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Normal form and KAM methods for higher dimensional linear PDEs

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Contents

Introduction

In its original form, KAM theory deals with perturbations of finite dimensional integrable systems and allows to prove that most of the invariant tori of the unperturbed system persist under perturbation.
The first extensions to PDEs have been obtained in the late eighties in [Kuk87] and [Way90]. Since then several authors, among which we mention [CW93, Bou94, CY00, BBM14, BBHM18], have contributed to the development of a satisfactory theory that allows to prove persistence of quasiperiodic motions in quasi-integrable Hamiltonian PDEs on one dimensional domains. For higher dimensional Hamiltonian PDEs, the situation is completely different, and essentially amounts to isolated examples. First of all we have the theory developed by Bourgain in [Bou98] and [Bou04], but it only allows to deal with nonlinear Schrödinger and nonlinear wave equations on the square torus $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z}^d)$, $d \geq 2$; Bourgain's method has subsequently been extended in [BCP15] to equations on Zoll manifolds and Lie groups, and in [BM19] to the case of arbitrary flat tori, but nothing is known on more general manifolds or domains: a key common ingredient in all these results is indeed a technical Lemma by Bourgain, allowing to prove "separation of resonant sites" on tori. A different technique has been developed by Kuksin and Eliasson in [EK10] (see also [PX13] and [EGK16]), but again it only applies to square tori: up to now, there is a lack of a general theory for equations on domains with spatial dimension higher than one.

In this thesis a new approach to higher dimensional problems is developed: the strategy is an extension of the methods originally elaborated in [BBM14] in the context of quasilinear one dimensional equations. To explain such a method, recall that a possible strategy to construct invariant tori exploits a Nash Moser type theorem (essentially based on a Newton algorithm). Here the key step is to construct and estimate the inverse of a linear operator:

more precisely, at each step of the iterative scheme, one linearizes the system at an approximate solution, and gets an operator of the form $\mathcal{L} = \partial_t - A(\omega t)$, where A is a linear operator with quasi-periodic dependence on time.

The operator $\mathcal{L}$ has to be inverted, and it is necessary to exhibit *tame estimates* on its inverse operator. A technique to show the invertibility is to consider the equation

$$\partial_t u - A(\omega t)u = 0 \tag{0.0.1}$$

and to conjugate it to a new equation of the form $\partial_t u - \widetilde{A}u = 0$, where $\widetilde{A}$ is a time independent linear operator, via an invertible map which depends on time in a quasi-periodic way; such a conjugating process is referred to as reducibility.
This thesis deals with such a problem, and also with a further one which is preliminary to reducibility, namely the spectral problem for periodic Schrödinger operators.
We remark that linear time dependent operators have been object of great interest per se and have been widely studied, most of all due to the major role they play in quantum mechanics; to this aim we recall the cornerstones [Kat70, Kat75, Kat85], providing existence and uniqueness results of smooth solutions for equations of the form $\partial_t u - A(t)u = 0$, and [Bou99b, Bou99a, Del10], where upper bounds on the growth of Sobolev norms are exhibited (see also [Wan08a, FHW14], and the more recent works [MR17, BGMR17]). The analysis of the dynamic behavior of linear time dependent problems is deeply related to the study of spectral problems: the second part part of this thesis is indeed entirely focused on the spectral analysis of a Schrödinger operator. This is also a widely studied subject, and some of the relevant related literature will be discussed later on.

Our starting point is the construction elaborated in [BBM14]. The main idea developed therein, in order to obtain reducibility in the one dimensional quasilinear case, consists in performing a preliminary step which, using techniques of pseudo-differential calculus, enables to conjugate the original operator to a smoothing perturbation of a "trivial operator" (typically, a diagonal, or block diagonal, operator with respect to the Fourier basis), and subsequently applying a suitable KAM algorithm to completely eliminate the time dependence.

In [Bam18] the pseudo-differential techniques by [BBM14] have been recognized to be the quantization of a classical normal form procedure, and a first partial generalization of these techniques to a higher dimensional case has been obtained in [BGMR18]. However, in the particular system of [BGMR18], the KAM part of the proof trivially follows from its classical counterpart, and this cannot be expected in the general case.

This is the origin of the work reported in the first part of this thesis, and published in [BLM19], where a higher dimensional equation of the form

$$\partial_t u(t,x) = (\nu + \varepsilon V(x,\omega t)) \cdot \nabla u(t,x) + \varepsilon \mathcal{W}(\omega t) u(t,x)\,, \quad x \in \mathbb{T}^d, \quad d \geq 2\,, \tag{0.0.2}$$

is considered. Here $\nu \in \mathbb{R}^d$, $\omega \in \mathbb{R}^{\mathfrak{n}}$, $V \in \mathcal{C}^\infty(\mathbb{T}^d \times \mathbb{T}^{\mathfrak{n}};\mathbb{R})$, and $\forall \varphi \in \mathbb{T}^{\mathfrak{n}}$ $\mathcal{W}(\varphi)$ is a pseudo-differential operator of order strictly less than 1, symmetric hyperbolic[1] and smoothly dependent on φ. For such an equation we prove reducibility. The following theorem is one of the main results of this thesis:

Theorem 0.0.1. *For any $\sigma \geq 0$ and $s \geq 0$ there exists ε_0 such that for any $|\varepsilon| < \varepsilon_0$ there exists a set $\Omega_\varepsilon \subset [1,2]^{d+\mathfrak{n}}$ with the following properties. For any $(\omega,\nu) \in \Omega_\varepsilon$, the system (0.0.2) is reducible, namely in the Sobolev space $\mathcal{H}^\sigma$ there exist an operator $\widetilde{A}$, diagonal on the Fourier basis, and a bounded linear operator with bounded inverse $\mathcal{U}(\varphi)$, continuously dependent on $\varphi \in \mathbb{T}^{\mathfrak{n}}$, such that, if $u = \mathcal{U}(\omega t)v$, v satisfies*

$$\partial_t v = \widetilde{A} v\,.$$

Furthermore, the complementary set of Ω_ε in $[1,2]^{d+\mathfrak{n}}$ has Lebesgue measure which tends to 0 as $\varepsilon \to 0$, and for any s there exists $\eta \geq 0$ such that the map $\varphi \mapsto \mathcal{U}(\varphi)$ is of class $\mathcal{C}^s$ from $\mathbb{T}^{\mathfrak{n}}$ to the set of bounded linear operators from $\mathcal{H}^{\sigma+\eta}$ to $\mathcal{H}^\sigma$.

The main interest of this part of the work lies in the fact that, in order to prove Theorem 0.0.1, after a preliminary application of the result in [FGMP19], one has to merge the regularization techniques of [BGMR17] and the reducibility scheme of [BBM14], thus developing an algorithm which is likely to be suitable for applications to much more general systems. As a result, one of the few examples of reducibility for an equation on a higher dimensional domain with an unbounded perturbation is obtained. Remark that, moreover, equation (0.0.2) corresponds to a case where the unperturbed

[1] see Definition 2.1.5 of Chapter 2

operator has eigenvalues whose differences are dense on the real axis; such a configuration is usually considered as particularly difficult to deal with. As anticipated, in order to construct KAM tori it is actually necessary to complement the reducibility result in Theorem 0.0.1 with suitable tame estimates; here we do not focus on this issue, but we will comment later on it.

Let us focus on the ideas lying behind the proof of reducibility for equation (0.0.2): as anticipated, the proof is based on two steps. Rewrite equation (0.0.2) in the form

$$\mathrm{i}\partial_t u = (A_0 + \varepsilon A_1(\omega t))\, u\,, \tag{0.0.3}$$

with

$$A_0 = \mathrm{i}\nu \cdot \nabla\,, \quad A_1(\omega t) = \mathrm{i}V(\omega t, x)\cdot\nabla + \mathrm{i}\mathcal{W}(\omega t)\,.$$

In the first step of the procedure, the goal is to reduce the *order* of the perturbation as a (pseudo)-differential operator, namely to map equation (0.0.3) into a new one of the form $\mathrm{i}\partial_t v = \left(\widetilde{A}_0 + \varepsilon\widetilde{A}_1(\omega t)\right) v$, where the new perturbation $\widetilde{A}_1$ is a smoothing operator of order N for some arbitrary $N > 0$, in the sense that it maps any Sobolev space $\mathcal{H}^\sigma$ into the Sobolev space $\mathcal{H}^{\sigma+N}$. We refer to such a procedure as regularization. Actually, in the case of (0.0.2) the two terms $\mathrm{i}V\cdot\nabla$ and $\mathrm{i}\mathcal{W}$ are treated separately. Indeed, the former is eliminated by means of a diffeomorphism on the torus, applying directly the result proven in [FGMP19]. The lower order term $\mathrm{i}\mathcal{W}$ is instead dealt with using tools from pseudo-differential calculus: the proof is based on an arbitrarily large, but finite, number of steps of a non convergent normal form, and it is actually a variant of the arguments contained in [BGMR17].
In the second part of the procedure, one uses a KAM type algorithm to reduce the *size* of the perturbation $\widetilde{A}_1(\omega t)$: an iterative and convergent scheme is implemented, and the operator $\widetilde{A}_0 + \widetilde{A}_1$ is conjugated to a diagonal, time independent one. Such a scheme requires to impose at any step some non resonance conditions, known in literature as *second order Mel'nikov conditions*. Here the higher dimensional setting forces us to assume very weak non resonance conditions; this yields the presence along the algorithm of small divisors which accumulate very fast to 0, and would cause a loss of regularity along the process. This is where the smoothing character of the perturbation $\widetilde{A}_1$ comes into play, balancing the loss of regularity due to the presence of small divisors. The scheme that is implemented here is a variant of the one developed in [BBHM18], and then extended in [Mon19].

Furthermore, we remark that here the algorithm is not performed in a self-adjoint setting. This is essentially due to the presence of the first order perturbation $\mathrm{i}V \cdot \nabla$, which violates such a structural hypothesis: this is why, instead, symmetric hyperbolicity is imposed and preserved along the iteration. Thus, in order to ensure the stability of the solutions of (0.0.2), the case where the system has some additional structural hypotheses is also studied: namely, reality and reversibility. If these two properties are satisfied, we prove that all solutions of (0.0.2) are almost-periodic. On the contrary, without assuming them, only one of the two following possibilities can occur, in analogy with Floquet theory: either all solutions are almost-periodic, or there exist solutions of (0.0.2) whose Sobolev norms exponentially diverge in time.

The main limitation of the techniques developed in [BGMR18], as in all previous works, and of the ones presented in Chapter 2, lies in the fact that the regularization normal form procedure developed therein is the quantization of a classical normal form which is *global* in phase space, in the sense that in both cases the conjugating map is defined on the whole phase space (namely, the cotangent bundle of $\mathbb{R}^d$ in [BGMR18] and the cotangent bundle of the $\mathbb{T}^d$ in Chapter 2). This is due to the fact that in both these two particular cases the frequencies of the unperturbed system do not depend on the point of the phase space.
For such a reason, the subsequent problem that is tackled in the present thesis is that of developing a local quantum normal form procedure, which could allow to deal with the more general case of an actual dependence of the unperturbed frequencies on the point of the phase space. By the way let us emphasize that the problem of quantizing the classical normal form procedure has been the object of extensive studies in the '90s, and also all the results obtained in that period were limited to systems where a global normal form was possible (see [GP87, BV90, Sjo92, BGP99]).
Remark that, since the focus here is on the first step of the procedure, namely the regularization process, time dependence does not play any fundamental role. Therefore the idea is that one can temporarily neglect time dependence, and start regularizing the problem with the time variable frozen.

Thus, as the simplest relevant model containing all the difficulties of the general case, in the second part of this thesis we consider

$$-\Delta + V(x) \tag{0.0.4}$$

on $L^2(\mathbb{T}^d_\Gamma)$, where $\mathbb{T}^d_\Gamma := \mathbb{R}^d/\Gamma$, and Γ is an arbitrary d dimensional lattice, and we adapt the quantum normal form procedure to deal with this operator. The aim is to develop a method which is not based on the structure of the torus, but is suitable for applications to more general situations.

First, recall that the classical Hamiltonian associated to the operator (0.0.4) is $h(\xi, x) = h_0(\xi) + V(x)$, where $h_0(\xi) = \|\xi\|^2$. In this context the classical Hamiltonian is called the symbol of the operator one is studying.

In classical perturbation theory it is well known that the kind of normal form one can obtain strongly depends on the resonance relations fulfilled at a given point of the phase space. Thus in this thesis we develop a local normal form theory that is suitable for quantization. The normal form method we obtain is actually a variant of the one developed in [PS10, PS12], which respectively deal with the proof of Bethe Sommerfeld conjecture and with the analysis of the integrated density of states of the operator (0.0.4) on $L^2(\mathbb{R}^d)$. However, we give a global formulation, which allows an analysis of the asymptotic behavior of the eigenvalues and makes possible applications to dynamical systems that are not possible with the approach of [PS10, PS12]. We will comment later on the connection with the methods presented therein (see also Chapters 4 and 5).

As a first application of our normal form result, we focus on "nonresonant eigenvalues", namely the ones corresponding to sites ξ which are far away from any resonances. In particular, in Chapter 4 we prove the following result:

Theorem 0.0.2. *Let* $0 < \delta < 1$ *and let* Γ^* *be the dual lattice of* Γ. [2] *There exist a set* $\Omega \subset \Gamma^*$ *of density one at infinity and a sequence* $\{m_j\}_{j\in\mathbb{N}}$ *of smooth functions* $m_j : \mathbb{R}^d \to \mathbb{R}$ *such that for all* $j \in \mathbb{N}$

$$\exists\ C_j > 0 \ s.t. \quad |m_j(\xi)| \leq C_j \langle\xi\rangle^{-2\delta j} \quad \forall \xi \in \mathbb{R}^d$$

with the following property: for any $\xi \in \Omega$ *the operator* (0.0.4) *has an eigenvalue of the form*

$$\lambda_\xi = \|\xi\|^2 + \sum_{j=1}^{n-1} m_j(\xi) + \mathcal{O}(\langle\xi\rangle^{-2\delta n}) \quad \forall n \in \mathbb{N}. \tag{0.0.5}$$

Furthermore, for any $K-$*uple* $\{\xi_1, \ldots, \xi_K\} \subset \Omega$, *the eigenspace generated by the eigenfunctions corresponding to* $\lambda_{\xi_1}, \ldots, \lambda_{\xi_K}$ *has multiplicity* K, *and for*

[2] Namely, $\Gamma^* = \{\ell \in \mathbb{R}^d \mid \ell \cdot \gamma \in 2\pi\mathbb{Z} \quad \forall \gamma \in \Gamma\}$.

any $\xi \in \Gamma^*$, *one has* $\xi \in \Omega$ *if and only if* $-\xi \in \Omega$*; in such a case,*

$$m_j(\xi) = m_j(-\xi) \quad \forall j \in \mathbb{N}\,. \tag{0.0.6}$$

(Here we have used the notation $\langle \xi \rangle = \sqrt{1 + \|\xi\|^2}\,$*.)*

The proof of this theorem, published in [BLM20], is based on a combination of the normal form we perform for the operator H and on a refined quasi-mode argument, which is a development of the one used in [BKP15]. This allows to deduce the one to one correspondence between points ξ in the non-resonant set Ω and eigenvalues λ_ξ satisfying (0.0.5). Density one at infinity is finally proved using measure estimates and exploiting the lattice structure of Γ^*. Actually we prove a stronger version of Theorem 0.0.2, and we give spectral asymptotics for eigenvalues of any operator of the form $(-\Delta)^{\frac{M}{2}} + V$, $M > 0$, where $(-\Delta)^{\frac{M}{2}}$ is any positive power of the Laplacian operator and V is an unbounded operator of order strictly less than M. We point out that the essential property needed for the proof is the fact that the symbol of the Laplacian is an integrable Hamiltonian. We will comment later on the connections with previous spectral results on periodic Schrödinger operators.

The result stated in Theorem 0.0.2 is not sufficient for our original purpose: to deal with the time dependent case, one needs to know in a quite precise way *all* the eigenvalues of the operator (0.0.4), and not only those related to the non resonant sites ξ.
In order to deal with the eigenvalues lying in the resonant regions, one has to make a more refined analysis, which is the content of Chapters 5 and 6. First, we perform for the Schrödinger operator a quantum normal form that is different in the various regions of the phase space, and varies in each region according to which resonance relations are fulfilled therein. Then we proceed with the analysis of the normalized operator. This is done by making a decomposition of the phase space inspired by the classical geometrical construction in the proof of Nekhoroshev Theorem. (This also led to an alternative proof, published in [BL20], of the classical Nekhoroshev result.) As a result, one gets a Structure Theorem, which is one of the main results we obtain in the present thesis; since it requires non trivial preliminary steps, its precise statement is postponed to Chapter 5 (see Theorem 5.1.10). According to such a Structure Theorem, the operator (0.0.4) is unitarily conjugated

to a smoothing perturbation of a normal form operator, and such a normal form operator is block diagonal, with the following structure:

- There is a largest block (the one described in Theorem 0.0.2) in which the operator is a Fourier multiplier
- there is one finite dimensional block, whose dimension is controlled by the initial parameters of the system
- in all the other blocks, the operator is a Schrödinger operator on a flat torus of lower dimension, with new Floquet boundary conditions.

In the last case, one can iterate the construction and in such a way after a finite number of steps (at most d) one is reduced to deal only with Fourier multipliers, or finite dimensional operators. Our Structure Theorem has strong analogies with the normal form of [PS10, PS12], which however does not provide an invariant partition of $L^2(\mathbb{T}^d)$ and furthermore, also due to some lack of uniformity, cannot be iterated, as needed for our purpose.

Finally, by means of a refined quasi-mode argument, we show that it is possible to deduce new spectral asymptotics for all the eigenvalues of (0.0.4). The asymptotic expansion that we get in the resonant sites is not in the parameter $\langle\xi\rangle$, but it is a "directional" asymptotic expansion. Roughly speaking, the result is the following: consider a subgroup $M \subseteq \Gamma^*$ (actually, a module) and assume that a sequence $\{\xi_j\}_{j\in\mathbb{N}} \subset \Gamma^*$ is such that all the points ξ_j are resonant with the vectors of a basis of M. Then, the corresponding eigenvalues λ_{ξ_j} admit an asymptotic expansion in the parameter $\langle(\xi_j)_M\rangle$, the subscript M denoting orthogonal projection on M (we refer to Theorem 6.0.5 of Chapter 6 for a precise statement).
We point out that the estimates in the asymptotic expansions that we give are uniform with respect to the choice of the modules M; this requires to have uniform estimates for the seminorms of the pseudo-differential operators involved in the normal form construction on all the blocks. In order to ensure this, here a formulation of pseudo-differential calculus is given where only intrinsic quantities are involved. (This is contained in Appendix C).

The spectrum of Schrödinger operators on $d-$dimensional flat tori has been widely studied. First of all we mention the works [FKT90] and [Fri90], where it was shown that, for a generic lattice Γ and for any ξ belonging to a density one subset of the dual lattice Γ^*, there are exactly two eigenvalues $\lambda_{\pm\xi}$ inside an interval of the form

$$\left[\|\xi\|^2 + [V] - \frac{1}{\|\xi\|^{2-\epsilon}},\ \|\xi\|^2 + [V] + \frac{1}{\|\xi\|^{2-\epsilon}}\right], \tag{0.0.7}$$

for $0 < \epsilon \ll 1$, where $[V]$ is the average of V. Such a property is referred to in [FKT90, Fri90] as stability. Actually, the result was proven for $d = 2, 3$ in [FKT90] and generalized to arbitrary dimension, with a slightly simpler proof, in [Fri90]. A refinement of such a result was proven in [Kar97] in the case where the periodic boundary conditions on the torus $\mathbb{T}^d_\Gamma$ are replaced with Floquet boundary conditions, namely one has that

$$u(x+\gamma) = e^{\mathrm{i}\kappa\cdot x}u(x) \quad \forall x \in \mathbb{R}^d, \gamma \in \Gamma$$

for a $\kappa \in \mathbb{R}^d$ which is called Floquet parameter. In [Kar97] asymptotic expansions of the form of the ones exhibited in (0.0.5) are given, in the cases $d = 2, 3$ and restricting to a large set of Floquet parameters κ (which leaves out the case of periodic boundary conditions).

As proven in [FKT91], not all the eigenvalues of $-\Delta$ are stable: there are also eigenvalues of (0.0.4) lying outside intervals of the form of (0.0.7) – in our language, *resonant* eigenvalues. The construction performed in order to prove the existence of such eigenvalues is obtained in [FKT91], following [ERT84], by approximating the unstable spectrum of the Schrödinger operator (0.0.4) with the spectra of the following lower dimensional operators:

$$-\Delta + V_\gamma, \quad V_\gamma(x) = \sum_{\substack{\xi\in\Gamma^* \\ \xi\cdot\gamma=0}} \hat{V}_\xi e^{\mathrm{i}\xi\cdot x},$$

as γ varies in (some suitable subset of) Γ. With analogous approximation techniques, some eigenvalues in the unstable part of the spectrum have also been studied in [Kar96], again only in the case $d = 3$ and for a large set of κ, in particular excluding the case of periodic boundary conditions. The result contained in the present thesis generalizes the previous ones, obtaining asymptotics for all the eigenvalues, all lattices, all Floquet parameters and all dimensions.

From the construction exhibited in this thesis it is also possible to deduce some information on the eigenfunctions: in particular we prove that their negative Sobolev norms decay faster than any negative power of their corresponding eigenvalue.

As already mentioned, constructions with strong analogies to the one presented in Chapter 4 and Chapter 5 have been performed in [Par08, PS09, PS10, PS12]. The techniques developed in such works are based on the conjugation of (0.0.4) to a new operator, which is a small perturbation of a normal form one leaving invariant suitable sets. Such techniques enable to determine an asymptotic development of the *total number* of resonant and non resonant eigenvalues lying in any suitable interval of the real axis; however, they are not sufficient to obtain the global asymptotic result that we find for the eigenvalues of the operator (0.0.4). More precisely, the constructions in [Par08, PS10, PS12] could be adapted in order to deduce the asymptotics that we find in Theorem 0.0.2 for non-resonant eigenvalues, but the same is not true for the directional asymptotics that we determine for resonant eigenvalues in Chapter 6. This is essentially due to the fact that the results of [Par08, PS10, PS12] do not enable to deduce an analogue of our Structure Theorem, nor to set up the iterative construction of dimensional reductions from which we deduce our spectral result.

As anticipated, the main application that we have in mind for the analysis of operator (0.0.4) performed in the present work is to reintroduce the time variable and to deduce reducibility for a linear Schrödinger equation of the form

$$\mathrm{i}\partial_t\psi = -\Delta\psi + V(x,\omega t)\psi \tag{0.0.8}$$

on $L^2(\mathbb{T}^d_\Gamma)$, where V is a small amplitude smooth function with quasi-periodic dependence on time and diophantine frequency $\omega \in \mathbb{R}^{\mathtt{n}}$. Unlike in the case of the transport equation (0.0.2), reducibility for such an equation does not immediately follow from the normal form result found in Chapter 5. This is due to the following fact: the operator (0.0.4) can be conjugated to a smoothing perturbation of a Fourier multiplier only on the block corresponding to non resonant eigenvalues, and this implies that the very weak second order Mel'nikov conditions of the form of the ones imposed in Chapter 2, which are associated to a loss of regularity in space, cannot be imposed for the whole set of frequencies. A tuning of the parameter ω would then be necessary to impose suitable non resonance conditions, working independently on each one of the invariant blocks and contemporarily preventing the set of allowed ω to shrink to a null measure one. This is why, aside reducibility, as an object of future studies we intend to investigate the weaker notion of *almost* *reducibility*[3] for (0.0.8).

Actually, further applications of our Structure Theorem are likely to be possible: for example, a more detailed description of the semiclassical measures [AFKM15] could be performed, or (following [Roy07]) a precise semiclassical expansion in $\hbar$ of the eigenvalues could be obtained.
Furthermore, since the normal form construction that is implemented here is based on an explicit algorithm, it is possible, at least in principle, to compute an arbitrary number of terms in the asymptotic expansions found for resonant and non-resonant eigenvalues.
This idea is exploited in Chapter 7 of the present thesis, which contains a further application of the non-resonant normal form result of Chapter 4 to the analysis of the spectral degeneracy of the operator (0.0.4).
In particular, let us focus on the square torus $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z}^d)$, where the Laplacian operator $-\Delta$ has eigenvalues $\|\xi\|^2$ with multiplicities growing as $\|\xi\|^{d-1}$. Consider then all the stable eigenvalues of the form (0.0.5) bifurcating from the same eigenvalue $\|\bar{\xi}\|^2$ of the Laplacian operator, namely the set $\{\lambda_\xi \mid \|\xi\|^2 = \|\bar{\xi}\|^2\}$: as a consequence of (0.0.6) of Theorem 0.0.2, the two eigenvalues $\lambda_\xi, \lambda_{-\xi}$ always lie inside the same interval of width $\mathcal{O}(\|\xi\|^{-\infty})$. We ask under which conditions on the potential V there are more than 2 eigenvalues of (0.0.4) inside such an interval. As a first result, we exhibit a class of potentials V such that the number of eigenvalues of (0.0.4) inside the same interval of width $\mathcal{O}(\|\xi\|^{-\infty})$ is higher than 2; then we investigate such a configuration under genericity conditions on the potential. We show that, for a generic potential V, the set of eigenvalues of (0.0.4) bifurcating from the same unperturbed eigenvalue $\|\xi\|^2$ splits in at least two separated groups and we refer to such phenomenon as *breaking of degeneracy* in the spectrum of (0.0.4). This is in opposition to what happens in the one dimensional case ([Mar86]).

The thesis is organized as follows: Part I consists of Chapter 1, which contains a brief review of the main topics and results in reducibility, and Chapter 2, where reducibility is proven for the transport equation (0.0.2). Part II is devoted to the analysis of the operator (0.0.4) on an arbitrary torus $\mathbb{T}^d_\Gamma$. In Chapter 3 we define our setting; in Chapter 4 we implement our

[3] See Section 1.2 for a brief digression on almost reducibility results in the finite dimensional case.

normal form construction and we show how Theorem 0.0.2 follows from it. Chapter 5 is devoted to the proof of the Structure Theorem à la Nekhoroshev and in Chapter 6 we deduce global spectral asymptotics for the spectrum of the operator (0.0.4). Finally, in Chapter 7 we investigate the breaking of degeneracy in the spectrum of (0.0.4) on the standard square torus, as an application of the spectral result given in Chapter 4. In order to simplify the presentation, some auxiliary results are collected in several Appendixes. In particular, Appendix A collects some basic facts about standard pseudo-differential calculus on the torus, Appendix B is devoted to technical estimates that are necessary for the reducibility scheme of Chapter 2, Appendix C contains a brief review of standard facts about pseudo-differential calculus on the torus based on symbols with a coordinate independent definition, and Appendix D contains the technical estimates that are needed to prove the Structure Theorem of Chapter 5.

Part I

A higher dimensional reducibility result

Chapter 1

Reducibility: a short review

In the present chapter we recall some known reducibility results. Although our focus here is on its applications in KAM theory for PDEs, reducibility has been object of interest in itself, already in finite dimension. Consider a system of the form

$$\partial_t u = (A + \varepsilon P(\omega t))\, u\,, \quad u \in \mathcal{H}\,, \tag{1.0.1}$$

where $\mathcal{H}$ is a vector space that can be either real or complex, finite or infinite dimensional, A is an operator with pure point spectrum, and P is a linear operator quasi-periodically dependent on time with Diophantine frequency $\omega \in \mathbb{R}^d$, namely the map $\varphi \mapsto P(\varphi)$ is regular (continuous, finitely or infinitely differentiable, or analytic) from $\mathbb{T}^d$ to a space of linear operators in $\mathcal{H}$. Here we only focus on perturbative results, thus we consider the case where $\varepsilon \ll 1$ is a small parameter, whose threshold is allowed to depend on A and on ω.
Roughly speaking, the system (1.0.1) is said to be reducible if it can be conjugated to a new system with constant coefficients, via a map Z which depends smoothly on the angles φ. According to the choice of the space $\mathcal{H}$, such a map may be defined on $\mathbb{T}^d$, or on a suitable covering of $\mathbb{T}^d$. This is why, in order to give a precise definition, we need to distinguish among different cases:

Definition 1.0.1 (Reducibility on a finite dimensional Lie algebra)**.**

1. (Real algebras) *If $A \in \mathfrak{g}$ and $P : \mathbb{T}^d \to \mathfrak{g}$, with $\mathfrak{g} = \mathrm{gl}(n;\mathbb{R}), \mathrm{sl}(n;\mathbb{R})$, or $\mathrm{so}(n;\mathbb{R})$, we say that* (1.0.1) *is reducible if there exist $\widetilde{A} \in \mathfrak{g}$ and a regular map $Z : \varphi \to Z(\varphi)$, defined from $2\mathbb{T}^d = \mathbb{R}^d/(4\pi\mathbb{Z}^d)$ to the Lie*

group G associated to $\mathfrak{g}$, such that the following holds: if $u = Z(\omega t)v$, v satisfies

$$\partial_t v = \widetilde{A}v\,. \tag{1.0.2}$$

2. (Complex algebras) *If $A \in \mathrm{u}(n)$ and $P : \mathbb{T}^d \to \mathrm{u}(n)$, we say that* (1.0.1) *is reducible if there exist $\widetilde{A} \in \mathrm{u}(n)$ and a regular map $Z : \varphi \to Z(\varphi)$, defined from $\mathbb{T}^d$ to the Lie group $U(n)$, such that, if $u = Z(\omega t)v$, v satisfies* (1.0.2).

In the infinite dimensional case, $\mathcal{H}$ is typically the space of L^2 functions on a given domain in $\mathbb{R}^n$, $n \geq 1$. Specially due to the relation between reducibility and KAM theory for PDEs, a whole scale of Sobolev spaces is usually considered, thus we require:

Definition 1.0.2 (Infinite dimensional reducibility)**.** *Let $\{\mathcal{H}^\sigma\}_{\sigma\in\mathbb{R}}$ be a scale of Hilbert spaces and $\mathcal{H}^0 = \mathcal{H}$. Then, given a real interval $S \subseteq \mathbb{R}^+$,* (1.0.1) *is reducible in $\{\mathcal{H}^\sigma\}_{\sigma\in S}$ if in $\mathcal{H}$ there exist a time independent operator $\widetilde{A}$ and a map $Z : \varphi \mapsto Z(\varphi)$, from $\mathbb{T}^d$ to the set of linear bounded isomorphisms of $\mathcal{H}^\sigma$ for any $\sigma \in S$ satisfying the following. If $u = Z(\omega t)v$, then v solves the differential equation*

$$\partial_t v = \widetilde{A}v\,; \tag{1.0.3}$$

furthermore, for any $\sigma \in S$ and for any $\psi \in \mathcal{H}^\sigma$ the map $\varphi \mapsto Z(\varphi)\psi$ is continuous from $\mathbb{T}^d$ to $\mathcal{H}^\sigma$, and for any $s \geq 0$ there exists $\eta \geq 0$ such that, for any $\sigma \in S$, the map $\varphi \mapsto Z(\varphi)$ is of class $\mathcal{C}^s$ as a map from $\mathbb{T}^d$ to the set of bounded linear operators from $\mathcal{H}^\sigma$ to $\mathcal{H}^{\sigma-\eta}$.

Remark that, if the system in (1.0.1) is reducible and the eigenvalues of $\widetilde{A}$ are all purely imaginary, then all its solutions are almost-periodic.

1.1 General scheme

In order to clarify the key steps and difficulties in proving reducibility for a system of the form (1.0.1), we start with giving a schematic outline of the general strategy for the proof. Usually reducibility is proven via an iterative process made of infinite steps, where the size of the time dependent perturbation is recursively reduced. So as to present the procedure, we focus on the first step: one starts with looking for a map Z which conjugates $A + \varepsilon P$ to a new operator $A^+ + P^+$, where A^+ is still time independent

(possibly diagonal), and $P^+ = \mathcal{O}(\varepsilon^2)$. Assume for simplicity that Z is of the form $Z = \mathbb{I} + \varepsilon X$.
If X solves an equation of the form

$$-\omega \cdot \partial_\varphi X(\varphi) + [A, X(\varphi)] + P(\varphi) = A' , \tag{1.1.1}$$

then one sets $A^+ = A + \varepsilon A'$. Suppose, again for simplicity, that A is diagonal with distinct eigenvalues $\{\lambda_j\}_j$ and that $\{e_j\}_j$ is a complete orthonormal set of eigenfunctions (in the finite dimensional case, simply a basis). Passing to Fourier modes with respect to the φ variable and to matrix elements with respect to $\{e_j\}_j$, namely writing, given a generic operator B, $B(\varphi) = \sum_{k\in\mathbb{Z}^d} \widehat{B}(k) e^{\mathrm{i}k\cdot\varphi}$, and

$$\widehat{B}_{i,j}(k) = \langle \widehat{B}(k) e_i, e_j \rangle_{\mathcal{H}} \quad \forall i, j ,$$

in order to solve (1.1.1) one is lead to consider an expression of the form

$$\widehat{X}_{ij}(k) = \frac{\widehat{P}_{ij}(k)}{(\mathrm{i}\omega \cdot k - \lambda_i + \lambda_j)} \tag{1.1.2}$$

for $k \neq 0$ or $k = 0$ and $i \neq j$, and to set $\widehat{X}(0) = 0$, $A' = [\widehat{P}(0)]$, where $[\widehat{P}(0)]$ is the diagonal part of $\widehat{P}(0)$.
Of course, in order to have a well defined solution X, this would require to have at disposal good lower bounds for the quantities appearing at the denominators in (1.1.2), and in order to iterate the procedure, also for the quantities

$$\mathrm{i}\omega \cdot k - \lambda_i^+ + \lambda_j^+ ,$$

if $\{\lambda_j^+\}_j$ are the eigenvalues of A^+. Such lower bounds are the ones well known in literature as *second order Mel'nikov conditions*, and play a crucial role along the reducibility process.
The case where the eigenvalues of A have multiplicities higher than one is more delicate, especially in the infinite dimensional case; see [CY00], where the case of well separated eigenvalues with double multiplicity has been treated, and the discussion in Subsection 1.3.2 concerning blockwise imposition of Mel'nikov conditions.

1.2 A glance at the finite dimensional case

Perturbative reducibility theory in finite dimension is up to now quite well understood. Not all finite dimensional quasi-periodic systems are reducible, but a weaker notion has been formulated and widely investigated together with reducibility: *almost reducibility*. Since these two notions describe all finite dimensional quasi-periodic systems close to a constant coefficients one (in a sense that now we are going to specify), before stating results let us briefly focus on the concept of almost reducibility.
Roughly speaking, the system (1.0.1) is said to be almost reducible if it can be conjugated to a new system that is not time independent, but it is arbitrarily close to a time independent one. We point out that, if a system is almost reducible, its solutions exhibit for an arbitrarily long time the same qualitative behavior of a reducible system; however, this does not prevent the system from being close to an ergodic one: see [Eli02], where unique ergodicity is shown in the case $\mathrm{so}(3;\mathbb{R})$.
For the sake of clarity, we specialize on the case of quasi-periodic systems belonging to the Lie algebra $\mathrm{gl}(n,\mathbb{R})$, and we give a quantitative definition of almost reducibility, following [Cha13, Eli01].

Definition 1.2.1. *Let $A \in \mathrm{gl}(n;\mathbb{R})$ and $P : \mathbb{T}^d \to \mathrm{gl}(n;\mathbb{R})$ be analytic in a complex neighborhood of $\mathbb{T}^d$ of width r for some $r > 0$, which we denote by $\mathbb{T}^d_r$. Let $|\cdot|_r$ be the norm in $\mathbb{T}^d_r$. (1.0.1) is said to be almost reducible if for any $\delta > 0$ there exist $r_\delta > 0$ and $Z_\delta : 2\mathbb{T}^d \to GL(n;\mathbb{R})$, with an analytic extension on $\mathbb{T}^d_{r_\delta}$, such that for all $\varphi \in 2\mathbb{T}^d$ $u = Z_\delta(\varphi)v$ solves on $\mathrm{gl}(n;\mathbb{R})$ the equation*

$$\partial_t v = \left(\widetilde{A}_\delta + F_\delta(\theta)\right) v\,, \tag{1.2.1}$$

where $\widetilde{A}_\delta \in \mathrm{gl}(n;\mathbb{R})$ and $F_\delta : 2\mathbb{T}^d \to \mathrm{gl}(n;\mathbb{R})$ satisfies $|F_\delta|_{r_\delta} < \delta$.

With no aim at all of being exhaustive, we just recall the following few basic facts on the behavior of systems in $\mathrm{gl}(n,\mathbb{R})$: fix A and $P \in \mathrm{gl}(n,\mathbb{R})$, let $\omega \in \mathbb{R}^d$ a Diophantine vector, namely

$$|\omega \cdot k| \geq \gamma |k|^{-\tau} \quad \forall k \in \mathbb{Z}^d \backslash \{0\} \tag{1.2.2}$$

for some $\gamma, \tau \in \mathbb{R}^+$, and suppose that $r > 0$ is such that $|P|_r < \infty$, where $|\cdot|_r$ is the sup norm in the complexification of the torus $\mathbb{T}^d$ of width r. Consider instead of (1.0.1) the one parameter family of systems

$$\partial_t u = (\lambda A + P(\omega t))\, u\,, \quad \lambda \in (0,1]\,. \tag{1.2.3}$$

There exists $\varepsilon_0 = \varepsilon_0(|A|_r, r, \omega)$ such that, if $|P|_r < \varepsilon_0$, the following holds:

- For all λ, the system (1.2.3) is almost reducible (see [Eli01, Cha13])
- For almost all λ, the system (1.2.3) is reducible (see [Kri95, Cha11])

The two above results hold also for the other aforementioned finite dimensional Lie algebras. Furthermore we recall that, at least in the case of $\mathrm{so}(3;\mathbb{R})$, although reducibility is a generic property from a measure theory point of view, it is not from a topological one: for any λ there exists a dense set inside $\{P \mid |P|_r < \varepsilon_0\}$ such that (1.2.3) is non reducible (see [Eli92]). (Here, density is w.r.t. the topology induced by $|\cdot|_r$).

1.3 Infinite dimensional reducibility

Coming to infinite dimensional systems, the theory is far from being complete. The development of reducibility techniques and results is deeply related to those for KAM theory for PDEs, and again most of the results are on one dimensional domains.

1.3.1 A first breakthrough

The first reducibility work is [Com88], concerning smoothing perturbations of the Harmonic oscillator on the real line; the techniques implemented therein were then extended in [Dv96, DLvV02] in order to treat the case of bounded perturbations of the Schrödinger operator with a super quadratic potential. Almost at the same time, the pioneering works in KAM for PDEs paved the way for a new approach: in [Kuk87] and [Way90], and later in [KP96, Pos96], the existence of periodic and quasi-periodic solutions to bounded perturbations of the Schrödinger and wave equations was shown in the case of Dirichlet boundary conditions, with techniques that can easily be adapted to the reducibility problem. Recall that their constructions exploit the existence of a large measure set of frequencies ω such that it is possible to impose second order Mel'nikov conditions of the form

$$|\mathrm{i}\omega \cdot k + \lambda_i - \lambda_j| \geq \gamma |k|^{-\tau}$$

for the eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ of the unperturbed operator, and to preserve such conditions along the iterative algorithm, as in the finite dimensional theory.

The same kind of techniques enabled to show in [Kuk98] the existence of quasi-periodic solutions for unbounded perturbations of the KdV equation (in particular, the perturbation contained one space derivative), as well as to prove, in [Kuk97], a Lemma which provided an a priori estimate on the solution χ of the following equation:

$$-\mathrm{i}\omega\cdot\partial_\varphi\chi(\varphi)+(E+Bh(\varphi))\,\chi(\varphi)=b(\varphi)\,,\quad \varphi\in\mathbb{T}^d\,. \tag{1.3.1}$$

Here h and b are analytic functions, $\omega\in\mathbb{T}^d$ is a Diophantine vector incommensurable with E, and the two real numbers E and B satisfy the following relation:

$$\exists C>0 \text{ and } \theta\in(0,1)\quad \text{s.t. } E^\theta\geq CB\,.$$

Kuksin's Lemma enabled to prove, in [BG01], the first reducibility result for unbounded perturbations of a Schrödinger operator on the real line with a super quadratic potential. In particular, the authors treated the case of an abstract system of the form

$$\mathrm{i}\partial_t\psi(t)=(A+\varepsilon P(\omega t))\psi(t)\,,\quad \psi\in\mathcal{H}\,,$$

with $\mathcal{H}$ a separable Hilbert space, $\omega\in[1,2]^d$ a Diophantine vector, A a self-adjoint operator with eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ growing at infinity as $\lambda_j\sim j^a$ for some $a>1$, and P a quasi-periodic perturbation of order δ, in the sense that its growth properties are controlled as follows: there exists $\delta<a-1$ such that the map

$$\mathbb{T}^d\ni\varphi\mapsto\sup_{\|u\|_{\mathcal{H}}=1}\|A^{-\frac{\delta}{a}}P(\varphi)u\|_{\mathcal{H}}$$

is analytic. The key point is that the gaps between the eigenvalues of A are increasing, namely one has $|\lambda_i-\lambda_j|\sim|i^a-j^a|$ as $i,j\to\infty$: this enables to impose and to preserve along the iterative process second order Mel'nikov conditions of the form

$$|-\mathrm{i}\omega\cdot k+\lambda_i-\lambda_j|\geq\gamma|i^a-j^a|(1+|k|^{-\tau})\,. \tag{1.3.2}$$

Then Kuksin's Lemma guarantees that the solution X of an homological equation of the form

$$(-\mathrm{i}\omega\cdot k+\lambda_i-\lambda_j)\,\widehat{X}_{ij}(k)=\widehat{P}_{ij}(k)$$

with $\omega,\lambda_i,\lambda_j$ satisfying (1.3.2) is well posed and bounded, notwithstanding the unboundedness of the perturbation P.

We point out that the constraint $\delta < a-1$ was then weakened to $\delta \leq a-1$ in [LY10], and for the Harmonic oscillator in dimension 1, the case of bounded perturbations going to 0 as $x \to \infty$ was treated in [GT11] and [Wan08b].

1.3.2 Increasing unboundedness of the perturbation

The above ideas do not apply to the case of more unbounded perturbations, but a further breakthrough to treat such a case was given by the new method proposed in [BBM14], which enabled to extend KAM and reducibility techniques to the quasilinear case.
The idea, originated from [IPT05] in the context of water waves theory, is to proceed in two steps and, before reducibility itself, to use tools from pseudo-differential calculus in order to conjugate the initial operator $A + \varepsilon P(\omega t)$, with P containing the same number of derivatives as A, to a new operator $A^{(1)} + \varepsilon P^{(1)}$, where $P^{(1)}$ is of lower order with respect to P; after a finite number of such steps, one is left with a new operator of the form $A^{(N)} + \varepsilon P^{(N)}$, with $P^{(N)}$ at least bounded, which fits the standard KAM approach for reducibility.
This led to the first reducibility results for quasi-linear perturbations of the Airy equation, KdV and mKdV equation (see [BBM14, BBM16a, BBM16b]), for quasi-linear reversible and Hamiltonian NLS equations [FP15, Feo16], for water waves equations [BM20b, BM17]. Furthermore, the same technique was extended in [Bam18, Bam17, BM18] in order to treat the case of unbounded perturbations of the Harmonic oscillator, reaching up to the value $\delta = a$ under suitable assumptions on the form of the perturbation.

1.3.3 Dealing with higher dimension

Apart from the unboundedness of the perturbation, a further typical obstacle in proving reducibility for a given system is represented by the case of multiplicities in the spectrum of the unperturbed operator A in (1.0.1). This configuration is typical of systems in dimension higher than one. In fact, there are only a few examples of reducibility, or KAM, results on higher dimensional domains, and most of them are proved in the case of tori.
First of all, recall the work of Eliasson and Kuksin [EK09] on the $d-$dimensional torus, for bounded quasi-periodic in time perturbations of the Schrödinger operator; such a reducibility result was obtained as a consequence of their

KAM result [EK10] for a Schrödinger equation of the form

$$\mathrm{i}\psi_t = -\Delta\psi + V * \psi + \varepsilon\partial_{\overline{\psi}}F(\psi, \overline{\psi}, x)\,, \quad x \in \mathbb{T}^d,\ \psi \in L^2(\mathbb{T}^d)\,. \tag{1.3.3}$$

(Here, $\overline{\psi}$ denotes the complex conjugate and $*$ is the convolution product). The construction performed therein is based on a *blockwise imposition* of second order Mel'nikov conditions, due to the high multiplicities of the eigenvalues of the unperturbed operator $-\Delta$.
Note that the unperturbed operator (corresponding to $\varepsilon = 0$) satisfies the Töpliz property, namely its matrix elements are such that $A_i^j = A_{i-j}$ for all $i, j \in \mathbb{Z}^d$: this makes possible to verify, at least at the first iterative step, second order Mel'nikov conditions of the following form:

$$|\mathrm{i}\omega \cdot k + \lambda_i - \lambda_k| \geq \kappa \quad \forall |k|,\ |i-j| < K\,, \tag{1.3.4}$$

for a suitable choice of K and κ.
However, the Töpliz property is not preserved along the iteration: this is why, in order to preserve Mel'nikov conditions of the form of (1.3.4) at any step, the authors actually exploit a weaker property, referred to as Töpliz-Lipschitz property. We also point out that a similar result, inspired by the techniques in [EK10], was obtained in [PX13], where the authors analyze a nonlinear perturbation of the Schrödinger operator which preserves momentum, and in order to ensure imposition and preservation of Mel'nikov conditions exploit an analogous property, which is instead closed with respect to the operation of taking Poisson brackets (quasi-Töpliz functions). See also the KAM result [PP15] and the reducibility result [PP16].
In all such results, the structure of the torus, and in particular of its dual lattice $\mathbb{Z}^d$, is deeply exploited, as well as the good separation properties of eigenvalues of the unperturbed operator $-\Delta$. This is what pushed us to tackle, in Part II of the present work, the case of flat tori, where such structural hypotheses are violated, and to analyze the spectral problem for a stationary Schrödinger operator as a preliminary step in order to investigate reducibility in the time dependent case.

We also recall [GP16a] as another result on the same lines of [EK10] but on a different domain, where blockwise imposition of the Mel'nikov conditions and good separation properties of the eigenvalues are again exploited in order to prove reducibility for the completely resonant quantum Harmonic oscillator on $\mathbb{R}^d$, perturbed by a time quasi-periodic potential. See also [GP16b], for the Klein Gordon equation on the sphere.

1.3.4 A new technique for the high dimensions

As anticipated, the new method introduced in [BBM14] has also been recently applied to obtain a few higher dimensional reducibility results.
In particular, in [BGMR18] reducibility is obtained for unbounded perturbations of the quantum Harmonic oscillator on $\mathbb{R}^d$ of the form

$$\mathrm{i}\partial_t\psi = \left(-\Delta + \sum_{j=1}^{d} \nu_j^2 x_j^2\right)\psi + \varepsilon\mathcal{W}(\omega t, x, -\mathrm{i}\nabla)\psi\,, \tag{1.3.5}$$

where $\{\nu_j\}_{j=1}^d$ are non resonant frequencies, $\omega \in \mathbb{R}^{\mathfrak{n}}$ and the smooth function $\mathcal{W} : \mathbb{T}^{\mathfrak{n}} \times \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathcal{W}(\theta, x, \xi)$ is a polynomial of degree 2 in the variables (x, ξ).
Furthermore, reducibility is shown in [FGMP19] for the transport-like equation on $\mathbb{T}^d$

$$\partial_t u = \nu \cdot \nabla u + \varepsilon a(\omega t, x) \cdot \nabla u\,, \tag{1.3.6}$$

where $\nu \in \mathbb{R}^d$, $\omega \in \mathbb{R}^{\mathfrak{n}}$ and $a : \mathbb{T}^{\mathfrak{n}} \times \mathbb{T}^d \to \mathbb{R}^d$ is a smooth function with quasi-periodic dependence on time, and in [Mon19] for linear waves equation on $\mathbb{T}^d$ of the form

$$\partial_{tt} u = -\Delta u + \varepsilon a(\omega t)(-\Delta)u + \varepsilon\mathcal{R}(\omega t)u\,, \tag{1.3.7}$$

where $\omega \in \mathbb{R}^{\mathfrak{n}}$, a is a quasi-periodic function with no dependence on space variable and $\mathcal{R}$ is a quasi-periodic finite rank operator. We point out that in [Mon19], along the lines of [BBHM18], very weak Mel'nikov conditions are assumed, of the form

$$|\mathrm{i}\omega \cdot k + \lambda_i - \lambda_j| \geq \gamma\langle k\rangle^{-\tau}\langle i\rangle^{-\tau}\langle j\rangle^{-\tau} \quad \forall k \neq 0, \quad \forall i \neq j\,, \tag{1.3.8}$$

and the preliminary regularization step is then used in order to balance the loss of regularity in space due to such very weak Mel'nikov conditions.
Non resonance conditions of the same form (1.3.8) are assumed in Chapter 2 of this thesis to prove Theorem 0.0.1, which is part of this group of results and, as we are about to expose in detail, combines techniques from [Mon19, FGMP19] and from [BGMR17].
We finally mention the very recent works [FGN20], about reducibility on Zoll manifolds for unbounded perturbations of order less than 1/2, and [BM20a], where the techniques that we present in Chapter 2 are extended in order to find quasi-periodic solutions in the case of three dimensional Euler equation.

Chapter 2

Reducibility of a transport equation on $\mathbb{T}^d$

In the present chapter we obtain reducibility on the d-dimensional torus $\mathbb{T}^d$, $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$, $d \geq 1$, for the following transport equation:

$$\partial_t u = \Big(\nu + \varepsilon V(\omega t, x)\Big) \cdot \nabla u + \varepsilon \mathcal{W}(\omega t)[u], \tag{2.0.1}$$

where the frequencies $\omega \in \mathbb{R}^{\mathfrak{n}}$ and $\nu \in \mathbb{R}^d$ play the role of parameters, $\varepsilon > 0$ is a small parameter, $V \in \mathcal{C}^\infty(\mathbb{T}^{\mathfrak{n}} \times \mathbb{T}^d, \mathbb{R}^d)$ is a real function and $\mathcal{W}(\varphi)$, $\varphi \in \mathbb{T}^{\mathfrak{n}}$ is a pseudo-differential operator of order $1 - \mathfrak{e}$, for some $\mathfrak{e} > 0$.
More precisely, our aim is to show that for *most* values of $\widetilde{\omega} = (\omega, \nu) \in \Omega := [1, 2]^{\mathfrak{n}+d}$ and for ε small enough, there exists a bounded and invertible transformation (acting on the scale of Sobolev spaces) which transforms the PDE (2.0.1) into another one whose vector field is a time independent diagonal operator. As a corollary, we give a characterization for all solutions of (2.0.1): either they are all almost periodic, and all their Sobolev norms remain bounded globally in time, or there exists at least one solution diverging exponentially in time.

In order to state precisely the main results of the present chapter, first of all we introduce some notations: for any $\sigma \in \mathbb{R}$ we consider the Sobolev space $\mathcal{H}^\sigma(\mathbb{T}^d)$ endowed by the norm

$$\|u\|_{\mathcal{H}^\sigma} := \Big(\sum_{\xi \in \mathbb{Z}^d} \langle \xi \rangle^{2\sigma} |\hat{u}_\xi|^2 \Big)^{\frac{1}{2}}$$

where $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$ and $\hat{u}_\xi$ are the Fourier coefficients of u.
Furthermore, given two Banach spaces X, Y we denote by $\mathcal{B}(X,Y)$ the space of bounded linear operators $X \to Y$ equipped by the standard operator norm; if $X = Y$, we simply write $\mathcal{B}(X)$ instead of $\mathcal{B}(X,X)$. We define the constant

$$s_0 := \left\lceil \frac{\mathfrak{n}}{2} \right\rceil + 1\,, \tag{2.0.2}$$

where $\lceil\cdot\rceil$ denotes the rounding to the next integer, and given a sequence $(\lambda_j)_{j\in\mathbb{Z}^d} \subset \mathbb{C}$, we define the diagonal operator $\operatorname{diag}(\lambda_j)$ by

$$[\operatorname{diag}(\lambda_j)u](x) := \sum_{j\in\mathbb{Z}^d} \lambda_j \hat{u}_j e^{\mathrm{i}x\cdot j}\ .$$

Finally, in the present chapter and throughout all the present work, given $\alpha,\ \beta \in \mathbb{R}$, we will write $\alpha \lesssim \beta$ if there exists $C > 0$ (independent of all the relevant quantities) such that $\alpha \le C\beta$. Sometimes we will write $\alpha \lesssim_{s_1,\dots,s_n} \beta$ if C depends on parameters $s_1, \cdots, s_n$.

2.1 Setting: pseudo-differential operators and structural hypotheses

As anticipated in the Introduction, our reducibility result exploits pseudo-differential calculus. Roughly speaking, pseudo-differential calculus enables to establish a correspondence between a given class of functions (symbols), and a class of operators (pseudo-differential operators). Here we specify the setting that we are going to assume in order to obtain our result:

Definition 2.1.1 (Symbols on the standard torus)**.** *Let $m \in \mathbb{R}$. We say that a function $a \in \mathcal{C}^\infty\left(\mathbb{T}^d \times \mathbb{R}^d; \mathbb{C}\right)$ is a symbol of class S^m if for any multiindex $\alpha,\ \beta \in \mathbb{N}^d$ there exists a constant $C_{\alpha,\beta} > 0$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-|\beta|}\,, \quad \forall (x,\xi) \in \mathbb{T}^d \times \mathbb{R}^d\,. \tag{2.1.1}$$

A symbol a defines univocally a linear operator A acting as

$$A[u](x) := \sum_{\xi\in\mathbb{Z}^d} a(x,\xi)\hat{u}_\xi e^{\mathrm{i}x\cdot\xi}\,, \qquad \forall u \in \mathcal{C}^\infty(\mathbb{T}^d)\,,$$

that we denote by $A = Op^{cl}\left(a\right)$.

Definition 2.1.2 (Classical quantization)**.** *An operator A is called a pseudo-differential operator of order m, namely $A \in OPS^m$, if there exists $a \in S^m$ such that*

$$A = Op^{cl}(a).$$

Remark 2.1.3. *The constants $C_{\alpha,\beta}$ of Definition 2.1.1 form a family of seminorms for S^m and for OPS^m.*

Remark 2.1.4. *The above correspondence $S^m \ni a \mapsto Op^{cl}(a)$ is well known in literature as* classical quantization*. Of course this is not the only possible way to put in correspondence symbols and operators. In Part II for instance,* Weyl quantization *is used. See for instance [Rob87] for an exposition about the different types of quantization. A few details about this are also given in Appendix A.*

In the following, we will consider pseudo-differential operators depending in a smooth way on the angles $\varphi \in \mathbb{T}^{\mathfrak{n}}$ and in a Lipschitz way on the frequencies $\widetilde{\omega} = (\omega, \nu) \in \Omega_0 \subseteq \Omega$. We will denote them by $\mathcal{L}ip\left(\Omega_0; \mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}}; OPS^m\right)\right)$. We refer to Appendix A for a quick survey on a few basic properties on such a class of pseudo-differential operators, and to the books [Tay91], [Rob87], [SV02] for an extensive analysis.

There is no self-adjoint (or anti self-adjoint) structure in the system of 2.0.1, due to the presence of the first order term $\varepsilon V(\omega t, x) \cdot \nabla$. However, we work in a symmetric hyperbolic context. In particular we assume that the perturbation $\mathcal{W}$ is symmetric hyperbolic, according to the following definition:

Definition 2.1.5 (Symmetric hyperbolicity)**.** *We say that $\mathcal{W} \in OPS^1$ is symmetric hyperbolic if $\mathcal{W} + \mathcal{W}^* \in OPS^0$.*

Furthermore, we will consider the following additional structural hypotheses on our system (2.0.1):

Definition 2.1.6 (Structural hypotheses)**.**

(i) *We say that $\mathcal{R} \in \mathcal{B}(L^2(\mathbb{T}^d))$ is a real operator if it maps real valued functions into real valued functions, namely*

$$u \in L^2(\mathbb{T}^d; \mathbb{R}) \Rightarrow \mathcal{R}[u] \in L^2(\mathbb{T}^d; \mathbb{R}).$$

Equivalently, we can say that $\mathcal{R}$ is a real operator if $\mathcal{R} = \overline{\mathcal{R}}$, where the operator $\overline{\mathcal{R}}$ is defined by $\overline{\mathcal{R}}[u] := \overline{\mathcal{R}[\overline{u}]}$, $u \in L^2(\mathbb{T}^d)$, and given $\alpha \in \mathbb{C}$, $\overline{\alpha}$ denotes its conjugate.

(ii) Let $\varphi \mapsto \mathcal{R}(\varphi), \mathcal{Q}(\varphi)$ be smooth φ-dependent families of real operators $\mathbb{T}^{\mathfrak{n}} \to \mathcal{B}\big(L^2(\mathbb{T}^d)\big)$; we say that $\mathcal{R}$ is reversible if

$$\mathcal{R}(\varphi) \circ S = -S \circ \mathcal{R}(-\varphi)\,, \quad \forall \varphi \in \mathbb{T}^{\mathfrak{n}}, \tag{2.1.2}$$

where S is the involution defined by

$$S : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)\,, \quad u(x) \mapsto u(-x). \tag{2.1.3}$$

On the other hand, we say that $\mathcal{Q}$ is reversibility preserving if

$$\mathcal{Q}(\varphi) \circ S = S \circ \mathcal{Q}(-\varphi)\,, \quad \forall \varphi \in \mathbb{T}^{\mathfrak{n}}. \tag{2.1.4}$$

We will also consider the case where V is even, namely one has

$$V(-\varphi, -x) = V(\varphi, x)\ .$$

This is the main result of the present chapter:

Theorem 2.1.7. *Let $V \in \mathcal{C}^\infty(\mathbb{T}^{\mathfrak{n}} \times \mathbb{T}^d, \mathbb{R}^d)$, $\mathcal{W} \in \mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}}; OPS^{1-\mathfrak{e}}\right)$ and assume that $\mathcal{W}$ is symmetric hyperbolic. For any $s \geq 0$ and $\sigma \geq 0$ there exist $\varepsilon^* > 0$ and $\eta = \eta_s > 0$ such that $\forall \varepsilon < \varepsilon^*$ there exists a closed set $\Omega_\varepsilon \subseteq \Omega$ of asymptotically full Lebesgue measure, i.e. $\lim_{\varepsilon \to 0} |\Omega \setminus \Omega_\varepsilon| = 0$, with the following properties. For any $\widetilde{\omega} = (\omega, \nu) \in \Omega_\varepsilon$ there exists a map $\mathbb{T}^{\mathfrak{n}} \ni \varphi \mapsto \mathcal{U}(\varphi) = \mathcal{U}(\varphi; \widetilde{\omega})$, with $\mathcal{U}(\varphi) \in \mathcal{B}(\mathcal{H}^\sigma)$ a linear bounded and invertible operator $\forall \varphi$, such that, if u solves (2.0.1), then v defined by $u = \mathcal{U}(\omega t) v$ solves*

$$\partial_t v = H_\infty v, \tag{2.1.5}$$

where

$$H_\infty = \operatorname{diag}(\lambda_j^{(\infty)}(\tilde{\omega}, \varepsilon))\,, \tag{2.1.6}$$

and $\mathcal{U}$ has the following properties:

(i) $\mathcal{U}^{\pm 1} \in \mathcal{C}^s\left(\mathbb{T}^{\mathfrak{n}}; \mathcal{B}(\mathcal{H}^{\sigma+\eta}, \mathcal{H}^\sigma)\right)$, and $\forall \psi \in \mathcal{H}^\sigma$ the map $\varphi \mapsto \mathcal{U}^{\pm 1}(\varphi)\psi$ is continuous.

(ii) $\exists \sigma_1 > 0$, independent of σ, such that $\|\mathcal{U}^{\pm 1}(\varphi) - \mathbb{I}\|_{\mathcal{B}(\mathcal{H}^{\sigma+\sigma_1}, \mathcal{H}^\sigma)} \lesssim_\sigma \varepsilon$ for all $\varphi \in \mathbb{T}^{\mathfrak{n}}$.

Furthermore, the eigenvalues $\{\lambda_j^{(\infty)}(\tilde{\omega}, \varepsilon)\}_{j\in\mathbb{Z}^d}$ *have the following structure:*

$$\lambda_j^{(\infty)}(\tilde{\omega}, \varepsilon) = \mathrm{i}\nu^{(0)} \cdot j + z(j) + O\big(\varepsilon \langle j \rangle^{-2m}\big) \ , \tag{2.1.7}$$

where $m := 2\tau + 2$, $z(\cdot) \in S^{1-\mathfrak{e}}$ *depends in a Lipschitz way on* $\tilde{\omega}$, *and* $\nu^{(0)} = \nu^{(0)}(\tilde{\omega})$ *fulfills*

$$\big|\nu^{(0)} - \nu\big| \le C\varepsilon \ .$$

Finally, assume that the following assumption holds:

$$V \textit{ is even and } \mathcal{W} \textit{ is real and reversible;} \tag{Sym}$$

then $\lambda_j^{(\infty)} \in \mathrm{i}\mathbb{R} \quad \forall\, j \in \mathbb{Z}^d$.

Remark 2.1.8. *Note that the Theorem above implies global well posedness of the equation* (2.0.1) *for all the frequencies* $(\omega, \nu) \in \Omega_\varepsilon$ *and for* ε *small enough. Actually, global well-posedness holds for all values of* ε, ω, ν, *by Proposition 0.8.A of [Tay91].*

Remark 2.1.9. *As anticipated in the Introduction, here we do not provide tame estimates enabling to treat the nonlinear problem. To this aim we refer to [BM20a], which is subsequent to the present work. In [BM20a] indeed the existence of quasi-periodic solutions close to constant vector fields for 3D Euler equation is proven, with a Nash Moser algorithm that requires to show reducibility with tame estimates for a small quasi-periodic in time perturbation of a transport equation with a structure very close to* (2.0.1)*. The main differences are that the linearized equation in [BM20a] is vector valued, and that the perturbation therein still has the form* $V(\varphi, x) \cdot \nabla + \mathcal{W}(\varphi)$, *but with a bounded operator* $\mathcal{W}$.

From the above Theorem 2.1.7 we can deduce information concerning the dynamics of the PDE (2.0.1).

Corollary 2.1.10. *Without assuming the hypothesis* (Sym), $\forall \sigma \ge 0$ *and* $\forall (\omega, \nu) \in \Omega_\varepsilon$, *only one of the following two possibilities occurs:*

(1) All the solutions of (2.0.1) *are almost periodic and*

$$u_0 \in \mathcal{H}^\sigma \implies \|u(t, \cdot)\|_{\mathcal{H}^\sigma} \lesssim \|u_0\|_{\mathcal{H}^\sigma} \tag{2.1.8}$$

uniformly w.r. to $t \in \mathbb{R}$.

(2) There exist $a, C > 0$ and some initial data u_0 s.t.

$$\|u(t,\cdot)\|_{\mathcal{H}^\sigma} \geq Ce^{a|t|}\|u_0\|_{\mathcal{H}^\sigma} \tag{2.1.9}$$

either for $t > 0$ or for $t < 0$ or for $t \in \mathbb{R}$.

On the contrary, if hypothesis (Sym) *holds, only possibility* (1) *occurs.*

The remaining part of the present chapter is organized as follows: in Section 2.2 we conjugate the vector field of the equation (2.0.1) to another one which is an arbitrarily smoothing perturbation of a diagonal operator, while in Section 2.3 we perform a KAM-reducibility scheme for vector fields which are smoothing perturbations of a diagonal one, by imposing second order Mel'nikov conditions of the form (1.3.8), which, as observed in Subsection 1.3.4, yield a loss of regularity in space into the system (see Theorem 2.3.8). Finally, in Section 2.4 we complete the proof of Theorem 2.1.7 and we deduce from it Corollary 2.1.10.

2.2 Regularization

As a first step, we regularize the vector field

$$H(\varphi) := \big(\nu + \varepsilon V(\varphi, x)\big)\cdot\nabla + \varepsilon\mathcal{W}^{(0)}(\varphi)\,, \quad \mathcal{W} \in OPS^{1-\mathfrak{e}} \tag{2.2.1}$$

namely we conjugate it to another one which is a smoothing perturbation of a time independent diagonal operator.
First remark that a linear invertible transformation $u = \Phi(\omega t)u'$ which depends on time in a quasiperiodic way, transforms the equation $\partial_t u = Hu$ into the equation $\partial_t u' = H'u'$, where

$$H' = \Phi_{\omega*}H := \Phi(\varphi)^{-1}[H\Phi(\varphi) - \omega\cdot\partial_\varphi\Phi(\varphi)]\,, \tag{2.2.2}$$

and we used $\partial_t \equiv \omega\cdot\partial_\varphi$.

Definition 2.2.1 (Lipschitz norm)**.** *Given a Banach space $(X, \|\cdot\|_X)$, a set $\Omega_0 \subset \Omega = [1,2]^{\mathtt{n}+d}$, $\gamma > 0$ and a Lipschitz function $f : \Omega_0 \to X$, we denote by $\|\cdot\|_X^{\mathrm{Lip}(\gamma)}$ the Lipschitz norm defined by*

$$\begin{aligned} &\|f\|_X^{\mathrm{Lip}(\gamma)} := \|f\|_X^{\sup} + \gamma\|f\|_X^{\mathrm{lip}}\,, \\ &\|f\|^{\sup} := \sup_{\widetilde{\omega}\in\Omega_0}\|f(\widetilde{\omega})\|_X\,, \quad \|f\|_X^{\mathrm{lip}} := \sup_{\substack{\widetilde{\omega}_1,\widetilde{\omega}_2\in\Omega_0 \\ \widetilde{\omega}_1\neq\widetilde{\omega}_2}}\frac{\|f(\widetilde{\omega}_1) - f(\widetilde{\omega}_2)\|_X}{|\widetilde{\omega}_1 - \widetilde{\omega}_2|}\,. \end{aligned} \tag{2.2.3}$$

In the case where $\gamma = 1$*, we simply write* $\|\cdot\|_X^{\mathrm{Lip}}$ *for* $\|\cdot\|_X^{\mathrm{Lip}(1)}$*. If* $X = \mathbb{C}$ *we write* $|\cdot|^{\mathrm{Lip}(\gamma)}, |\cdot|^{\mathrm{sup}}, |\cdot|^{\mathrm{lip}}$ *for* $\|\cdot\|_{\mathbb{C}}^{\mathrm{Lip}(\gamma)}, \|\cdot\|_{\mathbb{C}}^{\mathrm{sup}}, \|\cdot\|_{\mathbb{C}}^{\mathrm{lip}}$*.*

2.2.1 Reduction to constant coefficients of the highest order term

We consider a diffeomorphism of the torus $\mathbb{T}^d$ of the form

$$\mathbb{T}^d \to \mathbb{T}^d, \quad x \mapsto x + \alpha(\varphi, x)$$

where $\alpha \in \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}} \times \mathbb{T}^d, \mathbb{R}^d)$ is a function to be determined. It is well known that for $\|\alpha\|_{C^1}$ small enough such a diffeomorphism is invertible and its inverse has the form

$$\mathbb{T}^d \to \mathbb{T}^d, \quad y \mapsto y + \widetilde{\alpha}(\varphi, y)$$

with $\widetilde{\alpha} \in \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}} \times \mathbb{T}^d, \mathbb{R}^d)$. We then consider the transformation

$$\mathcal{A}(\varphi) : u(x) \mapsto u(x + \alpha(\varphi, x)), \quad \varphi \in \mathbb{T}^{\mathtt{n}} \tag{2.2.4}$$

whose inverse is given by

$$\mathcal{A}(\varphi)^{-1} : u(y) \mapsto u(y + \widetilde{\alpha}(\varphi, y)), \quad \varphi \in \mathbb{T}^{\mathtt{n}}. \tag{2.2.5}$$

A direct calculation shows that the *quasi-periodic push-forward* of the vector field $H(\varphi)$ (recall the formula (2.2.2)) is given by

$$H^{(0)}(\varphi) = \mathcal{A}_{\omega *} H(\varphi) = V^{(0)}(\varphi, x) \cdot \nabla + \varepsilon \mathcal{W}^{(0)}(\varphi) \tag{2.2.6}$$

where

$$\begin{aligned} V^{(0)}(\varphi, x) &:= \mathcal{A}(\varphi)^{-1}\Big(\omega \cdot \partial_\varphi \alpha + \nu + \varepsilon V + \big(\nu + \varepsilon V\big) \cdot \nabla \alpha\Big) \\ \mathcal{W}^{(0)}(\varphi) &:= \mathcal{A}(\varphi)^{-1} \mathcal{W}(\varphi) \mathcal{A}(\varphi). \end{aligned} \tag{2.2.7}$$

The following proposition is a direct consequence of Proposition 3.4 in [FGMP19] to which we refer for the proof. It allows to choose the function $\alpha(\varphi, x)$ so that the highest order term $V^{(0)}(\varphi, x) \cdot \nabla$ in (2.2.6) is reduced to constant coefficients. Recall that, throughout the present Chapter, we have defined $\Omega = [1, 2]^{\mathtt{n}+d}$.

Proposition 2.2.2. *Let $\gamma \in (0,1)$ and $\tau > \mathfrak{n} + d$. There exists a Lipschitz function $\nu^{(0)} : \Omega \to \mathbb{R}^d$, $\quad \tilde{\omega} = (\omega, \nu) \mapsto \nu^{(0)}(\tilde{\omega})$ such that*

$$|\nu^{(0)}(\tilde{\omega}) - \nu|^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon, \tag{2.2.8}$$

and, in the set

$$\Omega_{0,\gamma} := \left\{ \tilde{\omega} \in \Omega : \ |\omega \cdot l + \nu^{(0)}(\tilde{\omega}) \cdot j| > \frac{\gamma}{\langle l, j \rangle^\tau} \, , \ \forall (l,j) \in \mathbb{Z}^{\mathfrak{n}+d} \setminus \{0\} \right\} \, , \tag{2.2.9}$$

the following holds. There exists a map

$$\alpha : \mathbb{T}^{\mathfrak{n}+d} \times \Omega_{0,\gamma} \to \mathbb{R}^d \tag{2.2.10}$$

such that the map $\mathbb{T}^{\mathfrak{n}+d} \to \mathbb{T}^{\mathfrak{n}+d}$, $(\varphi, x) \mapsto (\varphi, x + \alpha(\varphi, x, \tilde{\omega}))$ is a diffeomorphism with inverse given by $(\varphi, y) \mapsto (\varphi, y + \widetilde{\alpha}(\varphi, y, \tilde{\omega}))$, and

$$\|\alpha\|^{\mathrm{Lip}(\gamma)}_{\mathcal{H}^s} \lesssim_s \varepsilon\gamma^{-1}, \quad \|\widetilde{\alpha}\|^{\mathrm{Lip}(\gamma)}_{\mathcal{H}^s} \lesssim_s \varepsilon\gamma^{-1} \, , \quad \forall s \geq 0 \, . \tag{2.2.11}$$

Moreover for any $\tilde{\omega} \in \Omega_{0,\gamma}$ $V^{(0)}$ reduces to a constant (as a function of x and φ), namely

$$V^{(0)} = \mathcal{A}^{-1}(\varphi)\Big(\omega \cdot \partial_\varphi \alpha + \nu + \varepsilon V + \big(\nu + \varepsilon V\big) \cdot \nabla \alpha\Big) = \nu^{(0)}(\tilde{\omega}). \tag{2.2.12}$$

Finally, if V is even, then α and $\widetilde{\alpha}$ are odd.

Remark 2.2.3. *By standard arguments one has $|\Omega \setminus \Omega_{0,\gamma}| \lesssim \gamma$. More precisely, on the one side one has that vectors which are Diophantine with constant γ have complement with measure of order γ, and on the other, Lipschitz maps preserve the order of magnitude of the measure of sets.*

Remark 2.2.4. *Using the definitions* (2.2.4), (2.2.5) *and the estimates* (2.2.11), *a direct calculation shows that the map $\mathbb{T}^{\mathfrak{n}} \mapsto \mathcal{B}(\mathcal{H}^s)$, $\varphi \mapsto \mathcal{A}(\varphi)^{\pm 1}$ is bounded for any $s \geq 0$ and*

$$\sup_{\varphi \in \mathbb{T}^{\mathfrak{n}}} \|\mathcal{A}(\varphi)^{\pm 1} - \mathbb{I}\|_{\mathcal{B}(\mathcal{H}^{s+1}, \mathcal{H}^s)} \lesssim_s \varepsilon\gamma^{-1}, \quad \forall s \geq 0 \, , \tag{2.2.13}$$

$$\sup_{\varphi \in \mathbb{T}^{\mathfrak{n}}} \|\partial_\varphi^\alpha \mathcal{A}(\varphi)^{\pm 1}\|_{\mathcal{B}(\mathcal{H}^{s+|\alpha|}, \mathcal{H}^s)} \lesssim_{s,\alpha} \varepsilon\gamma^{-1}, \quad \forall s \geq 0, \quad \forall \alpha \in \mathbb{N}^{\mathfrak{n}} \, . \tag{2.2.14}$$

Recalling (2.2.6), (2.2.7) and applying Proposition 2.2.2 one gets that the vector field $H^{(0)}(\varphi)$ takes the form

$$H^{(0)}(\varphi) = \nu^{(0)} \cdot \nabla + \varepsilon \mathcal{W}^{(0)}(\varphi) \, . \tag{2.2.15}$$

We now study the properties of $\mathcal{W}^{(0)}$.

Lemma 2.2.5. *One has that* $\mathcal{W}^{(0)} \in \mathcal{L}ip\Big(\Omega_{0,\gamma}, \mathcal{C}^\infty\Big(\mathbb{T}^{\mathfrak{n}}, OPS^{1-\mathfrak{e}}\Big)\Big)$*. Moreover* $\mathcal{W}^{(0)}$ *is symmetric hyperbolic. Furthermore, if* V *is even and* $\mathcal{W}$ *real and reversible, then* $\mathcal{W}^{(0)}$ *is real and reversible.*

Proof. Let $\Phi(\varphi) := \mathcal{A}(\varphi)^{-1}$, i.e. $\Phi(\varphi)[u](y) = u(y + \widetilde{\alpha}(\varphi, y, \tilde{\omega}))$. For any $\tau \in [0, 1]$ let $\psi(\tau, \varphi, y, \tilde{\omega}) := u(y + \tau\widetilde{\alpha}(\varphi, y, \tilde{\omega}))$: then $\psi(0, \varphi, y, \tilde{\omega}) = u(y)$ and $\psi(\tau, \varphi, y, \tilde{\omega})$ satisfies the differential equation

$$\partial_\tau \psi = a(\tau, \varphi, y, \tilde{\omega}) \cdot \nabla\psi\,, \quad a(\tau, \varphi, y, \tilde{\omega}) := \big(\mathbb{I} + \tau\nabla\widetilde{\alpha}(\varphi, y, \tilde{\omega})\big)^{-1}\widetilde{\alpha}(\varphi, y, \tilde{\omega})\,. \tag{2.2.16}$$

Then by Egorov Theorem (see Theorem A.0.9 in [Tay91]) it follows that $\mathcal{W}^{(0)} \in \mathcal{L}ip\Big(\Omega_{0,\gamma}, \mathcal{C}^\infty\Big(\mathbb{T}^{\mathfrak{n}}, OPS^{1-\mathfrak{e}}\Big)\Big)$, since $\mathcal{W}^{(0)} = \mathcal{A}(\varphi)^{-1}\mathcal{W}\mathcal{A}(\varphi)$ and $\mathcal{W} \in \mathcal{L}ip\Big(\Omega_{0,\gamma}, \mathcal{C}^\infty\Big(\mathbb{T}^{\mathfrak{n}}, OPS^{1-\mathfrak{e}}\Big)\Big)$.

We now show that $\mathcal{W}^{(0)}$ is symmetric hyperbolic. To shorten notations, we sometimes omit the dependence on $\varphi \in \mathbb{T}^{\mathfrak{n}}$ and on $\tilde{\omega} \in \Omega_{0,\gamma}$. Since by (2.2.10), (2.2.11) the functions $\alpha, \widetilde{\alpha}$ satisfy $\alpha, \widetilde{\alpha} = \mathcal{O}(\varepsilon\gamma^{-1})$ one has that

$$\det\big(\mathbb{I} + \nabla\alpha\big)\,, \det\big(\mathbb{I} + \nabla\widetilde{\alpha}\big) > 0$$

for $\varepsilon\gamma^{-1}$ small enough. Moreover, using that $y \mapsto y + \widetilde{\alpha}(y)$ is the inverse diffeomorphism of $x \mapsto x + \alpha(x)$ one gets that

$$\det\big(\mathbb{I} + \nabla\widetilde{\alpha}(y)\big) = \frac{1}{\det\big(\mathbb{I} + \nabla\alpha(x)\big)|_{x=y+\widetilde{\alpha}(y)}}\,, \quad \forall y \in \mathbb{T}^d \tag{2.2.17}$$

A direct calculation shows that

$$\mathcal{A}^* = \det\big(\mathbb{I} + \nabla\widetilde{\alpha}\big)\mathcal{A}^{-1}\,, \quad (\mathcal{A}^{-1})^* = \det\big(\mathbb{I} + \nabla\alpha\big)\mathcal{A}\,.$$

Then

$$\begin{aligned}(\mathcal{W}^{(0)})^* &= (\mathcal{A}^{-1}\mathcal{W}\mathcal{A})^* = \mathcal{A}^*\mathcal{W}^*(\mathcal{A}^{-1})^* \\ &= \det\big(\mathbb{I} + \nabla\widetilde{\alpha}\big)\mathcal{A}^{-1}\mathcal{W}^*\det\big(\mathbb{I} + \nabla\alpha\big)\mathcal{A} \\ &= \det\big(\mathbb{I} + \nabla\widetilde{\alpha}\big)\mathcal{A}^{-1}\det\big(\mathbb{I} + \nabla\alpha\big)\mathcal{W}^*\mathcal{A} \\ &\quad + \det\big(\mathbb{I} + \nabla\widetilde{\alpha}\big)\mathcal{A}^{-1}[\mathcal{W}^*\,, \det\big(\mathbb{I} + \nabla\alpha\big)]\mathcal{A}\,.\end{aligned} \tag{2.2.18}$$

Since $\mathcal{W}^* \in OPS^{1-\mathfrak{e}}$ one has that the commutator $[\mathcal{W}^*\,, \det\big(\mathbb{I} + \nabla\alpha\big)] \in OPS^{-\mathfrak{e}} \subset OPS^0$. Using again that $\mathcal{A}(\varphi)^{-1} = \Phi(\varphi)$ is the time 1 flow map of the PDE (2.2.16), by applying Egorov Theorem A.0.9 in [Tay91], one gets that $\det\big(\mathbb{I}+\nabla\widetilde{\alpha}\big)\mathcal{A}^{-1}[\mathcal{W}^*\,,\,\det\big(\mathbb{I}+\nabla\alpha\big)]\mathcal{A}\in OPS^0$. Hence

$$
\begin{aligned}
(\mathcal{W}^{(0)})^* &= \det\big(\mathbb{I}+\nabla\widetilde{\alpha}\big)\mathcal{A}^{-1}\det\big(\mathbb{I}+\nabla\alpha\big)\mathcal{W}^*\mathcal{A}+OPS^0\\
&= \det\big(\mathbb{I}+\nabla\widetilde{\alpha}\big)\det\big(\mathbb{I}+\nabla\alpha\big)|_{x=y+\widetilde{\alpha}(y)}\mathcal{A}^{-1}\mathcal{W}^*\mathcal{A}+OPS^0\\
&= \mathcal{A}^{-1}\mathcal{W}^*\mathcal{A}+OPS^0\,, \qquad (2.2.19)
\end{aligned}
$$

due to (2.2.17).
Finally, using that $\mathcal{W}$ is symmetric hyperbolic, i.e. $\mathcal{W}+\mathcal{W}^*\in OPS^0$, by (2.2.17) and applying again Egorov Theorem A.0.9 in [Tay91] to deduce that $\mathcal{A}^{-1}(\mathcal{W}+\mathcal{W}^*)\mathcal{A}\in OPS^0$, one gets that $\mathcal{W}^{(0)}+(\mathcal{W}^{(0)})^*\in OPS^0$.
Suppose now that V is even and that $\mathcal{W}$ is real and reversible. By Proposition 2.2.2, from the fact that V is even it follows that $\alpha,\widetilde{\alpha}$ are odd functions, implying that $\mathcal{A}$, $\mathcal{A}^{-1}$ are reversibility preserving operators. Hence one concludes that $\mathcal{W}^{(0)}=\mathcal{A}^{-1}\mathcal{W}\mathcal{A}$ is a reversible operator. □

2.2.2 Reduction of the lower order terms

The reduction of the lower order terms follows from an adaptation of Theorem 3.8 of [BGMR17] to a symmetric hyperbolic context. In order to state the result, consider the following definition:

Definition 2.2.6. *For all $j=1,\dots,d$ define $K_j=\mathrm{i}\partial_{x_j}$, and $K=(K_1,\dots,K_d)$.*

Remark that the so defined $K_1,\dots,K_d$ are self-adjoint commuting operators such that $K_m\in OPS^1$ $\forall m=1,\dots,d$.

Theorem 2.2.7. *$\forall$ $M>0$ there exists a sequence of symmetric hyperbolic maps $\{G_j(\varphi,\tilde{\omega})\}_{j=1}^M$ with $G_j(\varphi,\tilde{\omega})\in\mathcal{L}ip\,(\Omega_{0,\gamma};\mathcal{C}^\infty\,(\mathbb{T}^{\mathfrak{n}};OPS^{1-j\mathfrak{e}}))$ $\forall j$ such that the change of variables $\psi=e^{-\varepsilon G_1(\varphi,\tilde{\omega})}\dots\ e^{-\varepsilon G_M(\varphi,\tilde{\omega})}\phi$ transforms $H_0+\varepsilon\mathcal{W}^{(0)}(\varphi)$ into the operator*

$$
H^{(M)}(\varphi)=H_0+\varepsilon Z^{(M)}(\tilde{\omega})+\varepsilon\mathcal{W}^{(M)}(\varphi,\tilde{\omega}), \qquad (2.2.20)
$$

where $Z^{(M)}$ is a time independent Fourier multiplier, which in particular fulfills

$$
[Z^{(M)},K_m]=0,\quad m=1\dots,d, \qquad (2.2.21)
$$

and

$$
\begin{aligned}
Z^{(M)}(\tilde{\omega})&\in\mathcal{L}ip\left(\Omega_{0,\gamma};OPS^{1-\mathfrak{e}}\right),\\
\mathcal{W}^{(M)}(\varphi,\tilde{\omega})&\in\mathcal{L}ip\left(\Omega_{0,\gamma};\mathcal{C}^\infty(\mathbb{T}^{\mathfrak{n}};OPS^{1-M\mathfrak{e}})\right).
\end{aligned} \qquad (2.2.22)
$$

Furthermore, if $\mathcal{W}^{(0)}$ is real and reversible, then $Z^{(M)}$, $\mathcal{W}^{(M)}$ are real and reversible too.

The main step for the proof of Theorem 2.2.7 is the following lemma, which is a variant of Lemma 3.7 of [BGMR17]:

Lemma 2.2.8. *Let $\mathcal{W} \in \mathcal{L}ip\left(\Omega_{0,\gamma}; \mathcal{C}^\infty\left(\mathbb{T}^{\mathtt{n}}; OPS^\eta\right)\right)$, be given and consider the homological equation*

$$\omega \cdot \partial_\varphi G + [H_0,\ G] = \mathcal{W} - \langle \mathcal{W} \rangle \tag{2.2.23}$$

with

$$\langle \mathcal{W} \rangle := \frac{1}{(2\pi)^{\mathtt{n}+d}} \int_{\mathbb{T}^d} \int_{\mathbb{T}^{\mathtt{n}}} e^{\mathrm{i}\tau\cdot K} \mathcal{W} e^{-\mathrm{i}\tau\cdot K}\ d\varphi\ d\tau\ ;$$

then (2.2.23) *has a solution $G \in \mathcal{L}ip\left(\Omega_{0,\gamma}; \mathcal{C}^\infty\left(\mathbb{T}^{\mathtt{n}}; OPS^\eta\right)\right)$.*
If $\mathcal{W}$ is symmetric hyperbolic, G is symmetric hyperbolic. Moreover, if $\mathcal{W}$ is real and reversible, G is real and reversibility preserving; if $\mathcal{W}$ is anti self-adjoint, G is anti self-adjoint.

Proof. Define $\forall \tau \in \mathbb{T}^d$

$$\mathcal{W}(\tau) := e^{\mathrm{i}\tau\cdot K} \mathcal{W} e^{-\mathrm{i}\tau\cdot K},$$

then we look for G s.t.

$$G(\tau) := e^{\mathrm{i}\tau\cdot K} G e^{-\mathrm{i}\tau\cdot K}$$

solves

$$\omega \cdot \partial_\varphi G(\tau) + [H_0,\ G(\tau)] = \mathcal{W}(\tau) - \langle \mathcal{W} \rangle \quad \forall\ \tau \in \mathbb{T}^d. \tag{2.2.24}$$

Notice that, since $G = G(0)$ and $\mathcal{W} = \mathcal{W}(0)$, solving equation (2.2.24) $\forall\ \tau$ implies having solved (2.2.23).
Note that $\forall\ \eta \in \mathbb{R}$, $\forall\ A \in OPS^\eta$ the map

$$[-1,\ 1] \ni \tau \mapsto e^{-\mathrm{i}\tau\cdot K} A e^{\mathrm{i}\tau\cdot K} \in \mathcal{C}^\infty\left(\mathbb{T}^d; OPS^\eta\right) \tag{2.2.25}$$

(see Remark A.1.5 of Appendix A). We make a Fourier expansion both in φ and τ variables, namely

$$\mathcal{W}(\tilde{\omega}, \varphi, \tau) = \sum_{k\in\mathbb{Z}^d} \sum_{l\in\mathbb{Z}^{\mathtt{n}}} \widehat{W}_{k,l}(\tilde{\omega}) e^{\mathrm{i}\varphi\cdot l} e^{\mathrm{i}\tau\cdot k}, \tag{2.2.26}$$

and similarly for G. A direct calculation shows that

$$[H_0,\ G(\tau)] = \sum_{k,\ l} i\left(\nu^{(0)} \cdot k\right)\ \widehat{G}_{k,l} e^{\mathrm{i}\tau\cdot k} e^{\mathrm{i}\varphi\cdot l}\,.$$

Thus, taking the $(k,l)-$th Fourier coefficient of equation (2.2.24), one has

$$\mathrm{i}\left(\omega \cdot l + \nu^{(0)} \cdot k\right) \widehat{G}_{k,l} = \widehat{W}_{k,l} \quad \text{if } (k,l) \neq (0,0), \quad \widehat{G}_{0,0} = 0.$$

For $|k| + |l| \neq 0$,define

$$\widehat{G}_{k,l} := \frac{\widehat{W}_{k,l}}{\mathrm{i}(\omega \cdot l + \nu^{(0)} \cdot k)}\ .$$

Then, by regularity of the map $(\varphi, \tau) \mapsto \mathcal{W}(\varphi, \tau)$ all the seminorms of the operator $\widehat{W}_{k,l}$ decay faster than any power of $(|k|+|l|)$, and since the frequencies belong to $\Omega_{0,\gamma}$ (cf. (2.2.9)), it follows that the seminorms of the operator $\widehat{G}_{k,l}$ exhibit the same decay; hence the series defining $G(\tau)$ converges absolutely and $G = G(0) \in \mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}}; OPS^\eta\right)$.
Lipschitz regularity with respect to $\tilde{\omega} = (\omega, \nu) \in \Omega_{0,\gamma}$ follows observing that given $(\omega_1, \nu_1), (\omega_2, \nu_2) \in \Omega_{0,\gamma}$, one has that

$$\begin{aligned}\widehat{G}_{k,l}(\omega_1) - \widehat{G}_{k,l}(\omega_2) = \widehat{W}_{k,l}(\omega_1)&\frac{\mathrm{i}(\omega_2 - \omega_1)\cdot l + \mathrm{i}\big(\nu^{(0)}(\omega_2,\nu_2) - \nu^{(0)}(\omega_1,\nu_1)\big)\cdot k}{(\omega_1 \cdot l + \nu^{(0)}(\omega_1,\nu_1)\cdot k)(\omega_2 \cdot l + \nu^{(0)}(\omega_2,\nu_2)\cdot k)} \\ &+ \frac{\widehat{W}_{k,l}(\omega_1) - \widehat{W}_{k,l}(\omega_2)}{\mathrm{i}(\omega_2 \cdot l + \nu^{(0)}(\omega_2,\nu_2)\cdot k)}\,,\end{aligned}$$

using the fact that the map $(\omega, \nu) \mapsto \nu^{(0)}(\omega, \nu)$ is Lipschitz (see Proposition 2.2.2) and the diophantine estimate required in (2.2.9).
It remains to verify the structural hypotheses.
Symmetric hyperbolicity: We observe that

$$\mathcal{W}+\mathcal{W}^* = e^{-\mathrm{i}\tau\cdot K}\left(\mathcal{W}(\tau) + \mathcal{W}^*(\tau)\right)e^{\mathrm{i}\tau\cdot K}, \quad G+G^* = e^{-\mathrm{i}\tau\cdot K}\left(G(\tau) + G^*(\tau)\right)e^{\mathrm{i}\tau\cdot K}.$$

Hence $\mathcal{W}$ (resp., G) is symmetric hyperbolic if and only if $\mathcal{W}(\tau)$ (resp., $G(\tau)$) is symmetric hyperbolic.
Thus, arguing as before and using

$$\widehat{(W^*)}_{k,l} = \overline{\widehat{W}_{-k,-l}} \quad \forall\ k \in \mathbb{Z}^d,\ l \in \mathbb{Z}^{\mathfrak{n}},$$

it follows that if $\forall\ k \in \mathbb{Z}^d,\ l \in \mathbb{Z}^{\mathtt{n}}\ \widehat{W}_{k,l} + \overline{\widehat{W}_{-k,-l}}$ are the Fourier coefficients of an operator in OPS^0, then

$$\widehat{G}_{k,l} + \overline{\widehat{G}_{-k,-l}} = \frac{\widehat{W}_{k,l} + \overline{\widehat{W}_{-k,-l}}}{\mathrm{i}\,(\omega \cdot l + \nu \cdot k)}$$

are again Fourier coefficients of an operator in OPS^0.
Reversibility: We apply Lemma A.1.6 of Appendix A to deduce reversibility of $\mathcal{W}$ and we observe that an operator $A(\tau, \varphi)$ is reversible (resp. reversibility preserving) if and only if, developing in Fourier series as in (2.2.26), its coefficients satisfy

$$\widehat{A}_{k,l} \circ S = -S \circ \widehat{A}_{-k,-l} \qquad \left(\text{resp. } \widehat{A}_{k,l} \circ S = S \circ \widehat{A}_{-k,-l}\right),$$

so that $\forall\ k \in \mathbb{Z}^d,\ l \in \mathbb{Z}^{\mathtt{n}}$,

$$\widehat{G}_{k,l} \circ S = \frac{\widehat{W}_{k,l} \circ S}{\mathrm{i}(\omega \cdot l + \nu \cdot k)} = \frac{-S \circ \widehat{W}_{-k,-l}}{-\mathrm{i}(\omega \cdot (-l) + \nu \cdot (-k))} = S \circ \widehat{G}_{-k,-l}.$$

Hence $G(\tau)$ and thus G is reversibility preserving, again by Lemma A.1.6.
Reality: Reality condition in Fourier coefficients reads

$$\widehat{A}_{l,k} = \overline{\widehat{A}_{-l,-k}}.$$

We apply Lemma A.1.6 again to deduce that reality of $\mathcal{W}(\tau)$ (resp, $G(\tau)$) is equivalent to reality of $\mathcal{W}$ (resp, G) and we compute

$$\widehat{G}_{k,l} = \frac{\widehat{W}_{k,l}}{\mathrm{i}(\omega \cdot l + \nu \cdot k)} = \frac{\overline{\widehat{W}_{-k,-l}}}{-\mathrm{i}(\omega \cdot (-l) + \nu \cdot (-k))} = \overline{\widehat{G}_{-k,-l}}.$$

□

Proof of Theorem 2.2.7. Fix $M > 0$. We prove by induction that $\forall j = 0, \ldots, M-1$

$$H^{(j)}(\varphi) = H_0 + \varepsilon Z^{(j)}(\tilde{\omega}) + \varepsilon \mathcal{W}^{(j)}(\varphi, \tilde{\omega})$$

is mapped by the change of variables

$$u = e^{-\varepsilon G_j(\varphi, \tilde{\omega})} v \tag{2.2.27}$$

into

$$H^{(j+1)}(\varphi) = H_0 + \varepsilon Z^{(j+1)}(\tilde{\omega}) + \varepsilon \mathcal{W}^{(j+1)}(\varphi, \tilde{\omega}),$$

with

$$\begin{aligned} Z^{(j+1)}(\tilde{\omega}) &\in \mathcal{L}ip\Big(\Omega_{0,\gamma}; \mathcal{C}^\infty(\mathbb{T}^{\mathfrak{n}}; OPS^{1-\mathfrak{e}})\Big), \\ \mathcal{W}^{(j+1)} &\in \mathcal{L}ip\left(\Omega_{0,\gamma}; \mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}}; OPS^{1-(j+1)\mathfrak{e}}\right)\right), \end{aligned} \tag{2.2.28}$$

$\mathcal{W}^{(j+1)}$ symmetric hyperbolic and $Z^{(j+1)}(\tilde{\omega})$ a Fourier multiplier commuting with all the K_m.
If $j = 0$, the hypotheses are satisfied for $Z^{(0)} = 0,\ \mathcal{W}^{(0)} = \mathcal{W} \in \mathcal{L}ip\left(\Omega_{0,\gamma}; \mathcal{C}^\infty(\mathbb{T}^{\mathfrak{n}}; OPS^{1-\mathfrak{e}})\right)$.
Suppose now that $H^{(j)}$ satisfies the required hypotheses; the change of coordinates (2.2.27) maps $H^{(j)}$ into

$$H^{(j+1)}(\varphi, \tilde{\omega}) = H_0 + \varepsilon Z^{(j)}(\tilde{\omega}) + \varepsilon\langle \mathcal{W}^{(j)}\rangle \tag{2.2.29}$$

$$+ \varepsilon\left(-\omega \cdot \partial_\varphi G_j + [H_0, G_j] + \mathcal{W}^{(j)}(\varphi, \tilde{\omega}) - \langle \mathcal{W}^{(j)}\rangle\right) \tag{2.2.30}$$

$$+ e^{\varepsilon G_j(\varphi,\tilde{\omega})} H_0 e^{-\varepsilon G_j(\varphi,\tilde{\omega})} - H_0 - \varepsilon[H_0, G_j] \tag{2.2.31}$$

$$+ \varepsilon e^{\varepsilon G_j(\varphi,\tilde{\omega})} Z^{(j)}(\tilde{\omega}) e^{-\varepsilon G_j(\varphi,\tilde{\omega})} - \varepsilon Z^{(j)}(\tilde{\omega}) \tag{2.2.32}$$

$$+ \varepsilon e^{\varepsilon G_j(\varphi,\tilde{\omega})} \mathcal{W}^{(j)}(\varphi, \tilde{\omega}) e^{-\varepsilon G_j(\varphi,\tilde{\omega})} - \varepsilon \mathcal{W}^{(j)}(\varphi, \tilde{\omega}) \tag{2.2.33}$$

$$- \varepsilon \int_0^1 e^{-\varepsilon s G_j(\varphi,\tilde{\omega})} \omega \cdot \partial_\varphi G_j(\varphi, \tilde{\omega}) e^{\varepsilon s G_j(\varphi,\tilde{\omega})}\, ds + \varepsilon \omega \cdot \partial_\varphi G_j. \tag{2.2.34}$$

By Lemma 2.2.8, it is possible to find an operator $G_j \in \mathcal{L}ip\left(\Omega_{0,\gamma}; \mathcal{C}^\infty(\mathbb{T}^{\mathfrak{n}}; OPS^{1-j\mathfrak{e}})\right)$ such that G_j is symmetric hyperbolic and (2.2.30) equals zero. Since Lemma A.1.3 of Appendix A implies that

$$\begin{aligned} (2.2.31) &\in \mathcal{L}ip\left(\Omega_{0,\gamma}; \mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}}; OPS^{1-2j\mathfrak{e}}\right)\right), \\ (2.2.32) &\in \mathcal{L}ip\left(\Omega_{0,\gamma}; \mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}}; OPS^{1-(j+1)\mathfrak{e}}\right)\right), \\ (2.2.33) &\in \mathcal{L}ip\left(\Omega_{0,\gamma}; \mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}}; OPS^{1-2j\mathfrak{e}}\right)\right), \\ (2.2.34) &\in \mathcal{L}ip\left(\Omega_{0,\gamma}; \mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}}; OPS^{1-2j\mathfrak{e}}\right)\right), \end{aligned}$$

if we define

$$\begin{aligned} Z^{(j+1)}(\tilde{\omega}) &:= Z^{(j)}(\tilde{\omega}) + \langle \mathcal{W}^{(j)}\rangle, \\ \varepsilon \mathcal{W}^{(j+1)}(\varphi, \tilde{\omega}) &= (2.2.31) + (2.2.32) + (2.2.33) + (2.2.34), \end{aligned} \tag{2.2.35}$$

we have $\mathcal{W}^{(j+1)}(\varphi, \tilde{\omega}) \in \mathcal{L}ip\left(\Omega_{0,\gamma}; \mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}}; OPS^{1-(j+1)\mathfrak{e}}\right)\right)$.
We observe that (2.2.31) is of order ε, as can be seen performing a Taylor expansion of the operator $e^{-\varepsilon G_j(\varphi,\tilde{\omega})} H_0 e^{\varepsilon G_j(\varphi,\tilde{\omega})}$ as in Lemma A.1.3 of Appendix A.
Reality and reversibility of $\mathcal{W}^{(j+1)}(\varphi, \tilde{\omega})$ follow from Lemma A.1.1, whereas symmetric hyperbolicity of $\mathcal{W}^{(j+1)}(\varphi, \tilde{\omega})$ follows from Lemma A.1.4. $\square$

Remark 2.2.9. *For all* $j = 1, \ldots, M$ *and* $\forall\ \sigma \geq 0$ *we have* $e^{\varepsilon G_j} \in \mathcal{B}(\mathcal{H}^\sigma)$ *and*

$$\|e^{\varepsilon G_j} - \mathbb{I}\|_{\mathcal{B}(\mathcal{H}^\sigma, \mathcal{H}^{\sigma-(1-j\mathfrak{e})})} \lesssim \varepsilon \|G_j\|_{\mathcal{B}(\mathcal{H}^\sigma, \mathcal{H}^{\sigma-(1-j\mathfrak{e})})}.$$

Furthermore, from Lemma A.1.1, $\forall\ \alpha \in \mathbb{N}$ *we have*

$$\partial_\varphi^\alpha e^{\varepsilon G_j} \in \mathcal{B}\left(\mathcal{H}^\sigma, \mathcal{H}^{\sigma-(1-j\mathfrak{e})|\alpha|}\right).$$

Note that, since $Z^{(M)} \in \mathcal{L}ip\Big(\Omega_{0,\gamma}; \mathcal{C}^\infty(\mathbb{T}^\mathfrak{n}; OPS^{1-\mathfrak{e}})\Big)$ then $Z^{(M)} = \mathrm{Op}(z_\xi)$ with $z \in \mathcal{L}ip\Big(\Omega_{0,\gamma}; \mathcal{C}^\infty(\mathbb{T}^\mathfrak{n}; S^{1-\mathfrak{e}})\Big)$. Hence $\partial_\xi z \in \mathcal{L}ip\Big(\Omega_{0,\gamma}; \mathcal{C}^\infty(\mathbb{T}^\mathfrak{n}; S^{-\mathfrak{e}})\Big)$ and the following estimate holds

$$\sup_{\xi \in \mathbb{R}^d} \langle \xi \rangle^{\mathfrak{e}-1} |z|^{\mathrm{Lip}}, \ \sup_{\xi \in \mathbb{R}^d} \langle \xi \rangle^{1-\mathfrak{e}} |\partial_\xi z(\xi, \cdot)|^{\mathrm{Lip}} \lesssim \varepsilon; \tag{2.2.36}$$

Concerning the second of (2.2.36), we remark that we will only use the fact that $|\partial_\xi z(\xi, \cdot)|^{\mathrm{Lip}}$ is bounded.

2.3 Reducibility

2.3.1 Functional Setting

Given a linear operator $R : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$, we denote by $R_j^{j'}$ its matrix elements with respect to the exponential basis $\{e^{\mathrm{i} j \cdot x} : j \in \mathbb{Z}^d\}$, namely

$$R_j^{j'} := \int_{\mathbb{T}^d} R[e^{\mathrm{i} j' \cdot x}]\ e^{-\mathrm{i} j \cdot x}\, dx, \quad \forall j, j' \in \mathbb{Z}^d.$$

We define some families of operators related to $R \in \mathcal{B}(L^2(\mathbb{T}^d))$ that will be useful in our estimates; we point out that similar definitions to the ones given in the present subsection also appear in Section 2 of [BBHM18] and in Section 2 of [FGP19]:

Definition 2.3.1. *Given* $\beta \geq 0$ *and* $R \in \mathcal{B}(L^2(\mathbb{T}^d))$, *we define the operator* $\langle \nabla \rangle^\beta R$ *as*

$$(\langle \nabla \rangle^\beta R)_j^{j'} := \langle j - j' \rangle^\beta R_j^{j'}.$$

We remark that this operator is useful since, for any operator R and any function u, one has

$$\nabla R u = R \nabla u + [R; \nabla] u,$$

and

$$[R; \nabla] \simeq \langle \nabla \rangle R.$$

Definition 2.3.2. *We consider the space*

$$\mathcal{B}^{HS}(\mathcal{H}^{\sigma_1}, \mathcal{H}^{\sigma_2}) := \left\{ R \in \mathcal{B}\left(\mathcal{H}^{\sigma_1}, \mathcal{H}^{\sigma_2}\right) \,\middle|\, \|R\|^{HS}_{\sigma_1,\sigma_2} < +\infty \right\},$$

with

$$\left(\|R\|^{HS}_{\sigma_1,\sigma_2}\right)^2 := \sum_{k\in\mathbb{Z}^d} \sum_{k'\in\mathbb{Z}^d} \langle k\rangle^{2\sigma_2} |R_k^{k'}|^2 \langle k'\rangle^{-2\sigma_1}.$$

We consider operators $R(\varphi)$ depending on the angles $\varphi \in \mathbb{T}^{\mathtt{n}}$, with $R \in \mathcal{H}^s\left(\mathbb{T}^{\mathtt{n}};\ \mathcal{B}^{HS}(\mathcal{H}^{\sigma_1}, \mathcal{H}^{\sigma_2})\right)$. Thus we define the time Fourier coefficients of R : $\forall\, l \in \mathbb{Z}^{\mathtt{n}}$ $\widehat{R}(l)$ is the operator with matrix elements

$$(\widehat{R}(l))_j^{j'} := \frac{1}{(2\pi)^{\mathtt{n}}} \int_{\mathbb{T}^{\mathtt{n}}} R_j^{j'} e^{-\mathrm{i}l\cdot\varphi} \ \ d\varphi. \tag{2.3.1}$$

Definition 2.3.3 (Class of operators)**.** *Given $s, \sigma \geq 0$, we consider the space*

$$\mathcal{M}^s_{\sigma_1,\sigma_2} := \mathcal{H}^s\left(\mathbb{T}^{\mathtt{n}};\ \mathcal{B}^{HS}(\mathcal{H}^{\sigma_1}, \mathcal{H}^{\sigma_2})\right), \tag{2.3.2}$$

endowed with the norm

$$\|R\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}} := \Big(\sum_{l\in\mathbb{Z}^{\mathtt{n}}} \langle l\rangle^{2s} \big(\|\widehat{R}(l)\|^{HS}_{\sigma_1,\sigma_2}\big)^2 \Big)^{\frac{1}{2}}. \tag{2.3.3}$$

In the following, consider the case where the operator R has a further dependence on a parameter $\tilde{\omega} \in \Omega_0$, for a given set $\Omega_0 \subseteq \Omega$:

Definition 2.3.4 (Higher regularity norm)**.** *Let $\Omega_0 \subseteq \Omega$ and $R \in \mathcal{L}ip\left(\Omega_0; \mathcal{M}^s_{\sigma_1,\sigma_2}\right)$. Given $\beta > 0$, if $\forall \tilde{\omega} \in \Omega_0$ $R(\tilde{\omega})$ is such that*

$$R(\tilde{\omega}) \in \mathcal{L}ip\left(\Omega_0; \mathcal{M}^{s+\beta}_{\sigma_1,\sigma_2}\right), \quad \langle\nabla\rangle^\beta R(\tilde{\omega}) \in \mathcal{L}ip\left(\Omega_0; \mathcal{M}^s_{\sigma_1,\sigma_2}\right),$$

we define

$$\|R\|^{\mathrm{Lip}}_{\mathcal{W}^{s,\beta}_{\sigma_1,\sigma_2}} := \|R\|^{\mathrm{Lip}}_{\mathcal{M}^{s+\beta}_{\sigma_1,\sigma_2}} + \|\langle\nabla\rangle^\beta R\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_1,\sigma_2}}. \tag{2.3.4}$$

Definition 2.3.5 (Cutoffs)**.** *Given an operator $R : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$, for any $N \in \mathbb{N}$, we define the projector $\pi_N R$ as*

$$(\pi_N R)_j^{j'} := \begin{cases} R_j^{j'} & \text{if } |j-j'| < N \\ 0 & \text{if } |j-j'| \geq N \end{cases} \tag{2.3.5}$$

and we set $\pi_N^\perp R := R - \pi_N R$. For $R : \mathbb{T}^{\mathtt{n}} \to \mathcal{B}(L^2(\mathbb{T}^d))$, $\varphi \mapsto R(\varphi)$, we define $\Pi_N R$ as

$$\Pi_N R(\varphi) := \sum_{|l|\leq N} \pi_N \widehat{R}(l)\, e^{\mathrm{i}l\cdot\varphi}. \tag{2.3.6}$$

We then set $\Pi_N^\perp R := R - \Pi_N R$.

In the following lemma we point out a key estimate for the remainder $\Pi_N^\perp R$ of an operator R :

Lemma 2.3.6. *Let $R(\tilde\omega) \in \mathcal{M}^s_{\sigma_1,\sigma_2}$, $\tilde\omega \in \Omega_0 \subseteq \Omega$. Then for any $N > 0$,*

$$\|\Pi_N R\|^{\rm Lip}_{\mathcal{M}^s_{\sigma_1,\sigma_2}}, \ \|\Pi_N^\perp R\|^{\rm Lip}_{\mathcal{M}^s_{\sigma_1,\sigma_2}} \le \|R\|^{\rm Lip}_{\mathcal{M}^s_{\sigma_1,\sigma_2}}. \tag{2.3.7}$$

Moreover, let $\beta > 0$ and assume that for all $\tilde\omega \in \Omega_0$ $R(\tilde\omega)$ also satisfies $R(\tilde\omega) \in \mathcal{M}^{s+\beta}_{\sigma_1,\sigma_2}$, $\langle\nabla\rangle^\beta R(\tilde\omega) \in \mathcal{M}^s_{\sigma_1,\sigma_2}$. Then, for any $N \in \mathbb{N}$, one has $\Pi_N^\perp R(\tilde\omega) \in \mathcal{M}^s_{\sigma_1,\sigma_2}$ and

$$\|\Pi_N^\perp R\|^{\rm Lip}_{\mathcal{M}^s_{\sigma_1,\sigma_2}} \le N^{-\beta}\|R\|^{\rm Lip}_{\mathcal{W}^{s,\beta}_{\sigma_1,\sigma_2}} \tag{2.3.8}$$

Proof. Estimate (2.3.7) is a direct consequence of the definitions (2.3.3)-(2.3.6). We prove estimate (2.3.8). By (2.3.5), (2.3.6), one has

$$\begin{aligned} &\Pi_N^\perp R(\varphi) = R_{1,N}(\varphi) + R_{2,N}(\varphi)\,, \\ &R_{1,N}(\varphi) := \sum_{|l|\le N} \pi_N^\perp \widehat{R}(l) e^{{\rm i} l\cdot\varphi}\,, \quad R_{2,N}(\varphi) := \sum_{|l|>N} \widehat{R}(l) e^{{\rm i} l\cdot\varphi}\,. \end{aligned} \tag{2.3.9}$$

We estimate separately the two terms in the above formula.
Estimate of $R_{1,N}$. For any $l \in \mathbb{Z}^{\mathtt{n}}$, one has

$$\begin{aligned} \left(\|\pi_N^\perp \widehat{R}(l)\|^{HS}_{\sigma_1,\sigma_2}\right)^2 &= \sum_{\substack{k,k'\in\mathbb{Z}^d \\ |k-k'|>N}} |\widehat{R}(l)_k^{k'}|^2 \langle k\rangle^{2\sigma_2}\langle k'\rangle^{-2\sigma_1} \\ &\le N^{-2\beta} \sum_{k,k'\in\mathbb{Z}^d} \langle k-k'\rangle^{2\beta} |\widehat{R}(l)_k^{k'}|^2 \langle k\rangle^{2\sigma_2}\langle k'\rangle^{-2\sigma_1} \\ &= N^{-2\beta}\left(\|\langle\nabla\rangle^\beta \widehat{R}(l)\|^{HS}_{\sigma_1,\sigma_2}\right)^2. \end{aligned}$$

Therefore, recalling (2.3.3), one gets the estimate

$$\|R_{1,N}\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}} \le N^{-\beta}\|\langle\nabla\rangle^\beta R\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}}\,. \tag{2.3.10}$$

Estimate of $R_{2,N}$. The operator $R_{2,N}$ can be estimated as

$$\begin{aligned} \left(\|R_{2,N}\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}}\right)^2 &= \sum_{|l|>N} \langle l\rangle^{2s}\left(\|\ \widehat{R}(l)\|^{HS}_{\sigma_1,\sigma_2}\right)^2 \\ &\le N^{-2\beta}\sum_{l\in\mathbb{Z}^{\mathtt{n}}}\langle l\rangle^{2(s+\beta)}\left(\|\ \widehat{R}(l)\|^{HS}_{\sigma_1,\sigma_2}\right)^2 \\ &= N^{-2\beta}\left(\|R\|_{\mathcal{M}^{s+\beta}_{\sigma_1,\sigma_2}}\right)^2, \end{aligned}$$

implying that

$$\|R_{2,N}\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}} \leq N^{-\beta}\|R\|_{\mathcal{M}^{s+\beta}_{\sigma_1,\sigma_2}} \,. \tag{2.3.11}$$

The claimed inequality then follows by (2.3.4), (2.3.9), (2.3.10) and (2.3.11). □

2.3.2 Diagonalization

Fix $M > 0$ and consider the regularized operator $H^{(M)}$ of Theorem 2.2.7; it is of the form

$$H^{(M)} = A_0 + P_0(\varphi), \quad A_0 := D_0 + Z \,, \tag{2.3.12}$$

where $D_0 = \nu^{(0)}(\tilde{\omega}) \cdot \nabla$, $Z = \varepsilon Z^{(M)}$ and $P_0 = \mathcal{W}^{(M)}$.
Since D_0 and Z depend only on ∇ and not on the x variable, such operators remain diagonal if we pass to Fourier variables, so that we deal with the sum of an operator $A_0 = D_0 + Z$ which is diagonal with respect to the Fourier basis $\{e^{\mathrm{i}\xi\cdot x} \mid \xi \in \mathbb{T}^d\}$ and a perturbative term $P_0(\varphi)$ whose dependence on the angle φ we want to eliminate. More precisely

$$A_0 = \mathrm{diag}(\lambda_j^{(0)}), \quad \lambda_j^{(0)} := \mathrm{i}\nu^{(0)} \cdot j + z(j) \tag{2.3.13}$$

where we recall that $z \in \mathcal{L}ip(\Omega_{0,\gamma}; OPS^{1-\mathfrak{e}})$. Before stating the reducibility theorem, we fix some constants. Given $\tau > 0$ we define

$$\alpha := 12\tau + 7\,, \quad \beta := \alpha + 1\,, \quad m := 2\tau + 2 \tag{2.3.14}$$

Moreover, we fix the scale on which we perform the reducibility scheme as

$$N_k = N_0^{\left(\frac{3}{2}\right)^k} \quad \forall k \in \mathbb{N}, \quad N_{-1} := 1 \tag{2.3.15}$$

where for convenience we link N_0 and γ as

$$N_0 = \gamma^{-1}\,, \tag{2.3.16}$$

where γ is the constant appearing in the definition (2.2.9) of the set $\Omega_{0,\gamma}$ (see also (2.3.22) in the theorem below). We also fix the number M of regularization steps in Theorem 2.2.7 as

$$M = \lceil M'\mathfrak{e} - 1\rceil, \quad M' := 2m + 2\beta + d/2 + 1\,. \tag{2.3.17}$$

Remark 2.3.7. *By Theorem 2.2.7 one has that* $P_0 = \varepsilon \mathcal{W}^{(M)} \in \mathcal{C}^\infty(\mathbb{T}^\mathtt{n}; OPS^{-M'})$. *Since by* (2.3.17), $M' > 2m + 2\beta + \frac{d}{2}$, *by applying Lemma B.2.3, one has that*

$$\|P_0\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\,,\ \|P_0\|^{\rm Lip}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} \lesssim_{s,\sigma} \varepsilon\,, \quad \forall s \geq 0\,, \quad \forall \sigma \in \mathbb{R}\,. \tag{2.3.18}$$

Theorem 2.3.8. (KAM reducibility) *Consider the system* (2.2.20). *Let* $\gamma \in (0,1)$, $\tau > 0$. *Then for any* $s > s_0$ *(with* s_0 *defined as in* (2.0.2)*)and for any* $\sigma \geq 0$ *there exist constants* $C_0 = C_0(s,\sigma,\tau) > 0$ *large enough and* $\delta = \delta(s,\sigma,\tau) \in (0,1)$ *small enough such that, if*

$$N_0^{C_0}\varepsilon \leq \delta\,, \tag{2.3.19}$$

then, for all $k \geq 0$*:*

$(\mathbf{S1})_k$ *There exists a vector field*

$$H_k(\varphi) := A_k + P_k(\varphi)\,, \qquad \varphi \in \mathbb{T}^\mathtt{n}\,, \tag{2.3.20}$$

$$A_k = {\rm diag}(\lambda_j^{(k)}), \quad \lambda_j^{(k)}(\tilde\omega) = \lambda_j^{(0)}(\tilde\omega) + \rho_j^{(k)}(\tilde\omega) \tag{2.3.21}$$

defined for all $\widetilde\omega \in \mathcal{O}_{k,\gamma}$, *where we set* $\mathcal{O}_{0,\gamma} := \Omega_{0,\gamma}$ *(see* (2.2.9)*) and for* $k \geq 1$,

$$\mathcal{O}_{k,\gamma} := \Big\{ \widetilde\omega = (\omega,\nu) \in \mathcal{O}_{k-1,\gamma} : |{\rm i}\omega\cdot l + \lambda_j^{(k-1)}(\widetilde\omega) - \lambda_{j'}^{(k-1)}(\widetilde\omega)| \geq \frac{\gamma}{\langle l\rangle^\tau \langle j\rangle^\tau \langle j'\rangle^\tau} \\ \forall (l,j,j') \neq (0,j,j), \quad |l|, |j-j'| \leq N_{k-1} \Big\}\,. \tag{2.3.22}$$

For $k \geq 0$, *the Lipschitz functions* $\mathcal{O}_{k,\gamma} \to \mathbb{C}$, $\tilde\omega \mapsto \rho_j^{(k)}(\tilde\omega)$, $j \in \mathbb{Z}^d$ *satisfy*

$$\sup_{j\in\mathbb{Z}^d} \langle j\rangle^{2m} |\rho_j^{(k)}|^{\rm Lip} \lesssim_{s,\sigma} \varepsilon\,. \tag{2.3.23}$$

There exist a constant $C_* = C_*(s,\sigma,\beta,\tau,m) > 0$ *such that*

$$\|P_k\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \leq C_* N_{k-1}^{-\alpha}\varepsilon, \quad \|P_k\|^{\rm Lip}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} \leq C_* N_{k-1}\varepsilon\,. \tag{2.3.24}$$

Moreover, for $k \geq 1$,

$$H_k(\varphi) = (\Phi_{k-1})_{\omega *} H_{k-1}(\varphi)\,, \quad \Phi_{k-1} := \mathbb{I} + X_{k-1} \tag{2.3.25}$$

where the map X_{k-1} *satisfies the estimates*

$$\|X_{k-1}\|^{\rm Lip}_{\mathcal{M}^s_{\sigma\pm m,\sigma\pm m}} \lesssim_s N_k^{4\tau+2} N_{k-1}^{-\alpha}\varepsilon\,. \tag{2.3.26}$$

Moreover, if $P_0(\varphi)$ *is real and reversible, for any* $k \geq 1$, $P_k(\varphi)$ *is real and reversible and*

$$\lambda_j^{(k)} \in {\rm i}\mathbb{R} \quad \forall j \in \mathbb{Z}^d. \tag{2.3.27}$$

$(\mathbf{S2})_k$ *For all $j \in \mathbb{Z}^d$, there exists a Lipschitz extension to the set $\Omega_{0,\gamma}$ defined in (2.2.9), that we denote by $\widetilde{\lambda}_j^{(k)} : \Omega_{0,\gamma} \to \mathbb{C}$ of $\lambda_j^{(k)} : \mathcal{O}_{k,\gamma} \to \mathbb{C}$ satisfying, for $k \geq 1$,*

$$|\widetilde{\lambda}_j^{(k)} - \widetilde{\lambda}_j^{(k-1)}|^{\mathrm{Lip}} \lesssim \langle j \rangle^{-2m} \|P_{k-1}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \lesssim_{s,\sigma} \langle j \rangle^{-2m} N_{k-2}^{-\alpha} \varepsilon \,. \quad (2.3.28)$$

We remark that $(\mathbf{S2})_k$ will be used to construct the final eigenvalues $\lambda_j^{(\infty)}$. The procedure will be to show that as $k \to \infty$, the sequence $\lambda_j^{(k)}$ admits a limit on $\Omega_{0,\gamma}$ and then to use the final value $\lambda_j^{(\infty)}$ in order to define the set in which reducibility holds (c.f. eq. (2.3.54)).

2.3.3 Proof of Theorem 2.3.8

We start with verifying the inductive hypotheses:
Proof of $(\mathbf{Si})_0$, $i = 1, 2$. Properties (2.3.20)-(2.3.24) hold by setting $\rho_j^{(0)} = 0$ for any $j \in \mathbb{Z}^d$, $N_{-1} := 1$ and recalling the estimate (2.3.18).

$(\mathbf{S2})_0$ holds, since the constant $\lambda_j^{(0)}$ is already defined for all $\tilde{\omega} \in \Omega_{0,\gamma}$ and in the real and reversible case it satisfies $\lambda_j^{(0)} \in \mathrm{i}\mathbb{R}$ in force of Proposition 2.2.2. Thus we simply set $\rho_j^{(0)} = 0$ for any $j \in \mathbb{Z}^d$.

The reducibility step: proof of $(\mathbf{Si})_{k+1}$, $i = 1, 2$.

Proof of $(\mathbf{S1})_{k+1}$.
We now describe the inductive step, showing how to define a transformation $\Phi_k := \mathbb{I} + X_k$ so that the transformed vector field $H_{k+1}(\varphi) = (\Phi_k)_{\omega *} H_k(\varphi)$ has the desired properties. If we perform a change of coordinates of the form $u' := \Phi_k(\varphi) u$, $\Phi_k(\varphi) = \mathbb{I} + X_k(\varphi)$, then $H_{k+1}(\varphi) = (\Phi_k)_{\omega *} H_k(\varphi)$ takes the form

$$\begin{aligned} H_{k+1}(\varphi) = A_k + \Phi_k(\varphi)^{-1}\big(\Pi_{N_k} P_k(\varphi) + [X_k(\varphi), A_k] - \omega \cdot \partial_\varphi X_k(\varphi)\big) \\ + \Phi_k(\varphi)^{-1}\big(\Pi_{N_k}^{\perp} P_k(\varphi) + P_k(\varphi) X_k(\varphi)\big), . \end{aligned}$$

We look for a transformation $X_k(\varphi)$ solving the *homological equation*

$$\Pi_{N_k} P_k(\varphi) + [X_k(\varphi), A_k] - \omega \cdot \partial_\varphi X_k(\varphi) = [P_k] \,, \quad (2.3.29)$$

where $[P_k]$ is a diagonal operator. Then we set

$$\begin{aligned} &A^{k+1} = A_k + \overline{P_k}, \quad P^{k+1} = \Pi_{N_k}^{\perp} P_k + P_k X_k + (\Phi_k^{-1} - \mathbb{I})\big([P_k] + \Pi_{N_k}^{\perp} P_k + P_k X_k\big) \,, \\ &[P_k] := \mathrm{diag}_{j \in \mathbb{Z}} (\widehat{P}_k)_j^j(0) \,. \end{aligned} \quad (2.3.30)$$

By formula (2.3.30) one obtains that

$$A_{k+1} := \mathrm{diag}_{j\in\mathbb{Z}^d}\lambda_j^{(k+1)}\,,$$

where for any $j \in \mathbb{Z}^d$

$$\begin{aligned} \lambda_j^{(k+1)} &:= \lambda_j^{(k)} + \widehat{P}_k(0)_j^j = \mathrm{i}\nu^{(0)}\cdot j + \varepsilon z(j) + \rho_j^{(k+1)}\,,\\ \rho_j^{(k+1)} &:= \rho_j^{(k)} + \widehat{P}_k(0)_j^j\,. \end{aligned} \tag{2.3.31}$$

In the real and reversible case, since P_k is real and reversible, by Lemma B.1.1 one has $\widehat{P}_k(0)_j^j \in \mathrm{i}\mathbb{R}$, and since $\lambda_j^{(k)}, \rho_j^{(k)} \in \mathrm{i}\mathbb{R}$, then one has that $\lambda_j^{(k+1)}, \rho_j^{(k+1)} \in \mathrm{i}\mathbb{R}$.

First of all, we prove that (2.3.23) holds at the step $k+1$. By the definition (2.3.31), applying Lemma B.2.4 and using the estimate (2.3.24), one gets that for any $j \in \mathbb{Z}^d$ for any $i \in \{0, 1, \ldots, k\}$

$$\begin{aligned} |\lambda_j^{(i+1)} - \lambda_j^{(i)}|^{\mathrm{Lip}} &= |\rho_j^{(i+1)} - \rho_j^{(i)}|^{\mathrm{Lip}} = |(\widehat{P}_i)_{jj}(0)|^{\mathrm{Lip}}\\ &\lesssim \langle j\rangle^{-2m}\|P_i\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \lesssim_{s,\sigma} \langle j\rangle^{-2m}N_{i-1}^{-\alpha}\varepsilon\,. \end{aligned} \tag{2.3.32}$$

By using a telescoping argument, recalling that $\rho_j^{(0)} = 0$ for any $j \in \mathbb{Z}^d$, one gets that

$$|\rho_j^{(k+1)}|^{\mathrm{Lip}} \leq \sum_{i=0}^{k} |\rho_j^{(i+1)} - \rho_j^{(i)}|^{\mathrm{Lip}} \overset{(2.3.32)}{\lesssim_{s,\sigma}} \langle j\rangle^{-2m}\varepsilon\sum_{i=0}^{\infty} N_{i-1}^{-\alpha} \lesssim \langle j\rangle^{-2m}\varepsilon \tag{2.3.33}$$

since the series $\sum_{i=0}^{\infty} N_{i-1}^{-\alpha}$ is convergent (see the definition of N_i at (2.3.15)). Hence (2.3.23) is verified at the step $k+1$.

In the next lemma we will show how to solve the homological equation (2.3.29). This is the main lemma of the section.

Lemma 2.3.9. *Let $m > 2\tau + 1$ and let $\mathcal{O}_{k+1,\gamma}$ as in* (2.3.22). *Then for any $\tilde{\omega} \in \mathcal{O}_{k+1,\gamma}$ the homological equation*

$$[A_k, X_k] + \omega\cdot\partial_\varphi X_k = \Pi_{N_k}P_k - [P_k]\,, \tag{2.3.34}$$

with

$$[P_k] = \mathrm{diag}_{j\in\mathbb{Z}^d}\,\widehat{P_k}(0)_j^j\,, \tag{2.3.35}$$

has a solution X_k defined on $\mathcal{O}_{k,\gamma}$ and satisfying the estimates

$$\|X_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma\pm m,\sigma\pm m}} \lesssim N_k^{4\tau+2}\|P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}, \tag{2.3.36}$$

$$\|\langle\nabla\rangle^\beta X_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma\pm m,\sigma\pm m}} \lesssim N_k^{4\tau+2}\|\langle\nabla\rangle^\beta P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}. \tag{2.3.37}$$

Furthermore, if P_k is real and reversible then X_k is real and reversibility preserving.

Proof. To simplify notations, here we drop the index k, namely we write A, P, X, λ_j, ρ_j instead of A_k, P_k, X_k, $\lambda_j^{(k)}$, $\rho_j^{(k)}$. Taking the $(j, j')-$th matrix element and the $l-$th Fourier coefficient of (2.3.34) we get:

$$\begin{aligned}
&(\mathrm{i}\omega \cdot l + \lambda_j - \lambda_{j'})\ \widehat{X}(l)_j^{j'} = \widehat{P}(l)_j^{j'} \quad \text{if } 0 < |j - j'| < N,\ 0 < |l| < N \\
&\widehat{X}(l)_j^{j'} = 0 \quad \text{otherwise}
\end{aligned}$$

Since $\tilde{\omega} \in \mathcal{O}_{k+1,\gamma}$ one has

$$|\widehat{X}(l)_j^{j'}| \le \frac{|\widehat{P}(l)_j^{j'}||j|^\tau |j'|^\tau |l|^\tau}{\gamma}, \tag{2.3.38}$$

hence

$$\begin{aligned}
|\widehat{X}(l)_j^{j'}| &\lesssim \gamma^{-1} |\widehat{P}(l)_j^{j'}||l|^\tau \langle j'\rangle^\tau \Big(\langle j'\rangle^\tau + |j - j'|^\tau\Big) \\
&\le \gamma^{-1} |\widehat{P}(l)_j^{j'}| N^\tau \langle j'\rangle^\tau \Big(\langle j'\rangle^\tau + N^\tau\Big) \\
&\lesssim \gamma^{-1} |\widehat{P}(l)_j^{j'}| N^{2\tau} \langle j'\rangle^{2\tau},
\end{aligned} \tag{2.3.39}$$

Similarly, one gets

$$|\widehat{X}(l)_j^{j'}| \lesssim \gamma^{-1} |\widehat{P}(l)_j^{j'}| N^{2\tau} \langle j\rangle^{2\tau}. \tag{2.3.40}$$

Thus, recalling that $\tau < m$, (see (2.3.14)) the norm $\|X\|_{\mathcal{M}^s_{\sigma+m,\sigma+m}}$ is estimated by:

$$\begin{aligned}
\Big(\|X\|_{\mathcal{M}^s_{\sigma+m,\sigma+m}}\Big)^2 &= \sum_{l\in\mathbb{Z}^\mathfrak{n}} \langle l\rangle^{2s} \sum_{j,j'\in\mathbb{Z}^d} \langle j\rangle^{2(\sigma+m)} |\widehat{X}(l)_j^{j'}(l)|^2 \langle j'\rangle^{-2(\sigma+m)} \\
&\lesssim \gamma^{-2} N^{4\tau} \sum_{l\in\mathbb{Z}^\mathfrak{n}} \langle l\rangle^{2s} \sum_{j,j'\in\mathbb{Z}^d} \langle j\rangle^{2(\sigma+m)} |\widehat{P}(l)_j^{j'}|^2 \langle j'\rangle^{4\tau} \langle j'\rangle^{-2(\sigma+m)} \\
&\le \gamma^{-2} N^{4\tau} \sum_{l\in\mathbb{Z}^\mathfrak{n}} \langle l\rangle^{2s} \sum_{j,j'\in\mathbb{Z}^d} \langle j\rangle^{2(\sigma+m)} |\widehat{P}(l)_j^{j'}|^2 \langle j'\rangle^{-2(\sigma-m)} \\
&= \gamma^{-2} N^{4\tau} \Big(\|P\|_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\Big)^2.
\end{aligned} \tag{2.3.41}$$

Similarly, one obtains

$$\Big(\|X\|_{\mathcal{M}^s_{\sigma-m,\sigma-m}}\Big)^2 \lesssim \gamma^{-2} N^{4\tau} \Big(\|P\|_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\Big)^2. \tag{2.3.42}$$

To estimate the norm of the operator $\langle\nabla\rangle^\beta X$, we argue as in (2.3.39), (2.3.40) to get

$$\begin{aligned}\langle j-j'\rangle^\beta|\widehat{X}(l)_j^{j'}| &\lesssim N^{2\tau}\langle j\rangle^{2\tau}\langle j-j'\rangle^\beta|\widehat{P}(l)_j^{j'}|,\\ \langle j-j'\rangle^\beta|\widehat{X}(l)_j^{j'}| &\lesssim N^{2\tau}\langle j'\rangle^{2\tau}\langle j-j'\rangle^\beta|\widehat{P}(l)_j^{j'}|;\end{aligned} \tag{2.3.43}$$

hence we repeat the same argument of (2.3.41), (2.3.42) to get (2.3.37). Concerning Lipschitz estimates, recall that the eigenvalues λ_j have the expansion

$$\lambda_j(\tilde\omega) = \lambda_j^{(0)}(\tilde\omega) + \rho_j(\tilde\omega) = \mathrm{i}\nu^{(0)}(\tilde\omega)\cdot j + z(\tilde\omega, j) + \rho_j(\tilde\omega) \quad \forall j\in\mathbb{Z}^d\,.$$

By (2.2.8), (2.2.36) and the induction hypotheses (2.3.23), one has that for any $\widetilde\omega_1,\widetilde\omega_2\in\Omega_\gamma$ and any $j,j'\in\mathbb{Z}^d$,

$$|(\lambda_j-\lambda_{j'})(\widetilde\omega_1) - (\lambda_j-\lambda_{j'})(\widetilde\omega_2)| \lesssim \varepsilon\gamma^{-1}\langle j-j'\rangle|\widetilde\omega_1-\widetilde\omega_2|\,. \tag{2.3.44}$$

Hence one uses $|l|, |j-j'|\le N$, (2.3.38), (2.3.44) and the inequality

$$|l|^{2\tau+1}|j|^{2\tau}|j'|^{2\tau} \lesssim_\tau N^{2\tau+1}|j|^{2\tau}\left(|j|^{2\tau}+N^{2\tau}\right) \lesssim N^{4\tau+1}\langle j\rangle^{4\tau}$$

to deduce the Lipschitz estimates as usual. By Remark B.1.1 of Appendix B, if $A = \mathrm{diag}_{j\in\mathbb{Z}^d}\lambda_j$ and P are real and reversible one immediately gets that X is real and reversible too. $\square$

The estimate (2.3.26) at the step $k+1$ then follows combining (2.3.36) and (2.3.24) at step k. Moreover, using that by (2.3.14), $\alpha > 6\tau+3$ and by using the smallness condition (2.3.19), one gets that

$$\|X_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma\pm m,\sigma\pm m}} \le \delta(s) \tag{2.3.45}$$

for some $\delta(s)\in(0,1)$ small enough. Therefore, one can apply Lemma B.1.4 and deduce that, by (2.3.36) and (2.3.37),

$$\begin{aligned}&\|\Phi_k^{-1}-\mathbb{I}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma\pm m,\sigma\pm m}} \lesssim_{s,\sigma} \|X_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma\pm m,\sigma\pm m}} \lesssim_s N_k^{4\tau+2}\|P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\\ &\|\langle\nabla\rangle^\beta(\Phi_k^{-1}-\mathbb{I})\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma\pm m,\sigma\pm m}} \lesssim_{s,\beta} \|\langle\nabla\rangle^\beta X_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma\pm m,\sigma\pm m}} \lesssim_{s,\beta} N_k^{4\tau+2}\|\langle\nabla\rangle^\beta P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\,.\end{aligned} \tag{2.3.46}$$

It remains to prove that (2.3.24) holds at the step $k+1$. To this aim, in the next lemma we obtain key estimates for the remainder term P_{k+1} defined in (2.3.30).

Lemma 2.3.10. *There exists a constant $C = C(s, \sigma, \tau) > 0$ such that the operator $P_{k+1}(\varphi)$ defined in* (2.3.30) *fulfills*

$$\begin{aligned} \|P_{k+1}\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} &\leq C\Big(N_k^{4\tau+2}\big(\|P_k\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\big)^2 + N_k^{-\beta}\|P_k\|^{\rm Lip}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}}\Big), \\ \|P_{k+1}\|^{\rm Lip}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} &\leq C\|P_k\|^{\rm Lip}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}}. \end{aligned} \tag{2.3.47}$$

Furthermore, if $P_k(\varphi)$ is real and reversible then $P_{k+1}(\varphi)$ is real and reversible too.

Proof. By recalling the definition of P_{k+1} given in (2.3.30), using the inductive estimates (2.3.36), (2.3.37), and the estimate (2.3.46), by applying Lemma 2.3.6 and Lemma B.1.3 in Appendix B, which gives an estimate of the product of operators, we get

$$\begin{aligned} \|P_{k+1}\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \lesssim_{s,\sigma} & \; N_k^{4\tau+2}\big(\|P_k\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\big)^2 \\ & + N_k^{-\beta}\left(\|P_k\|^{\rm Lip}_{\mathcal{M}^{s+\beta}_{\sigma-m,\sigma+m}} + \|\langle\nabla\rangle^\beta P_k\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\right), \end{aligned} \tag{2.3.48}$$

$$\|P_{k+1}\|^{\rm Lip}_{\mathcal{M}^{s+\beta}_{\sigma-m,\sigma+m}} \lesssim_{s,\sigma} N_k^{4\tau+2}\|P_k\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\|P_k\|^{\rm Lip}_{\mathcal{M}^{s+\beta}_{\sigma-m,\sigma+m}} + \|P_k\|^{\rm Lip}_{\mathcal{M}^{s+\beta}_{\sigma-m,\sigma+m}}, \tag{2.3.49}$$

$$\begin{aligned} \|\langle\nabla\rangle^\beta P_{k+1}\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \lesssim_{s,\sigma} & \; \|\langle\nabla\rangle^\beta P_k\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \\ & + N_k^{4\tau+2}\|P_k\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\|\langle\nabla\rangle^\beta P_k\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}. \end{aligned} \tag{2.3.50}$$

Recalling that $\|\cdot\|^{\rm Lip}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} = \|\cdot\|^{\rm Lip}_{\mathcal{M}^{s+\beta}_{\sigma-m,\sigma+m}} + \|\langle\nabla\rangle^\beta\cdot\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}$ and summing up the contribution of (2.3.49), (2.3.50), we get

$$\begin{aligned} \|P_{k+1}\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} &\lesssim N_k^{4\tau+2}\big(\|P_k\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\big)^2 + N_k^{-\beta}\|P_k\|^{\rm Lip}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}}, \\ \|P_{k+1}\|^{\rm Lip}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} &\lesssim N_k^{4\tau+2}\|P_k\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\|P_k\|^{\rm Lip}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} + \|P_k\|^{\rm Lip}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}}. \end{aligned} \tag{2.3.51}$$

Furthermore, by using the smallness condition (2.3.19), recalling the definition (2.3.15), using that $\alpha > 6\tau + 3$, taking N_0 large enough and ε small enough one gets that

$$N_k^{4\tau+2}\|P_k\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \lesssim N_k^{4\tau+2}N_{k-1}^{-\alpha}\varepsilon \leq 1$$

and then (2.3.51) implies the claimed estimate (2.3.47).

Finally, if P_k is real and reversible, then by Lemma 2.3.9, the operator X_k (and hence $\Phi_k = \mathbb{I} + X_k$ and Φ_k^{-1}) is real and reversibility preserving. By the definition (2.3.30), one concludes that P_{k+1} is real and reversible. □

By Lemma 2.3.10 and by (2.3.24), one has

$$\|P_{k+1}\|^{\rm Lip}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} \leq C\|P_k\|^{\rm Lip}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} \leq CC_*\varepsilon N_{k-1} \leq C_*\varepsilon N_k$$

provided $CN_{k-1} \leq N_k$ for any $k \geq 0$. This latter condition is verified by taking $N_0 > 0$ large enough. Furthermore, by (2.3.24), one has

$$\begin{aligned}\|P_{k+1}\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} &\leq CN_k^{4\tau+2}\big(\|P_k\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\big)^2 + CN_k^{-\beta}\|P_k\|^{\rm Lip}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} \\ &\leq CN_k^{4\tau+2}C_*^2\varepsilon^2 N_{k-1}^{-2\alpha} + CN_k^{-\beta}C_*N_{k-1}\varepsilon \leq C_*\varepsilon N_k^{-\alpha}\,,\end{aligned}$$

provided

$$2CN_k^{\alpha+4\tau+2}N_{k-1}^{-2\alpha}\varepsilon \leq 1\,, \quad ,2CN_k^{\alpha-\beta}N_{k-1} \leq 1 \quad \forall k \geq 0\,.$$

The above conditions are verified by (2.3.14), the smallness condition (2.3.19), recalling the definition (2.3.15) and taking ε small enough and N_0 large enough. Hence the estimate (2.3.24) is proved at the step $k+1$, and the proof of $(\mathbf{S1})_{k+1}$ is then concluded.

Proof of $(\mathbf{S2})_{k+1}$. By the estimate (2.3.32), on the set $\mathcal{O}_{k,\gamma}$,

$$\delta_j^{(k)} := \rho_j^{(k+1)} - \rho_j^{(k)}$$

satisfies $|\delta_j^{(k)}|^{\rm Lip} \lesssim \langle j\rangle^{-2m}\|P_k\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \lesssim_{s,\sigma} \langle j\rangle^{-2m}N_{k-1}^{-\alpha}\varepsilon$ for any $j \in \mathbb{Z}^d$. By the Kirszbraun Theorem (see Lemma M.5 in [KP03]), we extend the function $\delta_j^{(k)} : \mathcal{O}_{k,\gamma} \to \mathbb{C}$ to a function $\widetilde{\delta}_j^{(k)} : \Omega_{0,\gamma} \to \mathbb{C}$ which still satisfies the estimate $|\widetilde{\delta}_j^{(k)}|^{\rm Lip} \lesssim \langle j\rangle^{-2m}\|P_k\|^{\rm Lip}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \lesssim_{s,\sigma} \langle j\rangle^{-2m}N_{k-1}^{-\alpha}\varepsilon$. Therefore, $(\mathbf{S2})_{k+1}$ follows by defining $\widetilde{\rho}_j^{(k+1)} := \widetilde{\rho}_j^{(k)} + \widetilde{\delta}_j^{(k)}$ and $\widetilde{\lambda}_j^{(k+1)} = \lambda_j^{(0)} + \widetilde{\rho}_j^{(k+1)}$ (note that $\lambda_j^{(0)}$ is already defined on $\Omega_{0,\gamma}$). Furthermore we observe that, in the real and reversible case, one has that $\rho_j^{(k)}, \lambda_j^{(k)} \in {\rm i}\mathbb{R}$, $\widetilde{\rho}_j^{(k)}, \widetilde{\lambda}_j^{(k)} \in {\rm i}\mathbb{R}$, and $\delta_j^{(k)} \in {\rm i}\mathbb{R}$, thus also $\widetilde{\lambda}_j^{(k+1)}$, $\widetilde{\rho}_j^{(k+1)} \in {\rm i}\mathbb{R}$.

2.3.4 Passing to the limit

By Theorem 2.3.8-$(\mathbf{S2})_k$, using a telescoping argument, for any $j \in \mathbb{Z}^d$, the sequence $(\widetilde{\rho}_j^{(k)})_{k\geq 0}$ is a Cauchy sequence w.r. to the norm $|\cdot|^{\rm Lip}$ in $\Omega_{0,\gamma}$, and hence it converges to $\rho_j^{(\infty)}$. Moreover, the following estimates hold:

$$|\widetilde{\rho}_j^{(k)} - \rho_j^{(\infty)}|^{\mathrm{Lip}} \lesssim_{s,\sigma} \langle j \rangle^{-2m} N_{k-1}^{-\alpha} \varepsilon \,, \qquad |\rho_j^{(\infty)}|^{\mathrm{Lip}} \lesssim_{s,\sigma} \langle j \rangle^{-2m} \varepsilon \,. \tag{2.3.52}$$

Note that, as observed in the proof of $(\mathbf{S2})_{k+1}$, in the real and reversible case one has $\rho_j^{(\infty)} : \Omega_{0,\gamma} \to \mathrm{i}\mathbb{R}$ for any $j \in \mathbb{Z}^d$.
We then define the *final eigenvalues* $\lambda_j^{(\infty)} : \Omega_{0,\gamma} \to \mathbb{C}$ as

$$\lambda_j^{(\infty)} := \lambda_j^{(0)} + \rho_j^{(\infty)} = \mathrm{i}\nu^{(0)} \cdot j + z(j) + \rho_j^{(\infty)} \,, \quad j \in \mathbb{Z}^d \,, \tag{2.3.53}$$

where in the last equality we have used the definition of $\lambda_j^{(0)}$ as in (2.3.13). We then define

$$\mathcal{O}_{\infty,\gamma} := \Big\{ \widetilde{\omega} = (\omega,\nu) \in \Omega_{0,\gamma} \;:\; |\mathrm{i}\omega \cdot l + \lambda_j^{(\infty)}(\widetilde{\omega}) - \lambda_{j'}^{(\infty)}(\widetilde{\omega})| \geq \frac{2\gamma}{\langle l \rangle^\tau \langle j \rangle^\tau \langle j' \rangle^\tau} \quad \forall (l,j,j') \neq (0,j,j) \quad \Big\} \,. \tag{2.3.54}$$

The following lemma holds for such a set:

Lemma 2.3.11. *One has $\mathcal{O}_{\infty,\gamma} \subseteq \cap_{k\geq 0} \mathcal{O}_{k,\gamma}$.*

Proof. We prove by induction that for any $k \geq 0$ one has $\mathcal{O}_{\infty,\gamma} \subseteq \mathcal{O}_{k,\gamma}$. For $k = 0$, it follows by definition that $\mathcal{O}_{\infty,\gamma} \subseteq \mathcal{O}_{0,\gamma}$ since $\mathcal{O}_{0,\gamma} = \Omega_{0,\gamma}$. Then assume that $\mathcal{O}_{\infty,\gamma} \subseteq \mathcal{O}_{k,\gamma}$ for some $k \geq 0$: we come to show that $\mathcal{O}_{\infty,\gamma} \subseteq \mathcal{O}_{k+1,\gamma}$. Let $\widetilde{\omega} = (\omega,\nu) \in \mathcal{O}_{\infty,\gamma}$. Since by the inductive hypothesis $\widetilde{\omega} \in \mathcal{O}_{k,\gamma}$, item $(\mathbf{S1})_k$ of Theorem 2.3.8 entails that $\lambda_j^{(k)}(\widetilde{\omega})$ is well defined, and by $(\mathbf{S2})_k$ of Theorem 2.3.8 one has that $\widetilde{\lambda}_j^{(k)}(\widetilde{\omega}) = \lambda_j^{(k)}(\widetilde{\omega})$ and $\widetilde{\rho}_j^{(k)}(\widetilde{\omega}) = \rho_j^{(k)}(\widetilde{\omega})$ (recall that $\lambda_j^{(k)} = \lambda_j^{(0)} + \rho_j^{(k)}$ and $\widetilde{\lambda}_j^{(k)} = \lambda_j^{(0)} + \widetilde{\rho}_j^{(k)}$). We then have that for any $(l,j,j') \neq (0,j,j)$ such that $|l|, |j-j'| \leq N_k$,

$$\begin{aligned} |\mathrm{i}\omega \cdot l + \lambda_j^{(k)}(\widetilde{\omega}) - \lambda_{j'}^{(k)}(\widetilde{\omega})| &\geq |\mathrm{i}\omega \cdot l + \lambda_j^{(\infty)}(\widetilde{\omega}) - \lambda_{j'}^{(\infty)}(\widetilde{\omega})| \\ &\quad - |\widetilde{\rho}_j^{(k)}(\widetilde{\omega}) - \rho_j^{(\infty)}(\widetilde{\omega})| - |\widetilde{\rho}_{j'}^{(k)}(\widetilde{\omega}) - \rho_{j'}^{(\infty)}(\widetilde{\omega})| \,. \end{aligned}$$

By the first estimate in (2.3.52), it follows that there exists a constant $C > 0$ such that

$$|\mathrm{i}\omega \cdot l + \lambda_j^{(k)}(\widetilde{\omega}) - \lambda_{j'}^{(k)}(\widetilde{\omega})| \geq \frac{2\gamma}{\langle l \rangle^\tau \langle j \rangle^\tau \langle j' \rangle^\tau} - \frac{C\varepsilon}{N_{k-1}^{\alpha} \min\{\langle j \rangle, \langle j' \rangle\}^{2m}}$$

thus

$$|\mathrm{i}\omega \cdot l + \lambda_j^{(k)}(\widetilde{\omega}) - \lambda_{j'}^{(k)}(\widetilde{\omega})| \geq \frac{\gamma}{\langle l \rangle^\tau \langle j \rangle^\tau \langle j' \rangle^\tau} \,,$$

provided

$$\frac{C\varepsilon\langle l\rangle^\tau\langle j\rangle^\tau\langle j'\rangle^\tau}{\gamma N_{k-1}^\alpha \min\{\langle j\rangle,\langle j'\rangle\}^{2m}} \le 1\,. \tag{2.3.55}$$

Using that $|l|, |j-j'| \le N_k$, $m > \tau$ and since

$$\begin{aligned}\langle j\rangle\langle j'\rangle &\le \big(\langle j-j'\rangle + \min\{\langle j\rangle,\langle j'\rangle\}\big)^2\\ &\lesssim \langle j-j'\rangle^2 + \min\{\langle j\rangle,\langle j'\rangle\}^2\\ &\lesssim N_k^2 + \min\{\langle j\rangle,\langle j'\rangle\}^2\,,\end{aligned}$$

one gets that

$$\frac{\langle l\rangle^\tau\langle j\rangle^\tau\langle j'\rangle^\tau}{\min\{\langle j\rangle,\langle j'\rangle\}^{2m}} \lesssim N_k^{3\tau}\,. \tag{2.3.56}$$

Therefore

$$\frac{C\varepsilon\langle l\rangle^\tau\langle j\rangle^\tau\langle j'\rangle^\tau}{\gamma N_{k-1}^\alpha \min\{\langle j\rangle,\langle j'\rangle\}^{2m}} \le C'\varepsilon\gamma^{-1}N_k^{3\tau}N_{k-1}^{-\alpha} \le 1$$

since $\alpha > \frac{9}{2}\tau$ (see (2.3.14)) and by taking ε small enough (see the smallness condition (2.3.19) and recall that $\gamma^{-1} = N_0$). Condition (2.3.55) is then verified and hence $\widetilde{\omega} \in \mathcal{O}_{k+1,\gamma}$. This concludes the proof of the lemma. □

For any $k \ge 0$, $\widetilde{\omega} \in \mathcal{O}_{\infty,\gamma}$ we define the map

$$\mathcal{V}_k(\varphi,\widetilde{\omega}) \equiv \mathcal{V}_k(\varphi) := \Phi_0(\varphi)\circ\Phi_1(\varphi)\circ\ldots\circ\Phi_k(\varphi)\,. \tag{2.3.57}$$

Note that by Lemma 2.3.11 and Theorem 2.3.8 all the maps $\Phi_k(\varphi)$ are well defined for $\widetilde{\omega} \in \mathcal{O}_{\infty,\gamma}$.
Furthermore, the following lemma holds:

Lemma 2.3.12. *The sequence $(\mathcal{V}_k)_{k\ge 0}$ converges to an invertible operator $\mathcal{V}_\infty$ in $\mathcal{L}ip\Big(\mathcal{O}_{\infty,\gamma}; \mathcal{H}^s\big(\mathbb{T}^{\mathtt{n}}; \mathcal{B}(\mathcal{H}^{\sigma\pm m}, \mathcal{H}^{\sigma\pm m})\big)\Big)$ and for any $\sigma \ge 0$ the operator $\mathcal{V}_\infty^{\pm 1} - \mathbb{I}$ satisfies the estimate*

$$\|\mathcal{V}_\infty^{\pm1} - \mathbb{I}\|^{\mathrm{Lip}}_{H^s\left(\mathbb{T}^{\mathtt{n}},\mathcal{B}(\mathcal{H}^\sigma,\mathcal{H}^\sigma)\right)} \lesssim_{s,\sigma} N_0^{4\tau+2}\varepsilon\,.$$

Moreover in the real and reversible case, $\mathcal{V}_\infty^{\pm1}$ is real and reversibility preserving.

Proof. The proof is based on standard arguments and therefore it is omitted (see for instance the proof of Corollary 4.1 in [Mon19]). The presence of $N_0^{4\tau+2}$ in front of ε in the claimed inequality is due to the fact that (2.3.26) for $k = 0$ gives $\|\Phi_0 - \mathbb{I}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma\pm m,\sigma\pm m}} \lesssim_{s,\sigma} N_0^{4\tau+2}\varepsilon$. □

Lemma 2.3.13. *For any $\widetilde{\omega} \in \mathcal{O}_{\infty,\gamma}$, one has that $(\mathcal{V}_\infty)_{\omega *}(A_0 + P_0) = H_\infty$ (recall (2.3.12)) where the operator H_∞ is given by $H_\infty = \mathrm{diag}_{j \in \mathbb{Z}^d} \lambda_j^{(\infty)}$. Furthermore in the real and reversible case, the eigenvalues $\lambda_j^{(\infty)}$ are purely imaginary.*

Proof. By (2.3.25) and recalling the definition (2.3.57), one gets that for any $k \geq 1$

$$(\mathcal{V}_{k-1})_{\omega *}(A_0 + P_0(\varphi)) = H_k(\varphi) = A_k + P_k(\varphi)\,.$$

The claimed statement then follows by passing to the limit in the above identity, recalling the definition of A_k given in (2.3.21), the definition (2.3.53), the estimates (2.3.24), (2.3.52) and Lemma 2.3.12. □

2.3.5 Measure Estimates

In this section we show that the set $\mathcal{O}_{\infty,\gamma}$ defined in (2.3.54) has *large* Lebesgue measure. We actually prove the following:

Proposition 2.3.14. *One has $|\Omega \setminus \mathcal{O}_{\infty,\gamma}| \lesssim \gamma$.*

Since $\Omega \setminus \mathcal{O}_{\infty,\gamma} = (\Omega \setminus \Omega_{0,\gamma}) \cup (\Omega_{0,\gamma} \setminus \mathcal{O}_{\infty,\gamma})$ and by Remark 2.2.3 one has that $|\Omega \setminus \Omega_{0,\gamma}| \lesssim \gamma$, it is enough to estimate the measure of the set $\Omega_{0,\gamma} \setminus \mathcal{O}_{\infty,\gamma}$. By the definition (2.3.54), one has that

$$\Omega_{0,\gamma} \setminus \mathcal{O}_{\infty,\gamma} = \bigcup_{\substack{(l,j,j') \in \mathbb{Z}^{\mathrm{n}} \times \mathbb{Z}^d \times \mathbb{Z}^d \\ (l, j-j') \neq (0,0)}} \mathcal{R}_{ljj'}(\gamma)\,, \tag{2.3.58}$$

with

$$\mathcal{R}_{ljj'}(\gamma) := \Big\{ \tilde{\omega} = (\omega, \nu) \in \Omega_{0,\gamma} \ \Big| \ |\mathrm{i}\omega \cdot l + \lambda_j^{(\infty)}(\omega,\nu) - \lambda_{j'}^{(\infty)}(\omega,\nu)| < \frac{2\gamma}{\langle l \rangle^\tau \langle j \rangle^\tau \langle j' \rangle^\tau} \Big\}\,. \tag{2.3.59}$$

Lemma 2.3.15. *For any $(l, j, j') \in \mathbb{Z}^{\mathrm{n}} \times \mathbb{Z}^d \times \mathbb{Z}^d$ such that $l \neq 0$ and $j - j' \neq 0$, one has $|\mathcal{R}_{ljj'}(\gamma)| \lesssim \gamma \langle l \rangle^{-\tau} \langle j \rangle^{-\tau} \langle j' \rangle^{-\tau}$.*

Proof. By the definition of $\lambda_j^{(\infty)}$ as in (2.3.53), one has that for any $j \in \mathbb{Z}^d$

$$\lambda_j^{(\infty)}(\omega,\nu) = \mathrm{i}\nu^{(0)}(\omega,\nu) \cdot j + z(j,\omega,\nu) + \rho_j^{(\infty)}(\omega,\nu)\,,$$

where by the estimates (2.2.8), (2.2.36), one has

$$|\nu^{(0)} - \nu|^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon, \quad \sup_{j\in\mathbb{Z}^d} |\partial_\xi z(j)|^{\mathrm{lip}} \lesssim \varepsilon$$

(see Definition 2.2.1). Then the map

$$\Psi : \Omega_{0,\gamma} \to \Psi(\Omega_{0,\gamma})$$
$$(\omega, \nu) \mapsto (\omega, \nu^{(0)}(\omega, \nu))$$

is a Lipschitz homeomorphism with inverse given by

$$\Psi^{-1} : \Psi(\Omega_{0,\gamma}) \to \Omega_{0,\gamma}$$
$$(\omega, \zeta) \mapsto \Psi^{-1}(\omega, \zeta)$$

and satisfying

$$|\Psi^{-1} - \mathbb{I}|^{\sup} \lesssim \varepsilon\,, \quad |\Psi^{-1} - \mathbb{I}|^{\mathrm{lip}} \lesssim \varepsilon\gamma^{-1}\,. \tag{2.3.60}$$

Defining

$$a_j^{(\infty)}(\omega, \zeta) := \lambda_j^{(\infty)}(\Psi^{-1}(\omega, \zeta)), \quad j \in \mathbb{Z}^d$$

and

$$\widetilde{\mathcal{R}}_{ljj'}(\gamma) := \Big\{(\omega, \zeta) \in \Psi(\Omega_{0,\gamma}) : |\mathrm{i}\omega\cdot l + a_j^{(\infty)}(\omega, \zeta) - a_{j'}^{(\infty)}(\omega, \zeta)| < \frac{2\gamma}{\langle l\rangle^\tau \langle j\rangle^\tau \langle j'\rangle^\tau}\Big\}$$

one has that

$$|\mathcal{R}_{ljj'}(\gamma)| \simeq |\widetilde{\mathcal{R}}_{ljj'}(\gamma)|, \tag{2.3.61}$$

thus it is sufficient to estimate the measure of the set $\widetilde{\mathcal{R}}_{ljj'}(\gamma)$. The functions $a_j^{(\infty)}$ admit the expansion

$$a_j^{(\infty)}(\omega, \zeta) = \mathrm{i}\zeta \cdot j + z_\Psi(j, \omega, \zeta) + r_j^{(\infty)}(\omega, \zeta)$$

where

$$z_\Psi(j, \omega, \zeta) := z(j, \Psi^{-1}(\omega, \zeta)), \quad r_j^{(\infty)}(\omega, \zeta) := \rho_j^{(\infty)}(\Psi^{-1}(\omega, \zeta))\,.$$

By the estimate (2.3.60) and using the estimates (2.2.36), (2.3.52) on z and $\rho_j^{(\infty)}$, for $\varepsilon\gamma^{-1}$ small enough, one can easily deduce that

$$\sup_{j\in\mathbb{Z}^d} |\partial_\xi z_\Psi(j, \cdot)|^{\mathrm{Lip}} \lesssim \varepsilon, \quad \sup_{j\in\mathbb{Z}^d} \langle j\rangle^{2m} |r_j^{(\infty)}|^{\mathrm{Lip}} \lesssim \varepsilon\,. \tag{2.3.62}$$

Since $(l, j-j') \neq (0,0)$, we write

$$(\omega, \zeta) = (\omega(s), \zeta(s))) = \frac{(l, j-j')}{|(l, j-j')|} s + w, \quad w \in \mathbb{R}^{\mathtt{n}+d}, \quad w \cdot (l, j-j') = 0$$

and we consider

$$\begin{aligned} f_{ljj'}(s) &:= \mathrm{i}\omega(s) \cdot l + a_j^{(\infty)}(\omega(s), \zeta(s)) - a_{j'}^{(\infty)}(\omega(s), \zeta(s)) \\ &= \mathrm{i}|(l, j-j')| s + z_\Psi(j, \omega(s), \zeta(s)) - z_\Psi(j', \omega(s), \zeta(s)) \\ &+ r_j^{(\infty)}(\omega(s), \zeta(s)) - r_{j'}^{(\infty)}(\omega(s), \zeta(s)) \,. \end{aligned}$$

Using the estimates (2.3.62) and recalling that $|j-j'| \leq |(l, j-j')$, one obtains that

$$\begin{aligned} |f_{ljj'}(s_1) - f_{ljj'}(s_2)| &\geq \Big(|(l, j-j')| - C\varepsilon|j-j'| - C\varepsilon \Big) |s_1 - s_2| \\ &\geq \Big((1 - C\varepsilon)|(l, j-j')| - C\varepsilon \Big) |s_1 - s_2| \qquad (2.3.63) \\ &\geq \frac{1}{2} |s_1 - s_2| \qquad (2.3.64) \end{aligned}$$

by taking ε small enough. This implies that

$$\Big| \Big\{ s : |f_{ljj'}(s)| < \frac{2\gamma}{\langle l \rangle^\tau \langle j \rangle^\tau \langle j' \rangle^\tau} \Big\} \Big| \lesssim \frac{\gamma}{\langle l \rangle^\tau \langle j \rangle^\tau \langle j' \rangle^\tau} \,.$$

By a Fubini argument one gets that $|\widetilde{\mathcal{R}}_{ljj'}(\gamma)| \lesssim \gamma \langle l \rangle^{-\tau} \langle j \rangle^{-\tau} \langle j' \rangle^{-\tau}$. The claimed statement then follows by recalling (2.3.61). □

Proof of Proposition 2.3.14. By (2.3.58) and Lemma 2.3.15 one gets that

$$|\Omega_{0,\gamma} \setminus \mathcal{O}_{\infty,\gamma}| \lesssim \gamma \sum_{l \in \mathbb{Z}^{\mathtt{n}}, j, j' \in \mathbb{Z}^d} \langle l \rangle^{-\tau} \langle j \rangle^{-\tau} \langle j' \rangle^{-\tau} \lesssim \gamma$$

since $\tau > \max\{\mathtt{n}\,, d\}$. The claimed statement then follows by recalling that $|\Omega \setminus \Omega_{0,\gamma}| \lesssim \gamma$ and that $\Omega \setminus \mathcal{O}_{\infty,\gamma} = (\Omega \setminus \Omega_{0,\gamma}) \cup (\Omega_{0,\gamma} \setminus \mathcal{O}_{\infty,\gamma})$.

2.4 Proof of the main results

In this section we prove Theorem 2.1.7 and 2.1.10.

Proof of Theorem 2.1.7. We consider the composition

$$\mathcal{U}(\varphi) = \mathcal{V}(\varphi) \circ \mathcal{V}_\infty(\varphi), \quad \mathcal{V}(\varphi) := \mathcal{A}(\varphi) \circ e^{-\varepsilon G_1(\varphi,\tilde{\omega})} \circ \cdots \circ \ e^{-\varepsilon G_M(\varphi,\tilde{\omega})},$$

where $\mathcal{A}(\varphi)$ is defined in Section 2.2.1, the maps $e^{-\varepsilon G_j}$ are constructed in Section 2.2.2 (see Theorem 2.2.7) and $\mathcal{V}_\infty$ is given in Lemma 2.3.12. By Section 2.2.1, Theorem 2.2.7 and Lemma 2.3.13, for any $\widetilde{\omega} \in \mathcal{O}_{\infty,\gamma}$, the map $\mathcal{U}(\varphi)$ conjugates the equation (2.0.1) to the equation $\partial_t u = H_\infty u$ where H_∞ is the diagonal operator with eigenvalues $(\lambda_j^{(\infty)})_{j\in\mathbb{Z}^d}$. Let $0 < \mathfrak{a} < \frac{1}{C_0}$ and $N_0 := \frac{1}{\varepsilon^{\mathfrak{a}}}$ so that the smallness condition (2.3.19), i.e. $N_0^{C_0}\varepsilon \le \delta$, becomes

$$N_0^{C_0}\varepsilon = \varepsilon^{1-C_0\mathfrak{a}} \le \delta\,,$$

which is satisfied for ε small enough. Since $\gamma = N_0^{-1} = \varepsilon^{\mathfrak{a}}$, setting $\Omega_\varepsilon := \mathcal{O}_{\infty,\gamma}$, Proposition 2.3.14 implies that $\lim_{\varepsilon\to 0}|\Omega \setminus \Omega_\varepsilon| = 0$. Concerning the properties of the map $\mathcal{U}$, by (2.2.14) of Remark 2.2.4, for any $\alpha \in \mathbb{N}^{\mathfrak{n}}$ one has $\partial_\varphi^\alpha \mathcal{A} \in \mathcal{B}\left(\mathcal{H}^{\sigma+|\alpha|}, \mathcal{H}^\sigma\right)$, as well as $\partial_\varphi^\alpha e^{-\varepsilon G_j} \in \mathcal{B}\left(\mathcal{H}^\sigma, \mathcal{H}^{\sigma-(1-j\mathfrak{e})|\alpha|}\right)$ $\forall j = 1,\dots,M$, as shown in Remark 2.2.9, so that the map $\mathcal{V} : \varphi \mapsto \mathcal{V}(\phi)$ satisfies

$$\mathcal{V}^{\pm 1} \in \mathcal{C}^s\left(\mathbb{T}^{\mathfrak{n}}; \mathcal{B}\left(\mathcal{H}^{\sigma+(M(1-\mathfrak{e})+1)(s+1)}, \mathcal{H}^\sigma\right)\right) \quad \forall s \ge 0,\ \forall \sigma \ge 0\,. \tag{2.4.1}$$

Moreover, by Lemma 2.3.12, one deduces

$$\mathcal{V}_\infty^{\pm 1} \in \mathcal{H}^s\left(\mathbb{T}^{\mathfrak{n}}; \mathcal{B}\left(\mathcal{H}^\sigma, \mathcal{H}^\sigma\right)\right) \quad \forall s \ge 0,\ \sigma \ge 0\,. \tag{2.4.2}$$

Then (2.4.1) and (2.4.2) imply that the map $\varphi \mapsto \mathcal{U}(\varphi)$ satisfies $\forall s \ge 0$ and $\forall \sigma \ge 0$

$$\mathcal{U}^{\pm 1} \in \mathcal{C}^s\left(\mathbb{T}^{\mathfrak{n}}; \mathcal{B}\left(\mathcal{H}^{\sigma+\eta_s}, \mathcal{H}^\sigma\right)\right), \tag{2.4.3}$$

for some positive η_s depending on s only. Furthermore, by (2.4.3), with a standard density argument one also obtains that for any $\psi \in \mathcal{H}^\sigma$ and for any $\sigma \ge 0$ the map $\mathbb{T}^{\mathfrak{n}} \ni \varphi \mapsto \mathcal{U}^{\pm 1}(\varphi)\psi \in \mathcal{H}^\sigma$ is continuous. Finally, again by Remarks 2.2.4, 2.2.9 and by Lemma 2.3.12 one also obtains the existence of a positive σ_1, independent of σ, such that $\|\mathcal{U}^{\pm 1}(\varphi) - \mathbb{I}\|_{\mathcal{B}(\mathcal{H}^{\sigma+\sigma_1},\mathcal{H}^\sigma)} \lesssim_\sigma \varepsilon$. The proof is therefore concluded. □

Finally, from Theorem 2.1.7 we immediately deduce Corollary 2.1.10 as follows:

Proof of Corollary 2.1.10. By Theorem 2.1.7, for any $\widetilde{\omega} = (\omega, \nu) \in \Omega_\varepsilon$ under the change of coordinates $u = \mathcal{U}(\omega t)v$, the Cauchy problem

$$\begin{cases} \partial_t u = \Big(\nu + \varepsilon V(\omega t, x)\Big) \cdot \nabla u + \varepsilon \mathcal{W}(\omega t)[u] \\ u(0,x) = u_0(x), \end{cases} \qquad u_0 \in \mathcal{H}^\sigma(\mathbb{T}^d) \tag{2.4.4}$$

is transformed into

$$\begin{cases} \partial_t v = H_\infty v \\ v(0) = v_0, \end{cases} \qquad v_0 := \mathcal{U}(0)^{-1} u_0 \,. \tag{2.4.5}$$

Using that for any $\widetilde{\omega} = (\omega, \nu) \in \Omega_\varepsilon$, $\mathcal{U}(\varphi)$ is bounded and invertible on $\mathcal{H}^\sigma$ one gets that

$$\|\psi\|_{\mathcal{H}^\sigma} \lesssim_\sigma \|\mathcal{U}(\varphi)^{\pm 1}\psi\|_{\mathcal{H}^\sigma} \lesssim_\sigma \|\psi\|_{\mathcal{H}^\sigma}, \quad \forall \psi \in \mathcal{H}^\sigma(\mathbb{T}^d) \tag{2.4.6}$$

uniformly w.r. to $\varphi \in \mathbb{T}^{\mathfrak{n}}$.
CASE (1). If all the eigenvalues $\lambda_j^{(\infty)}$, $j \in \mathbb{Z}^d$ of the operator H_∞ are purely imaginary, the solution of the Cauchy problem (2.4.5) satisfies

$$\|v(t,\cdot)\|_{\mathcal{H}^\sigma} = \|v_0\|_{\mathcal{H}^\sigma}$$

for any $t \in \mathbb{R}$. By the estimate (2.4.6) and recalling that $u = \mathcal{U}(\omega t)v$, one obtains the desired bound on the solution $u(t,x)$ of (2.4.4).
CASE (2) Let $j \in \mathbb{Z}^d$ so that $\mathrm{Re}(\lambda_j^{(\infty)}) \neq 0$. Then for any $\alpha \in \mathbb{C}$, the solution v of the Cauchy problem (2.4.5) with initial datum $v_0(x) = \alpha e^{\mathrm{i} j \cdot x}$ is given by

$$v(t,x) = \alpha e^{\lambda_j^{(\infty)} t} e^{\mathrm{i} j \cdot x} \,.$$

Hence, setting $u_0 := \mathcal{U}(0)[\alpha e^{\mathrm{i} j \cdot x}] = \alpha \mathcal{U}(0)[e^{\mathrm{i} j \cdot x}]$, one has that the solution of the Cauchy problem (2.4.4) with such an initial datum u_0 is given by

$$u(t,x) = \mathcal{U}(\omega t)[\alpha e^{\lambda_j^{(\infty)} t} e^{\mathrm{i} j \cdot x}] = \alpha e^{\lambda_j^{(\infty)} t} \mathcal{U}(\omega t)[e^{\mathrm{i} j \cdot x}] \,.$$

Recalling (2.4.6) one gets that

$$\|u(t,\cdot)\|_{\mathcal{H}^\sigma} \simeq_\sigma C_j e^{\mathrm{Re}(\lambda_j^{(\infty)})t} \,.$$

This gives the growth for $t > 0$ if $\mathrm{Re}\lambda_j^{(\infty)} > 0$ or for $t < 0$ if $\mathrm{Re}\lambda_j^{(\infty)} > 0$. If there exists $\lambda_j^{(\infty)}$ with $\mathrm{Re}\lambda_j^{(\infty)} > 0$ and $\lambda_{j'}^{(\infty)}$ with $\mathrm{Re}\lambda_{j'}^{(\infty)} < 0$ then the solution with initial datum $\alpha e^{\mathrm{i} j \cdot x} + \beta e^{\mathrm{i} j' \cdot x}$ grows both as $t > 0$ and as $t < 0$. □

Part II

Towards a more general model: Schrödinger operators on flat tori

Chapter 3

Setting and pseudo-differential calculus

In the second part of this thesis we study spectral properties of a Schrödinger operator of the form

$$-\Delta + V \quad \text{on} \quad L^2(\mathbb{T}^d_\Gamma)\,, \tag{3.0.1}$$

where $V \in \mathcal{C}^\infty(\mathbb{T}^d_\Gamma; \mathbb{R})$ is a smooth real potential and $\mathbb{T}^d_\Gamma$ is an arbitrary flat torus, namely $\mathbb{T}^d_\Gamma = \mathbb{R}^d/\Gamma$ with Γ a maximal dimensional lattice in $\mathbb{R}^d$ generated by the vectors $\alpha_1, \ldots, \alpha_d$:

$$\Gamma = \left\{ \gamma \in \mathbb{R}^d \;\middle|\; \gamma = \sum_{i=1}^{d} n_i \alpha_i\,,\ n_i \in \mathbb{Z} \right\}. \tag{3.0.2}$$

We consider the case of Floquet boundary conditions,

$$u(x+\gamma) = e^{\mathrm{i}\kappa\cdot\gamma} u(x) \quad \forall x \in \mathbb{T}^d_\Gamma, \quad \forall \gamma \in \Gamma\,, \tag{3.0.3}$$

for an arbitrary $\kappa \in \mathbb{R}^d$. For such an operator we give two types of result: first we implement a quantum normal form, from which we deduce spectral asymptotics for *most* eigenvalues of the operator (3.0.1). This is the content of Chapter 4; actually, all the results contained therein are stated and proven even in the case where the Laplacian operator $-\Delta$ is replaced by an arbitrary positive power $(-\Delta)^{\frac{M}{2}}$, and if V is a pseudo-differential operator of order strictly less than M– see Section 3.2 of the present chapter for a precise definition. Then we prove a Structure Theorem according to which H is unitary conjugated (up to a smoothing operator) to a block diagonal operator, which acts in the majority of the blocks as a Fourier multiplier, and in all the others as a lower dimensional Schrödinger operator, and we deduce a spectral result giving an asymptotic expansion of *all* the eigenvalues of the operator. This is contained in Chapters 5 and 6. In the present chapter we define the setting where we are going to work in order to get the aforementioned results.

3.1 Setting: a few preparatory steps

First of all, we reduce to the case of an operator on the standard torus $\mathbb{T}^d$ with periodic boundary conditions. In order to get rid of the Floquet boundary conditions, we perform the Gauge transformation

$$u(x) = e^{\mathrm{i}\kappa\cdot x}\widetilde{u}(x)\,, \tag{3.1.1}$$

which conjugates the operator (3.0.1) to

$$\sum_{j=1}^{d} (D_j + \kappa_j)^2 + V\,, \quad D_j = \mathrm{i}\partial_{x_j}\,, \tag{3.1.2}$$

with periodic boundary conditions on $\mathbb{T}^d_\Gamma$. Furthermore, by introducing on $\mathbb{T}^d_\Gamma$ the basis of the vectors $\{\alpha_1, \dots, \alpha_d\}$, the operator (3.0.1) takes the form

$$-\Delta_{\mathsf{g},\kappa} + V\,, \quad -\Delta_{\mathsf{g},\kappa} = \sum_{A,B=1}^{d} \mathsf{g}^{AB}\left(D_A + \kappa_A\right)\left(D_B + \kappa_B\right)\,, \tag{3.1.3}$$

where $\forall A, B = 1, \dots, d$

$$\mathsf{g}_{AB} = \alpha_A \cdot \alpha_B \tag{3.1.4}$$

and, as usual, by the matrix with coefficients g^{AB} we denote the inverse of the matrix with coefficients g_{AB}, namely

$$\sum_{C=1}^{d} \mathsf{g}_{AC}\mathsf{g}^{CB} = \delta_A^B\,. \tag{3.1.5}$$

In the following, we will only deal with scalar products and norms with respect to the metric g. We write

$$\langle x; y\rangle_{\mathsf{g}} := \mathsf{g}_{AB}x^A y^B\,, \quad \langle \xi; \eta\rangle_{\mathsf{g}*} := \mathsf{g}^{AB}\xi_A\eta_B \tag{3.1.6}$$

the scalar product with respect to this metric of two vector x, y or two covectors ξ, η. Correspondingly we will denote

$$\|x\|_{\mathtt{g}}^2 := \langle x; x\rangle_{\mathtt{g}} \ , \quad \|\xi\|_{\mathtt{g}*}^2 := \langle \xi; \xi\rangle_{\mathtt{g}*} \ . \tag{3.1.7}$$

Furthermore, we denote by $d\mu_{\mathtt{g}}(x)$ the volume form corresponding to $\mathtt{g}$.
We finally define the following quantity, which plays a relevant role in our estimates:

$$\mathfrak{c} = \inf_{k\in\mathbb{Z}^d\setminus\{0\}} \|k\|_{\mathtt{g}*}^2 \ . \tag{3.1.8}$$

Given s linearly independent vectors $\{u_1, \ldots, u_s\}$ in $\mathbb{Z}^d$, denote by $\mathrm{Vol}_{\mathtt{g}^*}\{u_1|\cdots|u_s\}$ the $s-$ dimensional volume of the parallelepiped in $\mathbb{R}^d$ with edges given by $\{u_1, \ldots, u_s\}$, calculated with respect to the metric $\mathtt{g}^*$. A second relevant quantity is

$$\mathfrak{C} = \min_{1\le s\le d} \min_{u_1,\ldots,u_s\in\mathbb{Z}^d} \mathrm{Vol}_{\mathtt{g}^*}\{u_1|\cdots|u_s\} \ . \tag{3.1.9}$$

Remark 3.1.1. *In Lemma D.0.2 of Appendix D we will prove that $\mathfrak{C}$ is strictly positive.*

Definition 3.1.2. *In the following we will refer to the constants $\mathfrak{c}$, $\mathfrak{C}$ as the constants of the metric.*

Consider the operator $-\Delta_{\mathtt{g},\kappa}$ defined as in (3.1.3): as for the case of the Laplace Beltrami operator $-\Delta_{\mathtt{g}}$, Weyl law holds for its spectrum. Albeit the following result is standard, here we give it for the sake of completeness:

Lemma 3.1.3 (Weyl law for $-\Delta_{\mathtt{g},\kappa}$)**.** *The spectrum of the operator $-\Delta_{\mathtt{g},\kappa}$ is given by $\{\|\xi+\kappa\|_{\mathtt{g}^*}^2 \mid \xi \in \mathbb{Z}^d\}$. Furthermore,*

$$\#\left\{\xi \mid \|\xi+\kappa\|_{\mathtt{g}^*}^2| \le R^2\right\} \le \left(\frac{2}{\mathfrak{c}}\right)^d R^d \ . \tag{3.1.10}$$

Proof. An estimate of the quantity (3.1.10) is the number of points $\xi \in \mathbb{Z}^d$ contained in a ball centered at $-\kappa$ and having radius R, of course taking distances with respect to the metric $\mathtt{g}^*$. For any $\xi \in \mathbb{Z}^d$, consider a ball $B_{\mathfrak{c}/2}(\xi)$ of radius $\mathfrak{c}/2$ and center ξ. Then, as ξ varies, such balls do not intersect: thus the "volume occupied" by n points of the lattice is at least $n\mathrm{Vol}\left(B_{\mathfrak{c}/2}(\xi)\right) = n\omega_d(\mathfrak{c}/2)^d$, with ω_d the volume of the unitary ball in $\mathbb{R}^d$. It follows that for the number n of points in the ball of radius R the following inequality holds:

$$nC_d(\mathfrak{c}/2)^d \le \mathrm{Vol}B_0(R) = C_d R^d \ ,$$

from which the estimate follows. □

3.2 Pseudo-differential calculus

The normal form we exhibit in Chapter 4 is based on regularization and exploits techniques from pseudo-differential calculus. Thus, in order to precisely state our result, in this section we present our setting of pseudo-differential operators.
Let $u \in \mathcal{C}^k(\mathbb{T}^d)$. For $l \leq k$ and $x \in \mathbb{T}^d$, consider its l-th differential at x, which is a multilinear form denoted by $d^l u(x)$. We define

$$\|d^l u(x)\| := \sup_{\left\|h^{(1)}\right\|_{\mathtt{g}}=1,\dots,\left\|h^{(l)}\right\|_{\mathtt{g}}=1} \left| d^l u(x) \left[h^{(1)}, ..., h^{(l)} \right] \right| , \tag{3.2.1}$$

where $h^{(j)} \in \mathbb{R}^d$.

As usual, for positive integer s, we define $H^s(\mathbb{T}^d)$ as the completion of $\mathcal{C}^\infty(\mathbb{T}^d)$ in the norm

$$|u|^2_{H^s} := \int_{\mathbb{T}^d} \left(|u(x)|^2 + \|d^s u(x)\|^2 \right) d\mu_{\mathtt{g}}(x) . \tag{3.2.2}$$

Given $u \in L^2(\mathbb{T}^d) \equiv H^0(\mathbb{T}^d)$, we define as usual its Fourier series by

$$u(x) = \sum_{\xi \in \mathbb{Z}^d} \hat{u}_\xi \; e^{\mathrm{i}\xi \cdot k} ,$$

where $\xi \cdot k = \xi_A x^A$ is the usual pairing between a vector and a covector.
Let $\kappa \in \mathbb{R}^d$; then the norm (3.2.2) is equivalent to the norm

$$\|u\|^2_{H^s} = \sum_{\xi \in \mathbb{Z}^d} \|\xi + \kappa\|^{2s}_{\mathtt{g}*} |\hat{u}_\xi|^2 , \tag{3.2.3}$$

where the shift by κ in the weight of the Sobolev norm (3.2.3) has been introduced for future convenience. In a way analogous to (3.2.1), given a function $a \in C^\infty(T^*\mathbb{T}^d)$, we define, exploiting the equivalence $T^*\mathbb{T}^d \simeq \mathbb{T}^d \times \mathbb{R}^d$,

$$\|d^M_x d^N_\xi a(x_0, \xi_0)\| = \sup_{\substack{\left\|h^{(i)}\right\|_{\mathtt{g}}=1 \\ \left\|k^{(j)}\right\|_{\mathtt{g}*}=1}} \left| d^M_x d^N_\xi a(x_0, \xi_0) \left[h^{(1)}, \dots, h^{(M)}, k^{(1)}, \dots, k^{(N)} \right] \right| . \tag{3.2.4}$$

Definition 3.2.1. *Let $a \in \mathcal{C}^\infty\left(T^*\mathbb{T}^d;\mathbb{C}\right)$, $m \in \mathbb{R}$, $\delta > 0$. Furthermore, fix a d-dimensional parameter $\kappa \in \mathbb{R}^d$. We say that $a \in S^{m,\delta}$ is a symbol of order m, if $\forall\ N_1, N_2 \in \mathbb{N}$, there exists a constant $C_{N_1,N_2} > 0$ such that*

$$\|d_x^{N_1} d_\xi^{N_2} a(x,\xi)\| \leq C_{N_1,N_2} \langle \xi + \kappa \rangle_{\mathtt{g}^*}^{m-\delta|N_2|} \quad \forall x \in \mathbb{T}^d,\ \xi \in \mathbb{R}^d$$

where $\langle \xi \rangle_{\mathtt{g}^*} := \left(1 + \|\xi\|_{\mathtt{g}*}^2\right)^{1/2}$. We also define $S^{-\infty,\delta} := \cap_m S^{m,\delta}$.

Remark 3.2.2. *The parameter κ which appears in the definition of symbol and as a weight in the Sobolev norms 3.2.3 has been introduced only because it appears in a natural way in the iterative construction of Chapter 5. In particular, it is needed in order to obtain uniform estimates along the iterative construction performed therein.*

Notice that there are two main differences between the standard definition of symbols given in Definition 2.1.1 and the one given in Definition 3.2.1: the first is the presence of the weight δ, which in the present chapter and in the following ones we are forced to assume strictly less than 1. The second one is that Definition 3.2.1 only intrinsic quantities are involved: thus in particular the sequence $\{C_{N_1,N_2}\}_{N_1,N_2}$ does not depend on the choice of coordinates on $T^*\mathbb{T}^d$. The latter difference is not relevant for what concerns the contents of Chapter 4, but it will play a significant role in Chapters 5, 6. Moreover, the following holds:

Remark 3.2.3. *Fix a basis $\{e_1, \dots, e_d\}$ on $\mathbb{T}^d$ and let $\{\epsilon^1, \dots, \epsilon^d\}$ be its dual basis; then $a \in S^{m,\delta}$ according to Definition 3.2.1 if and only if for any $N_1, N_2 \in \mathbb{N}$ there exists a constant $D_{N_1,N_2} > 0$ such that*

$$\sup_{|\alpha|=N_1} \sup_{|\beta|=N_2} \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leq D_{N_1,N_2} \langle \xi + \kappa \rangle_{\mathtt{g}^*}^{m-\delta N_2} \quad \forall (x,\xi) \in T^*\mathbb{T}^d .$$

Clearly, the constants $\{D_{N_1,N_2}\}_{N_1,N_2\in\mathbb{N}}$ depend on the choice of the basis $\{e_1, \dots, e_d\}$.

To a symbol $a \in S^{m,\delta}$ we can associate an operator as follows:

Definition 3.2.4. *If $a \in S^{m,\delta}$, its* Weyl quantization *is the linear operator $A \equiv Op^W(a)$ by*

$$\left(Op^W(a)[u]\right)(x) = \sum_{\xi\in\mathbb{Z}^d} \sum_{h\in\mathbb{Z}^d} \hat{a}_h \left(\xi + \frac{h}{2}\right) \hat{u}_\xi\ e^{\mathrm{i}(\xi+h)\cdot x}, \tag{3.2.5}$$

where $\forall k \in \mathbb{Z}^d$ *and* $\forall \xi \in \mathbb{R}^d$

$$\hat{a}_k(\xi) = \frac{1}{\mu_{\mathtt{g}}(\mathbb{T}^d)} \int_{\mathbb{T}^d} a(x,\xi) e^{\mathrm{i}\xi\cdot x} d\mu_{\mathtt{g}}(x)\,.$$

Definition 3.2.5. *Let A be a linear operator on $L^2(\mathbb{T}^d)$, we say that it is a pseudodifferential operator of class $OPS^{m,\delta}$ if there exists $a \in S^{m,\delta}$, such that $A = Op^W(a)$.*

Definition 3.2.6 (Seminorms). *Let $a \in S^{m,\delta}$ and $N_1, N_2 \in \mathbb{N}$. We define*

$$C_{N_1,N_2}(a) := \sup_{(x,\xi)\in\mathbb{T}^d\times\mathbb{R}^d} \langle \xi+\kappa\rangle_{\mathtt{g}^*}^{\delta N_2 - m} \| d_x^{N_1} d_\xi^{N_2} a(x,\xi)\|\,.$$

Equivalently, if $A = Op^W(a)$, we set $C_{N_1,N_2}(A) = C_{N_1,N_2}(a)$.

Remark 3.2.7. *$\{C_{N_1,N_2}(\cdot)\}_{N_1,N_2\in\mathbb{N}}$ is a family of seminorms on $S^{m,\delta}$, and we will also refer to $\{C_{N_1,N_2}(A)\}_{N_1,N_2\in\mathbb{N}}$ as the family of seminorms of the operator A.*

In the following chapters we will often estimate the family of seminorms of a given operator (or symbol) in terms of the family of seminorms of another operator (or symbol).

Definition 3.2.8. *Given two pseudo-differential operators A and B, when we say that the family of seminorms of A only depends on the family of seminorms of B, it is also understood that in particular, in order to give a bound on a finite number of seminorms of the operator A, it is only necessary to have a bound on a finite (and, in general, higher) number of seminorms of the operator B.*

Definition 3.2.9. *Given a sequence of symbols $\{f_j\}_{j\geq 0}$ with $f_j \in S^{m-\rho j,\delta}$ for some $m \in \mathbb{R}$ and $\rho > 0$, and a function $f(x,\xi)$, possibly defined only on $\mathbb{T}^d \times \mathbb{Z}^d$, we write*

$$f \sim \sum_j f_j\,, \tag{3.2.6}$$

if for any N there exists C_N s.t.

$$\left| f(x,\xi) - \sum_{j=0}^{N} f_j(x,\xi) \right| \leq C_N \langle \xi+\kappa\rangle_{\mathtt{g}^*}^{m-(N+1)\rho}\,. \tag{3.2.7}$$

If f is defined only on $\mathbb{T}^d \times \mathbb{Z}^d$ then eq. (3.2.7) is valid in such a set.

Taking symbols as in Definition 3.2.1, all the standard properties of pseudo-differential calculus hold; in particular, the composition and the commutator of pseudo-differential operators are still pseudo-differential operators, and one has Egorov Theorem and Calderon Vaillancourt Theorem. We refer to Appendix C for a quick review of such standard results.

Chapter 4

Non resonant eigenvalues

Let $M > 0$. Consider the operator

$$H := (-\Delta_{\mathtt{g},\kappa})^{M/2} + V \quad \text{on} \quad L^2(\mathbb{T}^d)\,, \tag{4.0.1}$$

assume that V is a self-adjoint operator and that there exist $\mathfrak{e} > 0$ and $0 < \delta < 1$ such that

$$V \in OPS^{\,M-\mathfrak{e},\delta}\,, \quad \max\left\{\frac{1}{2},\ 1 - \frac{\mathfrak{e}}{2}\right\} < \delta\,. \tag{4.0.2}$$

Define furthermore

$$\begin{aligned} &\rho = \min\{4\delta - 2,\ -2 + \mathfrak{e} + 2\delta\} \quad \text{if} \quad M \neq 2\,, \\ &\rho = -2 + \mathfrak{e} + 2\delta \quad \text{if } M = 2\,. \end{aligned} \tag{4.0.3}$$

Denote by $B_R(x)$ the open ball of $\mathbb{R}^d$ having radius R and center x, $B_R := B_R(0)$ and denote by $\sharp E$ the cardinality of a set E. The main objective of the present chapter is to develop a normal form construction for the operator (4.0.1), and to deduce from such a normal form the following spectral result:

Theorem 4.0.1. *Consider the operator*

$$H := (-\Delta_{\mathtt{g},\kappa})^{M/2} + V\,, \tag{4.0.4}$$

with $V = Op^W(v)$ *self-adjoint and fulfilling* (4.0.2). *There exists a set* $\Omega \subset \mathbb{R}^d$*, such that* $\Omega \cap \mathbb{Z}^d$ *has density one, more precisely one has*

$$1 - \frac{\sharp(\Omega \cap \mathbb{Z}^d \cap B_R)}{\sharp(B_R \cap \mathbb{Z}^d)} = \mathcal{O}(R^{\delta-1})\,, \tag{4.0.5}$$

and a sequence of symbols $z_j \in S^{M-\mathfrak{e}-j\rho}$*, which depend on* ξ *only, with the following property:*

(i) *for any* $\xi \in \Omega \cap \mathbb{Z}^d$ *there exists an eigenvalue* λ_ξ *of* (4.0.4) *which admits the asymptotic expansion*

$$\lambda_\xi \sim \|\xi + \kappa\|_{\mathsf{g}^*}^M + \sum_{j \geq 0} z_j(\xi) \ , \quad \xi \in \Omega \cap \mathbb{Z}^d \ . \tag{4.0.6}$$

(ii) *For any* $K \in \mathbb{N}$ *and any* $\xi_1, \dots, \xi_K \in \Omega \cap \mathbb{Z}^d$ *such that* $\xi_i \neq \xi_j$ *for all* i, j*, the eigenspace generated by* $\lambda_{\xi_1}, \dots, \lambda_{\xi_K}$ *has dimension at least* K

(iii) *if* $\kappa = 0$ *and the symbol* $v(x,\xi)$ *of* V *is even with respect to* ξ*, namely* $v(x,\xi) = v(x,-\xi)$*, then* $\xi \in \Omega$ *implies* $-\xi \in \Omega$ *and one also has* $z_j(\xi) = z_j(-\xi)$, $\forall j$.

Furthermore, the constants $\{C_N\}_{N\in\mathbb{N}}$ *of* (3.2.7) *in the asymptotic expansion* (4.0.6) *only depend on the family of seminorms of the symbol* v.

Remark 4.0.2. *Item* (*iii*) *applies in particular to the operator* $-\Delta + V(x)$ *with* $\kappa = 0$. *It implies the existence of a sequence* $\{C_N\}_{N\in\mathbb{N}}$, *depending on* V *only, such that for all* $\xi \in \Omega$

$$|\lambda_\xi - \lambda_{-\xi}| \leq C_N \|\xi\|_{\mathsf{g}^*}^{-N} \quad \forall N \in \mathbb{N} \,. \tag{4.0.7}$$

In such a case, we simply write

$$\lambda_\xi - \lambda_{-\xi} = \mathcal{O}(\|\xi\|_{\mathsf{g}^*}^{-\infty}) \ . \tag{4.0.8}$$

Remark 4.0.3. *Property* (4.0.8) *is well known to hold in dimension* 1; *see for instance [MO75]. In Chapter 7 we will investigate the possibility that there are other couples* $(\xi, \xi') \neq (\xi, -\xi)$*, with* $\xi, \xi' \in \Omega$*, such that*

$$\lambda_\xi - \lambda_{\xi'} = \mathcal{O}(\|\xi\|_{\mathsf{g}^*}^{-\infty}) \,.$$

Remark 4.0.4. *Item* (*ii*) *enables to establish an injective correspondence between non-resonant sites* $\xi \in \Omega$ *and eigenvalues* λ_ξ *of* H *satisfying* (4.0.6)*, counted with their multiplicities. Its proof requires to develop a refined quasi-mode argument, which is performed in Section 4.4 below.*

We now prove Theorem 4.0.1. In particular, the remaining part of the chapter is structured as follows: Section 4.1 contains a rough exposition of the strategy we follow for our proof; in Section 4.2 we state and prove our normal form result (see Theorem 4.2.1), and in Sections 4.3, 4.4 we show how Theorem 4.0.1 can be deduced from it. From now on, in the present chapter we omit the dependence on g in norms and scalar products, namely we simply write $\langle \cdot ; \cdot \rangle$, $\langle \cdot \rangle$ and $\| \cdot \|$ instead of, respectively, $\langle \cdot ; \cdot \rangle_{\mathsf{g}}$, $\langle \cdot \rangle_{\mathsf{g}^*}$ and $\| \cdot \|_{\mathsf{g}^*}$.

4.1 Scheme of the proof: a normal form construction

The idea of the proof is to perform a "semiclassical normal form" (see e.g. [Bam05]) working on the symbol of H, namely to quantize the classical normal form procedure for the symbol of H.

To explain the algorithm we consider the simple case in which

$$H = -\Delta + V(x) \ ,$$

and $\kappa = 0$. In this case H is the Weyl quantization of the classical Hamiltonian

$$h(x, \xi) := \|\xi\|^2 + V(x) \ . \tag{4.1.1}$$

We are interested in studying the system in the region $\langle \xi \rangle \gg 1$, in which the potential is a perturbation of the term $\|\xi\|^2$. Taking this point of view the perturbative parameter is $\langle \xi \rangle^{-1}$. Remark also that this corresponds to an expansion in lower order pseudodifferential operators after quantization.

The classical normal form procedure consists of looking for an auxiliary Hamiltonian function g s.t. the corresponding time 1 flow ϕ_g^1 (namely the time one flow of the corresponding Hamiltonian system), conjugates h to a new Hamiltonian $h \circ \phi_g^1$ in which the dependence on the angles x is pushed to higher order. It is well known that this can be done only in the nonresonant regions of the action space. The definition of the resonant regions is a key step of our procedure, hence we are now going to describe it.

By a formal computation one has

$$h \circ \phi_g^1 = h + \{\|\xi\|^2; g\} + V + \text{lower order terms} \ ,$$

where $\{\cdot; \cdot\}$ is the Poisson bracket. The idea is to determine g in such a way that

$$\{\|\xi\|^2; g\} + V = \text{function of } \xi \text{ only} \ . \tag{4.1.2}$$

Expanding g and V in Fourier series in x, equation (4.1.2) takes the form

$$2\mathrm{i}\, \langle \xi; k \rangle\, \hat{g}_k(\xi) = \hat{V}_k(\xi) \iff \hat{g}_k(\xi) = \frac{\hat{V}_k(\xi)}{2\mathrm{i}\, \langle k; \xi \rangle} \ , \quad \forall k \in \mathbb{Z}^d \setminus \{0\} \ , \quad \xi \in \mathbb{R}^d \ , \tag{4.1.3}$$

so that the corresponding function g would turn out to be singular at the dense subset

$$\bigcup_{k \in \mathbb{Z}^d \setminus \{0\}} \left\{ \xi \in \mathbb{R}^d \ : \ \langle k; \xi \rangle = 0 \right\} \ .$$

In classical mechanics it is well known how to solve this problem: first take advantage of the decay of $|\hat{V}_k|$ with $\|k\|$ in order to restrict the union to a finite subset of $\mathbb{Z}^d$, and then remove from the phase space a neighborhood of the so obtained set.

Since in our case the small parameter is $\langle\xi\rangle^{-1}$, and we are in a C^∞ context, so that $|\hat{V}_k(\xi)|$ decays faster then any inverse power of $\|k\|$, we can proceed as follows: we fix some $\epsilon > 0$ and define

$$g(x,\xi) := \sum_{0<\|k\|<\langle\xi\rangle^\epsilon} \frac{\hat{V}_k(\xi)}{2\mathrm{i}\,\langle k;\xi\rangle} \mathrm{e}^{\mathrm{i}k\cdot x} \;, \tag{4.1.4}$$

but only on the set

$$\left\{ \xi \in \mathbb{R}^d \ : \ |\,\langle k;\xi\rangle\,| > \frac{\mu}{\|k\|^\tau}, \quad \forall k \in \mathbb{Z}^d, \quad 0 < \|k\| < N = \langle\xi\rangle^\epsilon \,, \right\}. \tag{4.1.5}$$

However, even if g is well defined and smooth on the set (4.1.5), this choice still has a problem: g does not decay as $\langle\xi\rangle \to \infty$, since the k-th term of the sum (4.1.4) decays only in the direction k. The last remark for the classical case is that in the domain $|\langle k;\xi\rangle| \geq C\langle\xi\rangle^\delta$, with some $\delta > 0$, the k-th term at r.h.s. of (4.1.4) decays as $\langle\xi\rangle^{-\delta}$. This leads to the choice

$$\mu = \langle\xi\rangle^\delta$$

in the formula (4.1.5). Thus the function g that we use is actually of the form

$$g(x,\xi) := \sum_{0<\|k\|<\langle\xi\rangle^\epsilon} \frac{\hat{V}_k(\xi)}{2\mathrm{i}\,\langle k;\xi\rangle} \chi\left(\frac{\langle\xi;k\rangle}{\langle\xi\rangle^\delta \|k\|^{-\tau}}\right) \mathrm{e}^{\mathrm{i}k\cdot x} \;,$$

where $\chi(t)$ is a smooth cut-off function supported where $|t| \geq 1$. Finally, we observe that if we restrict to

$$\Omega^{(0)} := \left\{ \xi \in \mathbb{R}^d \ : \ |\,\langle k;\xi\rangle\,| > \frac{\langle\xi\rangle^\delta}{\|k\|^\tau}, \quad \forall k \in \mathbb{Z}^d, \quad 0 < \|k\| < \langle\xi\rangle^\epsilon \right\}, \tag{4.1.6}$$

then the contributions from all the terms $\hat{V}_k(\xi)$ with $k \neq 0$ can be eliminated.

This is the classical procedure that we quantize. As usual in semiclassical normal form theory, the main remark is that, if $g \in S^{m,\delta}$ is a real valued symbol with $m < \delta$ and $G = Op^w(g)$, then $\mathrm{e}^{-\mathrm{i}G}$ is unitary, the operator $\mathrm{e}^{\mathrm{i}G} H \mathrm{e}^{-\mathrm{i}G}$ is pseudodifferential and is given by

$$\mathrm{e}^{\mathrm{i}G} H \mathrm{e}^{-\mathrm{i}G} = -\Delta - \mathrm{i}[-\Delta, G] + V + l.h.t.$$

whose symbol has the form

$$|\xi|^2 + \{\|\xi\|^2; g\}_{\mathcal{M}} + V + l.h.t. \ ,$$

where $\{\|\xi\|^2; g\}_{\mathcal{M}}$ is the Moyal bracket, namely the symbol of the operator $-\mathrm{i}[-\Delta, G]$ (see its definition in (C.2.6) of Appendix C). Furthermore, since $\|\xi\|^2$ is quadratic, as stated in Lemma C.2.5 the Moyal bracket coincides with the Poisson bracket, so the function g constructed in (4.1.4) is suitable (after localization) in order to perform the semiclassical normal form of H.

Using $Op^W g$ in order to transform H and iterating the procedure, we conjugate H to an operator with symbol

$$\|\xi\|^2 + z(\xi) + z^{(\mathrm{res})}(x, \xi) + \mathcal{O}(\|\xi\|^{-N}) \ , \tag{4.1.7}$$

with some arbitrarily large N. Here $z^{(\mathrm{res})}$ is a symbol localized in the complement of $\Omega^{(0)}$.

As a last step, we exploit the equivalence between Weyl quantization and classical quantization in order to show that the operator obtained by quantizing (4.1.7) acts on $\mathrm{e}^{\mathrm{i}x\cdot\xi}$ with $\xi \in \mathbb{Z}^d \cap \Omega^{(0)}$ as a multiplication by $\|\xi\|^2 + z(\xi)$ plus an operator which is smoothing of order N. Thus $\mathrm{e}^{\mathrm{i}\xi\cdot x}$ is a quasimode for the quantization of (4.1.7) and Item (i) of Theorem 4.0.1 follows, at least in the case of Sturm-Liouville type operators. The case of a general flat torus with Floquet boundary conditions is totally analogous to the case just considered, up to replacing ξ with $\xi + \kappa$, and the case where the main operator is $\|\xi + \kappa\|^M$ is easily obtained by just remarking that the resonant zones of $\|\xi + \kappa\|^M$ are the same as those of $\|\xi + \kappa\|^2$. The proof of Item (ii) requires instead to refine the standard quasi-mode argument, in order to guarantee that to the quasi-eigenfunctions $e^{\mathrm{i}\xi_1\cdot x}, \ldots, e^{\mathrm{i}\xi_K\cdot x}$ correspond eigenvalues $\lambda_{\xi_1}, \ldots, \lambda_{\xi_K}$ of multiplicity K in the spectrum of H. The argument we develop is based on the fact that is possible to group the eigenvalues of the Laplacian into clusters, whose separation properties and sizes are estimated using Weyl's law. Once one has done this, it remains to apply our refined quasi-mode argument separately in each one of these clusters.

4.2 The normal form theorem and its proof

In this section we state and prove the normal form result lying at the basis of the proof of Theorem 4.0.1. We will use the constant ρ defined by eq.

(4.0.3), we fix γ s.t.

$$0 < 2\gamma < \sqrt{\mathfrak{c}} = \inf_{x \neq y \in \mathbb{Z}^d} \|x - y\| \tag{4.2.1}$$

(recall definition (3.1.8) of Chapter 3), $\delta, \epsilon, \tau > 0$ such that

$$0 < \epsilon(\tau + 1) + \delta < 1 \,, \quad \tau > d \,, \tag{4.2.2}$$

and we define

$$\Omega := \left\{ \xi \in \mathbb{R}^d \;\middle|\; |\langle k; \xi + \kappa \rangle| > \frac{\langle \xi + \kappa \rangle^{\delta}}{|k|^{\tau}} \quad \forall k \in \mathbb{Z}^d, \quad 0 < |k| < \langle \xi + \kappa \rangle^{\epsilon} \right\} . \tag{4.2.3}$$

Theorem 4.2.1 (Normal form result)**.** *Let $\mathfrak{c} > 0$ and let δ, ρ be two constants satisfying (4.0.2), (4.0.3). Then there exists a sequence of self-adjoint operators $\{G_n\}_{n \geq 1}$ with $G_n = Op^W(g_n) \in OPS^{2-\mathfrak{c}-\delta-n\rho,\delta}$ for any $n \geq 1$, such that for any integer $n \geq 1$, the operator*

$$\mathcal{U}_n := \mathrm{e}^{\mathrm{i}G_n} \circ \ldots \circ e^{\mathrm{i}G_1} \tag{4.2.4}$$

conjugates H to a pseudodifferential operator H_n with symbol h_n of the form

$$h_n = h^0 + z^{(n)} + v_n \;, \tag{4.2.5}$$

where

$$h^0(\xi) = \|\xi + \kappa\|^M \quad \forall \xi \in \mathbb{Z}^d, \tag{4.2.6}$$

$$v_n \in S^{M-\mathfrak{c}-n\rho,\delta},$$

and $z^{(n)} \in S^{M-\mathfrak{c},\delta}$ has the following properties: $z^{(n)} = [z^{(n)}] + z^{(n,res)}$, where

$$z^{(n,res)}(x, \xi) = 0 \quad \forall \xi \in \Omega,$$

$[z^{(n)}]$ is independent of x and for any integer $j \geq 0$ there exists a symbol $z_j \in S^{M-\mathfrak{c}-\rho j,\delta}$ such that

$$[z^{(n)}](\xi) = \sum_{j=0}^{n-1} z_j(\xi) \,. \tag{4.2.7}$$

Furthermore, for any $n \in \mathbb{N}$ the seminorms of the symbols $[z^{(n)}], z^{(n,res)}$, v_n, g_n only depend on the family of seminorms of V (in the sense of Definition 3.2.8), on $n, d, \mathfrak{c}$, and on the parameters δ, ϵ, τ involved in the definition of Ω.
Finally, if $\kappa = 0$ and V is even in ξ, then the same is true for v_n and z_n, whereas $g_1, \ldots, g_n$ are odd functions in ξ.

The rest of the section is split into few subsections and is devoted to the proof of this Theorem.

4.2.1 Localizing on non-resonant regions: cut-offs

Let us consider an even cut-off function $\chi \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp}(\chi) \subseteq [-1,1]$, $0 \le \chi \le 1$ and $\chi(t) = 1$ for any $t \in [-\frac{1}{2}, \frac{1}{2}]$. With its help we define, for any $k \in \mathbb{Z}^d$,

$$
\begin{aligned}
\chi_k(\xi) &:= \chi\Big(\frac{2\|k\|^\tau \langle \xi+\kappa; k\rangle}{\langle \xi+\kappa\rangle^\delta}\Big) , \\
\eta_k(\xi) &:= \frac{1}{\langle \xi+\kappa; k\rangle}(1-\chi_k(\xi)) , \\
\tilde{\chi}_k(\xi) &:= \chi\left(\frac{\|k\|}{\langle \xi+\kappa\rangle^\epsilon}\right) .
\end{aligned} \tag{4.2.8}
$$

Given $m \in \mathbb{R}$, $\delta > 0$, $a \in S^{m,\delta}$, we define

$$
\begin{aligned}
[a](\xi) &:= \frac{1}{\mu_{\mathtt{g}}(\mathbb{T}^d)} \int_{\mathbb{T}^d} a(x,\xi)\, d\mu_{\mathtt{g}}(x) , \\
a^{(\mathrm{res})}(x,\xi) &:= \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \chi_k(\xi)\tilde{\chi}_k(\xi)\hat{a}_k(\xi)e^{\mathrm{i}k\cdot x} , \\
a^{(\mathrm{nr})}(x,\xi) &:= \sum_{k\in\mathbb{Z}^d\setminus\{0\}} (1-\chi_k(\xi))\tilde{\chi}_k(\xi)\hat{a}_k(\xi)e^{\mathrm{i}k\cdot x} , \\
a^{(S)}(x,\xi) &:= \sum_{k\in\mathbb{Z}^d\setminus\{0\}} (1-\tilde{\chi}_k(\xi))\hat{a}_k(\xi)e^{\mathrm{i}k\cdot x} ,
\end{aligned} \tag{4.2.9}
$$

so that one has

$$
a = [a] + a^{(nr)} + a^{(res)} + a^{(S)} . \tag{4.2.10}
$$

Lemma 4.2.2. *Let $a \in S^{m,\delta}$. Then $[a]$, $a^{(\mathrm{nr})}$, $a^{(\mathrm{res})} \in S^{m,\delta}$ and $a^{(S)} \in S^{-\infty,\delta}$, and the families of the seminorms of the symbols $[a], a^{(\mathrm{nr})}, a^{(\mathrm{res})}, a^{(S)}$ only depend on the seminorms of a, on d, on $\mathfrak{c}$ and on the parameters δ, ϵ, τ. Moreover, in the periodic case $\kappa = 0$, if a is even in ξ, namely $a(x,\xi) = a(x,-\xi)$ for all $(x,\xi) \in \mathbb{T}^d \times \mathbb{R}^d$, then $a^{(\mathrm{nr})}, a^{(\mathrm{res})}, [a]$ and $a^{(S)}$ are even in ξ too.*

The following result follows by a direct computation and is useful to prove Lemma 4.2.2.

Lemma 4.2.3. *Let $c > 0$. For any $N \in \mathbb{N}$ one has*

$$
\left\| d_\xi^N \left(\langle \xi+\kappa\rangle^{-c}\right)\right\| \lesssim_{c,N} \langle \xi+\kappa\rangle^{-c-N} .
$$

Proof. Fix a set of coordinates $\{\xi_1, \dots, \xi_d\}$. Then one has that, for all $j = 1, \dots, d$,

$$\partial_{\xi_j} \left(\langle \xi + \kappa \rangle^{-c} \right) = (-c) \sum_{i=1}^{d} \langle \xi + \kappa \rangle^{-c-2} \mathrm{g}^{ij} \left(\xi_i + \kappa_i \right) .$$

Thus, if $h \in \mathbb{R}^d$ is such that $\|h\| = 1$, one has

$$\begin{aligned} \left| d_\xi \left(\langle \xi \rangle^{-c} \right) [h] \right| &= \left| (-c) \langle \xi + \kappa \rangle^{-c-2} \langle \xi + \kappa ; h \rangle \right| \\ &\leq c \langle \xi + \kappa \rangle^{-c-2} \| \xi + \kappa \| \, \| h \| \\ &\leq c \langle \xi + \kappa \rangle^{-c-1} . \end{aligned}$$

The estimates of higher order differentials follows arguing in analogous way. □

Proof of Lemma 4.2.2. : The statement about $[a]$ is immediate. We start with proving the thesis for $a^{(\mathrm{nr})}$. To this aim, we observe that for any $h \in \mathbb{R}^d$

$$d_\xi \left(\langle \xi + \kappa ; k \rangle \right) [h] = \langle k ; h \rangle \ ,$$

and that $d_\xi^N \left(\langle \xi + \kappa ; k \rangle \right) = 0$ for all $N > 1$; this, together with Lemma 4.2.3, enables to prove that for any $k \in \mathbb{Z}^d \backslash \{0\}$ and for any $N \in \mathbb{N}$ one has

$$\left\| d_\xi^N \left(\frac{\|k\|^\tau \langle \xi + \kappa ; k \rangle}{\langle \xi + \kappa \rangle^\delta} \right) \right\| \lesssim_{N,\delta} \|k\|^{\tau+1} \langle \xi + \kappa \rangle^{-\delta-(N-1)} ,$$

hence, being $\delta < 1$,

$$\left\| d_\xi^N \left(\frac{\|k\|^\tau \langle \xi + \kappa ; k \rangle}{\langle \xi + \kappa \rangle^\delta} \right) \right\| \lesssim_{N,\delta} \|k\|^{\tau+1} \langle \xi + \kappa \rangle^{-N\delta} . \tag{4.2.11}$$

Applying again Lemma 4.2.3 and recalling $\delta < 1$, analogously one gets that

$$\left\| d_\xi^N \left(\frac{\|k\|}{\langle \xi + \kappa \rangle^\epsilon} \right) \right\| \lesssim_{N,\epsilon} \|k\| \langle \xi + \kappa \rangle^{-\epsilon-(N-1)} \lesssim_{N,\epsilon} \|k\| \langle \xi + \kappa \rangle^{-\delta N} . \tag{4.2.12}$$

Recall the definition of χ_k and $\tilde{\chi}_k$ as in (4.2.8): by (4.2.11) and (4.2.12), since χ is a smooth and compactly supported function, one gets that for any $N \in \mathbb{N}$ and $\xi \in \mathbb{R}^d$ one has

$$\begin{gathered} \left\| d_\xi^N \left(\chi_k(\xi) \right) \right\| \lesssim_{N,\delta,\epsilon} \|k\|^{N(\tau+1)} \langle \xi + \kappa \rangle^{-\delta N} , \\ \left\| d_\xi^N \left(\tilde{\chi}_k(\xi) \right) \right\| \lesssim_{N,\delta,\epsilon} \|k\|^{N} \langle \xi + \kappa \rangle^{-\delta N} . \end{gathered} \tag{4.2.13}$$

In order to show that $a^{(\mathrm{nr})}$ is a symbol of order m, we use Lemma C.1.1 of Appendix C to estimate its seminorms in terms of the seminorms of the symbol a. Let indeed $\{C'_{N_1,N_2}(a))\}_{N_1,N_2\in\mathbb{N}}$ be as in (C.1.1): by Lemma C.1.1, for any $k \in \mathbb{Z}^d$ and $\xi \in \mathbb{R}^d$ one has

$$\left\| d_\xi^{N_2} \hat{a}_k(\xi) \right\| \leq C'_{N_1,N_2}(a) \langle k \rangle^{-N_1} \langle \xi + \kappa \rangle^{m - \delta N_2}, \quad \forall N_1, N_2 \in \mathbb{N}.$$

Thus, by (4.2.13), one gets that, for any N_1, N_1' and $N_2 \in \mathbb{N}$ and for all $\xi \in \mathbb{R}^d$ and $k \in \mathbb{Z}^d$,

$$\left\| d_\xi^{N_2} \left(\hat{a}_k(\xi) \left(1 - \chi_k(\xi)\right) \tilde{\chi}_k(\xi) \right) \right\| \lesssim_{N_2,\delta,\epsilon} C'_{N_1',N_2} \langle \xi + \kappa \rangle^{m - \delta N_2} \|k\|^{N_2(\tau+2)} \langle k \rangle^{-N_1'} \tag{4.2.14}$$

thus, choosing $N_1' = N_1 + \lceil N_2(\tau + 2) \rceil$, one gets

$$C'_{N_1,N_2}\left(a^{(\mathrm{nr})}\right) \lesssim_{N_1,N_2,\delta,\epsilon} C'_{N_1',N_2}(a). \tag{4.2.15}$$

The estimate about $a^{(\mathrm{res})}$ follows analogously. Concerning $a^{(S)}$, we apply estimate (4.2.13) to deduce that for any N_1' and $N_2 \in \mathbb{N}$ and for all $\xi \in \mathbb{R}^d$ and $k \in \mathbb{Z}^d$ one has

$$\left\| d_\xi^{N_2} \left(\hat{a}_k(\xi) \left(1 - \tilde{\chi}_k(\xi)\right) \right) \right\| \lesssim_{N_2,\delta,\epsilon} C'_{N_1',N_2}(a) \langle \xi + \kappa \rangle^{m - \delta N_2} \|k\|^{N_2} \langle k \rangle^{-N_1'}.$$

Since, by definition of $\tilde{\chi}_k$, the function $1 - \tilde{\chi}_k(\xi)$ and its N_2-th order differentials $d_\xi^{N_2}(1 - \tilde{\chi}_k(\xi))$ for all N_2 are supported where

$$\|k\| \geq \frac{\langle \xi + \kappa \rangle^\epsilon}{2}$$

(see (4.2.8)), for any N_1', N_2 and $K \in \mathbb{N}$ one also has

$$\left\| d_\xi^{N_2} \left(\hat{a}_k(\xi) \left(1 - \tilde{\chi}_k(\xi)\right) \right) \right\| \lesssim_{N_2,\delta,\epsilon} C'_{N_1',N_2}(a) \langle \xi + \kappa \rangle^{-K} \langle k \rangle^{-(N_1' - N_2 - (K - m - \delta N_2)\epsilon^{-1})}.$$

Choosing $N_1' = N_1 + N_2 + \lceil (K - m - \delta N_2)\epsilon \rceil$, this entails

$$\left\| d_\xi^{N_2} \left(\hat{a}_k(\xi) \left(1 - \tilde{\chi}_k(\xi)\right) \right) \right\| \lesssim_{N_2,\delta,\varepsilon} C'_{N_1',N_2}(a) \langle \xi + \kappa \rangle^{-K} \langle k \rangle^{-N_1}$$

for any $K, N_1, N_2 \in \mathbb{N}$. By Lemma C.1.1, this proves that $a^{(S)} \in S^{-K,\delta}$ for any $K \in \mathbb{N}$, with constants $\{C'_{N_1,N_2}(a^{(S)})\}_{N_1,N_2\in\mathbb{N}}$ such that

$$C'_{N_1,N_2}\left(a^{(S)}\right) \lesssim_{N_1,N_2,\delta,\epsilon} C'_{N_1',N_2}(a). \tag{4.2.16}$$

□

We come to the normal form procedure. First, in order to regularize the possible singularity at $-\kappa$ of the derivatives, we substitute $\|\xi+\kappa\|^M$ with

$$h^0(\xi) := \psi(\xi)\,\|\xi+\kappa\|^M\,, \tag{4.2.17}$$

where $\psi(\xi) = \widetilde{\psi}(\|\xi+\kappa\|^2)$, and $\widetilde{\psi} \in C^\infty(\mathbb{R})$ is an even cut-off function with the following properties: $0 \le \widetilde{\psi} \le 1$, $\widetilde{\psi}(t) = 0$ for any t such that $|t| \le \gamma_\star^2$, and $\widetilde{\psi}(t) = 1$ for any $|t| \ge (2\gamma_\star)^2$, and $\gamma_\star \in \mathbb{R}^+$ is defined as follows:

$$\gamma_\star := \inf\left\{\tfrac{1}{2}\mathrm{dist}(-\kappa, \mathbb{Z}^d)\,, \gamma\right\} \qquad \text{if } \kappa \notin \mathbb{Z}^d\,,$$
$$\gamma_\star := \gamma \qquad \text{if } \kappa \in \mathbb{Z}^d\,,$$

with γ the constant defined in (4.2.1).

Remark 4.2.4. *If h^0 is defined as in* (4.2.17)*, for any function $u \in L^2(\mathbb{T}^d)$ one has that*

$$Op^W(h^0)[u](x) = \sum_{\xi\in\mathbb{Z}^d} \|\xi+\kappa\|^M \hat{u}_\xi e^{\mathrm{i}\xi\cdot x}\,.$$

Remark 4.2.5. *Since the differentials of ψ at any order are different from* 0 *only for $\|\xi+\kappa\| < 2\gamma_\star < 2\gamma < \sqrt{\mathfrak{c}}$, for any $N \in \mathbb{N}$ and $M' \in \mathbb{R}$, a direct computation shows that*

$$\left\| d_\xi^N \left(\psi(\xi)\|\xi+\kappa\|^{M'}\right)\right\| \lesssim_{N,\mathfrak{c}} \|\xi+\kappa\|^{M'-N} \quad , \quad \forall \xi \in \mathbb{R}^d\,. \tag{4.2.18}$$

In particular, taking $M' = M$, one has

$$\left\| d_\xi^N h^0(\xi)\right\| \lesssim_{N,\mathfrak{c}} \|\xi+\kappa\|^{M-N} \quad , \quad \forall \xi \in \mathbb{R}^d\,. \tag{4.2.19}$$

4.2.2 The normal form construction

Then, given $a \in S^{m,\delta}$, consider

$$g(x,\xi) := \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \psi(\xi)\tilde{\chi}_k(\xi)\,\|\xi+\kappa\|^{2-M}\,\frac{(-\mathrm{i})}{M}\eta_k(\xi)\hat{a}_k(\xi)e^{\mathrm{i}k\cdot x}\,, \tag{4.2.20}$$

where we recall the definitions given in (4.2.8). The following Lemma is easily seen to hold:

Lemma 4.2.6. *The symbol g defined in* (4.2.20) *belongs to class $S^{2+m-M-\delta,\delta}$, and the family of its seminorms only depends on the family of the seminorms of the symbol a, and on $d, \delta, \epsilon, \tau, \mathfrak{c}$. Moreover, g satisfies*

$$\left\{h^0; g\right\} + a^{(\mathrm{nr})} \in S^{-\infty,\delta} \tag{4.2.21}$$

and in the case $\kappa = 0$, if a is even in ξ, then g is odd.

Proof. Recall the definition of h^0 in (4.2.17). Since the cut-off function ψ is a smooth function with compact support, one has that $\nabla_\xi(\psi(\xi)\|\xi+\kappa\|^M) = M\|\xi+\kappa\|^{M-2}\xi\psi(\xi) + S^{-\infty,\delta}$. Therefore

$$\left\{h^0; g\right\}(x,\xi) = -M\|\xi+\kappa\|^{M-2}\psi(\xi)\left\langle \xi; \nabla_x g(x,\xi)\right\rangle + S^{-\infty,\delta}. \tag{4.2.22}$$

In order to solve the equation (4.2.21), it is enough to solve

$$-M\|\xi+\kappa\|^{M-2}\psi(\xi)\left\langle \xi; \nabla_x g\right\rangle + a^{(nr)} \in S^{-\infty,\delta}. \tag{4.2.23}$$

Recalling the definition of $a^{(\mathrm{nr})}$ given in (4.2.9) and the definitions given in (4.2.8), a solution of the equation (4.2.23) is then given by g defined in (4.2.20). It remains to prove that g as in (4.2.20) is a symbol. This is proved arguing as in Lemma 4.2.2. In particular, one observes that for any $N \in \mathbb{N}$

$$\left\| d_\xi^N \left(\frac{1}{\langle \xi+\kappa; k\rangle} \right) \right\| \lesssim_N \frac{\|k\|^N}{|\left\langle \xi+\kappa; k\right\rangle|^{N+1}},$$

and that, by definition of the function χ_k, $(1-\chi_k(\xi))$ and all its $N-$th order differentials $d_\xi^N(1-\chi_k)$ are supported where

$$|\left\langle \xi+\kappa; k\right\rangle| \geq \frac{\langle \xi+\kappa\rangle^\delta}{2}.$$

Combining the above estimates with the first of (4.2.13), an explicit calculation then enables to deduce that, for any $N \in \mathbb{N}$,

$$\left\| d_\xi^N \eta_k(\xi) \right\| \lesssim_{N,\delta,\epsilon} \|k\|^{\tau(N+1)+N} \langle \xi+\kappa\rangle^{-N\delta}. \tag{4.2.24}$$

This, arguing as to obtain the estimate (4.2.14) exhibited in the proof of Lemma 4.2.2 and recalling Remark 4.2.5, implies that for any N_1, N_1' and $N_2 \in \mathbb{N}$

$$\|d_\xi^{N_2}\hat{g}_k(\xi)\| \lesssim_{N_2,\kappa,\mathfrak{c}} C'_{N_1',N_2}(a)\langle \xi+\kappa\rangle^{2-M+m-\delta-\delta N_2}\langle k\rangle^{-(N_1'-N_2(\tau+3))},$$

which, choosing $N_1' = N_1 + (3+\tau)N_2$, gives the thesis. Finally, if $\kappa = 0$ and a is even in ξ, using that $\widetilde{\chi}_k$ and ψ are even and η_k is odd, one gets that g is odd in ξ and the proof is concluded. □

Remark 4.2.7. *The regularization function ψ has been introduced in order to ensure that g as in (4.2.20) is actually a symbol, due to the presence of the term $\|\xi+\kappa\|^{M-2}$. If $M = 2$, there is no need to introduce it, thus one simply has*

$$g(x,\xi) = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \tilde{\chi}_k(\xi)\frac{(1-\chi_k(\xi))}{2\mathrm{i}\left\langle \xi+\kappa; k\right\rangle}\hat{a}_k(\xi)e^{\mathrm{i}k\cdot x}. \tag{4.2.25}$$

Proof of Theorem 4.2.1. We describe the induction step of the normal form procedure which allows to prove Theorem 4.2.1. Let $0 \leq j < n$ and assume that $H_j = Op^W(h_j)$ has the form given in (4.2.5), namely $z_j = [z_j] + z_j^{(\mathrm{res})} \in S^{M-\mathfrak{e},\delta}$ and $v_j \in S^{M-\mathfrak{e}-j\rho,\delta}$. Suppose furthermore that v_j has the form $v_j = v_{1,j} + v_{2,j}$, with $v_{2,j} \in S^{M-\mathfrak{e}-n\rho,\delta}$, and in the case with symmetry, assume that z_j and $v_{1,j}$ are even in ξ. Notice that such an inductive hypothesis is immediately verified for $j = 0$, with $v_{1,0} = v$ and $z_0 = v_{2,0} = 0$. By Lemma 4.2.6, there exists a solution

$$g_{j+1} \in S^{2-\mathfrak{e}-\delta-j\rho,\delta} \tag{4.2.26}$$

of the homological equation

$$\left\{h^0; g_{j+1}\right\} + v_{1,j}^{(\mathrm{nr})} \in S^{-\infty,\delta}\,. \tag{4.2.27}$$

Moreover since the symbol $v_{1,j}$ is real valued, then also the symbol $v_{1,j}^{(\mathrm{nr})}$ is real valued and therefore g_{j+1} is real valued too. Then, if we define $G_{j+1} := Op^W\left(g_{j+1}\right)$, G_{j+1} is self-adjoint, since the symbol g_{j+1} is real valued. Since, by (4.0.2), $\delta > 1 - \frac{\mathfrak{e}}{4} > 1 - \frac{\mathfrak{e}}{2}$, one obtains that g_{j+1} is a symbol of order $2 - \mathfrak{e} - \delta - j\rho < \delta < 1$. Hence by Lemma C.2.6 of Appendix C, $e^{\mathrm{i}G_j}, e^{-\mathrm{i}G_j}$ are well defined linear operators in $\mathcal{B}(H^s)$ for any $s \geq 0$ and $H_{j+1} = Op^W(h_{j+1}) = e^{\mathrm{i}G_{j+1}} H_j e^{-\mathrm{i}G_{j+1}} \in OPS^{M,\delta}$. Furthermore, by (C.2.12) of Lemma C.2.6, (with $a = h_j$, $g = g_{j+1}$, $m = M$, $\eta = 2 - \mathfrak{e} - \delta - n\rho$), h_{j+1} admits the expansion

$$h_{j+1} = h_j + \{h_j; g_{j+1}\}_{\mathcal{M}} + S^{M+4-2\mathfrak{e}-4\delta-2j\rho,\delta}\,.$$

By the definition of ρ (see (4.0.3)), one gets that $M + 4 - 2\mathfrak{e} - 4\delta - 2j\rho \leq M - \mathfrak{e} - (j+1)\rho$, implying that $S^{M+4-2\mathfrak{e}-4\delta-2j\rho,\delta} \subseteq S^{M-\mathfrak{e}-(j+1)\rho,\delta}$. Hence

$$h_{j+1} = h_j + \{h_j; g_{j+1}\}_{\mathcal{M}} + S^{M-\mathfrak{e}-(j+1)\rho,\delta}\,. \tag{4.2.28}$$

Moreover, recalling that $h_j = h^0 + z_j + v_{1,j} + v_{2,j}$ and splitting the symbol $v_{1,j}$ according to (4.2.9), one has

$$\begin{aligned} h_j + \{h_j; g_{j+1}\}_{\mathcal{M}} = h^0 + z_j + [v_{1,j}] + v_{1,j}^{(\mathrm{res})} + \left(v_{1,j}^{(\mathrm{nr})} + \left\{h^0; g_{j+1}\right\}\right) \\ + \{z_j; g_{j+1}\}_{\mathcal{M}} + \{v_j; g_{j+1}\}_{\mathcal{M}} \\ + \left\{h^0; g_{j+1}\right\}_{\mathcal{M}} - \left\{h^0; g_{j+1}\right\} \\ + v_{2,j} + v_{1,j}^{(S)}\,. \end{aligned} \tag{4.2.29}$$

We now estimate all the remainder terms in the left hand side of (4.2.29) one by one. Since $z_j \in S^{M-\mathfrak{e},\delta}$, $v_{1,j} \in S^{M-\mathfrak{e}-j\rho,\delta} \subseteq S^{M-\mathfrak{e},\delta}$, and by (4.2.26) $g_{j+1} \in S^{2-\mathfrak{e}-\delta-j\rho,\delta}$, one gets

$$\{z_j + v_j; g_{j+1}\}_{\mathcal{M}} \in S^{M-\mathfrak{e}-j\rho-(2\delta+\mathfrak{e}-2),\delta}\,.$$

Furthermore, if $M = 2$, by (C.2.8) of Lemma C.2.5 of Appendix C one has

$$\left\{h^0; g_{j+1}\right\}_{\mathcal{M}} = \left\{h^0; g_{j+1}\right\}\,,$$

while for all other values of M (C.2.6) of the same Lemma C.2.5 implies

$$\left\{h^0; g_{j+1}\right\}_{\mathcal{M}} - \left\{h^0; g_{j+1}\right\} \in S^{M-\mathfrak{e}-j\rho-(4\delta-2),\delta}\,.$$

Thus, by definition of ρ as in (4.0.3), it follows that $\{z_j + v_j; g_{j+1}\}_{\mathcal{M}} \in S^{M-\mathfrak{e}-(j+1)\rho,\delta}$ and $\{h^0; g_{j+1}\}_{\mathcal{M}} - \{h^0; g_{j+1}\} \in S^{M-\mathfrak{e}-(j+1)\rho,\delta}$. Since $v_{2,j} \in S^{M\mathfrak{e}-n\rho,\delta}$ and $v_{1,j}^{(S)} \in S^{-\infty,\delta}$ by Lemma 4.2.2, (4.2.28), (4.2.29) imply that

$$h_{j+1} = h^0 + z_{j+1} + v_{j+1}\,, \quad v_{j+1} \in S^{M-\mathfrak{e}-(j+1)\rho,\delta}$$

where

$$z_{j+1} := [z_{j+1}] + z_{j+1}^{(\mathrm{res})}\,,$$
$$[z_{j+1}] := [z_j] + [v_{1,j}]\,, \quad z_{j+1}^{(\mathrm{res})} := z_j^{(\mathrm{res})} + v_{1,j}^{(\mathrm{res})}\,.$$

The expansions (4.2.5), (4.2.7) are then proved at the step $j+1$. In the case without symmetry, it is sufficient to set $v_{2,j+1} = v_{2,j} + v_{1,j}^{(S)}$ and $v_{1,j+1} = v_{j+1} - v_{2,j+1}$. On the contrary, if $\kappa = 0$ and $v_{1,j}$, z_j are even in ξ, then Lemma 4.2.6 implies that g_{j+1} is odd in ξ. Thus by Lemma C.2.6 and Remark C.2.7 of Appendix C, the decomposition of h_{j+1} given by (4.2.28) is of the form

$$h_{j+1} = h_j + \{h_j; g_{j+1}\}_{\mathcal{M}} + \tilde{v}_{1,j+1} + \tilde{v}_{2,j+1}\,,$$

with $\tilde{v}_{1,j+1} \in S^{M-\mathfrak{e}-(j+1)\rho,\delta}$ still even and $\tilde{v}_{2,j+1} \in S^{M-\mathfrak{e}-n\rho,\delta}$. Then the inductive step is verified, setting $v_{2,j+1} = v_{2,j} + \tilde{v}_{2,j+1}$.

We finally observe that, by Lemma 4.2.6, Lemma 4.2.2, Lemma C.2.5 and Lemma C.2.6, for all $j \geq 0$ the families of seminorms of $[z^{(j+1)}]$, $z^{(j+1,\mathrm{res})}$, v_{j+1} and g_{j+1} only depend on the families of seminorms of $[z^{(j)}]$, $z^{(j,\mathrm{res})}$, v_j, and in the case $j = 0$ the family of seminorms of $[z^{(0)}] = [v]$, $z^{(0,\mathrm{res})} = v^{(\mathrm{res})}$ and $v_0 = v$ depends again only on the family of seminorms of v, due to the same Lemmas 4.2.6 and 4.2.2.

□

4.2.3 Measure estimates of the non resonant set

In this section we prove that the non resonant set Ω introduced in Theorem 4.0.1 is of density one at infinity. Recall the definition of Ω as

$$\Omega := \left\{ \xi \in \mathbb{R}^d \;\middle|\; |\langle k; \xi + \kappa \rangle| > \frac{\langle \xi + \kappa \rangle^\delta}{\|k\|^\tau} \quad \forall k \in \mathbb{Z}^d, \quad 0 < \|k\| < \langle \xi + \kappa \rangle^\epsilon \right\}$$

according to Definition (4.2.3). In particular, we prove the following:

Proposition 4.2.8. *Assume $\epsilon \leq \frac{\delta}{1+\tau}$, $\tau > d$. Then there exists $R_0 > 0$ such that, for any $R \geq R_0$, one has*

$$1 - \frac{\sharp(\Omega \cap \mathbb{Z}^d \cap B_R(0))}{\sharp(\mathbb{Z}^d \cap B_R(0))} = \mathcal{O}(R^{\delta - 1}) \ . \tag{4.2.30}$$

Given a (measurable) set $\mathcal{A}$, and a positive parameter r we will denote

$$\mathcal{A}_r := \bigcup_{x \in \mathcal{A}} B_r(x) \ . \tag{4.2.31}$$

We start by a few remarks that will be useful in order to estimate the cardinality of $\Omega \cap \mathbb{Z}^d$.

Remark 4.2.9. *There exists a constant C s.t.*

$$\sharp(\mathbb{Z}^d \cap B_R) \geq C R^d \ .$$

Remark 4.2.10. *Let $E = \{x_1, \dots, x_N\} \subset \mathbb{R}^d$, be a finite subset and for $r := \frac{\sqrt{\mathfrak{c}}}{2}$, consider the set E_r (defined according to (4.2.31)). Then*

$$N \equiv \sharp E = \frac{|E_r|}{|B_r(0)|} = \frac{|E_r|}{|B_1(0)|\, r^d} \ .$$

Recall the definition of r as in (4.2.1); clearly, one has that for any $\xi_0 \in \mathbb{Z}^d$, $B_r(\xi_0) \cap (\mathbb{Z}^d \setminus \{\xi_0\}) = \emptyset$.

Remark 4.2.11. *Given a measurable set, $\mathcal{A}$, we have*

$$\sharp(\mathcal{A} \cap \mathbb{Z}^d) \leq \frac{|\mathcal{A}_r|}{|B_r(0)|} \ . \tag{4.2.32}$$

Remark 4.2.12. *By the above remark one also has*

$$\sharp(\mathbb{Z}^d \cap B_R(0)) \lesssim R^d \ . \tag{4.2.33}$$

Let

$$\Omega^{(R)} := \Omega \cap B_R \quad \text{and} \quad \Omega^{(R,c)} := B_R \setminus \Omega^{(R)}\,; \tag{4.2.34}$$

In order to estimate the cardinality of $\Omega^{(R,c)} \cap \mathbb{Z}^d$ we estimate the measure of $\Omega^{(R,c)}_r$. To this end we remark that the following holds:

Lemma 4.2.13. *Assume $\epsilon \leq \frac{\delta}{1+\tau}$. There exist positive constant C and R_0, depending on $\epsilon, \delta, \tau, \mathfrak{c}$, such that for all $R \geq R_0$*

$$\Omega^{(R,c)}_r \subset \bigcup_{0<\|k\|<CR^\epsilon} \widetilde{A}_k\,, \quad \widetilde{A}_k := \left\{\xi \in B_{2R} : |\,\langle \xi; k\rangle\,| < \frac{CR^\delta}{\|k\|^\tau}\right\}_r\,, \tag{4.2.35}$$

where $\Omega^{(R,c)}_r$ is the extension of $\Omega^{(R,c)}$ according to (4.2.31).

Proof. If $\xi \in \Omega^{(R,c)}$, there exists $k \in \mathbb{Z}^d$, $0 < \|k\| < \langle \xi + \kappa\rangle^\epsilon$, such that $|\,\langle \xi + \kappa; k\rangle\,| \leq \langle \xi + \kappa\rangle^\delta \|k\|^{-\tau}$. As stated in Remark C.1.2 of Appendix C, there exists a positive constant C_ϵ, depending only on ϵ, such that

$$\|k\| \leq \langle \xi + \kappa\rangle^\epsilon \leq C_\epsilon\left(\langle \xi\rangle^\epsilon + \|\kappa\|^\epsilon\right) \leq 2C_\epsilon R^\epsilon\,, \tag{4.2.36}$$

and again by Remark C.1.2, there exists a positive constant C_δ depending only on δ such that

$$|\,\langle \xi + \kappa; k\rangle\,| \leq \langle \xi + \kappa\rangle^\delta \|k\|^{-\tau} \leq C_\delta\left(\langle \xi\rangle^\delta + \|k\|^\delta\right)\|k\|^{-\tau} \leq 2C_\delta R^\delta \|k\|^{-\tau}\,.$$

Let then $\xi' \in \Omega^{(R,c)}_r$ and let $\xi \in \Omega^{(R,c)}$ and $h \in \mathbb{R}^d$, $\|h\| \leq r$, be such that $\xi' = \xi + h$. Then $\|\xi'\| \leq \|\xi\| + \|h\| \leq 2R$, if $R \geq r$, and one observes that

$$\begin{aligned} |\,\langle \xi'; k\rangle\,| &\leq |\,\langle \xi + \kappa; k\rangle\,| + |\,\langle h; k\rangle\,| + |\,\langle \kappa; k\rangle\,| \\ &\leq \left(4\gamma C_\delta R^\delta + \left(\|\kappa\| + \|h\|\right)\|k\|^{\tau+1}\right)\|k\|^{-\tau} \\ &\leq \left(4\gamma C_\delta + (2C_\epsilon)^{\tau+1}\right) R^\delta \|k\|^{-\tau}\,, \end{aligned}$$

since $\|k\|^{\tau+1} \leq (2C_\epsilon)^{\tau+1} R^{\varepsilon(\tau+1)}$ due to equation (4.2.36), and $\|\kappa\| \leq R^{\delta-\epsilon(\tau+1)}$ provided $R \geq R_0 = \|\kappa\|^{1/(\delta-\epsilon(\tau+1))}$.
Then the thesis follows, taking $C = \max\{2C_\epsilon, 4\gamma C_\delta + (2C_\epsilon)^{\tau+1}\}$. □

Proposition 4.2.14. *Assume $\epsilon \leq \frac{\delta}{1+\tau}$, $\tau > d$ and $R > R_0$, with R_0 as in Lemma 4.2.13. Then there exists a positive constant C' such that*

$$\left|\Omega^{(R,c)}_r\right| \leq C' R^{d+\delta-1}\,. \tag{4.2.37}$$

Proof. The proof is standard, we give it here for the sake of completeness. Since $\widetilde{A}_k$ as defined in (4.2.35) is the intersection of a layer of thickness $C\|k\|R^\delta/\,\|k\|^\tau$ with a sphere of radius $2R$, we have

$$\left|\widetilde{A}_k\right| \lesssim \frac{R^\delta}{\|k\|^{\tau+1}} R^{d-1} \ ,$$

thus, having fixed some large R_1, we have

$$\begin{aligned} \left|\bigcup_{\|k\|\le CR^\epsilon} \widetilde{A}_k\right| &\le \left|\bigcup_{k\in\mathbb{Z}^d\setminus\{0\}} \widetilde{A}_k\right| \le \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \left|\widetilde{A}_k\right| \\ \lesssim \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{R^\delta}{\|k\|^{\tau+1}} R^{d-1} &\lesssim R^{\delta+d-1} \sum_{l=0}^{+\infty} \sum_{lR_1<\|k\|\le(l+1)R_1} \frac{1}{\|k\|^\tau} \\ \lesssim R^{\delta+d-1} &\left(\sum_{0<\|k\|\le R_1} \frac{1}{\|k\|^\tau} + \sum_{l=1}^{+\infty} \sum_{lR_1<\|k\|\le(l+1)R_1} \frac{1}{(lR_1)^\tau} \right) . \end{aligned} \tag{4.2.38}$$

Exploiting again Remark 4.2.11, we get, for $l \ge 1$,

$$\sharp[(B_{(l+1)R_1} \setminus B_{lR_1}) \cap \mathbb{Z}^d] \lesssim \left|B_{(l+1)R_1} \setminus B_{lR_1}\right| \lesssim l^{d-1} R_1^d \ ,$$

and thus, recalling $\tau-(d-1) > 1$, the bracket at r.h.s. of (4.2.38) is bounded by a constant and the proposition holds. □

We finally show the following

Proposition 4.2.15. *Let R_0 as in Lemma 4.2.13. For any $R \ge R_0$, one has that*

$$\frac{\sharp\Big(\Omega \cap B_R \cap \mathbb{Z}^d\Big)}{\sharp\Big(B_R \cap \mathbb{Z}^d\Big)} = 1 - \mathcal{O}(R^{\delta-1})\,. \tag{4.2.39}$$

As a consequence, since $0 < \delta < 1$,

$$\lim_{R\to+\infty} \frac{\sharp\Big(\Omega \cap B_R \cap \mathbb{Z}^d\Big)}{\sharp\Big(B_R \cap \mathbb{Z}^d\Big)} = 1\,.$$

Proof. By recalling the formula (4.2.34), one has that

$$\sharp\Big(\Omega \cap B_R \cap \mathbb{Z}^d\Big) = \sharp\Big(\Omega^{(R)} \cap \mathbb{Z}^d\Big) = \sharp\Big(B_R \cap \mathbb{Z}^d\Big) - \sharp\Big(\Omega^{(R,c)} \cap \mathbb{Z}^d\Big)\,. \tag{4.2.40}$$

By Remark 4.2.11 and Lemma 4.2.14, one obtains that

$$\sharp\Big(\Omega^{(R,c)} \cap \mathbb{Z}^d\Big) \lesssim R^{d+\delta-1}$$

and therefore, using Remark 4.2.33 and the formula (4.2.40), one obtains the claimed estimate (4.2.39). □

4.2.4 Comparison of our normal form with [PS10, PS12]

We now briefly recall the analogous normal form constructions of [PS10, PS12]. In such papers, the authors work separately in annuli $\mathcal{A}_\rho$ of $\mathbb{R}^d$; more precisely, they fix a positive parameter $\rho \gg 1$ and consider

$$\mathcal{A}_\rho = \{\xi \in \mathbb{R}^d \mid \|\xi\|^2 \in [\rho^2 - 20\|V\|\,,\ \rho^2 + 20\|V\|]\}\,. \tag{4.2.41}$$

(Here for concreteness we refer to the construction performed in [PS10]; the one of [PS12] is analogous, but with a different choice of the parameters involved in the construction.) Then, the normal form is performed in each annulus: precisely, their non-resonant set is

$$\widetilde{\Omega}^{(0)} := \Big\{\xi \in \mathcal{A}_\rho \ :\ |\,\langle k;\xi\rangle\,| > \tilde{\mu}, \quad \forall k \in \mathbb{Z}^d, \quad 0 < \|k\| < \tilde{N}\Big\}\,, \tag{4.2.42}$$

with

$$\tilde{\mu} = \|k\|\rho^{\alpha}\,, \tag{4.2.43}$$

for some positive parameter $\alpha \in (0, \frac{1}{3})$. Note that in [PS10, PS12] $\tilde{\mu} = \tilde{\mu}_\rho$ and $\tilde{N} = \tilde{N}_\rho$ depend on the parameter ρ only, whereas in the definition of the non-resonant set that we use (see (4.1.6)) μ and N are smooth functions of ξ.

As a result, in [PS10, PS12] for any ρ one conjugates the operator H to a new operator with a symbol

$$\|\xi\|^2 + z_\rho(\xi) + z_\rho^{(\mathrm{res})}(x,\xi) + r_\rho(x,\xi)\,,$$

where $r_\rho(x,\xi)$ has a small size, but only for $\xi \in \mathcal{A}_\rho$. Of course, this implies that such normal form requires a localization on the annulus $\mathcal{A}_\rho$, in order to be able to consider $Op\,(r_\rho)$ as a remainder term. Instead, in our construction the choice $\mu = \langle\xi\rangle^{\delta}\,,\ N = \langle\xi\rangle^{\epsilon}$ enables to avoid localization in annuli, and the normal form that we obtain is valid on the whole space $L^2(\mathbb{T}^d)$.

4.3 Theorem 4.0.1: proof of Item (i)

The estimate (4.0.5) about the density one set Ω follows by Proposition 4.2.15. We start with proving Item (i): as anticipated in Section 4.1, this is done by showing that for $\xi \in \Omega$, $e^{\mathrm{i}\xi\cdot x}$ is a quasimode for H_n.
By the normal form Theorem 4.2.1, for any $n \in \mathbb{N}$, there exists a unitary map $\mathcal{U}_n \in \mathcal{B}(H^s)$ for any $s \geq 0$ such that the $H_n = Op^W(h_n) = \mathcal{U}_n H \mathcal{U}_n^{-1}$ satisfies the expansion given in (4.2.5), namely

$$h_n = h^0 + [z_n] + z_n^{(\mathrm{res})} + v_n$$

with $[z_n], z_n^{(\mathrm{res})} \in S^{M-\mathfrak{e},\delta}$, $v_n \in S^{M-\mathfrak{e}-\rho n,\delta}$ and $\mathrm{supp}(z_n^{(\mathrm{res})}) \cap \Omega = \emptyset$. By applying Lemma C.2.1 and Remark C.2.2 of Appendix C, which relate classical and Weyl quantization of a symbol, one deduces that there exist $\widetilde{z}_n^{(\mathrm{res})} \in S^{M-\mathfrak{e},\delta}$, $\widetilde{v}_n \in S^{M-\mathfrak{e}-\rho n,\delta}$ and $\widetilde{w}_n \in S^{-\infty,\delta}$ such that

$$\begin{aligned} &Op^W(z_n^{(\mathrm{res})}) = Op^{cl}(\widetilde{z}_n^{(\mathrm{res})}) + Op^{cl}(\widetilde{w}_n), \quad Op^W(v_n) = Op^{cl}(\widetilde{v}_n), \\ &\mathrm{supp}(\widetilde{z}_n^{(\mathrm{res})}) = \mathrm{supp}(z_n^{(\mathrm{res})}) \subseteq \mathbb{R}^d \setminus \Omega. \end{aligned} \tag{4.3.1}$$

Therefore, given $\xi \in \Omega \cap \mathbb{Z}^d$, one gets

$$Op^{cl}(\widetilde{z}_n^{(\mathrm{res})})[e^{\mathrm{i}\xi\cdot x}] = 0 \tag{4.3.2}$$

since due to the second equation of (4.3.1) $\widetilde{z}_n^{(res)}(x,\xi) = 0$ for any $\xi \in \Omega$. Moreover, since in particular $\tilde{v}_n, \tilde{w}_n \in S^{M-\mathfrak{e}-\rho n,\delta}$, one has that

$$\begin{aligned} Op^{cl}(\widetilde{w}_n)[e^{\mathrm{i}\xi\cdot x}] + Op^W(v_n)[e^{\mathrm{i}\xi\cdot x}] &= Op^{cl}(\widetilde{w}_n)[e^{\mathrm{i}\xi\cdot x}] + Op^{cl}(\widetilde{v}_n)[e^{\mathrm{i}\xi\cdot x}] \\ &= (\widetilde{w}_n(x,\xi) + \widetilde{v}_n(x,\xi))\, e^{\mathrm{i}\xi\cdot x} \\ &= \mathcal{O}(\langle \xi+\kappa\rangle^{M-\mathfrak{e}-\rho n}). \end{aligned} \tag{4.3.3}$$

Hence (4.3.1)-(4.3.3) imply that for any $\xi \in \Omega \cap \mathbb{Z}^d$

$$\begin{aligned} H_n[e^{\mathrm{i}\xi\cdot x}] &= \lambda_n(\xi) e^{\mathrm{i}\xi\cdot x} + \mathcal{O}(\langle \xi+\kappa\rangle^{M-\mathfrak{e}-\rho n}), \\ \lambda_n(\xi) &:= \|\xi+\kappa\|^M + [z_n](\xi). \end{aligned} \tag{4.3.4}$$

Then the existence of one eigenvalue $\mathcal{O}(\langle \xi+\kappa\rangle^{M-\mathfrak{e}-\rho n})$ close to $\lambda_n(\xi)$ follows by the standard quasimode argument, and this concludes the proof of Item (i).

Finally remark that, defining $\varphi_{n,\xi} := \mathcal{U}_n^{-1}[e^{\mathrm{i}x\cdot\xi}]$, it is a quasimode for the original Hamiltonian. Indeed one has

$$H_0\varphi_{n,\xi} = \mathcal{U}_n^{-1} H_n \mathcal{U}_n[\varphi_{n,\xi}] = \lambda_n(\xi)\varphi_{n,\xi} + \mathcal{O}(\langle \xi+\kappa\rangle^{M-\mathfrak{e}-\rho n}).$$

4.4 End of the proof: a refined quasi-mode argument

In order to conclude the proof of Theorem 4.0.1, we need a refined version of the standard quasi-mode argument. The situation of double eigenvalues, which is actually sufficient to prove Item (*iii*) of our Theorem 4.0.1, was studied for instance in [BKP15]. According to Proposition 5.1, statement (ii) of that paper, in such a case the result follows from the fact that $\langle e^{\mathrm{i}\xi\cdot x}, e^{\mathrm{i}\xi'\cdot x}\rangle_{L^2} = 0$. Here we give a generalization of Proposition 5.1 of [BKP15] that is suitable to deal with the case of higher multiplicities, and that enables us to prove Item (*ii*):

Proposition 4.4.1. *Let A a self-adjoint operator on a Hilbert space $\mathcal{H}$ and suppose that there exist $K \in \mathbb{N}$, $\mu_1 \leq \cdots \leq \mu_K \in \mathbb{R}$ and $\psi_1, \ldots, \psi_K \in \mathcal{H}$ such that the following holds:*

$$\langle \psi_k, \psi_l \rangle = \delta_{k,l} \quad \forall k, l = 1, \ldots, K \tag{4.4.1}$$

and, given

$$\varepsilon := \max_{j=1,\ldots,K} \{\|A\psi_j - \mu_j \psi_j\|_{L^2}\} , \tag{4.4.2}$$

$D \in \mathbb{R}^+$ is such that

$$D^2 \geq \frac{16}{\pi} K^2 \varepsilon \left(|\mu_K - \mu_1| + D\right) . \tag{4.4.3}$$

Then there exist at least K eigenvalues of A lying in the interval

$$\mathcal{I} = (-D + \mu_1,\ \mu_K + D) .$$

Proof. Suppose by contradiction that there are $K' < K$ eigenvalues of A contained inside the interval $\mathcal{I}$. Then in particular there are less than $K' < K$ eigenvalues of A inside the intervals

$$\mathcal{I}^+ = \left(\mu_K + \frac{D}{2},\ \mu_K + D\right) , \quad \mathcal{I}^- = \left(\mu_1 - D,\ \mu_1 - \frac{D}{2}\right) .$$

Since I^+ has length $\frac{D}{2}$, there exists at least one interval $J^+ \subset I^+$ such that $|J^+| \geq \frac{D}{2K}$ which contains no eigenvalues of A, analogously for I^- : there exists at least an interval $J^- \subset I^-$ containing no eigenvalues of A and having length $|J^-| \geq \frac{D}{2K}$. Consider then a square closed path γ in the complex

plane intersecting the real axis at the middle points of J^+ and J^-. Again by the contradictory assumption, there are less than K eigenvalues of A in the segment of the real axis contained inside γ.
Furthermore, by construction

$$\mathrm{dist}\,(\gamma, \sigma(A)) \geq \frac{D}{4K}\,, \tag{4.4.4}$$

and

$$\mathrm{dist}\,(\gamma, \{\mu_j\}) \geq \frac{D}{2} \quad \forall j\,. \tag{4.4.5}$$

Moreover, the length $\ell(\gamma)$ of γ fulfills

$$\ell(\gamma) \leq 4|\mathcal{I}| < 4(2D + |\mu_K - \mu_1|)\,. \tag{4.4.6}$$

Denote by $R(z) = (A - z\mathbb{I})^{-1}$ the resolvent of A: then if P is the projection operator on the eigenspace corresponding to the eigenvalues of A contained inside γ (which is not empty, due to the standard quasi-mode argument), one has

$$P = \frac{1}{2\pi \mathrm{i}} \int_\gamma R(z)\; dz\,,$$

and for all j and for any $z \in \mathbb{C}$, by the identity

$$\psi_j = (z\mathbb{I} - A)^{-1}(z\mathbb{I} - A)\psi_j = (z\mathbb{I} - A)^{-1}(z - \mu_j)\psi_j + (z\mathbb{I} - A)^{-1}(\mu_j\mathbb{I} - A)\psi_j$$

one obtains

$$(z\mathbb{I} - A)^{-1}\psi_j = (z - \mu_j)^{-1}\psi_j - (z\mathbb{I} - A)^{-1}(z - \mu_j)^{-1}(\mu_j\mathbb{I} - A)\psi_j\,.$$

Integrating along γ, since by Cauchy Integral Theorem one has

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma (z - \mu_j)^{-1}\; dz\; \psi_j = \psi_j\,,$$

by the very definition of P this entails

$$P\psi_j = \psi_j + r_j\,, \quad r_j = -\frac{1}{2\pi i} \int_\gamma R(z)(z - \mu_j)^{-1}\; dz(\mu_j\mathbb{I} - A)\psi_j\,.$$

Due to (4.4.4), (4.4.5), (4.4.6), and by the definition of ε, one has

$$\|r_j\| \leq \frac{\ell(\gamma)}{2\pi} \frac{\|(A - \mu_j\mathbb{I})\psi_j\|_{L^2}}{\mathrm{dist}(\gamma, \sigma(A))\mathrm{dist}(\gamma, \mu_j)} \leq \frac{16K(2D + |\mu_K - \mu_1|)\varepsilon}{\pi D^2}\,. \tag{4.4.7}$$

We are going to show that the vectors $\{P\psi_1, \dots, P\psi_K\}$ are linearly independent, which is against the assumption $K' < K$. Assume that

$$\sum_{j=1}^{K} \beta_j P\psi_j = 0\,.$$

Then one has

$$\sum_{j=1}^{K} \beta_k P\psi_k = \sum_{k=1}^{M} \beta_k \left(\psi_k + r_k\right) = 0\,;$$

in particular,

$$\sum_{k=1}^{M} \beta_k \left(\langle \psi_k\,,\ \psi_j\rangle + \langle r_k\,,\ \psi_j\rangle\right) = 0 \quad \forall\, j\,.$$

Defining $\beta = (\beta_1, \dots, \beta_k) \in \mathbb{R}^K$ and letting B be the K dimensional matrix with matrix elements given by $B_k^j = \langle \psi_k, r_j\rangle$, this reads $(\mathbb{I} + B)\,\beta = 0$. Since, by (4.4.7),

$$\|B\| \le K \sup_{i,j}\{|B_{i,j}|\} \le \frac{16K^2(2D + |\mu_K - \mu_1|)\varepsilon}{\pi D^2}\,,$$

hypothesis (4.4.3) ensures $\|B\| < 1$, so that $\mathbb{I} + B$ is invertible and thus $\beta_j = 0\ \forall j$. This shows that $\{P\psi_k\}_{j=1}^K$ form a set of K linearly independent eigenfunctions, which contradicts the hypothesis that there is only a set of multiplicity $K' < K$ of eigenvalues of A inside the interval $(\mu_1 - D\,, \mu_K + D)$.
□

The idea to prove Item *(ii)* of Theorem 2.1.7 is to show that the set $\mathcal{A} = \{\lambda_n(\xi) = \|\xi + \kappa\|_{\mathtt{g}^*}^M + [z_n](\xi) \mid \xi \in \Omega\}$ can be grouped in separated subsets of finite cardinality, and then, for each one of those separated subgroups $\{\lambda_n(\xi_1), \dots, \lambda_n(\xi_K)\}$, to apply the above Proposition 4.4.1 for the operator $A = H_n$, with quasi-modes $\lambda_n(\xi_1), \dots, \lambda_n(\xi_K)$ and quasi-eigenfunctions $e^{\mathrm{i}\xi_1\cdot x}, \dots, e^{\mathrm{i}\xi_K\cdot x}$. In order to do this, we exploit the following results:

Lemma 4.4.2. *Let* $\mathtt{z} = \sup_{\xi\in\Omega\cap\mathbb{Z}^d} |[z_n](\xi)|$ *and* $R \ge \left((2^M - 1)^{-1}\mathtt{z}\right)^{1/M}$. *Then one has*

$$\sharp\{\xi \in \Omega \cap \mathbb{Z}^d \mid |\lambda_n(\xi)| \le R^M\} \le \left(\frac{4}{\mathfrak{c}}\right)^d R^d\,.$$

Proof. By Lemma 3.1.3 Weyl law holds for the eigenvalues of the operator $-\Delta_{g,\kappa}$, and in particular one has that for any positive R

$$\sharp\{\xi \in \mathbb{Z}^d \mid \|\xi+\kappa\|^M \leq R^M\} = \sharp\{\xi \in \mathbb{Z}^d \| \ \|\xi+\kappa\|^2 \leq R^2\} \leq \left(\frac{2}{\mathfrak{c}}\right)^d R^d \,.$$

Recall that $\lambda_n(\xi) = \|\xi+\kappa\|^M + z_n(\xi)$ for all $\xi \in \Omega$: since for all $\xi \in \Omega \cap \mathbb{Z}^d$ $|z_n(\xi)| \leq \mathsf{z}$, and we are assuming $(2^M-1)R^M \geq \mathsf{z}$, we have that $\big|\|\xi+\kappa\|^M + [z_n](\xi)\big| \leq R^M$ implies $\|\xi+\kappa\|^M \leq R^M + \mathsf{z} \leq 2^M R^M$. Thus it follows

$$\sharp\{\xi \in \Omega\cap\mathbb{Z}^d \mid |\lambda_n(\xi)| \leq R^M\} \leq \sharp\{\xi \in \mathbb{Z}^d \mid \|\xi+\kappa\|^2 \leq (2R)^M\} \leq \left(\frac{4}{\mathfrak{c}}\right)^d R^d \,.$$

□

The above result enables us to group the elements of $\mathcal{A}$ as follows:

Lemma 4.4.3. *Let* $\mathsf{z} = \max_{\xi\in\Omega} |[z_n](\xi)|$. *For any* $0 < L < \mathsf{z}$ *and any* $N > \frac{d}{M}$ *there exist a sequence of intervals*

$$E_j = [a_j, b_j]\,, \quad j \in \mathbb{N}$$

and a positive constant C *such that* $a_j, b_j \in \mathcal{A}$ $\forall j$ *and the following holds:*

$$\mathcal{A} \subset \left[-\mathsf{z}, a_1 - \frac{1}{a_1^N}\right] \cup \left(\bigcup_{j\in\mathbb{N}} E_j\right) \tag{4.4.8}$$

$$|b_j - a_j| \leq L \tag{4.4.9}$$

$$d(E_j, E_{j+1}) = |a_{j+1} - b_j| \geq \frac{L}{b_j^N} \tag{4.4.10}$$

$$\sharp\,(\mathcal{A}\cap E_j) \leq C b_j^{\frac{d}{M}}\,. \tag{4.4.11}$$

Proof. Let a_1 be the first element in $\mathcal{A}$ such that

$$\begin{aligned} a_1 &\geq (2^M-1)^{-1/M}\mathsf{z}^{1/M} \quad \text{and} \\ a_1 &\geq \left((\tfrac{4}{c})^d + 1\right)^{N-d/M} \,. \end{aligned} \tag{4.4.12}$$

Since $a_1 + L \leq 2a_1$, by Lemma 4.4.2 there are no more than $K = 2^{d/M}(\frac{4}{\mathfrak{c}})^d a_1^{d/M}$ elements of $\mathcal{A}$ inside the interval $[a_1, a_1+L]$. Thus there exists at least an interval inside $[a_1, a_1+L]$ which has length greater or equal than $L/(K+1)$

and does not contain elements of $\mathcal{A}$. Let b_1 its left endpoint and a_2 its right endpoint. Due to the second of (4.4.12), one has

$$|a_2 - b_1| \geq L/(K+1) \geq L/a_1^N \geq L/b_1^N \,,$$

thus (4.4.10) holds in the case $j = 1$. Furthermore, $b_1 - a_1 \leq L$, thus (4.4.9) holds for $j = 1$, and again by the definition of a_1 one has that $b_1 \leq a_1 + L \leq 2a_1$, which implies

$$\sharp(\mathcal{A} \cap [a_1, b_1]) \leq (\tfrac{4}{\mathfrak{c}})^d a_1^{d/M} \leq 2^{d/M} (\tfrac{4}{\mathfrak{c}})^d b_1^{d/M} \,,$$

namely also (4.4.11) is verified for $j = 1$. One then argues analogously, with a_2 in place of a_1, to define the points b_2, a_3, and so on for all $j \in \mathbb{N}$.
It remains to verify that (4.4.8) holds. To this aim notice that, by Lemma 4.4.2, there are at most $K' = (\frac{4}{\mathfrak{c}})^d a_1^{d/M}$ elements of $\mathcal{A}$ lying inside the interval $[0, a_1]$. Since $[0, a_1]$ has length $|a_1| \geq 1$, there exists at least an interval of length not less than $|a_1|/(K'+1)$ which does not contain any element of $\mathcal{A}$, and again by the definition of a_1 one has that $|a_1|/(K'+1) \geq 1/a_1^N$. □

Then, in order to prove Item (ii) of Theorem 4.0.1, one chooses $L = 1$ and $N = n$ in Lemma 4.4.3 and applies Proposition 4.4.1 for any interval E_j, with $K = \sharp\,(\mathcal{A} \cap E_j)$, and $\{\mu_1, \dots, \mu_K\} = \mathcal{A} \cap E_j$. Indeed, for any $k = 1, \dots, K$ μ_k is of the form $\mu_k = \lambda_n(\xi_k)$ for some $\xi_k \in \Omega \cap \mathbb{Z}^d$, and $\psi_k = \frac{1}{\mu_{\mathrm{g}}(\mathbb{T}^d)} e^{\mathrm{i}\xi_k \cdot x}$ is such that

$$\|H_n \psi_k - \lambda_n(\xi_k)\psi_k\| = \mathcal{O}(\langle \xi_k + \kappa \rangle^{-n}) \,.$$

Lemma 4.4.3 entails that for all $j \in \mathbb{N}$ one has $K \lesssim \langle \xi_K + \kappa \rangle^{d/M)}$, and due to the fact that the intervals E_j have length bounded by 1, one also has that

$$\varepsilon = \max_{k=1,\dots,K} \|H_n \psi_k - \lambda_n(\xi_k)\psi_k\|$$

satisfies (4.4.3), up to choosing $D = \mathcal{O}(\langle \xi_K + \kappa \rangle^{-(n-2d/M)})$. Then Item (ii) follows by the arbitrariety of n and recalling that, since E_j has length 1, one has $\mathcal{O}(\langle \xi_k + \kappa \rangle^{-n}) = \mathcal{O}(\langle \xi_l + \kappa \rangle^{-n})$ for any $k, l = 1, \dots, K$.

Chapter 5

Structure Theorem à la Nekhoroshev

In this chapter we give a Structure Theorem concerning the operator

$$H = -\Delta_{\mathsf{g},\kappa} + V \quad \text{on} \quad L^2(\mathbb{T}^d)\,, \tag{5.0.1}$$

where $\mathbb{T}^d$ is the d dimensional torus equipped with a flat Riemannian metric g as described in Section 3.1 of Chapter 3, $-\Delta_{\mathsf{g},\kappa}$ is the operator defined as in (3.1.3) and $V \in OPS^{\,0,\delta}$ is a self-adjoint operator. Of course a case of particular interest is $V \in \mathcal{C}^\infty(\mathbb{T}^d, \mathbb{R})$, but with a little abuse of notations, we will refer to H as a *Schrödinger operator in dimension d with Floquet boundary conditions* even in the case of a generic self-adjoint $V \in OPS^{\,0,\delta}$.

Here we prove that, up to a regularizing remainder, H as in (5.0.1) is unitarily equivalent to a block diagonal operator, which in each block is either diagonal or, up to a gauge transformation, it is a Schrödinger operator on a lower dimensional torus.

The block decomposition that we perform on $L^2(\mathbb{T}^d)$ corresponds to a block decomposition on the space of Fourier modes $\mathbb{Z}^d$, which we obtain as a quantum analogue of the classical decomposition of the action space $\mathbb{R}^d$ exhibited in the proof of the geometric part of Nekhoroshev Theorem ([Nek77, Nek79]; actually here we refer to the construction as explained in [Gio03]). As the geometric part of the proof of Nekhoroshev Theorem relies on an analytic part, namely a normal form lemma, in order to obtain our block decomposition we exploit the normal form result exhibited in Theorem 4.2.1 of Chapter 4.

The geometric decomposition à la Nekhoroshev presented here differs from the ones developed in [Par08, PS10, PS12] in several aspects. The main ones are that, in order to get a global partition of $L^2(\mathbb{T}^d)$ and an iterable construction, we give a different definition of resonances; as a consequence, also the invariant blocks are defined differently. As anticipated in the Introduction, the main consequence is that this enables us to show that in each block the normal form operator turns out to have the same structure than the original one. More detailed comments about the connections with this series of works are postponed to Section 5.2 below: see in particular Subsection 5.1.2.

5.1 Setting and statement of the Structure Theorem

In the present chapter, as in the previous one, we omit the dependence on the metric $\mathtt{g}$ from scalar products, norms and angled brackets, namely we keep on writing $\langle\cdot;\cdot\rangle$, $\|\cdot\|$, $\langle\cdot\rangle$ instead of $\langle\cdot;\cdot\rangle_{\mathtt{g}*}$, $\|\cdot\|_{\mathtt{g}*}$, $\langle\cdot\rangle_{\mathtt{g}}$.

Definition 5.1.1. *Given $E \subseteq \mathbb{Z}^d$, we denote*

$$\mathcal{E} = \overline{\mathrm{span}\{e^{\mathrm{i}\xi\cdot x} \mid \xi \in E\}}\,, \tag{5.1.1}$$

where the bar denotes the closure in $L^2(\mathbb{T}^d)$, and we will call such a subspace subspace generated by E.

Definition 5.1.2. *We will denote by $\Pi_{\mathcal{E}} : L^2(\mathbb{T}^d) \to \mathcal{E}$ the orthogonal projector on $\mathcal{E}$ and, given a linear (pseudodifferential) operator F, we will write*

$$F_{\mathcal{E}} := \Pi_{\mathcal{E}} F \Pi_{\mathcal{E}}\,. \tag{5.1.2}$$

The block decomposition that we perform is related to the submoduli of $\mathbb{Z}^d$: for this reason we recall some properties of the bases of the moduli.

Definition 5.1.3. *A subgroup M of $\mathbb{Z}^d$ is called a submodulus if $\mathbb{Z}^d \cap \mathrm{span}_{\mathbb{R}} M = M$. Here and below,* $\mathrm{span}_{\mathbb{R}} M$ *is the subspace generated by taking linear combinations with real coefficients of elements of M.*

Given a discrete submodule M of $\mathbb{R}^d$ (or of $\mathbb{Z}^d$) it is well known that it admits a basis, namely that there exist d' independent vectors $\mathbf{v}^1, ..., \mathbf{v}^{d'}$ such that

$$M = \mathrm{span}_{\mathbb{Z}}\{\mathbf{v}^1, ..., \mathbf{v}^{d'}\} := \left\{ w \in \mathbb{Z}^d \ : \ w = \sum_{k=1}^{d'} n_k \mathbf{v}^k, \quad n_1, \dots, n_{d'} \in \mathbb{Z} \right\}. \tag{5.1.3}$$

Furthermore, if $M \subset \mathbb{Z}^d$ and $d' < d$ then $\mathbf{v}^1, ..., \mathbf{v}^{d'}$ can be completed to a basis of $\mathbb{Z}^d$, namely there exist $\mathbf{v}^{d'+1}, ..., \mathbf{v}^d$ such that the whole collection $\mathbf{v}^1, ..., \mathbf{v}^d$ generates $\mathbb{Z}^d$. Such a basis will be called a basis *adapted to* M.

In what follows, given a collection of such vectors $\{\mathbf{v}^{d'+1}, \ldots, \mathbf{v}^d\}$, we will denote

$$M^c := \text{span}_{\mathbb{Z}}\{\mathbf{v}^{d'+1}, ..., \mathbf{v}^d\} \ ; \tag{5.1.4}$$

if $M = \mathbb{Z}^d$ then of course $M^c = \{0\}$ and if $M = \{0\}$ then $M^c = \mathbb{Z}^d$.

Of course, in general M^c is not unique, but this will not affect our construction. Consider now the basis $\{\mathbf{u}_j\}_{j=1,...,d}$ of $\mathbb{R}^d$ dual to $\{\mathbf{v}^j\}_{j=1,...,d}$. Then the coordinates z^j introduced by

$$x = z^j \mathbf{u}_j \tag{5.1.5}$$

are good coordinates on $\mathbb{T}^d$ (in the sense that they respect the 2π periodicity of the torus). These coordinates will be called *coordinates adapted to* M.

Given a covector $w \in \mathbb{R}^d$ and a module M, we will have to decompose it in a component along M and a component in the orthogonal direction (see Section 5.5), and this has to be done in a compatible way with the lattice structure of $\mathbb{Z}^d$ and with the Floquet parameter.

First we give the following definition.

Definition 5.1.4. *Given a basis* $\{\mathbf{v}^A\}_{A=1,...,d}$ *of* $\mathbb{Z}^d$ *and a vector* $w = w_A \mathbf{v}^A \in \mathbb{R}^d$*, we denote*

$$\lfloor w \rfloor := \lfloor w_A \rfloor \mathbf{v}^A \ ,$$

with $\lfloor w_A \rfloor$ *the integer part of* w_A*, and*

$$\{w\} := w - \lfloor w \rfloor \, .$$

Given a module M, we consider the orthogonal decomposition

$$\mathbb{R}^d = \text{span}_{\mathbb{R}} M \oplus (\text{span}_{\mathbb{R}} M)^{\perp}.$$

Correspondingly, given a vector $w \in \mathbb{R}^d$, we will decompose it as

$$w = w_M + w_{M^\perp} \ , \quad w_M \in \text{span}_{\mathbb{R}} M \ , \quad w_{M^\perp} \in (\text{span}_{\mathbb{R}} M)^{\perp} \ .$$

Definition 5.1.5. *Given a vector* $\xi \in \mathbb{Z}^d$*, a modulus* M *and a Floquet parameter* κ*, having introduced a basis adapted to* M*, we define the following two objects:*

$$\begin{aligned} \widetilde{\xi} &:= \xi - \lfloor (\xi + \kappa)_M \rfloor \ , \\ \kappa_\xi &:= \{(\xi + \kappa)_M\} \ . \end{aligned} \tag{5.1.6}$$

Remark 5.1.6. *If we denote* $\zeta := \lfloor (\xi + \kappa)_M \rfloor$*, one has*

$$(\xi + \kappa)_M = \zeta + \kappa_\xi \ , \quad (\xi + \kappa)_{M^\perp} = (\widetilde{\xi} + \kappa)_{M^\perp} \ . \tag{5.1.7}$$

Given a vector $\beta \in M^c$, we will have to consider the space

$$M + \beta := \left\{ \xi \in \mathbb{R}^d \mid \exists v \in M \text{ s.t. } \xi = v + \beta \right\} \ . \tag{5.1.8}$$

Remark 5.1.7. *Notice that, for any* $\xi \in M + \beta$*, one has*

$$\tilde{\xi} = \tilde{\beta} \, , \quad \kappa_\xi = \kappa_\beta \, ,$$

thus the quantities $\widetilde{\xi}$ *and* κ_ξ *defined in* (5.1.6) *are constant along the set* $M + \beta$.

Remark 5.1.8. *The set* $M + \beta$ *defined as in* (5.1.8) *is clearly an affine module isomorphic to* M*. A convenient way to identify the two spaces* $M + \beta$ *and* M *is to subtract* $\widetilde{\beta}$ *to a vector* $w \in M + \beta$.
Correspondingly, the subspace of $L^2(\mathbb{T}^d)$ *generated by* $M + \beta$ *(in the sense of Definition 5.1.1) is isomorphic to the subspace generated by* M*. Explicitly, the isomorphism is realized by using the Gauge transformation* U_β *defined by*

$$U_\beta u := e^{-\mathrm{i} x \cdot \tilde{\beta}} u \ . \tag{5.1.9}$$

Definition 5.1.9. *Given a module* M*, a vector* $\beta \in M^c$ *and a set* $W \subset M + \beta$*, we denote* $W^t := W - \widetilde{\beta}$ *and* $\mathcal{W}^t := U_\beta \mathcal{W} \subset L^2(\mathbb{T}^d)$.

5.1.1 Structure Theorem: a qualitative description

Before giving a rigorous statement of our Structure Theorem (which is Theorem 5.1.10 below), we give a rough description of the result. For simplicity, we restrict to the case $d = 2$, we consider as g the Euclidean metric and we assume periodic boundary conditions, namely $\kappa = 0$. The Structure Theorem states that it is possible to conjugate the operator H as in (5.0.1), up to a smoothing remainder, to a block diagonal operator $\widetilde{H}$. Each block is of the form $\mathcal{W}_{M,\beta} = \mathrm{span}\{e^{\mathrm{i}\xi \cdot x} \mid \xi \in W_{M,\beta}\}$ for some $W_{M,\beta} \subset \mathbb{Z}^2$; we are now going to describe the structure of such sets $W_{M,\beta}$ and how they emerge in the construction. The blocks $W_{M,\beta}$ are labeled by all the possible submodules $M \in \mathbb{Z}^2$ and by a further index $\beta \in \widetilde{M}$; roughly speaking, a block is labeled by M if it contains sites ξ which are affected by resonances along

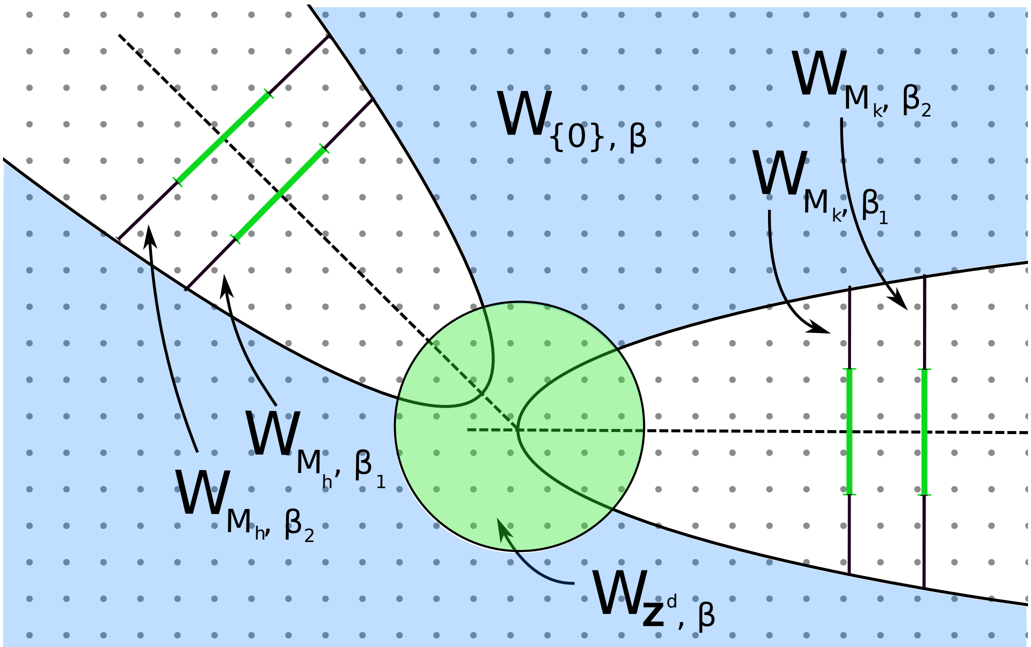

Figure 5.1: A cartoon of the block decomposition described in Theorem 5.1.10 below.

the directions contained in M. Thus if M is a submodule of $\mathbb{Z}^2$ it can be $\{0\}, \mathbb{Z}^2$ itself, or one dimensional, namely of the form $M = M_k = \mathrm{span}_{\mathbb{Z}}\{k\}$ for some $k \in \mathbb{Z}^2$. In Figure 5.1 there is a cartoon of such a block decomposition, where for simplicity only the blocks related to $M = \{0\}$ (in blue), $M = \mathbb{Z}^2$ (in green), $M = M_k$ with $k = \mathbf{e}_2$ and $M = M_h$ with $h = \mathbf{e}_1 + \mathbf{e}_2$ are represented (here as usual $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the standard basis in $\mathbb{R}^2$). More precisely:

- The blue block labeled by $M = \{0\}$ corresponds to the region described in Theorem 4.2.1; it coincides with non-resonant sites.

- On the opposite, the green block labeled by $M = \mathbb{Z}^2$ corresponds to those sites $\xi \in \mathbb{Z}^2$ which have the maximum number of linearly independent resonances, namely $d' = 2$: the Structure Theorem 5.1.10 claims that it is finite dimensional, and its cardinality only depends on some fixed parameters of the system.

- The resonant regions corresponding to sites which are affected by resonances in only one direction are represented in white: if the block is labeled by M_k, such a resonant direction is indeed the one parallel to k. As shown in Figure 5.1, these resonant regions roughly correspond to parabolic domains, with the axes of the parabolas orthogonal to the vector k.

Inside each one of these white regions corresponding to a resonance with the vector k, the operator $\widetilde{H}$ turns out to have non vanishing matrix elements $\langle \widetilde{H} e^{\mathrm{i}\xi\cdot x}, e^{\mathrm{i}\zeta\cdot x}\rangle$ only if the two sites ξ, ζ belong to the same plane parallel to k; this is the analogue to the classical phenomenon of motion along *fast drift planes* in Nekhoroshev's theory, which shows that the motion inside a resonance region happens only in the directions parallel to the resonances affecting such a region. Here comes the role played by the second label β: each white resonant region corresponding to a module M_k can be intersected with an affine plane parallel to k, namely of the form $M_k + \beta$, and such blocks, pictured in black in Figure 5.1, are then left invariant by $\widetilde{H}$.
The Structure Theorem also claims that the restriction of $\widetilde{H}$ to any one of the blocks $\mathcal{W}_{M_k,\beta}$ is still (unitarily equivalent to) a one dimensional Schrödinger operator: geometrically, the Gauge map realizing the equivalence is just the translation of the hyperplane $M_k + \beta$ at the origin (see Remark 5.1.8).
One is then left with a collection of Schrödinger operators in dimension 1, which we call $H^{(1)}_{M,\beta}$; we finally point out that it is possible to apply again the Structure Theorem to any one of these operators. Due to the fact that such operators are now one dimensional, this time only two possibilities occur: either the sites are non-resonant (in black in Figure 5.1), or they have the maximum possible number of resonances (in green in Figure 5.1), thus their dimension is finite and only depends on the parameters of the system.

5.1.2 Structure Theorem: rigorous statement

The following theorem is the main result of the present section:

Theorem 5.1.10 (Structure Theorem). *Given $\epsilon, \delta \in \mathbb{R}^+$ and $\tau > d$ such that*

$$\delta + d(d+\tau+1)\epsilon < 1\,, \quad \epsilon(\tau+1) \leq \delta\,, \tag{5.1.10}$$

a Floquet parameter κ and a flat metric g, there exists a partition of $\mathbb{Z}^d$:

$$\mathbb{Z}^d = \bigcup_{M \subseteq \mathbb{Z}^d} \bigcup_{\beta \in \widetilde{M}} W_{M,\beta} \tag{5.1.11}$$

where M runs over the submoduli of $\mathbb{Z}^d$ and $\widetilde{M}$ is a subset of M^c. The set $E_{\{0\}} := \bigcup_\beta W_{\{0\},\beta}$ has density one at infinity, $W_{\mathbb{Z}^d,\{0\}}$ has cardinality bounded *by an integer n_* which depends on the constants of the metric (see definitions* (3.1.8) *and* (3.1.9) *given in Chapter 3) and on d, δ, ϵ, τ only.*

Consider the operator (5.0.1) *and assume that $V \in OPS^{0,\delta}$, then $\forall n > 0$ there exists a unitary transformation U, which fulfills*

$$U - \mathbb{I}\,,\ U^{-1} - \mathbb{I} \in OPS^{-\delta,\delta} \tag{5.1.12}$$

and is s.t.

$$UHU^{-1} = \widetilde{H} + R\ , \tag{5.1.13}$$

where

1. $R \in OPS^{-2n\delta,\delta}$
2. $\widetilde{H}$ *leaves invariant the subspaces generated by $W_{M,\beta}$ (according to Definition 5.1.1) for all M and $\beta \in \widetilde{M}$.*
3. *Furthermore*
   (a) $\forall\beta$, $\widetilde{H}_{\mathcal{W}_{\{0\},\beta}} \equiv \widetilde{H}\big|_{\mathcal{W}_{\{0\},\beta}}$ *is a Fourier multiplier*
   (b) *$\forall M$ proper submodule and $\forall\beta \in \widetilde{M}$, one has that $H^{(1)}_{M,\beta} := U^*_\beta \tilde{H}_{\mathcal{W}_{M,\beta}} U_\beta$ is a* Schrödinger operator of dimension $d' = dimM$, *in the sense that introducing coordinates adapted to M, it takes the form*

$$H^{(1)}_{M,\beta} = \Pi_{\mathcal{W}^t_{M,\beta}} \left(-\Delta_{\mathsf{g},\kappa_\beta} + V_{M,\beta} + \left\| (\widetilde{\beta} + \kappa)_{M^\perp} \right\|^2_{\mathsf{g}*} \right) \Pi_{\mathcal{W}^t_{M,\beta}}\ . \tag{5.1.14}$$

Here $-\Delta_{\mathsf{g},\kappa_\beta}$ is the d' dimensional Laplacian computed with respect to the restriction of the metric g^* to $\text{span}_{\mathbb{R}} M$ and with Floquet parameter $\kappa_\beta = \{(\beta + \kappa)_M\}$, and $V_{M,\beta}$ is a pseudodifferential operator of order 0 *(in d' dimensions)*

Furthermore, the seminorms of the operators U, R and $V_{M,\beta}$ only depend on the constants of the metric $\mathfrak{c}, \mathfrak{C}$, and on the seminorms of V.

Remark 5.1.11. *The partition of $\mathbb{Z}^d$ does not depend on the operator* (5.0.1)*, but only on the properties of the metrics, and on κ.*

Remark 5.1.12. *The theorem holds also if the initial operator* (5.0.1) *is replaced by the restriction of a Schrödinger operator to any finite subset of $\mathbb{Z}^d$, according to Definition 5.1.1, with the only exception that in such a case* *the set $E_{\{0\}}$ does not have, of course, density one at infinity. This is useful for an iteration of the construction, that will be actually performed in the following Chapter 6.*

Remark 5.1.13. *The restriction of the metric g^* to a modulus M has new constants which are controlled by the constants $\mathfrak{c}$ and $\mathfrak{C}$ of the initial metric (see respectively Remark C.0.2 and Lemma D.0.2 of the Appendix). This is a further useful fact for the iteration of the construction performed in Chapter 6.*

5.1.3 Comparison of the Structure Theorem with the results in [Par08, PS10, PS12]

The aforementioned papers [Par08, PS10, PS12] contain a weaker version of our Structure Theorem 5.1.10; here we briefly enlighten the main differences. As explained in Subsection 4.2.4 of Chapter 4, their construction requires to localize into annuli $\mathcal{A}_\rho$ of growing radius ρ (see for instance (4.2.41) of Chapter 4 for the precise definition given in [PS10]). The operator H is then conjugated to the sum of a normal form one plus a remainder term, and their normal form operator also admits a block diagonal decomposition. However, both the normal form operator and its decomposition in invariant blocks depend on the annulus $\mathcal{A}_\rho$, and the remainder term is actually negligible only when one localizes the operator on the subspace generated by $\mathcal{A}_\rho$, in the sense of Definition 5.1.1. When summing up the contributions coming from all annuli $\mathcal{A}_\rho$, one has that two blocks associated to close values of ρ are not necessarily disjoint, thus they do not provide a partition: as a consequence, their construction does not enable to conjugate H to the sum of a global normal form operator, admitting an invariant decomposition, and a remainder term, which is negligible on the whole space $L^2(\mathbb{T}^d)$.

Most of all, our result has two main differences with respect to all previous constructions: first, the approach of the papers [Par08, PS10, PS12] does not allow to describe the normal form operator in each block as a new Schrödinger operator on a lower dimensional torus (as stated in Item 3 of the Structural Theorem 5.1.10). Furthermore, we are able to show that in our construction the size of the finite dimensional block $W_{\mathbb{Z}^d,\{0\}}$ and the seminorms of all the operators involved in the normalizing process (the conjugating map U, the block diagonal operator $\widetilde{H}$, the remainder term R) only depend on quantities which do not change when the dimensional reduction is performed; see

Remark 5.1.13. This is what enables us, in Chapter 6 of the present thesis, to set up an iterative construction and to deduce global spectral asymptotics for the spectrum of H.
To this aim we point out that a further difference with the methods of [PS10, PS12] is that our construction is deeply based on the lattice structure of the set Γ^*, and on the module structure of its resonance subsets M: this plays a fundamental role in the identification of the normal form operator inside the resonant blocks with a Schrödinger operator on a lower dimensional torus (see Section 5.5).
Although this issue is not tackled in the present work, we finally remark that our result also has some further applications in the time dependent case, for which both the global nature of the normal form and its iterability are exploited. Indeed, if a smooth time dependence is added in the potential V, namely one considers a time dependent Schrödinger equation of the form

$$\mathrm{i}\partial_t \psi = (-\Delta + V(t,x))\,\psi \quad \text{on} \quad L^2(\mathbb{T}^d_\Gamma)\,, \tag{5.1.15}$$

our Structure Theorem enables to deduce the bounds on the growth in time of the Sobolev norms of the solutions of (5.1.15) that were proven in [BM19]. Furthermore, in the case of the standard torus $\mathbb{T}^d$, or of a rational torus, and assuming that V in (5.1.15) has a quasi-periodic dependence on time, we also expect that it enables to obtain an alternative proof of the reducibility result for (5.1.15) and of the existence of invariant tori for the associated nonlinear problem that were exhibited in [EK10].

5.1.4 Structure of the proof

The remaining part of the chapter is devoted to the proof of our Structure Theorem 5.1.10; more precisely, in Section 5.2 we exhibit the partition of $\mathbb{Z}^d$ which generates the invariant blocks of $L^2(\mathbb{T}^d)$, in Section 5.3 we apply Theorem 4.2.1 of Chapter 4 in order to conjugate the operator H to a suitable normal form operator, and in Section 5.4 we show the invariance of the blocks exhibited in Section 5.2. This proves Items 1, 2 and 3a of Theorem 5.1.10. Finally, in Section 5.5 we conclude the proof of Theorem 5.1.10 by proving Item 3b, namely that the restriction of the normalized operator on each of the invariant blocks can be conjugated to a Schrödinger operator on a lower dimensional torus.

5.2 Construction of the partition

We are now giving the construction of the sets $W_{M,\beta}$. This is a quantum analogue of the classical geometrical construction of Nekhoroshev theorem; a direct classical counterpart can be found in [BL20]. Roughly speaking, given a submodulus M of dimension s, the sets

$$E_M^{(s)} := \bigcup_{\beta \in \widetilde{M}} W_{M,\beta} \tag{5.2.1}$$

are composed by the points $\xi \in \mathbb{Z}^d$ which are resonant only with the integer vectors of M. In order to make the construction precise consider the classical symbol of $-\Delta_{\mathtt{g},\kappa}$, namely

$$h_0(\xi) = \|\xi + \kappa\|^2 \ ; \tag{5.2.2}$$

the frequencies of the corresponding classical motion are

$$\omega_j = \xi_j + \kappa_j \ ,$$

so that a point ξ is (exactly) resonant with some integer k if

$$\langle (\xi + \kappa); k \rangle = 0 \ .$$

The theory developed in Chapter 4 shows that a point can be considered non resonant with an integer vector k if

$$|\langle (\xi + \kappa); k \rangle| \geq \frac{\langle \xi + \kappa \rangle^{\delta}}{\|k\|^{\tau}} \ ;$$

furthermore, due to the decay of the Fourier coefficients of a smooth function, it is enough to consider the vectors k s.t.

$$\|k\| \leq \langle \xi \rangle^{\epsilon}$$

for some positive small ϵ.

So, in principle $E_M^{(s)}$ should be the set of the ξ which resonate with the vectors k belonging to M and having a not too large modulus. However this has to be modified, due to the translation by $\frac{k}{2}$ present in the definition of Weyl quantization (see Definition (3.2.4)). Furthermore, one has to modify the construction both in order to get that the sets $E_M^{(s)}$ do not overlap and in order to obtain invariant sets.

To start with we define the resonance zones, in which the following notation will be used:

Definition 5.2.1. *Given* $\xi \in \mathbb{R}^d$ *and* $k \in \mathbb{Z}^d$, *and a Floquet parameter* κ, *we define*

$$\xi^\kappa := \xi + \kappa \ , \tag{5.2.3}$$

$$\xi_k := \xi + \kappa + \frac{k}{2} \equiv \xi^\kappa + \frac{k}{2} \, . \tag{5.2.4}$$

Definition 5.2.2 (Resonant zones)**.** *Fix* δ, ϵ, τ *as in the statement of Theorem 5.1.10; fix also constants fufilling:*

$$\begin{gathered} \delta_0 = \delta \, , \\ \delta_{s+1} = \delta_s + (d+\tau+1)\epsilon \quad \forall s = 0, \dots, d-1 \, , \\ 1 = D_0 < D_1 < \cdots D_{d-1} \, , \\ 1 = C_0 < C_1 < \cdots C_{d-1} \, , \end{gathered}$$

then we define the following sets:

1.

$$\mathcal{Z}^{(0)} = \left\{ \xi \in \mathbb{Z}^d \ \middle| \ | \langle \xi_k ; k \rangle | > \langle \xi_k \rangle^\delta \|k\|^{-\tau} \quad \forall k \in \mathbb{Z}^d \ \ s. \ t. \ \|k\| \leq \langle \xi_k \rangle^\epsilon \right\} \tag{5.2.5}$$

2. *Given* $M \subseteq \mathbb{Z}^d$ *a resonance module of dimension* $s \geq 1$ *and* s *linearly independent vectors* $\{k_1, k_2, \dots, k_s\} \subset M$, *we define*

$$\mathcal{Z}_{k_1,\dots,k_s} = \Big\{ \xi \in \mathbb{Z}^d \ \Big| \ \forall j = 1, \dots s \quad | \langle \xi_{k_1} ; k_j \rangle | \leq C_{j-1} \langle \xi_{k_1} \rangle^{\delta_{j-1}} \|k_j\|^{-\tau} \\ \text{and } \|k_j\| \leq D_{j-1} \langle \xi_{k_1} \rangle^\epsilon \Big\} \tag{5.2.6}$$

and

$$\mathcal{Z}^{(s)}_M = \bigcup_{\substack{\{k_1,\dots,k_s\} \subset M \\ lin. \ ind.}} \mathcal{Z}_{k_1,\dots,k_s} \, . \tag{5.2.7}$$

Example 5.2.3. *Let* $d = 2$, $\kappa = 0$ *and* g *the Euclidean metric. Let furthermore* $k = \mathbf{e}_1$ *and* $M = M_k = \mathrm{span}_{\mathbb{Z}}\{k\}$ *: then one has*

$$\mathcal{Z}^{(1)}_{M_k} = \left\{ \xi \in \mathbb{Z}^2 \mid \exists n \in \mathbb{Z} \ s. \ t. \ \left| n \left(\xi_1 + \frac{n}{2} \right) \right| \leq \left(\left(\xi_1 + \frac{n}{2} \right)^2 + (\xi_2)^2 \right)^{\frac{\delta}{2}} |n|^{-\tau} \right. \\ \left. \text{and } |n| \leq \left(\left(\xi_1 + \frac{n}{2} \right)^2 + (\xi_2)^2 \right)^{\frac{\epsilon}{2}} \right\} .$$

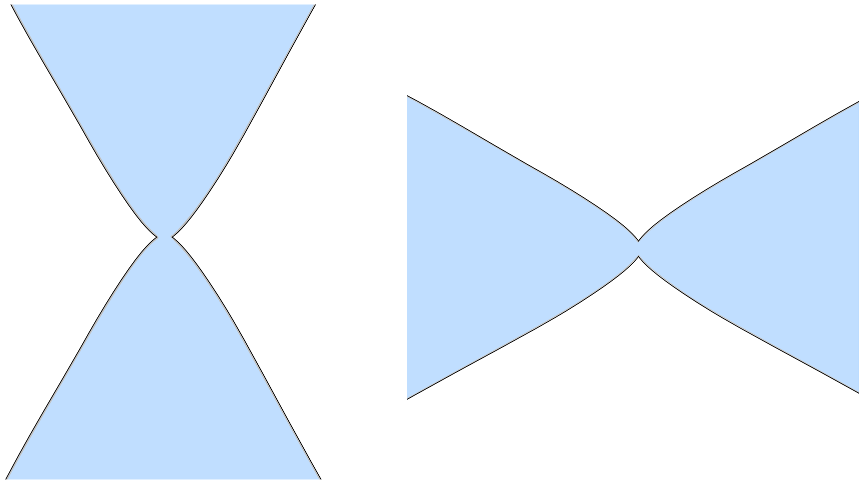

Figure 5.2: Qualitative representation of the one dimensional resonant zones $\mathcal{Z}^{(1)}_{M_k}$*, with* $M_k = \mathrm{span}_{\mathbb{Z}}\{\mathbf{e}_1\}$ *on the left and* $M_k = \mathrm{span}_{\mathbb{Z}}\{\mathbf{e}_2\}$ *on the right, in the case* $d = 2$*, with* $\kappa = 0$ *and* g *the Euclidean metric.*

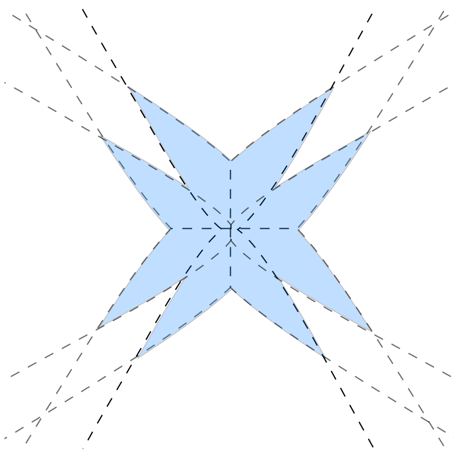

Figure 5.3: A cartoon of the two dimensional resonant zone $\mathcal{Z}^{(2)}_{\mathbb{Z}^d}$*; actually in the picture only the set* $\mathcal{Z}^{(2)}_{\mathbf{e}_1,\mathbf{e}_2} \cup \mathcal{Z}^{(2)}_{\mathbf{e}_2,\mathbf{e}_1}$ *has been represented for simplicity.*

Moreover, as follows from Lemma D.0.6 of the Appendix, there exist positive constants $C > 0$ *and* $D > 0$ *such that*

$$\mathcal{Z}^{(1)}_{M_k} \subseteq \left\{ \xi \in \mathbb{Z}^2 \mid \exists n \in \mathbb{Z} \ s.\ t.\ \ |n\xi_1| \le C\langle\xi\rangle^{\delta}|n|^{-\tau} \ and\ |n| \le D\langle\xi\rangle^{\epsilon} \right\}.$$

Remark 5.2.4. $\mathcal{Z}^{(0)}$ *defined as in* (5.2.5) *is our set of non resonant points. Recall that, in Chapter 4, the non resonant set* $\Omega \cap \mathbb{Z}^d$ *was defined as the set of points* ξ *such that*

$$|\langle\xi^{\kappa}; k\rangle| > \langle\xi^{\kappa}\rangle\|k\|^{-\tau} \quad \forall k \in \mathbb{Z}^d \quad s.t.\ 0 < \|k\| < \langle\xi^{\kappa}\rangle^{\epsilon},$$

namely requiring that the non resonance condition holds for $\xi^{\kappa} = \xi + \kappa$ *instead of the term* $\xi_k = \xi + \kappa + \frac{k}{2}$ *(see* (4.2.3) *). This is due to the fact that, in order to deduce the spectral Theorem 4.0.1, we exploited the correspondence between classical and Weyl quantization, up to smoothing operators. Here we do not exploit such a correspondence, since the proof of our structural result requires to work with self-adjoint operators, and classical quantization does not ensure self-adjointness of real symbols.*

Remark 5.2.5. *By* (5.2.5) *and* (5.2.6), $\forall s \ge 1$ *and* $\forall M$ *one has* $\mathcal{Z}^{(s)}_M \cap \mathcal{Z}^{(0)} = \emptyset$.

Lemma 5.2.6. *If* $1 \le r < s$*, then for any* M *with dim* $M = s$*, one has*

$$\mathcal{Z}^{(s)}_M \subseteq \bigcup_{\substack{M' \subset M \\ dim.M' = r}} \mathcal{Z}^{(r)}_{M'}.$$

Proof. If $\xi \in \mathcal{Z}^{(s)}_M$, then there exist $\{k_1, \dots, k_s\} \subset M$ such that $\xi \in \mathcal{Z}_{k_1,\dots,k_s}$. Let $r < s$: then it is sufficient to consider only the first r vectors $\{k_1, \dots, k_r\}$ of the $s-$uple $\{k_1, \dots, k_s\}$ and to observe that $\xi \in \mathcal{Z}_{k_1,\dots,k_r}$, which proves that $\xi \in \mathcal{Z}^{(r)}_{M'}$, with $M' = \text{span}\{k_1, \dots, k_r\} \subset \text{span}\{k_1, \dots, k_s\} \subseteq M$. □

The regions $\mathcal{Z}^{(s)}_M$ contain points $\xi \in \mathbb{Z}^d$ which are in resonance with *at least* s linearly independent vectors in M. Thus such regions are clearly not reciprocally disjoint, as pointed out in Lemma 5.2.6 above. We identify now the points $\xi \in \mathbb{Z}^d$ which admit *exactly* s linearly independent resonance relations.

Definition 5.2.7 (Resonant blocks)**.** *Consider the following sets:*

1.

$$B^{(d)} := \mathcal{Z}^{(d)}_{\mathbb{Z}^d}$$

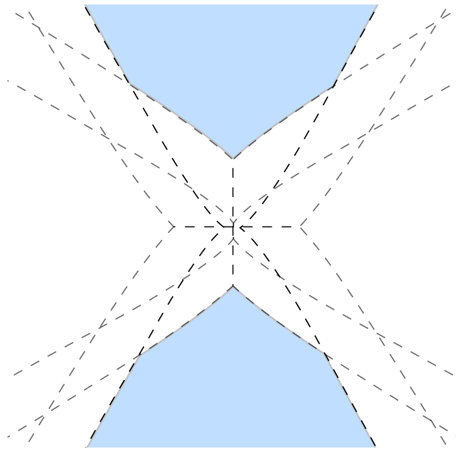

Figure 5.4: A cartoon of the resonant block $B^{(1)}_{M_k}$ with $M_k = \mathrm{span}_{\mathbb{Z}}\{\mathbf{e}_1\}$: one has to remove from the resonant zone $Z^{(1)}_{M_k}$ all the vectors which belong to $Z^{(2)}_{\mathbb{Z}^2}$.

2. *Given $M \subset \mathbb{Z}^d$ a resonance module of dimension $s \in \{1, \dots, d-1\}$,*

$$B^{(s)}_M := \mathcal{Z}^{(s)}_M \setminus \left\{ \bigcup_{M' s.t. \dim M' = s+1} \mathcal{Z}^{(s+1)}_{M'} \right\}$$

3.

$$B^{(0)} := \mathcal{Z}^{(0)} .$$

We say that B is a resonant block if $B = B^{(d)}$, $B = B^{(0)}$, or $B = B^{(s)}_M$ for some module M of dimension s.

Remark 5.2.8. *The resonant blocks form a covering of $\mathbb{Z}^d$.*

As proven below in Lemma 5.4.6 there exists a suitable choice of the constants $C_s,\ D_s,\ \delta_s$ such that two blocks $B^{(s)}_M,\ B^{(s)}_{M'}$ are disjoint, provided $M,\ M'$ are two distinct subspaces of equal dimension. Such a property plays a fundamental role in our construction.

Still the blocks defined in Definition 5.2.7 do not provide a suitable partition of $\mathbb{Z}^d$, since they are not left invariant by the operator $\widetilde{H}$ of eq. (5.1.13).

Recall now that, given two sets A and B, their Minkowski sum $A + B$ is defined by:

$$A + B := \{a + b \quad | \quad a \in A\,,\ b \in B\} .$$

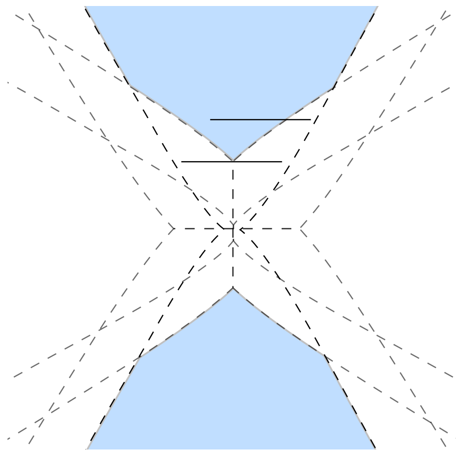

Figure 5.5: A cartoon of the resonant block $B^{(1)}_{M_k}$ with $M_k = \mathrm{span}_{\mathbb{Z}}\{\mathbf{e}_1\}$: the segments of the form $\xi + k$ with $k = \mathbf{e}_1$ and $\xi \in B^{(1)}_{M_k}$ are not included inside the block $B^{(1)}_{M_k}$.

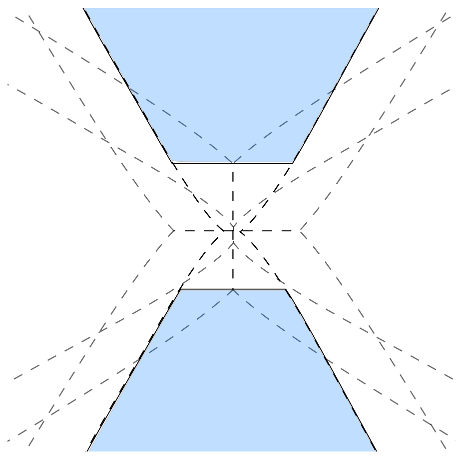

Figure 5.6: A cartoon of the extended block $E^{(1)}_{M_k}$ with $M_k = \mathrm{span}_{\mathbb{Z}}\{\mathbf{e}_1\}$.

Definition 5.2.9 (Extended blocks)**.**

1. $E^{(0)} := B^{(0)} \equiv \mathcal{Z}^{(0)}$

2. *Given a resonance module* M *of dimension* 1 *we define*

$$E_M^{(1)} := \left\{ B_M^{(1)} + M \right\} \cap \mathcal{Z}_M^{(1)} ,$$

$$E^{(1)} := \bigcup_{M \text{ of dim. } 1} E_M^{(1)}$$

3. *Given a resonance module* M *of dimension* s*, with* $2 \leq s \leq d$*, we define*

$$E_M^{(s)} := \left\{ B_M^{(s)} + M \right\} \cap \mathcal{Z}_M^{(s)} \cap \left(E^{(s-1)} \right)^c \cdots \cap \cdots \cap \left(E^{(1)} \right)^c ;$$

correspondingly we define

$$E^{(s)} := \bigcup_{M \text{ of dim. } s} E_M^{(s)} .$$

Remark 5.2.10. *The blocks* $\{E_M^{(s)}\}_{M,s}$*,* $E^{(0)}$*,* $E^{(d)}$ *form a covering of* $\mathbb{Z}^d$*. Actually, as shown in Theorem 5.4.8 below, they form a partition of* $\mathbb{Z}^d$*.*

The decomposition $\mathbb{Z}^d = \bigcup_M E_M^{(s)}$, as we are going to show in Theorem 5.4.10 below, is also left invariant by the operator $\widetilde{H}$ of Theorem 5.1.10. Furthermore the sets $E_M^{(s)}$ can still be decomposed in invariant subsets which are given by

$$W_{M,\beta} := E_M^{(s)} \cap \{\beta + M\} . \tag{5.2.8}$$

Definition 5.2.11. *The set of the* $\beta \in M^c$ *s.t. the set* (5.2.8) *is not empty is denoted by* $\widetilde{M}$*.*

Remark 5.2.12. *As will be proven in Section 5.4, the sets* $W_{M,\beta}$ *of Theorem 5.1.10 are the sets defined by equation* (5.2.8) *.*

Remark 5.2.13. *We point out that there are strong similarities between the block decomposition of* $\mathbb{Z}^d$ *that we give in the present section and the* stratification by cuts *exhibited in [PP15] (see also [PX13]). Roughly speaking, our resonant blocks of dimension* s *correspond to points* having a cut at s, *according to Definition 3.6 of [PX13]. However, since we are interested in exhibiting a partition in subsets that are left invariant by a normal form operator, we introduce extended blocks and their lower dimensional subsets* $W_{M,\beta}$*.*

5.3 Analytic part

In this section we exploit the normal form construction of Chapter 4 in order to deal also with the resonant regions of the phase space.

Definition 5.3.1. *A symbol* $z(x,\xi)=\sum_{k\in\mathbb{Z}^d}\hat{z}_k(\xi)e^{\mathrm{i}k\cdot x}\in S^{0,\delta}$ *is said to be in (resonant) normal form if,* $\forall k\in\mathbb{Z}^d\backslash\{0\}$ *,*

$$supp(\hat{z}_k)\subseteq\left\{\xi\in\mathbb{R}^d\ \middle|\ |\langle\xi+\kappa;k\rangle|\le\langle\xi+\kappa\rangle^{\delta}\|k\|^{-\tau}\ and\quad\|k\|\le\langle\xi+\kappa\rangle^{\epsilon}\right\}. \tag{5.3.1}$$

Remark 5.3.2. *According to the above Definition 5.3.1, the symbol* $z^{(n)}$ *in* (4.2.5) *of Theorem 4.2.1 is in normal form. Indeed, recall that one has* $z^{(n)}=[z^{(n)}]+z^{(n,res)}$*, with* $z^{(n,res)}(x,\xi)=0$ *on* Ω*. By definition of* Ω *as in* (4.2.3)*, for any* $k\in\mathbb{Z}^d\backslash\{0\}$ *the set in the right hand side of* (5.3.1) *is contained in* Ω^C *.*

Definition 5.3.3. *Let* $M\subset\mathbb{Z}^d$ *be a module, then a symbol* $z\in S^{0,\delta}$ *is said to be in* normal form with respect to M *if it is in normal form and furthermore its Fourier transform is given by*

$$z(x,\xi)=\sum_{k\in M}\hat{z}_k(\xi)e^{\mathrm{i}k\cdot x}\,. \tag{5.3.2}$$

Definition 5.3.4. *A pseudodifferential operator will be said to be in normal form (resp. normal form with respect to a module* M*) if the corresponding symbol is in normal form (resp. normal form with respct to a module* M*).*

The following result follows from Theorem 4.2.1 of Chapter 4:

Theorem 5.3.5. *Consider the operator* $H=-\Delta_{\mathtt{g},\kappa}+V$ *, with* $V=Op^W(v)\in OPS^{0,\delta}$ *. For all* $n>0$ *there exists a unitary transformation* $U=U_n$ *such that*

1)

$$U_n-\mathbb{I}\,,\ U_n^{-1}-\mathbb{I}\in OPS^{-\delta,\delta} \tag{5.3.3}$$

$$U_nHU_n^{-1}=\mathcal{L}_n=\widetilde{H}_n+R_n \tag{5.3.4}$$

with $R_n\in OPS^{-2\delta n,\delta}$*, and*

$$\widetilde{H}_n=-\Delta_{\mathtt{g},\kappa}+Z^{(n)}\,, \tag{5.3.5}$$

where $Z^{(n)} \in OPS^{0,\delta}$ is in resonant normal form.
Furthermore, the families of seminorms of the operators $Z^{(n)}, R_n, U_n$ only depend on the family of seminorms of the operator V and on the constants of the metric, as well as on n, on d and on the parameters δ, ϵ, τ .

2) Let $E \subset \mathbb{Z}^d$ be a subset and let $\mathcal{E}$ be the space it generates according to (5.1.1)*. If V leaves $\mathcal{E}$ invariant, namely $[V, \Pi_{\mathcal{E}}] = 0$, then one has*

$$[U_n \, , \Pi_{\mathcal{E}}] = 0 \, . \tag{5.3.6}$$

Remark 5.3.6. *Assume that V leaves invariant a subspace $\mathcal{E}$ of the form* (5.1.1)*. Then, by Item 2 of Theorem 5.3.5, for any $n \in \mathbb{N}$ one also has*

$$U_n \Pi_{\mathcal{E}} H \Pi_{\mathcal{E}} U_n^{-1} = \Pi_{\mathcal{E}} H_n \Pi_{\mathcal{E}} \, .$$

Proof of Thm 5.3.5. Item 1 follows from Theorem 4.2.1 of Chapter 4, recalling that, as observed in Remark 5.3.2, for all j $Z^{(j)} = Op\,(z^{(j)})$ is a normal form operator. To prove Item 2, namely the commutation relation (5.3.6), we proceed inductively. First of all we observe that, in the case $j = 0$, $Z^{(0)} = 0$ and

$$[R_0, \Pi_{\mathcal{E}}] = [V, \Pi_{\mathcal{E}}] = 0 \, .$$

Let us now fix some $j \geq 0$ and suppose that $Z^{(j)}$ and R_j commute with $\Pi_{\mathcal{E}}$.

Given a self-adjoint operator A, since $\mathcal{E}$ has the form $\mathcal{E} = \overline{\text{span}\{e^{\mathrm{i}k\cdot x} \mid k \in E\}}$ for some $E \subseteq \mathbb{Z}^d$, first of all we observe that condition $[A, \Pi_{\mathcal{E}}] = 0$ holds if and only if

$$\left(k \in E \quad A_k^{k'} \neq 0\right) \Rightarrow k' \in E \, , \tag{5.3.7}$$

where $A_k^{k'} = \dfrac{1}{\mu_g(\mathbb{T}^d)} \langle A e^{\mathrm{i}k\cdot x}, \; e^{\mathrm{i}k'\cdot x} \rangle$ are the matrix elements of A with respect to the basis of Fourier modes. Furthermore, by definition of Weyl quantization one has that, if $A = Op\,(a)$,

$$A_k^{k'} = \hat{a}_{k'-k} \left(\frac{k + k'}{2} \right) \, . \tag{5.3.8}$$

Due to definitions (4.2.9) of the symbols $r_j^{(\mathrm{nr})}$ and $r_j^{(\mathrm{res})}$, equation (5.3.8) immediately implies that

$$\left(Op\,\left(r_j^{(\mathrm{nr})}\right)\right)_k^{k'} \neq 0 \, , \text{ or } \; \left(Op\,\left(r_j^{(\mathrm{res})}\right)\right)_k^{k'} \neq 0 \text{ for some } k, k' \in \mathbb{Z}^d \, ,$$
$$\Longrightarrow (R_j)_k^{k'} \neq 0 \, ,$$

Similarly, recall that U_n has the form $U_n = e^{\mathrm{i}G_n} \circ \ldots \circ e^{\mathrm{i}G_1}$, with $G_j = Op\,(g_j)$ and g_j defined

$$(G_j)_k^{k'} \neq 0 \quad \Longrightarrow (R_j)_k^{k'} \neq 0\,.$$

This, together with condition (5.3.7), enables to conclude that G_j commutes with $\Pi_{\mathcal{E}}$, and so do $Op\,(r_j^{(\mathrm{res})})$ and $Op\,(r_j^{(\mathrm{nr})})$. Hence $e^{\mathrm{i}G_j}$ commutes with $\Pi_{\mathcal{E}}$, since G_j does. The same holds for $Z^{(j+1)}$, being

$$[Z^{(j+1)}, \Pi_{\mathcal{E}}] = [Z^{(j)}, \Pi_{\mathcal{E}}] + [Op\,([r_j]), \Pi_{\mathcal{E}}] + [Op\,\left(r_j^{(\mathrm{res})}\right), \Pi_{\mathcal{E}}] = 0\,,$$

and

$$R_{j+1} = e^{\mathrm{i}G_j} H_j e^{-\mathrm{i}G_j} - \left(-\Delta_{\mathsf{g},\kappa} + Z^{(j+1)}\right)\ .$$

□

5.4 Geometric part

In order to iterate Theorem 5.1.10 we will have to work in a subspace $\mathcal{E}$ of $L^2(\mathbb{T}^d)$ generated by some subset $E \subset \mathbb{Z}^d$. For such a reason, in the present section we fix $E \subseteq \mathbb{Z}^d$ and we replace the extended blocks $E_M^{(s)}$ of Definition (5.2.9) with $E_M^{(s)} \cap E$, which we still denote by $E_M^{(s)}$. We do the same for the blocks $B_M^{(s)}$ and for the zones $Z_M^{(s)}$ of Definitions 5.2.2, 5.2.7.

5.4.1 Properties of the extended blocks $E_M^{(s)}$: non overlapping of resonances.

We show here that the extended blocks $E_M^{(s)}$ form a partition of $\mathbb{Z}^d$ and prove some properties which are needed in order to show that they are left invariant by an operator in normal form. As in the proof of the classical Nekhoroshev Theorem, the following Lemma plays a fundamental role.

Lemma 5.4.1. *Let $s \in \{1\,, \ldots\,, d\}$ and let $\{u_1\,, \ldots u_s\}$ be linearly independent vectors in $\mathbb{R}^d$. Let $w \in \mathrm{span}\{u_1\,, \ldots u_s\}$ be any vector. If $\alpha\,, N$ are such that*

$$\|u_j\| \leq N \quad \forall j = 1\,, \ldots s\,,$$
$$|\,\langle w; u_j\rangle\,| \leq \alpha \quad \forall j = 1\,, \ldots s\,,$$

then

$$\|w\| \leq \frac{sN^{s-1}\alpha}{Vol_{\mathsf{g}}\{u_1\,|\cdots\,|u_s\}}\,.$$

This is just a coordinate free formulation of Lemma 5.7 of [Gio03], which is recalled in the appendix as Lemma D.0.1. By (3.1.9), one also has

$$\|w\| \leq sN^{s-1}\alpha\mathfrak{C}^{-1} . \tag{5.4.1}$$

We state now a couple of simple properties of the extended blocks.

Lemma 5.4.2. *The extended block $E^{(d)}$ is finite dimensional; in particular, there exists a positive $n_* = n_*(\mathfrak{c}, \mathfrak{C}, \epsilon, \tau, \delta_{d-1}, C_{d-1}, D_{d-1})$ such that*

$$E^{(d)} \subseteq \left\{\xi \in \mathbb{R}^d \ \mid \|\xi + \kappa\| \leq n_*\right\} .$$

Proof. If $\xi \in E^{(d)}$, in particular there exist $\{k_1, \ldots, k_d\} \subset \mathbb{Z}^d$ linear independent vectors such that

$$\begin{aligned} \|k_1\| &\leq D_0\langle\xi_{k_1}\rangle^{\epsilon} , \\ \|k_j\| &\leq D_{j-1}\langle\xi_{k_1}\rangle^{\epsilon} \leq D_{d-1}\langle\xi_{k_1}\rangle^{\epsilon} , \\ |\left\langle\xi_{k_1}; k_j\right\rangle| &\leq C_{d-1}\langle\xi_{k_1}\rangle^{\delta_{d-1}}\|k_j\|^{-\tau} . \end{aligned}$$

In order to eliminate the indexes k_1 from ξ, we apply Lemma D.0.6, with $\varsigma = \eta = \xi^{\kappa}$, $l = 0$, $h = k_j$ and $k = \frac{k_1}{2}$ to deduce that there exist constants

$$C' = C'(\mathfrak{c}, \epsilon, \tau, \delta_{d-1}, D_{d-1}, C_{d-1}) , \quad D' = D'(\mathfrak{c}, \epsilon, \tau, \delta_{d-1}, D_{d-1}, C_{d-1})$$

such that

$$|\left\langle\xi^{\kappa}; k_j\right\rangle| \leq C'\langle\xi^{\kappa}\rangle^{\delta}\|k_j\|^{-\tau} , \quad \|k_j\| \leq D'\langle\xi^{\kappa}\rangle^{\epsilon} .$$

Recalling that $\mathfrak{c}$ is such that, for all $h \in \mathbb{Z}^d$, $\|h\|^2 \geq \mathfrak{c}$ and using Lemma 5.4.1, and Eq. (5.4.1) we have

$$\|\xi^{\kappa}\| \leq d\mathfrak{c}^{-\tau/2}\mathfrak{C}^{-1}C' \left(D'\right)^{d-1}\langle\xi^{\kappa}\rangle^{d\epsilon+\delta} ,$$

which, applying Remark D.0.4 with $a = d\varepsilon + \delta < 1$, implies the existence of a constant $n_* = n_*(\delta, \epsilon, \tau, C', D', \mathfrak{c}, \mathfrak{C})$ such that $\|\xi^{\kappa}\| < \bar{N}$. □

Lemma 5.4.3. *If $E = \mathbb{Z}^d$, the set $E^{(0)}$ is of density one at infinity, namely*

$$\lim_{R\to\infty} \frac{\sharp\left(E^{(0)} \cap B_R(0)\right)}{\sharp\left(\mathbb{Z}^d \cap B_R(0)\right)} = 1 .$$

Proof. We exploit the fact that a set is of density one at infinity if and only if its complementary set is of density zero, and we analyze the complementary set of $E^{(0)}$. Recall that $E^{(0)} = \mathcal{Z}^{(0)}$ so that, then, by Definition 5.2.2, its complementary set is

$$\mathbb{Z}^d \backslash E^{(0)} = \bigcup_{M \text{ of dim. } 1} \mathcal{Z}_M^{(1)} = \{\xi \in \mathbb{Z}^d \mid \exists k \in \mathbb{Z}^d \text{ s. t. } |\langle \xi_k ; k \rangle| \leq \langle \xi_k \rangle^{\delta} \|k\|^{-\tau},\ \|k\| \leq \langle \xi_k \rangle^{\epsilon}\}.$$

By Lemma D.0.6 there exists constants C', D' depending only on $\delta, \epsilon, \tau, \mathfrak{c}, \mathfrak{C}$ such that

$$\mathbb{Z}^d \backslash E^{(0)} \subseteq \{\xi \in \mathbb{Z}^d \mid \exists k \in \mathbb{Z}^d \text{ s. t. } |\langle \xi ; k \rangle| \leq C' \langle \xi \rangle^{\delta} \|k\|^{-\tau},\ \|k\| \leq D' \langle \xi \rangle^{\epsilon}\}.$$

But the latter is the complementary set to

$$\widetilde{\Omega} = \{\xi \in \mathbb{Z}^d \mid |\langle \xi_k ; k \rangle| > C' \langle \xi_k \rangle^{\delta} \|k\|^{-\tau} \quad \forall k \in \mathbb{Z}^d \ \text{ s. t. } \|k\| \leq D' \langle \xi_k \rangle^{\epsilon}\}.$$

Then the result follows arguing as in Section 4.2.3 of Chapter 4, in order to prove that Ω defined as in (4.2.3) is a density one set. □

We now analyze the other blocks.

First remark that, if $s' \neq s$, then two extended blocks $E_M^{(s)}$ and $E_{M'}^{(s')}$ are disjoint. Then we have to prove that two different extended blocks of the same dimension do not intersect. To this end a further geometric analysis is required.

Lemma 5.4.4. *If $\xi \in \mathcal{Z}_M^{(s)}$ then there exists a positive constant K depending only on $\mathfrak{c}, \mathfrak{C}, d, \epsilon, \tau, \delta_{s-1}, C_{s-1}, D_{s-1}$, such that*

$$\|(\xi^{\kappa})_M\| \leq K \langle \xi^{\kappa} \rangle^{\delta_{s-1} + d\epsilon}. \tag{5.4.2}$$

Proof. Since $\xi \in \mathcal{Z}_M^{(s)}$, there exist $\{k_1, \dots, k_s\} \subset M$ linearly independent vectors such that for all $j = 1, \dots, s$

$$|\langle (\xi_{k_1})_M ; k_j \rangle| = |\langle \xi_{k_1} ; k_j \rangle| \leq C_{j-1} \langle \xi_{k_1} \rangle^{\delta_{j-1}} \|k_j\|^{-\tau}, \quad \|k_j\| \leq D_{j-1} \langle \xi_{k_1} \rangle^{\epsilon}. \tag{5.4.3}$$

Then, by Lemma D.0.6 one can substitute in the above formulae ξ^{κ} to ξ_{k_1}; precisely, there exist two positive constants $C',\ D' = C', D'(\mathfrak{c}, \epsilon, \delta_{s-1}, C_{s-1}, D_{s-1})$, such that,

$$|\langle (\xi^{\kappa})_M ; k_j \rangle| = |\langle \xi^{\kappa} ; k_j \rangle| \leq C' \langle \xi^{\kappa} \rangle^{\delta_{s-1}} \|k_j\|^{-\tau} \leq C' \mathfrak{c}^{-\tau/2} \langle \xi^{\kappa} \rangle^{\delta_{s-1}},$$
$$\|k_j\| \leq D' \langle \xi^{\kappa} \rangle^{\epsilon}.$$

By Lemma 5.4.1, there exists $C = C(d)$ such that

$$\|(\xi^\kappa)_M\| \leq C(d) \frac{(D')^d \langle \xi^\kappa \rangle^{d\varepsilon}}{\mathrm{Vol}_{\mathfrak{g}}(k_1|\cdots|k_s)} C' \mathfrak{c}^{-\tau/2} \langle \xi^\kappa \rangle^{\delta_{s-1}} ,$$

and therefore, recalling that $\mathrm{Vol}_{\mathfrak{g}}(k_1|\cdots|k_s) \geq \mathfrak{C}$ (see the definition of $\mathfrak{C}$ as in (3.1.9)), the thesis holds. □

By definition, the points belonging to a block $B_M^{(s)}$ are resonant only with vectors $k \in M$. A priori, this property does not hold true for points in the extended block $E_M^{(s)}$. So we need an estimate of the distance between $E_M^{(s)}$ and $B_M^{(s)}$.

Lemma 5.4.5. *Let $\delta_{s-1} + d\epsilon < 1$ and M with $dimM = s$; if $\zeta \in E_M^{(s)}$ then there exists $\xi \in B_M^{(s)}$ and a positive constant F depending only on $\mathfrak{c}, \mathfrak{C}, d, \epsilon, \tau, \delta_{s-1}, C_{s-1}, D_{s-1}$ such that*

$$\|\xi - \zeta\| \leq F \langle \xi^\kappa \rangle^{\delta_{s-1}+\epsilon d} , \quad \|\xi - \zeta\| \leq F \langle \zeta^\kappa \rangle^{\delta_{s-1}+\epsilon d} \tag{5.4.4}$$

Proof. If $\zeta \in E_M^{(s)}$, then in particular $\zeta \in \mathcal{Z}_M^{(s)}$ and there exists a point $\xi \in B_M^{(s)}$ such that $\zeta = \xi + \upsilon$, with $\upsilon \in M$. In particular, $(\xi)_M^\perp = (\zeta)_M^\perp$, hence one has

$$\|\xi - \zeta\| = \| \,(\xi - \zeta)_M\, \| \leq \|(\xi^\kappa)_M\|\| + \|(\zeta^\kappa)_M\| .$$

Since $\xi \in \mathcal{Z}_M^{(s)}$ and $\zeta \in \mathcal{Z}_M^{(s)}$, due to Lemma 5.4.4, there exists K, such that

$$\|(\xi^\kappa)_M\| \leq K \langle \xi^\kappa \rangle^{d\epsilon+\delta_{s-1}} , \quad \|(\zeta^\kappa)_M\| \leq K \langle \zeta^\kappa \rangle^{d\epsilon+\delta_{s-1}} . \tag{5.4.5}$$

Exploiting Remark D.0.4 of Appendix D with $a = \delta_{s-1} + \epsilon d$, one gets

$$\langle \zeta^\kappa \rangle^a = \langle \xi + \kappa + \zeta - \xi \rangle^a \leq K' (\langle \xi^\kappa \rangle^a + \|\zeta - \xi\|^a)$$

and, exploiting Lemma D.0.5 of Appendix D, we immediately get

$$\|\zeta - \xi\| \leq F \langle \xi^\kappa \rangle^a .$$

Inverting the role of ξ and ζ one gets the other estimate. □

The next two lemmata ensure that, if the parameters C_j, D_j are suitably chosen for all j, an extended block $E_M^{(s)}$ is far from every resonant zone associated to a lower dimensional modulus M' which is not contained in M.

Lemma 5.4.6. *[Non overlapping of resonances] For all $s = 1, \dots d-1$ there exist positive constants $\bar{C}_s$ and $\bar{D}_s$, depending only on $\mathfrak{c}, \mathfrak{C}, d, C_{s-1}, D_{s-1}, \epsilon, \delta_{s-1}, \tau$, such that the following holds: suppose that M and M' are two distinct resonance modules of respective dimensions s and s' with $s' \leq s$ and $M' \nsubseteq M$. If*

$$C_s > \bar{C}_s \, , \quad D_s > \bar{D}_s \, ,$$

then

$$E_M^{(s)} \cap Z_{M'}^{(s')} = \emptyset \, .$$

Proof. Assume by contradiction, that there exists $\zeta \in E_M^{(s)} \cap \mathcal{Z}_{M'}^{(s')}$ then there exists $\xi \in B_M^{(s)}$ s.t. (5.4.4) holds.

Since $\zeta \in \mathcal{Z}_{M'}^{(s')}$, there exist s' integer vectors, $k_1, ..., k_{s'} \in M'$ *among which at least one does not belong to* M s.t.

$$|\langle \zeta_{k_1}; k_j \rangle| \leq C_{j-1} \langle \zeta_{k_1} \rangle^{\delta_{j-1}} \|k_j\|^{-\tau} \, , \quad \|k_j\| \leq D_{j-1} \langle \zeta_{k_1} \rangle^{\epsilon} \, . \tag{5.4.6}$$

Let $k_{\bar{\jmath}}$ be the vector which does not belong to M; the idea is to show that the resonance relation of ζ with $k_{\bar{\jmath}}$ implies an analogous relation for ξ, but this will be in contradiction with the fact that $\xi \in B_M^{(s)}$ (which contains vectors that are *only* resonant with M).

To start with remark that, since $\xi \in B_M^{(s)} \subset \mathcal{Z}_M^{(s)}$, there exist $l_1, ..., l_s \in M$, linearly independent, s.t.

$$|\langle \xi_{l_1}; l_j \rangle| \leq C_{j-1} \langle \xi_{l_1} \rangle^{\delta_{j-1}} \|l_j\|^{-\tau} \, , \quad \|l_j\| \leq D_{j-1} \langle \xi_{l_1} \rangle^{\epsilon} \, . \tag{5.4.7}$$

We now apply Lemma D.0.6 of Appendix D with $h := k_{\bar{\jmath}}/2$, $\ell := l_1/2$, $\varsigma := \zeta + \kappa$, $\eta := \xi + \kappa$. So, (D.0.12) implies

$$|\langle \xi_{l_1}; k_{\bar{\jmath}} \rangle| \leq K' \langle \xi_{l_1} \rangle^{\delta_{s-1} + \epsilon(d+\tau+1)} \|k_{\bar{\jmath}}\|^{-\tau} \, , \quad \|k_{\bar{\jmath}}\| \leq D' \langle \xi_{l_1} \rangle^{\epsilon} \, .$$

But, if $C_s > K'$, $D_s > D'$ and $\delta_s \geq \delta_{s-1} + \epsilon(d + \tau + 1)$, this means that ξ is also resonant with $k_{\bar{\jmath}}$, and thus it belongs to $\mathcal{Z}_{M''}^{(s+1)}$ with $M'' := span_{\mathbb{Z}}(M, k_{\bar{\jmath}})$, but this contradicts the fact that $\xi \in B_M^{(s)}$. □

Lemma 5.4.7. *[Separation of resonances] There exist positive constants $\tilde{C}_s$ and $\tilde{D}_s$ depending only on $\mathfrak{c}, \mathfrak{C}, d, \epsilon, \tau, \delta_{s-1}, C_{s-1}, D_{s-1}$ such that, if*

$$C_s > \tilde{C}_s \, , \quad D_s > \tilde{D}_s ,$$

then the following holds true. Let $\zeta \in E_M^{(s)}$ *for some* M *of dimension* $s = 1, \ldots, d-1$*, and let* k' *be such that*

$$\|k'\| \leq \langle \zeta_{k'} \rangle^{\epsilon} ,$$

then $\forall M' \not\subset M$ *s. t.* $s' := \dim M' \leq s$ *one has*

$$\zeta + k' \notin \mathcal{Z}_{M'}^{(s')} \quad .$$

Proof. The proof is very similar to that of Lemma 5.4.6. Assume by contradiction that $\zeta + k' \in \mathcal{Z}_{M'}^{(s')}$ for some $M' \neq M$. It follows that there exist s integer vectors, $k_1, ..., k_{s'} \in M'$ *among which at least one does not belong to* M s.t.

$$|\langle \zeta_{k_1} + k'; k_j \rangle| \leq C_{j-1} \langle \zeta_{k_1} + k' \rangle^{\delta_{j-1}} \|k_j\|^{-\tau} , \quad \|k_j\| \leq D_{j-1} \langle \zeta_{k_1} + k' \rangle^{\epsilon} . \tag{5.4.8}$$

Let $k_{\bar{\jmath}}$ be the vector which does not belong to M. By (5.4.4) there exists $\xi \in B_M^{(s)}$ s.t. $\|\xi - \zeta\| \leq F \langle \xi^{\kappa} \rangle^{\delta_{s-1} + \epsilon d}$. Since in particular $\xi \in \mathcal{Z}_M^{(s)}$ there exist $l_1, ..., l_s \in M$, linearly independent, s.t.

$$|\langle \xi_{l_1}; l_j \rangle| \leq C_{j-1} \langle \xi_{l_1} \rangle^{\delta_{j-1}} \|l_j\|^{-\tau} , \quad \|l_j\| \leq D_{j-1} \langle \xi_{l_1} \rangle^{\epsilon} . \tag{5.4.9}$$

We now apply Lemma D.0.6 of Appendix D with $h := k_{\bar{\jmath}}/2$, $\ell := l_1/2$, $\varsigma := \zeta + \kappa + k'$, $\eta := \xi + \kappa$. The only nontrivial assumption of Lemma D.0.6 to verify is the first of (D.0.10). One has

$$\|\xi - \zeta - k'\| \leq \|\xi - \zeta\| + \|k'\| \leq F \|\xi^{\kappa}\|^{\delta_{s-1} + \epsilon d} + \|k'\| .$$

To estimate $\|k'\|$ we proceed as follows:

$$\|k'\| \leq D_0 \left\langle \zeta + \kappa + \frac{k'}{2} \right\rangle^{\epsilon} \leq D_0 K \left(\langle \zeta + \kappa \rangle^{\epsilon} + \frac{1}{2^{\epsilon}} \langle k' \rangle^{\epsilon} \right) ,$$

where we used eq. (D.0.4). Using Lemma D.0.5, we get $\|k'\| \leq K'' \langle \zeta + \kappa \rangle^{\epsilon}$ and therefore

$$\|\xi - \zeta - k'\| \leq K \|\xi^{\kappa}\|^{\delta_{s-1} + \epsilon d}$$

Thus (D.0.12) implies

$$|\langle \xi_{l_1}; k_{\bar{\jmath}} \rangle| \leq K' \langle \xi_{l_1} \rangle^{\delta_{s-1} + (d + \tau + 1)\epsilon} \|k_{\bar{\jmath}}\|^{-\tau} , \quad \|l_1\| \leq D' \langle \xi_{l_1} \rangle^{\epsilon} .$$

But, if $C_s > K'$, $D_s > D'$, this means that ξ is also resonant with $k_{\bar{\jmath}}$, and thus it belongs to $\mathcal{Z}_{M''}^{(s+1)}$ with $M'' := span_{\mathbb{Z}}(M, k_{\bar{\jmath}})$, and this contradicts the fact that $\xi \in B_M^{(s)}$. □

The following theorem summarizes the result of this subsection:

Theorem 5.4.8. *Under the hypotheses of Theorem 5.4.10, the blocks* $E^{(0)}$, $E^{(d)}$, $\{E_M^{(s)}\}_{s,M}$ *are a partition of* E. *Furthermore,* $E^{(d)}$ *has dimension less then* $n_* < \infty$, *with* n_* *only depending on* $\mathfrak{c}, \mathfrak{C}, \delta, \epsilon, \tau$ *, and if* $E = \mathbb{Z}^d$ $E^{(0)}$ *is of density 1 at infinity.*

Proof. Let M_1 and M_2 be two submoduli of respective dimension s_1 and s_2. If $s_1 > s_2$, by definition of the extended blocks one has $E_{M_1}^{(s_1)} \cap E_{M_2}^{(s_2)} = \emptyset$. Let then $s_1 = s_2$: by Lemma 5.4.6,

$$E_{M_1}^{(s_1)} \cap \mathcal{Z}_{M_2}^{(s_2)} = \emptyset\,,$$

hence, being $E_{M_2}^{(s_2)} \subseteq \mathcal{Z}_{M_2}^{(s_2)}$, it follows that $E_{M_1}^{(s_1)}$ and $E_{M_2}^{(s_2)}$ have no intersection. □

5.4.2 Invariance of the sets $E_M^{(s)}$.

Consider now an operator of the form

$$\mathcal{L} = \widetilde{H} + R\,, \tag{5.4.10}$$

$$\widetilde{H} := -\Delta_{\mathtt{g},\kappa} + Z\,, \quad R \in OPS^{-2\delta n,\delta} \tag{5.4.11}$$

with Z in resonant normal form. Since a Fourier multiplier like $-\Delta_{\mathtt{g},\kappa}$, leaves invariant any set of the form (5.1.1), we focus on Z only.

Remark that, in order to study if a set is invariant, we have to study the couples $\xi, \zeta \in \mathbb{Z}^d$ s.t.

$$\langle Ze^{\mathrm{i}\xi\cdot x}\,, e^{\mathrm{i}\zeta\cdot x}\rangle_{L^2(\mathbb{T}^d)} \neq 0\,.$$

Lemma 5.4.9. *Let* $Z = Op\,(z)$, $z(x,\xi) = \sum_{k\in\mathbb{Z}^d} \hat{z}_k(\xi) e^{\mathrm{i}k\cdot x}$, *be a normal form operator; let* M *be a submodulus with dim*$M \geq 1$, *then*

$$\xi \in E_M^{(s)} \quad \Longrightarrow Z[e^{\mathrm{i}\xi\cdot x}] = \sum_{k\in M} \hat{z}_k\left(\xi + \frac{k}{2}\right) e^{\mathrm{i}(k+\xi)\cdot x}\ . \tag{5.4.12}$$

Proof. By the definition of Weyl quantization one has

$$Z[e^{\mathrm{i}\xi\cdot x}] = \sum_{k\in\mathbb{Z}^d} \hat{z}_k\left(\xi + \frac{k}{2}\right) e^{\mathrm{i}(\xi+k)\cdot x}\,.$$

In particular, given $\xi \in \mathbb{Z}^d$,

$$\langle Z[e^{\mathrm{i}\xi\cdot x}]\,, e^{\mathrm{i}(\xi+k)\cdot x}\rangle_{L^2(\mathbb{T}^d)} \neq 0$$

implies that either $k = 0$, or

$$\left(\xi + \frac{k}{2}\right) \in \mathrm{supp}(\hat{z}_k)\,.$$

Assume now by contradiction that $\exists k \notin M$ s.t. $\hat{z}_k\left(\xi + \frac{k}{2}\right) \neq 0$; since Z is in normal form this implies in particular

$$|\langle \xi_k; k\rangle| \leq \langle \xi_k\rangle^\delta \ , \quad \|k\| \leq \langle \xi_k\rangle^\epsilon \ ,$$

which means that, defining $M' := \mathrm{span}_{\mathbb{Z}}\{k\}$, that $\xi \in \mathcal{Z}^{(1)}_{M'}$, with $M' \not\subset M$. This conclusion however is in contradiction with the conclusion of Lemma 5.4.6. □

The main result of this subsection is the following theorem.

Theorem 5.4.10. *Let $E \subseteq \mathbb{Z}^d$ and let $\mathcal{E} \subseteq L^2(\mathbb{T}^d)$ be the corresponding subset of L^2. There exists a choice of the constants $C_1\,, \ldots, C_{d-1}\,, D_1\,, \ldots, D_{d-1}$ in Definition 5.2.2 and in Equation* (5.3.1) *such that $\forall s, M$ the set $E^{(s)}_M$ is left invariant by an operator Z in normal form, namely: if $\zeta \in E^{(s)}_M$ and $\langle Z[e^{\mathrm{i}\zeta\cdot x}], e^{\mathrm{i}\xi\cdot x}\rangle_{L^2(\mathbb{T}^d)} \neq 0$, then $\xi \in E^{(s)}_M$. Furthermore, in such a case one has*

$$\zeta - \xi \in M\,. \tag{5.4.13}$$

Furthermore, the above constants $C_1, \ldots, C_{d-1}$ and $D_1, \ldots, D_{d-1}$depend only on the parameters $d, \epsilon, \delta, \tau, \mathfrak{c}, \mathfrak{C}$.

Proof. Take $\zeta \in E^{(s)}_M$, assume that ξ is such that

$$\langle e^{\mathrm{i}\xi\cdot x}; Z[e^{\mathrm{i}\zeta\cdot x}]\rangle_{L^2(\mathbb{T}^d)} \neq 0\ . \tag{5.4.14}$$

First we remark that, by Lemma 5.4.9, one has

$$Z[e^{\mathrm{i}\zeta\cdot x}] = \sum_{k\in M} \hat{z}_k\left(\zeta + \frac{k}{2}\right) e^{\mathrm{i}(\zeta+k)\cdot x}\ ,$$

so, in particular

$$(5.4.14) \implies \xi - \zeta \in M$$

and also

$$\xi = \zeta + k \ , \quad \|k\| \leq \langle \zeta_k \rangle^{\epsilon} \ . \tag{5.4.15}$$

We now proceed in proving that (5.4.14) also implies $\xi \in E^{(s)}_M$.

First, if $M = \{0\}$, then, by the very definition of normal form, Z acts as a Fourier multiplier on $E^{(0)}$, and thus in particular it is diagonal and leaves it invariant. Furthermore, $E^{(0)}$ decomposes into invariant subspaces each of which is just a single point of $\mathbb{Z}^d$.

In order to prove the result for higher values of s, we first remark that

$$E^{(s)}_M = \left(\left\{ B^{(s)}_M + M \right\} \cap \mathcal{Z}^{(s)}_M \right) \setminus \left(\bigcup_{r<s} E^{(s)} \right) \ .$$

From (5.4.15) it follows that $\xi \in E^{(s)}_M + M \subset B^{(s)}_M + M$. We are going to prove by induction on s that $\xi \in \mathcal{Z}^{(s)}_M$ and that it also belongs to the complement of $\bigcup_{r<s} E^{(s)}$.

We know the result is true for $s = 0$. By induction we have that if $\zeta \in E^{(s-1)}_M$ then $\xi \in E^{(s-1)}_M$, and therefore also $\xi \in \mathcal{Z}^{(s-1)}_M$; we prove now that if $\zeta \in E^{(s)}_M$ then $\xi \in \mathcal{Z}^{(s)}_M$. Assume by contradiction that this is not true. Since the sets $\{E^{(\tilde{s})}_{\tilde{M}}\}_{\tilde{s},\tilde{M}}$ form a partition, then there exists s', and $M' \neq M$ s.t. $\xi \in E^{(s')}_{M'} \subset \mathcal{Z}^{(s')}_{M'}$.

There are three cases

1) $s' = s$. Then, by (5.4.15), one can apply Lemma 5.4.7, which implies

$$\xi \notin \mathcal{Z}^{(s)}_{M'} \ , \quad \text{unless } M = M' \ .$$

   Thus this case is not possible.

2) $s' > s$. By Remark (5.2.6), and item 1), this implies $\xi \in \mathcal{Z}^{(s)}_M$, against the contradiction assumption.

3) $s' < s$. Just remark that (5.4.14) is equal to

$$\langle e^{\mathrm{i}\xi\cdot x}; Z[e^{\mathrm{i}\zeta\cdot x}] \rangle_{L^2(\mathbb{T}^d)} = \langle Z[e^{\mathrm{i}\xi\cdot x}]; e^{\mathrm{i}\zeta\cdot x} \rangle_{L^2(\mathbb{T}^d)} \neq 0 \ , \tag{5.4.16}$$

   but the inductive assumptions says that $E^{(s')}_{M'}$ is invariant for $s' < s$, thus (5.4.16) implies $\zeta \in E^{(s')}_{M'}$ which is impossible since the extended blocks form a partition.

Thus we have $\zeta \in E_M^{(s)}$ then $\xi \in \left\{B_M^{(s)} + M\right\} \cap \mathcal{Z}_M^{(s)}$. Then by induction, using (5.4.16), $\xi \in E_{M'}^{(s')}$, $s' < s$, implies $\zeta \in E_{M'}^{(s')}$ and thus $\zeta \in E_M^{(s)}$ implies $\xi \notin E_{M'}^{(s')}$, $\forall s' < s$, and this concludes the proof. □

By equation (5.4.13), each extended block is foliated in equivalence classes left invariant by an operator in normal form. *We define the sets $W_{M,\beta}$ of Theorem 5.1.10 to be such equivalence classes.* We are now going to show that they are labelled by β in a subset of M^c. First remark that, if $\xi \in E_M^{(s)}$, there exists $W_{M,\beta}$ s.t. $\xi \in W_{M,\beta}$ and then one has

$$W_{M,\beta} \subset \xi + M \ .$$

Introduce now a basis adapted to M, then, since $\mathbb{Z}^d = M + M^c$, for any equivalence class there exists $\beta \in M^c$ s.t. $W_{M,\beta} \subset \beta + M$. Conversely, given $\beta \in M^c$ we define

$$W_{M,\beta} := \{\beta + M\} \cap E_M^{(s)} \ ,$$

which is possibly empty. Following Definition 5.2.11, $\widetilde{M}$ is the subset of the β's s.t. $W_{M,\beta}$ is not empty.

We have thus established that the following Corollary holds:

Corollary 5.4.11. *The partition $\{W_{M,\beta}\}_{M \subseteq \mathbb{Z}^d, \beta \in M^c}$ just defined is left invariant by any operator in normal form.*

5.5 Dimensional reduction

We analyze now the restriction of $\widetilde{H}$ to each invariant set, in order to show that the gauge map (5.1.9) conjugates it to a lower dimensional Schrödinger operator. This concludes the proof of Theorem 4.0.1. Thus consider

$$\widetilde{H}_{M,\beta} \equiv \Pi_{\mathcal{W}_{M,\beta}} \left(-\Delta_{\mathsf{g},\kappa} + Z_M\right) \Pi_{\mathcal{W}_{M,\beta}} \,, \tag{5.5.1}$$

with

$$Z_M = Op\left(z_M\right) , \quad z_M(x,\xi) = \sum_{k \in M} \hat{z}_k(\xi) e^{\mathrm{i}k \cdot x} \,, \tag{5.5.2}$$

in normal form.

Given $\xi \in W_{M,\beta}$, let $\tilde{\xi}$ and κ_ξ be defined as in (5.1.6), namely

$$\tilde{\xi} = \xi - \lfloor (\xi + \kappa)_M \rfloor \,, \quad \kappa_\xi = \{(\xi + \kappa)_M\} \ ,$$

and recall that, as pointed out in Remark 5.1.7, one has $\tilde{\xi} = \tilde{\beta}$, $\kappa_\xi = \kappa_\beta$. Thus, defining

$$\zeta := \lfloor (\xi + \kappa)_M \rfloor \ , \quad b^2 := \|(\tilde{\beta} + \kappa)_{M^\perp}\|^2 \ , \tag{5.5.3}$$

one has

$$\xi = \zeta + \tilde{\beta} \ , \quad (\xi + \kappa)_M = \zeta + \kappa_\beta \ , \tag{5.5.4}$$
$$(\xi + \kappa)_{M^\perp} = (\tilde{\beta} + \kappa)_{M^\perp} \ , \tag{5.5.5}$$
$$\|\xi + \kappa\|^2 = \|\zeta + \kappa_\beta\|^2 + b^2 \ . \tag{5.5.6}$$

Remark 5.5.1. *Consider the translation $W_{M,\beta} \ni \xi \mapsto \zeta = \xi - \tilde{\beta} \in W^t_{M,\beta} \subset M$; as pointed out in Remark 5.1.8, its quantization is the Gauge transformation $U_{\tilde{\beta}} = e^{i\tilde{\beta}\cdot x}$. By standard pseudodifferential calculus, given a symbol $a(x,\xi)$ one has that the symbol of $U_{\tilde{\beta}}^{-1} Op\,(a) U_{\tilde{\beta}}$ is*

$$a^{trasl}(x,\zeta) := a(x, \zeta + \tilde{\beta}) \ , \tag{5.5.7}$$

which, if a is in normal form, is a function on $T^*\mathbb{T}^s$.

Precisely, we have the following lemma

Lemma 5.5.2. *With the above notations, assume that $z_M \in S^{m,\delta}$ with $m \leq 0$, is in normal form with respect to M, then, in coordinates adapted to M, one has*

$$U_{\tilde{\beta}}^{-1} \left(-\Delta_{\mathtt{g},\kappa} + Z_M\right)\big|_{\mathcal{W}_{M,\beta}} U_{\tilde{\beta}} = -\Delta_{\mathtt{g},\kappa_\beta} + Z'_M + b^2 \ , \tag{5.5.8}$$

where $-\Delta_{g,\kappa_\beta}$ is the Laplacian (in s dimensions) with respect to the restriction of the metric to M and $Z'_M = Op\,(z'_M)$, with

$$z'_M(x,\zeta) = z_M(x, \zeta + \tilde{\beta})$$

of class $S^{m,\delta}$ (as a symbol on $\mathbb{T}^s$), with seminorms bounded by the seminorms of z_M.

Proof. First remark that, by (5.5.6) the transformation of the Laplacian is $-\Delta_{\mathtt{g},\kappa_\beta} + b^2$.

We come to the transformation of Z_M. First remark that, since it is in normal form with respect to M its symbol has the structure

$$z_M(x,\xi) = \sum_{k \in M} \hat{z}_k(\xi) e^{ik\cdot x} \ .$$

Furthermore, introducing a basis $\mathbf{v}^A$ adapted to M, and denoting by $\mathbf{u}_A$ its dual basis, one has, for $k \in M$,

$$k \cdot x = \sum_{a=1}^{d'} x^a k_a$$

(since the coordinates k_A, $A = d'+1, ..., d$ of a vector in M vanish). Thus one gets that the symbol z'_M of the transformed operator is

$$z'_M(\zeta, \hat{z}) = \sum_{k \in \mathbb{Z}^{d'}} \hat{z}_{k_a \mathbf{v}^a}(\zeta + \widetilde{\beta}) e^{\mathrm{i} x^a k_a} = z_M(\zeta' + \widetilde{\beta}, \hat{x}) \ , \quad \hat{x} := (x^1, ..., x^{d'})$$

Remark that, denoting $M_R := \mathrm{span}_R(\mathbf{v}_1, ..., \mathbf{v}_{d'})$ and $M_R^* := \mathrm{span}_R(\mathbf{u}_1, ..., \mathbf{u}_{d'})$, one has

$$\begin{aligned}
\|d_{\hat{x}}^{N_2} d_{\zeta'}^{N_1} z'_M(\hat{x}, \zeta')\| &= \sup_{\substack{\|h^{(j)}\|=1,\ h^{(j)} \in M_R^* \\ \|k^{(j)}\|=1,\ k^{(j)} \in M_R}} |d_{\hat{x}}^{N_2} d_{\zeta'}^{N_1} z'_M(\zeta', \hat{z}) \left[h^{(1)}, \cdots, h^{(M)}, k^{(1)}, \cdots, k^{(N)}\right]| \\
&\leq \sup_{\substack{\|h^{(j)}\|=1,\ h^{(j)} \in \mathbb{R}^d \\ \|k^{(j)}\|=1,\ k^{(j)} \in \mathbb{R}^d}} |d_{x}^{N_2} d_{\xi}^{N_1} z_M(\zeta' + \widetilde{\beta}, \hat{z}) \left[h^{(1)}, \cdots, h^{(M)}, k^{(1)}, \cdots, k^{(N)}\right]| \\
&= \|d_x^{N_2} d_\xi^{N_1} z_M(\hat{x}, \zeta' + \widetilde{\beta})\| \leq C\langle \zeta' + \widetilde{\beta} + \kappa \rangle^{m - N_1 \delta} \leq C\langle (\zeta' + \widetilde{\beta} + \kappa)_M \rangle^{m - N_1 \delta} \\
&= C\langle \zeta' + \kappa_\beta \rangle^{m - N_1 \delta} \ .
\end{aligned}$$

which is the thesis. □

In order to deduce the spectral result, the following corollary will be useful

Corollary 5.5.3. *Let* $\|\zeta + \kappa_\beta\|^2 + m(\zeta)$ *be an eigenvalue of* $(-\Delta_{\mathbf{g},\kappa_\beta} + Z'_M)\big|_{\mathcal{W}^t_{M,\beta}}$ *with eigenfunction* $\phi^{(\zeta)}$*. Then* $\|\xi + \kappa\|^2 + m(\xi - \tilde{\beta})$ *is an eigenvalue of* $(-\Delta_{\mathbf{g},\kappa} + Z_M)|_{\mathcal{W}_{M,\beta}}$ *with eigenfunction* $\psi^{(\xi)} := e^{\mathrm{i}\tilde{\beta} \cdot x} \phi^{(\zeta)}$*.*

Remark 5.5.4. *By* (5.5.4)*, in the particular case where* $\phi^{(\zeta)} = e^{\mathrm{i}\zeta \cdot x}$*, one has* $\psi^{(\xi)} = e^{\mathrm{i}\xi \cdot x}$*.*

Chapter 6

Global spectral asymptotics

Consider the operator $H = -\Delta_{\mathbf{g},\kappa} + V$ as in (5.0.1): the spectral Theorem 4.0.1 of Chapter 4 enables us to deduce that there is a density one set of eigenvalues λ_ξ of H which admits an asymptotic expansion of the form

$$\lambda_\xi = \|\xi + \kappa\|^2 + \sum_{j=0}^{n-1} z_j(\xi) + \mathcal{O}(\|\xi + \kappa\|^{-2\delta n}), \quad z_j \in S^{-2\delta j,\delta} \quad \forall j. \qquad (6.0.1)$$

In this chapter we provide asymptotics for all the other eigenvalues. Recall that, as pointed out in [FKT91], an asymptotic expansion of the form of (6.0.1) does not hold for the whole spectrum of H. Instead, we obtain asymptotic expansions with a *directional decay*: roughly speaking, to the points ξ which are in resonance (in the sense of Definition 5.2.9) with a given module $M \subset \mathbb{Z}^d$ correspond eigenvalues λ_ξ with an asymptotic expansion in powers of $\|(\xi)_M\|^{-2\delta}$, instead of $\|\xi\|^{-2\delta}$ (see Theorem 6.0.5 for a precise formulation). This is done combining an iterative application of the Structure Theorem 5.1.10 and the of the quasi-modes argument of Section 4.4, Chapter 4.

In particular, we argue as follows. Theorem 5.1.10 allows to conjugate the operator H to a sequence of lower dimensional Schrödinger operators, the majority of which are trivial (there are infinitely many Fourier multipliers and one finite dimensional operator). In order to study the nontrivial Schrödinger operators one can apply again Theorem 5.1.10 to the operators of eq. (5.1.14). In this way one can conjugate each of these operators to Schrödinger operators of lower dimension. Iterating further and further, one is finally reduced either to finite dimensional operators, or to Fourier multipliers.

Remark 6.0.1. *For any $M \subseteq \mathbb{Z}^d$ of dimension d' and $\beta \in \widetilde{M}$, the Schrödinger operator of eq.* (5.1.14) *acts on $\mathbb{T}^{d'}$ and its corresponding symbol, written in coordinates adapted to M depends only on the first d' variables (both x and ξ). Thus if one looks at such a symbol as a symbol of an operator on the original torus, namely as a function in $\mathcal{C}^\infty(T^*\mathbb{T}^d)$, then one has that taking derivatives with respect to the ξ variables does not improve the decay in the directions of the variables which are not present in the symbol, namely $(\xi^{d'+1}, ..., \xi^d)$. For this reason we will get that some eigenvalues (these are the unstable eigenvalues of [FKT91]) have asymptotics with only a directional decay.*

Directional decay is captured by the following couple of definitions:

Definition 6.0.2. *Let $m \leq 0$, and let $M \subseteq \mathbb{Z}^d$ be a submodule. We say that $a \in \mathcal{C}^\infty(T^*\mathbb{T}^d)$ is a symbol of order m in direction M if $\forall N_1, N_2 \in \mathbb{N}^d$ there exists a constant $C_{N_1,N_2} > 0$ such that*

$$\|d_x^{N_1} d_\xi^{N_2} a(x,\xi)\| \leq C_{N_1,N_2} \langle (\xi+\kappa)_M \rangle_{\mathtt{g}^*}^{m-\delta N_2} \quad \forall x \in \mathbb{T}^d,\ \forall \xi \in \mathbb{R}^d . \tag{6.0.2}$$

In this case we will write $a \in S_M^{m,\delta}$.

Definition 6.0.3. *Given a modulus $M \subset \mathbb{Z}^d$, a sequence of symbols $z_j \in S_M^{-2j\delta,\delta}$, $j \geq 0$, depending only on ξ and a function $z(\xi)$, possibly defined only on $\mathbb{Z}^d$ or on a subset E of $\mathbb{Z}^d$, we write*

$$z \overset{M}{\sim} \sum_j z_j \ , \tag{6.0.3}$$

if for any N there exists C_N s.t.

$$\left| z(\xi) - \sum_{j=0}^{N} z_j(\xi) \right| \leq \frac{C_N}{\langle (\xi+\kappa)_M \rangle_{\mathtt{g}^*}^{(N+1)2\delta}} \ . \tag{6.0.4}$$

Theorem 4.0.1 implies that to any invariant block $\mathcal{W}_{M,\beta}$ is associated a resonance module M (together with a vector β.) Thus each time one applies Theorem 4.0.1 and focuses on an invariant block, a new resonance module is selected. An iterative application of Theorem 4.0.1 leads to consider finite sequences of modules with the following structure:

Definition 6.0.4. *A sequence of moduli*

$$\mathbb{Z}^d \supset M^{(1)} \supset ... \supset M^{(r-1)} \supseteq M^{(r)} \ , \quad \dim M^{(j)} = d_j \ , \tag{6.0.5}$$

will be said to be admissible if

$$d_r \leq d_{r-1} < d_{r-2} < ... < d_1 < d \ , \tag{6.0.6}$$

and either $d_r = d_{r-1}$ *or* $d_r = 0$ *(namely the sequence ends when either the last module coincides with the previous one or it consists of* $\{0\}$*).*

The number r *will be called the length of the sequence.*

We will denote by $\mathcal{M}od$ *the set of all admissible sequences of moduli.*

We also denote $\vec{M} := (M^{(1)}, ..., M^{(k)})$.

Let now $\vec{M} \in \mathcal{M}od$, then for any j consider a modulus $M^{(j),c}$ complementary to $M^{(j)}$ in $M^{(j-1)}$, namely a modulus such that

$$M^{(j)} + M^{(j),c} = M^{(j-1)} \ , \quad M^{(j)} \cap M^{(j),c} = \{0\}$$

then the above construction forces to use also subsets

$$\widetilde{M^{(j)}} \subset M^{(j),c} \ .$$

We denote

$$\vec{M}^{\sim} := (\widetilde{M^{(1)}}, ..., \widetilde{M^{(k)}}) \ ,$$

then the sequence of normalizations that one performs is determined by the couple $(\vec{M}, \vec{\beta})$ with $\vec{\beta} \equiv (\beta_1, ..., \beta_k) \in \vec{M}^{\sim}$. With the above definitions, the following is the main result of the present chapter:

Theorem 6.0.5. *There exists a partition*

$$\mathbb{Z}^d = \bigcup_{\vec{M} \in \mathcal{M}od} \bigcup_{\vec{\beta} \in \vec{M}^{\sim}} W_{\vec{M},\vec{\beta}} \ ,$$

and for any $(\vec{M}, \vec{\beta})$ *with* $\vec{M} \in \mathcal{M}od$ *and* $\vec{\beta} \in \vec{M}^{\sim}$ *there exists a sequence of* x *independent symbols* $\{z^{(j)}_{\vec{M},\vec{\beta}}\}_{j\in\mathbb{N}}$, $z^{(j)}_{\vec{M},\vec{\beta}} \in S^{-2\delta j,\delta}_{M^{(r-1)}}$ $\forall j$*, with the following property. If* $\xi \in W_{\vec{M},\vec{\beta}}$*, then there exists a unique corresponding eigenvalue* λ_ξ *which admits the asymptotic expansion*

$$\lambda_\xi \overset{M^{(r-1)}}{\sim} \|\xi + \kappa\|^2_{\mathtt{g}*} + \sum_{j\in\mathbb{N}} z^{(j)}_{\vec{M},\vec{\beta}}(\xi) \ , \tag{6.0.7}$$

where r *is the length of the sequence* $\vec{M}$*. The operator* H *does not have other eigenvalues. Furthermore, the constants* C_N *of* (3.2.7) *are uniform with respect to the choice of the couple* $(\vec{M}, \vec{\beta})$*.*

From now on, as done in the previous chapters, in order to simplify notations we omit again the dependence on the metric g in scalar products, angled brackets, and norms.

6.1 A spectral result by quasi-modes

The proof of Theorem 6.0.5 is essentially based on an iterative application of the quasi-mode argument exhibited in Proposition 4.4.1 of Chapter 4. We will apply such a proposition to a smoothing perturbation of an operator H_0 acting on a subspace $\mathcal{E} \subseteq L^2(\mathbb{T}^d)$ generated by some $E \subseteq \mathbb{Z}^d$ and satisfying the following assumptions:

1.
$$\begin{gathered} \sigma(H_0) = \{h_0(\xi) \mid \xi \in E\} \,, \\ h_0(\xi) = \|\xi + \kappa\|^2 + z(\xi) \quad \forall \xi \in E \,, \\ \sup_{\xi \in E} |z(\xi)| = \mathtt{Z} < \infty \end{gathered} \tag{6.1.1}$$

2. There exists a complete set of normalized eigenfunctions $\{\phi_\xi\}_{\xi \in E}$ and positive constants a, n, C_n such that any eigenvalue $h_0(\xi) \neq 0$ has related eigenfunction ϕ_ξ satisfying
$$\|\phi_\xi\|_{H^{-n}} \leq C_n |h_0(\xi)|^{-an} \,. \tag{6.1.2}$$

Remark 6.1.1. *If H_0 is a Fourier multiplier with eigenvalues of the form 6.1.1, then the above assumptions trivially hold. If E is a finite dimensional set, the same is true, with constants $\mathtt{Z}$ and C_n depending on the dimension of E. In particular, by the Structure Theorem 5.1.10, this is the case of the operator $\widetilde{H}_{M,\beta}$ if $M = \mathbb{Z}^d$ (in such a case, $W_{M,\beta} = E^{(d)}$ has finite dimension n^*), or if $M = \{0\}$ (in such a case, $\widetilde{H}_{M,\beta}$ is a Fourier multiplier).*

The following result holds for the eigenvalues of H_0 satisfying assumptions 1 and 2 as above:

Lemma 6.1.2. *Suppose that $\sup_{\xi \in E} |z(\xi)| \leq \mathtt{Z}$ and let $R > \sqrt{3\mathtt{Z}}$. Then one has*
$$\# \left\{ \xi \ : \ |h_0(\xi)| \leq R^2 \right\} \leq \left(\frac{4}{\mathfrak{c}_1} \right)^d R^d \,. \tag{6.1.3}$$

Proof. The proof exploits the fact that Weyl law holds for the eigenvalues of $-\Delta_{\mathtt{g},\kappa}$ (see Lemma 3.1.3 of Chapter 3) and is done arguing as in the proof of Lemma 4.4.2 of Chapter 4. □

Furthermore, arguing again as to prove Lemma 4.4.3 of Chapter 4, as a consequence of the Weyl's law stated in Lemma 6.1.2 one has the following:

Corollary 6.1.3. *For any $N > \frac{d}{2}$ and $0 < L < \mathtt{M}$, there exists a sequence of intervals*

$$E_j = [a_j, b_j] \ , \quad j \in \mathbb{N} \tag{6.1.4}$$

and a positive constant C, with the following properties:

$$\sigma(H_0) \subset \left[-\mathtt{Z},\ a_1 - \frac{1}{a_1^N}\right] \bigcup \left(\bigcup_j E_j \right) \ , \tag{6.1.5}$$

$$|b_j - a_j| \equiv |E_j| \leq 2L \tag{6.1.6}$$

$$d(E_j, E_{j+1}) \equiv a_{j+1} - b_j \geq \frac{L}{b_j^N} \tag{6.1.7}$$

$$\# \left(\sigma(H_0) \cap E_j\right) \leq C b_j^{d/2} \ . \tag{6.1.8}$$

Our proof relies on the following results:

Lemma 6.1.4. *Consider an operator $H_0 + R$, with $R \in OPS^{-n,\delta}$ for some $n > 0$ and assume that*

1. *$\exists C$ and d s.t. the spectrum of H_0 satisfies a Weyl's law of the form*

$$\sharp\{\lambda^{(0)} \in \sigma(H_0) \mid \lambda^{(0)} \leq r\} \leq C r^{\frac{d}{2}} \ . \tag{6.1.9}$$

2. *There exist $a > 0$ and C_1 such that any normalized eigenfunction ψ relative to an eigenvalue $\lambda^{(0)}$ of H_0 fulfills*

$$\|\psi\|_{H^{-n}} \leq C_1 |\lambda^{(0)}|^{-an} \ . \tag{6.1.10}$$

Then there exists $\Lambda, C_1' > 0$ which depend on $C, C_1, d, \|R\|_{\mathcal{B}(H^{-N}, H^0)}$ only, with the following properties: any normalized eigenfunction ϕ of $H_0 + R$ which corresponds to an eigenvalue $\lambda > \Lambda$ fulfills

$$\|\phi\|_{H^{-n}} \leq C_1' |\lambda|^{\frac{d}{2} - an} \ . \tag{6.1.11}$$

Proof. First remark that, by Calderon Vaillancourt theorem, one has

$$\|R\psi\|_{L^2} \le \|R\|_{\mathcal{B}(H^{-n},H^0)}\|\psi\|_{H^{-n}} \le \frac{\|R\|_{\mathcal{B}(H^{-n},H^0)}C_1}{|\lambda^{(0)}|^{an}} \ . \tag{6.1.12}$$

Fix $\mathsf{c}_1 < \lambda/2$ and decompose

$$\phi = \phi_0 + \phi_1$$

with

$$\phi_0 \in \mathcal{Q} = \mathrm{span}\left\{\psi \ \mid H_0\psi = \lambda_\psi\psi\,, \quad |\lambda_\psi - \lambda| \le \mathsf{c}_1\right\} \ ;$$

and $\phi_1 \in \mathcal{Q}^\perp$. We analyze the eigenvalue equation

$$(H_0 + R)\,\phi = \lambda\phi\,.$$

by using the method of Lyapunov Schmidt decomposition. Denote by Π the orthogonal projector on $\mathcal{Q}$ and by $\Pi^\perp$ the orthogonal projector on $\mathcal{Q}^\perp$. Inserting the decomposition of ϕ in the eigenvalue equation, applying $\Pi^\perp$ and taking into account that the projector commutes with H_0, we get (reorganizing the terms)

$$\left[\left(\Pi^\perp H_0\Pi^\perp - \lambda\right) + \Pi^\perp R\right]\phi_1 = -\Pi^\perp R\phi_0 \ .$$

By definition of $\mathcal{Q}^\perp$, the operator in square brackets is invertible and the norm of its inverse is bounded by 2, provided $\mathsf{c}_1 \ge 2\|R\|_{\mathcal{B}(H^{-n},H^0)}$. It follows that

$$\|\phi_1\|_{L^2} \le 2\|R\phi_0\|_{L^2} \ .$$

To estimate $\|R\phi_0\|_{L^2}$ we decompose ϕ_0 in eigenfunctions of H_0 and use assumption (6.1.10). First remark that by construction ϕ_0 has components only on eigenfunctions corresponding to eigenvalues between $\lambda - \mathsf{c}_1 > \lambda/2$ and $\lambda + \mathsf{c}_1 < 2\lambda$. So it has at most $J \le 2^{d/2}C\lambda^{\mathsf{d}/2}$ components:

$$\phi_0 = \sum_{j=1}^{J}\alpha_j\psi_j\,.$$

It follows that the H^{-n} norm of ϕ_0 is bounded by $2^{na}J/\lambda^{an}$. Concerning ϕ_1, we show that its L^2 norm, which bounds all the negative Sobolev norms, is small. One has

$$\|R\phi_0\|_{L^2} \le \sum_{j=1}^{J}|\alpha_j|\|R\psi_j\|_{L^2} \le \left(\sum_{j=1}^{J}|\alpha_j|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{J}\|R\psi_j\|_{L^2}^2\right)^{\frac{1}{2}} \le J^{1/2}\frac{\|R\|_{\mathcal{B}(H^{-n},H^0)}}{(\lambda/2)^{an}}\,,$$

where we used that the norm of ϕ_0 is smaller than the norm of ϕ and therefore is smaller than 1. From this the thesis follows. $\square$

The following lemma is useful in order to verify the assumptions of Lemma 6.1.4:

Lemma 6.1.5. *Let $M \subset \mathbb{Z}^d$ be a modulus, and let u be a function of the form*

$$u(x) = \sum_{\zeta \in M} \hat{u}_\zeta e^{\mathrm{i}\zeta \cdot x}$$

be such that

$$\|u\|_{H^{-n}} \leq K\,. \tag{6.1.13}$$

Let $\beta \in M^c$ and consider $\tilde{\beta}$ defined as in (5.1.6)*. Then one has*

$$\|e^{\mathrm{i}\tilde{\beta}\cdot x} u\|_{H^{-2n}} \leq \frac{K}{\langle(\tilde{\beta}+\kappa)_{M^\perp}\rangle^n}\ . \tag{6.1.14}$$

Proof. One has

$$\|e^{\mathrm{i}\tilde{\beta}\cdot x} u\|^2_{H^{-2n}} = \sum_{\zeta \in M} \langle \tilde{\beta}+\kappa+\zeta\rangle^{-2n} \left|\hat{u}_\zeta\right|^2\ . \tag{6.1.15}$$

We analyse, using (5.5.4) and (5.5.6), the term

$$\begin{aligned}
\langle \tilde{\beta}+\kappa+\zeta\rangle^2 &= 1 + (\zeta+\tilde{\beta}+\kappa)^2_M + (\zeta+\tilde{\beta}+\kappa)^2_{M^\perp} \\
= 1 + (\zeta+\kappa')^2 + (\tilde{\beta}+\kappa)^2_{M^\perp} &= \frac{1}{2} + (\zeta+\kappa')^2 + \frac{1}{2} + (\tilde{\beta}+\kappa)^2_{M^\perp} \\
\geq 2\sqrt{\frac{1}{2}+(\zeta+\kappa')^2}\sqrt{\frac{1}{2}+(\tilde{\beta}+\kappa)^2_{M^\perp}} &\geq \langle\zeta+\kappa'\rangle\langle(\tilde{\beta}+\kappa)_{M^\perp}\rangle\ .
\end{aligned}$$

Inserting in (6.1.15), one immediately gets the thesis. $\square$

The following Lemma enables to relate the spectrum and the structure of eigenfunctions of the two operators $H^{(1)}_{M,\beta}$ and $\widetilde{H}_{M,\beta}$ of Theorem 5.1.10, for any $M \subset \mathbb{Z}^d$ and $\beta \in \widetilde{M}$:

Lemma 6.1.6. *For any M, β, consider the operator $-\Delta_{\mathtt{g},\kappa_\beta} + V_{M,\beta}$ as in* (5.1.14) *of Theorem 5.1.10, and assume that its eigenvalues are given by*

$$\lambda_\zeta = h_{M,\beta}(\zeta) = \|\zeta+\kappa_\beta\|^2 + z_{M,\beta}(\zeta)\,, \quad \zeta \in M, \tag{6.1.16}$$

with $\sup_{M,\beta}\sup_\zeta |z_{M,\beta}(\zeta)| \le \mathtt{Z}$. *Assume that there exist positive constants* $a < \frac{1}{2}$, $n \in \mathbb{N}$ *and* C *such that, given any eigenvalue* $\lambda_\zeta \neq 0$, *the corresponding eigenfunction* $\phi^{(\zeta)}$ *fulfills*

$$\|\phi^{(\zeta)}\|_{H^{-n}} \le \frac{C}{\lambda_\zeta^{an}} \quad \forall \zeta \in M\,. \tag{6.1.17}$$

Then the eigenvalues of $U_\beta\left(-\Delta_{\mathtt{g},\kappa_\beta} + V_{M,\beta}\right)U_\beta^* + b^2$ *are given by*

$$\lambda_\xi = h_0(\zeta) = \|\xi+\kappa\|^2 + z_{M,\beta}(\xi - \tilde\beta)\ , \quad \xi = \zeta + \tilde\beta \tag{6.1.18}$$

and, if $\lambda_\xi \neq 0$, *there exists* $C' > 0$, *depending only on* $a, \mathtt{Z}, n, C$, *such that the corresponding eigenfunction* $\psi^{(\xi)}$ *fulfills*

$$\|\psi^{(\xi)}\|_{H^{-2n}} \le \frac{C'}{\lambda_\xi^{an}}\ . \tag{6.1.19}$$

Proof. The form of the eigenvalues is a direct consequence of eq. (5.5.6). Concerning the eigenfunctions, the unitary map U_β transforms them in $\psi^{(\xi)} := e^{\mathrm{i}\tilde\beta\cdot x}\phi^{(\zeta)}$, which, by Lemma 6.1.5, are estimated by

$$\|\psi^{(\xi)}\|_{H^{-2n}} \le \frac{C}{\lambda_\zeta^{an}}\frac{1}{\langle(\tilde\beta+\kappa)_{M^\perp}\rangle^n}\ . \tag{6.1.20}$$

Then one has

$$\lambda_\zeta^a\langle(\tilde\beta+\kappa)_{M^\perp}\rangle \ge \left(\lambda_\zeta^{1/2}\langle(\tilde\beta+\kappa)_{M^\perp}\rangle\right)^{2a}\ ,$$

since $2a < 1$. Then, provided λ_ζ is large enough, $\lambda_\zeta^{1/2} \ge \langle(\zeta+\kappa')\rangle/2$, from which

$$\lambda_\zeta^{1/2}\langle(\tilde\beta+\kappa)_{M^\perp}\rangle \ge \frac{1}{2}\langle\zeta+\kappa'\rangle\langle(\tilde\beta+\kappa)_{M^\perp}\rangle = \frac{1}{2}\langle(\xi+\kappa)_M\rangle\langle(\xi+\kappa)_{M^\perp}\rangle \ge \frac{1}{2}\langle\xi+\kappa\rangle\ , \tag{6.1.21}$$

where the last inequality follows from the trivial remark that for any real x, y, one has $(1+x^2)(1+y^2) \ge 1+x^2+y^2$. Collecting the results and remarking that, for λ_ξ large enough, $\lambda_\xi < 2\langle\xi+\kappa\rangle^2$, one gets the thesis for large eigenvalues. In order to cover all the nonvanishing eigenvalues, just remark that the number of eigenvalues smaller than any threshold is finite, so that the claimed estmates trivially hold. □

Lemma 6.1.7. *Assume that all the operators* (5.1.14) *fulfill the assumptions of Lemma 6.1.6: then the properties* (6.1.18) *and* (6.1.19) *hold, also for the eigenvalues and the eigenfunctions of the operator* (5.0.1), *but with new constants depending only on the seminorms of* V *and on the constants of the metric, and with a new function* $z'_{M,\beta}$ *such that*

$$z'_{M,\beta}(\xi) = z_{M,\beta}(\xi) + r_\xi\,, \quad |r_\xi| \le C\|\xi+\kappa\|^{-an} \quad \forall \xi\,.$$

Proof. First, by Theorem 5.1.10, for any $n' \in \mathbb{N}$, the operator $-\Delta_{\mathsf{g},\kappa} + V$ is unitarily equivalent, through a pseudodifferential operator U of order 0, to $\widetilde{H}_{n'} + R_{n'}$. Fix $n \in \mathbb{N}$, let $n' = \frac{n}{2}$ and from now on drop the dependence on n' by the operators $\widetilde{H}_{n'}, R_{n'}$. By Lemma 6.1.6 the eigenvalues of $\widetilde{H}$ fulfill (6.1.18) and (6.1.19) with $2n$ replaced by n, due to the choice of n'. Concerning the eigenfunctions, we observe that, by (6.1.17), Lemma 6.1.4 ensures that there exists a constant $C'' > 0$ such that any eigenvalue λ_ξ of $\widetilde{H} + R$ with $\lambda_\xi \neq 0$ has a related normalized eigenfunction ψ_ξ satisfying

$$\|\psi_\xi\|_{H^{-n}} \le C''|\lambda_\xi|^{\frac{d}{2} - a\frac{n}{2}}\,, \tag{6.1.22}$$

thus (6.1.19) still holds for the eigenfunctions of $\widetilde{H} + R$. It remains to prove (6.1.18). We split $\sigma(\widetilde{H})$ according to Corollary 6.1.3, choosing $L = 1$ and $n = n/3$ and in each of the intervals E_j we apply Lemma 4.4.1. To this end, remark that for all eigenvalues $\lambda \in E_j$ one has $\lambda/2 < a_j < b_j < 2\lambda$. Let ϕ be the eigenfunction of $\widetilde{H}$ corresponding to λ: then by Calderon Vaillancourt Theorem and since the eigenfunctions of $\widetilde{H}$ satisfy eq. (6.1.19), one has

$$\|R\phi\|_{L^2} \le \|R\|_{\mathcal{B}(H^{-n},H^0)} \frac{2^n C'}{\lambda^{a\frac{n}{2}}}\,.$$

Thus an application of Lemma 4.4.1 with $A = \widetilde{H} + R$ ensures that, if for all $j \in \mathbb{N}$ one defines $D_j^- = a_j^{-n/2}, D_j^+ = b_j^{-n/2}$ and $M_j = \sharp\left(\sigma(\widetilde{H}) \cap E_j\right)$, then there are $M_j' \ge M_j$ eigenvalues of $\widetilde{H} + R$ inside the interval

$$\widetilde{E}_j = \left[a_j - \frac{1}{4}D_j^-\,,\ b_j + \frac{1}{4}D_j^+\right] \supset E_j\,.$$

We prove now that there are no eigenvalues of $\widetilde{H} + R$ outside the intervals $\widetilde{E}_j$. Assume by contradiction that $\bar\lambda$ is an eigenvalue of $\widetilde{H} + R$ with $\bar\lambda \notin \bigcup_j \widetilde{E}_j$. Let $\bar{j}$ be the positive integer such that $b_{\bar j} < \bar\lambda < a_{\bar j + 1}$. Since the eigenfunction ψ of $\widetilde{H} + R$ related to $\bar\lambda$ satisfies (6.1.22), one has

$$\|R\psi\|_{L^2} \lesssim \bar\lambda^{\frac{d}{2} - a\frac{n}{2}} \lesssim {a_{\bar j+1}}^{\frac{d}{2} - a\frac{n}{2}}\,,$$

which implies that ψ is a quasi-mode for $\widetilde{H}$ with approximated eigenvalue $\bar{\lambda}$. In particular (up to choosing a_1 big enough), this implies that there exists an exact eigenvalue $\lambda = \bar{\lambda} + O(a_{\bar{j}+1}{}^{\frac{d}{2}-a\frac{n}{2}})$ of $\widetilde{H}$ such that $b_{\bar{j}} < \lambda < a_{\bar{j}+1}$, which is absurd, by definition of the intervals E_j.

We prove now that $M'_j = M_j$ for all $j \in \mathbb{N}$. Arguing as before, we can apply the quasi-mode argument of Lemma 4.4.1 with $A = \widetilde{H}$, $M = M'_j$ and $[\lambda_1^{(0)}, \lambda_{M'_j}^{(0)}] = \widetilde{E}_j$ to deduce that, since all the eigenfunctions of $\widetilde{H}+R$ related to the eigenvalues contained inside $\widetilde{E}_j$ satisfy (6.1.22), then there are $M''_j \geq M'_j$ eigenvalues of $\widetilde{H}$ inside a slight enlargement of the interval $\widetilde{E}_j$. But there are exactly M_j eigenvalues of $\widetilde{H}$ inside $E_j \subset \widetilde{E}_j$, thus $M''_j = M_j$, which proves that *all* the eigenvalues of $\widetilde{H} + R$ are of the form (6.1.18). We finally observe that, since for any eigenvalue λ' of $\widetilde{H} + R$ the corresponding eigenfunction ψ fulfills again equation (6.1.19) with updated constants, the corresponding eigenfunction $U\psi$ of $-\Delta_{\mathtt{g},\kappa} + V$ fulfills again equation (6.1.19), due to the fact that U is a bounded operator onto H^{-n}, since U is a pseudo-differential operator of order 0. □

By combining Remark 6.1.1 with an iterative application of the above Lemma, one gets the proof of Theorem 6.0.5.

Remark 6.1.8. *From the above Lemma it follows in particular that all the eigenvalues and eigenfunctions of $\widetilde{H} + R$ are constructed through our quasi-mode procedure.*

Chapter 7

On persistence of spectral degeneracies

Consider the operator

$$H = -\Delta_{\mathsf{g}} + V\,, \quad V \in \mathcal{C}^\infty(\mathbb{T}^d;\mathbb{R}) \tag{7.0.1}$$

on $L^2(\mathbb{T}^d)$ with periodic boundary conditions, where $-\Delta_{\mathsf{g}} = -\Delta_{\mathsf{g},0}$ is defined as in (3.1.3).In Chapter 4 we proved that in the periodic case, which corresponds to the case of Floquet parameter $\kappa = 0$, for any $\xi \in \Omega \cap \mathbb{Z}^d$ there exists an eigenvalue λ_ξ of H of the form

$$\lambda_\xi = \|\xi\|^2 + \sum_{j=0}^{n-1} z_j(\xi) + \mathcal{O}(\|\xi\|^{-2\delta n})\,. \tag{7.0.2}$$

Moreover, in analogy with what happens in the case $d = 1$ (see for instance [MO75]), as a consequence of Theorem 4.0.1 we proved that one has

$$\lambda_\xi - \lambda_{-\xi} = \mathcal{O}(\|\xi\|^{-\infty}) \tag{7.0.3}$$

(see Remark 4.0.2).
Concerning the case of a higher dimensional flat torus, we observe that each flat metric g is represented by the matrix with coefficients $\{\mathsf{g}^{A,B}\}_{A,B=1}^d$ defined as in (3.1.5) of Chapter 3.

Remark 7.0.1. *Assume that the coefficients of the symmetric bilinear form $\langle\cdot;\cdot\rangle_{\mathsf{g}*}$ form a rationally independent vector in $\mathbb{R}^D$, with $D = \frac{d(d+1)}{2}$. Then for any $\xi, \xi' \in \mathbb{Z}^d$, $\|\xi\|^2 = \|\xi'\|^2$ implies $\xi = \pm\xi'$. In particular, all the non null eigenvalues of the Laplacian operator $-\Delta_{\mathsf{g}}$ on $\mathbb{T}^d$ have multiplicities exactly equal to* 2.

The set of vectors with rationally independent components has full Lebesgue measure. Thus in the case of a generic metric $\mathtt{g}$, there exists a density one subset Ω of $\mathbb{Z}^d$ such that, for any $\xi, \xi' \in \Omega$,

$$\|\xi\|^2 = \|\xi'\|^2 \quad \Longrightarrow \quad \lambda_\xi - \lambda_{\xi'} = \mathcal{O}(\|\xi\|^{-\infty})\,,$$

for any choice of the smooth potential V.
Here instead we focus on the particular case of the standard torus, where $\mathtt{g}$ is the Euclidean metric, namely $\forall A, B = 1, \ldots, d$ its matrix elements with respect to the canonical basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ in (3.1.4) are defined by $\mathtt{g}_{AB} = \delta_{AB}$. In such a case the multiplicities of an eigenvalue $\|\xi\|^2$ in the spectrum of the Laplacian operator are well known to grow as $\|\xi\|^{d-1}$: thus it is natural to ask if one can have that the eigenvalues in (7.0.2) satisfy

$$\lambda_\xi - \lambda_{\xi'} = \mathcal{O}(\|\xi\|^{-\infty})$$

for more integer points (ξ, ξ'), namely points such that $\|\xi\|^2 = \|\xi'\|^2$, with $\xi' \neq \pm\xi$. More precisely, we give the following definitions:

Definition 7.0.2. *Given two unbounded sequences $\{\xi_n\}_{n\in\mathbb{N}}, \{\xi'_n\}_{n\in\mathbb{N}}$, if for any $n_0 \in \mathbb{N}$ there exists $n \geq n_0$ such that $\xi'_n \neq \xi_n$ and $\xi'_n \neq -\xi_n$, we simply say that $\{\xi'_n\}_{n\in\mathbb{N}}$ and $\{\pm\xi_n\}_{n\in\mathbb{N}}$ are asymptotically different.*

Definition 7.0.3. *If there exist two unbounded sequences $\{\xi_n\}_{n\in\mathbb{N}}, \{\xi'_n\}_{n\in\mathbb{N}} \subseteq \Omega \cap \mathbb{Z}^d$ with the following properties: $\{\pm\xi_n\}_{n\in\mathbb{N}}$ and $\{\xi'_n\}_{n\in\mathbb{N}}$ are asymptotically different and fulfill*

$$\|\xi_n\|^2 = \|\xi'_n\|^2 \quad \forall n \in \mathbb{N}\,, \tag{7.0.4}$$

and there exists a sequence of positive constants $\{C_N\}_{N\in\mathbb{N}}$ such that

$$\left|\lambda_{\xi_n} - \lambda_{\xi'_n}\right| \leq C_N \langle \xi_n \rangle^{-N} \quad \forall N, n \in \mathbb{N}\,, \tag{7.0.5}$$

we say that the degeneracy (at $\{\xi_n\}_{n\in\mathbb{N}}, \{\xi'_n\}_{n\in\mathbb{N}}$) in the spectrum of H persists up to any order.

Definition 7.0.4. *Given two unbounded sequences $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{\xi'_n\}_{n\in\mathbb{N}}$ in $\Omega \cap \mathbb{Z}^d$ asymptotically different and such that* (7.0.4) *holds, we say that the degeneracy (at $\{\xi_n\}_{n\in\mathbb{N}}, \{\xi'_n\}_{n\in\mathbb{N}}$) in the spectrum of H* is broken *if the degeneracy at $\{\xi_n\}_{n\in\mathbb{N}}, \{\xi'_n\}_{n\in\mathbb{N}}$ does not persist up to any order, namely there exists a positive integer $\bar{N}$ such that for any $C > 0$ there exists $n \in \mathbb{N}$ such that*

$$\left|\lambda_{\xi_n} - \lambda_{\xi'_n}\right| > C \langle \xi_n \rangle^{-\bar{N}}\,.$$

Before stating the main results of the present chapter, we fix some notations: taking Ω as in (4.2.3) of Chapter 4, we define

$$\widetilde{\Omega} := \Omega \cap \mathbb{Z}^d = \left\{ \xi \in \mathbb{Z}^d \;\middle|\; |\langle \xi; k \rangle| \geq \langle \xi \rangle^\delta \|k\|^{-\tau} \quad \forall 0 < \|k\| < \langle \xi \rangle^\epsilon \right\} . \quad (7.0.6)$$

Given $\xi \in \widetilde{\Omega}$, remark that in this chapter we always denote with λ_ξ an eigenvalue satisfying the asymptotic expansion (7.0.2); when we want to emphasize the dependence of λ_ξ on the potential V, we write λ_ξ^V instead of λ_ξ . Furthermore, we also assume that the parameters δ, ϵ, τ satisfy

$$\delta > \max \left\{ \tfrac{4}{5} + \tfrac{\epsilon}{5}(\tau + 2) \, , \; \epsilon(\tau + 2) \right\} . \quad (7.0.7)$$

As a first result, we exhibit a class of potentials such that there actually is some persistence of degeneracy up to any order in the spectrum of H:

Theorem 7.0.5 (Persistence of degeneracy)**.** *Suppose that there exists a unitary and unimodular map $U : \mathbb{R}^d \to \mathbb{R}^d$ such that*

$$V(x) = V(Ux) \quad \forall x \in \mathbb{T}^d . \quad (7.0.8)$$

Then the following holds:

1. *For any $\xi \in \mathbb{Z}^d$ one has $\xi \in \widetilde{\Omega}$ if and only if $U\xi \in \widetilde{\Omega}$*
2. *There exists a sequence $\{C_N\}_{N\in\mathbb{N}}$ of positive constants, depending only on the potential V, such that, for any unbounded sequence $\{\xi_n\}_{n\in\mathbb{N}} \subseteq \widetilde{\Omega}$,*

$$|\lambda_{\xi_n} - \lambda_{U\xi_n}| \leq C_N \langle \xi_n \rangle^{-N} \quad \forall N, \; n \in \mathbb{N} . \quad (7.0.9)$$

As a consequence, for any unbounded sequence $\{\xi_n\}_{n\in\mathbb{N}} \subset \widetilde{\Omega}$ such that $\forall n \in \mathbb{N}$ $\xi_n \neq \pm U\xi_n$, the degeneracy at $\{\xi_n\}_{n\in\mathbb{N}}, \{U\xi_n\}_{n\in\mathbb{N}}$ in the spectrum of H persists up to any order.

Actually, unitary and unimodular maps on $\mathbb{R}^d$ are a very small class. More precisely, we have the following:

Remark 7.0.6. *If U is a unimodular map, then its matrix elements $U\mathbf{e}_i \cdot \mathbf{e}_j =: U_i^j$ with respect to the standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ have to be integers. If U is also unitary, then its inverse matrix coincides with its transpose, which in particular entails that $\sum_{j=1}^d \left(U_i^j\right)^2 = 1 \; \forall i$. Thus the class of maps U described by* *the Theorem reduces to exchanges of axes and inversions in their orientation, namely maps of the form*

$$\xi = (\xi_1, \ldots, \xi_i, \ldots, \xi_j \ldots, \xi_d) \longmapsto (\xi_1, \ldots, \xi_j, \ldots, \xi_i, \ldots, \xi_d)\,,$$
$$\xi = (\xi_1, \ldots, \xi_i, \ldots, \xi_d) \longmapsto (\xi_1, \ldots, -\xi_i, \ldots, \xi_d)\,,$$

and their compositions. Some standard combinatorics then entails that there are $2^d d!$ maps of such form.

Remark 7.0.7. *The class of potentials described in Theorem 7.0.5 has strong similarities with the class of* separable potentials *exhibited in [GK91] and in [ERT84], within the context of the analysis of isospectral potentials. More precisely, separable potentials are built as the sum of d potentials $V_1, \ldots, V_d$, where for all $j = 1, \ldots, d$ V_j depends only on the variable x_j. In particular, each one of the potentials V_j satisfies the hypotheses of Theorem 7.0.5, since it is symmetrical with respect to the exchange of any two coordinates x_i, x_k with $i, k \neq j$. However, neither our class of potential is contained inside the set of separable potentials, nor all separable potentials are symmetric with respect to a unitary and unimodular map. Consider, in $d = 2$, $V(x_1, x_2) = \cos(x_1 + x_2)$, as a potential which is symmetric with respect to the exchange among coordinates x_1, x_2, but is not separable, and $V(x_1, x_2) = \cos(x_1 + \frac{\pi}{3}) + 2\cos(x_2 + \frac{\pi}{3})$ as an example of a separable potential which does not have any unimodular and unitary symmetry.*

The second result we give in the present chapter deals instead with breaking of degeneracy. Let indeed

$$\mathcal{B}_{\{\xi_n\},\{\xi'_n\}} = \big\{V \in \mathcal{C}^\infty(\mathbb{T}^d;\mathbb{R}) \;\big|\; \text{the degeneracy in the spectrum of } H = -\Delta + V \text{ at } \{\xi_n\}_{n\in\mathbb{N}}, \{\xi'_n\}_{n\in\mathbb{N}} \text{ is broken}\big\}\,; \tag{7.0.10}$$

we prove the following:

Theorem 7.0.8 (Breaking of degeneracy)**.** *There exists $\mathtt{N} > 0$ such that for any $s > \mathtt{N}$ the following holds. Consider any couple of unbounded sequences $\{\xi_n\}_{n\in\mathbb{N}}, \{\xi'_n\}_{n\in\mathbb{N}}$ contained in $\widetilde{\Omega}$ such that (7.0.4) holds and $\{\pm\xi_n\}_{n\in\mathbb{N}}$ and $\{\xi'_n\}_{n\in\mathbb{N}}$ are asymptotically different: then $\mathcal{B}_{\{\xi_n\},\{\xi'_n\}}$ is a generic set in $\mathcal{C}^\infty(\mathbb{T}^d;\mathbb{R})$ with respect to the topology induced by the norm $\|\cdot\|_s$, namely it is the complementary of a countable union of closed and nowhere dense sets.*

Of course, Theorem 7.0.8 is not sufficient to deduce that, for a generic potential, the only sequences such that $\lambda_{\xi_n} - \lambda_{\xi'_n} = \mathcal{O}(\langle \xi_n \rangle^{-\infty})$ are (asymptotically) of the form $\{\xi_n\}_{n\in\mathbb{N}}, \{-\xi_n\}_{n\in\mathbb{N}}$, namely all possible degeneracies are broken. Theorem 7.0.8 ensures indeed that, if both the sequences $\{\xi_n\}_{n\in\mathbb{N}}, \{\xi'_n\}_{n\in\mathbb{N}}$ are fixed, the degeneracy at $\{\xi_n\}_{n\in\mathbb{N}}, \{\xi'_n\}_{n\in\mathbb{N}}$ is broken by any potential V in $\mathcal{B}_{\{\xi_n\},\{\xi'_n\}}$, but nothing is claimed about the fact that the same potential V also breaks the degeneracy at $\{\xi_n\}_{n\in\mathbb{N}}, \{\xi''_n\}_{n\in\mathbb{N}}$, if a third sequence $\{\xi''_n\}_{n\in\mathbb{N}}$ is considered. The strategy we follow to prove Theorem 7.0.8 cannot be adapted in a straightforward way to prove such a result, essentially due to the unboundedness of the multiplicities in the spectrum of the Laplacian operator; thus we leave such a question open.

The remaining part of the chapter is divided into two sections: Section 7.1, where Theorem 7.0.5 is proven, and Section 7.2, devoted to the proof of Theorem 7.0.8.

7.1 Persistence of degeneracy

In the present section we prove Theorem 7.0.5. The idea is that the higher degeneracy described in Theorem 7.0.5 is associated to the preservation, along the iterative process given by the normal form of Chapter 4, of the following symmetry: given U a unitary and unimodular transformation on $\mathbb{R}^d$,

$$f(x,\xi) = f(Ux, U\xi) \quad \forall x \in \mathbb{T}^d, \quad \forall \xi \in \mathbb{R}^d\,. \tag{Sym}$$

More precisely, Theorem 7.0.5 is a consequence of the following couple of results:

Lemma 7.1.1. *Let U be an unitary and unimodular map. Then $\xi \in \widetilde{\Omega}$ if and only if $U\xi \in \widetilde{\Omega}$.*

Proof. First of all, we observe that, since U is unimodular, it maps points $\xi \in \mathbb{Z}^d$ into points $U\xi \in \mathbb{Z}^d$. Assume now that $\xi \in \widetilde{\Omega} = \Omega \cap \mathbb{Z}^d$; then this implies $U\xi \in \mathbb{Z}^d$ and we are left to show that $U\xi \in \Omega$. By definition of the non resonant set Ω (4.2.3), since $\xi \in \Omega$ one has that

$$|\,\langle \xi; k\rangle\,| \geq \langle \xi\rangle^{\delta} \|k\|^{-\tau} \quad \forall k \;\text{ s.t. } 0 < \|k\| \leq \langle \xi \rangle^{\epsilon}\,.$$

Let $h \in \mathbb{Z}^d$ and assume that $0 < \|h\| \leq \langle U\xi\rangle^{\epsilon} = \langle \xi \rangle^{\epsilon}$; then the unitariety of U implies that its inverse U^{-1} is unitary too, hence $U^{-1}h \in \mathbb{Z}^d$ and

$\|U^{-1}h\| = \|h\|$. Hence, again by unitariety and by the fact that ξ is non resonant, one has that

$$|\langle U\xi; h\rangle| = |\langle \xi; U^{-1}h\rangle| \geq \langle\xi\rangle^{\delta}\|U^{-1}h\|^{-\tau} = \langle U\xi\rangle^{\delta}\|h\|^{-\tau},$$

which proves $U\xi$ is non resonant. □

Proposition 7.1.2. *Assume the hypotheses of Theorem 4.2.1. If there exists a unimodular and unitary map U such that the potential V has the symmetry property*

$$V(x) = V(Ux) \quad \forall x \in \mathbb{T}^d, \tag{7.1.1}$$

then $\forall n \geq 0$ the symbols $h_n, v_n, z_n, z^{(n)}$ and g_{n+1}, whose existence is ensured by Theorem 4.2.1, satisfy the symmetry property (Sym)*. In particular, z_n satisfies*

$$z_n(\xi) = z_n(U\xi) \quad \forall \xi \in \mathbb{R}^d. \tag{7.1.2}$$

Assuming Proposition 7.1.2, whose proof is postponed to the following subsection, Theorem 7.0.5 is deduced as follows.

Proof of Thm 7.0.5. Since $\{\xi_n\}_{n\in\mathbb{N}}$ is contained in $\widetilde{\Omega}$, by Lemma 7.1.1 the sequence $\{U\xi_n\}_{n\in\mathbb{N}}$ is contained inside $\widetilde{\Omega}$. Then, since for any $n \in \mathbb{N}$ λ_{ξ_n} and $\lambda_{U\xi_n}$ satisfy the asymptotic expansion (4.0.6) proven in Theorem 4.0.1 of Chapter 4, for any $N \in \mathbb{N}$ there exists a positive constant C_N such that

$$\left|\lambda_{\xi_n} - \|\xi_n\|^2 - \sum_{j=1}^{N-1} z_j(\xi_n)\right| \leq C_N \langle\xi_n\rangle^{-2\delta N},$$

as well as

$$\left|\lambda_{U\xi_n} - \|\xi_n\|^2 - \sum_{j=1}^{N-1} z_j(U\xi_n)\right| \leq C_N \langle\xi_n\rangle^{-2\delta N}.$$

By Proposition 7.1.2, $z_j(\xi_n) = z_j(U\xi_n)$ $\forall n \in \mathbb{N}$ and $\forall j \in \mathbb{N}$, thus this implies

$$|\lambda_{\xi_n} - \lambda_{U\xi_n}| \leq 2C_N\langle\xi_n\rangle^{-2\delta N} \quad \forall n,$$

which gives the thesis. □

7.1.1 Symmetry preservation: proof of Proposition 7.1.2

Passing to Fourier variables, a symbol $f(x,\xi) = \sum_{k\in\mathbb{Z}^d} \hat{f}_k(\xi)e^{\mathrm{i}k\cdot x}$ satisfies to the symmetry property (Sym) if and only if for all $k \in \mathbb{Z}^d$ one has

$$\hat{f}_k(\xi) = \frac{1}{|\mathbb{T}^d|}\int_{\mathbb{T}^d} f(x,\xi)e^{-\mathrm{i}k\cdot x}\,dx = \frac{1}{|\mathbb{T}^d|}\int_{\mathbb{T}^d} f(Ux,U\xi)e^{-\mathrm{i}k\cdot x}\,dx\,.$$

Since U is unitary, for all $k \in \mathbb{Z}^d$ and $x \in \mathbb{T}^d$ this is equivalent to

$$\hat{f}_k(\xi) = \frac{1}{|\mathbb{T}^d|}\int_{\mathbb{T}^d} f(Ux,U\xi)e^{-\mathrm{i}Uk\cdot Ux}\,dx\,,$$

and, by the fact that U is unimodular,

$$\begin{aligned}\hat{f}_k(\xi) &= \frac{1}{|\mathbb{T}^d|}\int_{\mathbb{T}^d} f(x',U\xi)e^{-\mathrm{i}Uk\cdot x'}\,dx' \\ &= \hat{f}_{Uk}(U\xi) \quad \forall k\in\mathbb{Z}^d,\ \forall \xi\in\mathbb{R}^d\,.\end{aligned} \tag{7.1.3}$$

This leads to the following:

Remark 7.1.3. *A symbol f satisfies the symmetry property* (Sym) *if and only if its Fourier coefficients satisfy*

$$\hat{f}_{Uk}(U\xi) = \hat{f}_k(\xi) \quad \forall \xi\in\mathbb{R}^d, \forall k\in\mathbb{Z}^d\,.$$

The characterization exhibited in Remark 7.1.3 enables us to prove the following results.

Lemma 7.1.4. *Given $f \in S^{m,\delta}$ a symbol of order $m\in\mathbb{R}$, let $f^{(\mathrm{nr})}, f^{(\mathrm{res})}, [f]$ and $f^{(\mathrm{S})}$ be defined according to Definition 4.2.9. Let furthermore g be the symbol solving the homological equation $\{\|\xi\|^2, g\} + f^{(\mathrm{nr})} = 0$. If f satisfies the symmetry property* (Sym)*, then $f^{(\mathrm{nr})}, f^{(\mathrm{res})}, [f], f^{(\mathrm{S})}$ and g satisfy the same property* (Sym).

Proof. If $f(x,\xi) = \sum_{k\in\mathbb{Z}^d} \hat{f}_k(\xi)e^{\mathrm{i}k\cdot x}$ satisfies (Sym), it immediately follows that $[f](\xi) = \hat{f}_0(\xi)$ satisfies the same property (Sym), by Remark 7.1.3.
We now prove the claim for $f^{(\mathrm{res})}$, the ones about $f^{(\mathrm{nr})}$ and $f^{(\mathrm{S})}$ following in analogous way. Recall the definition of $f^{(\mathrm{res})}(x,\xi)$ as

$$f^{(\mathrm{res})}(x,\xi) = \sum_{k\neq 0} \hat{f}_k(\xi)\chi_k(\xi)\tilde{\chi}_k(\xi)e^{\mathrm{i}k\cdot x}\,,$$

with

$$\chi_k(\xi) = \chi\left(\frac{\langle \xi; k\rangle \, \|k\|^{-\tau}}{\langle \xi\rangle^{\delta}}\right), \quad \tilde{\chi}_k(\xi) = \chi\left(\frac{\|k\|}{\langle \xi\rangle^{\epsilon}}\right).$$

(see Definition 4.2.8 of Chapter 4). Since the map U is unitary, one has

$$\chi_{Uk}(U\xi) = \chi\left(\frac{\langle U\xi; Uk\rangle \, \|Uk\|^{-\tau}}{\langle U\xi\rangle^{\delta}}\right) = \chi\left(\frac{\langle \xi; k\rangle \, \|k\|^{-\tau}}{\langle \xi\rangle^{\delta}}\right) = \chi_k(\xi),$$

and analogously $\tilde{\chi}_{Uk}(U\xi) = \tilde{\chi}_k(\xi)$. Thus we can conclude that

$$\begin{aligned} f^{(\mathrm{res})}(x,\xi) &= \sum_{0\neq k\in\mathbb{Z}^d} \hat{f}_k(\xi)\chi_k(\xi)\tilde{\chi}_k(\xi)e^{\mathrm{i}k\cdot x} \\ &= \sum_{0\neq k\in\mathbb{Z}^d} \hat{f}_{Uk}(U\xi)\chi_{Uk}(U\xi)\tilde{\chi}_{Uk}(U\xi)e^{\mathrm{i}Uk\cdot Ux} \\ &= \sum_{0\neq k'\in\mathbb{Z}^d} \hat{f}_{k'}(U\xi)\chi_{k'}(U\xi)\tilde{\chi}_{k'}(U\xi)e^{\mathrm{i}k'\cdot Ux} \\ &= f^{(\mathrm{res})}(Ux, U\xi). \end{aligned}$$

Concerning the solution g of the homological equation $\{\|\xi\|^2, g\} + f^{(\mathrm{nr})} = 0$, remark that, as pointed out in Remark 4.2.7 of Chapter 4, g has the form

$$g(x,\xi) = -\sum_{0\neq k\in\mathbb{Z}^d} \hat{f}_k(\xi)\frac{(1-\chi_k(\xi))}{2\mathrm{i}\,\langle \xi; k\rangle}\tilde{\chi}_k(\xi)e^{\mathrm{i}k\cdot x}.$$

Thus we observe that, since U is unitary, it is

$$\langle U\xi; Uk\rangle = \langle \xi; k\rangle \quad \forall \xi\in\mathbb{R}^d, \quad \forall k\in\mathbb{Z}^d,$$

in order to deduce that

$$\begin{aligned} g(x,\xi) &= -\sum_{0\neq k\in\mathbb{Z}^d} \hat{f}_{Uk}(U\xi)\frac{(1-\chi_{Uk}(U\xi))}{2\mathrm{i}\,\langle U\xi; Uk\rangle}\tilde{\chi}_{Uk}(U\xi)e^{\mathrm{i}k\cdot x} \\ &= g(Ux, U\xi), \end{aligned}$$

and thus that g satisfies (SYM). □

Lemma 7.1.5. *Let f and g two symbols satisfying* (SYM). *Then the symbols of their composition $f\sharp g$ and Moyal brackets $\{f,g\}_{\mathcal{M}}$ satisfy* (SYM) *too.*

Proof. Recall that $\{f\,,g\}_{\mathcal{M}} = \frac{1}{\mathrm{i}}\,(f\sharp g - g\sharp f)\,.$ Thus it is sufficient to prove that $f\sharp g$ satisfies (Sym), and (Sym) for $g\sharp f$ follows analogously.
By Lemma C.2.5 of Appendix C, for any $(x,\xi)\in\mathbb{T}^d\times\mathbb{R}^d$ one has

$$(f\sharp g)\,(x,\xi) = \sum_{k,k'\in\mathbb{Z}^d} \hat{f}_{k-k'}\left(\xi+\frac{k'}{2}\right)\hat{g}_{k'}\left(\xi-\frac{k-k'}{2}\right)e^{\mathrm{i}k\cdot x}$$

and, if f and g satisfy (Sym) , by Remark 7.1.3, $\forall k\in\mathbb{Z}^d$ and $\xi\in\mathbb{R}^d$ one has

$$\hat{f}_k(\xi) = \hat{f}_{Uk}(U\xi)\,,\quad \hat{g}_k(\xi) = \hat{g}_{Uk}(U\xi)\,.$$

Hence it is sufficient to compute

$$\begin{aligned}(f\sharp g)\,(x,\xi) &= \sum_{k,k'\in\mathbb{Z}^d} \hat{f}_{k-k'}\left(\xi+\frac{k'}{2}\right)\hat{g}_{k'}\left(\xi-\frac{k-k'}{2}\right)e^{\mathrm{i}k\cdot x}\\ &= \sum_{k,k'\in\mathbb{Z}^d} \hat{f}_{Uk-Uk'}\left(U\xi+\frac{Uk'}{2}\right)\hat{g}_{Uk'}\left(U\xi-\frac{Uk-Uk'}{2}\right)e^{\mathrm{i}Uk\cdot Ux}\\ &= \sum_{h,h'\in\mathbb{Z}^d} \hat{f}_{h-h'}\left(U\xi+\frac{h'}{2}\right)\hat{g}_{h'}\left(U\xi-\frac{h-h'}{2}\right)e^{\mathrm{i}h\cdot Ux}\\ &= (f\sharp g)\,(Ux,U\xi)\,.\end{aligned}$$

□

Lemma 7.1.6. *Let $G = Op\,(g)\in OPS^{0,\delta}$. Then, as stated in Lemma C.2.6 of Appendix C, $e^{\mathrm{i}G}\in OPS^{0,\delta}$. Let $\sigma\in S^{0,\delta}$ be its symbol; if g satisfies the symmetry property* (Sym)*, σ satisfies* (Sym) *too.*

Proof. The above result follows from the fact that, if we define $\forall(x,\xi)\in\mathbb{T}^d\times\mathbb{R}^d$

$$\sigma(x,\xi) = \sum_{n\geq 0}\frac{\mathrm{i}^n g_n(x,\xi)}{n!}\,, \tag{7.1.4}$$

where $\{g_n\}_{n\in\mathbb{N}}$ is the sequence defined by $g_n = g\sharp g_{n-1}\quad \forall n>0$ and $g_0 = g\,,$ then we have $\sigma\in S^{0,\delta}$. Then, if g satisfies (Sym), the symmetry property (Sym) of the symbols g_n for all n follows from an iterative application of Lemma 7.1.5, while the convergence of the series in (7.1.4) ensures that property (Sym) holds, as well as the fact that $e^{\mathrm{i}G} = Op\,(\sigma)\,.$ □

We are now in position to prove Proposition 7.1.2:

Proof of Proposition 7.1.2. The proof is done by induction on the number n of normal form steps which have been performed in Theorem 4.2.1.
INDUCTIVE BASIS: In the case $n = 0$, we have $z^{(0)} = z_0 = 0$ and $v_0 = v$, thus (SYM) is automatically satisfied by $z^{(0)}, z_0$ and v_0. Since $h_0(\xi, x) = \|\xi\|^2 + v_0(x)$, and $\|\xi\|^2 = \|U\xi\|^2$ due to the fact that U a unitary map, it follows that h_0 satisfies (SYM) too.
Since g_1 solves the equation $\{\|\xi\|^2, g_1\} + v_0^{(\mathrm{nr})} = 0$, Lemma 7.1.4 entails that g_1 satisfies (SYM).
INDUCTIVE STEP: Suppose that h_n, $z^{(n)}$, v_n, z_n, satisfy (SYM). Then Lemma 7.1.4 together with the inductive hypothesis on v_n implies that $v_n^{(\mathrm{res})}$, $v_n^{(\mathrm{nr})}$, $v_n^{(S)}$, $[v_n]$ satisfy (SYM) and so does the solution g_{n+1} of the equation $\{\|\xi\|^2, g_1\} + v_n^{(\mathrm{nr})} = 0$. Thus, since $z^{(n+1)} = z^{(n)} + [v_n] + v_n^{(\mathrm{res})}$ and $z_{n+1} = z_n + [z_n]$, property (SYM) is satisfied by $z^{(n+1)}$ and z_{n+1} too. It remains to prove that h_{n+1} and v_{n+1} satisfy (SYM).
By Lemma 7.1.6, g_{n+1} satisfying (SYM) implies that $e^{\pm \mathrm{i} G_{n+1}}$ has a symbol $\sigma_n^{\pm}$ satisfying the same property (SYM). Recalling that $H_{n+1} = Op\ (h_{n+1})$, with

$$h_{n+1} = \left(\sigma_n^{+} \sharp\ h_n\right) \sharp\ \sigma_n^{-},$$

using the inductive hypothesis on h_n and what proven for g_{n+1}, we apply twice Lemma 7.1.5 to get that h_{n+1} satisfies (SYM). It is finally sufficient to observe that v_{n+1} is given by

$$v_{n+1}(x, \xi) = h_{n+1}(x, \xi) - \|\xi\|^2 - z^{(n+1)}(x, \xi),$$

in order to deduce that, analogously, v_{n+1} satisfies (SYM).
Notice that here there is no need to perform the splitting $v_n = v_{1,n} + v_{2,n}$, with $v_{1,n}$ symmetric and $v_{2,n}$ regularizing enough, as done in the proof of Theorem 4.2.1 in the case of a generic perturbation V: this is due to the fact that, since here V is a bounded symbol, the symbols v_n are bounded $\forall n \in \mathbb{N}$, thus also the symbols g_n are bounded. This makes possible to apply Lemma 7.1.6, and thus to ensure that property (SYM) is satisfied by the whole symbol h_n, at every step of the iteration. □

7.2 Breaking of degeneracy

In the present section we prove Theorem 7.0.8, as a consequence of the following:

Proposition 7.2.1. *Let* $\{\xi_n\}_{n\in\mathbb{N}}, \{\xi'_n\}_{n\in\mathbb{N}} \subseteq \widetilde{\Omega}$ *be two divergent sequences such that* $\|\xi_n\|^2 = \|\xi'_n\|^2$ *and* $\forall n \in \mathbb{N}\ \ \xi_n \neq \pm\xi'_n$*. Then there exists* $\mathtt{N} > 0$ *such that for any* $s > \mathtt{N}$ *and for any* $R, C > 0$

$$\mathcal{V}_{C,R} = \left\{ V \in \mathcal{C}^\infty(\mathbb{T}^d;\mathbb{R}) \;\middle|\; \|V\|_{\mathtt{N}} \leq R \quad \text{and} \quad \left|\lambda^V_{\xi_n} - \lambda^V_{\xi'_n}\right| \leq C\langle\xi_n\rangle^{-4\delta} \quad \forall n \in \mathbb{N} \right\} \tag{7.2.1}$$

is a nowhere dense set in $\mathcal{C}^\infty(\mathbb{T}^d;\mathbb{R})$ *with respect to the topology induced by the norm* $\|\cdot\|_s$ *.*

We postpone to the next subsection the proof of the above proposition, and we now prove Theorem 7.0.8.

Proof of Theorem 7.0.8. : Since $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{\xi'_n\}_{n\in\mathbb{N}}$ are unbounded sequences, up to reducing to a subsequence we can assume that the two sequences diverge. Furthermore, since $\{\pm\xi_n\}_{n\in\mathbb{N}}$ and $\{\xi'_n\}_{n\in\mathbb{N}}$ do not asymptotically coincide, we can reduce to a couple of subsequences such that $\xi_n \neq \pm\xi'_n$ for all n. We analyze the complementary set of $\mathcal{B}_{\{\xi_n\},\{\xi'_n\}}$ in $\mathcal{C}^\infty(\mathbb{T}^d;\mathbb{R})$, namely the set of potentials V such that in the spectrum of $H = -\Delta + V$ there is persistence of degeneracy up to any order at $\{\xi_n\}_{n\in\mathbb{N}}, \{\xi'_n\}_{n\in\mathbb{N}}$. Then it is sufficient to observe that, for such potentials V, in particular (taking $N > 4\delta$ in (7.0.5)) there exists a positive constant C such that for all n $|\lambda^V_{\xi_n} - \lambda^V_{\xi'_n}| \leq C\langle\xi_n\rangle^{-4\delta}$. Since for any $C \in \mathbb{R}$

$$\begin{aligned}&\left\{ V \in \mathcal{C}^\infty(\mathbb{T}^d;\mathbb{R}) \;\middle|\; |\lambda^V_{\xi_n} - \lambda^V_{\xi'_n}| \leq C\langle\xi_n\rangle^{-4\delta} \quad \forall n \in \mathbb{N} \right\} \\ &\qquad \subseteq \left\{ V \in \mathcal{C}^\infty(\mathbb{T}^d;\mathbb{R}) \;\middle|\; |\lambda^V_{\xi_n} - \lambda^V_{\xi'_n}| \leq \lceil C \rceil\langle\xi_n\rangle^{-4\delta} \quad \forall n \in \mathbb{N} \right\} ,\end{aligned} \tag{7.2.2}$$

one has that

$$\begin{aligned}&\{V \in \mathcal{C}^\infty(\mathbb{T}^d;\mathbb{R}) \mid \exists C > 0 \quad \text{s.t.} \quad \left|\lambda^V_{\xi_n} - \lambda^V_{\xi'_n}\right| \leq C\langle\xi_n\rangle^{-4\delta} \quad \forall n\} \\ &= \bigcup_{R\in\mathbb{N}, K\in\mathbb{N}} \left\{V \in \mathcal{C}^\infty(\mathbb{T}^d;\mathbb{R}) \mid \|V\|_{\mathtt{N}} < R \quad \text{and} \quad \left|\lambda^V_{\xi_n} - \lambda^V_{\xi'_n}\right| \leq K\langle\xi_n\rangle^{-4\delta} \quad \forall n \right\} .\end{aligned} \tag{7.2.3}$$

Then, by Proposition 7.2.1, there exists $n > 0$ such that the right hand side in 7.2.3 is a countable union of nowhere dense sets with respect to the topology induced by $\|\cdot\|_s$, for any $s \geq \mathtt{N}$. □

The remaining part of the section is devoted to the proof of Proposition 7.2.1.

7.2.1 Second order expansion of the eigenvalues

The proof of Proposition 7.2.1 requires a careful analysis of the first two terms z_0 and z_1 in the asymptotic expansion (4.0.6) of the eigenvalues associated to non resonant points. In the present subsection we prove the following:

Lemma 7.2.2. *Let $H = -\Delta + V$ as in* (5.0.1). *There exist a positive integer $\mathtt{N}$ and a constant $\mathtt{C}_V > 0$, depending only on the Sobolev norm $\|V\|_{\mathtt{N}}$ of the potential V, on d and on the parameters δ, τ, ϵ, such that for all $\xi \in \widetilde{\Omega}$*

$$\lambda_\xi^V = \|\xi\|^2 + [V] + \frac{1}{2} \sum_{\substack{k \in \mathbb{Z}^d \\ 0 < \|k\| \leq \langle \xi \rangle^\epsilon}} \frac{|\hat{V}_k|^2}{\langle \xi + \frac{k}{2}; k \rangle} + \ell_\xi^V \,, \quad \left|\ell_\xi^V\right| \leq \mathtt{C}_V \langle \xi \rangle^{-4\delta} . \quad (7.2.4)$$

Proof. Consider the asymptotic expansion (7.0.2) with $N = 2$: for any $\xi \in \widetilde{\Omega}$, there exists a constant $\mathtt{C}_1$, such that

$$\lambda_\xi = z_1(\xi) + z_2(\xi) + s_\xi \,, \quad |s_\xi| \leq \mathtt{C}_1 \langle \xi \rangle^{-4\delta} . \quad (7.2.5)$$

Let $\{z_n\}_{n \in \mathbb{N}}$ as in Theorem 4.0.1 of Chapter 4 and let $\{C_N\}_{N \in \mathbb{N}}$ be the sequence of positive constants such that

$$\left| \lambda_\xi^V - \|\xi\|^2 - \sum_{j=1}^{N} z_j(\xi) \right| \leq C_N \langle \xi \rangle^{-2\delta(N+1)} \quad \forall N \in \mathbb{N} .$$

Then, by the same Theorem 4.0.1, for all $j \in \mathbb{N}$ the family of seminorms of the symbols z_j depends only on the family of seminorms of V, and the same holds for the sequence $\{C_N\}_{N \in \mathbb{N}}$. Furthermore, since V depends on x only, the family of seminorms of V only depends on the sequence of its Sobolev norms. In particular, this entails that the following holds: for any $j \in \mathbb{N}$ there exists $N' \in \mathbb{N}$ (depending on j) such that $C_{0,0}(z_j)$ only depends on $\|V\|_{N'}$, as well as on the parameters $\delta, \epsilon, \tau, d$. Analogously, for any $N \in \mathbb{N}$ there exists $N'' \in \mathbb{N}$ such that C_N only depends on $\|V\|_{N''}$, and on the parameters $\delta, \epsilon, \tau, d$.
This in particular implies that there exists a positive integer N_1 such that the constant $\mathtt{C}_1$ in (7.2.5) only depends on $\|V\|_{N_1}$. Recall that, with the notations of Theorem 4.2.1 of Chapter 4, one has $z_1 = [v_1]$, with v_1 defined by

$$h_1 = \|\xi\|^2 + z^{(1)} + v_1 \,,$$

where $H_1 = Op\,(h_1) = e^{\mathrm{i}G_1} H e^{-\mathrm{i}G_1}$ and $z^{(1)} = [V] + V^{(\mathrm{res})}$. Thus an application of Lemma C.2.6 implies that

$$\begin{aligned} v_1 &= \frac{1}{2}\left\{\left\{\|\xi\|^2, g_1\right\}_{\mathcal{M}}, g_1\right\}_{\mathcal{M}} + \left\{V^{(\mathrm{nr})}, g_1\right\}_{\mathcal{M}} + \left\{z^{(1)}, g_1\right\}_{\mathcal{M}} + w_1 \\ &= \frac{1}{2}\left\{V^{(\mathrm{nr})}, g_1\right\}_{\mathcal{M}} + \left\{V^{(\mathrm{res})}, g_1\right\}_{\mathcal{M}} + w_1\,, \end{aligned} \tag{7.2.6}$$

with $w_1 \in S^{-4\delta,\delta}$. Then by Lemma 4.2.6 the family of seminorms of g_1 solving the equation $\{\|\xi\|^2, g_1\} + V^{(\mathrm{nr})} = 0$ only depends on the family of seminorms of V, and by Lemma 4.2.2 the same holds for the seminorms of the symbols $V^{(\mathrm{nr})}$, $V^{(\mathrm{res})}$. Thus arguing as above, by the same Egorov Theorem C.2.6 we deduce the existence of an integer N_2 such that $\sup_{\xi\in\mathbb{Z}^d} |[w_1](\xi)| \le \mathtt{C}_2\langle\xi\rangle^{-4\delta}$, for a positive constant $\mathtt{C}_2$ that again depends only on $\|V\|_{N_2}$.
Then Lemma 7.2.9 of Subsection 7.2.3 below ensures the existence of $N_3 \in \mathbb{N}$ and $\mathtt{C}_3 > 0$, depending again only on $\|V\|_{N_3}$, such that

$$[v_1](\xi) = \frac{1}{2}\sum_{\substack{k\in\mathbb{Z}^d \\ 0<\|k\|\le\langle\xi\rangle^{\epsilon}}} \frac{|\hat{V}_k|^2}{\langle \xi + \frac{k}{2}; k\rangle} + \frac{1}{2} r_\xi + \left[\{V^{(\mathrm{res})}, g_1\}_{\mathcal{M}}\right](\xi) + [w_1](\xi)\,,$$

with

$$\left|\frac{1}{2} r_\xi + \left[\{V^{(\mathrm{res})}, g_1\}_{\mathcal{M}}\right](\xi)\right| < \mathtt{C}_3\langle\xi\rangle^{-4\delta}\,.$$

Thus the thesis follows taking $\mathtt{N} = \max\{N_1, N_2, N_3\}$ and $\mathtt{C}_V = \mathtt{C}_1 + \mathtt{C}_2 + \mathtt{C}_3$. □

From now on, we will always denote by $\mathtt{N}$ the positive integer defined as in Lemma 7.2.2. Furthermore, for any $R > 0$ we define

$$B_R^{\mathtt{N}} = \{V \in \mathcal{C}^\infty(\mathbb{T}^d;\mathbb{R}) \mid \|V\|_{\mathtt{N}} \le R\}\,. \tag{7.2.7}$$

Observe then that the following holds:

Remark 7.2.3. *Since, with the notations of Lemma 7.2.2, the constant $\mathtt{C}_V$ only depends on the Sobolev norm $\|V\|_{\mathtt{N}}$ of the potential V, we deduce that there exists a finite, positive constant $\mathtt{K}_R$, depending only on R, such that*

$$\sup\{\mathtt{C}_V \mid V \in B_R^{\mathtt{N}}\} \le \mathtt{K}_R\,.$$

7.2.2 Proof of Proposition 7.2.1

Fix two non resonant sequences $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{\xi'_n\}_{n\in\mathbb{N}}$ satisfying the hypotheses of Proposition 7.2.1 and positive constants R and C. The strategy in order to prove Proposition 7.2.1 consists in exploiting the explicit expression of the second order expansion for the eigenvalues $\lambda^V_{\xi_n}$ given by Lemma 7.2.2, in order to determine a set

$$\mathcal{B} \supseteq \mathcal{V}_{R,C} = \left\{ V \in B^{\mathtt{N}}_R \;\middle|\; |\lambda^V_{\xi_n} - \lambda^V_{\xi'_n}| \le C\langle \xi_n\rangle^{-4\delta} \quad \forall n \in \mathbb{N} \right\} ,$$

and then in showing that such a $\mathcal{B}$ is a closed and nowhere dense set.
In particular, the following holds:

Lemma 7.2.4. *For any couple of non resonant sequences $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{\xi'_n\}_{n\in\mathbb{N}}$ and any positive constant R, let $\mathcal{V}_{R,C}$ as in* (7.2.1)*. Then the set defined by*

$$\mathcal{B} = \left\{ V \in B^{\mathtt{N}}_R \;:\; \forall n \in \mathbb{N} \left| \frac{1}{2} \sum_{0<\|k\|\le\langle\xi_n\rangle^{\epsilon}} |\hat{V}_k|^2 \frac{\langle \xi'_n - \xi_n; k\rangle}{\left\langle \xi_n + \frac{k}{2}; k\right\rangle \left\langle \xi'_n + \frac{k}{2}; k\right\rangle} \right| \le (2\mathtt{K}_R{+}C)\langle\xi_n\rangle^{-4\delta} \right\} \tag{7.2.8}$$

contains $\mathcal{V}_{R,C}$.

Proof. Observe that, by Lemma 7.2.2, for any $n \in \mathbb{N}$ one has

$$\begin{gathered} \lambda^V_{\xi_n} - \lambda^V_{\xi'_n} = \frac{1}{2} \sum_{0<\|k\|<\langle\xi_n\rangle^{\epsilon}} |\hat{V}_k|^2 \frac{\langle \xi'_n - \xi_n; k\rangle}{\left\langle \xi_n + \frac{k}{2}; k\right\rangle \left\langle \xi'_n + \frac{k}{2}; k\right\rangle} + \ell^V_{\xi_n} - \ell^V_{\xi'_n} , \\ \left|\ell^V_{\xi_n}\right| , \ \left|\ell^V_{\xi'_n}\right| \le \mathtt{C}_V \langle\xi_n\rangle^{-4\delta} . \end{gathered} \tag{7.2.9}$$

By Remark 7.2.3, if $V \in B^{\mathtt{N}}_R$, one has that $\left|\ell^V_{\xi_n} - \ell^V_{\xi'_n}\right| \le 2\mathtt{K}_R\langle\xi_n\rangle^{-4\delta}$. Thus, by (7.2.9), we deduce that, for any $n \in \mathbb{N}$, if V is such that $\left|\lambda^V_{\xi_n} - \lambda^V_{\xi'_n}\right| \le C\langle\xi_n\rangle^{-4\delta}$, then

$$\begin{aligned} \left| \frac{1}{2} \sum_{0<\|k\|<\langle\xi_n\rangle^{\epsilon}} |\hat{V}_k|^2 \frac{\langle \xi'_n - \xi_n; k\rangle}{\left\langle \xi_n + \frac{k}{2}; k\right\rangle \left\langle \xi'_n + \frac{k}{2}; k\right\rangle} \right| &\le \left|\lambda^V_{\xi_n} - \lambda^V_{\xi'_n}\right| + \left|\ell^V_{\xi_n} - \ell^V_{\xi'_n}\right| \\ &\le (2\mathtt{K}_R + C)\langle\xi_n\rangle^{-4\delta} . \end{aligned}$$

□

To prove Proposition 7.2.1, it remains then to show that $\mathcal{B}$ is a closed and nowhere dense set.

Lemma 7.2.5 ($\mathcal{B}$ is closed)**.** *Let $\mathcal{B}$ be defined as in* (7.2.8). *Then for any $s > \mathtt{N}$ the set $\mathcal{B}$ is closed with respect to the topology induced on $\mathcal{C}^\infty(\mathbb{T}^d;\mathbb{R})$ by the norm $\|\cdot\|_s$.*

Proof. Fix $s > \mathtt{N}$. We show that, for any $n \in \mathbb{N}$, the set

$$\mathcal{A}_n = \left\{ V \in \mathcal{C}^\infty(\mathbb{T}^d;\mathbb{R}) \ : \ \left| \frac{1}{2} \sum_{0<\|k\|\le\langle\xi_n\rangle^\epsilon} |\hat{V}_k|^2 \frac{\langle \xi_n - \xi_n'; k\rangle}{\left\langle \xi_n + \frac{k}{2}; k\right\rangle \left\langle \xi_n' + \frac{k}{2}; k\right\rangle} \right| > (2\mathtt{K}_R + C)\langle\xi_n\rangle^{-4\delta} \right\}$$

is open in $\mathcal{C}^\infty(\mathbb{T}^d;\mathbb{R})$. Then the thesis follows, since $\mathcal{B}$ is the intersection of $B_R^{\mathtt{N}}$ with the complementary set of $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ in $\mathcal{C}^\infty(\mathbb{T}^d;\mathbb{R})$ and, due to the fact that $s \ge \mathtt{N}$, $B_R^{\mathtt{N}}$ is a closed set. To show that $\mathcal{A}_n$ is open, fix $V \in \mathcal{A}_n$, define

$$0 < r_V := \left| \frac{1}{2} \sum_{0<\|k\|\le\langle\xi_n\rangle^\epsilon} |\hat{V}_k|^2 \frac{\langle \xi_n - \xi_n'; k\rangle}{\left\langle \xi_n + \frac{k}{2}; k\right\rangle \left\langle \xi_n' + \frac{k}{2}; k\right\rangle} \right| - (2\mathtt{K}_R + C)\langle\xi_n\rangle^{-4\delta}$$

and assume that $W = V + Z \in \mathcal{C}^\infty(\mathbb{T}^d;\mathbb{R})$: we are going to show the existence of a positive constant ϵ such that, if $\|Z\|_s < \epsilon$, then $W \in \mathcal{A}_n$.
Since for any $k \in \mathbb{Z}^d$ one has

$$|\hat{W}_k|^2 = |\hat{V}_k|^2 + |\hat{Z}_k|^2 + 2\mathrm{Re}(\hat{V}_k \hat{Z}_k),$$

one can compute

$$\begin{aligned} &\left| \frac{1}{2} \sum_{0<\|k\|\le\langle\xi_n\rangle^\epsilon} |\hat{W}_k|^2 \frac{\langle \xi_n - \xi_n'; k\rangle}{\left\langle \xi_n + \frac{k}{2}; k\right\rangle \left\langle \xi_n' + \frac{k}{2}; k\right\rangle} \right| - (2\mathtt{K}_R + C)\langle\xi_n\rangle^{-4\delta} \\ &\qquad \ge r_V - \frac{1}{2} \sum_{0<\|k\|\le\langle\xi_n\rangle^\epsilon} \left| |\hat{Z}_k|^2 + 2\mathrm{Re}\left(\hat{V}_k \hat{Z}_k\right) \right| \left| \frac{\langle \xi_n - \xi_n'; k\rangle}{\left\langle \xi_n + \frac{k}{2}; k\right\rangle \left\langle \xi_n' + \frac{k}{2}; k\right\rangle} \right| . \end{aligned} \tag{7.2.10}$$

We are going to estimate from above the sum in the right hand side of (7.2.10). To this purpose, observe now that, since ξ_n, ξ_n' belong to $\widetilde{\Omega}$, and $\epsilon(\tau+2) > \delta$, for any $k \in \mathbb{Z}^d$ such that $\|k\| \le \langle\xi_n\rangle^\epsilon$ one has

$$\begin{aligned} \left| \left\langle \xi_n + \frac{k}{2}; k \right\rangle \right| &\ge |\langle \xi_n; k\rangle| - \frac{1}{2}\|k\|^2 \\ &\ge \langle\xi_n\rangle^\delta \|k\|^{-\tau} - \frac{1}{2}\|k\|^2 \\ &\ge \langle\xi_n\rangle^{\delta-\epsilon\tau} - \frac{1}{2}\langle\xi_n\rangle^{2\epsilon} \ge \frac{1}{2}\langle\xi_n\rangle^{\delta-\epsilon\tau}, \end{aligned}$$

and recalling that $\|\xi_n\|^2 = \|\xi_n'\|^2$, the same holds for ξ_n', namely

$$\left|\left\langle \xi_n' + \frac{k}{2}; k \right\rangle\right| \geq \frac{1}{2}\langle \xi_n \rangle^{\delta - \epsilon\tau} .$$

Thus we obtain that, for any k with $\|k\| \leq \langle \xi_n \rangle^{\epsilon}$,

$$\left| \frac{\langle \xi_n - \xi_n'; k \rangle}{\left\langle \xi_n + \frac{k}{2}; k \right\rangle \left\langle \xi_n' + \frac{k}{2}; k \right\rangle} \right| \leq \frac{(\|\xi_n\| + \|\xi_n'\|)\, \|k\|}{\frac{1}{4}\langle \xi_n \rangle^{2\delta - 2\epsilon\tau}} \leq \frac{8\langle \xi_n \rangle^{1+\epsilon}}{\langle \xi_n \rangle^{2\delta - 2\epsilon\tau}} \leq 8 ,$$

recalling that $\delta > \frac{1}{2} + \epsilon(\tau + 1)$. This in turn implies that

$$\begin{aligned} \sum_{0<\|k\|\leq\langle\xi_n\rangle^{\epsilon}} \left| |\hat{Z}_k|^2 + 2\mathrm{Re}\left(\hat{V}_k \hat{Z}_k\right) \right| & \left| \frac{\langle \xi_n - \xi_n'; k \rangle}{\left\langle \xi_n + \frac{k}{2}; k \right\rangle \left\langle \xi_n' + \frac{k}{2}; k \right\rangle} \right| \\ & \leq 8 \sum_{0<\|k\|<\langle\xi_n\rangle^{\epsilon}} \left| |\hat{Z}_k|^2 + 2\mathrm{Re}\left(\hat{V}_k \hat{Z}_k\right) \right| \\ & \leq 8 \sum_{0<\|k\|<\langle\xi_n\rangle^{\epsilon}} \left(|\hat{Z}_k|^2 + 2|\hat{V}_k||\hat{Z}_k| \right) \\ & \leq 8\|Z\|_s^2 + 16\|V\|_s\|Z\|_s . \end{aligned} \tag{7.2.11}$$

Thus, combining (7.2.10) and (7.2.11), we obtain

$$\begin{aligned} \left| \frac{1}{2} \sum_{0<\|k\|\leq\langle\xi_n\rangle^{\epsilon}} |\hat{W}_k|^2 \frac{\langle \xi_n - \xi_n'; k \rangle}{\left\langle \xi_n + \frac{k}{2}; k \right\rangle \left\langle \xi_n' + \frac{k}{2}; k \right\rangle} \right| & - (2\mathtt{K}_R + C)\langle \xi_n \rangle^{-4\delta} \\ & \geq r_V - 4\|Z\|_s^2 - 8\|V\|_s\|Z\|_s . \end{aligned} \tag{7.2.12}$$

Assume now that $\|Z\|_s < \varepsilon = \min\{\|V\|_s,\ \frac{1}{16} r_V \|V\|_s^{-1}\}$: by (7.2.12) we deduce that

$$\left| \frac{1}{2} \sum_{0<\|k\|\leq\langle\xi_n\rangle^{\epsilon}} |\hat{W}_k|^2 \frac{\langle \xi_n - \xi_n'; k \rangle}{\left\langle \xi_n + \frac{k}{2}; k \right\rangle \left\langle \xi_n' + \frac{k}{2}; k \right\rangle} \right| - (2\mathtt{K}_R + C)\langle \xi_n \rangle^{-4\delta} > 0 ,$$

which implies $W = V + Z \in \mathcal{A}_n$. $\square$

We are only left to prove that the set $\mathcal{B}$ is nowhere dense. Such a result exploits the following auxiliary lemma:

Lemma 7.2.6. *Let $0 < \epsilon, \delta < 1$ and $\tau > d$ be such that $\epsilon(\tau + 2) < \delta$. There exists a positive constant R' (depending only on d) such that, for any couple of non resonant points ξ, ξ' with $\|\xi\|^2 = \|\xi'\|^2 > (R')^{\frac{2}{\epsilon}}$ and $\xi \neq \pm\xi'$, $B_{R'}(0)$ contains at least a vector $k \in \mathbb{Z}^d$ such that*

$$\left| \frac{\langle \xi - \xi'; k \rangle \langle \xi + \xi'; k \rangle}{\left(\|k\|^4 - 4\left(\langle \xi; k \rangle\right)^2\right)\left(|k|^4 - 4\left(\langle \xi'; k \rangle\right)^2\right)} \right| \geq \frac{2}{25} \langle \xi \rangle^{\delta - 4 - \epsilon(\tau+4)} . \tag{7.2.13}$$

Proof. We proceed in two steps: first we select a ball $B_{R'}(0)$ big enough to contain as many vectors as are needed to guarantee that at least one of them is such that $\langle \xi + \xi'; k \rangle \langle \xi - \xi'; k \rangle \neq 0$; then we use the non resonant properties of ξ, ξ' in order to prove that, if the quantity at the left hand side of (7.2.13) does not vanish, then (7.2.13) actually holds. Let $D = 2(d-1)+1$ and consider[1]

$$R' = \min\Big\{ R > 0 \;\Big|\; \text{there exist } \{h_1, \ldots, h_D\} \subset B_R(0) \cap \mathbb{Z}^d \text{ s. t. } \{h_{i_1}, \ldots, h_{i_d}\} \text{ are linearly independent vectors } \forall \{i_1, \ldots, i_d\} \subset \{1, \ldots, D\} \Big\} . \tag{7.2.14}$$

Fix then a set $\mathcal{F} = \{h_1, \ldots h_D\} \subset B_{R'}(0) \cap \mathbb{Z}^d$ such that any subset of d vectors inside $\mathcal{F}$ is made of linearly independent vectors: since $\xi - \xi' \neq 0$, there exist at most $d-1$ vectors in $\mathcal{F}$ which are orthogonal to $\xi - \xi'$, and analogously for $\xi + \xi'$. Thus, since $\mathcal{F}$ contains at least $2(d-1)+1$ vectors, we can conclude that there exists at least one $k \in \mathcal{F} \subset B_{R'}(0)$ such that

$$\langle \xi + \xi'; k \rangle \neq 0 \quad \text{and} \quad \langle \xi - \xi'; k \rangle \neq 0 . \tag{7.2.15}$$

Assume now that $\|\xi\|^2, \|\xi'\|^2 > (R')^{\frac{2}{\epsilon}}$ and consider the integer quantities $\langle \xi; k \rangle$ and $\langle \xi'; k \rangle$. Two cases hold: either they have the same sign, or they have opposite sign. In the first case, since ξ and ξ' are non resonant and $\|\xi\| = \|\xi'\| > (R')^{\frac{1}{\epsilon}} \geq \|k\|^{\frac{1}{\epsilon}}$, one has

$$| \langle \xi + \xi'; k \rangle | = | \langle \xi; k \rangle | + | \langle \xi'; k \rangle | \geq 2\langle \xi \rangle^{\delta} \|k\|^{-\tau} \geq 2\langle \xi \rangle^{\delta - \epsilon\tau} .$$

[1]The set in (7.2.14) is clearly non empty. It is easy to see how it can be inductively constructed: suppose one has $d \leq D' < D$ vectors $\{h_1, \ldots, h_{D'}\}$ such that any subset of it of cardinality d is made of linearly independent vectors. Then it is sufficient to consider the union of the hyperplanes generated by the all possible choices of $d-1$ vectors in $\{h_1, \ldots, h_{D'}\}$ and to select a vector $h_{D'+1}$ not belonging to such a union of hyperplanes.

Arguing in the same way, in the second case it is

$$|\langle \xi - \xi'; k\rangle| = |\langle \xi; k\rangle| + |\langle \xi'; k\rangle| \geq 2\langle \xi\rangle^{\delta-\epsilon\tau}.$$

Thus, recalling that $\langle \xi \pm \xi'; k\rangle$ are integer quantities, in both cases we get

$$|\langle \xi + \xi'; k\rangle \langle \xi - \xi'; k\rangle| \geq 2\langle \xi\rangle^{\delta-\epsilon\tau}.$$

Then, again by the fact that $\|\xi\|^2 = \|\xi'\|^2 \geq \|k\|^{\frac{2}{\epsilon}}$, we obtain that

$$\begin{aligned}\left|\frac{\langle \xi + \xi'; k\rangle \langle \xi - \xi'; k\rangle}{\left(\|k\|^4 - 4\left(\langle \xi; k\rangle\right)^2\right)\left(\|k\|^4 - 4\left(\langle \xi'; k\rangle\right)^2\right)}\right| &\geq \frac{2\langle \xi\rangle^{\delta-\epsilon\tau}}{\left(4\|\xi\|^2\|k\|^2 + \|k\|^4\right)\left(4\|\xi'\|^2\|k\|^2 + \|k\|^4\right)} \\ &\geq \frac{2\langle \xi\rangle^{\delta-\epsilon\tau}}{\left(5\|\xi\|^{2+2\epsilon}\right)^2} = \frac{2}{25}\langle \xi\rangle^{\delta-4-\epsilon(\tau+4)}.\end{aligned}$$

□

We are now ready to prove the following result:

Lemma 7.2.7 ($\mathcal{B}$ is nowhere dense)**.** *For any $R > 0$ and for any $s > 0$ the set $\mathcal{B}$ defined as in* (7.2.8) *has empty interior in $\mathcal{C}^\infty(\mathbb{T}^d; \mathbb{R})$ with respect to the topology induced by the metric $\|\cdot\|_s$.*

Proof. Fix $s > 0$ and $R > 0$; for any $V \in \mathcal{B}$ we show that there exists a sequence $\{W_n\}_{n\in\mathbb{N}}$ contained in $\mathcal{C}^\infty(\mathbb{T}^d; \mathbb{R})\backslash\mathcal{B}$, such that $W_n \overset{\|\cdot\|_s}{\to} V$ as $n \to \infty$. In particular, for any $n \in \mathbb{N}$ big enough we construct a potential $W_n \in \mathcal{C}^\infty(\mathbb{T}^d; \mathbb{R})$ such that

$$\left|\frac{1}{2}\sum_{0<\|k\|\leq\langle \xi_n\rangle^\epsilon} |(\hat{W}_n)_k|^2 \frac{\langle \xi_n - \xi_n'; k\rangle}{\left\langle \xi_n + \frac{k}{2}; k\right\rangle \left\langle \xi_n' + \frac{k}{2}; k\right\rangle}\right| > (2\mathrm{K}_R + C)\langle \xi_n\rangle^{-4\delta}, \quad (7.2.16)$$

which implies $W_n \notin \mathcal{B}$, and

$$\|W_n - W\|_s \leq K\langle \xi_n\rangle^{-b} \quad (7.2.17)$$

for some positive constants b and K that do not depend on n. Letting $n \to \infty$, (7.2.17) of course implies $W_n \to V$ with respect to the norm $\|\cdot\|_s$.
Fix $V \in \mathcal{B}$ and $n \in \mathbb{N}$. We look for $W_n \in \mathcal{C}^\infty(\mathbb{T}^d; \mathbb{R})$ of the form

$$W_n(x) = V(x) + \hat{z}_{k_n} e^{\mathrm{i}k_n\cdot x} + \hat{z}_{k_n} e^{-\mathrm{i}k_n\cdot x} \quad (7.2.18)$$

for some $\hat{z}_{k_n} \in \mathbb{R}$ and $k_n \in \mathbb{Z}^d$ such that $\|k_n\| \leq \langle \xi_n \rangle^{\epsilon}$. We start with ensuring that (7.2.16) is verified.
Observe that for such a W_n one has $\left|(\hat{W}_n)_k\right|^2 = \left|\hat{V}_k\right|^2$ if $k \neq \pm k_n$, while

$$\left|(\hat{W}_n)_{-k_n}\right|^2 = \left|(\hat{W}_n)_{k_n}\right|^2 = \left|\hat{V}_{k_n}\right|^2 + 2\mathrm{Re}(\hat{V}_{k_n})\hat{z}_{k_n} + |\hat{z}_{k_n}|^2 \,.$$

Thus an immediate calculation entails

$$\begin{aligned} \frac{1}{2} \sum_{0<\|k\|\leq \langle \xi_n \rangle^{\epsilon}} |(\hat{W}_n)_k|^2 \frac{\langle \xi_n - \xi_n'; k\rangle}{\left\langle \xi_n + \frac{k}{2}; k\right\rangle \left\langle \xi_n' + \frac{k}{2}; k\right\rangle} \\ = \frac{1}{2} \sum_{0<\|k\|\leq \langle \xi_n \rangle^{\epsilon}} |\hat{V}_k|^2 \frac{\langle \xi_n - \xi_n'; k\rangle}{\left\langle \xi_n + \frac{k}{2}; k\right\rangle \left\langle \xi_n' + \frac{k}{2}; k\right\rangle} + T\,, \end{aligned} \tag{7.2.19}$$

with

$$T = 2\left(|\hat{z}_{k_n}|^2 + 2\mathrm{Re}(\hat{V}_{k_n})\hat{z}_{k_n}\right) \frac{\langle \xi_n - \xi_n'; k_n\rangle \langle \xi_n + \xi_n'; k_n\rangle}{\left(\|k_n\|^4 - 4\left(\langle \xi_n; k_n\rangle\right)^2\right)\left(\|k_n\|^4 - 4\left(\langle \xi_n'; k_n\rangle\right)^2\right)}\,. \tag{7.2.20}$$

By equation (7.2.19) and recalling that $V \in \mathcal{B}$ implies

$$\left| \frac{1}{2} \sum_{0<\|k\|\leq \langle \xi_n \rangle^{\epsilon}} |\hat{V}_k|^2 \frac{\langle \xi_n - \xi_n'; k\rangle}{\left\langle \xi_n + \frac{k}{2}; k\right\rangle \left\langle \xi_n' + \frac{k}{2}; k\right\rangle} \right| \leq (2\mathrm{K}_R + C)\langle \xi_n \rangle^{-4\delta}\,,$$

it is sufficient to show that $|T| > 2(2\mathrm{K}_R + C)\langle \xi_n \rangle^{-4\delta}$, in order to ensure that (7.2.16) holds.
Let $R' > 0$ be as in Lemma 7.2.6 and suppose that $n \geq n_0$, with $n_0 \in \mathbb{N}$ such that $\|\xi_n\|^2 = \|\xi_n'\|^2 > (R')^{\frac{2}{\epsilon}}$ for any $n \geq n_0$. Then an application of the same Lemma 7.2.6 yields that it is possible to choose k_n in (7.2.18) such that $\|k_n\| \leq R'$ and

$$\left| \frac{\langle \xi_n - \xi_n'; k_n\rangle \langle \xi_n + \xi_n'; k_n\rangle}{\left(\|k_n\|^4 - 4\left(\langle \xi_n; k_n\rangle\right)^2\right)\left(\|k_n\|^4 - 4\left(\langle \xi_n'; k_n\rangle\right)^2\right)} \right| \geq \frac{2}{25}\langle \xi_n \rangle^{\delta - 4 - \epsilon(\tau+4)}\,.$$

Recalling the definition of T as in (7.2.20), this implies

$$|T| \geq \frac{4}{25}\left| |\hat{z}_{k_n}|^2 + 2\mathrm{Re}(\hat{V}_{k_n})\hat{z}_{k_n} \right| \langle \xi_n \rangle^{\delta - 4 - \epsilon(\tau+4)}\,. \tag{7.2.21}$$

Remark that, by (7.0.7), the exponent $a = \delta - 4 - \epsilon(\tau + 4)$ is greater than -4δ. Then it is sufficient to choose

$$\hat{z}_{k_n} = \langle \xi_n \rangle^{\frac{-4\delta - a}{2}} \quad \text{if} \quad \left| \mathrm{Re}(\hat{V}_{k_n}) \right| > 14(2\mathtt{K}_R + C)\langle \xi_n \rangle^{\frac{-4\delta - a}{2}} ,$$
$$\hat{z}_{k_n} = 42(2\mathtt{K}_R + C)\langle \xi_n \rangle^{\frac{-4\delta - a}{2}} \quad \text{if} \quad \left| \mathrm{Re}(\hat{V}_{k_n}) \right| \leq 14(2\mathtt{K}_R + C)\langle \xi_n \rangle^{\frac{-4\delta - a}{2}} , \tag{7.2.22}$$

in order to ensure that $\left| |\hat{z}_{k_n}|^2 + 2\mathrm{Re}(\hat{V}_{k_n})\hat{z}_{k_n} \right| > 14(\mathtt{K}_R + C)\langle \xi_n \rangle^{-4\delta - a}$ and thus, by (7.2.21),

$$|T| > \frac{4}{25} 14(\mathtt{K}_R + 1)\langle \xi_n \rangle^{-4\delta - a} \langle \xi_n \rangle^{a} > 2(\mathtt{K}_R + C)\langle \xi_n \rangle^{-4\delta} ,$$

which implies that (7.2.16) holds.
It remains to check that (7.2.17) is verified for some positive K and b. This is a consequence of the fact that $a > -4\delta$. Indeed, by (7.2.18),

$$W_n - V = \hat{z}_{k_n} e^{\mathrm{i} k_n \cdot x} + \hat{z}_{k_n} e^{-\mathrm{i} k_n \cdot x} ,$$

and by (7.2.22)

$$\| W_n - V \|_s^2 \leq 2 \| k_n \|^{2s} |\hat{z}_{k_n}|^2 \leq 2(42)^2 (R')^{2s} (2\mathtt{K}_R + C)^2 \langle \xi_n \rangle^{-4\delta - a} ,$$

which entails that (7.2.17) holds with $b = \frac{4\delta + a}{2} > 0$ and $K = \sqrt{2}42(R')^s(2\mathtt{K}_R + C)$.
□

7.2.3 Second order expansion of non resonant eigenvalues

The present subsection contains the explicit calculations required in order to determine the leading term in the expression of the symbol z_1 of the asymptotic expansion (4.0.6) exhibited in Theorem 4.0.1 of Chapter 4, and exploited in the proof of the above Lemma 7.2.2.
In the following, given $a, b \in \mathbb{R}^+$, we will simply write $a \lesssim b$ if there exists a constant $C > 0$, depending only on d and an the choice of the parameters δ, ϵ, τ (which are fixed along our construction), such that $a \leq Cb$, omitting the dependence on such parameters.

Remark 7.2.8. *For any $k \in \mathbb{Z}^d$ let $\chi_k, \tilde{\chi}_k : \mathbb{R}^d \to \mathbb{R}$ be defined as in* (4.2.8). *Then, as shown in the proof of Lemma 4.2.2 of Chapter 4, estimate* (4.2.13) *holds, namely* $\forall N \in \mathbb{N}^d$ *there exists a positive constant* C_N*, depending on* N*, such that for all* $\xi \in \mathbb{R}^d$

$$\left\| d_\xi^N \chi_k(\xi) \right\| \le C_N \|k\|^{N(\tau+1)} \langle \xi \rangle^{-N\delta}\,, \tag{7.2.23}$$

$$\left\| d_\xi^N \tilde{\chi}_k(\xi) \right\| \le C_N \|k\|^{N} \langle \xi \rangle^{-\delta N}\,. \tag{7.2.24}$$

Lemma 7.2.9. *Fix* $\epsilon > 0, \delta, \tau, d > 0$ *satisfying* (4.2.2)*. There exists a positive constant* C *such that, if* $\xi \in \Omega$ *and* $\|k\| \le \langle \xi \rangle^\epsilon$ *, then the following hold:*

(i)

$$\left| \chi_{\pm k}(\xi + \tfrac{k}{2}) \tilde{\chi}_{\pm k}(\xi + \tfrac{k}{2}) \right| \le C \|k\|^{4(2\delta+\tau+4)} \langle \xi \rangle^{-4\delta}\,, \tag{7.2.25}$$

(ii)

$$\begin{aligned} \left| \left(1 - \chi_{\pm k}(\xi + \tfrac{k}{2})\right) \tilde{\chi}_{\pm k}(\xi + \tfrac{k}{2}) \right| &= 1 + s_k^{\pm}(\xi)\,, \\ |s_k^{\pm}(\xi)| &\le C \|k\|^{4(2\delta+\tau+4)} \langle \xi \rangle^{-4\delta}\,, \end{aligned} \tag{7.2.26}$$

(iii)

$$\begin{aligned} \left| \frac{\left(1 - \chi_{\pm k}(\xi + \frac{k}{2})\right)}{\pm \left\langle \xi + \frac{k}{2}; k \right\rangle} \tilde{\chi}_{\pm}(\xi + \tfrac{k}{2}) \right| &= \frac{1}{\pm \left\langle \xi + \frac{k}{2}; k \right\rangle} + t_k^{\pm}(\xi)\,, \\ |t_k^{\pm}(\xi)| &\le C \|k\|^{8(\delta+\tau+2)} \langle \xi \rangle^{-4\delta}\,. \end{aligned} \tag{7.2.27}$$

Proof. Let us start by considering the function χ_k. A Taylor development up to 4th order yields

$$\begin{aligned} \chi_{\pm}(\xi + \tfrac{k}{2}) = \chi_{\pm k}(\xi) &+ \sum_{0<|\alpha|<4} \frac{1}{\alpha!} \partial_\xi^\alpha \chi_{\pm k}(\xi) \left(\tfrac{k}{2}\right)^\alpha \\ &+ \sum_{|\alpha|=4} \frac{4}{\alpha!} \int_0^1 (1-\tau)^{|\alpha|-1} \partial_\xi^\alpha \chi_{\pm k}(\xi + \tau \tfrac{k}{2})\, d\tau \left(\tfrac{k}{2}\right)^\alpha\,. \end{aligned}$$

We have that $\chi_k \equiv 0$ on Ω. Thus all its derivatives at any order vanish on Ω, and we deduce that

$$\chi_{\pm k}(\xi + \tfrac{k}{2}) = \sum_{|\alpha|=4} \frac{4}{\alpha!} \int_0^1 (1-\tau)^{|\alpha|-1} \partial_\xi^\alpha \chi_{\pm k}(\xi + \tau \tfrac{k}{2})\, d\tau \left(\tfrac{k}{2}\right)^\alpha\,.$$

By the first of (7.2.23), this implies

$$\begin{aligned} \left| \chi_{\pm k}(\xi + \tfrac{k}{2}) \right| &\lesssim \sup_{|\tau| \le 1} \langle \xi + \tau \tfrac{k}{2} \rangle^{-4\delta} \|k\|^{4+4(\tau+1)} \\ &\lesssim \langle \xi \rangle^{-4\delta} \|k\|^{4(\delta+\tau+2)}\,. \end{aligned} \tag{7.2.28}$$

We argue analogously for $\tilde{\chi}_k$: we have $\tilde{\chi}_k \equiv 1$ on Ω, thus a 4th order Taylor development entails that, if $\xi \in \Omega$,

$$\tilde{\chi}_{\pm k}(\xi + \tfrac{k}{2}) = 1 + \sum_{|\alpha|=4} \frac{4}{\alpha!} \int_0^1 (1-\tau)^{|\alpha|-1} \partial_\xi^\alpha \tilde{\chi}_{\pm k}(\xi + \tau \tfrac{k}{2})\, d\tau \left(\tfrac{k}{2}\right)^\alpha ,$$

and by the second estimate in (7.2.23) we have

$$\begin{aligned} \left|\tilde{\chi}_{\pm k}(\xi + \tfrac{k}{2}) - 1\right| &\lesssim \sup_{|\tau|\le 1} \langle \xi + \tau \tfrac{k}{2} \rangle^{-4\delta} \|k\|^8 \\ &\lesssim \langle \xi \rangle^{-4\delta} \|k\|^{8+4\delta} . \end{aligned} \tag{7.2.29}$$

Combining (7.2.28) and (7.2.29), one gets

$$\begin{aligned} |\chi_{\pm k}(\xi + \tfrac{k}{2}) \tilde{\chi}_{\pm k}(\xi + \tfrac{k}{2})| &\le |\chi_{\pm k}(\xi + \tfrac{k}{2})| + |\chi_{\pm k}(\xi + \tfrac{k}{2})||1 - \tilde{\chi}_{\pm k}(\xi + \tfrac{k}{2})| \\ &\lesssim \langle \xi \rangle^{-4\delta} \|k\|^{4(2\delta+\tau+4)} , \end{aligned}$$

which proves (7.2.25). Analogously, observing that

$$\left|\left(1 - \chi_{\pm k}(\xi + \tfrac{k}{2})\right) \tilde{\chi}_{\pm k}(\xi + \tfrac{k}{2})\right| \le 1 + s_k^\pm(\xi) ,$$

with $s_k^\pm(\xi) = |\chi_{\pm k}(\xi + \frac{k}{2})| + \left(1 + |\chi_{\pm k}(\xi + \frac{k}{2})|\right) |1 - \tilde{\chi}_{\pm k}(\xi + \frac{k}{2})|$, and using again (7.2.28) and (7.2.29) to deduce that

$$|s_k^\pm(\xi)| \lesssim \langle \xi \rangle^{-4\delta} \|k\|^{4(2\delta+\tau+4)} , \tag{7.2.30}$$

one proves (7.2.26). Finally, (7.2.27) is deduced observing that, by the very definition of the function χ_k,

$$\frac{\left(1 - \chi_{\pm k}(\xi + \frac{k}{2})\right)}{\pm \left\langle \xi + \frac{k}{2}; k \right\rangle} \ne 0 \text{ only if } \left|\left\langle \xi + \tfrac{k}{2}; k \right\rangle\right| \ge \langle \xi + \tfrac{k}{2} \rangle^\delta \|k\|^{-\tau} , \tag{7.2.31}$$

and, in such a case,

$$\frac{\left(1 - \chi_{\pm k}(\xi + \frac{k}{2})\right)}{\pm \left\langle \xi + \frac{k}{2}; k \right\rangle} \tilde{\chi}_{\pm k}(\xi + \tfrac{k}{2}) = \frac{1}{\pm \left\langle \xi + \frac{k}{2}; k \right\rangle} + \frac{s_k^\pm(\xi)}{\pm \left\langle \xi + \frac{k}{2}; k \right\rangle} .$$

Then, by (7.2.30) and (7.2.31), one deduces

$$\begin{aligned} \left| \frac{s_k^\pm(\xi)}{\left\langle \xi + \frac{k}{2}; k \right\rangle} \right| &\le \frac{|s_k^\pm(\xi)|}{\langle \xi + \frac{k}{2} \rangle^\delta} \|k\|^\tau \\ &\le |s_k^\pm(\xi)| \|k\|^\tau \lesssim \langle \xi \rangle^{-4\delta} \|k\|^{8(\delta+\tau+2)} , \end{aligned}$$

which gives (7.2.27). □

Lemma 7.2.10. *Given a smooth potential V, let $V^{(\mathrm{nr})}, V^{(\mathrm{res})}$ be as in Definition 4.2.9 and let g be defined as in* (4.2.25)*, with the symbol a replaced by the potential V. Then there exists a positive constant C, depending only on $\|V\|_{N'}$ for some positive integer N', such that, for all $\xi \in \Omega$,*

$$\left[\left\{V^{(\mathrm{nr})}, g\right\}_{\mathcal{M}}\right](\xi) = \sum_{\substack{k\in\mathbb{Z}^d \\ 0<\|k\|\le\langle\xi\rangle^{\epsilon}}} \frac{|\hat{V}_k|^2}{\left\langle \xi + \frac{k}{2}; k\right\rangle} + r_\xi\,, \quad |r_\xi| \le C\langle\xi\rangle^{-4\delta}\,, \tag{7.2.32}$$

and

$$\left|\left[\left\{V^{(\mathrm{res})}, g\right\}_{\mathcal{M}}\right](\xi)\right| \le C\langle\xi\rangle^{-4\delta}\,. \tag{7.2.33}$$

Proof. We start by proving (7.2.32). Recalling the explicit definition of $V^{(\mathrm{nr})}$ and g, we deduce that

$$\left[\{V^{(\mathrm{nr})},\ g\}_{\mathcal{M}}\right](\xi) = -\mathrm{i}\left(\left[V^{(\mathrm{nr})}\sharp\ g\right](\xi) - \left[g\sharp\ V^{(\mathrm{nr})}\right](\xi)\right)\,,$$

and by Lemma C.2.5 of Appendix C one has

$$\begin{aligned}\left[V^{(\mathrm{nr})}\sharp\ g\right](\xi) &= \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \hat{V}^{(\mathrm{nr})}_{-k}(\xi+\tfrac{k}{2})\hat{g}_k(\xi+\tfrac{k}{2}) \\ &= -\sum_{k\in\mathbb{Z}^d\setminus\{0\}} \hat{V}_{-k}\left(1-\chi_{-k}(\xi+\tfrac{k}{2})\right)\tilde{\chi}_{-k}(\xi+\tfrac{k}{2})\hat{V}_k\frac{\left(1-\chi_k(\xi+\frac{k}{2})\right)}{2\mathrm{i}\left\langle\xi+\frac{k}{2};k\right\rangle}\tilde{\chi}_k(\xi+\tfrac{k}{2})\end{aligned}$$

and

$$\left[g\sharp\ v^{(\mathrm{nr})}\right](\xi) = -\sum_{k\in\mathbb{Z}^d\setminus\{0\}} \hat{V}_{-k}\frac{\left(1-\chi_{-k}(\xi+\frac{k}{2})\right)}{2\mathrm{i}\left\langle\xi+\frac{k}{2};-k\right\rangle}\tilde{\chi}_{-k}(\xi+\tfrac{k}{2})\hat{V}_k\left(1-\chi_k(\xi+\tfrac{k}{2})\right)\tilde{\chi}_k(\xi+\tfrac{k}{2})\,.$$

Let us analyze $\left[V^{(\mathrm{nr})}\sharp\ g\right](\xi)$. The idea is to split the sum into a term M_2 containing only high Fourier modes, namely those corresponding to $\|k\| > \langle\xi\rangle^{\epsilon}$, whose contribution is neglectable due to the regularity of V, and a term M_1 containing only low Fourier modes, namely those related to a vector k with $\|k\| \le \langle\xi\rangle^{\epsilon}$, where we can apply the previous Lemma 7.2.9.
Indeed, we compute

$$\begin{aligned}\left[V^{(\mathrm{nr})}\sharp\ g\right](\xi) &= -\sum_{\substack{k\in\mathbb{Z}^d \\ 0<\|k\|\le\langle\xi\rangle^{\epsilon}}} |\hat{V}_k|^2\left(1-\chi_{-k}(\xi+\tfrac{k}{2})\right)\tilde{\chi}_{-k}(\xi+\tfrac{k}{2})\frac{\left(1-\chi_k(\xi+\frac{k}{2})\right)}{2\mathrm{i}\left\langle\xi+\frac{k}{2};k\right\rangle}\tilde{\chi}_k(\xi+\tfrac{k}{2}) \\ &\quad - \sum_{\substack{k\in\mathbb{Z}^d \\ \|k\|>\langle\xi\rangle^{\epsilon}}} |\hat{V}_k|^2\left(1-\chi_{-k}(\xi+\tfrac{k}{2})\right)\tilde{\chi}_{-k}(\xi+\tfrac{k}{2})\frac{\left(1-\chi_k(\xi+\frac{k}{2})\right)}{2\mathrm{i}\left\langle\xi+\frac{k}{2};k\right\rangle}\tilde{\chi}_k(\xi+\tfrac{k}{2}) \\ &=: M_1(\xi) + M_2(\xi)\,. \end{aligned} \tag{7.2.34}$$

To estimate M_2, we argue as in the proof of Lemma 7.2.9 to deduce that, if $1 - \chi_k(\xi + \frac{k}{2}) \neq 0$, one has

$$|(\xi + \tfrac{k}{2}) \cdot k| \geq \langle \xi + \tfrac{k}{2} \rangle^{\delta} \|k\|^{-\tau} \geq \|k\|^{-\tau} .$$

Thus, since the functions $1 - \chi_k$ and $\tilde{\chi}_k$ are bounded by 1, we get

$$|M_2(\xi)| \leq \sum_{\substack{k \in \mathbb{Z}^d \\ \|k\| > \langle \xi \rangle^{\epsilon}}} |\hat{V}_k|^2 \|k\|^{\tau} .$$

Recalling that, since $V \in \mathcal{H}^s$ for any s, its Fourier coefficients satisfy

$$|\hat{V}_k| \leq \|V\|_s \langle k \rangle^{-s} \quad \forall s > 0 , \tag{7.2.35}$$

and choosing $s = \lceil 4\delta\epsilon^{-1} \rceil + \tau + d + 1$, we obtain that

$$\begin{aligned} |\hat{V}_k| &\leq \|V\|_s \langle k \rangle^{-4\delta\epsilon^{-1}} \langle k \rangle^{-(\tau+d+1)} \\ &\leq \|V\|_s \langle \xi \rangle^{-4\delta} \langle k \rangle^{-(\tau+d+1)} \end{aligned}$$

for all those k such that $\|k\| > \langle \xi \rangle^{\epsilon}$. Thus, we deduce

$$\begin{aligned} |M_2(\xi)| &\leq \sum_{\substack{k \in \mathbb{Z}^d \\ \|k\| > \langle \xi \rangle^{\epsilon}}} |\hat{V}_k|^2 \|k\|^{\tau} \\ &\leq \|V\|_s^2 \sum_{\substack{k \in \mathbb{Z}^d \\ \|k\| > \langle \xi \rangle^{\epsilon}}} \langle \xi \rangle^{-4\delta} \langle k \rangle^{-(\tau+d+1)} \|k\|^{\tau} \leq C_2 \langle \xi \rangle^{-4\delta} , \end{aligned}$$

with $C_2 = \|V\|_s^2 \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-(d+1)}$.
The estimate of $M_1(\xi)$ is an application of Lemma 7.2.9: since

$$\begin{aligned} \big(1 - \chi_{-k}(\xi + \tfrac{k}{2})\big)\, \tilde{\chi}_{-k}(\xi + \tfrac{k}{2}) \frac{\big(1 - \chi_k(\xi + \frac{k}{2})\big)}{2\mathrm{i} \left\langle \xi + \frac{k}{2}; k \right\rangle} \tilde{\chi}_k(\xi + \tfrac{k}{2}) \\ = \big(1 - s_k^-(\xi)\big) \left(\frac{1}{2\mathrm{i} \left\langle \xi + \frac{k}{2}; k \right\rangle} + t_k^+(\xi) \right) , \end{aligned}$$

with $|s_k^-(\xi)| \leq C \|k\|^{4(2\delta+\tau+4)} \langle \xi \rangle^{-4\delta}$ and $|t_k^+(\xi)| \leq C \|k\|^{8(\delta+\tau+2)} \langle \xi \rangle^{-4\delta}$ for some positive constant C, we have

$$M_1(\xi) = \sum_{\substack{k \in \mathbb{Z}^d \\ 0 < \|k\| \leq \langle \xi \rangle^{\epsilon}}} |\hat{V}_k|^2 \left(\frac{1}{2\mathrm{i} \left\langle \xi + \frac{k}{2}; k \right\rangle} + s_k^-(\xi) + t_k^+(\xi) + s_k^-(\xi) t_k^+(\xi) \right) ,$$

and

$$\left| \sum_{\substack{k\in\mathbb{Z}^d \\ 0<\|k\|\leq\langle\xi\rangle^\epsilon}} |\hat{V}_k|^2 \left(s_k^-(\xi) + t_k^+(\xi) + s_k^-(\xi)t_k^+(\xi)\right) \right| \leq 3C^2 \sum_{\substack{k\in\mathbb{Z}^d \\ 0<\|k\|\leq\langle\xi\rangle^\epsilon}} |\hat{V}_k|^2 \|k\|^{16\delta+12\tau+24} \langle\xi\rangle^{-4\delta}$$
$$\leq C_1 \langle\xi\rangle^{-4\delta} \,,$$

with $C_1 = 3C^2\|V\|_{s'}^2 \sum_{k\in\mathbb{Z}^d} \langle k\rangle^{-(d+1)}$ and $s' = \lceil 16\delta + 12\tau \rceil + 24 + d + 1$. This enables to conclude that

$$\begin{gathered} \left[V^{(\mathrm{nr})}\sharp\, g\right](\xi) = -\sum_{\substack{k\in\mathbb{Z}^d \\ 0<\|k\|\leq\langle\xi\rangle^\epsilon}} |\hat{V}_k|^2 \frac{1}{2\mathrm{i}\left\langle \xi + \frac{k}{2}; k \right\rangle} + r_{1,\xi} \,, \\ |r_{1,\xi}| \leq (C_1 + C_2)\langle\xi\rangle^{-4\delta} \,. \end{gathered} \tag{7.2.36}$$

Arguing analogously to estimate the term $\left[g\sharp\, V^{(\mathrm{nr})}\right](\xi)$, we get that

$$\begin{gathered} \left[g\sharp\, V^{(\mathrm{nr})}\right](\xi) = -\sum_{\substack{k\in M \\ 0<\|k\|\leq\langle\xi\rangle^\epsilon}} |\hat{V}_k|^2 \frac{1}{2\mathrm{i}\left\langle \xi + \frac{k}{2}; -k \right\rangle} + r_{2,\xi} \,, \\ |r_{2,\xi}| \leq (C_1 + C_2)\langle\xi\rangle^{-4\delta} \,. \end{gathered} \tag{7.2.37}$$

Recalling the definition of $\{f, g\}_{\mathcal{M}}$ as $-\mathrm{i}\,(f\sharp g - g\sharp f)$ and combining the contributions of (7.2.36) and (7.2.37), we obtain (7.2.32).
Concerning (7.2.33), the estimate of $\left[\{V^{(\mathrm{res})}, g\}_{\mathcal{M}}\right](\xi)$ follows arguing in an analogous way: one has

$$\left[\{V^{(\mathrm{res})},\ g\}_{\mathcal{M}}\right](\xi) = \frac{1}{\mathrm{i}} \left(\left[V^{(\mathrm{res})}\sharp\, g\right](\xi) - \left[g\sharp\, V^{(\mathrm{res})}\right](\xi) \right) \,,$$

with

$$\left[V^{(\mathrm{res})}\sharp\, g\right](\xi) = -\sum_{k\in\mathbb{Z}^d\setminus\{0\}} |\hat{V}_k|^2 \chi_{-k}(\xi+\tfrac{k}{2}) \tilde{\chi}_{-k}(\xi+\tfrac{k}{2}) \frac{\left(1 - \chi_k(\xi + \frac{k}{2})\right)}{2\mathrm{i}\left\langle \xi + \frac{k}{2}; k \right\rangle} \tilde{\chi}_k(\xi+\tfrac{k}{2}) \tag{7.2.38}$$

and

$$\left[g\sharp\, V^{(\mathrm{res})}\right](\xi) = -\sum_{k\in\mathbb{Z}^d\setminus\{0\}} |\hat{V}_k|^2 \frac{\left(1 - \chi_{-k}(\xi + \frac{k}{2})\right)}{2\mathrm{i}\left\langle \xi + \frac{k}{2}; -k \right\rangle} \tilde{\chi}_{-k}(\xi+\tfrac{k}{2}) \chi_k(\xi+\tfrac{k}{2}) \tilde{\chi}_k(\xi+\tfrac{k}{2}) \,. \tag{7.2.39}$$

Then one splits each one of the sums in (7.2.38), (7.2.39) in two pieces, the first one supported on high Fourier modes, satisfying the condition $\|k\| > \langle \xi \rangle^{\epsilon}$, which are dealt with as the term M_1 of (7.2.34), and the second one supported on Fourier modes satisfying $\|k\| \leq \langle \xi \rangle^{\epsilon}$, which is dealt with exploiting Lemma 7.2.9, as the term M_2 defined in (7.2.34). □

Appendix A

Pseudo-differential calculus and structural hypotheses

This chapter is devoted to a brief exposition of a few basic notions of pseudo-differential calculus, which is a fundamental tool for the reducibility result of Chapter 2. In particular, pseudo-differential calculus on the $d-$ dimensional torus $\mathbb{T}^d$ is used here, with classical quantization of the operators and with the standard coordinate dependent definition on symbols. A specific focus is given here on the preservation of structural hypotheses considered in Chapter 2, namely symmetric hyperbolicity, reality and reversibility. Since pseudo-differential calculus is widely used even in Chapters 4 –7, but with a slightly different setting, we point out that in Appendix C another class of symbols is analyzed, which is suitable to work on the torus $\mathbb{T}^d$ equipped with a non Euclidean Riemannian flat metric; in particular intrinsic definitions are given therein, and Weyl quantization of symbols is used.
Recall that the following definition of symbols and their quantization is considered here:

Definition A.0.1. *Let $m \in \mathbb{R}$. We say that $a \in \mathcal{C}^\infty\left(\mathbb{T}^d \times \mathbb{R}^d; \mathbb{C}\right)$ is a symbol of class S^m if for any multiindex $\alpha, \beta \in \mathbb{N}^d$ there exists a positive constant $C_{\alpha,\beta}$ such that*

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le C_{\alpha,\beta}\langle\xi\rangle^{m-|\beta|} \quad \forall (x,\xi) \in \mathbb{T}^d \times \mathbb{R}^d\,.$$

Definition A.0.2 (Classical quantization)**.** *Let $m \in \mathbb{R}$ and $a \in S^m$. Then we say that $A = Op^{cl}(a)$ is a pseudo-differential operator of order m, $A \in$*

OPS^m, *if* A *acts on* $u \in L^2(\mathbb{T}^d)$ *as*

$$A : u = \sum_{\xi \in \mathbb{Z}^d} \hat{u}_\xi e^{\mathrm{i}\xi \cdot x} \longmapsto Au = \sum_{\xi \in \mathbb{Z}^d} a(x,\xi) \hat{u}_\xi e^{\mathrm{i}\xi \cdot x} . \tag{A.0.1}$$

Actually, the one described in Definition A.0.2 is not the only possible way of putting in correspondence operators and symbols. Given $a \in \mathcal{C}^\infty(\mathbb{T}^d \times \mathbb{R}^d; \mathbb{C})$, define its Fourier coefficients with respect to the variable $x \in \mathbb{T}^d$ as

$$\hat{a}_k(\xi) = \int_{\mathbb{T}^d} a(x,\xi) e^{-\mathrm{i}\xi \cdot x} \, dx \quad \forall k \in \mathbb{Z}^d, \quad \xi \in \mathbb{R}^d ; \tag{A.0.2}$$

then for all $t \in [0,1]$ one can define an operator $Op^{\,t}(a)$ of a symbol $a \in S^m$, acting on a generic function $u \in L^2(\mathbb{T}^d)$ as

$$Op^{\,t}(a)u(x) = \sum_{\xi \in Z^d} \hat{u}_\xi e^{\mathrm{i}\xi \cdot x} \longmapsto Op^{\,t}(a)u(x) = \sum_{\xi \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \hat{a}_k \left(\xi + tk\right) \hat{u}_k e^{\mathrm{i}(k+\xi)\cdot x}$$

(see for instance [Rob87]). Notice that with these notations classical quantization, namely the symbol-operator associaton $a \mapsto Op^{cl}(a)$ exhibited in the above Definition A.0.2, also referred to as left quantization, corresponds to $t = 0$, whereas Weyl quantization, which is used in Chapters 4–7 below, corresponds to $t = \frac{1}{2}$.
Pseudo-differential operators are closed under the operation of composition. Of course this implies that, given two pseudo-differential operators A and B, their commutator is again a pseudo-differential operator. In particular, the following standard result holds (see [Tay91]):

Lemma A.0.3. *Let* $A \in \mathcal{L}ip\left(\Omega_0; \mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}}; OPS^m\right)\right)$ *and* $B \in \mathcal{L}ip\left(\Omega_0; \mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}}; OPS^{m'}\right)\right)$.

(i) Then their composition AB *is such that* $AB \in \mathcal{L}ip\left(\Omega_0; \mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}}; OPS^{m+m'}\right)\right)$

(ii) Their commutator $[A,B] = AB - BA$ *satisfies* $[A,B] \in \mathcal{L}ip\left(\Omega_0; \mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}}; OPS^{m+m'-1}\right)\right)$.

It immediately follows by the definition of reality that the composition of two real operators is real, as well as that the composition of a reversible and a reversibility preserving operator is reversible. In the following section we describe the preservation of such properties under conjugation with a particular class of pseudo-differential operators.

A.1 Flow of operators of order $\eta \leq 1$ and Egorov Theorem

To regularize (2.0.1), we make use of operators that are the flow at time $\tau \in [-1,\ 1]$ of the PDE

$$\partial_\tau u = G(\varphi) u$$

for a given pseudo differential operator $G(\varphi) \in OPS^\eta,\ \eta \leq 1$. An operator of this sort is denoted by $e^{\tau G}$. Here we state some of its main properties.

Lemma A.1.1. *Let $\eta \leq 1$ and $G(\varphi) \in \mathcal{C}^\infty(\mathbb{T}^\mathfrak{n}; OPS^\eta)$ be such that $G(\varphi) + G(\varphi)^* \in OPS^0$ and let $e^{\tau G}$ be the flow of the autonomous PDE $\partial_\tau u = G(\varphi)u, \quad \tau \in [-1, 1]$.*
Then the following results hold $\forall \sigma \geq 0$:
(i) $e^{\tau G}(\varphi) \in \mathcal{B}(\mathcal{H}^\sigma, \mathcal{H}^\sigma)$
(ii) $\forall\ \alpha \in \mathbb{N}^\mathfrak{n},\ \partial_\varphi^\alpha e^{\tau G}(\varphi) \in \mathcal{B}\left(\mathcal{H}^\sigma, \mathcal{H}^{\sigma - \eta|\alpha|}\right)$
(iii) If $G \in \mathcal{L}ip\left(\Omega;\ \mathcal{C}^\infty(\mathbb{T}^\mathfrak{n}; OPS^\eta)\right),\ \partial_\varphi^\alpha e^{\tau G}(\varphi, \omega) \in \mathcal{L}ip\left(\Omega;\ \mathcal{B}\left(\mathcal{H}^\sigma, \mathcal{H}^{\sigma - \eta|\alpha| - \eta}\right)\right)$ $\forall\ \alpha \in \mathbb{N}^\mathfrak{n}$
Furthermore, if G is reversibility preserving (or real), $e^{\tau G}$ is reversibility preserving (resp. real) too.

Proof. Item (i) is a well known result. It is proved trough a Galerkin type approximation on the subspace E_N of the compact supported sequences $\{\hat{u}_k\}_{k \in \mathbb{Z}^d}$ such that $\hat{u}_k = 0 \quad \text{if } |k| > N$. See [Tay91], Section 0.8, for details.
Items (ii) and (iii) are obtained arguing as in Lemma A.3 of [BM20b]; here we exhibit the proof for the sake of completeness. Item (ii) is proved by induction. The thesis for $|\alpha| = 0$ follows from (i); suppose now that $\forall\ \alpha' \in \mathbb{N}^\mathfrak{n}$ with $|\alpha'| \leq |\alpha| \quad \partial_\varphi^\alpha e^{\tau G}(\varphi) \in \mathcal{B}\left(\mathcal{H}^\sigma, \mathcal{H}^{\sigma - \eta|\alpha|}\right)$.
If $|\alpha^+| = |\alpha| + 1,\ \partial_\varphi^{\alpha^+} e^{\tau G}$ solves the differential equation

$$\partial_\tau\left(\partial_\varphi^{\alpha^+} e^{\tau G}\right) = G\ \left(\partial_\varphi^{\alpha^+} e^{\tau G}\right) + \sum_{|\alpha'| \leq |\alpha|} \binom{\alpha}{\alpha'} \left(\partial_\varphi^{\alpha'} G\right)\left(\partial_\varphi^{\alpha - \alpha'} e^{\tau G}\right).$$

Hence, using Duhamel representation formula, we get

$$\partial_\varphi^{\alpha^+} e^{\tau G} = \int_0^\tau e^{(\tau - \tau')G} \left(\sum_{|\alpha'| \leq |\alpha|} \binom{\alpha}{\alpha'} \left(\partial_\varphi^{\alpha'} G\right)\left(\partial_\varphi^{\alpha - \alpha'} e^{\tau' G}\right)\right)\ d\tau';$$

since $\forall\ \sigma$ one has $\partial_\varphi^{\alpha'} G \in \mathcal{B}\left(\mathcal{H}^\sigma, \mathcal{H}^{\sigma-\eta}\right)$, $\partial_\varphi^{\alpha-\alpha'} e^{\tau' G} \in \mathcal{B}\left(\mathcal{H}^\sigma, \mathcal{H}^{\sigma-\eta(|\alpha|-|\alpha'|)}\right)$ and $e^{(\tau-\tau')G} \in \mathcal{B}\left(\mathcal{H}^\sigma, \mathcal{H}^\sigma\right)$, we have

$$\|\partial_\varphi^{\alpha^+} e^{\tau G}\|_{\mathcal{B}(\mathcal{H}^\sigma,\mathcal{H}^{\sigma-\eta|\alpha^+|})} \leq \sum_{|\alpha'|\leq|\alpha|} \binom{\alpha}{\alpha'} \|\partial_\varphi^{\alpha'} G\|_{\mathcal{B}(\mathcal{H}^\sigma,\mathcal{H}^{\sigma-\eta})} \|\partial_\varphi^{\alpha-\alpha'} e^{\tau' G}\|_{\mathcal{B}(\mathcal{H}^{\sigma-\eta},\mathcal{H}^{\sigma-\eta(1+|\alpha|-|\alpha'|)})}$$
$$\cdot \|e^{(\tau-\tau')G}\|_{\mathcal{B}(\mathcal{H}^{\sigma-\eta(1+|\alpha|-|\alpha'|)},\mathcal{H}^{\sigma-\eta(1+|\alpha|-|\alpha'|)})}.$$

Item (iii) is proved as item (ii), hence we omit the details. Arguing again by induction, if $|\alpha^+| = |\alpha| + 1$, it is sufficient to write the differential equation solved by $\Delta_\omega \partial_\varphi^{\alpha^+} e^{\tau G} := \partial_\varphi^{\alpha^+} e^{\tau G(\omega_2,\varphi)} - \partial_\varphi^{\alpha^+} e^{\tau G(\omega_1,\varphi)}$ and to use Duhamel representation formula again to get

$$\Delta_\omega \partial_\varphi^{\alpha^+} e^{\tau G(\omega,\varphi)} = \int_0^\tau e^{\tau G(\omega_1,\varphi)} \Bigg[\left(\partial_\varphi^{\alpha^+} e^{sG(\omega_1,\varphi)}\right) \Delta_\omega G(\omega,\varphi)$$
$$+ \sum_{0\leq|k|\leq|\alpha|} \binom{\alpha}{k} \left(\left(\Delta_\omega \partial_\varphi^k e^{\tau G(\omega,\varphi)}\right) \left(\partial_\varphi^{\alpha-k} e^{sG(\omega_2,\varphi)}\right) + \left(\partial_\varphi^k G(\omega_1,\varphi)\right) \left(\Delta_\omega \partial_\varphi^{\alpha-k} e^{sG(\omega,\varphi)}\right)\right) \Bigg] ds.$$

The thesis then follows using item (ii) and observing that $\forall\ \sigma$

$$\sup_{\omega_1,\omega_2\in\Omega} (\Delta\omega)^{-1} \Delta_\omega G(\omega,\varphi) \in \mathcal{B}\left(\mathcal{H}^\sigma, \mathcal{H}^{\sigma-\eta}\right),$$
$$\sup_{\omega_1,\omega_2\in\Omega} (\Delta\omega)^{-1} \Delta_\omega \partial_\varphi^k e^{\tau G(\omega,\varphi)} \in \mathcal{B}\left(\mathcal{H}^\sigma, \mathcal{H}^{\sigma-\eta|k|-\eta}\right) \quad \forall k \in \mathbb{N}^{\mathfrak{n}}.$$

Reversibility preserving property: We remark that since

$$S \circ \partial_t = \partial_t \circ S,$$

one both has

$$\partial_t [S \circ e^{\tau G(\varphi)}] u = S \circ \partial_t \circ e^{\tau G(\varphi)} u = S \circ G(\varphi) e^{\tau G(\varphi)} u = G(-\varphi) \circ S\ u$$

and

$$\partial_t [e^{\tau G(-\varphi)} \circ S] u = G(-\varphi) \circ S\ u.$$

Since $S \circ e^{\tau G(\varphi)}$ and $e^{\tau G(-\varphi)} \circ S$ solve the same initial value problem for all the functions $u(x)$, they must coincide. Thus we can deduce the reversibility preserving property for $e^{\tau G(\varphi)}$.

Reality: The proof can be done arguing similarly, using the fact that, since $G = \overline{G}$, then $e^{\tau G(\varphi)}$ and $\overline{e^{\tau G(\varphi)}}$ solve the same initial value problem. □

In the following we restrict to an analysis of the flow of a pseudo-differential operator of order $\eta < 1$. The following Remark is an immediate consequence of Lemma A.0.3:

Remark A.1.2. *Let* $A(\varphi) \in \mathcal{L}ip\,(\Omega;\ \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}}; OPS^m))$ *and* $G \in \mathcal{L}ip\,(\Omega;\ \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}}; OPS^\eta))$, *with* $\eta < 1$. *If* $\forall\ j \in \mathbb{N}$ *we define*

$$Ad^0_G A = A, \quad Ad^{j+1}_G A = [G, Ad^j_G A],$$

then

$$Ad^j_G A \in \mathcal{L}ip\left(\Omega;\ \mathcal{C}^\infty\left(\mathbb{T}^{\mathtt{n}}; OPS^{m-j(1-\eta)}\right)\right) \quad \forall\ j \in \mathbb{N}.$$

The following version of Egorov Theorem holds:

Proposition A.1.3. *Let* $A(\varphi) \in \mathcal{L}ip\,(\Omega;\ \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}}; OPS^m))$ *and* $G \in \mathcal{L}ip\,(\Omega;\ \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}}; OPS^\eta))$, *with* $\eta < 1$ *and* G *such that* $G(\varphi) + G(\varphi)^* \in OPS^0$. *Then* $\forall \tau \in [-1, 1]$

$$e^{\tau G} A e^{-\tau G} \in \mathcal{L}ip\,(\Omega;\ \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}}; OPS^m))\ .$$

Proof. This version of Egorov Theorem is actually simpler than the one stated in Theorem A.0.9 in [Tay91]. The reason is that the order of G is strictly smaller than one and hence one has the asymptotic expansion

$$e^{\tau G} A e^{-\tau G} \sim \sum_{j=0}^{\infty} \frac{\tau^j}{j!} Ad^j_G A\,,$$

with $Ad^j_G A \in OPS^{m-j(1-\eta)}$ (see Remark A.1.2). □

The following result states the preservation of symmetric hyperbolicity property under conjugation with the flow of a symmetric hyperbolic operator G:

Lemma A.1.4. *Let* $\eta < 1$, $G \in \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}}; OPS^\eta)$ *with* $G + G^* \in OPS^0$ *and* $A \in \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}}; OPS^1)$. *Then*

(i)

$$Ad^k_G A + Ad^k_{-G^*} A^* \in OPS^{-(k-1)(1-\eta)} \quad \forall\ k \geq 1;$$

(ii) In particular,

$$\left(e^G A e^{-G} - A\right) + \left(e^G A e^{-G} - A\right)^* \in OPS^0.$$

Proof. Proof of (i).We argue by induction: if $k=1$, one has

$$\begin{aligned}[G,A]-[G^*,\ A^*] &= [G,A]+[A^*,G^*]\\ &= [G,A+A^*]+[A^*,\ G+G^*]\in OPS^0.\end{aligned}$$

Assume that for some $k\geq 1$

$$Ad_G^kA+(Ad_G^kA)^*\in OPS^{-(k-1)(1-\eta)}\,.$$

A direct calculation shows that

$$Ad_G^{k+1}A+(Ad_G^{k+1}A)^*=[G+G^*,Ad_G^kA]-[G^*,Ad_G^kA+(Ad_G^kA)^*]\,.$$

Since by Remark A.1.2 $Ad_G^kA,(Ad_G^kA)^*\in OPS^{1-k(1-\eta)}$ and using the induction hypothesis and that $G^*\in OPS^\eta$, $G+G^*\in OPS^0$, one obtains that $Ad_G^{k+1}A+(Ad_G^{k+1}A)^*\in OPS^{-k(1-\eta)}$.
Proof of (ii). $\forall\ M>0$ one computes

$$e^GAe^{-G}-A=\sum_{k=1}^{M}\frac{Ad_G^kA}{k!}+\int_0^1\frac{(1-s)^{M+1}}{(M+1)!}e^{sG}Ad_G^{M+1}Ae^{-sG}.$$

By applying Remark A.1.2, choosing M large enough such that $\eta-(1-M)(1-\eta)<0$, one gets that

$$e^GAe^{-G}-A+\left(e^GAe^{-G}-A\right)^*=\sum_{k=1}^{M}\frac{Ad_G^kA+(Ad_G^kA)^*}{k!}+OPS^0\,,$$

and thus $e^GAe^{-G}-A+\left(e^GAe^{-G}-A\right)^*\in OPS^{\,0}$, by Item (i). □

Finally, the following couple of results ensures the preservation of symmetry properties under the conjugation of a pseudo-differential operator with the operator $e^{\mathrm{i}\tau\cdot K}$, where $\tau\in\mathbb{T}^d$ and $K=(K_1,\dots,K_d)$ as in Definition 2.2.6.

Remark A.1.5. *By Theorem A.0.9 in [Tay91], one has that if* $A\in\mathcal{L}ip\left(\Omega;\ \mathcal{C}^\infty(\mathbb{T}^{\mathfrak{n}};OPS^m)\right)$, *then* $e^{i\tau\cdot K}Ae^{-i\tau\cdot K},\ \partial_\tau^\alpha\left(e^{i\tau\cdot K}Ae^{-i\tau\cdot K}\right)\in\mathcal{L}ip\left(\Omega;\ \mathcal{C}^\infty(\mathbb{T}^{\mathfrak{n}};OPS^m)\right)\ \ \forall\alpha\in\mathbb{N}^d$.

Lemma A.1.6. *Given S acting as $S:u(x)\mapsto u(-x)$, a linear operator $A(\varphi)$ satisfies the reversibility condition*

$$A(\varphi)\circ S=-S\circ A(-\varphi)$$

if and only if $\mathcal{A}(\tau,\varphi) := e^{\mathrm{i}\tau\cdot K} A(\varphi) e^{-\mathrm{i}\tau\cdot K}$ *satisfies the reversibility condition*

$$\mathcal{A}(\tau,\varphi) \circ S = -S \circ \mathcal{A}(-\tau,-\varphi).$$

Analogously, $A(\varphi)$ *satisfies the reversibility preserving condition*

$$A(\varphi) \circ S = S \circ A(-\varphi)$$

if and only if $\mathcal{A}(\tau,\varphi) := e^{\mathrm{i}\tau\cdot K} A(\varphi) e^{-\mathrm{i}\tau\cdot K}$ *satisfies the reversibility preserving condition*

$$\mathcal{A}(\tau,\varphi) \circ S = S \circ \mathcal{A}(-\tau,-\varphi).$$

Furthermore, $A(\varphi)$ *is real if and only if* $\mathcal{A}(\tau,\varphi)$ *is real.*

Proof. We only prove the statement about reversibility, the one concerning reversibility preserving property following in an analogous way. Recalling the definition of K as in (2.2.6), it is sufficient to check that

$$\begin{aligned}\left(e^{\mathrm{i}\tau\cdot K} \circ S\right)[u] &= e^{\mathrm{i}\tau\cdot K}[u(-x)] \\ &= e^{\mathrm{i}\tau\cdot K}\Big[\sum_{k\in\mathbb{Z}^d} \hat{u}_k e^{-\mathrm{i}k\cdot x}\Big] \\ &= \sum_{k\in\mathbb{Z}^d} e^{-\mathrm{i}\tau\cdot k}\hat{u}_{-k} e^{\mathrm{i}k\cdot x}\end{aligned}$$

and

$$\begin{aligned}\left(S \circ e^{\mathrm{i}\tau\cdot K}\right)[u(x)] &= S\Big[\sum_{k\in\mathbb{Z}^d} e^{\mathrm{i}\tau\cdot k}\hat{u}_k e^{\mathrm{i}k\cdot x}\Big] \\ &= \sum_{k\in\mathbb{Z}^d} e^{\mathrm{i}\tau\cdot k}\hat{u}_k e^{-\mathrm{i}k\cdot x} \\ &= \sum_{k\in\mathbb{Z}^d} e^{-\mathrm{i}\tau\cdot k}\hat{u}_{-k} e^{\mathrm{i}k\cdot x}\end{aligned}$$

to get that $e^{\mathrm{i}\tau\cdot K} \circ S = S \circ e^{\mathrm{i}\tau\cdot K}$, namely $e^{\mathrm{i}\tau\cdot K}$ is reversibility preserving. Hence, if $A(\varphi)$ is φ-reversible, $\mathcal{A}(\tau,\varphi)$ is reversible, since the composition of a reversibility preserving operator and a reversible one is still reversible.
Vice versa, $\mathcal{A}(\tau,\varphi) \circ S = -S \circ \mathcal{A}(-\tau,-\varphi)$ implies

$$A(\varphi) \circ S = \mathcal{A}(0,\varphi) \circ S = -S \circ \mathcal{A}(0,-\varphi) = -S \circ A(-\varphi).$$

Concerning reality, it is sufficient to observe that since $e^{\mathrm{i}\tau\cdot K}$ has matrix elements

$$[e^{\mathrm{i}\tau\cdot K}]_k^{k'} = e^{\mathrm{i}\tau\cdot k}\delta_{k,k'},$$

it satisfies the reality condition

$$[e^{\mathrm{i}\tau\cdot K}]_k^{k'} = \left([e^{\mathrm{i}\tau\cdot K}]_{-k}^{-k'}\right)^*,$$

and the same holds for $e^{-\mathrm{i}\tau\cdot K}$. Hence $e^{\pm\mathrm{i}\tau\cdot K}$ are invertible real operators; since the composition of real operators is real, it follows that $A(\varphi)$ is real if and only if $\mathcal{A}(\tau, \varphi)$ is real. $\square$

Appendix B

Technical estimates on $\mathcal{M}^s_{\sigma_1,\sigma_2}$

B.1 Tame estimates in $\mathcal{M}^s_{\sigma_1,\sigma_2}$

Remark B.1.1. *Let $P \in \mathcal{M}^s_{\sigma_1,\sigma_2}$ and $\forall\ k,k' \in \mathbb{Z}^d$, $\forall\ l \in \mathbb{Z}^{\mathtt{n}}$ let $[\widehat{P}(l)]_k^{k'}$ be the $(k,k')-$th matrix element with respect to the basis $\{e^{\mathrm{i}k\cdot x} \mid k \in \mathbb{Z}^d\}$ of the operator $\widehat{P}(l)$ defined as in (2.3.1). The following conditions hold:*

(a) $P(\varphi)$ is real if and only if

$$[\widehat{P}(l)]_k^{k'} = \overline{[\widehat{P}(-l)]_{-k}^{-k'}};$$

(b) $P(\varphi)$ is reversible if and only if

$$[\widehat{P}(l)]_k^{k'} = -[\widehat{P}(-l)]_{-k}^{-k'};$$

(c) $P(\varphi)$ is reversibility preserving if and only if

$$[\widehat{P}(l)]_k^{k'} = [\widehat{P}(-l)]_{-k}^{-k'}\,.$$

Lemma B.1.2. (i) *Let $\sigma_1,\sigma_2,\sigma_3 \in \mathbb{R}$ and let us assume that $\mathcal{R}$, $\mathcal{P}$ are linear operators such that*
$\mathcal{P} \in \mathcal{B}^{HS}(\mathcal{H}^{\sigma_1},\mathcal{H}^{\sigma_2})$, $\mathcal{R} \in \mathcal{B}^{HS}(\mathcal{H}^{\sigma_2},\mathcal{H}^{\sigma_3})$, Then $\mathcal{R}\mathcal{P} \in \mathcal{B}^{HS}(\mathcal{H}^{\sigma_1},\mathcal{H}^{\sigma_3})$ with

$$\|\mathcal{R}\mathcal{P}\|^{HS}_{\sigma_1,\sigma_3} \le \|\mathcal{R}\|^{HS}_{\sigma_2,\sigma_3}\|\mathcal{P}\|^{HS}_{\sigma_1,\sigma_2}.$$

(ii) *Let $\sigma_1,\sigma_2,\sigma_3 \in \mathbb{R}$, $\beta \ge 0$ and assume that $\langle\nabla\rangle^\beta\mathcal{P}, \mathcal{P} \in \mathcal{B}^{HS}(\mathcal{H}^{\sigma_1},\mathcal{H}^{\sigma_2})$., $\langle\nabla\rangle^\beta\mathcal{R}, \mathcal{R} \in \mathcal{B}^{HS}(\mathcal{H}^{\sigma_2},\mathcal{H}^{\sigma_3})$, Then $\langle\nabla\rangle^\beta\mathcal{R}\mathcal{P} \in \mathcal{B}^{HS}(\mathcal{H}^{\sigma_1},\mathcal{H}^{\sigma_3})$ with*

$$\|\langle\nabla\rangle^\beta\mathcal{R}\mathcal{P}\|^{HS}_{\sigma_1,\sigma_3} \lesssim_\beta \|\langle\nabla\rangle^\beta\mathcal{R}\|^{HS}_{\sigma_2,\sigma_3}\|\mathcal{P}\|^{HS}_{\sigma_1,\sigma_2} + \|\mathcal{R}\|^{HS}_{\sigma_2,\sigma_3}\|\langle\nabla\rangle^\beta\mathcal{P}\|^{HS}_{\sigma_1,\sigma_2}$$

Proof. We prove the estimate (ii). The estimate (i) can be proved by similar arguments (and it is actually simpler). We have that

$$\begin{aligned}
\left(\|\langle\nabla\rangle^\beta \mathcal{R}\mathcal{P}\|^{HS}_{\sigma_1,\sigma_3}\right)^2 &= \sum_{k,k',\in\mathbb{Z}^d} \langle k\rangle^{2\sigma_3}\langle k'\rangle^{-2\sigma_1} \Big|\sum_{j\in\mathbb{Z}^d} \langle k-k'\rangle^\beta \mathcal{R}^j_k \mathcal{P}^{k'}_j\Big|^2 \\
&\lesssim_\beta \sum_{k,k',\in\mathbb{Z}^d} \langle k\rangle^{2\sigma_3}\langle k'\rangle^{-2\sigma_1} \Big[\sum_{j\in\mathbb{Z}^d} \left(\langle k-j\rangle^\beta + \langle j-k'\rangle^\beta\right) |\mathcal{R}^j_k \mathcal{P}^{k'}_j|\Big]^2 \\
&\lesssim_\beta \sum_{k,k',\in\mathbb{Z}^d} \langle k\rangle^{2\sigma_3}\langle k'\rangle^{-2\sigma_1} \Big[\sum_{j\in\mathbb{Z}^d} |\left(\langle\nabla\rangle^\beta\mathcal{R}\right)^j_k|\, |\mathcal{P}^{k'}_j|\Big]^2 \\
&+ \sum_{k,k',\in\mathbb{Z}^d} \langle k\rangle^{2\sigma_3}\langle k'\rangle^{-2\sigma_1} \Big[\sum_{j\in\mathbb{Z}^d} |\mathcal{R}^j_k|\, |\left(\langle\nabla\rangle^\beta\mathcal{P}\right)^{k'}_j|\Big]^2 \\
&\lesssim_\beta \sum_{k,j\in\mathbb{Z}^d} \langle k\rangle^{2\sigma_3} |\left(\langle\nabla\rangle^\beta\mathcal{R}\right)^j_k|^2 \langle j\rangle^{-2\sigma_2} \sum_{j,k'\in\mathbb{Z}^d} \langle j\rangle^{2\sigma_2} |\mathcal{P}^{k'}_j|^2 \langle k'\rangle^{-2\sigma_1} \\
&+ \sum_{k,j\in\mathbb{Z}^d} \langle k\rangle^{2\sigma_3} |\mathcal{R}^j_k|^2 \langle j\rangle^{-2\sigma_2} \sum_{j,k'\in\mathbb{Z}^d} \langle j\rangle^{2\sigma_2} |\left(\langle\nabla\rangle^\beta\mathcal{P}\right)^{k'}_j|^2 \langle k'\rangle^{-2\sigma_1} \\
&\lesssim_\beta \left(\|\langle\nabla\rangle^\beta\mathcal{R}\|^{HS}_{\sigma_2,\sigma_3}\right)^2 \left(\|\mathcal{P}\|^{HS}_{\sigma_1,\sigma_2}\right)^2 + \left(\|\mathcal{R}\|^{HS}_{\sigma_2,\sigma_3}\right)^2 \left(\|\langle\nabla\rangle^\beta\mathcal{P}\|^{HS}_{\sigma_1,\sigma_2}\right)^2 .
\end{aligned}$$

□

Lemma B.1.3.
(i) Let $s \geq s_0$, $\sigma_1,\sigma_2,\sigma_3 \in \mathbb{R}$, $\mathcal{P}(\lambda) \in \mathcal{M}^s_{\sigma_1,\sigma_2}$, $\mathcal{R}(\lambda) \in \mathcal{M}^s_{\sigma_2,\sigma_3}$. *Then* $\mathcal{R}\mathcal{P}(\lambda) \in \mathcal{M}^s_{\sigma_1,\sigma_3}$ *and*

$$\|\mathcal{R}\mathcal{P}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_1,\sigma_3}} \lesssim_s \|\mathcal{R}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_2,\sigma_3}} \|\mathcal{P}\|^{\mathrm{Lip}}_{\mathcal{M}^{s_0}_{\sigma_1,\sigma_2}} + \|\mathcal{R}\|^{\mathrm{Lip}}_{\mathcal{M}^{s_0}_{\sigma_2,\sigma_3}} \|\mathcal{P}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_1,\sigma_2}}.$$

(ii) Let $\beta \geq 0$, $s \geq s_0$, $\sigma_1,\sigma_2,\sigma_3 \in \mathbb{R}$. *Assume that* $\langle\nabla\rangle^\beta\mathcal{P}(\lambda) \in \mathcal{M}^s_{\sigma_1,\sigma_2}$, $\langle\nabla\rangle^\beta\mathcal{R}(\lambda) \in \mathcal{M}^s_{\sigma_2,\sigma_3}$. *Then* $\langle\nabla\rangle^\beta\mathcal{R}\mathcal{P}(\lambda) \in \mathcal{M}^s_{\sigma_1,\sigma_3}$, *and*

$$\|\langle\nabla\rangle^\beta\mathcal{R}\mathcal{P}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_1,\sigma_3}} \lesssim_{s,\beta} \|\langle\nabla\rangle^\beta\mathcal{R}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_2,\sigma_3}} \|\mathcal{P}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_1,\sigma_2}} + \|\mathcal{R}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_2,\sigma_3}} \|\langle\nabla\rangle^\beta\mathcal{P}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_1,\sigma_2}}.$$

Proof. Estimate (i). By applying Lemma B.1.2-(i), one computes

$$\begin{aligned}
\left(\|(\mathcal{R}\mathcal{P})\|_{\mathcal{M}^s_{\sigma_1,\sigma_3}}\right)^2 &\leq \sum_{l\in\mathbb{Z}^n}\langle l\rangle^{2s}\left[\sum_{l'\in\mathbb{Z}^n}\|\widehat{\mathcal{R}}(l-l')\|^{HS}_{\sigma_2,\sigma_2}\|\widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_2}\right]^2\\
&\lesssim_s \sum_{l\in\mathbb{Z}^n}\left[\sum_{l'\in\mathbb{Z}^n}\langle l'\rangle^{s}\|\widehat{\mathcal{R}}(l-l')\|^{HS}_{\sigma_2,\sigma_3}\|\widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_2}\right]^2\\
&+\sum_{l\in\mathbb{Z}^n}\left[\sum_{l'\in\mathbb{Z}^n}\langle l-l'\rangle^{s}\|\widehat{\mathcal{R}}(l-l')\|^{HS}_{\sigma_2,\sigma_3}\|\widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_2}\right]^2\\
&\leq \sum_{l,l'\in Z^n}\langle l'\rangle^{2s}\langle l-l'\rangle^{2s_0}\left(\|\widehat{\mathcal{R}}(l-l')\|^{HS}_{\sigma_2,\sigma_3}\right)^2\left(\|\widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_2}\right)^2\\
&+\sum_{l,l'\in Z^n}\langle l'\rangle^{2s_0}\langle l-l'\rangle^{2s}\left(\|\widehat{\mathcal{R}}(l-l')\|^{HS}_{\sigma_2,\sigma_3}\right)^2\left(\|\widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_2}\right)^2\\
&=\left(\|\mathcal{R}\|_{\mathcal{M}^{s_0}_{\sigma_2,\sigma_3}}\right)^2\left(\|\mathcal{P}\|_{\mathcal{M}^{s}_{\sigma_1,\sigma_2}}\right)^2+\left(\|\mathcal{R}\|_{\mathcal{M}^{s}_{\sigma_2,\sigma_3}}\right)^2\left(\|\mathcal{P}\|_{\mathcal{M}^{s_0}_{\sigma_1,\sigma_2}}\right)^2.
\end{aligned}$$

To get the required estimate in Lipschitz norm, it is sufficient to decompose

$$(\mathcal{R}\mathcal{P})(\lambda_2)-(\mathcal{R}\mathcal{P})(\lambda_1)=\mathcal{R}(\lambda_2)\left(\mathcal{P}(\lambda_2)-\mathcal{P}(\lambda_1)\right)+\left(\mathcal{R}(\lambda_2)-\mathcal{R}(\lambda_1)\right)\mathcal{P}(\lambda_1)$$

and to apply the above inequality to both the terms of the right-hand side, taking respectively

$$\mathcal{R}(\lambda_2)\text{ as }\mathcal{R},\quad \mathcal{P}(\lambda_2)-\mathcal{P}(\lambda_1)\text{ as }\mathcal{P}$$

and

$$\mathcal{R}(\lambda_2)-\mathcal{R}(\lambda_1)\text{ as }\mathcal{R},\quad \mathcal{P}(\lambda_1)\text{ as }\mathcal{P}.$$

Estimate (ii). Arguing as before, one has

$$\begin{aligned}
\left(\|\langle\nabla\rangle^{\beta}(\mathcal{R}\mathcal{P})\|_{\mathcal{M}^s_{\sigma_1,\sigma_3}}\right)^2 &= \sum_{l\in Z^n}\langle l\rangle^{2s}\left(\|\langle\nabla\rangle^{\beta}\widehat{(\mathcal{R}\mathcal{P})}(l)\|^{HS}_{\sigma_1,\sigma_3}\right)^2\\
&\leq \sum_{l\in\mathbb{Z}^n}\langle l\rangle^{2s}\left(\sum_{l'\in\mathbb{Z}^n}\|\langle\nabla\rangle^{\beta}\widehat{\mathcal{R}}(l-l')\widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_3}\right)^2\\
&\lesssim_s \sum_{l,l'\in\mathbb{Z}^n}\langle l'\rangle^{2s}\langle l-l'\rangle^{2s}\left(\|\langle\nabla\rangle^{\beta}\widehat{\mathcal{R}}(l-l')\widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_3}\right)^2
\end{aligned}\tag{B.1.1}$$

where in the last inequality, we have used that

$$\langle l\rangle^{2s} \lesssim_s \langle l'\rangle^{2s} + \langle l-l'\rangle^{2s} \lesssim_s \langle l'\rangle^{2s}\langle l-l'\rangle^{2s}\,.$$

By applying Lemma B.1.2-(ii) (to estimate $\|\langle\nabla\rangle^\beta\widehat{\mathcal{R}}(l-l')\widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_3}$) one obtains that

$$\begin{aligned}\Big(\|\langle\nabla\rangle^\beta(\mathcal{R}\mathcal{P})\|_{\mathcal{M}^s_{\sigma_1,\sigma_3}}\Big)^2 \lesssim_{s,\beta} & \sum_{l,l'\in\mathbb{Z}^{\mathtt{n}}}\langle l'\rangle^{2s}\langle l-l'\rangle^{2s}\Big(\|\langle\nabla\rangle^\beta\widehat{\mathcal{R}}(l-l')\|^{HS}_{\sigma_2,\sigma_3}\Big)^2\Big(\|\widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_2}\Big)^2\\ & + \sum_{l,l'\in\mathbb{Z}^{\mathtt{n}}}\langle l'\rangle^{2s}\langle l-l'\rangle^{2s}\Big(\|\widehat{\mathcal{R}}(l-l')\|^{HS}_{\sigma_2,\sigma_3}\Big)^2\Big(\|\langle\nabla\rangle^\beta\widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_2}\Big)^2\\ & \lesssim_{s,\beta}\Big(\|\langle\nabla\rangle^\beta\mathcal{R}\|_{\mathcal{M}^s_{\sigma_2,\sigma_3}}\Big)^2\Big(\|\mathcal{P}\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}}\Big)^2\\ & \quad + \Big(\|\mathcal{R}\|_{\mathcal{M}^s_{\sigma_2,\sigma_3}}\Big)^2\Big(\|\langle\nabla\rangle^\beta\mathcal{P}\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}}\Big)^2\,.\end{aligned}$$

Concerning the Lipschitz estimates, as in the proof of (i) we write

$$\langle\nabla\rangle^\beta\left(\mathcal{R}\mathcal{P}(\lambda_2)-\mathcal{R}\mathcal{P}(\lambda_1)\right) = \langle\nabla\rangle^\beta\mathcal{R}(\lambda_2)\left(\mathcal{P}(\lambda_2)-\mathcal{P}(\lambda_1)\right) + \langle\nabla\rangle^\beta\left(\mathcal{R}(\lambda_2)-\mathcal{R}(\lambda_1)\right)\mathcal{P}(\lambda_1)$$

and we repeat the same argument with

$$\mathcal{R}(\lambda_2) \text{ as } \mathcal{R}, \quad \mathcal{P}(\lambda_2)-\mathcal{P}(\lambda_1) \text{ as } \mathcal{P}$$

and

$$\mathcal{R}(\lambda_2)-\mathcal{R}(\lambda_1) \text{ as } \mathcal{R}, \quad \mathcal{P}(\lambda_1) \text{ as } \mathcal{P}.$$

□

Iterating the estimates of Lemma B.1.3, one gets for any $s\geq s_0$, $\sigma\in\mathbb{R}$, $n\geq 1$

$$\begin{aligned}&\|\mathcal{R}^n\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}} \leq C(s)^n\|\mathcal{R}\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}}\big(\|\mathcal{R}\|^{\rm Lip}_{\mathcal{M}^{s_0}_{\sigma,\sigma}}\big)^{n-1}\,,\\ &\|\langle\nabla\rangle^\beta(\mathcal{R}^n)\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}} \leq C(s,\beta)^n\|\langle\nabla\rangle^\beta\mathcal{R}\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}}\big(\|\mathcal{R}\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}}\big)^{n-1}\,.\end{aligned}\tag{B.1.2}$$

The following lemma holds:

Lemma B.1.4. *Let* $s\geq s_0$, $\sigma\in\mathbb{R}$, $\beta\geq 0$ *and* $X(\lambda),\langle\nabla\rangle^\beta X(\lambda)\in\mathcal{M}^s_{\sigma,\sigma}$. *Then there exists* $\delta(s,\beta)\in(0,1)$ *such that if* $\|X\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}}\leq\delta(s,\beta)$, *then* $\Phi:=\mathbb{I}+X$ *is invertible and its inverse* Φ^{-1} *satisfies the estimates*

$$\|\Phi^{-1}-\mathbb{I}\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}}\lesssim_s\|X\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}}\,,\quad \|\langle\nabla\rangle^\beta(\Phi^{-1}-\mathbb{I})\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}}\lesssim_s\|\langle\nabla\rangle^\beta X\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}}\,.$$

Proof. By the Neumann series one has $\Phi^{-1}-\mathbb{I}=\sum_{n\geq1}(-1)^nX^n$. Then, applying the estimates (B.1.2) to each term X^n, the claimed statement follows. □

B.2 Other estimates in $\mathcal{M}^s_{\sigma_1,\sigma_2}$

Lemma B.2.1. *(i) Let $\sigma_1, \sigma_2 \in \mathbb{R}$ and $A \in \mathcal{B}\left(\mathcal{H}^{\sigma_1-\eta}, \mathcal{H}^{\sigma_2}\right)$, $\eta > \frac{d}{2}$, then*

$$\|A\|^{HS}_{\sigma_1,\sigma_2} \lesssim_\eta \|A\|_{\mathcal{B}(\mathcal{H}^{\sigma_1-\eta},\mathcal{H}^{\sigma_2})},$$

(ii) Let $\sigma_1, \sigma_2 \in \mathbb{R}$, $\beta \geq 0$, $\eta > \frac{d}{2}$. Then if $A \in \mathcal{B}(\mathcal{H}^{\sigma_1-\beta-\eta}, \mathcal{H}^{\sigma_2+\beta})$, one has

$$\|\langle\nabla\rangle^\beta A\|^{HS}_{\sigma_1,\sigma_2} \lesssim_\beta \|A\|_{\mathcal{B}(\mathcal{H}^{\sigma_1-\beta-\eta},\mathcal{H}^{\sigma_2+\beta})}.$$

Proof. Proof of (i). Let us consider $\forall\ k' \in \mathbb{Z}^d$ $u^{(k')} \in \mathcal{H}^{\sigma_1}$ defined by

$$\hat{u}^{(k')}_h = \begin{cases} \langle k'\rangle^{-(\sigma_1-\eta)} & \text{if } h = k' \\ 0 & \text{if } h \neq k'; \end{cases}$$

We have that

$$\begin{aligned} \sum_k \langle k\rangle^{2\sigma_2}|A^{k'}_k|^2\langle k'\rangle^{-2(\sigma_1-\eta)} &= \|Au^{(k')}\|^2_{\mathcal{H}^{\sigma_2}} \\ &\leq \|A\|^2_{\mathcal{B}\left(\mathcal{H}^{\sigma_1-\eta},\mathcal{H}^{\sigma_2}\right)}\|u^{(k')}\|^2_{\mathcal{H}^{\sigma_1-\eta}} \\ &= \|A\|^2_{\mathcal{B}\left(\mathcal{H}^{\sigma_1-\eta},\mathcal{H}^{\sigma_2}\right)}, \end{aligned}$$

since $\|u^{(k')}\|_{\mathcal{H}^{\sigma_1-\eta}} = 1$. Thus we deduce that $\forall\ k'$

$$\sum_k \langle k\rangle^{2\sigma_2}|A^{k'}_k|^2 \leq \|A\|^2_{\mathcal{B}\left(\mathcal{H}^{\sigma_1-\eta},\mathcal{H}^{\sigma_2}\right)}\langle k'\rangle^{2(\sigma_1-\eta)}. \tag{B.2.1}$$

Let now u be a generic function in $\mathcal{H}^{\sigma_1}$: from (B.2.1) it follows that

$$\begin{aligned} \left(\|A\|^{HS}_{\sigma_1,\sigma_2}\right)^2 &= \sum_{k,k'\in\mathbb{Z}^d} \langle k\rangle^{2\sigma_2}|A^{k'}_k|^2\langle k'\rangle^{-2\sigma_1} \\ &\leq \sum_{k'\in\mathbb{Z}^d} \langle k'\rangle^{2(\sigma_1-\eta)}\|A\|^2_{\mathcal{B}\left(\mathcal{H}^{\sigma_1-\eta},\mathcal{H}^{\sigma_2}\right)}\langle k'\rangle^{-2\sigma_1} \\ &\lesssim_{\sigma_0} \|A\|^2_{\mathcal{B}\left(\mathcal{H}^{\sigma_1-\eta},\mathcal{H}^{\sigma_2}\right)}. \end{aligned}$$

Proof of (ii). Using that for any $j, j' \in \mathbb{Z}^d$,

$$\langle j-j'\rangle^{2\beta} \lesssim_\beta \langle j\rangle^{2\beta} + \langle j'\rangle^{2\beta} \lesssim_\beta \langle j\rangle^{2\beta}\langle j'\rangle^{2\beta},$$

one gets that

$$\begin{aligned}\left(\|\langle\nabla\rangle^\beta A\|^{HS}_{\sigma_1,\sigma_2}\right)^2 &= \sum_{j,j'\in\mathbb{Z}^d}\langle j\rangle^{2\sigma_2}\langle j-j'\rangle^{2\beta}|A_j^{j'}|^2\langle j'\rangle^{-2\sigma_1}\\ &\lesssim_\beta \sum_{j,j'\in\mathbb{Z}^d}\langle j\rangle^{2(\sigma_2+\beta)}|A_j^{j'}|^2\langle j'\rangle^{-2(\sigma_1-\beta)} = \left(\|A\|^{HS}_{\sigma_1-\beta,\sigma_2+\beta}\right)^2.\end{aligned} \tag{B.2.2}$$

The claimed statement follows by applying item (i) (replacing σ_1 with $\sigma_1-\beta$ and σ_2 with $\sigma_2+\beta$). ☐

Lemma B.2.2. *(i) Let $A\in\mathcal{C}^s\Big(\mathbb{T}^{\mathtt{n}};\mathcal{B}\left(\mathcal{H}^{\sigma_1-\eta},\ \mathcal{H}^{\sigma_2}\right)\Big)$, $\sigma_1,\sigma_2\in\mathbb{R}$, $\eta>\frac{d}{2}$ and $s\geq 0$. Then*

$$\|A\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}}\lesssim\|A\|_{\mathcal{C}^s\left(\mathbb{T}^{\mathtt{n}};\mathcal{B}\left(\mathcal{H}^{\sigma_1-\eta},\ \mathcal{H}^{\sigma_2}\right)\right)}.$$

(ii) Let $s\geq 0$, $\sigma_1,\sigma_2\in\mathbb{R}$, $\beta\geq 0$, $\eta>\frac{d}{2}$ and $A\in\mathcal{C}^s\Big(\mathbb{T}^{\mathtt{n}};\mathcal{B}(\mathcal{H}^{\sigma_1-\beta-\eta},\mathcal{H}^{\sigma_2+\beta})\Big)$. Then

$$\|\langle\nabla\rangle^\beta A\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}}\lesssim_\beta\|A\|_{\mathcal{C}^s\left(\mathbb{T}^{\mathtt{n}};\mathcal{B}(\mathcal{H}^{\sigma_1-\beta-\eta},\mathcal{H}^{\sigma_2+\beta})\right)}$$

Proof. The claimed statement follows recalling that $\mathcal{M}^s_{\sigma_1,\sigma_2}=\mathcal{H}^s\Big(\mathbb{T}^{\mathtt{n}};\mathcal{B}^{HS}(\mathcal{H}^{\sigma_1},\mathcal{H}^{\sigma_2})\Big)$, by applying Lemma B.2.1 and using that for every Banach space X one has that $\|\cdot\|_{H^s(\mathbb{T}^{\mathtt{n}};X)}\leq\|\cdot\|_{\mathcal{C}^s(\mathbb{T}^{\mathtt{n}};X)}$. ☐

Lemma B.2.3. *(i) Let $m\geq 0$, $A\in\mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}},OPS^{-\kappa})$, $\kappa>2m+\frac{d}{2}$. Then for any $\sigma\in\mathbb{R}$,*
$A\in\mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}};\mathcal{B}\left(\mathcal{H}^{\sigma+m-\kappa},\mathcal{H}^{\sigma+m}\right))$ and for any $s\geq 0$

$$\|A\|_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\lesssim\|A\|_{\mathcal{C}^s(\mathbb{T}^{\mathtt{n}};\mathcal{B}(\mathcal{H}^{\sigma+m-\kappa},\mathcal{H}^{\sigma+m}))}$$

(ii) Let $m,\beta\geq 0$ and $A\in\mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}};OPS^{-\kappa})$, $\kappa>2m+2\beta+\frac{d}{2}$. Then for any $\sigma\in\mathbb{R}$,
$A\in\mathcal{C}^\infty\Big(\mathbb{T}^{\mathtt{n}};\mathcal{B}\big(\mathcal{H}^{\sigma+m+\beta-\kappa},\mathcal{H}^{\sigma+m+\beta}\big)\Big)$ and for any $s\geq 0$

$$\|\langle\nabla\rangle^\beta A\|_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\lesssim_\beta\|A\|_{\mathcal{C}^s(\mathbb{T}^{\mathtt{n}};\mathcal{B}(\mathcal{H}^{\sigma+m+\beta-\kappa},\mathcal{H}^{\sigma+m+\beta}))}.$$

Proof. The statement (i) follows by applying Lemma B.2.2-(i) with $\sigma_1=\sigma-m,\ \sigma_2=\sigma+m,\ \eta=\kappa-2m$.
The statement (ii) follows by applying Lemma B.2.2-(ii) with $\sigma_1=\sigma-m,\ \sigma_2=\sigma+m,\ \eta=\kappa-2m-2\beta$. ☐

Lemma B.2.4. *Let $\sigma \in \mathbb{R}$, $\kappa \geq 0$, $P(\lambda) \in \mathcal{B}^{HS}(\mathcal{H}^\sigma, \mathcal{H}^{\sigma+\kappa})$, $\lambda \in \Omega_0 \subseteq \mathbb{R}^{\mathtt{n}+d}$. Then $\forall j \in \mathbb{Z}^d$ its matrix elements P_j^j satisfy*

$$|P_j^j| \leq \|P\|^{HS}_{\sigma,\sigma+\kappa} \langle j \rangle^{-\kappa}, \quad |P_j^j|^{\mathrm{Lip}} \leq \|P\|^{HS,\mathrm{Lip}}_{\sigma,\sigma+\kappa} \langle j \rangle^{-\kappa}.$$

Proof. For any $j \in \mathbb{Z}^d$, one has

$$(\|P\|^{HS}_{\sigma,\sigma+\kappa})^2 = \sum_{k,k' \in \mathbb{Z}^d} \langle k \rangle^{2(\sigma+\kappa)} |P_k^{k'}| \langle k' \rangle^{-2\sigma} \geq \langle j \rangle^{2(\sigma+\kappa)} |P_j^j| \langle j \rangle^{-2\sigma} = \langle j \rangle^{\kappa} |P_j^j| \,.$$

The Lipschitz estimate follows arguing similarly by estimating $\frac{\|P(\lambda_1)-P(\lambda_2)\|^{HS}_{\sigma,\sigma+\kappa}}{|\lambda_1-\lambda_2|}$ for any $\lambda_1, \lambda_2 \in \Omega_0$, $\lambda_1 \neq \lambda_2$. $\square$

Appendix C

Pseudo-differential calculus: intrinsic formulation

In this section we recall some basic facts about pseudo-differential calculus, with the aim of stressing the fact that, with the definitions given in Chapter 4, all the involved quantities are intrinsic, and that all the seminorms of the symbols that we will consider only depend on the metric $\mathtt{g}^*$ through constants that are invariant with respect to the operation of restricting $\mathtt{g}^*$ to a subspace M of $\mathbb{Z}^d$. In particular, the following quantities play a relevant role: the coercivity constant $\mathfrak{c}$ defined as in (3.1.8), namely

$$\mathfrak{c} = \inf_{0 \neq k \in \mathbb{Z}^d} \|k\|^2_{\mathtt{g}^*} ,$$

and

$$\mathfrak{s}_p = \sum_{k \in \mathbb{Z}^d} \frac{1}{\langle k \rangle^p_{\mathtt{g}^*}} , \quad \forall p > d+1 . \tag{C.0.1}$$

With start with remarking the following couple of immediate properties:

Lemma C.0.1. *If $p > d+1$ the quantity $\mathfrak{s}_p$ is finite and only depends on p and on the constant $\mathfrak{c}$.*

Proof. Recall that, as stated in Lemma 3.1.3 (in the case $\kappa = 0$), Weyl's law holds for the Laplacian operator $-\Delta_{\mathtt{g}}$ on $\mathbb{T}^d$, namely for all $R > 0$ one has

$$\sharp \left\{ k \in \mathbb{Z}^d \;\middle|\; \|k\|^2_{\mathtt{g}^*} \leq R^2 \right\} \lesssim_{\mathfrak{c},d} R^d . \tag{C.0.2}$$

Thus

$$\mathfrak{s}_p = \sum_{k\in\mathbb{Z}^d} \langle k\rangle_{\mathtt{g}^*}^{-p} \leq 1 + \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \|k\|_{\mathtt{g}^*}^{-p} \leq 1 + \sum_{n=1}^{\infty} \sum_{\substack{k\in\mathbb{Z}^d \\ \lfloor\|k\|_{\mathtt{g}^*}\rfloor = n}} n^{-p}\,.$$

By (C.0.2), one has $\mathfrak{s}_p \lesssim_{\mathfrak{c},d} \sum_{n\in\mathbb{N}} n^{d-p}$, which is a finite quantity, since $p > d+1$.

□

Notice that the quantity $\mathfrak{s}_p$ defined as in (C.0.1) is finite also in the case $d < p \leq d+1$, however here we restrict to $p > d+1$ in order to easily ensure (by Weyl's estimate (C.0.2)) that the dependence on the metric $\mathtt{g}$ is only controlled by $\mathfrak{c}$.

Remark C.0.2. *Let $M \subseteq \mathbb{Z}^d$ a module of $\mathbb{Z}^d$ of dimension s and consider the metric $\mathtt{g}^*_M$ induced by $\mathtt{g}^*$ on M. Then, if $\iota : \mathbb{Z}^s \cong M \to \mathbb{Z}^d$ denotes the standard inclusion map, one has*

$$\inf_{k\in\mathbb{Z}^s\setminus\{0\}} \|k\|^2_{\mathtt{g}^*_M} \geq \inf_{k\in\mathbb{Z}^s\setminus\{0\}} \|\iota k\|^2_{\mathtt{g}^*} \geq \inf_{k'\in\mathbb{Z}^d\setminus\{0\}} \|k'\|^2_{\mathtt{g}^*} = \mathfrak{c}\,.$$

C.1 An equivalent definition of symbols

The following Lemma gives a characterization of a symbol which is useful for future estimates:

Lemma C.1.1. *Let $m \in \mathbb{R}$ and $\delta > 0$ and let $a(x,\xi) = \sum_{k\in\mathbb{Z}^d} \hat{a}_k(\xi) e^{\mathrm{i}k\cdot x}$. Then one has that $a \in S^{m,\delta}$ if and only if for all $N_1,\ N_2 > 0$*

$$\sup_{\widetilde{N}_1\leq N_1,\ \widetilde{N}_2\leq N_2} \sup_{k\in\mathbb{Z}^d} \sup_{\xi\in\mathbb{R}^d} \|d_\xi^{\widetilde{N}_2} \hat{a}_k(\xi)\| \langle k\rangle_{\mathtt{g}^*}^{\widetilde{N}_1} \langle \xi+\kappa\rangle_{\mathtt{g}^*}^{-\left(m-\widetilde{N}_2\delta\right)} =: C'_{N_1,N_2}(a) < +\infty\,, \tag{C.1.1}$$

where for any $\xi_0 \in \mathbb{R}^d$ and $k \in \mathbb{Z}^d$

$$\|d_\xi^{N_2} \hat{a}_k(\xi_0)\| = \sup_{\substack{\|l^{(j)}\|_{\mathtt{g}^*}=1 \\ j=1,\dots,N_2}} \left| d_\xi^{N_2} \hat{a}_k(\xi_0) \left(l^{(1)},\dots,l^{(N_2)}\right)\right|\,. \tag{C.1.2}$$

In particular, if $\mathfrak{c}$ is defined as in (3.1.8) we have that $\forall N_1, N_2 \in \mathbb{N}$

$$\begin{aligned} C'_{N_1,N_2}(a) &\lesssim_{\mathfrak{c},N_1} \sup_{\widetilde{N}_1\leq N_1,\ \widetilde{N}_2\leq N_2} C_{\widetilde{N}_1,\widetilde{N}_2}(a)\,, \\ \sup_{\widetilde{N}_1\leq N_1,\ \widetilde{N}_2\leq N_2} C_{\widetilde{N}_1,\widetilde{N}_2}(a) &\lesssim_{\mathfrak{c}} C'_{N_1+d+2,N_2}(a)\,. \end{aligned} \tag{C.1.3}$$

The following technical results are useful for the proof of Lemma C.1.1, as well as for proving the subsequent results of the present chapter.

Remark C.1.2. *Let $a \in \mathbb{R}^+$. For all $\xi,\ k \in \mathbb{R}^d$ there exists a positive constant K depending only on a such that*

$$\langle \xi + k \rangle_{\mathfrak{g}^*}^a \le K \left(\langle \xi \rangle_{\mathfrak{g}^*}^a + \|k\|_{\mathfrak{g}^*}^a \right) . \tag{C.1.4}$$

Lemma C.1.3. *Let $m \in \mathbb{R}$. There exists a constant K' depending only on m and $\mathfrak{c}$ such that $\forall \xi \in \mathbb{R}^d$, $\forall k \in \mathbb{Z}^d \backslash \{0\}$, and $\forall \epsilon \in \mathbb{R}$ with $|\epsilon| \le 1$,*

$$\langle \xi + \epsilon k \rangle_{\mathfrak{g}^*}^m \le K' \|k\|_{\mathfrak{g}^*}^{|m|} \langle \xi \rangle_{\mathfrak{g}^*}^m .$$

Proof. If $m = 0$ the claim is immediate. If $m > 0$, by Remark C.1.2 one deduces the existence of a $K > 0$ (depending on m) such that

$$\langle \xi + \epsilon \rangle_{\mathfrak{g}^*}^m \le K \left(\langle \xi \rangle_{\mathfrak{g}^*}^m + \|k\|_{\mathfrak{g}^*}^m \right) .$$

Then it is sufficient to observe that if a, b are two positive numbers such that $a, b > 1$, then $a + b \le 2ab$, to deduce that

$$\begin{aligned} \langle \xi + \epsilon \rangle_{\mathfrak{g}^*}^m &\le K \left(1 + \sqrt{\mathfrak{c}}^m \right) \left(\langle \xi \rangle_{\mathfrak{g}^*}^m + \frac{\|k\|_{\mathfrak{g}^*}^m}{\sqrt{\mathfrak{c}}^m} \right) \\ &\le K \frac{2^m \left(1 + \sqrt{\mathfrak{c}}^m \right)}{\sqrt{\mathfrak{c}}^m} \langle \xi \rangle_{\mathfrak{g}^*}^m \|k\|_{\mathfrak{g}^*}^m . \end{aligned}$$

Let instead $m < 0$. By Remark C.1.2, there exists a constant K such that

$$\begin{aligned} \frac{\langle \xi \rangle_{\mathfrak{g}^*}}{\langle \xi + \epsilon k \rangle_{\mathfrak{g}^*}} &\le K \left(\frac{\langle \xi + \epsilon k \rangle_{\mathfrak{g}^*}}{\langle \xi + \epsilon k \rangle_{\mathfrak{g}^*}} + \frac{\|k\|_{\mathfrak{g}^*}}{\langle \xi + \epsilon k \rangle_{\mathfrak{g}^*}} \right) \\ &\le K \left(1 + \|k\|_{\mathfrak{g}^*} \right) . \end{aligned}$$

Since $\|k\|_{\mathfrak{g}^*} \ge \sqrt{\mathfrak{c}}$, we obtain

$$\frac{\langle \xi \rangle_{\mathfrak{g}^*}}{\langle \xi + \epsilon k \rangle_{\mathfrak{g}^*}} \le K \left(\frac{1}{\sqrt{\mathfrak{c}}} + 1 \right) \|k\|_{\mathfrak{g}^*} .$$

Thus, in the case $m < 0$, the estimate follows with $K' = \left(K \left(\frac{1}{\sqrt{c}} + 1 \right) \right)^{-m}$. ∎

Remark C.1.4. *For all $k \in \mathbb{Z}^d \backslash \{0\}$ it is immediate to see that*

$$\|k\|_{\mathfrak{g}^*}^2 \le \langle k \rangle_{\mathfrak{g}^*}^2 \le 2 \max \left\{ \frac{1}{\mathfrak{c}}, 1 \right\} \|k\|_{\mathfrak{g}^*}^2 ,$$

with $\mathfrak{c}$ defined as in (3.1.8).

We now prove Lemma C.1.1. Although it is a very standard result, here we include its proof, in order to stress that all calculations are intrinsic and that the only involved quantity depending on the metric is the constant $\mathfrak{c}$.

Proof of Lemma C.1.1. Let $a \in S^{m,\delta}$, fix coordinates $\{e_1, \dots, e_d\}$ on $\mathbb{T}^d$ and let $\{\epsilon^1, \dots, \epsilon^d\}$ be its dual basis. We observe that for any smooth function $f(x) : \mathbb{T}^d \to \mathbb{C}$ and for any $k \in \mathbb{Z}^d$ one has

$$\sup_{x\in\mathbb{T}^d} |f(x)| \geq \frac{1}{\mu_{\mathsf{g}}(\mathbb{T}^d)} \left| \int_{\mathbb{T}^d} f(x) e^{-\mathrm{i}k\cdot x} \, d\mu_{\mathsf{g}}(x) \right|$$

in order to deduce that, for any N_1 and N_2 in $\mathbb{N}$, for any normalized vectors $h^{(1)}, \dots, h^{(N_1)}$, $l^{(1)}, \dots, l^{(N_2)}$, and for any $\xi \in \mathbb{R}^d$ and $k \in \mathbb{Z}^d$

$$\begin{aligned} &\sup_{x\in\mathbb{T}^d} \left| d_x^{N_1} d_\xi^{N_2} a(x,\xi) \left[h^{(1)}, \dots h^{(N_1)}, l^{(1)}, \dots l^{(N_2)} \right] \right| \\ &\quad \geq \frac{1}{\mu_{\mathsf{g}}(\mathbb{T}^d)} \left| \int_{\mathbb{T}^d} d_x^{N_1} d_\xi^{N_2} a(x,\xi) \left[h^{(1)}, \dots h^{(N_1)}, l^{(1)}, \dots l^{(N_2)} \right] e^{-\mathrm{i}k\cdot x} d\mu_{\mathsf{g}}(x) \right| . \end{aligned}$$

Thus, in coordinates, an N_1-times integration by parts yields that for any ξ and k

$$\begin{aligned} &\sup_{x\in\mathbb{T}^d} \left| d_x^{N_1} d_\xi^{N_2} a(x,\xi) \left[h^{(1)}, \dots h^{(N_1)}, l^{(1)}, \dots l^{(N_2)} \right] \right| \\ &\geq \frac{1}{\mu_{\mathsf{g}}(\mathbb{T}^d)} \left| \int_{\mathbb{T}^d} \partial^{N_2}_{\xi_{j_1}\cdots\xi_{j_{N_2}}} a(x,\xi) k_{i_1} \cdots k_{i_{N_1}} e^{-\mathrm{i}k\cdot x} \, l^{(1)}_{j_1} \cdots l^{N_2}_{j_{N_2}} h^{(1)}_{i_1} \cdots h^{(N_1)}_{i_{N_1}} \, d\mu_{\mathsf{g}}(x) \right| , \end{aligned}$$

where (here and in the following) the symbol of summation over the repeated indexes $j_1, \dots, j_{N_2}$ and $i_1, \dots, i_{N_1}$ is omitted. Recalling that for all $k \in \mathbb{Z}^d$

$$\hat{a}_k(\xi) = \frac{1}{\mu_{\mathsf{g}}(\mathbb{T}^d)} \int_{\mathbb{T}^d} a(x,\xi) e^{-\mathrm{i}k\cdot x} \, d\mu_{\mathsf{g}}(x) ,$$

one obtains that for all $\xi \in \mathbb{R}^d$ and $k \in \mathbb{Z}^d$

$$\begin{aligned} &\sup_{x\in\mathbb{T}^d} \left| d_x^{N_1} d_\xi^{N_2} a(x,\xi) \left[h^{(1)}, \dots h^{(N_1)}, l^{(1)}, \dots l^{(N_2)} \right] \right| \\ &\qquad\qquad \geq \left| \partial^{N_2}_{\xi_{j_1}\cdots\xi_{j_{N_2}}} \hat{a}_k(\xi) k_{i_1} \cdots k_{i_{N_2}} \, l^{(1)}_{j_1} \cdots l^{(N_2)}_{j_{N_2}} h^{(1)}_{i_1} \cdots h^{(N_1)}_{i_{N_1}} \right| . \end{aligned}$$

Choose then $h^{(1)} = \cdots h^{(N_1)} = h$, with $h = \bar{h} \left\| \bar{h} \right\|_{\mathsf{g}}^{-1}$ and $\bar{h}$ defined by $\bar{h}_j = \sum_{i=1}^d \mathsf{g}^{ij} k_i \ \forall j$: a direct computation shows

$$\left| k_{i_1} \cdots k_{i_{N_1}} h^{(1)}_{i_1} \cdots h^{(N_1)}_{i_{N_1}} \right| = \|k\|_{\mathsf{g}*}^{2N_1} ,$$

thus for any $k \in \mathbb{Z}^d$ it follows

$$\begin{aligned}\|d_x^{N_1} d_\xi^{N_2} a(x,\xi)\| &\geq \sup_{\substack{\|l^{(j)}\|_{\mathtt{g}*}=1 \\ j=1,\ldots,N_2}} \left| \partial^{N_2}_{\xi_{j_1}\cdots\xi_{j_{N_2}}} \hat{a}_k(\xi)\; l^{(1)}_{j_1} \cdots l^{(N_2)}_{j_{N_2}} \right| \|k\|^{2N_1}_{\mathtt{g}*} \\ &= \|d_\xi^{N_2} \hat{a}_k(\xi)\| \, \|k\|^{2N_1}_{\mathtt{g}*} \; .\end{aligned}$$

Then we have

$$\begin{aligned}C_{N_1,N_2}(a) &\geq \sup_{\xi\in\mathbb{R}^d} \langle \xi+\kappa\rangle_{\mathtt{g}^*}^{-(m-\delta N_2)} \|d_x^{N_1} d_\xi^{N_2} a(x,\xi)\| \\ &\geq \sup_{\xi\in\mathbb{R}^d} \langle \xi+\kappa\rangle_{\mathtt{g}^*}^{-(m-N_2\delta)} \|d_\xi^{N_2} \hat{a}_k(\xi)\| \, \|k\|^{2N_1}_{\mathtt{g}*} \; .\end{aligned}$$

Thus is sufficient to apply Remark C.1.4 to deduce the existence of a constant $K = K(\mathfrak{c}, N_1) > 0$ such that $\forall k \neq 0$

$$\begin{aligned}C_{N_1,N_2}(a) &\geq K(\mathfrak{c}, N_1) \sup_{\xi\in\mathbb{R}^d} \langle \xi+\kappa\rangle_{\mathtt{g}^*}^{-(m-\delta N_2)} \|d_\xi^{N_2} \hat{a}_k(\xi)\| \langle k\rangle^{2N_1}_{\mathtt{g}^*} \\ &\geq K(\mathfrak{c}, N_1) \sup_{\xi\in\mathbb{R}^d} \langle \xi+\kappa\rangle_{\mathtt{g}^*}^{-(m-\delta N_2)} \|d_\xi^{N_2} \hat{a}_k(\xi)\| \langle k\rangle^{N_1}_{\mathtt{g}^*} \; .\end{aligned}$$

Replacing N_1 with $\widetilde{N}_1$ and N_2 with $\widetilde{N}_2$ and taking the supremum on all $\widetilde{N}_1 \leq N_1$ and $\widetilde{N}_2 \leq N_2$, this implies

$$C'_{N_1,N_2}(a) \lesssim_{N_1,\mathfrak{c}} \sup_{\widetilde{N}_1\leq N_1,\ \widetilde{N}_2\leq N_2} C_{\widetilde{N}_1,\widetilde{N}_2}(a) \, ,$$

namely the first estimate in (C.1.3).

On the contrary, suppose that (C.1.1) is satisfied. For any N_1 and $N_2 \in \mathbb{N}$ and any choice of normalized vectors $h^{(1)}, \ldots, h^{(N_1)},\ l^{(1)}, \ldots, l^{(N_2)}$, one has

$$\begin{aligned}&d_x^{N_1} d_\xi^{N_2} a(x,\xi) \left[h^{(1)}, \ldots, h^{(N_1)}, l^{(1)}, \ldots, l^{(N_2)}\right] \\ &\qquad = \sum_{k\in\mathbb{Z}^d} d_x^{N_1} d_\xi^{N_2} \left(\hat{a}_k(\xi) e^{\mathrm{i}k\cdot x}\right) \left[h^{(1)}, \ldots, h^{(N_1)}, l^{(1)}, \ldots, l^{(N_2)}\right] .\end{aligned}$$

In coordinates, and again omitting summation symbol over repeated indexes, we have

$$\begin{aligned}&d_x^{N_1} d_\xi^{N_2} a(x,\xi) \left[h^{(1)}, \ldots, h^{(N_1)}, l^{(1)}, \ldots, l^{(N_2)}\right] \\ &\qquad = \sum_{k\in\mathbb{Z}^d} \partial^{N_2}_{\xi_{j_1}\cdots\xi_{j_{N_2}}} \hat{a}_k(\xi) \mathrm{i}^{N_1} k_{i_1} \cdots k_{i_{N_1}} e^{\mathrm{i}k\cdot x} h^{(1)}_{i_1} \cdots h^{(N_1)}_{i_{N_1}} l^{(1)}_{j_1} \cdots l^{(N_2)}_{j_{N_2}} \; .\end{aligned}$$

Thus, recalling that for any $l = 1, \ldots, N_1$

$$\left| \sum_{i_l=1}^{d} k_{i_l} h_{i_l}^{(l)} \right| \leq \left\| h^{(l)} \right\|_{\mathtt{g}} \|k\|_{\mathtt{g}^*} \leq \langle k \rangle_{\mathtt{g}^*}$$

since the vectors $h^{(l)}$, $l = 1, \ldots, N_1$, are normalized, one gets

$$\begin{aligned} \|d_x^{N_1} d_\xi^{N_2} a(x,\xi)\| &\leq \sup_{\substack{\left\|h^{(i)}\right\|_{\mathtt{g}}=1, \left\|l^{(j)}\right\|_{\mathtt{g}^*}=1 \\ i=1,\ldots N_2,\ j=1,\ldots,N_1}} \sum_{k\in\mathbb{Z}^d} \left| d_\xi^{N_2} \hat{a}_k(\xi) \left[l^{(1)}, \ldots, l^{(N_2)} \right] \right| \langle k \rangle_{\mathtt{g}^*}^{N_1} \\ &\leq C'_{N_1+d+2,N_2}(a) \left\langle \xi + \kappa \right\rangle_{\mathtt{g}^*}^{m-\delta N_2} \sum_{k\in\mathbb{Z}^d} \frac{1}{\langle k \rangle_{\mathtt{g}^*}^{d+2}} \, . \end{aligned}$$

Recalling the definition of $\mathfrak{s}_{d+2}$ as in (C.0.1), one has that $\forall (x, \xi)$

$$\|d_x^{N_1} d_\xi^{N_2} a(x,\xi)\| \leq C'_{N_1+d+2,N_2}(a) \langle \xi + \kappa \rangle_{\mathtt{g}^*}^{m-N_2\delta} \mathfrak{s}_{d+2} \, .$$

Then the thesis follows recalling that, by Lemma C.0.1, $\mathfrak{s}_{d+2}$ only depends on d and on $\mathfrak{c}$. □

C.2 A few basic properties about quantization

Recall that, given a symbol $a \in S^{m,\delta}$ as in Definition 3.2.1, its Weyl quantization has been defined as the operator

$$Op^W(a) : u(x) = \sum_{\xi\in\mathbb{Z}^d} \hat{u}_\xi e^{\mathrm{i}\xi\cdot x} \longmapsto \sum_{\xi\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} \hat{a}_k \left(\xi + \frac{k}{2} \right) \hat{u}_\xi e^{\mathrm{i}(\xi+k)\cdot x} \quad \forall u \in L^2(\mathbb{T}^d, \mu_{\mathtt{g}}) \, .$$

Of course other quantizations are possible: for instance, given $a \in S^{m,\delta}$, one can define its classical quantization as

$$Op^{cl}(a) : u(x) = \sum_{\xi\in\mathbb{Z}^d} \hat{u}_\xi e^{\mathrm{i}\xi\cdot x} \longmapsto \sum_{\xi\in\mathbb{Z}^d} a(x,\xi) \hat{u}_\xi e^{\mathrm{i}\xi\cdot x} \, . \tag{C.2.1}$$

(This is the same quantization we have used in Chapter 2, although in a slightly different context). Then the following result holds (see Theorem II-27 of [Rob87]):

Lemma C.2.1. *Let $a \in S^{m,\delta}$. Then the symbol b defined by*

$$b(x,\xi) = \sum_{k\in\mathbb{Z}^d} \hat{a}_k\left(\xi + \frac{k}{2}\right) e^{\mathrm{i}k\cdot x}$$

is such that $b \in S^{m,\delta}$, $Op^W(a) = Op^{cl}(b)$, and the following asymptotic expansion holds:

$$b \sim \sum_{\alpha\in\mathbb{N}^d} \frac{1}{\mathrm{i}^{|\alpha|}\alpha!2^{|\alpha|}} \partial_x^\alpha \partial_\xi^\alpha a\,. \tag{C.2.2}$$

Remark C.2.2. *In particular, the asymptotic expansion* (C.2.2) *of Lemma C.2.1 implies that there exist symbols $\widetilde{b} \in S^{m,\delta}$ and $b_S \in S^{-\infty,\delta}$ such that $b = \widetilde{b} + b_S$, and $supp(\widetilde{b}) = supp(a)$. Moreover, the family of seminorms of the symbols $\widetilde{b}$ and b_S only depend on the family of seminorms of a.*

Calderon-Vaillancourt Theorem is well known to hold for a pseudo-differential operator $A \in OPS^{m,\delta}$ defined as in (3.2.4). In particular, one has the following:

Lemma C.2.3 (Calderon Vaillancourt). *Let $A \in OPS^{m,\delta}$. Then A is a bounded linear operator from H^σ to $H^{\sigma-m}$ for any $\sigma\in\mathbb{R}$. In particular, for any σ there exists N, depending only on the parameters $m,\sigma,d,\mathfrak{c}$, such that $\|A\|_{\mathcal{B}(H^\sigma;H^{\sigma-m})} \lesssim_{\sigma,m,\mathfrak{c}} C'_{N,0}(a)$.*

Proof. The proof is standard; we only repeat it to underline that the dependence on the metric is only through the constant $\mathfrak{c}$. Let $u \in H^{\sigma+m}$; then, since by Lemma C.1.3 for any $h \in \mathbb{Z}^d\backslash\{0\}$ one has

$$\langle \xi+\kappa\rangle_{\mathrm{g}^*}^{2\sigma} \lesssim_\sigma \|h\|_{\mathrm{g}*}^{2|\sigma|} \langle \xi+\kappa-h\rangle_{\mathrm{g}^*}^{2\sigma}\,,$$

it follows

$$\begin{aligned}\left\|Op^W(a)u\right\|_\sigma^2 &= \sum_{\xi\in\mathbb{Z}^d}\langle\xi+\kappa\rangle_{\mathrm{g}^*}^{2\sigma}\left(\sum_{h\in\mathbb{Z}^d}\left|\hat{a}_h\left(\xi-\frac{h}{2}\right)\right|\,|\hat{u}_{\xi-h}|\right)^2\\ &\lesssim_\sigma \sum_{\xi\in\mathbb{Z}^d}\left(\sum_{h\in\mathbb{Z}^d}\langle\xi+\kappa-h\rangle_{\mathrm{g}^*}^{\sigma}\langle h\rangle_{\mathrm{g}^*}^{|\sigma|}\left|\hat{a}_h\left(\xi-\frac{h}{2}\right)\right|\,|\hat{u}_{\xi-h}|\right)^2.\end{aligned}$$

Let $\mathfrak{s}_{d+2} = \sum_{h\in\mathbb{Z}^d}\langle h\rangle_{\mathrm{g}^*}^{-(d+2)}$ as in (C.0.1) and recall that, by Lemma C.0.1, it is a finite quantity and it only depends on $\mathfrak{c}$. By Hölder inequality, if

$\sigma_0 = \frac{d+2}{2}$, one has

$$\left\|Op^W(a)u\right\|_\sigma^2 \lesssim_{\mathfrak{c},\sigma} \sum_{\xi\in\mathbb{Z}^d}\sum_{h\in\mathbb{Z}^d} \langle \xi+\kappa-h\rangle_{\mathtt{g}^*}^{2\sigma} \left|\hat{a}_h\left(\xi-\frac{h}{2}\right)\right|^2 |\hat{u}_{\xi-h}|^2 \langle h\rangle_{\mathtt{g}^*}^{2(|\sigma|+\sigma_0)}$$
$$= \sum_{\xi\in\mathbb{Z}^d} \langle \xi+\kappa\rangle_{\mathtt{g}^*}^{2\sigma} |\hat{u}_\xi|^2 \sum_{h\in\mathbb{Z}^d} \langle h\rangle_{\mathtt{g}^*}^{2(|\sigma|+\sigma_0)} \left|\hat{a}_h\left(\xi+\frac{h}{2}\right)\right|^2 .$$

By Lemma C.1.1, for any $N\in\mathbb{N}$ we have

$$\left\|Op^W(a)u\right\|_\sigma^2 \lesssim_{\sigma,\mathfrak{c}} C'_{N,0}(a)^2 \sum_{\xi\in\mathbb{Z}^d} \langle \xi+\kappa\rangle_{\mathtt{g}^*}^{2\sigma} |\hat{u}_\xi|^2 \sum_{h\in\mathbb{Z}^d} \langle h\rangle_{\mathtt{g}^*}^{2(|\sigma|+\sigma_0)} \|h\|_{\mathtt{g}^*}^{-2N} \left\langle \xi+\kappa+\frac{h}{2}\right\rangle_{\mathtt{g}^*}^{2m}$$

and, again by Lemma C.1.3,

$$\left\|Op^W(a)u\right\|_\sigma^2 \lesssim_{\sigma,\mathfrak{c},m} C'_{N,0}(a)^2 \sum_{\xi\in\mathbb{Z}^d} \langle \xi+\kappa\rangle_{\mathtt{g}^*}^{2\sigma} |\hat{u}_\xi|^2 \sum_{h\in\mathbb{Z}^d} \langle h\rangle_{\mathtt{g}^*}^{2(|\sigma|+\sigma_0+|m|)} \|h\|_{\mathtt{g}^*}^{-2N} \langle \xi+\kappa\rangle_{\mathtt{g}^*}^{2m} .$$

Thus, choosing $N = \lceil 2\sigma_0 + |\sigma| + |m|\rceil$ and summing over $h\in\mathbb{Z}^d$, we obtain

$$\left\|Op^W(a)u\right\|_\sigma^2 \lesssim_{\sigma,\mathfrak{c},m} C'_{N,0}(a)^2 \sum_{\xi\in\mathbb{Z}^d} \langle \xi+\kappa\rangle_{\mathtt{g}^*}^{2(\sigma+m)} |\hat{u}_\xi|^2 .$$

□

As every cotangent bundle, $T^*\mathbb{T}^d$ carries a natural symplectic structure. Denote by $\mathcal{J}$ the corresponding Poisson tensor, given two symbols a and b, their Poisson brackets is defined by

$$\{a,b\} = da\,(X_b)\ , \quad X_b = \mathcal{J} db\,. \tag{C.2.3}$$

The following standard results hold about the Poisson brackets of two symbols, and about their Moyal brackets and the symbol of their composition operator:

Lemma C.2.4. *Let $a\in S^{m,\delta}$ and $b\in S^{m',\delta}$; then $\{a,b\}\in S^{m+m'-\delta,\delta}$. In particular, for all $N_1,N_2\in\mathbb{N}$ there exists a positive constant $K=K_{N_1,N_2}$ depending only on N_1 and N_2 such that*

$$C_{N_1,N_2}(\{a,b\}) \le K_{N_1,N_2} C_{N_1+1,N_2+1}(a) C_{N_1+1,N_2+1}(b)\,.$$

Lemma C.2.5. *Let $A = Op^W(a)\in OPS^{m,\delta}$ and $B = Op^W(b)\in OPS^{m',\delta}$; then*

1. *$AB \in OPS^{m+m',\delta}$, with $AB = Op^W(a\sharp b)$,*

$$(a\sharp b)(x,\xi) = \sum_{k\in\mathbb{Z}^d} \hat{c}_k(\xi) e^{ik\cdot x}\,, \quad \hat{c}_k(\xi) = \sum_{h\in\mathbb{Z}^d} \hat{a}_{k-h}\left(\xi + \frac{h}{2}\right) \hat{b}_h\left(\xi + \frac{h-k}{2}\right). \tag{C.2.4}$$

In particular, for any $N_1, N_2 \in \mathbb{N}$, there exists $N_1' > N_1$ such that one has

$$C'_{N_1,N_2}(a\sharp b) \lesssim_{m,\delta,N_1,N_2,\mathfrak{c}} C'_{N_1',N_2}(a) C'_{N_1',N_2}(b)\,. \tag{C.2.5}$$

2. *Let $A = Op^W(a) \in OPS^{m,\delta}$ and $B = Op^W(b) \in OPS^{m',\delta}$. Then $[A,B] \in OPS^{m+m'-\delta,\delta}$, with*

$$\frac{1}{\mathrm{i}}[A,B] = Op\left(\{a,b\}_{\mathcal{M}}\right), \quad \{a,b\}_{\mathcal{M}} = \{a,b\} + S^{m+m'-3\delta,\delta}\,. \tag{C.2.6}$$

In particular, for all N_1 and $N_2 \in \mathbb{N}$ there exist $N_1' > N_1$ such that

$$C'_{N_1,N_2}(\{a,b\}_{\mathcal{M}}) \lesssim_{m,\delta,N_1,N_2,\mathfrak{c},} C'_{N_1',N_2+1}(a) C'_{N_1',N_2+1}(b)\,. \tag{C.2.7}$$

3. *If a is a quadratic polynomial in ξ, independent of x, then*

$$\{a,b\}_{\mathcal{M}} = \{a,b\}\,. \tag{C.2.8}$$

Proof. The above results are standard, and again we only prove them to stress that the only dependence on the metric g is given by $\mathfrak{c}$. Actually we only prove Item 1, since proof of Item 2 following in analogous way and Item 3 follows from a direct calculation.
The explicit expression (C.2.4) of $a\sharp b$ follows by a direct computation. To prove $a\sharp b \in S^{m+m',\delta}$, observe that, due to Lemma C.1.1 and Lemma C.1.3, for any $k, h \in \mathbb{Z}^d$ and for any $N_1, N_2 \in \mathbb{N}$

$$\begin{aligned}\left\| d_\xi^{N_2} \hat{a}_{k-h}\left(\xi + \frac{h}{2}\right)\right\| &\le C'_{N_1,N_2}(a) \left\langle \xi + \kappa + \frac{h}{2}\right\rangle_{\mathsf{g}^*}^{m-N_2\delta} \langle k-h\rangle_{\mathsf{g}^*}^{-N_1} \\ &\lesssim_{m,N_2,\delta,\mathfrak{c},N_1} C'_{N_1,N_2}(a) \left\langle \xi + \kappa\right\rangle_{\mathsf{g}^*}^{m-N_2\delta} \|h\|_{\mathsf{g}^*}^{|m-N_2\delta|} \langle h-k\rangle_{\mathsf{g}^*}^{-N_1}\,,\end{aligned} \tag{C.2.9}$$

and analogously

$$\left\| d_\xi^{N_2} \hat{b}_h\left(\xi + \frac{h-k}{2}\right)\right\| \lesssim_{m',N_2,\delta,\mathfrak{c},N_1} C'_{N_1,N_2}(b) \left\langle \xi + \kappa\right\rangle_{\mathsf{g}^*}^{m-N_2\delta} \|h-k\|_{\mathsf{g}^*}^{|m'-N_2\delta|} \langle h\rangle_{\mathsf{g}^*}^{-N_1}\,. \tag{C.2.10}$$

Thus we have that, for any N_1 and $N_2 \in \mathbb{N}$,

$$d_\xi^{N_2} \hat{c}_k(\xi) = \sum_{h\in\mathbb{Z}^d} \sum_{0\le \widetilde{N}_2 \le N_2} \mathsf{c}_{\widetilde{N}_2} d_\xi^{\widetilde{N}_2} \hat{a}_{k-h}\left(\xi + \frac{h}{2}\right) d_\xi^{N_2-\widetilde{N}_2} \hat{b}_h \left(\xi + \frac{h-k}{2}\right)$$

for suitable constants $\{\mathsf{c}_j\}_{j=0}^{N_2}$. By (C.2.9) and (C.2.10), if

$$N_1' = N_1 + d + 2 + \lceil \max\{|m|, |m'|\} + \delta N_2 \rceil ,$$

one has

$$\begin{aligned}\|d_\xi^{N_2} \hat{c}_k(\xi)\| \lesssim_{m,N_2,\delta} \sum_{h\in\mathbb{Z}^d} \sum_{\widetilde{N}_2 \le N_2} C'_{N_1',\widetilde{N}_2}(a) C'_{N_1',N_2-\widetilde{N}_2}(b) \left\langle \xi+\kappa \right\rangle_{\mathsf{g}^*}^{m-\widetilde{N}_2\delta} \left\langle h \right\rangle_{\mathsf{g}^*}^{-N_1'+|m-\widetilde{N}_2\delta|} \\ \left\langle \xi+\kappa \right\rangle_{\mathsf{g}^*}^{m'-(N_2-\widetilde{N}_2)\delta} \left\langle h-k \right\rangle_{\mathsf{g}^*}^{|m'-(N_2-\widetilde{N}_2)\delta|-N_1'} ,\end{aligned}$$

which gives

$$\begin{aligned}\|d_\xi^{N_2} \hat{c}_k(\xi)\| \lesssim_{m,N_2,\delta} C'_{N_1',N_2}(a) C'_{N_1',N_2}(b) \left\langle \xi+\kappa \right\rangle_{\mathsf{g}^*}^{m+m'-N_2\delta} \\ \sum_{h\in\mathbb{Z}^d} \left\langle h-k \right\rangle_{\mathsf{g}^*}^{-N_1'+|m'|+N_2\delta} \left\langle h \right\rangle_{\mathsf{g}^*}^{-N_1'+|m|+N_2\delta} .\end{aligned}$$

Observing that, by definition of N_1', one has

$$\left\langle h-k \right\rangle_{\mathsf{g}^*}^{-N_1'+|m'|+N_2\delta} \left\langle h \right\rangle_{\mathsf{g}^*}^{-N_1'+|m|+N_2\delta} \lesssim_{N_1} \left\langle k \right\rangle_{\mathsf{g}^*}^{-N_1} \left\langle h-k \right\rangle_{\mathsf{g}^*}^{-(d+2)} ,$$

we deduce

$$\begin{aligned}\|d_\xi^{N_2} \hat{c}_k(\xi)\| &\lesssim_{N_1,N_2,m,\delta,\mathfrak{c}} C'_{N_1',N_2}(a) C'_{N_1',N_2}(b) \left\langle \xi+\kappa \right\rangle_{\mathsf{g}^*}^{m+m'-N_2\delta} \left\langle k \right\rangle_{\mathsf{g}^*}^{-N_1} \mathfrak{s}_{d+2} \\ &\lesssim_{N_1,N_2,m,\delta,\mathfrak{c}} C'_{N_1',N_2}(a) C'_{N_1',N_2}(b) \left\langle \xi+\kappa \right\rangle_{\mathsf{g}^*}^{m+m'-N_2\delta} \left\langle k \right\rangle_{\mathsf{g}^*}^{-N_1} ,\end{aligned}$$

which, again by Lemma C.1.1, entails $a \sharp b \in S^{m+m',\delta}$, together with (C.2.5). □

Concerning Egorov type Theorems, we finally remark that the following result holds:

Lemma C.2.6 (Egorov Theorem)**.** *Let* $\eta < 1$, $\delta > 0$, $m \in \mathbb{R}$, $G := Op^W(g) \in OPS^{\eta,\delta}$ *a self-adjoint operator and* $A := Op^W(a) \in OPS^{m,\delta}$*. Then the following holds.*

1. *For any* $\tau \in [-1,1]$ $e^{\mathrm{i}\tau G} \in \mathcal{B}(H^s)$, $\forall s \in \mathbb{R}$ *and* $e^{\mathrm{i}\tau G}$ *is a unitary operator on* $L^2(\mathbb{T}^d)$. *Furthermore,* $H := e^{\mathrm{i}G} A e^{-\mathrm{i}G} \in OPS^{m,\delta}$ *and its symbol* $h(x,\xi)$ *admits an asymptotic expansion of the form*

$$h = a + \sum_{j=1}^{k-1} ad_g^j(a) + \widetilde{a}\,, \quad \widetilde{a} \in S^{m+k(\eta-\delta),\delta} \quad \forall k \in \mathbb{N}\,,$$
$$ad_g(a) = \mathrm{i}\{g,a\}_{\mathcal{M}}\,, \quad ad_g^{\,k+1}(a) = \frac{\mathrm{i}}{k+1}\left\{g, ad_g^{\,k}(a)\right\}_{\mathcal{M}} \quad \forall k \in \mathbb{N}\,. \tag{C.2.11}$$

2. *If* $\eta \le 0$, $e^{\mathrm{i}\tau G} \in OPS^{0,\delta}$ *and its symbol* σ *satisfies for all* $N_1, N_2 \in \mathbb{N}$

$$C'_{N_1,N_2}(\sigma) \lesssim_{N_1,N_2,\delta,\mathfrak{c}} e^{KC'_{N'_1,N_2}(g)}\,, \tag{C.2.12}$$

where again the positive constant K *and the integer* N'_1 *only depend on* $N_1, N_2, \delta, \mathfrak{c}$. *Moreover, the family of seminorms of the symbol* $\widetilde{a}$ *in* (C.2.11) *only depends on the family of seminorms of* a *and* g.

Proof. The proof of Item 1 is standard; in the general case $\eta < 1$ the claim follows for instance from Theorem 1.2 of [MR17]. Equation (C.2.11) follows from a Taylor expansion of order k (see for instance the proof of Lemma A.1.4, where analogous calculations are made in the case of the standard torus).
In the case $\eta \le 0$, with analogous calculations to the ones exhibited to prove Lemma C.2.5, one obtains that, defining $g_0 = g$ and $g_n = g \sharp g_{n-1}$ $\forall n > 0$, the series $\sum_{n\ge 0} \frac{\mathrm{i}^n g_n(x,\xi)}{n!}$ converges to a symbol $\sigma \in S^{0,\delta}$, whose seminorms $\{C'_{N_1,N_2}(\sigma)\}_{N_1\in\mathbb{N},N_2\in\mathbb{N}}$ satisfy estimate (C.2.12). This proves Item 2. □

Remark C.2.7. *A direct computation shows that, if* $a \in S^{m,\delta}$ *is even in* ξ *and* $g \in S^{m',\delta}$ *is odd in* ξ, *then* $(a\sharp g)(x,-\xi) = -(g\sharp a)(x,-\xi)$, *which in particular entails that* $\{a,g\}_{\mathcal{M}}$ *is odd in* ξ. *Analogously, if* a *is an even symbol in* ξ *and* g *is odd, then all the symbols* $ad_g^j(a)$ *defined in* (C.2.11) *are even symbols.*

Appendix D

Technical lemmas

We first recall Lemma 5.7 of [Gio03].

Lemma D.0.1 (Lemma 5.7 of [Gio03]). *Let $s \in \{1, \dots, d\}$ and let $\{u_1, \dots u_s\}$ be linearly independent vectors in $\mathbb{R}^d$ equipped with the euclidean metric $|\cdot|$. Denote by $Vol\{u_1 | \cdots | u_s\}$ the $s-$ dimensional volume of the parallelepiped with sides $u_1, \dots, u_s$. Let moreover $w \in \mathrm{span}\{u_1, \dots u_s\}$ be any vector. If there exists positive constants α, N such that*

$$
\begin{aligned}
|u_j| &\le N \quad \forall j = 1, \dots s, \\
|w \cdot u_j| &\le \alpha \quad \forall j = 1, \dots s,
\end{aligned}
$$

then

$$
|w| \le \frac{s N^{s-1} \alpha}{Vol\{u_1 | \cdots | u_s\}}.
$$

We remark that, since all the quantities involved in the statement are coordinate independent, Lemma 5.4.1 immediately follows from it.

Lemma D.0.2. *Let $\{e_1, \dots, e_d\}$ be the vectors of the standard basis in $\mathbb{R}^d$. There exists a positive constant $\mathfrak{C}$, depending only on*

$$
\mathfrak{c}_2 = \max_{j=1,\dots,d} \|e_j\| \tag{D.0.1}
$$

and

$$
\mathfrak{v} = \int_{\mathbb{T}^d} d\mu_{\mathtt{g}}(x) \equiv \mu_{\mathtt{g}}\left(\mathbb{T}^d\right), \tag{D.0.2}
$$

such that for any $s \in \{1, \dots, d\}$ and for any set $\{u_1, \dots u_s\}$ of linearly independent vectors in $\mathbb{Z}^d$

$$
Vol_{\mathtt{g}}\{u_1 | \cdots | u_s\} \ge \mathfrak{C}.
$$

Proof. We observe that, if $\{e_1, \dots, e_d\}$ is the canonical basis of $\mathbb{Z}^d$, there exists a subset $\{u'_{s+1}, \dots u'_d\} \subset \{e_1, \dots, e_d\}$ such that

$$\{u_1, \dots, u_s, u'_{s+1}, \dots u'_d\}$$

is a set of linearly independent vectors in $\mathbb{Z}^d$. Hence one has that, if M is the linear subspace generated by $\{u_1, \dots, u_s\}$,

$$\begin{aligned} \mathrm{Vol}_{\mathtt{g}}\{u_1|\dots|u_s|u'_{s+1}|\dots|u'_d\} &\le \|u'_{s+1}\|\cdots\|u'_d\| \ \mathrm{Vol}_{\mathtt{g}}\{u_1|\dots|u_s\} \\ &\le (\mathfrak{c}_2)^d \mathrm{Vol}_{\mathtt{g}}\left(\{u_1|\dots|u_s\}\right) , \end{aligned}$$

by the definition of $\mathfrak{c}_2$ as in (D.0.1). In particular, one has that

$$\mathrm{Vol}_{\mathtt{g}}\left(\{u_1|\dots|u_s\}\right) \ge (\mathfrak{c}_2)^{-d} \ \mathrm{Vol}_{\mathtt{g}}\{u_1|\dots|u_s|u'_{s+1}|\dots|u'_d\} . \tag{D.0.3}$$

Write

$$u_j = \sum_{k=1}^{d} n_{j,k} e_k , \quad u'_j = \sum_{k=1}^{d} n_{j,k} e_k ,$$

and if $\forall k = 1, \dots, d$ $\tilde{e}_k$ is the vector of the components of e_k with respect to an orthonormal basis for the inner product $\langle \cdot ; \cdot \rangle_{\mathtt{g}*}$, then

$$\begin{aligned} \mathrm{Vol}_{\mathtt{g}}\left(\{u_1|\dots|u_s|u'_{s+1}|\dots|u'_d\}\right) &= \mathrm{Vol}_{\mathtt{g}}\left(\left\{\sum_{k=1}^{d} n_{1,k} e_k|\dots|\sum_{k=1}^{d} n_{d,k} e_k\right\}\right) \\ &= \mathrm{Vol}\left(\left\{\sum_{k=1}^{d} n_{1,k} \tilde{e}_k|\dots|\sum_{k=1}^{d} n_{d,k} \tilde{e}_k\right\}\right) \\ &\ge \mathrm{Vol}\left(\tilde{e}_1|\cdots|\tilde{e}_d\right) \\ &= \mathrm{Vol}_{\mathtt{g}}\left(e_1|\cdots|e_d\right) = \mathfrak{v} . \end{aligned}$$

Thus (D.0.3) implies that

$$\mathrm{Vol}_{\mathtt{g}}\{u_1 \mid \cdots \mid u_s\} \ge (\mathfrak{c}_2)^{-d} \mathfrak{v} =: \mathfrak{C} .$$

□

Remark D.0.3. *Let $a > 0$. By studying the function $(1+x^2)^{a/2}$ it is easy to see that there exists a constant K s.t. $\forall \xi, \eta \in \mathbb{R}^d$ one has*

$$\langle \xi + \eta \rangle^a \le K(\langle \xi \rangle^a + \langle \eta \rangle^a) . \tag{D.0.4}$$

Furthermore, since, for any $C > 0$

$$\sup_{y>C} \frac{\langle y \rangle}{y} < \infty \ ,$$

one also has $\exists K' = K'(a, C)$ s.t.

$$\langle \xi + \eta \rangle^a \leq K'(\langle \xi \rangle^a + \|\eta\|^a) \ , \quad \forall \eta \ : \|\eta\| \geq C \ . \tag{D.0.5}$$

Remark D.0.4. *If $\|\xi - \eta\| \leq F\langle \xi \rangle^a$, with $a < 1$, one has*

$$\langle \xi \rangle \leq K(1 + F)\langle \eta \rangle \ .$$

A further useful lemma is the following one

Lemma D.0.5. *Let $N \geq 1$, $a < 1$, $K \geq 2^{-a}$ be positive real numbers, Then*

$$x - Kx^a \leq N \implies x \leq (2K)^{\frac{1}{1-a}} N \ . \tag{D.0.6}$$

Proof. If $Kx^a \leq \frac{x}{2}$, which is equivalent to

$$x \geq (2K)^{\frac{1}{1-a}} \ , \tag{D.0.7}$$

then the assumed inequalities implies

$$\frac{1}{2}x \leq x - Kx^a \leq N \implies x < 2N \ ,$$

but, by assumption, the r.h.s is smaller than $(2K)^{\frac{1}{1-a}}$, and therefore the thesis holds in this case. On the contrary, the converse of (D.0.7), implies

$$x < (2K)^{\frac{1}{1-a}} \leq (2K)^{\frac{1}{1-a}} N \ ,$$

which again implies the thesis. □

Lemma D.0.6. *Let $1 > a > \epsilon > 0$ and $1 > \delta > 0$ be parameters. Let ς, η, k, ℓ be vectors. Assume that there exist constants C, F, D, D_0 s.t.*

$$|\langle \varsigma + k; h \rangle| \leq C\langle \varsigma + k \rangle^\delta |h|^{-\tau} \ , \tag{D.0.8}$$

$$\|k\| \leq D\langle \varsigma + k \rangle^\epsilon \ , \quad \|h\| \leq D_0\langle \varsigma + k \rangle^\epsilon \tag{D.0.9}$$

$$\|\eta - \varsigma\| \leq F\langle \eta \rangle^a \ , \quad \|\ell\| \leq D\langle \eta + \ell \rangle^\epsilon \ ; \tag{D.0.10}$$

then there exists K' and D' (which depends on the above constants), s.t.

$$\langle \varsigma + k \rangle \leq D'\langle \eta + \ell \rangle \ , \tag{D.0.11}$$

$$|(\eta + \ell, h)| \leq K'\langle \eta + \ell \rangle^{\max\{\delta, a+\epsilon(\tau+1)\}} |h|^{-\tau} \ . \tag{D.0.12}$$

Proof. Start by writing

$$\varsigma + k = \eta + \ell + v \tag{D.0.13}$$

$$v := k - \ell + \varsigma - \eta \; ; \tag{D.0.14}$$

then we estimate v (with $\eta + \ell$). One has

$$\begin{aligned} \|v\| &\leq D\langle \varsigma + k\rangle^{\epsilon} + D\langle \eta + \ell\rangle^{\epsilon} + F\langle \eta\rangle^{a} \\ &= D\langle \eta + \ell + v\rangle^{\epsilon} + D\langle \eta + \ell\rangle^{\epsilon} + F\langle \eta + \ell - \ell\rangle^{a} \\ &\leq DK\left(\langle \eta + \ell\rangle^{\epsilon} + \langle v\rangle^{\epsilon}\right) + D\langle \eta + \ell\rangle^{\epsilon} + FK\left(\langle \eta + \ell\rangle^{a} + \langle \ell\rangle^{a}\right) \\ &\leq D(K+1)\langle \eta + \ell\rangle^{\epsilon} + FK\langle \eta + \ell\rangle^{a} + FK(1+D)\langle \eta + \ell\rangle^{a\epsilon} + DK\langle v\rangle^{\epsilon} \; . \end{aligned}$$

Using $a > \epsilon$ and $a > a\epsilon$, (and exploiting $\langle x\rangle \leq 1 + x$, which holds for all positive x) we get

$$\begin{aligned} \langle v\rangle \leq \left(D(K+1) + FK + FK(1+D) + 1\right)\langle \eta + \ell\rangle^{a} \\ + DK\langle v\rangle^{\epsilon} \; . \end{aligned}$$

Applying Lemma D.0.5 with N equal to the first line, we get that there exists a constant K'' (explicitely computable), s.t.

$$\langle v\rangle \leq K''\langle \eta + \ell\rangle^{a} \; . \tag{D.0.15}$$

Exploiting this and using again (D.0.13), we immediately get (D.0.11). We are now ready for the final estimate:

$$\begin{aligned} |(\eta + \ell, h)| &\leq |(\varsigma + k, h)| + |(v, h)| \, \|h\|^{\tau}\|h\|^{-\tau} \\ &\leq C\langle \varsigma + k\rangle^{\delta}\|h\|^{-\tau} + K''\langle \eta + \ell\rangle^{a} D_0\langle \varsigma + k\rangle^{\epsilon} D_0^{\tau}\langle \varsigma + k\rangle^{\varepsilon\tau}\|h\|^{-\tau} \\ &\leq C(D')^{\delta}\langle \eta + \ell\rangle^{\delta}\|h\|^{-\tau} + K''\langle \eta + \ell\rangle^{a} D_0^{\tau+1}(D')^{\varepsilon(\tau+1)}\langle \eta + \ell\rangle^{\epsilon(\tau+1)}\|h\|^{-\tau} \, , \end{aligned}$$

from which the thesis immediately follows. □

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