \mathbb{R}$, as required.
As an example of the necessary tests to prove the previous statement, let us consider any $\mathtt{t} \in \mathbb{R}$ and show that $\chi(\mathtt{t})$ defined by Eq. (E.12) is an element of $\mathfrak{D} \cap \mathfrak{D}^{1/2}$. Let us start showing that $\chi(\mathtt{t}) \in \mathfrak{D}$. Due to the descriptions (C.2,C.3) for $\mathfrak{H}$ and (C.20) for $\mathfrak{D}$, $\chi(\mathtt{t})$ in Eq. (E.12) is in fact in $\mathfrak{D}$ if we are able to prove the following for $i = 1, 2$ (and for fixed $\mathtt{t}$, as already indicated):

$$\lambda \mapsto \left[\langle e_{i\lambda}|q\rangle \cos(\lambda^{1/2}\mathtt{t}) + \langle e_{i\lambda}|p\rangle \frac{\sin(\lambda^{1/2}\mathtt{t})}{\lambda^{1/2}}\right] \quad \in L^2((0,+\infty), d\lambda)\,, \tag{E.15}$$

$$\begin{aligned} \lambda \mapsto & \lambda \left[\langle e_{i\lambda}|q\rangle \cos(\lambda^{1/2}\mathtt{t}) + \langle e_{i\lambda}|p\rangle \frac{\sin(\lambda^{1/2}\mathtt{t})}{\lambda^{1/2}}\right] \\ &= \lambda \langle e_{i\lambda}|q\rangle \cos(\lambda^{1/2}\mathtt{t}) + \lambda^{1/2}\langle e_{i\lambda}|p\rangle \sin(\lambda^{1/2}\mathtt{t}) \quad \in L^2((0,+\infty), d\lambda)\,. \end{aligned} \tag{E.16}$$

Indeed, Eq. (E.15) follows noting that

$$\begin{aligned} \lambda \mapsto \cos(\lambda^{1/2}\mathtt{t})\,,\ \lambda \mapsto \frac{\sin(\lambda^{1/2}\mathtt{t})}{\lambda^{1/2}} &\quad \in L^\infty((0,+\infty), d\lambda)\,, \\ \lambda \mapsto \langle e_{i\lambda}|q\rangle\,,\ \lambda \mapsto \langle e_{i\lambda}|p\rangle &\quad \in L^2((0,+\infty), d\lambda) \end{aligned} \tag{E.17}$$

(the statements on q, p in (E.17) are correct, since Eq. (E.13) obviously implies $q, p \in \mathfrak{H}$). Moreover, Eq. (E.16) follows noting that

$$\begin{aligned} \lambda \mapsto \cos(\lambda^{1/2}\mathtt{t})\,,\ \lambda \mapsto \sin(\lambda^{1/2}\mathtt{t}) &\quad \in L^\infty((0,+\infty), d\lambda)\,; \\ \lambda \mapsto \lambda\langle e_{i\lambda}|q\rangle\,,\ \lambda \mapsto \lambda^{1/2}\langle e_{i\lambda}|p\rangle &\quad \in L^2((0,+\infty), d\lambda) \end{aligned} \tag{E.18}$$

(the statements on q, p in (E.18) are correct, due to the assumption (E.13) and to the characterizations (C.20) for $\mathfrak{D}$, (C.26) for $\mathfrak{D}^{1/2}$).
We now show that $\chi(\mathtt{t}) \in \mathfrak{D}^{1/2}$; this is equivalent to show that

$$\begin{aligned} \lambda \mapsto & \lambda^{1/2} \left[\langle e_{i\lambda} | q \rangle \cos(\lambda^{1/2} \mathtt{t}) + \langle e_{i\lambda} | p \rangle \frac{\sin(\lambda^{1/2}\mathtt{t})}{\lambda^{1/2}} \right] \\ & = \lambda^{1/2} \langle e_{i\lambda} | q \rangle \cos(\lambda^{1/2}\mathtt{t}) + \langle e_{i\lambda} | p \rangle \sin(\lambda^{1/2}\mathtt{t}) \quad \in L^2((0,+\infty), d\lambda) \,. \end{aligned} \tag{E.19}$$

The previous equation can be verified using the statements (E.17) on q, p, the statements (E.18) on the trigonometric functions and observing that for every $\lambda \geq 0$

$$|\lambda^{1/2} \langle e_{i\lambda} | q \rangle| \leq |(1+\lambda) \langle e_{i\lambda} | q \rangle| \leq |\langle e_{i\lambda} | q \rangle| + |\lambda \langle e_{i\lambda} | q \rangle|$$

(the later is square integrable again for Eqs. (E.17,E.18)).

The results of the previous remark are summed up in the following

Proposition 17 *Under the assumption (E.13) on the initial data, the Cauchy problem (E.5), arising from the gauge-invariant linear stability analysis of the Torii-Shinkai wormhole, has a unique solution $\chi(\mathtt{t})$ defined for every $\mathtt{t} \in \mathbb{R}$ as in (E.14); moreover the solution $\chi(\mathtt{t})$ can be decomposed with respect to the generalized orthonormal basis made up of proper and improper eigenfunctions of the operator (D.1) as in Eq. (E.12).*

E.1.2 Solution of the master equation (5.26) with high regularity

Let us keep all notations of Section (5.1). Therein, we have proved that if $\Gamma(\mathtt{t},\mathtt{x}), \mathcal{R}(\mathtt{t},\mathtt{x}), \Psi(\mathtt{t},\mathtt{x}) \in C^\infty(\mathbb{R}^2, \mathbb{R})$ are solutions of the linearized Einstein equations (5.6-5.9), then $\Gamma(\mathtt{t},\mathtt{x}), \Psi(\mathtt{t},\mathtt{x})$ are given by Eqs. (5.17,5.20) where $\mathcal{R}(\mathtt{t},\mathtt{x})$ satisfies the master equation (5.21); note that Eqs. (5.17,5.20,5.21) involve the initial data $\Gamma_i, \mathcal{R}_i$ [Eq. (5.18)] through Eqs. (5.19,5.23).
In Remark 40, we have introduced an Hilbertian framework for the master equation (5.21) with the initial condition (5.18), yielding the system (5.26,5.30), that is

$$\begin{gathered} \ddot{\mathcal{R}}(\mathtt{t}) + H\mathcal{R}(\mathtt{t}) = \mathcal{J}_0 + \mathtt{t}\,\mathcal{J}_1 \qquad H \text{ as in Eq. (D.1)}\,, \\ \mathcal{R}(0) = \mathcal{R}_0\,, \qquad \dot{\mathcal{R}}(0) = \mathcal{R}_1\,, \end{gathered} \tag{E.20}$$

where the unknown is a function

$$\mathbb{R} \ni \mathtt{t} \mapsto \mathcal{R}(\mathtt{t}) \in \mathfrak{D}\,.$$

The subsequent calculations are purely formal and they assume that the initial data $\mathcal{R}_0, \mathcal{R}_1$ are in suitable spaces to be specified later. Applying $\langle e_1| \;\rangle$ and $\langle e_{i\lambda}| \;\rangle$ to the differential equation in Eq. (E.20) we get (see Eqs. (E.6,E.7) with $\mathcal{R}(\mathtt{t})$ in place of $\chi(\mathtt{t})$ to threat the left hand sides)

$$\left(\frac{d^2}{d\mathtt{t}^2} + \mu_1\right) \langle e_1|\mathcal{R}(\mathtt{t})\rangle = \langle e_1 \,|\, \mathcal{J}_0\rangle + \mathtt{t}\,\langle e_1 \,|\, \mathcal{J}_1\rangle \tag{E.21}$$

and

$$\left(\frac{d^2}{d\mathtt{t}^2} + \lambda\right) \langle e_{i\lambda}|\mathcal{R}(\mathtt{t})\rangle = \langle e_{i\lambda} \,|\, \mathcal{J}_0\rangle + \mathtt{t}\,\langle e_{i\lambda} \,|\, \mathcal{J}_1\rangle \qquad (i = 1, 2\,; \lambda > 0)\,. \tag{E.22}$$

These are two wave equations with source, respectively in the unknowns $y_1(\mathtt{t}) := \langle e_1|\mathcal{R}(\mathtt{t})\rangle$ and $y_{i\lambda}(\mathtt{t}) := \langle e_{i\lambda}|\mathcal{R}(\mathtt{t})\rangle$, and with the initial conditions given by Eq. (E.20), i.e.

$$\begin{cases} y_1(0) = \langle e_1|\mathcal{R}(0)\rangle = \langle e_1|\mathcal{R}_0\rangle\,, \\ \dot{y}_1(0) = \langle e_1|\dot{\mathcal{R}}(0)\rangle = \langle e_1|\mathcal{R}_1\rangle\,, \end{cases} \tag{E.23}$$

$$\begin{cases} y_{i\lambda}(0) = \langle e_{i\lambda}|\mathcal{R}(0)\rangle = \langle e_{i\lambda}|\mathcal{R}_0\rangle & (i = 1, 2\,;\; \lambda > 0)\,, \\ \dot{y}_{i\lambda}(0) = \langle e_{i\lambda}|\dot{\mathcal{R}}(0)\rangle = \langle e_{i\lambda}|\mathcal{R}_1\rangle & (i = 1, 2\,;\; \lambda > 0)\,. \end{cases} \tag{E.24}$$

From the previous initial conditions (and by using Eq. (E.2) with $E = \mu_1 < 0$ and Eq. (E.4) with $E = \lambda > 0$ and the source coefficients $J_0 = \langle e_{i\lambda} \,|\, \mathcal{J}_0\rangle$, $J_1 = \langle e_{i\lambda} \,|\, \mathcal{J}_1\rangle$, $J_2 = 0$), the wave equations (E.6,E.7) have the solutions

$$\langle e_1|\mathcal{R}(\mathtt{t})\rangle = \langle e_1|\mathcal{R}_0\rangle \cosh(|\mu_1|^{1/2}\mathtt{t}) + \langle e_1|\mathcal{R}_1\rangle \frac{\sinh(|\mu_1|^{1/2}\mathtt{t})}{|\mu_1|^{1/2}} \tag{E.25}$$

$$+ \langle e_1|\mathcal{J}_0\rangle \frac{\cosh(|\mu_1|^{1/2}\mathtt{t} - 1}{|\mu_1|} + \langle e_1|\mathcal{J}_1\rangle \frac{\sinh(|\mu_1|^{1/2}\mathtt{t}) - |\mu_1|^{1/2}\mathtt{t}}{|\mu_1|^{3/2}}\,, \tag{E.26}$$

$$\langle e_{i\lambda}|\mathcal{R}(\mathtt{t})\rangle = \langle e_{i\lambda}|\mathcal{R}_0\rangle \cos(\lambda^{1/2}t) + \langle e_{i\lambda}|\mathcal{R}_1\rangle \frac{\sin(\lambda^{1/2}\mathtt{t})}{\lambda^{1/2}} \tag{E.27}$$

$$+ \langle e_{i\lambda}|\mathcal{J}_0\rangle \frac{1 - \cos(\lambda^{1/2}\mathtt{t})}{\lambda} + \langle e_{i\lambda}|\mathcal{J}_1\rangle \frac{\lambda^{1/2}\mathtt{t} - \sin(\lambda^{1/2}\mathtt{t})}{\lambda^{3/2}}\,, \quad (i = 1, 2\,; \lambda > 0)\,; \tag{E.28}$$

the spectral decompositions of the function $\mathcal{R}(\mathtt{t}) \in \mathfrak{H}$ [Proposition 16] implies that the system (E.20) has the solution

$$\mathcal{R}(\mathtt{t}) = \Bigg[\langle e_1|\mathcal{R}_0\rangle \cosh(|\mu_1|^{1/2}\mathtt{t}) + \langle e_1|\mathcal{R}_1\rangle \frac{\sinh(|\mu_1|^{1/2}\mathtt{t})}{|\mu_1|^{1/2}}$$
$$+ \langle e_1|\mathcal{J}_0\rangle \frac{\cosh(|\mu_1|^{1/2}\mathtt{t} - 1}{|\mu_1|} + \langle e_1|\mathcal{J}_1\rangle \frac{\sinh(|\mu_1|^{1/2}\mathtt{t}) - |\mu_1|^{1/2}\mathtt{t}}{|\mu_1|^{3/2}}\Bigg] e_1$$
$$+ \sum_{i=1,2} \int_0^{+\infty} \Bigg[\langle e_{i\lambda}|\mathcal{R}_0\rangle \cos(\lambda^{1/2}t) + \langle e_{i\lambda}|\mathcal{R}_1\rangle \frac{\sin(\lambda^{1/2}\mathtt{t})}{\lambda^{1/2}}$$
$$+ \langle e_{i\lambda}|\mathcal{J}_0\rangle \frac{1 - \cos(\lambda^{1/2}\mathtt{t})}{\lambda} + \langle e_{i\lambda}|\mathcal{J}_1\rangle \frac{\lambda^{1/2}\mathtt{t} - \sin(\lambda^{1/2}\mathtt{t})}{\lambda^{3/2}}\Bigg] e_{i\lambda}\, d\lambda\,, \tag{E.29}$$

thus providing a formal justification for Eq. (5.31). Let us recall that the above expression has to be intended in the sense explained in Remark 100.

Remark 102 For subsequent use, we introduce the function space

$$\mathcal{E}((a,b),\mathbb{K}) := \{f \quad | \quad f, Hf, H^2 f, \ldots \in L^2((a,b),\mathbb{K})\}\,, \qquad \mathbb{K} := \mathbb{R},\mathbb{C}\,, \qquad -\infty \le a < b \le +\infty\,, \tag{E.30}$$

which is a Fréchet space [59] with the countably many norms

$$f \mapsto \big\{\|f\|\,,\ \|Hf\|\,,\ \|H^2 f\|\,,\ \ldots\big\}\,; \tag{E.31}$$

note that the eigenfunctions $e_n \in \mathcal{E}(\mathbb{R},\mathbb{R})$ for each $n \in \mathbb{N}$. By means of some Sobolev imbeddings (see Ref. [59], Theorem 7.25), one shows that

$$\mathcal{E}((a,b),\mathbb{K}) = \{f \in C^\infty((a,b),\mathbb{K}) \quad | \quad f, f_{\mathtt{x}}, f_{\mathtt{xx}}, \ldots \in L^2((a,b),\mathbb{K})\}$$

and that the family of norms in Eq. (E.31) is topologically equivalent to the family of (semi-)norms

$$f \mapsto \big\{\|f\|\,,\ \|f_{\mathtt{x}}\|\,,\ \|f_{\mathtt{xx}}\|\,,\ \ldots\big\}\,.$$

Note that, obviously, $\mathcal{E}(\mathbb{R},\mathbb{R}) \subset \mathfrak{D}$ and $\mathcal{E}(\mathbb{R},\mathbb{R}) \subset C^\infty(\mathbb{R},\mathbb{R})$.

With some effort, one can prove the following proposition [2].

Proposition 18 *Under the assumption*

$$\mathcal{R}_j, \Gamma_j \in C^\infty(\mathbb{R},\mathbb{R}) \quad : \quad \mathcal{R}_j, \mathcal{J}_j \in \mathcal{E}(\mathbb{R},\mathbb{R}) \qquad (j = 0,1)\,,$$

where the $\mathcal{J}_j$'s are defined by Eqs. (5.235.19), the linearized Einstein equations (5.6-5.9), arising from the gauge-dependent linear stability analysis of the Torii-Shinkai wormhole, has a unique solution $(\mathcal{R}(\mathtt{t},\mathtt{x}), \Gamma(\mathtt{t},\mathtt{x}), \Psi(\mathtt{t},\mathtt{x}))$*, defined for every* $(\mathtt{t},\mathtt{x}) \in \mathbb{R}^2$*, such that:*

(a) $\mathcal{R}(\mathtt{t},\mathtt{x}), \Gamma(\mathtt{t},\mathtt{x}), \Psi(\mathtt{t},\mathtt{x}) \in C^\infty(\mathbb{R}^2,\mathbb{R})$*;*

(b) $\mathcal{R}(\mathtt{t},\mathtt{x}) \equiv \mathcal{R}(\mathtt{t}) \in C^\infty(\mathbb{R},\mathcal{E}(\mathbb{R},\mathbb{R}))$*;* [74]

(c) $\mathcal{R}_j, \Gamma_j$ *(*$j = 0, 1$*) are the initial data for* $\mathcal{R}$ *and* Γ*, in the sense of Eq. (5.18);*

(d) Γ, Ψ *can be expressed in terms of the function* $\mathcal{R}$ *via Eqs. (5.17,5.20,5.19);*

moreover, the solution $\mathcal{R}(\mathtt{t},\mathtt{x}) \equiv \mathcal{R}(\mathtt{t})$ *can be decomposed with respect to the generalized orthonormal basis made up of proper and improper eigenfunctions of the operator (D.1) as in Eq. (E.29).*

E.2 Solution of the master equation of the non reflection symmetric Ellis-Bronnikov wormhole

We now deal with the master equation (7.40) in the non reflection symmetric case; hence, throughout this section we stipulate

$$\gamma_1 \neq 0\,.$$

In a similar way to the Torii-Shinkai case, in order to obtain the solution of this equations one has to consider the spectral properties of the operator (D.35) in the Hilbert space (D.34).

Remark 103 In Section D.2 we have shown that, in the non reflection symmetric case $\gamma_1 \neq 0$, the operator in Eq. (D.35) has two simple proper eigenvalues $\mu_1 < 0, \mu_2 = 0$ and a continuous spectrum $\sigma_c(H) = (0, +\infty)$; therefore, applying Theorem 15, one can construct a generalized orthonormal basis of the Hilbert space $\mathfrak{H}$, in the sense of Definition 10 with $N = 2$ and $I = 2$, which is made up of:

(i) two normalized eigenfunction e_1, e_2 for the eigenvalues $\mu_1 < 0$ and $\mu_2 = 0$, i.e.

$$e_1, e_2 \in \mathfrak{D} \quad : \quad He_1 = \mu_1 e_1\,,\ He_2 = 0\,, \quad \|e_1\| = \|e_2\| = 1$$

(e_1, e_2 are proved to be $C^\infty(\mathbb{R})$);

(ii) two suitably chosen and linearly independent "improper eigenfunctions" $e_{i\lambda}$ ($i = 1, 2$) for each improper eigenvalue $\lambda \in \sigma_c(H) = (0, +\infty)$, i.e.,

$$e_{i\lambda} \in C^\infty(\mathbb{R}) \setminus \mathfrak{D} \quad : \quad He_{i\lambda} = \lambda e_{i\lambda} \qquad (i = 1, 2\,; \lambda > 0)\,.$$

[74] This means that each for each $\mathtt{t} \in \mathbb{R}$ the map $\mathcal{R}(\mathtt{t},\cdot) : \mathtt{x} \mapsto \mathcal{R}(\mathtt{t},\mathtt{x})$ is an element of $\mathcal{E}(\mathbb{R},\mathbb{R})$, and that the map $\mathtt{t} \mapsto \mathcal{R}(\mathtt{t},\cdot)$ is C^∞ from $\mathbb{R}$ to $\mathcal{E}(\mathbb{R},\mathbb{R})$.

E.2.1 Solution of the master equation (7.40) with low regularity

Let us consider the equation (7.40) in the non reflection symmetric case $\gamma_1 \neq 0$, with the initial conditions (7.44); after introducing the Hilbertian framework of Remark 71, one can rewrite the master equation as

$$\begin{gathered} \ddot{\chi}(t) + H\chi(t) = 0 \qquad H \text{ as in Eq. (D.35)}\,, \\ \chi(0) = q\,, \qquad \dot{\chi}(0) = p\,; \end{gathered} \tag{E.32}$$

the unknown of the previous Cauchy problem is a function

$$\mathbb{R} \ni t \mapsto \chi(t) \in \mathfrak{D}\,.$$

Analogously to Torii-Shinkai case [Subsection E.1.1], one formally applies $\langle e_1 |\ \rangle$, $\langle e_{i\lambda} |\ \rangle$ and $\langle e_2 |\ \rangle$ to the differential equation in Eq. (E.32) assuming that the initial data q, p are in suitable spaces to be specified later. In this way, one obtains, respectively, two wave equations analogous to Eqs. (E.6,E.7) and the equation

$$0 = \left\langle e_2 \,\middle|\, \frac{d^2}{dt^2}\chi(t) + H\chi(t) \right\rangle = \frac{d^2}{dt^2}\langle e_2 \,|\, \chi(t)\rangle + \langle e_2 \,|\, H\chi(t)\rangle = \frac{d^2}{dt^2}\langle e_2 | \chi(t)\rangle\,; \tag{E.33}$$

the first two equations leads to two solutions similar to the solutions in Eqs. (E.10,E.11), while Eq. (E.33), which can be as well regarded as a wave equation in the unknown $y_2(t) := \langle e_2 \,|\, \chi(t)\rangle$ with the initial conditions

$$\begin{cases} y_2(0) = \langle e_2 | \chi(0)\rangle = \langle e_2 | q\rangle\,, \\ \dot{y}_2(0) = \langle e_2 | \dot{\chi}(0)\rangle = \langle e_2 | p\rangle\,, \end{cases} \tag{E.34}$$

has the trivial solution

$$\langle e_1 | \chi(t)\rangle = \langle e_2 | q\rangle + \langle e_2 | p\rangle t\,. \tag{E.35}$$

Moreover, again as in the reflection symmetric case, one can define two Hilbert structures for the domains $\mathfrak{D}$, $\mathfrak{D}^{1/2}$ of the operators H, $|H|^{1/2}$ as in Remark 91. In this way, by using Proposition 16 for the decomposition of the solution $\chi(t) \in \mathfrak{H}$, one can prove the following

Proposition 19 *Under the assumption*

$$q \in \mathfrak{D}\,, \quad p \in \mathfrak{D}^{1/2}\,,$$

on the initial data, the Cauchy problem (E.32), arising from the gauge-invariant linear stability analysis of the non reflection symmetric Ellis-Bronnikov wormhole, has a unique solution $\chi(t)$ *defined for every* $t \in \mathbb{R}$ *such that*

$$\chi \in C^2(\mathbb{R}, \mathfrak{H}) \cap C^1(\mathbb{R}, \mathfrak{D}^{1/2}) \cap C(\mathbb{R}, \mathfrak{D})\,;$$

moreover the solution $\chi(t)$ *can be decomposed with respect to the generalized orthonormal basis made up of proper and improper eigenfunctions of the operator (D.35) as*

$$\chi(t) = \left[\langle e_1|q\rangle \cosh(|\mu_1|^{1/2}t) + \langle e_1|p\rangle \frac{\sinh(|\mu_1|^{1/2}t)}{|\mu_1|^{1/2}}\right] e_1 + \Big[\langle e_2|q\rangle + \langle e_2|p\rangle t\Big] e_2 + \sum_{i=1}^{2} \int_0^{+\infty} \left[\langle e_{i\lambda}|q\rangle \cos(\lambda^{1/2}t) + \langle e_{i\lambda}|p\rangle \frac{\sin(\lambda^{1/2}t)}{\lambda^{1/2}}\right] e_{i\lambda}\, d\lambda\,. \quad \text{(E.36)}$$

The latter proposition provides a justification of Eq. (7.46). Let us finally underline the fact that, as explained in Remark 100, the above expression has to be suitably intended.

E.3 Solution of the master equations of the AdS wormhole

Let us consider the master equations (5.66,7.54); analogously to the previous examples, the solution of these equations can be reduced to the spectral analysis of the operator (D.40) in the Hilbert space (D.41).

Remark 104 In Section D.3 we have shown that the operator in Eq. (D.40) has a purely discrete spectrum $\{\mu_n\}_{n\in\mathbb{N}}$ with a single negative eigenvalue μ_1 and an increasing sequence of positive eigenvalues $\mu_2 < \mu_3 < \dots$; hence, the spectral theorem 14, tells us that one can get an orthonormal basis of the Hilbert space $\mathfrak{H}$ [Definition 9] using the normalized eigenfunctions $\{e_n\}_{n\in\mathbb{N}}$ of H, i.e.

$$e_n \in \mathfrak{D} \quad : \quad \|e_n\| = 1\,, \quad He_n = \mu_n e_n \qquad (n \in \mathbb{N})$$

(e_n is proved to be $C^\infty(\mathbb{R})$ for all $n \in \mathbb{N}$).

E.3.1 Solution of the master equation (7.54) with low regularity

Let us consider the master equation as written in Eq. (7.54) with the initial conditions given therein [Eq. 7.56], in the suitable Hilbertian framework introduced in Remark 48, that is

$$\begin{aligned} \ddot{\chi}(s) + H\chi(s) = 0 \qquad & H \text{ as in Eq. (D.40)} \\ \chi(0) = q\,, \qquad & \dot{\chi}(0) = p\,; \end{aligned} \tag{E.37}$$

the unknown of this system is a function

$$\mathbb{R} \ni s \mapsto \chi(s) \in \mathfrak{D}\,.$$

As in the previous cases, the spaces containing the data q, p have to be specified and, in the meanwhile, we proceed formally: applying $\langle e_n | \ \rangle$ to the differential equation in Eq. (E.37) we get

$$\begin{aligned} 0 = \left\langle e_n \,\middle|\, \frac{d^2}{ds^2}\chi(s) + H\chi(s) \right\rangle &= \frac{d^2}{ds^2}\langle e_n \,|\, \chi(s)\rangle + \langle e_n H\chi(s)\rangle \\ &= \left(\frac{d^2}{ds^2} + \mu_n\right)\langle e_n|\chi(s)\rangle \qquad (n \in \mathbb{N})\,, \end{aligned} \tag{E.38}$$

which are a countable set of wave equations in the unknowns $y_n(s) := \langle e_n \,|\, \chi(s)\rangle$. Taking into account the initial conditions

$$\begin{cases} y_n(0) = \langle e_n|\chi(0)\rangle = \langle e_n|q\rangle & (n \in \mathbb{N})\,, \\ \dot{y}_n(0) = \langle e_n|\dot{\chi}(0)\rangle = \langle e_n|p\rangle & (n \in \mathbb{N})\,, \end{cases} \tag{E.39}$$

and the fact that $\mu_1 < 0 < \mu_2 < \mu_3 < \ldots$ (and by using Eq. (E.2) with $E = \mu_1 < 0$ and Eq. (E.4) with $E = \mu_n > 0$ ($n \geq 2$), both with vanishing source coefficients $J_i = 0$ for $i = 0, 1, 2$), we conclude that

$$\begin{aligned} \langle e_1|\chi(s)\rangle &= \langle e_1|q\rangle \cosh(|\mu_1|^{1/2}s) + \langle e_1|p\rangle \frac{\sinh(|\mu_1|^{1/2}s)}{|\mu_1|^{1/2}}\,, \\ \langle e_n|\chi(s)\rangle &= \langle e_n|q\rangle \cos(\mu_n^{1/2}s) + \langle e_n|p\rangle \frac{\sin(\mu_n^{1/2}s)}{\mu_n^{1/2}} \qquad (n = 2, 3, \ldots)\,. \end{aligned}$$

Thus, having defined the operator $|H|^{1/2}$ and the corresponding domain $D^{1/2}$ as in Remark 89 and recalling the spectral decompositions of the function $\chi(s) \in \mathfrak{H}$ [Proposition 15], one can see (with a little effort) that the following proposition holds:

Proposition 20 *Under the assumption*

$$q \in \mathfrak{D}\,, \quad p \in \mathfrak{D}^{1/2}\,,$$

on the initial data, the Cauchy problem (E.37), arising from the gauge-invariant linear stability analysis of the AdS wormhole, has a unique solution $\chi(s)$ *defined for every* $s \in \mathbb{R}$ *such that*

$$\chi \in C^2(\mathbb{R}, \mathfrak{H}) \cap C^1(\mathbb{R}, \mathfrak{D}^{1/2}) \cap C(\mathbb{R}, \mathfrak{D})\,;$$

moreover the solution $\chi(s)$ *can be decomposed with respect to the orthonormal basis made up of the normalized eigenfunctions of the operator (D.40) as*

$$\chi(s) = \left[\langle e_1|q\rangle \cosh(|\mu_1|^{1/2}s) + \langle e_1|p\rangle \frac{\sinh(|\mu_1|^{1/2}s)}{|\mu_1|^{1/2}}\right] e_1 + \sum_{n=2}^{+\infty} \left[\langle e_n|q\rangle \cos(\mu_n^{1/2}s) + \langle e_n|p\rangle \frac{\sin(\mu_n^{1/2}s)}{\mu_n^{1/2}}\right] e_n\,. \quad \text{(E.40)}$$

Note that the previous expression provides a justification of Eq. (7.58) and that, unlike in the case of the Torii-Shinkai and the Ellis-Bronnikov wormholes, the inner products $\langle\cdot|\cdot\rangle$ appearing in the decomposition on the solution of the master equation are the usual inner product in $\mathfrak{H}$.

E.3.2 Solution of the master equation (5.66) with high regularity

Let us keep all notations of Section (5.2). Therein, we have shown that if $\Gamma(s,u), \mathcal{R}(s,u), \Psi(s,u) \in C^\infty(\mathbb{R} \times (-\pi,\pi), \mathbb{R})$ are solutions of the linearized Einstein equations (5.43-5.46), then $\Gamma(s,u), \Psi(s,u)$ are given by Eqs. (5.53,5.56) where $\mathcal{R}(s,u)$ satisfies the master equation (5.59); note that Eqs. (5.53,5.56,5.59) involve the initial data $\Gamma_i, \mathcal{R}_i, \Psi_0$ [Eqs. (5.54,5.57)] through Eqs. (5.55,5.58,5.61-5.63).
In Remark 48, we have introduced an Hilbertian framework for the master equation (5.59) with the initial condition (5.54), yielding the system (5.66,5.70), that is

$$\begin{gathered}\ddot{\mathcal{R}}(s) + H\mathcal{R}(s) = \mathcal{J}_0 + s\,\mathcal{J}_1 + s^2\,\mathcal{J}_2 \qquad H \text{ as in Eq. (D.40)}\,,\\ \mathcal{R}(0) = \mathcal{R}_0\,, \qquad \dot{\mathcal{R}}(0) = \mathcal{R}_1\,,\end{gathered} \quad \text{(E.41)}$$

where the unknown is a function

$$\mathbb{R} \ni s \mapsto \mathcal{R}(s) \in \mathfrak{D}\,.$$

We now make some computations that are purely formal as they assume that the initial data $\mathcal{R}_0, \mathcal{R}_1$ are in suitable spaces that have to be recovered a posteriori. For every $n \in \mathbb{N}$, let us apply $\langle e_n | \ \rangle$ to the differential equation in Eq. (E.41); this yields (see Eq. (E.38) with $\mathcal{R}(s)$ in place of $\chi(s)$ to threat the left hand sides)

$$\left(\frac{d^2}{ds^2} + \mu_n\right) \langle e_n | \mathcal{R}(s) \rangle = \langle e_n \,|\, \mathcal{J}_0 \rangle + s \, \langle e_1 \,|\, \mathcal{J}_1 \rangle + s^2 \, \langle e_n \,|\, \mathcal{J}_2 \rangle \qquad (n \in \mathbb{N}) \,. \tag{E.42}$$

These are a countable set of wave equations with source, in the unknowns $y_n(s) := \langle e_n | \mathcal{R}(s) \rangle$ and with the initial conditions given by Eq. (E.41), namely

$$\begin{cases} y_n(0) = \langle e_n | \mathcal{R}(0) \rangle = \langle e_n | \mathcal{R}_0 \rangle & (n \in \mathbb{N}) \,, \\ \dot{y}_n(0) = \langle e_n | \dot{\mathcal{R}}(0) \rangle = \langle e_n | \mathcal{R}_1 \rangle & (n \in \mathbb{N}) \,. \end{cases} \tag{E.43}$$

From the previous initial conditions and from Eq. (E.2) with $E = \mu_1 < 0$ and Eq. (E.4) with $E = \mu_n > 0$ ($n \geq 2$), both with source coefficients $J_0 = \langle e_n \,|\, \mathcal{J}_0 \rangle$, $J_1 = \langle e_n \,|\, \mathcal{J}_1 \rangle$, $J_2 = \langle e_n \,|\, \mathcal{J}_2 \rangle$, it turns out that the wave equations (E.42) have the solutions

$$\begin{aligned} \langle e_1 | \mathcal{R}(s) \rangle =& \langle e_1 | \mathcal{R}_0 \rangle \cosh(|\mu_1|^{1/2} s) + \langle e_1 | \mathcal{R}_1 \rangle \frac{\sinh(|\mu_1|^{1/2} s)}{|\mu_1|^{1/2}} \\ &+ \langle e_1 \,|\, \mathcal{J}_0 \rangle \, \frac{\cosh(|\mu_1|^{1/2} s) - 1}{|\mu_1|} + \langle e_1 \,|\, \mathcal{J}_1 \rangle \, \frac{\sinh(|\mu_1|^{1/2} s) - |\mu_1|^{1/2} s}{|\mu_1|^{3/2}} \\ &+ \langle e_1 \,|\, \mathcal{J}_2 \rangle \, \frac{2\cosh(|\mu_1|^{1/2} s) - |\mu_1| s^2 - 2}{|\mu_1|^2} \,, \\ \langle e_n | \mathcal{R}(s) \rangle =& \langle e_n | \mathcal{R}_0 \rangle \cos(\mu_n^{1/2} s) + \langle e_n | \mathcal{R}_1 \rangle \frac{\sin(\mu_n^{1/2} s)}{\mu_n^{1/2}} \\ &+ \langle e_n \,|\, \mathcal{J}_0 \rangle \, \frac{1 - \cos(\mu_n^{1/2} s)}{\mu_n} + \langle e_n \,|\, \mathcal{J}_1 \rangle \, \frac{\mu_n^{1/2} s - \sin(\mu_n^{1/2} s)}{\mu_n^{3/2}} \\ &+ \langle e_n \,|\, \mathcal{J}_2 \rangle \, \frac{2\cos(\mu_n^{1/2} s) + \mu_n s^2 - 2}{\mu_n^2} \qquad (n = 2, 3, \ldots) \,; \end{aligned}$$

the spectral decompositions of the function $\mathcal{R}(s) \in \mathfrak{H}$ [Proposition 15] implies that the system (E.41) has the solution

$$\mathcal{R}(s) = \Bigg[\langle e_1|\mathcal{R}_0\rangle \cosh(|\mu_1|^{1/2}s) + \langle e_1|\mathcal{R}_1\rangle \frac{\sinh(|\mu_1|^{1/2}s)}{|\mu_1|^{1/2}} \tag{E.44}$$

$$+ \langle e_1 \,|\, \mathcal{J}_0\rangle \, \frac{\cosh(|\mu_1|^{1/2}s) - 1}{|\mu_1|} + \langle e_1 \,|\, \mathcal{J}_1\rangle \, \frac{\sinh(|\mu_1|^{1/2}s) - |\mu_1|^{1/2}s}{|\mu_1|^{3/2}} \tag{E.45}$$

$$+ \langle e_1 \,|\, \mathcal{J}_2\rangle \, \frac{2\cosh(|\mu_1|^{1/2}s) - |\mu_1|s^2 - 2}{|\mu_1|^2}\Bigg] e_1 \tag{E.46}$$

$$+ \sum_{n=2}^{+\infty} \Bigg[\langle e_n|q\rangle \cos(\mu_n^{1/2}s) + \langle e_n|p\rangle \frac{\sin(\mu_n^{1/2}s)}{\mu_n^{1/2}} \tag{E.47}$$

$$+ \langle e_n \,|\, \mathcal{J}_0\rangle \, \frac{1 - \cos(\mu_n^{1/2}s)}{\mu_n} + \langle e_n \,|\, \mathcal{J}_1\rangle \, \frac{\mu_n^{1/2}s - \sin(\mu_n^{1/2}s)}{\mu_n^{3/2}} \tag{E.48}$$

$$+ \langle e_n \,|\, \mathcal{J}_2\rangle \, \frac{2\cos(\mu_n^{1/2}s) + \mu_n s^2 - 2}{\mu_n^2}\Bigg] e_n \,, \tag{E.49}$$

thus providing a formal justification for Eq. (5.71).

Remark 105 Consider the function space $\mathcal{E}((a,b),\mathbb{K})$ defined in Eq. (E.30) with the operator H as in Eq. (D.40); then one can see that $e_n \in \mathcal{E}((-\pi,\pi),\mathbb{R})$ for each $n \in \mathbb{N}$ and that, obviously, $\mathcal{E}((-\pi,\pi),\mathbb{R}) \subset \mathfrak{D}$ and $\mathcal{E}((-\pi,\pi),\mathbb{R}) \subset C^\infty((-\pi,\pi),\mathbb{R})$.

With some effort one can prove the following proposition (which is analogue to Proposition 18).

Proposition 21 *Under the assumption*

$$\Psi_0 \in \mathbb{R}\,, \qquad \mathcal{R}_j, \Gamma_j \in C^\infty((-\pi,\pi),\mathbb{R}) \;\; : \tag{E.50}$$

$$\mathcal{R}_j, \mathcal{J}_i \in \mathcal{E}((-\pi,\pi),\mathbb{R}) \qquad (i, = 0,1,2\,,;j = 0,1) \tag{E.51}$$

where the $\mathcal{J}_i$*'s are defined by Eqs. (5.61-5.63,5.55,5.58) the linearized Einstein equations (5.43-5.46), arising from the gauge-dependent linear stability analysis of the AdS wormhole, has a unique solution* $(\mathcal{R}(s,u), \Gamma(s,u), \Psi(s,u))$*, defined for every* $(s,u) \in \mathbb{R} \times (-\pi,\pi)$*, such that:*

(a) $\mathcal{R}(s,u), \Gamma(s,u), \Psi(s,u) \in C^\infty(\mathbb{R} \times (-\pi,\pi),\mathbb{R})$*;*

(b) $\mathcal{R}(s,u) \equiv \mathcal{R}(u) \in C^\infty(\mathbb{R}, \mathcal{E}((-\pi,\pi),\mathbb{R}))$*;* [75]

[75] This means that each for each $s \in \mathbb{R}$ the map $\mathcal{R}(s,\cdot) : u \mapsto \mathcal{R}(s,u)$ is an element of $\mathcal{E}((-\pi,\pi),\mathbb{R})$, and that the map $s \mapsto \mathcal{R}(s,\cdot)$ is C^∞ from $\mathbb{R}$ to $\mathcal{E}((-\pi,\pi),\mathbb{R})$.

(c) $\Psi_0, \mathcal{R}_j, \Gamma_j$ *($j = 0, 1$) are the initial data for* Ψ, $\mathcal{R}$, Γ, *in the sense of Eqs. (5.57,5.54);*

(d) Γ, Ψ *can be expressed in terms of the function* $\mathcal{R}$ *via Eqs. (5.53,5.56, 5.55,5.58);*

moreover, the solution $\mathcal{R}(s, u) \equiv \mathcal{R}(s)$ *can be decomposed with respect to the orthonormal basis made up of the normalized eigenfunctions of the operator (D.40) as in Eq. (E.49).*

E.4 Solution of the master equations of the dS wormhole

In this section we deal with the master equation (7.61); we will show that in this case the solution of the master equation can be reduced to the spectral analysis of the operator (D.53) in the Hilbert space (D.52).

Remark 106 In Section D.4 we have stated that the spectrum $\sigma(H)$ the operator in Eq. (D.53) is made up of a simple proper eigenvalue $\mu_1 < 0$ and a continuous spectrum $\sigma_c(H) = [0, +\infty)$; hence the spectral theorem 15 ensures that one can construct a generalized orthonormal basis of the Hilbert space $\mathfrak{H}$, in the sense of Definition 10 with $N = 1$ and $I = 2$, which is made up of:

(i) a normalized eigenfunction e_1 for the eigenvalue $\mu_1 < 0$, i.e.

$$e_1 \in \mathfrak{D} \quad : \quad \|e_1\| = 1 \,, \quad He_1 = \mu_1 e_1$$

(e_1 is proved to be $C^\infty(\mathbb{R})$);

(ii) two suitably chosen and linearly independent "improper eigenfunctions" $e_{i\lambda}$ ($i = 1, 2$) for each $\lambda \in \mathring{\sigma}_c(H) = (0, +\infty)$, i.e.,

$$e_{i\lambda} \in C^\infty(\mathbb{R}) \setminus \mathfrak{D} \quad : \quad He_{i\lambda} = \lambda e_{i\lambda} \qquad (i = 1, 2 \,; \lambda > 0) \,.$$

E.4.1 Solution of the master equation (7.61) with low regularity

Let us consider Eq. (7.61), with the initial conditions (7.65), in the suitable Hilbertian framework introduced in Remark 79, namely

$$\begin{gathered} \ddot{\chi}(s) + H\chi(s) = 0 \qquad H \text{ as in Eq. (D.53)}\,, \\ \chi(0) = q\,, \qquad \dot{\chi}(0) = p\,; \end{gathered} \tag{E.52}$$

the unknown of the previous Cauchy problem is a function

$$\mathbb{R} \ni s \mapsto \chi(s) \in \mathfrak{D}\,.$$

The situation is exactly analogue to that of the Torii-Shinkai case [see Subsection E.1.1]: defining the Hilbert space structures for the domains $\mathfrak{D}$, $\mathfrak{D}^{1/2}$ of the operators H, $|H|^{1/2}$ as in Remark 91, and using the spectral decompositions of the function $\chi(s) \in \mathfrak{H}$ [Proposition 16], one can prove the following

Proposition 22 *Under the assumption*

$$q \in \mathfrak{D}\,, \quad p \in \mathfrak{D}^{1/2}\,,$$

on the initial data, the Cauchy problem (E.52), arising from the gauge-invariant linear stability analysis of the dS wormhole, has a unique solution $\chi(s)$ *defined for every* $s \in \mathbb{R}$ *such that*

$$\chi \in C^2(\mathbb{R}, \mathfrak{H}) \cap C^1(\mathbb{R}, \mathfrak{D}^{1/2}) \cap C(\mathbb{R}, \mathfrak{D})\,;$$

moreover the solution $\chi(s)$ *can be decomposed with respect to the generalized orthonormal basis made up of proper and improper eigenfunctions of the operator (D.53) as*

$$\begin{aligned}\chi(s) = &\left[\langle e_1|q\rangle \cosh(|\mu_1|^{1/2}s) + \langle e_1|p\rangle \frac{\sinh(|\mu_1|^{1/2}s)}{|\mu_1|^{1/2}}\right] e_1 \\ &+ \sum_{i=1}^{2} \int_0^{+\infty} \left[\langle e_{i\lambda}|q\rangle \cos(\lambda^{1/2}s) + \langle e_{i\lambda}|p\rangle \frac{\sin(\lambda^{1/2}s)}{\lambda^{1/2}}\right] e_{i\lambda}\, d\lambda\,. \quad \text{(E.53)}\end{aligned}$$

The latter proposition provides a justification of Eq. (7.67); let us finally underline the fact that, as explained in Remark 100, the symbols $\langle\cdot|\cdot\rangle$ and the integrals over λ have to be suitably intended.

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