Non uniqueness of non runaway solutions of Abraham–Lorentz–Dirac equation in an external laser pulse

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Abstract

In the paper [6] it was shown that, for motions on a line under the action of a potential barrier, the third-order Abraham–Lorentz–Dirac equation presents the phenomenon of non uniqueness of non runaway solutions. Namely, at least for a sufficiently steep barrier, the physical solutions of the equation are not determined by the "mechanical state" of position and velocity, and knowledge of the initial acceleration too is required. Due to recent experiments, both in course and planned, on the interactions between strong laser pulses and ultra relativistic electrons, it becomes interesting to establish whether such a non uniqueness phenomenon extends to the latter case, and for which ranges of the parameters. In the present work we will consider just the simplest model, i.e., the case of an electromagnetic plane wave, and moreover the Abraham–Lorentz–Dirac equation will be dealt with in the non relativistic approximation. The result we found is that the non uniqueness phenomenon occurs if, at a given frequency of the incoming wave, the field intensity is sufficiently large. An analytic estimate of such a threshold is also given. At the moment it is unclear whether such a phenomenon applies also in the full relativistic case, which is the one of physical interest.

Keywords: radiation reaction, Abraham–Lorentz–Dirac equation, non uniqueness.

1 Introduction

The effects of radiation reaction acting on a accelerating charge was shown to be relevant in a series of recent experiments of interactions between beam of

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ultra relativistic electrons with strong laser pulses (see [7], [17]). Abraham in ref. [1] and Lorentz in ref. [15, 16] (for the later Dirac relativistic version see ref. [9]), proposed the following equation in order to describe the motion of a radiating electron:

$$m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}) + \frac{2e^2}{3c^3}\dddot{\mathbf{x}} \ , \tag{1}$$

where m, e and c are the mass of the electron, its charge and the speed of light respectively, while $\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$ is the Lorentz force due to an incoming electromagnetic wave. Notice that, by considering quantum models of charged particles interacting with the quantized electromagnetic field, taking the electrons described in the non relativistic approximation, the expectation value of the electron position operator (in the single particle state) satisfy equation (1) in a suitable limit (see [18] or the more recent [4, 3]).

As it is well known, the solution of the equation (1) for generic initial data diverge, for $t \to +\infty$, as $\exp(t/\varepsilon)$, where we have defined the characteristic time $\varepsilon \stackrel{\text{def}}{=} 2e^2/3mc^3$. So, they are physically absurd, since they keep continuing to accelerate also if the force (i.e. the electromagnetic pulse) vanishes. To overcome such a difficulty some proposal were advanced. Dirac himself in [9] proposed to consider only solutions whose acceleration $\ddot{\mathbf{x}}$ vanishes for $t \to +\infty$. From the mathematical point of view, one has then to deal no more with a Cauchy problem, for which existence and uniqueness of solutions are granted, but with a boundary value problem, in which are given the mechanical data of position and velocity at $-\infty$, and the acceleration at $+\infty$. For boundary problems uniqueness is not granted, i.e., having fixed the mechanical data before the interaction, there might exist several solutions which satisfy the non runaway condition $\ddot{\mathbf{x}} \to 0$.

The proposal of Dirac led to interesting developments: as the non runaway solutions form a sub manifold in the phase space of the problem, a proposal was to find second order equations restricted to the non runaway manifold whose solutions would be the non runaway solutions of the Abraham–Lorentz–Dirac equation (see [20], [10]). An other approach was followed in papers [11, 2], in which the equation of motion are formulated in terms of integral–differential equations without runaway solutions.

There exist a more drastic approach in which the Abraham–Lorentz–Dirac equations are replaced by second order differential equation of motion without runaway, the so called Sokolov equation (see [19, 5]).

A more pragmatic attitude can be taken. Following Landau–Lifschitz (see ref. [14], or the more recent ref. [8]), due to the smallness of the radiation reaction, one can think to approximate the term $\dddot{\mathbf{x}}$ with $\dot{\mathbf{F}}/m$ ending up with a second order equation, the so called Landau–Lifschitz (more on the derivation will be said in Section 5). On the other hand such an equation can be obtained also by taking the classical limit of the first perturbative order of Quantum Electrodynamics (QED) (see for example [13, 12]).

The Landau–Lifschitz approximation was very recently tested in some experiments of interaction between a beam of ultra relativistic electrons with strong laser pulses. The agreement between theoretical prediction and experimental data was not completely satisfactory (see ref. [7, 17]). In particular, it seems that the Landau–Lifschitz approximation overestimate the energy emitted by the scattered electrons.

One might think that the use of the original equation (1) could give better agreement. Some physical reasons will be given in Section 5.

In any case, approximations, such as the Landau–Lifschitz one, which admit just one solution for given initial data of position and velocity, will be a poor one in a regime in which Abraham–Lorentz–Dirac equation admit more then one non runaway ones. So, if such a regime exists, the difference between the two equations would became apparent. This show the importance to understand whether, and eventually in what regime, the non uniqueness of the non runaway solutions shows up for the Abraham–Lorentz–Dirac equation.

In this paper we investigate such a problem for the non relativistic version (1) of the Abraham–Lorentz–Dirac equation, in the case of an incoming electromagnetic plane wave. We will show through numerical computations that there exists a threshold in the intensity of the field, above which non uniqueness occurs. Some numerical checks are also performed, to control whether below threshold the Landau–Lifschitz approximation is sound. It seems that, also well below the threshold, the two equations lead to very different behaviors, because the electron energy loss, computed according the Abraham–Lorentz–Dirac equation, appears much smaller than the one computed using the Landau–Lifschitz approximation.

The paper is organized as follows. In Section 2 we describe the model studied, while in Section 3 we give an analytic estimate of the region of parameters in which non uniqueness is expected to occur. In Section 4 we illustrate the numerical results, and in Section 5 a comparison with the Landau-Lifschitz approximation is given. The conclusions follow.

2 The model

We consider the case of the interaction of an electron, described by the Abraham–Lorentz–Dirac equation, with an external electromagnetic linearly polarized plane wave. We will take the x axis as the direction of the wave propagation, the y axis as the direction of electric field and finally the z axis as the direction of the magnetic field. In the Coulomb gauge, the scalar potential vanishes, while the vector potential $\mathbf{A}$ takes the form $\mathbf{A} = (0, F(x-ct), 0)$, being $F(x-ct)$ an arbitrary function, and c the speed of light. To be definite, we model the electromagnetic pulse by choosing $F(\xi) = A\exp(-\xi^2/2\sigma)\cos(k\xi)$, although every choice with F vanishing suf-

ficiently fast at infinity would give the same qualitative results. So, the electromagnetic field takes the form:

$$\begin{cases} \mathbf{E}(\mathbf{r},t) = -F'(x-ct)\,\hat{\mathbf{e}}_y \\ \mathbf{B}(\mathbf{r},t) = \quad F'(x-ct)\,\hat{\mathbf{e}}_z \ , \end{cases}$$

where $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z$ are unit vectors directed as the y and z axis respectively, while F' denotes the derivative of F with respect to its argument. Denoting by $\mathbf{x}(t) = \big(x(t), y(t), z(t)\big)$ the electron trajectory, the Abraham–Lorentz–Dirac equation takes the form:

$$\begin{cases} m\ddot{x} = \dfrac{e}{c}F'(x-ct)\dot{y} + m\varepsilon\dddot{x} \\ m\ddot{y} = eF'(x-ct) - \dfrac{e}{c}\dot{x}F'(x-ct) + m\varepsilon\dddot{y} \\ m\ddot{z} = m\varepsilon\dddot{z} \ , \end{cases}$$

where, we recall, ε denotes the constant $2e^2/3mc^3$. Notice that the equation for z decouples, and that the only non runaway solutions are $z(t) = z_0 + v_z t$, i.e., uniform motions. From now on, we consider just the first two equations, which, by defining $\xi \stackrel{\text{def}}{=} x - ct$, can be put in the following form:

$$\begin{cases} \ddot{\xi} = \dfrac{e}{mc}F'(\xi)\dot{y} + \varepsilon\dddot{\xi} \\ \ddot{y} = -\dfrac{e}{mc}F'(\xi)\dot{\xi} + \varepsilon\dddot{y} \ . \end{cases} \tag{2}$$

The phase space corresponding to such an equation is six–dimensional, but the system can be reduced to a four dimensional one exploiting the invariance by translation along the y–axis. In fact, the second equation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{y} - \frac{e}{mc}F(\xi) - \varepsilon\ddot{y}\right) = 0 \ ,$$

i.e., the Abraham–Lorentz–Dirac equation reduces to

$$\begin{cases} \ddot{\xi} = \dfrac{e}{mc}F'(\xi)\dot{y} + \varepsilon\dddot{\xi} \\ \dot{y} = -\dfrac{e}{mc}F(\xi) + \varepsilon\ddot{y} + C \ , \end{cases}$$

where C is an integration constant which depends on the initial data. We can include the constant C in the potential, thus defining the "effective potential" $F_C(\xi) = F(\xi) + C$, and introduce the new variable $v \stackrel{\text{def}}{=} \dot{y}$: in such a way, one gets the equation

$$\begin{cases} \dddot{\xi} = \dfrac{1}{\varepsilon}\left(\ddot{\xi} - \dfrac{e}{mc}F_C'(\xi)v\right) \\ \dot{v} = \dfrac{1}{\varepsilon}\left(v + \dfrac{e}{mc}F_C(\xi)\right) \ , \end{cases} \tag{3}$$

i.e., an equation in a four–dimensional phase space.

To discuss the solution of this equation, consider first the "mechanical case" $\varepsilon = 0$, i.e., the case in which emission is neglected. So one gets

$$\begin{cases} \ddot{\xi} = \dfrac{e}{mc} F'_C(\xi) v \\ v = -\dfrac{e}{mc} F_C(\xi) \ , \end{cases}$$

which reduces to the one dimensional Newton's equation

$$\ddot{\xi} = -\frac{e^2}{m^2 c^2} F'_C(\xi) F_C(\xi) \ ,$$

with a potential $V_C(\xi) = e^2 F_C^2(\xi)/2m^2c^2$. The solutions are readily found. In particular, for motions of scattering type, if the initial "kinetic energy" $\dot{\xi}^2/2$ is larger then the maximum of $V_C(\xi)$, the electron will pass the barrier, while it will be reflected if the initial kinetic energy will be smaller. In addition, it is easily checked that the zero of $F_C(\xi)$ gives stable equilibrium points, while maxima of the modulus $|F_C(\xi)|$ will give unstable equilibrium. Return now to the full Abraham–Lorentz–Dirac equation (3). As recalled in the introduction, we look for "exceptional" initial data which correspond to solutions having an asymptotically vanishing acceleration. In other terms, given the initial value ξ_0 and $\dot{\xi}_0$, we want to find whether initial data v_0 and $\ddot{\xi}_0$ exist such that the corresponding solutions of (3) are non runaway. Since we are considering a scattering problem, this can be implemented in a straightforward way by numerically integrating backward in time the equations of motion. In other terms, one fixes the final data outside the interaction zone and integrates backwards in time: in such a way the Dirac manifold (the subset of phase space spanned by the non runaway solutions) becomes an attractor, and after a small transient the orbit practically will lie on such a manifold. Once the electron did come back into the non interacting zone, one gets the initial data which gives rise to a non runaway solution. Numerical evidence suggests that, if the electromagnetic field $F(\xi)$ is "strong" enough, then, having fixed the mechanical data ξ_0 and $\dot{\xi}_0$, there exist several initial v_0 and $\ddot{\xi}_0$ which give rise to non runaway different trajectories. In geometric terms, the Dirac manifold is folded. Such non uniqueness phenomenon will be discussed in the next Section.

3 The non uniqueness phenomenon

Following ref. [6], in order to discover whether there exist several non runaway solutions corresponding to the same initial mechanical state, we start investigating the unstable equilibrium point. By rescaling time by $t \to \varepsilon t$,

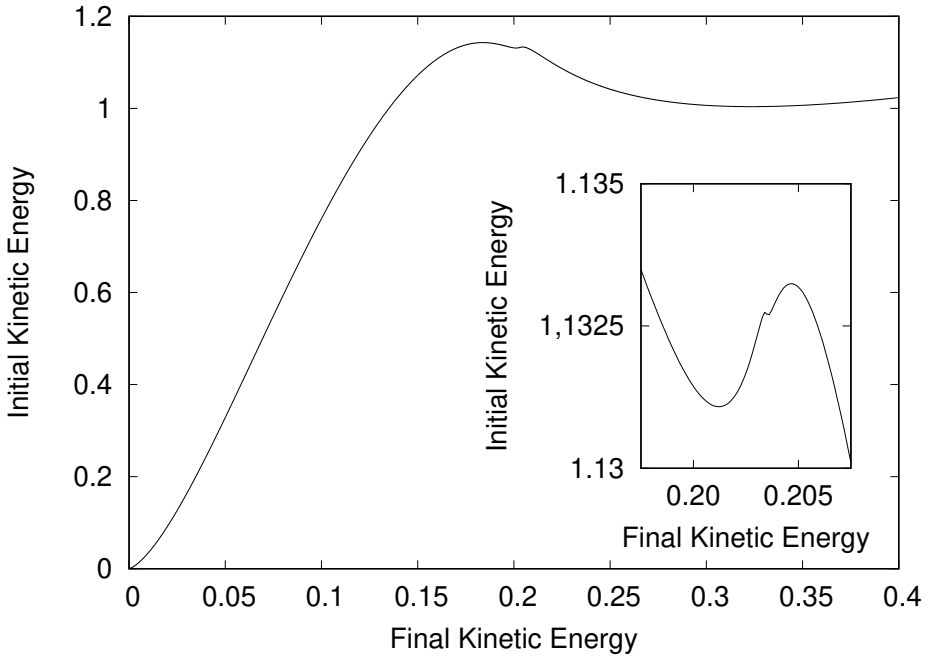

Figure 1: Plot of the initial kinetic energy vs. the final one for field amplitude $A = 1$, and vanishing wave vector. The map is not one to one, and this implies non uniqueness of the non runaway solutions. Indeed, drawing a horizontal line at energy about 1.13, one immediately checks that to a given initial energy there correspond different final ones. The inset hints at the complex structure of the maxima and minima of such a curve.

the equations (3) becomes

$$\begin{cases} \dddot{\xi} = \dfrac{e\varepsilon^2}{mc} F_C'(\xi) v + \ddot{\xi} \\ \dot{v} = -\dfrac{e}{mc} F_C(\xi) + v \ . \end{cases} \tag{4}$$

The equilibrium points of such an equation can be subdivided into two classes:

- The point(s) $v = 0, \xi = \xi^*$ with $F_C(\xi^*) = 0$ (and obviously $\dot{\xi} = \ddot{\xi} = 0$). Such points corresponds to the stable equilibrium points of the mechanical case, and are not interesting for the scattering states. In fact, the non runaway solutions are the ones which fall on the equilibrium point, and thus they do not describe scattering states.
- Points $v = v^*, \xi = \xi^*$ with $F_C'(\xi^*) = 0$ and $v^* = \frac{e}{mc} F_C(\xi^*)$.

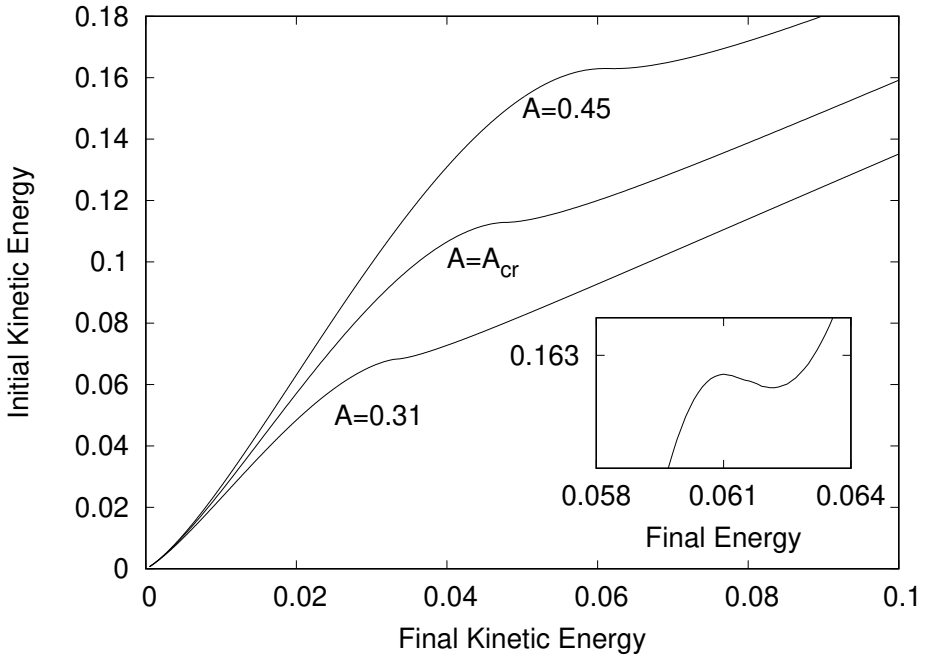

Figure 2: Plot of the initial kinetic energy vs. the final one for three field amplitudes: $A = 0.31$ smaller than $A_{cr} \simeq 0.38$, the critical one and $A = 0.45$ larger then the critical one. The wave vector vanishes. One sees that for $A = 0.45$ there exists a very weak local maximum (see the inset), which entails the non uniqueness of the non runaway solutions.

We consider only equilibrium points of the second type, more precisely we considers points such that ξ^* is a maximum for $F_C^2(\xi)$. It turns out that, as the parameters are changed, such points exhibit a bifurcation from a saddle to a saddle-focus, the same which drives the non uniqueness phenomenon in the one dimensional case (see ref. [6]). In fact, putting $\chi = \xi - \xi^*$ and $u = v - v^*$, the equation (4) to the first order becomes

$$\begin{cases} \dddot{\chi} = -k^2\chi + \ddot{\chi} \\ \dot{u} = u \ , \end{cases} \tag{5}$$

were we defined

$$k^2 \stackrel{\text{def}}{=} \frac{e^2\varepsilon^2}{m^2c^2} F_C''(\xi^*)F_C(\xi^*) \ . \tag{6}$$

So the linearized equations decouple: the second one defines a direction which is always unstable, while the first one is the same one just studied for the one-dimensional case in ref. [6]. As shown in the quoted paper, there is

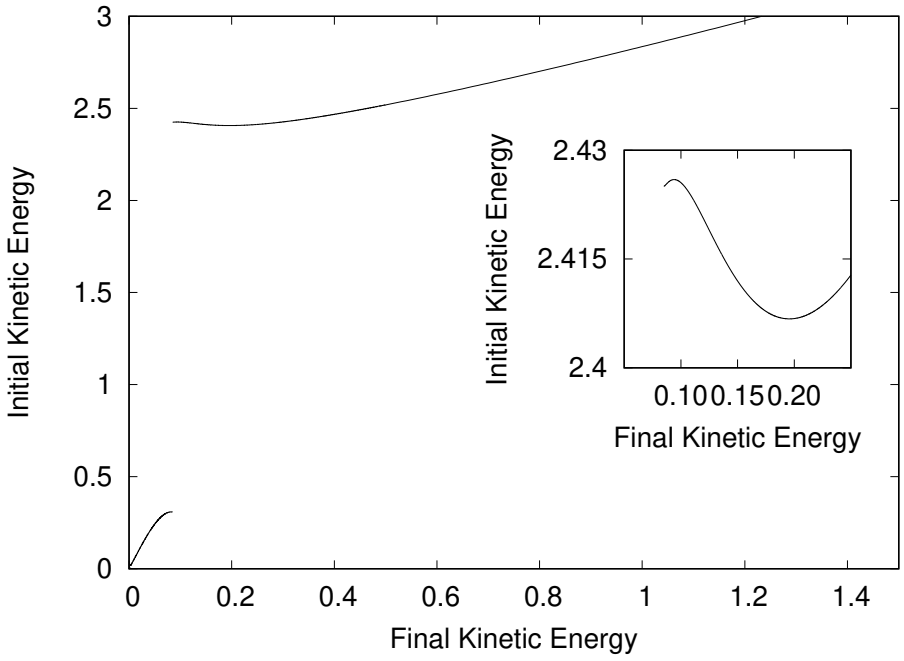

Figure 3: Plot of the initial kinetic energy vs. the final one for field amplitude $A = 1$, and non vanishing wave vector. Notice the jump. The inset is an enlargement of the curves around the minimum, which clearly exhibits the non uniqueness phenomenon. There are other jumps, not shown in the figure, at low energy. The jumps imply that, for some initial energies, there are no scattering solutions: for such energies the electron falls onto a stable equilibrium point.

a bifurcation value

$$k_{cr} = \frac{2\sqrt{3}}{9} \ . \tag{7}$$

For $k < k_{cr}$ the equilibrium point is an unstable saddle, with one stable direction and two unstable ones. Instead, for $k > k_{cr}$ one gets two complex eigenvectors, i.e., one has again a stable one-dimensional manifold (call it Σ^s), but the unstable manifold is indeed an unstable focus: the points spiral out from the origin going to infinity. This is the source of the non uniqueness behavior. In fact, one can argue as follows. Return to the nonlinear equation: the unstable manifold is three–dimensional, while the non runaway manifold, as recalled above, is a two-dimensional one, so that generically there will be a one-dimensional intersection $\gamma(t)$, which will be a solution belonging both to the unstable manifold and to the non runaway manifold: $\gamma(t)$ springs out spiraling from the unstable equilibrium point at $t = -\infty$, and goes to infinity with a vanishing acceleration for $t \to +\infty$. Consider now, at $t = +\infty$, the

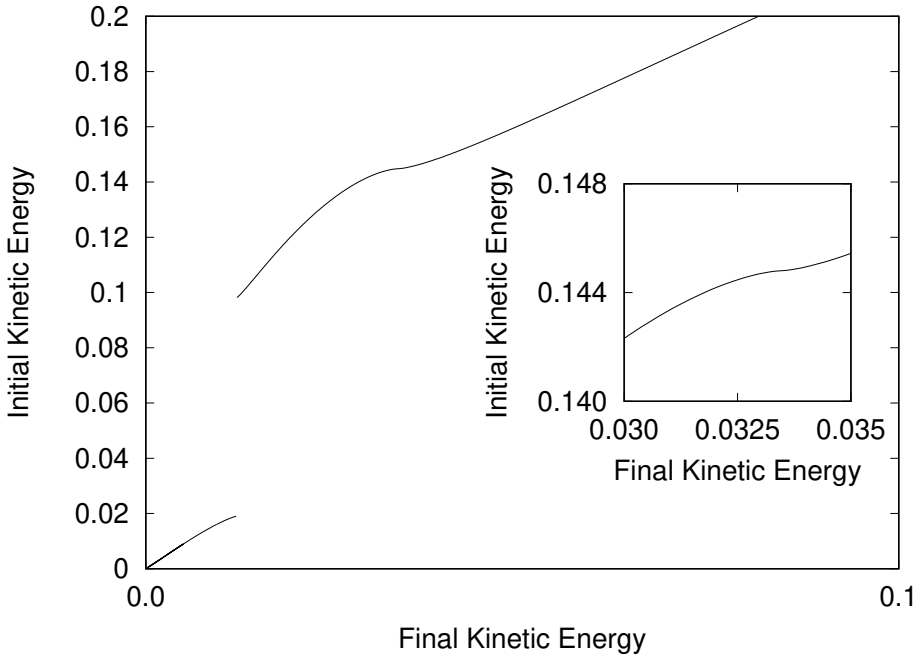

Figure 4: Same as figure 3, for $A = 0.3$, below the critical value. Now the insets show that the map is one to one, so that one has uniqueness.

non runaway solutions near to γ, and propagates them back in time: by continuity of solution of (4) with respect to the initial data, this solutions will follow $\gamma(t)$ near the equilibrium point spiraling about the stable one-dimensional manifold Σ^s. The backward-time flow turns the stable direction into the only unstable one, so that the orbits will finally follow the Σ^s manifold returning again to infinity. In other terms, the existence of an intersection between the unstable manifold and the Dirac one, entails that the Dirac manifold will be wrapped around the stable manifold Σ^s. This is the origin of the non uniqueness property. In fact, fix now $\xi = const$ sufficiently distant from the origin, and consider the intersection of the two-dimensional Dirac manifold (before scattering) with the three-dimensional hyperplane $\xi = const$: one would get a curve which projects on the plane of the initial "mechanical data" $(\dot{\xi}, v)$ like a (deformed) spiral. Letting C changing, the different spiral will have in general different center, so that they will intersect giving rise to different non runaway trajectory for the same mechanical initial data. These geometric considerations are obviously not a proof, but just an indication that the bifurcation of the unstable equilibrium points could drive the appearance of the non uniqueness behavior. In the next Section we will show, by numerical computations, that this is indeed the case.

4 Numerical results

The equations of motion (2) were integrated by a third order Runge–Kutta method which is easy to implement and sufficiently fast for our purposes. Moreover, we studied two case: either a simple Gaussian incoming wave

$$\frac{e\varepsilon}{mc}F(\xi) = A\exp\Big(-\frac{\xi^2}{\sigma^2}\Big)\ , \tag{8}$$

or the more complex wave form

$$\frac{e\varepsilon}{mc}F(\xi) = A\exp\Big(-\frac{\xi^2}{\sigma^2}\Big)\ \cos k\xi\ , \tag{9}$$

which allows one to investigate the role of the wave-length in the scattering process.

In the latter case one can rescale the distances by the wave length of the incoming laser pulse. Then, all the constants of the problem are resumed into only two parameters: the field intensity A and the width σ of the electromagnetic pulse. In the pure Gaussian case (8), we have taken $\sigma = 1$ and studied the behavior of the non runaway solutions as the field intensity A is changed. In particular, we find that $\xi = 0$ is an unstable equilibrium point, in fact the only equilibrium point. We compute the value A_{cr} which corresponds, through the formula (6) to the value of k_{cr}. For $\sigma = 1$ one finds $A_{cr} \simeq 0.38$.

In the case of the potential given by (9), we have taken a larger value $\sigma = 10$, and, by rescaling, $k = 1$. Such values of the parameters correspond to the values used in the actual experiments, in which the laser wave length is about 0.8 μm, while the duration of the pulse is about 40 fs, so that the laser wave length is an order of magnitude smaller than the pulse width.

In this latter case $\xi = 0$ is again an unstable equilibrium point, even if now there exists an infinite number of them (both stable and unstable). The point $\xi = 0$ gives however a lower value for A_{cr}, which in this case corresponds to $A_{cr} \simeq 0.36$. Moreover, as the peak intensity E of the electric field, i.e., the maximum of $F'(\xi)$, is given by

$$E \simeq \frac{mc}{e\varepsilon}kA = \frac{3m^2c^4}{2e^3}kA \tag{10}$$

to A_{cr} correspond a critical field E_{cr} (remember that here $k = 1$)

$$E_{cr} \simeq 0.54\frac{m^2c^4}{e^3}\ , \tag{11}$$

which is larger than the critical field of quantum electrodynamics by a factor of 70, i.e. 300 times larger than the maximum field experienced by the electrons in the experiments performed in [17]. Nevertheless, as the numerical

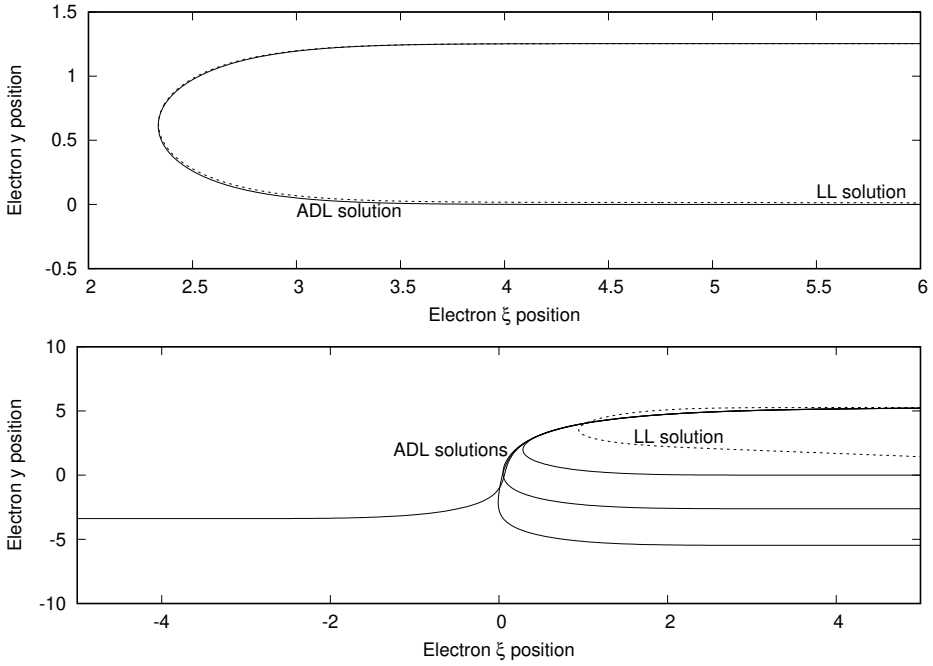

Figure 5: Comparison between orbits computed using the Landau–Lifschitz approximation (broken line), and the ones computed using the full Abraham–Lorentz–Dirac equation (full line) with the same initial mechanical data: upper panel refers to a pure Gaussian field with intensity $A = 0.31$ below the critical one and initial energy $E = 0.00295$; lower panel refers to a pure Gaussian field with $A = 1$ above the critical one, for an initial energy $E = 1.1322$ for which there are several non runaway solutions.

results of Section 5 show, the difference between the Abraham–Lorentz–Dirac equation and its Landau–Lifschitz approximation cannot be neglected also in that regime.

As remarked in Section 2, to obtain the non runaway solutions one integrate backwards in time. In such a way, one constructs a map from the "final data" $(\xi^f, y^f, \dot{\xi}^f, \dot{y}^f, \ddot{\xi}^f, \ddot{y}^f)$ to the initial one $(\xi^i, y^i, \dot{\xi}^i, \dot{y}^i, \ddot{\xi}^i, \ddot{y}^i)$. The only independent final parameters are the final velocities $\dot{\xi}^f$ and $\dot{y}^f$. In fact, as one is dealing with s scattering case, one has to considered ξ^f large (i.e. states in which the electron has left the interaction zone with the laser pulse), i.e., an arbitrary (but fixed) value for $|\xi^f| = R$ such that the force due to the electromagnetic field essentially vanishes. In such a case one is forced to fix $\ddot{\xi}^f = \ddot{y}^f = 0$, by the non runaway condition. Moreover, due to the invariance under translation along the y axes, one can fix arbitrarily $y^f = 0$.

Having fixed the final data, one starts integrating backwards up to a

time such that the electron, after having interacted with the electromagnetic wave, returns into a zone of vanishing field, for example again at $|\xi^i| = R$. At this moment one collects the initial value $\dot{\xi}^i, \dot{y}^i, \ddot{\xi}^i, \ddot{y}^i$. So defined, the map from the "final" to the "initial" data is one to one. The problem is whether the inverse map, i.e., the physical one which maps the "initial" data to the "final" ones, is one to one, or not. If it is one to one there is uniqueness, i.e., to a mechanical data of position and velocity corresponds just one non runaway solution; if it is one to many, to a single mechanical state, there corresponds different non runaway solutions with different asymptotic final states. In order to answer this question, we made the preliminary step of

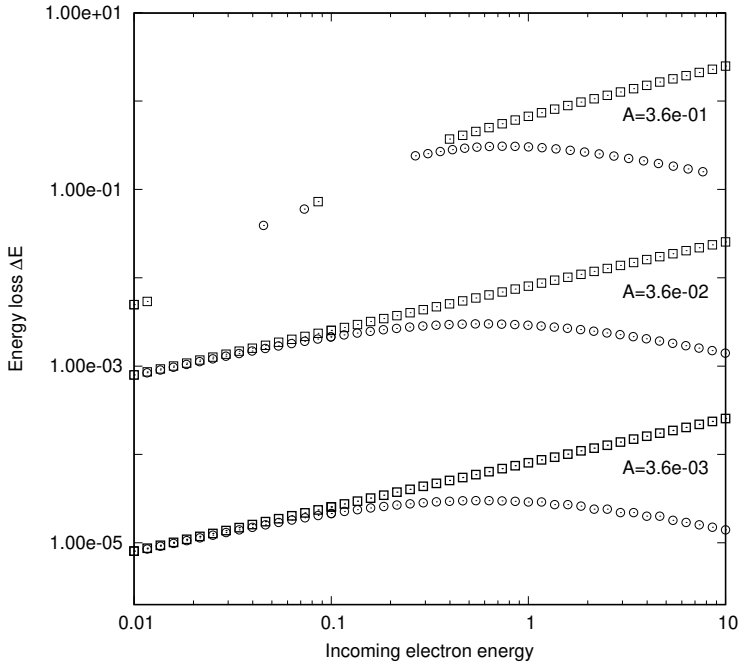

Figure 6: Energy loss in a collision versus the energy of the incoming electron in logarithmic scale, for three values of the field intensity, $A = 0.36$, 0.036, 0.0036. The potential is given by formula (9), with $k = 1$ and $\sigma = 10$. Squares are the losses computed according Landau-Lifschitz equation while circles are the losses computed according Abraham–Lorentz–Dirac equation. Notice that the Abraham–Lorentz–Dirac equation predicts a maximum for the energy loss, while, according to the Landau-Lifschitz equation, the loss keep to continuously increasing as the electron initial energy increases.In the case $A = 0.36$, missing points represent initial data in which the incoming electron fall on one of the potential minima (no scattering occur).

reducing to the case of a scattering normal to the plane wave, i.e., to the case in which the component $\dot{y}^i$ along the y axis of the initial velocity vanishes.

So, one has to solve the equation $\dot{y}^i(\dot{\xi}^f, \dot{y}^f) = 0$ (which is easily solved by the bisection method). This gives $\dot{y}^f$ as a function of $\dot{\xi}^f$, which remains the only free parameter. A curious feature of this equation, probably linked to the conservation of the y component of momentum in the mechanical case, is that $\dot{y}^f = 0$ gives a good approximation to the true solution. Now, by a simple inspection of the curve $\dot{\xi}^i$ as a function of $\dot{\xi}^f$ one can check whether the inverse map is one to one or not. Equivalently one can inspect the curves of the initial kinetic energy vs. final kinetic energy: in the non uniqueness case such map would show a non monotone behavior. In figure 1 this curve is drawn for the pure Gaussian potential (8) with $A = 1$, the inset showing details about the local maximum. There is evidence of a complex sequence of nested maxima, as in the case investigated in ref. [6]. So the map is not monotone, and thus the inverse map is not one to one. Figure 2 shows what happens when the field strength A is increased from below the critical value to above it: the curves of the initial kinetic energy are reported versus the final ones (always for fixed initial value $\dot{y}^i = 0$): if $A = 0.31$ the curve is monotone increasing, so that the inverse map is one to one and there is uniqueness of the non runaway solutions. For $A = A_{cr}$ the curve seems to have an inflection point but one can consider the inverse map again as one to one, i.e., uniqueness of non runaway solutions. Instead, a carefully inspection of the case $A = 0.41$, above the critical value, shows a weak local minimum for a final kinetic energy of $\simeq 0.06$ (more evident in the inset of the figure), so that uniqueness is lost. Figure 3 and 4 refers to the case of the potential given by (9). In figure 3 the initial kinetic energy is plotted versus the final one for $A = 1$: the inset is an enlargement about the minimum. Again, above the critical value, the map is not one to one, and one has the non uniqueness phenomenon. Notice that the map has a jump, i.e. the inverse map is not defined in a certain interval. It seem reasonable to assume that for such a value of the initial kinetic energy, the incoming particle falls onto one of the stable equilibrium points. This indeed happens for some value of the energy, but an analytical proof is lacking. A more detailed study at low final energies (too low to be appreciable in the figure) shows that there are other jumps. Instead, in figure 4, the initial kinetic energy is plotted versus the final one for $A = 0.3$, which is below the critical value. Now, the map appears to be one to one, and uniqueness recovered. As in the case of $A = 1$, the inverse map is not defined for some intervals of the initial kinetic energy. Again we think this is due to the fact that the particle be captured by one of the stable equilibrium points.

5 Comparison with the Landau–Lifschitz approximation

The Landau-Lifschitz approximation is obtained from the Abraham–Lorentz–Dirac equation using the following argument. For small ε one has

$$m\ddot{\mathbf{x}} \simeq \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}) \ ; \tag{12}$$

so that one can obtain an approximation of the third derivatives by

$$\dddot{\mathbf{x}} = \frac{\mathrm{d}}{\mathrm{d}t}\ddot{\mathbf{x}} \simeq \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{m}\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})\right) \ ,$$

which substituted into the Abraham–Lorentz–Dirac equation gives, neglecting the terms of order higher,

$$m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}) + \varepsilon\Big(\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{1}{m}\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}\mathbf{F}\Big) \ , \tag{13}$$

where we have replaced $\ddot{\mathbf{x}}$ again by its approximation (12). This equation does not have the problem of runaways and therefore of the choice of initial data. Using a third order Runge–Kutta methods, we integrate this equation, with the Gaussian vector potential as given by (8), for several values of the intensity A. Figure 5 show the orbits found: they are computed by first integrating the Abraham–Lorentz–Dirac equation backward for a certain amount of time, and then the Landau–Lifschitz equation forward in time, so that the initial mechanical data for the two equations agree. One can check that the orbits for low values of A essentially coincide, while they differ for higher field intensities as expected. A more meaningful comparison is given in figure 6, where it is reported, in logarithmic scale, the loss of energy $\Delta E \stackrel{\text{def}}{=} E_i - E_f$, i.e. the difference between the incoming electron energy E_i and its energy E_f after the scattering, as a function of the energy E_i. We take for the laser field the form (9) with $\sigma = 10$ and $k = 1$, which, we recall, are the typical value used in the actual experiment, and three different values for the field intensity A: 0.36, 0.036 and 0.0036. The largest of such values correspond to the the critical field for which non uniqueness shows up, while the smaller one, as explained, is of the order of magnitude of the field employed in the actual experiments. In the figure 6, the squares correspond to the loss computed according to the Landau–Lifschitz equation, while the circle to the one computed according to the Abraham–Lorentz–Dirac equation. The results are qualitatively different also for field's intensities well below the threshold, inasmuch as according Abraham–Lorentz–Dirac the loss has a well defined maximum at a definite energy, while according Landau–Lifschitz, at least in the range of energy we have explored, it keeps increasing without limit. From the figure, it seems that such energy loss

increases as a power of the incoming electron energy, being the log–log plot well approximated by a straight line. Fitting the power one gets a value very close to 1/2.

Moreover, the energy loss is systematically smaller for the Abraham–Lorentz–Dirac equation with respect to its approximation. This latter fact can be explained, on a physical ground, as follows. First of all, we recall that the Abraham–Lorentz–Dirac equation predict, for the power radiated by the particle, the classical Larmor expression $P_{Lar} = \frac{2e^2}{3c^3}|\mathbf{a}|^2$ being $\mathbf{a}$ the particle's acceleration. Instead, as it is easily verified[1], the Landau–Lifschitz equation predict the particle radiates a power given by $P_{LL} = \frac{2e^2}{3c^3}\frac{|\mathbf{F}|^2}{m^2}$, where $\mathbf{F}$ is the force due to the impinging laser pulse. Now, the particle's motion is damped by the radiation reaction, so its acceleration $|\mathbf{a}|$ is smaller with respect to the undamped case, i.e., with respect to $\frac{|\mathbf{F}|}{m}$. So, the Landau–Lifschitz equation overestimates the energy losses due to radiation, especially in cases in which the radiation reaction is important. This fact is acknowledged in reference [17] (see bottom of page 8, first column, and following lines), where, in Figure 4 panel b, the measured electron energy spectrum is reported together with the one computed from the Landau–Lifschitz equation. The authors ascribe such an overestimate to the quantum nature of the photon scattering process, but apparently, also the quantum corrections (see the same figure panel c) are not enough to account completely the experimental data. In this connection, the use of Abraham–Lorentz–Dirac equation which gives smaller energy losses could allow a better agreement with the experimental data.

6 Conclusion

We have show that, for the non relativistic Abraham–Lorentz–Dirac equation, there exists a threshold for the intensity of the incoming field, above which one has several non runaway solutions for the same mechanical initial data of position and velocity. Such a threshold agrees well with the bifurcation value of the main unstable point from saddle to saddle–focus. Above such a threshold the Landau-Lifschitz equation clearly differs from the full Abraham–Lorentz–Dirac equation, but we have checked that also well below the threshold the energy losses in a collision does not agree each other. As this is the main experimental observable, one may wonder whether the use of

[1] In fact, from the Landau–Lifschitz equation (13) written as $m\ddot{\mathbf{x}} = \mathbf{F} + \varepsilon\dot{\mathbf{F}}$, by multiplying by $\dot{\mathbf{x}}$ one finds $\dot{E} = \varepsilon\dot{\mathbf{F}}\cdot\dot{\mathbf{x}}$, where E is the mechanical energy. By integrating by parts the r.h.s., and expressing $\ddot{\mathbf{x}}$ again by the Landau–Lifschitz equation, one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E - \varepsilon F\cdot\dot{\mathbf{x}} + \frac{1}{2}\varepsilon^2\mathbf{F}\cdot\mathbf{F}\right) = \varepsilon\mathbf{F}\cdot\mathbf{F}\ .$$

In the bracket at the l.h.s. appears, besides the mechanical energy E, also two other terms which are the analog of the Schott term for the Abraham–Lorentz–Dirac equation.

the full Abraham–Lorentz–Dirac equations instead of the Landau–Lifschitz approximation might lead to an agreement between theory and experiment, better than the one found in ref. [17]. Answering such a question would require an analysis similar to that performed in this paper, for the full relativistic Abraham–Lorentz–Dirac equation, in the regime of interest for the experiments. This is a much more complex task, on which we hope to come back in the future.

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