Rev. Mat. Iberoam. **36** (2020), no. 4, 1167–1206 European Mathematical Society
DOI 10.4171/RMI/1163

Elliptic fibrations on K3 surfaces with a non-symplectic involution fixing rational curves and a curve of positive genus

Alice Garbagnati and Cecília Salgado

Abstract. In this paper we complete the classification of the elliptic fibrations on K3 surfaces which admit a non-symplectic involution acting trivially on the Néron–Severi group. We use the geometric method introduced by Oguiso and moreover we provide a geometric construction of the fibrations classified. If the non-symplectic involution fixes at least one curve of genus 1, we relate all the elliptic fibrations on the K3 surface with either elliptic fibrations or generalized conic bundles on rational elliptic surfaces. This description allows us to write the Weierstrass equations of the elliptic fibrations on the K3 surfaces explicitly and to study their specializations.

1. Introduction

The purpose of this paper is the classification of elliptic fibrations (with section) on several families of K3 surfaces. These families are characterized by the presence of a non–symplectic involution on their general member.

The families we are interested in were classified by Nikulin, in [15]: let X be a K3 surface over $\mathbb{C}$ and ι an involution on X which does not preserve the symplectic structure. The fixed locus of ι can be one of the following:

a) $\mathrm{Fix}(\iota) = \emptyset$.

b) $\mathrm{Fix}(\iota) = C$ or $\mathrm{Fix}(\iota) = C \coprod C_1 \cdots \coprod C_k$, where C is a curve of genus $g \geq 0$ and C_i are rational curves, $1 \leq i \leq k$ and $k \leq 9$.

c) $\mathrm{Fix}(\iota) = C \coprod D$, where C and D are both curves of genus 1.

The condition that X is generic among the K3 surfaces admitting a non–symplectic involution with a prescribed fixed locus is equivalent to the condition that ι acts trivially on its Néron–Severi group.

Mathematics Subject Classification (2010): 14J26, 14J27, 14J28.
Keywords: Elliptic fibrations, rational elliptic surfaces, K3 surfaces, double covers.

We classify all elliptic fibrations on X in terms of their trivial lattices. In particular one obtains that if $\mathrm{Fix}(\iota) = \emptyset$, then X does not admit elliptic fibrations with section and if $\mathrm{Fix}(\iota) = C \coprod D$ as in c) then X admits a unique elliptic fibration with section. Hence we concentrate ourselves on K3 surfaces with a non-symplectic involution whose fixed locus is as in b). The case $g = 0$ was already considered in [6]. Here we principally focus on the ones with $g = 1$, i.e., the non-symplectic involution ι fixes one curve of genus 1 and $1 \leq k \leq 9$ rational curves, but in Section 7 we also discuss all the other cases, completing the classification.

Several papers are devoted to the classifications of elliptic fibrations on K3 surfaces. The most classical are [16], where a lattice theoretic method is applied, and [17] where a more geometric technique is considered. More recently, the method used in [16] is applied, for example, in [4] and [5] and the one proposed in [17] is considered in [11], [6], [2]. A very deep recent result was obtained in [12], where the author classified the elliptic fibrations on the Kummer surface of a principally polarized Abelian surface without applying the previous method, which do not apply well in this situation.

In order to classify the elliptic fibrations on K3 surfaces, we first produce a list of the possible elliptic fibrations, applying the techniques developed by Oguiso in [17]. These techniques are based on the presence of a non-symplectic involution acting trivially on the Néron–Severi group and here we obtain the classifications on all the families for which this method applies. If $g = 1$, we construct explicitly the elliptic fibrations listed by means of the geometric constructions presented in our previous paper [8]. These constructions can be considered if the K3 surface is a 2-cover of a rational elliptic surface.

Once one has a classification of the elliptic fibrations on a K3 surface, it is natural to ask for the Weierstrass equations of such fibrations, see e.g. [25], [13], [14], [4]. The geometric realization that we provide for the elliptic fibrations allows us to obtain immediately the Weierstrass equations by applying an algorithm presented in [2]. Moreover, the knowledge of the equations of the classified elliptic fibrations allows one to consider specializations of the elliptic fibrations and thus of the underlying K3 surfaces. Hence we are able to find some values of the parameters of the considered families of K3 surfaces for which the transcendental lattice shrinks, i.e., its rank decreases, and to compute the new transcendental lattice for these values. In certain cases the elliptic fibrations specialize because the Mordell–Weil rank increases and our methods allow us to identify the new sections of the fibration.

This paper is organized as follows. In Section 2, we state the main theorem (Theorem 2.6) and give a list of all possible configurations of the trivial lattice of genus 1 fibrations on the K3 surfaces described above. Then we concentrate, in Sections 3, 4, 5 and 6 on the case $g = 1$. Section 3 is devoted to outlining which rational elliptic surfaces can arise as the quotient X/ι mentioned above. Section 4 contains a realization of the classification of the elliptic fibrations on X in terms of (generalized) conic bundles on X/ι. This allows one to compute, in Section 5, the Weierstrass equations of all the elliptic fibrations classified. In Section 6, we use the equations computed in order to describe several interesting specializations of the considered K3 surfaces and of their elliptic fibrations. Section 7 and the Appendix contain the classifications of elliptic fibrations on K3 surfaces which admit a non-symplectic involution acting trivially on the Néron–Severi group and fixing one curve of genus greater than 1 and complete the proof of our main theorem.

Basic definitions

In what follows we present the basic key definitions to this text.

Definition 1.1. Let X be a smooth projective algebraic surface and B a smooth projective algebraic curve. An elliptic fibration with base B on X is a flat morphism $\varepsilon\colon X \to B$ such that:

i) $\varepsilon^{-1}(t)$ is a smooth curve of genus one for all but finitely many $t \in B$.

ii) there is a section $\sigma\colon B \to X$, i.e., a map such that $\varepsilon \circ \sigma\colon B \to B$ is the identity map.

iii) $\varepsilon^{-1}(t)$ is singular for at least one $t \in B$.

iv) ε is relatively minimal, i.e., the fibers of ε do not admit (-1)-curves as components.

Condition ii) above assures that all but finitely many fibers of ε are elliptic curves with $\sigma(t)$ as its neutral element. In particular, this fibers admit an involution which we denote by $[-1]$ from now on. Condition iii) rules out surfaces of product type, i.e., $X \simeq C \times B$, where C is a curve. Finally, note that iv) above means that the surface is relatively minimal with respect to the fibration, but it does not imply that X is a minimal surface as (-1)-curves are allowed outside of the fibers of ε. If ε satisfies i), ii) and iii) but does not satisfy iv), then it is be called a *non-relatively minimal elliptic fibration*.

We will denote the Mordell–Weil group of ε, i.e., the group of the sections of the fibration, by $\mathrm{MW}(\varepsilon)$.

Definition 1.2. A K3 surface is a smooth projective algebraic surface, say X, such that

i) $q := h^1(X, \mathcal{O}_X) = 0$, i.e., X is regular,

ii) $K_X \simeq 0$, i.e., the canonical divisor of X is trivial.

If $\varepsilon\colon X \to B$ is an elliptic fibration and X is either a K3 surface or a rational surface, then $B \simeq \mathbb{P}^1$.

Definition 1.3. Let X be a K3 surface, then $H^{2,0}(X) \simeq \mathbb{C} \cdot \omega_X$, where ω_X is a nowhere vanishing symplectic form. An involution ι on X is called non-symplectic if it does not preserve the symplectic structure on X, i.e., $\iota(\omega_X) = -\omega_X$.

2. The non-symplectic involution and admissible fibrations

The aim of this section is to classify the elliptic fibrations which appear on surfaces X as in the following assumption.

Assumption 2.1. *Let X be a* K3 *surface and ι a non–symplectic involution of X which acts trivially on the Néron–Severi group.*

This assumption is very natural, since this means that the K3 surface is generic in the family of the K3 surfaces with a non-symplectic involution with a prescribed fixed locus. Nice and easy examples of these K3 surfaces are provided by double covers of $\mathbb{P}^2$ branched along a (possible singular or reducible) sextic, and by double covers of a rational elliptic surface such that all the reducible fibers are reduced and contained in the branch locus.

Given a non–symplectic involution ι we call *special for* ι the curves which are fixed by it. Throughout this note, we simply call these curves *special* since the dependence on the involution is clear.

Proposition 2.2. *Let (X,ι) be as in Assumption 2.1. Let C be a smooth rational curve in X. Then either C is special, or it meets the fixed locus of ι in exactly two points.*

Let C_1 and C_2 be two smooth rational curves which are not special. Then $C_1 \cdot C_2 \equiv 0 \mod 2$.

Proof. This follows immediately from results of Oguiso [17] and Kloosterman [11]. It is due to the fact that the class of each rational curve is mapped to itself by ι. □

Proposition 2.3. *Let (X,ι) be as in Assumption 2.1 and $\mathcal{E}\colon X \to \mathbb{P}^1$ an elliptic fibration on X. Then*

(1) *either ι maps each fiber of $\mathcal{E}$ to itself,*

(2) *or ι maps at least one fiber of $\mathcal{E}$ to another fiber of $\mathcal{E}$.*

If ι is as in (1)*, then it acts as the identity on the basis of $\mathcal{E}$ and as $[-1]$ on the fibers.*

If ι is as in (2)*, then it acts as an involution on the basis of $\mathcal{E}$ and it preserves two fibers.*

The involution ι is as in (1) *if and only if there is a section of $\mathcal{E}$ which is a special curve.*

Proof. Since ι acts as the identity on the Néron–Severi group, it maps the class of the fiber of $\mathcal{E}$ to itself. So it maps each fiber either to itself or to another fiber of the same fibration. In the first case, the automorphism induced by ι on the basis is the identity, but since it is not the identity on X, it acts on the fibers. Since ι is non-symplectic, it is not a translation on each fiber. Hence it is the elliptic involution $[-1]$ on the fibers, possibly composed with a translation by a torsion point. The class of the zero section is the class of an irreducible rational curve on a K3 surface. So ι preserves the zero section. Since it preserves each fiber and the zero section, the latter is a special curve for ι, and ι is $[-1]$ on the fibers.

If ι does not preserves all fibers, then it does not give the identity on the basis, and thus it gives an involution on the basis, with two fixed points p_1 and p_2. Its fixed locus is necessarily contained in the two fibers over the points p_1 and p_2, and so there are no sections among the special curves. □

Definition 2.4. An elliptic fibration is of type 1 (resp. of type 2) with respect to ι if it is as in case (1) of Proposition 2.3 (resp. as in case (2) of Proposition 2.3).

Proposition 2.5. *Let (X, ι) be as in Assumption 2.1, and let $\mathcal{E}\colon X \to \mathbb{P}^1$ be an elliptic fibration on X. Then the following hold*:

(1) *If $\mathcal{E}$ is of type 1 with respect to ι, then $\mathrm{MW}(\mathcal{E}) \subset (\mathbb{Z}/2\mathbb{Z})^2$.*

(2) *If $\mathcal{E}$ is of type 1 with respect to ι, and there is at least one non-rational special curve, then $\mathrm{MW}(\mathcal{E}) \subset \mathbb{Z}/2\mathbb{Z}$.*

(3) *If there is at least one special curve C of genus greater than 1, then $\mathrm{MW}(\mathcal{E}) \subset \mathbb{Z}/2\mathbb{Z}$, and in this case if $\mathrm{MW}(\mathcal{E}) = \mathbb{Z}/2\mathbb{Z}$, then C is hyperelliptic.*

(4) *If there is at least one special curve of genus greater than 3, then $\mathrm{MW}(\mathcal{E})$ is trivial.*

Proof. The sections of an elliptic fibration are rational curves, hence ι maps each section to itself. Indeed, as rational curves are rigid on a K3 surface, each rational curve is alone in its divisor class. By Assumption 2.1, ι acts trivially on the Néron–Severi group and therefore fixes each divisor class, fixing, in particular, each rational curve. If ι is the identity on the basis, then each section of the fibration is a fixed curve. In particular each section of an elliptic fibration of type 1 is fixed by ι. If $\mathcal{E}$ is of type 1 with respect to ι, then ι is the elliptic involution and in particular it fixes the zero section, the (possibly reducible) trisection passing through the 2-torsion points and no other sections. We conclude that if $\mathcal{E}$ is of type 1 with respect to ι, then $\mathrm{MW}(\mathcal{E}) \subset (\mathbb{Z}/2\mathbb{Z})^2$. If moreover there is a non-rational curve fixed by ι, then it is a component of the trisection passing through the 2-torsion points and thus this trisection is either irreducible (and so $\mathrm{MW}(\mathcal{E}) = \{0\}$) or it is a bisection (and so $\mathrm{MW}(\mathcal{E}) = \mathbb{Z}/2\mathbb{Z}$).

We recall that if $\mathcal{E}$ is of type 2 with respect to ι, then the special curves are contained in two fibers, so there are no special curves with genus higher than 1. Hence, if there is at least one special curve C of genus greater than 1, then $\mathcal{E}$ is of type 1 with respect to ι, and we conclude that either C is the trisection of the 2-torsion points, or the trisection splits into a section and a bisection. In the latter case C is the bisection and by definition it is hyperelliptic.

Given a lattice L, its length is the minimal number of generators of the discriminant group $L^\vee/L$. If ι fixes one curve of genus higher than 3, then, denote by r the rank of the Néron–Severi group and by a the length of the Néron–Severi group, it follows by [15] that $(22 - r - a)/2 > 3$. This implies that $r + a < 16$. By [9], if $r + a < 16$, then X cannot admit a symplectic involution. On the other hand, if a K3 surface admits an elliptic fibration with a 2-torsion section, the translation by this section is a symplectic involution on the K3 surface. We conclude that if $r + a > 16$, there are no elliptic fibrations on X with a 2-torsion section. □

Theorem 2.6. *Let X be a K3 surface which admits at least one elliptic fibration and let (X, ι) be as in Assumption 2.1. The following hold*:

(i) $\mathrm{Fix}_\iota(X) \neq \emptyset$.

(ii) *If $\mathrm{Fix}_\iota(X) = C \coprod D$ with $g(C) = g(D) = 1$, then there is a unique elliptic fibration on X, which is $\varphi_{|C|}\colon X \to \mathbb{P}^1$ and it is of type 2.*

(iii) *If* $\mathrm{Fix}(\iota) = C \coprod C_1 \cdots \coprod C_k$*,* $i = 1, \ldots, k$*,* $g(C_i) = 0$*, and* $g(C) > 1$*, then all the fibrations on* X *are of type* 1 *and they are given in Proposition* 7.1*.*

(iv) *If* $\mathrm{Fix}(\iota) = C \coprod C_1 \cdots \coprod C_k$*,* $i = 1, \ldots, k$*,* $g(C_i) = 0$*, and* $g(C) = 1$*, then there exists one fibration of type* 2*,* $\varphi_{|C|} \colon X \to \mathbb{P}^1$*, and all the other fibrations are of type* 1*. All the fibrations on* X *are given in Proposition* 2.9*.*

(v) *If* $\mathrm{Fix}(\iota) = C \coprod C_1 \cdots \coprod C_k$*,* $i = 1, \ldots, k$ *and* $g(C_i) = g(C) = 0$ *then there exists both fibrations of type* 2 *and of type* 1*. All the fibrations on* X *are given in* [6]*.*

Proof. If $\mathcal{E}$ is an elliptic fibration of type 1 with respect to ι, then the zero section is a special curve $\mathcal{E}$. If $\mathcal{E}$ is an elliptic fibration of type 2, then it preserves two fibers of the fibration and the zero section. So ι cannot be a translation on these fibers and thus $\mathrm{Fix}_\iota(X) \neq \emptyset$.

Let us assume that ι fixes at least one genus 1 curve E. Then $\varphi_{|E|} \colon X \to \mathbb{P}^1$ is a genus 1 fibration and ι fixes at least one of the fibers. If ι would act as the identity on the base of the fibration it cannot act as the identity also on the fibers of the fibration (otherwise it is the identity). So, if ι is the identity on the fiber E of $\varphi_{|E|} \colon X \to \mathbb{P}^1$, it is not the identity on the base, i.e., $\varphi_{|E|}$ is of type 2 with respect to ι. In particular it is an involution of the basis, which fixes two points on the basis, and thus it preserves two fibers of $\varphi_{|E|} \colon X \to \mathbb{P}^1$. The special curve E is one of these fibers.

Assume that X admits other genus 1 fibrations, which are not $\varphi_{|E|}$. Denote by $\mathcal{E} \colon X \to \mathbb{P}^1$ one of these. The special curve E cannot be a fiber of $\mathcal{E}$ as otherwise $\mathcal{E}$ would coincide with $\varphi_{|E|}$. So E is a horizontal curve and $\mathcal{E}$ is of type 1 with respect to ι. Since E is not a rational curve, it is neither a section nor an irreducible component of a reducible fiber. Thus it is a multisection meeting the fibers of $\mathcal{E}$ in 2-torsion points, i.e., it is either a trisection or a bisection. In both the cases there can not be another genus 1 curve in the fixed locus.

If there is a special curve of genus bigger than 1, then it is not contained in a fiber and thus there are no fibrations of type 2 with respect to ι.

Thanks to the results in [17], [11] and [6], to conclude the proof it remains to classify the elliptic fibrations which appear in the cases $\mathrm{Fix}(\iota) = C \coprod C_1 \cdots \coprod C_k$ with $g(C) \geq 1$. This is done in Propositions 2.9 and 7.1. □

We observe that most of the elliptic fibrations that we are looking for are of type 1. For these fibrations the Mordell–Weil group is extremely simple, so that one has to classify principally the reducible fibers which can appear. This is the purpose of the following proposition, where we reformulate the results by Oguiso, see [17], to deal with surfaces in our setting.

Proposition 2.7. *Let* (X, ι) *be as in Assumption* 2.1 *and let* $\mathcal{E} \colon X \to \mathbb{P}^1$ *be an elliptic fibration on* X *of type* 1 *with respect to* ι*. Then the reducible fibers which appear in* $\mathcal{E}$ *are among the ones contained in the following table, where the number* s *of special rational curves that they contain and the number* c *of components not meeting the zero section are given.*

fiber	s	c	Dynkin
I_2	0	1	A_1
I_{2n}^*	$n+1$	$2n+4$	D_{2n+4}
III^*	3	7	E_7
II^*	4	8	E_8

Proof. This is a consequence of Propositions 2.2 and 2.5. In particular, by the assumptions, ι fixes the zero section of the fibration.

To illustrate the kind of arguments, we show why fibers of type IV^* are not allowed. The arguments for the other cases are analogous and somewhat simpler. Suppose that there exists such a fiber. We denote by Θ_i its components: Θ_0 is the components meeting the zero section. If $i, j \in \{0, 1, \dots, 4\}$, then $\Theta_i \cdot \Theta_j = 1$ if and only if $|i - j| = 1$ and $\Theta_i \cdot \Theta_j = 0$ otherwise. The component Θ_5 meets only the component Θ_2 in one point and the component Θ_6 in another point. The component Θ_6 meets only the component Θ_5. Since ι acts trivially on the Néron–Severi group, each component Θ_i is sent to itself by ι. In particular the intersection points between Θ_2 and Θ_i, for $i = 1, 3, 5$ are fixed points. But then Θ_2 is a fixed curve (the involution ι acts on the rational curve Θ_2 with 3 fixed points). The fixed locus of ι is smooth, so Θ_1, Θ_3 and Θ_5 are not fixed. The intersection point between Θ_1 and Θ_0 is a fixed point and it is a singular point of the fiber IV^*. Neither a section nor the trisection of the 2-torsion points pass through this singular point of IV^*. A non-symplectic involution on a K3 surface cannot admit an isolated fixed point, so there is a curve passing through the intersection point between Θ_1 and Θ_0. So Θ_0 is a fixed curve. Analogously Θ_4 and Θ_6 are fixed. But these curves are the unique simple components of the fiber of type IV^* and we know that ι fixes at least one section. This means that there exist two special curves which intersect, namely a section of the fibration and a component of a fiber, which is impossible by the smoothness of the fixed locus of ι. □

Proposition 2.8 (See [15]). *Let (X, ι) be as in Assumption 2.1. Let us assume that ι fixes a curve of genus 1 and precisely k rational curves. Then:*

i) *the Néron–Severi group of X has rank $r = 10 + k$ and its discriminant is $(\mathbb{Z}/2\mathbb{Z})^a$, where $a = 20 - r (= 10 - k)$.*

ii) *The pair (r, a) determines $\mathrm{NS}(X)$ if $r \neq 14$ and $r \neq 18$. If $r = 14$ or $r = 18$, there are two different possibilities for $\mathrm{NS}(X)$, which depend on the values of $\delta \in \{0, 1\}$. If the discriminant form of $\mathrm{NS}(X)$ takes values in $\mathbb{Z}$, then $\delta = 0$, otherwise $\delta = 1$.*

iii) *The triple (r, a, δ) uniquely determines $\mathrm{NS}(X)$.*

Proposition 2.9. *Let (X, ι) be as in Proposition 2.8 and let $\mathcal{E}\colon X \to \mathbb{P}^1$ be an elliptic fibration on X. Then there are two possibilities:*

i) *$\mathcal{E}$ is the unique fibration of type 2 and the configuration for the reducible fibers appears among the first lines in Table 1.*

ii) *$\mathcal{E}$ is of type 1. The admissible configurations of the reducible fibers are listed in Table 1, in all the other lines.*

TABLE 1.

$k = 9$, $r = 19$, $a = 1$

n^o	trivial lattice	$17 = \sum c_i + \text{rank(MW)}$	$9 = k = \sum s_i + \#\text{sections}$	$MW(\mathcal{E})$
9.1	$U \oplus A_{17}$	$17 + 0$	$9 + 0$	$\mathbb{Z}/3\mathbb{Z}$
9.2	$U \oplus E_8 \oplus E_8 \oplus A_1$	$8 + 8 + 1 + 0$	$4 + 4 + 0 + 1$	$\{1\}$
9.3	$U \oplus E_7 \oplus D_{10}$	$7 + 10 + 0$	$3 + 4 + 2$	$\mathbb{Z}/2\mathbb{Z}$
9.4	$U \oplus D_{16} \oplus A_1$	$16 + 1 + 0$	$7 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$

$k = 8$, $r = 18$, $a = 2$, $\delta = 0$

n^o	trivial lattice	$16 = \sum c_i + \text{rank(MW)}$	$8 = k = \sum s_i + \#\text{sections}$	$MW(\mathcal{E})$
8.1	$U \oplus A_{15}$	$15 + 1$	$8 + 0$	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
8.2	$U \oplus E_8 \oplus D_8$	$8 + 8 + 0$	$4 + 3 + 1$	$\{1\}$
8.3	$U \oplus E_7^{\oplus 2} \oplus A_1^{\oplus 2}$	$7 + 7 + 1 + 1 + 0$	$3 + 3 + 0 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$
8.4	$U \oplus D_{16}$	$16 + 0$	$7 + 1$	$\{1\}$
8.5	$U \oplus D_{12} \oplus D_4$	$12 + 4 + 0$	$5 + 1 + 2$	$\mathbb{Z}/2\mathbb{Z}$
8.6	$U \oplus D_8 \oplus D_8$	$8 + 8 + 0$	$3 + 3 + 2$	$\mathbb{Z}/2\mathbb{Z}$

$k = 8$, $r = 18$, $a = 2$, $\delta = 1$

n^o	trivial lattice	$16 = \sum c_i + \text{rank(MW)}$	$8 = k = \sum s_i + \#\text{sections}$	$MW(\mathcal{E})$
8.1	$U \oplus A_{15}$	$15 + 1$	$8 + 0$	$\mathbb{Z}$
8.2	$U \oplus E_8 \oplus E_7 \oplus A_1$	$8 + 7 + 1 + 0$	$4 + 3 + 0 + 1$	$\{1\}$
8.3	$U \oplus E_7 \oplus D_8 \oplus A_1$	$7 + 8 + 1 + 0$	$3 + 3 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$
8.4	$U \oplus D_{14} \oplus A_1^{\oplus 2}$	$14 + 1 + 1 + 0$	$6 + 0 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$
8.5	$U \oplus D_{10} \oplus D_6$	$10 + 6 + 0$	$4 + 2 + 2$	$\mathbb{Z}/2\mathbb{Z}$

$k = 7$, $r = 17$, $a = 3$

n^o	trivial lattice	$15 = \sum c_i + \text{rank(MW)}$	$7 = k = \sum s_i + \#\text{sections}$	$MW(\mathcal{E})$
7.1	$U \oplus A_{13}$	$13 + 2$	$7 + 0$	$(\mathbb{Z})^2$
7.2	$U \oplus E_8 \oplus D_6 \oplus A_1$	$8 + 6 + 1 + 0$	$4 + 2 + 0 + 1$	$\{1\}$
7.3	$U \oplus E_7 \oplus D_6 \oplus A_1^{\oplus 2}$	$7 + 6 + 1 + 1 + 0$	$3 + 2 + 0 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$
7.4	$U \oplus E_7 \oplus D_8$	$7 + 8 + 0$	$3 + 3 + 1$	$\{1\}$
7.5	$U \oplus E_7^{\oplus 2} \oplus A_1$	$7 + 7 + 1 + 0$	$3 + 3 + 0 + 1$	$\{1\}$
7.6	$U \oplus D_{14} \oplus A_1$	$14 + 1 + 0$	$6 + 0 + 1$	$\{1\}$
7.7	$U \oplus D_{12} \oplus A_1^{\oplus 3}$	$12 + 1 + 1 + 1 + 0$	$5 + 0 + 0 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$
7.8	$U \oplus D_{10} \oplus D_4 \oplus A_1$	$10 + 4 + 1 + 0$	$4 + 1 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$
7.9	$U \oplus D_8 \oplus D_6 \oplus A_1$	$8 + 6 + 1 + 0$	$3 + 2 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$

$k = 6$, $r = 16$, $a = 4$

n^o	trivial lattice	$14 = \sum c_i + \text{rank(MW)}$	$6 = k = \sum s_i + \#\text{sections}$	$MW(\mathcal{E})$
6.1	$U \oplus A_{11}$	$11 + 3$	$6 + 0$	$(\mathbb{Z})^3$
6.2	$U \oplus E_8 \oplus D_4 \oplus A_1^{\oplus 2}$	$8 + 4 + 1 + 1 + 0$	$4 + 1 + 0 + 0 + 1$	$\{1\}$
6.3	$U \oplus E_7 \oplus D_6 \oplus A_1$	$7 + 6 + 1 + 0$	$3 + 2 + 0 + 1$	$\{1\}$
6.4	$U \oplus E_7 \oplus D_4 \oplus A_1^{\oplus 3}$	$7 + 4 + 1 + 1 + 1 + 0$	$3 + 1 + 0 + 0 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$
6.5	$U \oplus D_{12} \oplus A_1^{\oplus 2}$	$12 + 1 + 1 + 0$	$5 + 0 + 0 + 1$	$\{1\}$
6.6	$U \oplus D_{10} \oplus D_4$	$10 + 4 + 0$	$4 + 1 + 1$	$\{1\}$
6.7	$U \oplus D_{10} \oplus A_1^{\oplus 4}$	$10 + 1 + 1 + 1 + 1 + 0$	$4 + 0 + 0 + 0 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$
6.8	$U \oplus D_8 \oplus D_6$	$8 + 6 + 0$	$3 + 2 + 0 + 0 + 1$	$\{1\}$
6.9	$U \oplus D_8 \oplus D_4 \oplus A_1^{\oplus 2}$	$8 + 4 + 1 + 1 + 0$	$3 + 1 + 0 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$
6.10	$U \oplus D_6^{\oplus 2} \oplus A_1^{\oplus 2}$	$6 + 6 + 1 + 1 + 0$	$2 + 2 + 0 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$

$k = 5$, $r = 15$, $a = 5$

n^o	trivial lattice	$13 = \sum c_i + \text{rank(MW)}$	$5 = k = \sum s_i + \#\text{sections}$	$\text{MW}(\mathcal{E})$
5.1	$U \oplus A_9$	$9+4$	$5+0$	$(\mathbb{Z})^4$
5.2	$U \oplus E_8 \oplus A_1^{\oplus 5}$	$8+1+1+1+1+1+0$	$4+0+0+0+0+1$	$\{1\}$
5.3	$U \oplus E_7 \oplus D_4 \oplus A_1^{\oplus 2}$	$7+4+1+1+0$	$3+1+0+0+1$	$\{1\}$
5.4	$U \oplus E_7 \oplus A_1^{\oplus 6}$	$7+1+1+1+1+1+1+0$	$3+0+0+0+0+0+0+2$	$\mathbb{Z}/2\mathbb{Z}$
5.5	$U \oplus D_{10} \oplus A_1^{\oplus 3}$	$10+1+1+1+0$	$4+0+0+0+1$	$\{1\}$
5.6	$U \oplus D_8 \oplus D_4 \oplus A_1$	$8+4+1+0$	$3+1+0+1$	$\{1\}$
5.7	$U \oplus D_8 \oplus A_1^{\oplus 5}$	$8+1+1+1+1+1+0$	$3+0+0+0+0+0+2$	$\mathbb{Z}/2\mathbb{Z}$
5.8	$U \oplus D_6^{\oplus 2} \oplus A_1$	$6+6+1+0$	$2+2+0+1$	$\{1\}$
5.9	$U \oplus D_6 \oplus D_4 \oplus A_1^{\oplus 3}$	$6+4+1+1+1+0$	$2+1+0+0+0+2$	$\mathbb{Z}/2\mathbb{Z}$

$k = 4$, $r = 14$, $a = 6$, $\delta = 0$

n^o	trivial lattice	$12 = \sum c_i + \text{rank(MW)}$	$4 = k = \sum s_i + \#\text{sections}$	$\text{MW}(\mathcal{E})$
4.1	$U \oplus E_6$	$6+6$	$4+0$	$(\mathbb{Z})^6$
4.2	$U \oplus D_6 \oplus A_1^{\oplus 6}$	$6+1+1+1+1+1+1+0$	$2+0+0+0+0+0+0+2$	$\mathbb{Z}/2\mathbb{Z}$
4.3	$U \oplus D_4^{\oplus 3}$	$4+4+4+0$	$1+1+1+1$	$\{1\}$

$k = 4$, $r = 14$, $a = 6$, $\delta = 1$

n^o	trivial lattice	$12 = \sum c_i + \text{rank(MW)}$	$4 = k = \sum s_i + \#\text{sections}$	$\text{MW}(\mathcal{E})$
4.1	$U \oplus A_7$	$7+5$	$4+0$	$(\mathbb{Z})^5$
4.2	$U \oplus E_7 \oplus A_1^{\oplus 5}$	$7+1+1+1+1+1+0$	$3+0+0+0+0+0+1$	$\{1\}$
4.3	$U \oplus D_8 \oplus A_1^{\oplus 4}$	$8+1+1+1+1+0$	$3+0+0+0+0+1$	$\{1\}$
4.4	$U \oplus D_6 \oplus D_4 \oplus A_1^{\oplus 2}$	$6+4+1+1+0$	$2+1+0+0+1$	$\{1\}$
4.5	$U \oplus D_6 \oplus A_1^{\oplus 6}$	$6+1+1+1+1+1+1+0$	$2+0+0+0+0+0+0+2$	$\mathbb{Z}/2\mathbb{Z}$
4.6	$U \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 4}$	$4+4+1+1+1+1+0$	$1+1+0+0+0+0+2$	$\mathbb{Z}/2\mathbb{Z}$

$k = 3$, $r = 13$, $a = 7$

n^o	trivial lattice	$11 = \sum c_i + \text{rank(MW)}$	$3 = k = \sum s_i + \#\text{sections}$	$\text{MW}(\mathcal{E})$
3.1	$U \oplus A_5$	$5+6$	$3+0$	$(\mathbb{Z})^6$
3.2	$U \oplus D_6 \oplus A_1^{\oplus 5}$	$6+1+1+1+1+1+0$	$2+0+0+0+0+0+1$	$\{1\}$
3.3	$U \oplus D_4^{\oplus 2} \oplus A_1^{\oplus 3}$	$4+4+1+1+1+0$	$1+1+0+0+0+1$	$\{1\}$
3.4	$U \oplus D_4 \oplus A_1^{\oplus 7}$	$4+1+1+1+1+1+$ $1+1+0$	$1+0+0+0+0+0+0+0+2$	$\mathbb{Z}/2\mathbb{Z}$

$k = 2$, $r = 12$, $a = 8$

n^o	trivial lattice	$10 = \sum c_i + \text{rank(MW)}$	$2 = k = \sum s_i + \#\text{sections}$	$\text{MW}(\mathcal{E})$
2.1	$U \oplus A_3$	$3+7$	$2+0$	$(\mathbb{Z})^7$
2.2	$U \oplus D_4 \oplus A_1^{\oplus 6}$	$4+1+1+1+1+1+1+0$	$1+0+0+0+0+0+0+1$	$\{1\}$
2.3	$U \oplus A_1^{\oplus 10}$	$10+0$	$0+\ldots+0+2$	$\mathbb{Z}/2\mathbb{Z}$

$k = 1$, $r = 11$, $a = 9$

n^o	trivial lattice	$9 = \sum c_i + \text{rank(MW)}$	$1 = k = \sum s_i + \#\text{sections}$	$\text{MW}(\mathcal{E})$
1.1	$U \oplus A_1$	$1+8$	$1+0$	$(\mathbb{Z})^8$
1.2	$U \oplus A_1^{\oplus 9}$	$9+0$	$0+1$	$\{1\}$

Proof. Assume first that none of the special curves are sections for $\mathcal{E}$. Then $\mathcal{E}$ is of type 2 and the quotient X/ι is a rational elliptic surface R equipped with the elliptic fibration $\mathcal{E}_R$. The elliptic fibration $\mathcal{E}$ is induced by $\mathcal{E}_R$ by a base change of order 2, as proved in Theorem 4.2 of [8], see also [27]. Moreover, if ι acts trivially on the Néron–Severi group, then the elliptic fibration on R has exactly one reducible fiber which is contained in the branch locus of the cover $X \to R$. The reducible fiber is determined by k and δ, and it is always I_k, except for $k = 4$, when it can be also IV. This last statement is proved in the beginning of Section 3.

If there is a special curve which is a section of $\mathcal{E}$, then $\mathcal{E}$ is of type 1 and in order to produce the list one makes the following observations:

- By Proposition 2.5, $\mathrm{MW}(\mathcal{E}) \subset \mathbb{Z}/2\mathbb{Z}$.
- Since $\mathrm{MW}(\mathcal{E}) \subset \mathbb{Z}/2\mathbb{Z}$, the trivial lattice of the fibration has rank $r = \mathrm{rank}(\mathrm{NS}(X))$. So the sum of the non-trivial components of the reducible fibers has to be $r - 2$.
- If $\mathrm{MW}(\mathcal{E}) = 0$, then the discriminant of the trivial lattice coincides with the discriminant of $\mathrm{NS}(X)$, so it is $(\mathbb{Z}/2\mathbb{Z})^a$. In this case there is a unique section (the zero section), and it is a special curve. So the sum of the special curves contained in the reducible fibers has to be $k - 1$.
- If $\mathrm{MW}(\mathcal{E}) = \mathbb{Z}/2\mathbb{Z}$, then the trivial lattice is a sublattice with index 2 in $\mathrm{NS}(X)$, so its discriminant is $(\mathbb{Z}/2\mathbb{Z})^{a+2}$. In this case there are two sections, and they are both special curves. So the sum of the special curves contained in the reducible fibers has to be $k - 2$.

The list given in the proposition is the list of all the trivial lattices which satisfy these conditions. In order to conclude that every elliptic fibration in the list really occurs we explicitly construct all of them in the next sections. Alternatively, one can observe that the fibrations of type 1 have Mordell–Weil group with rank 0, so they exist if and only if they appear in the list given in [22]. The existence of the ones of type 2 follows by the existence of the associated rational elliptic surface as in Section 3. □

Remark 2.10. A different way to obtain Table 1 is to apply the so called Nishiyama method, [16]. In order to apply the method one has to consider a negative definite lattice T with the same discriminant group and form of the transcendental lattice of X, T_X, and such that $\mathrm{rank}(T) = \mathrm{rank}(T_X)+4$. For the surfaces considered in Proposition 2.9, if $\delta = 1$, $T \simeq E_7 \oplus A_1^{9-k}$; if $\delta = 0$ and $k = 8$, $T \simeq D_8$; if $\delta = 0$ and $k = 4$, $T \simeq D_4 \oplus D_4 \oplus D_4$.

3. Classification of the admissible rational elliptic surfaces

We study the rational elliptic surface R which appears when taking the quotient X/ι, giving a realization of the first line of the tables in Proposition 2.9.

Given (X, ι) as in Proposition 2.9 let $\mathcal{E}$ be the fibration of type 2 on X. We denote by R the surface obtained after blowing down (-1)-curves that are components of the fibers on the non-relatively minimal fibration induced on X/ι by $\mathcal{E}$.

Vice versa, given a rational elliptic surface $\mathcal{E}_R\colon R \to \mathbb{P}^1$, let $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a 2:1 map branched over 0 and ∞. We consider the associated base change. This produces another elliptic fibration $\mathcal{E}_X\colon X \to \mathbb{P}^1$, where X is the eventual desingularization of the fiber product $R \times_{\mathbb{P}^1} \mathbb{P}^1$. The surface X naturally comes with an involution ι, such that X/ι is birational to R. We now require the following:

a) X is a K3 surface;

b) ι fixes one curve of genus 1 and $1 \leq k \leq 9$ rational curves;

c) ι acts trivially on $\mathrm{NS}(X)$.

Condition a) implies that the fibers over 0 and ∞ of $\mathcal{E}_R$ are reduced, so they can be I_n, $n \geq 0$, or II, or III, or IV (see [23]).

Condition b) implies that exactly one among the fibers over 0 and ∞ of $\mathcal{E}_R$ is of type I_0 (see [8]). Just to fix the notation, we assume that the fiber over 0 is of type I_0.

Condition c) implies that $\mathcal{E}_R$ has no reducible fibers except possibly the fiber over 0 and the fiber over ∞ (see [8]). Moreover, condition c) implies that $\mathcal{E}_R$ has no fibers of type II and III over 0 and ∞. Indeed let us assume that the fiber over ∞ is a fiber of type II, i.e., a cuspidal curve. Then the fiber over ∞ of $\mathcal{E}_X$ is a fiber of type IV which has 3 components: one of them is the double cover of the strict transform of the fiber of type II and it is fixed by ι. The others are two curves switched by ι (which in fact are mapped to the same curve on the blow up of R in the singular point of the cuspidal fiber). But if ι switches the components of a reducible fiber, it can not be the identity on $\mathrm{NS}(X)$. Similarly, if the fiber of $\mathcal{E}_R$ over ∞ is of type III, the corresponding fiber over $\mathcal{E}_X$ is of type I_0^*. Two of the simple components of the fiber I_0^* are fixed by ι, the multiple component is preserved, but not fixed by ι, an so the other two simple components are switched by ι. Therefore we can exclude also this case.

The admissible rational elliptic surfaces are described in Table 2 where we give information both on R and on (X,ι), namely the reducible fibers of R; the Mordell–Weil group of $\mathcal{E}_R$; (r,a,δ), which determines $\mathrm{NS}(X)$; k, the number of rational curves fixed by ι.

TABLE 2.

Surface R			Surface X	
Branch fibers	Other singular fibers	MW	(r,a,δ)	k
$I_0 + I_n,\ 1 \leq n \leq 8$	irreducible	$\mathbb{Z}^{9-n}$	$(10+n, 10-n, 1)$	n
$I_0 + I_9$	irreducible	$\mathbb{Z}/3\mathbb{Z}$	$(19,1,1)$	9
$I_0 + I_8$	irreducible	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$(18,2,0)$	8
$I_0 + IV$	irreducible	$\mathbb{Z}^6$	$(14,6,0)$	4

Notice that, by the list presented in [19], the rational elliptic surfaces with a unique reducible fiber which is of type I_n have no torsion sections with only two exceptions, $n = 8, 9$. If $n = 8$ there exist two different families of rational elliptic surfaces, the Mordell–Weil group of one of them is torsion free while the other has a 2-torsion section. If $n = 9$, then there is a unique rational surface with a reducible fiber I_9, it is extremal and the Mordell–Weil group is necessarily $\mathbb{Z}/3\mathbb{Z}$.

3.1. Equations of the pencil of cubics

The rational elliptic surface associated to the involution which fixes 9 rational curves and one curve of genus 1 is the unique extremal rational surface with a fiber of type I_9 and three fibers of type I_1. Rational elliptic surfaces having exactly four semi-stable singular fibers, the latter configuration included, were discussed by Beauville in [3]. The paper [24] by Top and Yui also contains, in Section 2.3., this (and similar) examples, as well as some K3 surfaces that are double covers of them. The rational elliptic surfaces with configuration $(I_9, 3I_1)$ is associated to a known pencil of cubics which is generated, for example, by the reducible cubics xyz and by the smooth cubic $\mathcal{C}_9 := \{x^2y + y^2z + z^2x = 0\}$. In the following we will call l_1 the line $x = 0$, l_2 the line $y = 0$ and l_3 the line $z = 0$. Moreover we will denote by P_{ij} the point $l_i \cap l_j$. The cubic $\mathcal{C}_9$ passes through the points P_{ij} and in P_{12} it is tangent to l_1, in P_{13} it is tangent to l_3 and in P_{23} it is tangent to l_2. Hence in each point P_{ij} the cubic $\mathcal{C}_9$ intersects the cubic $l_1 \cup l_2 \cup l_3$ with multiplicity 3 and all the base points of the pencil $x^2y + y^2z + z^2x + \mu xyz$ are P_{ij} and points infinitely near to P_{ij}. After blowing up the base points one obtains a reducible fiber of type I_9 over $\mu = \infty$ and no other reducible fiber.

The rational elliptic surface with one fiber of type I_8 and no torsion is obtained by deforming the previous example. Indeed, we can obtain a fiber of type I_8 over $\mu = \infty$ if we separate one of the base points infinitely near to P_{ij}. This is equivalent to require that in one point P_{ij} the cubic $\mathcal{C}_9$ deforms to cubics that still pass through P_{ij}, but are not tangent to a line between l_i and l_j. We can assume that in $P_{13} = (0 : 1 : 0)$ the deformation of $\mathcal{C}_9$ is not tangent to l_3. This gives the pencil $\mathcal{C}_8 := \{x^2y + y^2z + z^2x + a_9xy^2 = 0\}$.

Proceeding by iterations of the above, we obtain that the pencil

$$\begin{aligned}\mathcal{P}_1 := \{&x^2y + y^2z + z^2x + a_9xy^2 + a_8x^2z + a_7yz^2 \\ &+ a_6z^3 + a_5y^3 + a_4x^3 + \mu(xyz + a_3z^3 + a_2y^3)\}.\end{aligned} \tag{3.1}$$

corresponds to a rational elliptic surface with one fiber of type I_1 in $\mu = \infty$ and no reducible fibers.

Proposition 3.1. *Let $\mathcal{P}_k$ be the pencil of cubics obtained by choosing $a_2 = \cdots = a_k = 0$ for $2 \leq k \leq 9$ in (3.1). For a generic choice of the a_i's, $\mathcal{P}_k$ corresponds to an elliptic fibration with a fiber of type I_k over $\mu = \infty$ and no other reducible fibers.*

For a generic choice of b, the pencil of cubics $x^2z + y^2z + y^2x + bxz^2 + \mu(xyz)$ corresponds to an elliptic fibration with a fiber of type I_8 over $\mu = \infty$, no other reducible fibers and a 2-torsion section.

For a generic choice of the c_i's, the pencil of cubics $z^3 + c_1xy^2 + c_2x^2z + c_3xyz + c_4y^2z + c_5xz^2 + (-1 - c_1 - c_2 - c_3 - c_4 - c_5)yz^2 + \mu xy(x - y)$ corresponds to an elliptic fibration with a fiber of type IV over $\mu = \infty$ and no other reducible fibers.

Proof. This is straightforward once one considers the base points of each pencil $\mathcal{P}_k$.

Alternatively, one can compute the Weierstrass equation of the elliptic fibration induced by each of the pencils in the statement and consider the discriminant. □

4. Geometric construction of type 1 elliptic fibrations

The aim of this section is to provide geometric realizations of the elliptic fibrations of type 1 on X which are listed in Proposition 2.9.

This is done by considering linear systems on the quotient surface X/ι, which is a rational surface, denoted from now on by $\widetilde{R}$.

Definition 4.1. (See Definition 3.3 in [8]) A generalized conic bundle on $\widetilde{R}$ is a *nef* class D in $\mathrm{NS}(\widetilde{R})$ such that i) $D\cdot(-K_{\widetilde{R}})=2$; ii) $D^2=0$.

We observe that $\widetilde{R}$ is a blow up of the rational elliptic surface R and the previous definition generalizes the standard definition of conic bundles on R. Note that $\tilde{R}$ is endowed with a non-relatively minimal elliptic fibration induced by the elliptic fibration on R. Since R is a rational elliptic surface it comes with a map $R\to\mathbb{P}^2$ given by the blow up of the base points of the pencil of cubics described in Proposition 3.1. The map $\widetilde{R}\to R$ contracts (-1)-curves contained in fibers. Hence we have a contraction map $\widetilde{R}\to\mathbb{P}^2$. Some of the generalized conic bundles remain base point free systems on R, and define standard conic bundles on R. All the generalized conic bundles are mapped by $\widetilde{R}\to\mathbb{P}^2$ to pencils of rational plane curves.

Proposition 4.2. (See Proposition 3.8 in [8]) *Let $\mathcal{B}$ be a generalized conic bundle over $\widetilde{R}$. Let C be a section of $\mathcal{B}$. The pencil $\mathcal{B}$ induces a genus 1 fibration $\mathcal{E}_{\mathcal{B}}$ on the* K3 *surface X which is the double cover of $\widetilde{R}$. The pull back of the curve C is a section of the fibration $\mathcal{E}_{\mathcal{B}}$ if and only if C is a branch curve of the double cover $X\to\widetilde{R}$. Moreover, all the elliptic fibrations on X of type 1 are of this form.*

In the following subsections, for each value of k, we describe the pencil of cubics used to construct the rational surface R and then we summarize in tables the relations between the generalized conic bundles on R and the elliptic fibrations on X. More precisely, by Proposition 2.6, each elliptic fibration on X listed in Table 1 with only one exception for each k is induced by a generalized conic bundle on $\widetilde{R}$. In the following tables, we associate to each of these fibrations a generalized conic bundle inducing the elliptic fibration. The elliptic fibration is described in the first and in the last column: the first column shows the number of the fibration as given in the table of Proposition 2.9, whereas, in the last column, the reducible fibers of the fibration and its Mordell–Weil group are described. In the other columns, we provide a generalized conic bundle associated to each elliptic fibration. The description of the generalized conic bundle consists in giving the degree of the planes curves and the list of the base points, since a generalized conic bundle is a pencil of rational plane curves with some base points. We distinguish between conic bundles (cb) and generalized conic bundles which are not conic bundles (Gcb).

In order to associate to each generalized conic bundle an elliptic fibration, it suffices to find the reducible fibers of the generalized conic bundle and to apply Theorem 5.3 in [8] which allows one to find the singular fibers of the elliptic fibration associated to a conic bundle.

If some of the base points, say p_1 and p_1', of the pencil of cubic curves that induces the rational elliptic surface are infinitely near, we say that a curve passes through p_1 and p_1' to express that it shares the same tangent direction as the cubics in the rational elliptic pencil. This will be used in what follows.

From the following tables one obtains that it suffices to consider generalized conic bundles which map to pencils of plane curves of degree at most 3 in order to recover all the elliptic fibrations listed in Proposition 2.9.

4.1. The case $k = 9$

The rational elliptic surface $\mathcal{E}$ is isomorphic to the blow-up of $\mathbb{P}^2$ in 9 points $p_1, p_1', p_1'', p_2, p_2', p_2''', p_3, p_3', p_3'''$, where p_i, p_i', p_i'' are infinitely near points. Moreover the points p_1, p_2, p_3 are not collinear. We call l_1 the line through p_1 and p_2, the line joining p_1 and p_3 will be called l_3, while l_2 will be the line connecting p_2 and p_3. We assume that l_i is tangent at p_i to the cubics of the pencil that induces the elliptic fibration in $\mathcal{E}$.

n^o	deg	base points	type	elliptic fibrations
9.3	1	p_1	cb	(III^*, I_6^*), MW $= \mathbb{Z}/2\mathbb{Z}$
9.4	2	p_1, p_1', p_1'', p_3	cb	(I_{12}^*, I_2), MW $= \mathbb{Z}/2\mathbb{Z}$
9.2	2	p_2, tg to l_1, p_3 tg to l_3	Gcb	$(2II^*, I_2)$, MW $= \{1\}$

4.2. The case $k = 8$, $\delta = 1$

The rational elliptic surface $\mathcal{E}$ is isomorphic to the blow up of $\mathbb{P}^2$ in 9 points $p_1, p_1', p_1'', p_2, p_2', p_2'', p_3, p_3', p_4$, such that p_i and p_i' and p_i'', for $i = 1, 2, 3$, are infinitely near points. These points are such that there is one triple of collinear points $l_3 := \{p_2, p_3, p_4\}$. The line l_2 through p_1, p_3 is tangent at the point p_1, and the line l_1 through p_1 and p_2 is tangent at p_2 to all cubics of the pencil that gives the elliptic fibration on $\mathcal{E}$, and there is no other collinearity relation.

n^o	deg	base points	type	elliptic fibrations
8.3	1	p_1	cb	(III^*, I_4^*, I_2), MW $= \mathbb{Z}/2\mathbb{Z}$
8.5	1	p_2	cb	(I_6^*, I_2^*), MW $= \mathbb{Z}/2\mathbb{Z}$
8.4	2	$p_1,\ p_3,\ p_3',\ p_4$	cb	$(I_{10}^*, 2I_2)$, MW $= \mathbb{Z}/2\mathbb{Z}$
8.2	2	p_1, tg to $l_1,\ p_3,\ p_3'$	Gcb	(II^*, III^*, I_2), MW $= \{1\}$

4.3. The case $k = 8$, $\delta = 0$

The rational elliptic surface $\mathcal{E}$ is isomorphic to the blow up of $\mathbb{P}^2$ in 9 points $p_1, p_1', p_1'', p_2, p_2', p_2'', p_3, p_3', p_4$, such that p_i and p_i' and p_i'', for $i = 1, 2$ are infinitely near points, and so are p_3 and p_3'. These points are such that there is one triple of collinear points $l_1 := \{p_1, p_2, p_4\}$. The line l_2 through p_1, p_3 is tangent at the point p_1, and the line l_3 through p_2 and p_3 is tangent at p_2 to all cubics of the pencil that gives the elliptic fibration on $\mathcal{E}$, and there is no other collinearity relation.

n^o	deg	base points	type	elliptic fibrations
8.6	1	p_1	cb	$(2I_4^*)$, MW $= \mathbb{Z}/2\mathbb{Z}$
8.3	1	p_3	cb	$(2III^*, 2I_2)$, MW $= \mathbb{Z}/2\mathbb{Z}$
8.5	1	p_4	cb	(I_8^*, I_0^*), MW $= \mathbb{Z}/2\mathbb{Z}$
8.2	2	p_3, tg to l_3, p_1, p_4	Gcb	(II^*, I_4^*), MW $= \{1\}$
8.4	3	p_1, p_1', p_1'', p_3 tg to l_3, with a node in p_4	Gcb	(I_{12}^*), MW $= \{1\}$

4.4. The case $k = 7$

The rational elliptic surface $\mathcal{E}$ is isomorphic to the blow-up of $\mathbb{P}^2$ in 9 points $p_1, p_1', p_1'', p_2, p_2', p_3, p_3', p_4$ and p_5, where p_1, p_1', p_1'' are infinitely near points, and the same holds for the pairs p_2, p_2' and p_3, p_3'. These points are such that there are two triples of collinear points $l_2 := \{p_1, p_4, p_3\}$ and $l_3 := \{p_2, p_3, p_5\}$ and no other collinearity relation. We call l_1 the line through p_1 and p_2, and we assume that it is tangent to the cubics at p_1.

n^o	deg	base points	type	elliptic fibrations
7.9	1	p_1	cb	(I_4^*, I_2^*, I_2), MW $= \mathbb{Z}/2\mathbb{Z}$
7.3	1	p_2	cb	$(III^*, I_2^*, 2I_2)$, MW $= \mathbb{Z}/2\mathbb{Z}$
7.8	1	p_4	cb	(I_6^*, I_0^*, I_2), MW $= \mathbb{Z}/2\mathbb{Z}$
7.7	2	p_2, p_2', p_1, p_5	cb	$(I_8^*, 3I_2)$, MW $= \mathbb{Z}/2\mathbb{Z}$
7.5	2	p_1, p_1', p_3 tg to l_3	Gcb	$(2III^*, I_2)$, MW $= \{1\}$
7.2	2	p_2 tg to l_1, p_3, p_4	Gcb	(II^*, I_2^*, I_2), MW $= \{1\}$
7.4	2	p_2 tg to l_1, p_4, p_5	Gcb	(III^*, I_4^*), MW $= \{1\}$
7.6	3	p_2 tg to l_1, p_3, p_3', p_5, with a node in p_4	Gcb	(I_{10}^*, I_2), MW $= \{1\}$

4.5. The case $k = 6$

The rational elliptic surface $\mathcal{E}$ is isomorphic to the blow up of $\mathbb{P}^2$ in 9 points $p_1, p_1', p_2, p_2', p_3, p_3', p_4, p_5, p_6$, such that p_i and p_i', for $i = 1, 2, 3$, are infinitely near points. These points are such that there are three triples of collinear points $l_1 := \{p_1, p_2, p_4\}, l_2 := \{p_1, p_5, p_3\}, l_3 := \{p_2, p_3, p_6\}$, and no other collinearity relation (see [18]).

n^o	deg	base points	type	elliptic fibrations
6.10	1	p_1	cb	$(2I_2^*, 2I_2)$, MW $= \mathbb{Z}/2\mathbb{Z}$
6.9	1	p_4	cb	$(I_4^*, I_0^*, 2I_2)$, MW $= \mathbb{Z}/2\mathbb{Z}$
6.4	2	$p_1,\ p_2,\ p_5,\ p_6$	cb	$(III^*, I_0^*, 3I_2)$, MW $= \mathbb{Z}/2\mathbb{Z}$
6.7	2	$p_1,\ p_1',\ p_2,\ p_5$	cb	$(I_6^*, 4I_2)$, MW $= \mathbb{Z}/2\mathbb{Z}$
6.8	2	$p_1,\ p_1',\ p_5,\ p_6$	cb	(I_4^*, I_2^*), MW $= \{1\}$
6.2	3	$p_1,\ p_1',\ p_3$ tg to $l_3,\ p_6,$ with a node in p_4	Gcb	$(II^*, I_0^*, 2I_2)$, MW $= \{1\}$
6.6	3	$p_1,\ p_1',\ p_3$ tg to $l_2,\ p_6,$ with a node in p_4	Gcb	(I_6^*, I_0^*), MW $= \{1\}$
6.5	3	$p_1,\ p_1',\ p_3$ tg to $l_3,\ p_5,$ with a node in p_4	Gcb	$(I_8^*, 2I_2)$, MW $= \{1\}$
6.3	3	$p_1,\ p_3$ tg to $l_3,\ p_5,\ p_6$ with a node in p_4	Gcb	(III^*, I_2^*, I_2), MW $= \{1\}$

4.6. The case $k = 5$

The rational elliptic surface $\mathcal{E}$ is isomorphic to the blow up of $\mathbb{P}^2$ in 9 points $p_1, p_1', p_2, p_2', p_3, p_4, p_5, p_6, p_7$, such that p_1 and p_1' are infinitely near points, as are p_2 and p_2'. These points are such that there are three triples of collinear points $l_1 := \{p_1, p_2, p_3\}, l_2 := \{p_1, p_4, p_5\}, l_3 := \{p_2, p_6, p_7\}$, and no other collinearity relation (see [21]). We call q the intersection $l_2 \cap l_3$.

n^o	deg	base points	type	elliptic fibrations
5.9	1	p_1	cb	$(I_2^*, I_0^*, 3I_2)$, MW $= \mathbb{Z}/2\mathbb{Z}$
5.7	1	p_3	cb	$(I_4^*, 5I_2)$, MW $= \mathbb{Z}/2\mathbb{Z}$
5.8	1	q	Gcb	$(2I_2^*, I_2)$, MW $= \{1\}$
5.4	2	$p_2,\ p_3,\ p_4,\ p_5$	cb	$(III^*, 6I_2)$, MW $= \mathbb{Z}/2\mathbb{Z}$
5.3	2	$p_1,\ p_3,\ p_6,\ q$	Gcb	$(III^*, I_0^*, 2I_2)$, MW $= \{1\}$
5.6	2	$p_1,\ p_1',\ p_2,\ q$	Gcb	(I_4^*, I_0^*, I_2), MW $= \{1\}$
5.5	2	$p_1,\ p_1',\ p_3,\ q$	Gcb	$(I_6^*, 3I_2)$, MW $= \{1\}$
5.2	3	$p_1,\ p_1',\ p_6,\ p_7,\ q$ with a node in p_3	Gcb	$(II^*, 5I_2)$, MW $= \{1\}$

4.7. The case $k = 4$, $\delta = 0$

The rational elliptic surface $\mathcal{E}$ is isomorphic to the blow up of $\mathbb{P}^2$ in 9 points $p_1, \ldots, p_9$. These points are such that there are three triples of collinear points $l_1 := \{p_1, p_2, p_3\}$, $l_2 := \{p_4, p_5, p_6\}$, $l_3 := \{p_7, p_8, p_9\}$, and no other collinearity relation (see [7]). The lines l_1, l_2 and l_3 meet in a unique point q.

n^o	deg	base points	type	elliptic fibrations
4.2	1	p_1	cb	$(I_2^*, 6I_2)$, MW $= \mathbb{Z}/2\mathbb{Z}$
4.3	1	q	Gcb	$(3I_0^*)$, MW $= \{1\}$

4.8. The case $k = 4$, $\delta = 1$

The rational elliptic surface $\mathcal{E}$ is isomorphic to the blow up of $\mathbb{P}^2$ in 9 points $p_1, \ldots, p_8, p_8'$, such that p_8 and p_8' are infinitely near points. These points are such that there are three triples of collinear points $l_1 := \{p_1, p_2, p_3\}$, $l_2 := \{p_4, p_5, p_8\}$, $l_3 := \{p_6, p_7, p_8\}$, and no other collinearity relation (see [7]). We call q_1, resp. q_2, the intersection point of the first line l_1 with the line l_2, resp. l_3.

n^o	deg	base points	type	elliptic fibrations
4.6	1	p_1	cb	$(2I_0^*, 4I_2)$, MW $= \mathbb{Z}/2\mathbb{Z}$
4.5	1	p_4	cb	$(I_2^*, 6I_2)$, MW $= \mathbb{Z}/2\mathbb{Z}$
4.4	1	q_1	Gcb	$(I_2^*, I_0^*, 2I_2)$, MW $= \{1\}$
4.3	2	q_1, p_1, p_4, p_5	Gcb	$(I_4^*, 4I_2)$, MW $= \{1\}$
4.2	2	q_1, p_4, p_6, p_7	Gcb	$(III^*, 5I_2)$, MW $= \{1\}$

4.9. The case $k = 3$

The rational elliptic surface $\mathcal{E}$ is isomorphic to the blow up of $\mathbb{P}^2$ in 9 distinct points $p_1, \ldots, p_9$, such that there are three triples of collinear points $l_1 := \{p_1, p_2, p_3\}$, $l_2 := \{p_4, p_5, p_6\}$, $l_3 := \{p_7, p_8, p_9\}$, and no other collinearity relation (see [7]). We call q_1, q_2 and q_3 the three intersection points of the three pairs of these lines, with the assumption that $q_3 = l_2 \cap l_3$.

n^o	deg	base points	type	elliptic fibrations
3.4	1	p_1	cb	$(I_0^*, 7I_2)$, MW $= \mathbb{Z}/2\mathbb{Z}$
3.3	1	q_1	Gcb	$(2I_0^*, 3I_2)$, MW $= \{1\}$
3.2	2	p_1, p_2, p_4, q_3	Gcb	$(I_2^*, 5I_2)$, MW $= \{1\}$

4.10. The case $k = 2$

The rational elliptic surface $\mathcal{E}$ is isomorphic to the blow up of $\mathbb{P}^2$ in 9 points, such that 3 of them lie on a line l, say p_1, p_2, p_3, and the remaining 6, namely, $p_4, \ldots, p_9$, lie on a conic Q (see [7]). We denote by q_i, for $i = 1, 2$, the two intersection points $Q \cap l$.

n^o	deg	base points	type	elliptic fibrations
2.3	1	p_4	cb	$(10I_2)$, MW $= \mathbb{Z}/2\mathbb{Z}$
2.2	1	q_1	Gcb	$(I_0^*, 6I_2)$, MW $= \{1\}$

4.11. The case $k = 1$

The rational elliptic surface $\mathcal{R}$ is isomorphic to the blow up of $\mathbb{P}^2$ in 9 points $p_1, \ldots, p_9$ in general position. There are twelve nodal cubics in the pencil of cubics through $p_1, \ldots, p_9$. We choose one of them as branching fiber of the double cover and we denote by q its singular point.

n^o	deg	base points	type	elliptic fibrations
1.2	1	q	Gcb	$(9I_2)$, MW $= \{1\}$

5. Equations

In the paper [2], the authors give an algorithm to compute the Weierstrass equations of certain elliptic fibrations on K3 surfaces which are double covers of rational elliptic surfaces. The aim of this section is to recall such an algorithm, and to compute explicitly some of these equations. We observe that for each elliptic fibration described in the previous section one can apply the algorithm and therefore find explicitly the Weierstrass equations.

5.1. An example

Let us consider the rational elliptic surface R_k, for $k = 1, \dots, 9$, associated to the pencil of cubics $\mathcal{P}_k$ as in Proposition 3.1. Let X be the K3 surface branched on two fibers of the rational elliptic surface R_k. We chose as branch fibers the one over ∞ and another one, say over $\mu = \mu_1$. So the K3 surface is obtained as double cover of $\mathbb{P}^2$ branched on the sextic $\mathcal{S}_k$

$$(xyz + a_3z^3 + a_2y^3)(x^2y + y^2z + z^2x + a_9xy^2 + a_8x^2z + a_7yz^2 + a_6z^3 + a_5y^3 \\ + a_4x^3 + \mu_1(xyz + a_3z^3 + a_2y^3)),$$

with $a_2 = \cdots = a_k = 0$.

If $k \leq 3$, the point $(1:0:0)$ is the singular point of the fiber over ∞ and it is not a base point of the pencil $\mathcal{P}_k$. If $k > 3$, then $(1:0:0)$ is a base point of the pencil $\mathcal{P}_k$. For every k, the pencil of lines through $(1:0:0)$ induces a generalized conic bundle on $\widetilde{R}$, which is also a conic bundle on R if $k > 3$.

In order to find the Weierstrass equation of the elliptic fibration induced by this generalized conic bundle, one has to consider the pencil of lines through $(1:0:0)$, $y = mz$. Then one intersects it with the branch sextic $\mathcal{S}_k$ obtaining the following equation for the elliptic fibration on the K3 surface X:

$$Y^2 = z^2\left(m + z(a_3 + a_2m^3)\right)\left(a_4 + z(m + a_8) \right. \\ \left. + z^2(1 + a_9m^2 + \mu_1 m) + z^3\left(m^3(a_5 + \mu_1a_2) + m^2 + a_7m + a_6 + \mu_1a_3\right)\right).$$

By the change of coordinates $Y \mapsto Yz$, one obtains Y^2 equals to a polynomial of degree 4 in z with a section: if $a_2 = a_3 = 0$ the section is at infinity otherwise it is $(z(m), Y(m)) = (-m/(a_3 + a_2m^3), 0)$. So we obtain the equation of an elliptic fibration. If $a_2 = a_3 = 0$, after the change of coordinates

$$\begin{array}{lcl} Y & \mapsto & Y/\left(m(m^3(a_5 + \mu_1a_2) + m^2 + a_7m + a_6 + \mu_1a_3)^2\right), \\ z & \mapsto & z/\left(m\left(m^3(a_5 + \mu_1a_2) + m^2 + a_7m + a_6 + \mu_1a_3\right)\right), \end{array}$$

one obtains the Weierstrass form. If $k > 3$, putting $a_2 = \cdots = a_k = 0$ one obtains a Weierstrass equation for the fibrations n^o $(k.h)$ in Proposition 2.9, for the following pairs of values $(k, h) \in \{(4,6), (5,9), (6,10), (7,9), (8,3), (9,3)\}$. For $k = 1$ (resp. $k = 2$, $k = 3$) this is an equation for the fibration 1.2 (resp. 2.2, 3.3) in Proposition 2.9.

5.2. The algorithm

The aim of this algorithm is to generalize the previous computation in order to be able to obtain Weierstrass equations for all the elliptic fibrations classified in Proposition 2.9.

Setup. Let V be a K3 surface obtained by a base change of order 2 from a rational elliptic surface R. Therefore, V can be described as double cover of $\mathbb{P}^2$ branched on the union of two (possibly reducible) plane cubics from the pencil determining R. It has an equation of the form

$$w^2 = f_3(x_0 : x_1 : x_2)g_3(x_0 : x_1 : x_2).$$

Let $\mathcal{B}$ be a (generalized) conic bundle on $\widetilde{R}$ whose curves are parametrized by τ. Pushing forward to $\mathbb{P}^2$, $\mathcal{B}$ is given by a pencil of plane rational curves with equation $h(x_0 : x_1 : x_2, \tau)$. The polynomial $h(x_0 : x_1 : x_2, \tau)$ is homogeneous in x_0, x_1, x_2, say of degree $e \geq 1$ and linear in τ.

The adjunction formula implies that every curve with equation $h(x_0 : x_1 : x_2, \tau)$ meets both of the branch curves (the proper transforms on $\widetilde{R}$ of) $f_3 = 0$ and $g_3 = 0$ in two additional points not blown up by $\widetilde{R} \to \mathbb{P}^2$. It therefore meets (the proper transform of) their union $f_3 g_3 = 0$ in four points not blown up by $\widetilde{R} \to \mathbb{P}^2$. So the preimage in V is the double cover of a rational curve branched over 4 points, i.e., the standard presentation of an elliptic curve. For general τ, we must find an isomorphism of the curve $h(x_0 : x_1 : x_2, \tau) = 0$ with $\mathbb{P}^1$, and extract the images of the four intersection points with $f_3 g_3 = 0$.

When all curves in the conic bundle have a basepoint of degree $e - 1$, the projection away from this point provides the required isomorphism of the curve $h(x_0 : x_1 : x_2, \tau) = 0$ with $\mathbb{P}^1$. Up to acting by $\mathrm{PGL}_3(\mathbb{C})$, we may assume that this point is $(0 : 1 : 0)$.

Algorithm.

(1) Compute the resultant of the polynomials $f_3(x_0 : x_1 : x_2)g_3(x_0 : x_1 : x_2)$ and $h(x_0 : x_1 : x_2, \tau)$ with respect to the variable x_1. The result is a polynomial $r(x_0 : x_2, \tau)$ which is homogeneous in x_0 and x_2, corresponding to the images of *all* of the intersection points $\{f_3 g_3 = 0\} \cap \{h_\tau = 0\}$ after projection from $(0 : 1 : 0)$.

(2) Since $\mathcal{B}$ is a conic bundle, $r(x_0 : x_2, \tau)$ will be of the form $a(x_0 : x_2, \tau)^2 b(x_0 : x_2, \tau)c(\tau)$, where a and b are homogeneous in x_0 and x_2, the degree of a depends upon e and the degree of b in x_0 and x_2 is 4.

(3) The equation of V is now given by $w^2 = r(x_0 : x_2, \tau)$, which is birationally equivalent to

$$w^2 = c(\tau)\, b(x_0 : x_2, \tau), \tag{5.1}$$

by the change of coordinates $w \mapsto wa(x_0 : x_2, \tau)$. Since for almost every τ, (5.1) is the equation of a $2:1$ cover of $\mathbb{P}^1_{(x_0:x_2)}$ branched in 4 points, (5.1) is the equation of the genus 1 fibration on the K3 surface V induced by the conic bundle $\mathcal{B}$.

(4) If there is a section of fibration (5.1), then it is possible to obtain the Weierstrass form by standard transformations.

Remark 5.1. There are several conic bundles whose general member cannot be parametrized by lines. An algorithm for some of them is described in [2], but here we do not need it, since all the conic bundles listed in Section 4 are of degree at most 3.

6. Specializations

6.1. Specialization of a 1-dimensional family of K3 surfaces

Among the rational elliptic surfaces listed in Section 3 the one with the smallest Mordell–Weil group is the extremal rational elliptic surface with a fiber of type I_9, three fibers of type I_1 and $\mathrm{MW} = \mathbb{Z}/3\mathbb{Z}$.

As already noticed, it is induced by a pencil of cubics $\mathcal{P}_9$ on $\mathbb{P}^2$. By standard transformations one obtains the Weierstrass equation of the rational elliptic surface:

$$Y^2 = X^3 + X\left(-\frac{\mu^4}{2^4\,3} - \frac{\mu}{2}\right) + \frac{\mu^6}{2^5\,3^3} + \frac{\mu^3}{2^3\,3} + \frac{1}{2^2}. \tag{6.1}$$

The discriminant is $\frac{1}{16}\left(3+\mu\right)\left(\mu^2-3\mu+9\right)$.

Now let us consider the K3 surface X_{μ_1} obtained by a base change of order 2 of this rational elliptic surface branched over the fiber of type I_9 (corresponding to $\mu = \infty$) and a generic smooth fiber, say the fiber corresponding to μ_1.

Its Weierstrass equation is obtained by (6.1) by substituting μ with $\tau^2 + \mu_1$. For a generic choice of μ_1 this corresponds to an elliptic fibration with $I_{18} + 6I_1$ as reducible fibers and Mordell Weil group given by $\mathbb{Z}/3\mathbb{Z}$. The transcendental lattice of K3 surfaces in this family is $U \oplus \langle 2\rangle$, indeed they are the K3 surfaces with one involution acting trivially on the Néron–Severi group and fixing 9 rational curves and 1 curve of genus 1.

These K3 surfaces specialize to several K3 surfaces whose transcendental lattice has rank 2 (i.e., whose Picard number is 20). For example, since $\langle 2d\rangle$ is primitively embedded in U, the K3 surfaces whose transcendental lattice is $\langle 2d\rangle \oplus \langle 2\rangle$, $d > 0$, are special members of the same family.

Let us consider one of these specializations in detail. Consider the plane conic $\mathcal{C} := \{xz + y^2\}$. It intersects the cubic of the pencil corresponding to $\overline{\mu}$ in $(0 : 0 : 1)$ with multiplicity 3, in $(1 : 0 : 0)$ with multiplicity 1 and in $(-\overline{\mu} : \pm\sqrt{\overline{\mu}} : 1)$ with multiplicity 1. This conic is a bisection of the rational elliptic fibration. Indeed, it intersects the generic cubic of the pencil in exactly 2 points which are not base points. If $\overline{\mu} = 0$, then the points $(0 : 0 : 1)$ and $(-\overline{\mu} : \pm\sqrt{\overline{\mu}} : 1)$ collapse to the same point, thus the conic $\mathcal{C}$ intersects the cubic of the pencil corresponding to $\mu = 0$ in $(0 : 0 : 1)$ with multiplicity 5 and $(1 : 0 : 0)$ with multiplicity 1.

If now one considers the K3 surfaces X_{μ_1} obtained by the base change branched on $\mu = \infty$ and μ_1, generically the bisection of the rational elliptic surface induced by $\mathcal{C}$ induces a bisection of the elliptic fibration on the K3 surface. This does not happen if μ_1 is 0. Indeed, in this case, the bisection of the rational surface splits in the double cover so that it induces, on the elliptic fibration on X_0, two distinct sections. This is due to the fact that X_0 is the double cover of $\mathbb{P}^2$ branched on the sextic $xyz(x^2y + y^2z + z^2x)$ which intersects the conic $\mathcal{C}$ in $(0 : 0 : 1)$ with multiplicity 8 and $(0 : 0 : 1)$ with multiplicity 4, in particular always with an even multiplicity, so it splits in the double cover.

From the above discussion, we have that if the base change from the rational surface to the K3 surface is branched on $\mu = 0$ and $\mu = \infty$, then the Picard number of the K3 surface X_0 is not 19, but 20. So all the fibrations on the K3 surface X_0 specialize. This suggests a method to determine some specializations: we have many elliptic fibrations on the same K3 surface. By requiring that the discriminant of one of these fibrations has zeros with higher multiplicity, one obtains values of μ_1 for which the K3 surface specializes to one for which the rank of the Néron–Severi group is larger than the one of a general member of the given family of K3 surfaces. One is now able to find different specializations considering different elliptic fibrations. In the following we apply this idea to different families of K3 surfaces.

6.1.1. The elliptic fibrations on X_{μ_1} and their specializations. For a generic choice of μ_1, there are four types of elliptic fibrations on X_{μ_1}: one comes from the rational elliptic surface, the other from generalized conics bundles on $\widetilde{R}$.

(1) The elliptic fibration $\mathcal{E}_1$ coming from the rational elliptic surface has $I_{18} + 6I_1$ as singular fibers and $\mathrm{MW} = \mathbb{Z}/3\mathbb{Z}$. Its equation is

$$\begin{aligned} Y^2 = X^3 &+ \Big(-\frac{1}{48}(\tau^2+\mu_1)(\tau^6+3\tau^4\mu_1+3\tau^2\mu_1^2+\mu_1^3+24)\Big)X \\ &+ \frac{1}{864}(\tau^2+\mu_1)^6 + \frac{1}{24}(\tau^2+\mu_1)^3 + \frac{1}{4}, \end{aligned}$$

and the discriminant is

$$\Delta_1 := \frac{1}{16}\left(\tau^2+\mu_1+3\right)\left(\mu_1^2+2\tau^2\mu_1-3\mu_1+\tau^4-3\tau^2+9\right);$$

(2) The elliptic fibration $\mathcal{E}_2$ is associated to the conic bundle $y = mx$ (lines through $(0 : 0 : 1)$), its singular fibres are $III^* + I_6^* + 3I_1$, $\mathrm{MW} = \mathbb{Z}/2\mathbb{Z}$ and its equation is

$$\begin{aligned} Y^2 &= X^3 - \frac{1}{3}m^3(m^3+2m^2\mu_1+\mu_1^2m-3)X \\ &\quad - \frac{1}{27}m^5(m+\mu_1)(2m^3+4m^2\mu_1+2\mu_1^2m-9) \\ \Delta_2 &:= -m^9(-4+m^3+2m^2\mu_1+\mu_1^2m). \end{aligned}$$

(3) The elliptic fibration $\mathcal{E}_3$ is associated to the conic bundle $y^2 + bxy + xz$, its singular fibres are $I_{12}^* + I_2 + 4I_1$, $\mathrm{MW} = \mathbb{Z}/2\mathbb{Z}$ and its equation is

$$Y^2 = X^3 + A_3X + B_3,$$

with

$$A_3 := -(1/3)b^6 + (4/3)\,b^3 + (2/3)\,b^5\,\mu_1 - 1/3 - (4/3)\,\mu_1\,b^2 - (1/3)\,\mu_1^2\,b^4,$$
$$B_3 := (1/27)(-\mu_1\,b^2 - 2 + b^3)\,(2b^6 - 4b^5\mu_1 + 2\mu_1^2\,b^4 - 8b^3 + 8\mu_1\,b^2 - 1),$$

and

$$\Delta_3 := -b^2(-\mu_1 b^2 - 4 + b^3)(-\mu_1 + b).$$

(4) The elliptic fibration $\mathcal{E}_4$ is associated to the generalized conic bundle $ax^2 + yz$, its singular fibres are $2II^* + I_2 + 2I_1$, MW $= \{1\}$ and its equation is

$$Y^2 = X^3 - \frac{1}{3}\mu_1^2 a^4 X + \frac{a^5}{27}(27a^2 - 54a - 2\mu_1^3 a + 27),$$
$$\Delta_4 := a^{10}(-1 + a)^2\,(27a^2 - 4\mu_1^3 a - 54a + 27).$$

A very natural specialization for elliptic fibrations is obtained by requiring that certain singular fibers collapse to a unique one.

By considering the discriminant Δ_1 of $\mathcal{E}_1$ one obtains that possible specializations, under which the Picard number jumps to 20 are obtained by requiring

$$\mu_1 \in \left\{ -3, \frac{3 - 3\sqrt{3}i}{2}, \frac{3 + 3\sqrt{3}i}{2} \right\}$$

(in this case the second branch fiber is a fiber of type I_1 on the rational elliptic surface and thus gives a fiber of type I_2 on the K3 surface).

By considering the discriminant Δ_2 of $\mathcal{E}_2$ one obtains that the Picard numbers jump for same values of μ_1.

By considering the discriminants Δ_3 and Δ_4 of $\mathcal{E}_3$ and $\mathcal{E}_4$, respectively, one obtains that the Picard numbers jump for $\mu_1 = 0$ and the same values found for $\mathcal{E}_1$.

6.1.2. Considering explicitly the specializations. If

$$\mu_1 \in \left\{ -3, \frac{3 - 3\sqrt{3}i}{2}, \frac{3 + 3\sqrt{3}i}{2} \right\},$$

one obtains that $\mathcal{E}_1$ has a new reducible fiber of type I_2 (obtained by gluing together two fibers of type I_1). Thus the trivial lattice of the specialized elliptic fibration is $U \oplus A_{17} \oplus A_1$. The 3-torsion section generating the Mordell–Weil group does not change, and there cannot be other torsion sections (by [22] the torsion part of the Mordell–Weil group of an elliptic fibration with trivial lattice $U \oplus A_{17} \oplus A_1$ is either $\{1\}$ or $\mathbb{Z}/3\mathbb{Z}$). Moreover, by the Shioda–Tate formula, there is no section of infinite order, since the rank of the trivial lattice is the maximum admitted. Hence we have a set of generators for the Néron–Severi of the specialized surface, and thus we can also compute the transcendental lattice: it is $\langle 2\rangle \oplus \langle 2\rangle$.

Let us consider the specialization of the elliptic fibrations $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$ if $\mu_1 \in \{-3, (3-3\sqrt{3}i)/2, (3+3\sqrt{3}i)/2\}$. In all the cases an extra reducible fiber appears.

Indeed, the singular fibers of the fibration $\mathcal{E}_2$ become $III^* + I_6^* + I_2 + I_1$, the ones of $\mathcal{E}_3$ become $I_{12}^* + 2I_2 + 2I_1$ and $\mathcal{E}_4$ become $2II^* + 2I_2$.

The unique other interesting value of μ_1 found before is $\mu_1 = 0$. Let us denote by X_0 the K3 surface obtained for $\mu_1 = 0$.

The elliptic fibration $\mathcal{E}_4$ has a new reducible fiber if $\mu_1 = 0$. The reducible fibers are $2II^* + IV$ and thus the trivial lattice is $U \oplus E_8 \oplus E_8 \oplus A_2$. The Mordell–Weil is trivial and so $\mathrm{NS}(X_0) \simeq U \oplus E_8 \oplus E_8 \oplus A_2$ and $T_{X_0} \simeq A_2$. If $\mu_1 = 0$, the equation of $\mathcal{E}_4$ is

$$Y^2 = X^3 + \frac{a^5}{27}(27a^2 - 54a + 27),$$

which is clearly an isotrivial fibration whose generic fiber is isomorphic to the elliptic curve with complex multiplication of order 3. Let us denote by α the order 3 automorphism induced on X_0 by the complex multiplication on the fibers; it acts trivially on the basis of the fibration and with order 3 on each fiber.

We observe that if $\mu_1 = 0$, the elliptic fibration $\mathcal{E}_2$ has the equation

$$Y^2 = X^3 - \frac{1}{3}m^3(m^3 - 3)X - \frac{1}{27}m^6(2m^3 - 9),$$

which admits the new automorphism of order 3, $(x, y, m) \mapsto (x, y, \zeta_3 m)$. It coincides with α. The singular fibers of this fibration are unchanged, but now one can view this fibration as $3:1$ cover of the rational elliptic fibration whose equation is

$$Y^2 = X^3 - \frac{1}{3}n(n-3)X - \frac{1}{27}n^2(2n - 9)$$

and whose singular fibers are $I_2^* + III + I_1$ and $\mathrm{MW} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$. The branch fiber of the triple cover are I_2^* and III. The fibration $\mathcal{E}_2$ over X_0, admits an extra section of infinite order, induced by the one of the rational elliptic surface .

The action of α on the fibration $\mathcal{E}_3$ is similar to the previous one, α acts on the basis of the fibration. But, moreover, one observes that the discriminant Δ_3 changes if $\mu_1 = 0$, and indeed it is $-b^3(-4 + b^3)$. So the trivial lattice of $\mathcal{E}_3$ specialized to X_0 is $U \oplus D_{16} \oplus A_2$. The Mordell–Weil group remains unchanged.

The elliptic fibration $\mathcal{E}_1$ witnesses no changes in its singular fibers. It admits the following Weierstrass equation:

$$Y^2 = X^3 - \frac{1}{48}(u^2(u^6 + 24))X + \frac{1}{864}u^{12} + \frac{1}{24}u^6 + \frac{1}{4}\cdot$$

The elliptic fibration $\mathcal{E}_1$ acquires a new non-torsion section in X_0. It is induced by a bisection of the rational elliptic surface (6.1), which splits in the double cover. To find explicitly this new section, we observe that in the plane $\mathbb{P}^2_{(x:y:z)}$ which contains the pencil $\mathcal{P}_9$, the bisection of the rational elliptic fibration (6.1) which splits in the double cover corresponds to the conic $\mathcal{C} := \{xz + y^2\}$; indeed we already observed that this is conic tangent to the branch fibers if $\mu_1 = 0$. This gives the new non-torsion section of X_0, which is

$$(X(u), Y(u)) = \Big(\frac{1}{4}u^6 + \frac{1}{4}u^3 + \frac{1}{4}, \frac{1}{2}(u+1)(u^2 - u + 1)\Big).$$

The fibration $\mathcal{E}_1$ has also the three 3-torsion sections inherited from the rational elliptic fibration.

The results obtained are summarized in the following table.

μ_1	$T_{X_{\mu_1}}$	fibrations		
		$\mathcal{E}_i$	fibers	MW
$\{-3, (3-3\sqrt{3}i)/2, (3+3\sqrt{3}i)/2\}$	$\langle 2\rangle^2$	$\mathcal{E}_1$	$I_{18}+I_2+4I_1$	$\mathbb{Z}/3\mathbb{Z}$
		$\mathcal{E}_2$	$III^*+I_6^*+I_2+I_1$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_3$	$I_{12}^*+2I_2+2I_1$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_4$	$2II^*+2I_2$	$\{1\}$
		$\mathcal{E}_i$	fibers	MW
0	$A_2(-1)$	$\mathcal{E}_1$	$I_{18}+6I_1$	$\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}$
		$\mathcal{E}_2$	$III^*+I_6^*+3I_1$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}$
		$\mathcal{E}_3$	$I_{12}^*+I_3+3I_1$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_4$	$2II^*+IV$	$\{1\}$

In particular we obtained the“two most algebraic K3 surfaces” by Vinberg, see [26], as specializations of our K3 surface. These appeared already in other papers as for example in [10], where it occurs in the way presented in this note. We recall that the elliptic fibrations on a rigid K3 surface whose transcendental lattice is either $\langle 2\rangle\oplus\langle 2\rangle$ or $A_2(-1)$ were classified by Nishiyama, see [16].

6.2. Specializations of 2-dimensional families

6.2.1. $\boldsymbol{\delta = 1}$. We consider the 1-dimensional family of rational elliptic surfaces, parametrized by a, associated to the pencil of cubics:

$$x^2y+y^2z+z^2x+axy^2+\mu xyz.$$

The Weierstrass equation of the rational elliptic fibration $\mathcal{E}_R\colon R_a\to\mathbb{P}^1_\mu$ is

$$\begin{aligned}Y^2 = X^3 &+ \Big(-\frac{1}{48}\mu^4+\frac{1}{6}\mu^2a-\frac{1}{3}a^2-\frac{1}{2}\mu\Big)X\\ &-\frac{1}{864}\mu^6+\frac{1}{72}\mu^4a-\frac{1}{24}\mu^3-\frac{1}{18}\mu^2a^2+\frac{1}{6}\mu a+\frac{2}{27}a^3-\frac{1}{4}\end{aligned} \tag{6.2}$$

and the discriminant is

$$\Delta := -\frac{1}{16}\mu^4a+\frac{1}{16}\mu^3+\frac{1}{2}\mu^2a^2-\frac{9}{4}\mu a-a^3+\frac{27}{16}.$$

To obtain the K3 surface X_{a,μ_1} with the elliptic fibration $\mathcal{E}_X: X_{a,\mu_1}\to\mathbb{P}^1_\tau$ we apply the base change $\mu=\tau^2+\mu_1$ which is branched at infinity and at $\mu=\mu_1$.

On X_{a,μ_1} we have 5 elliptic fibrations: one of them is induced by $\mathcal{E}_R$ after the base change and its Weierstrass equation is obtained directly by (6.2), the others are associated to generalized conic bundles and their equations can be found applying the method described in Section 5.2. Since in the following we are mainly interested in the discriminant of these fibrations, we only write the equations of the conic bundles and the discriminant of the associated elliptic fibrations.

The 5 elliptic fibrations are:

(1) $\mathcal{E}_1$ induced by the elliptic fibration $\mathcal{E}_R$ on R_a, whose discriminant is

$$\Delta_1 := -\tfrac{1}{16}(\tau^2+\mu_1)^4 a + \tfrac{1}{16}(\tau^2+\mu_1)^3 + \tfrac{1}{2}(\tau^2+\mu_1)^2 a^2 - \tfrac{9}{4}(\tau^2+\mu_1)a - a^3 + \tfrac{27}{16}.$$

The singular fibers are $I_{16}+8I_1$ and $\mathrm{MW}=\mathbb{Z}$;

(2) $\mathcal{E}_2$ induced by the conic bundle $x-my$, whose discriminant is

$$\Delta_2 := m^{10}(a+m)^2\left(4m^2a - 1 + 4m^3 - m^2\mu_1^2 - 2m\mu_1\right).$$

The singular fibers are $III^* + I_4^* + I_2 + 3I_1$ and $\mathrm{MW}=\mathbb{Z}/2\mathbb{Z}$;

(3) $\mathcal{E}_3$ induced by the conic bundle $y-mz$, whose discriminant is

$$\Delta_3 = -m^{12}\left(2m^3\mu_1 a + m^2\mu_1^2 + m^4a^2 + 2m\mu_1 + 1 + 2m^2a - 4m^3\right).$$

The singular fibers are $I_6^* + I_2^* + 4I_1$ and $\mathrm{MW}=\mathbb{Z}/2\mathbb{Z}$;

(4) $\mathcal{E}_4$ induced by the conic bundle $x^2+axy+dxz+yz$, whose discriminant is

$$\Delta_4 := -d^2\left(-1+da\right)^2\left(d^4 - 2d^3\mu_1 + \mu_1^2d^2 + 2d^2a - 4d - 2a\mu_1 d + 4\mu_1 + a^2\right).$$

The singular fibers are $I_{10}^* + 2I_2 + 4I_1$ and $\mathrm{MW}=\mathbb{Z}/2\mathbb{Z}$;

(5) $\mathcal{E}_5$ induced by the conic bundle $bx^2+axy+yz$, whose discriminant is

$$\begin{aligned}\Delta_5 := b^9(-1+b)^2\big(&4ba^3 - 4a^3 - a^2b\mu_1^2 - 18b\mu_1 a\\ &+ 18b^2\mu_1 a - 4b^2\mu_1^3 - 54b^2 + 27b^3 + 27b\big).\end{aligned}$$

The singular fibers are $II^* + III^* + I_2 + 3I_1$ and $\mathrm{MW}=\{1\}$.

Specializations of the rational elliptic surface. If $a=0$, the rational elliptic surface R_a becomes the rigid rational elliptic surface with reducible fibers I_9+3I_1, so we go back to our previous case.

From now on we assume that $a\neq 0$. If

$$a \in \left\{-\frac{3}{8}\sqrt[3]{2}, \frac{3\left(\sqrt[3]{2} - i\sqrt{3}\sqrt[3]{2}\right)}{16}, \frac{3\left(\sqrt[3]{2} + i\sqrt{3}\sqrt[3]{2}\right)}{16}\right\},$$

then the discriminant Δ has a multiple zero and in particular if $a=-\frac{3}{8}\sqrt[3]{2}$ (resp. $a = 3\left(\sqrt[3]{2} \pm i\sqrt{3}\sqrt[3]{2}\right)/16$), then the fibration has a fiber of type II in $\mu_1 = -\frac{3}{2}\sqrt[3]{4}$ (resp. $\mu_1 = \mp 3i\sqrt[3]{4}\sqrt{3} + \sqrt[3]{4}/4$). The other fibers are unchanged and they are I_8+2I_1.

Specialization of the surface X_{a,μ_1}. Let us assume that $a\neq 0$ and that $a\notin\{-\frac{3}{8}\sqrt[3]{2}, 3\left(\sqrt[3]{2}\pm i\sqrt{3}\sqrt[3]{2}\right)/16\}$. If μ_1 is chosen to be one of the solutions of

$$\Delta_1 = -\frac{1}{16}\mu^4 a + \frac{1}{16}\mu^3 + \frac{1}{2}\mu^2a^2 - \frac{9}{4}\mu a - a^3 + \frac{27}{16} = 0,$$

then the K3 surface X_{a,μ_1} is constructed as a double cover of R_a, branched over a fiber of type I_8 and a fiber of type I_1. So $\mathcal{E}_1$ is an elliptic fibration with fibers

$I_{16} + I_2 + 6I_1$ and MW $= \mathbb{Z}$ and the K3 surface X_{a,μ_1} admits an involution acting trivially on the Néron–Severi group whose fixed locus consists of 9 rational curves. For this specialization $T_{X_{a,\mu_1}} \simeq U(2) \oplus \langle 2 \rangle$ (because of the presence of such an automorphism). Since the degree of Δ is 4, this happens for 4 values of μ_1.

If $\mu_1 = 1/a$ one observes that the discriminant of the elliptic fibration $\mathcal{E}_2$ becomes $m^{10}(a+m)^3(4m^2a^2 - a - m)$. This has a zero of multiplicity 3 at $m = -a$ (whereas this multiplicity equals 2 for general values of μ_1). Moreover now $m = -a$ is also a zero of the coefficients $c_4(m)$ and $c_6(m)$ in the short Weierstrass equation of $\mathcal{E}_2$ with multiplicities 1 and 2 respectively. This implies that the elliptic fibration $\mathcal{E}_2$ has now a fiber of type III and the singular fibers of this fibration are $III^* + I_4^* + III + 3I_1$. This has no effect on the rank of the Néron–Severi group.

One can consider the discriminant of the other elliptic fibrations, obtaining five values of μ_1 for which the Picard number of X_{a,μ_1} increases, four are solutions of $\Delta = 0$ and one is $-a^2/4$. If $\mu_1 = -a^2/4$, then the discriminant of $\mathcal{E}_5$ has a zero with multiplicity three (which is of multiplicity 2 if $\mu_1 \neq -a^2/4$), so the trivial lattice of $\mathcal{E}_5$ becomes $U \oplus E_8 \oplus E_7 \oplus A_2$ and since there can not be torsion section we conclude that this is also the Néron–Severi group. So the transcendental lattice is $T_{X_{a,\mu_1}} \simeq A_2(-1) \oplus \langle -2 \rangle$.

We put in a table the information about the specializations due to μ_1 which changes the Néron–Severi of the surface:

μ_1	$T_{X_{a,\mu_1}}$	fibrations		
		$\mathcal{E}_i$	fibers	MW
solutions $\Delta = 0$	$\langle 2 \rangle^2 \oplus \langle -2 \rangle$	$\mathcal{E}_1$	$I_{16} + I_2 + 6I_1$	$\mathbb{Z}$
		$\mathcal{E}_2$	$III^* + I_4^* + 2I_2 + I_1$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_3$	$I_6^* + I_2^* + I_2 + 2I_1$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_4$	$I_{10}^* + 3I_2 + 2I_1$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_5$	$II^* + III^* + 2I_2 + I_1$	$\{1\}$
		$\mathcal{E}_i$	fibers	MW
$-a^2/4$	$A_2(-1) \oplus \langle 2 \rangle$	$\mathcal{E}_1$	$I_{16} + 8I_1$	$\mathbb{Z} \times \mathbb{Z}$
		$\mathcal{E}_2$	$III^* + I_4^* + I_2 + 3I_1$	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_3$	$I_6^* + I_2^* + 4I_1$	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_4$	$I_{10}^* + I_3 + I_2 + 3I_1$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_5$	$II^* + III^* + I_3 + 2I_1$	$\{1\}$

From the previous table it is clear that for $a = 0$ one obtains the previous specializations to K3 surfaces with transcendental lattice either $\langle 2 \rangle^2$ or $A_2(-1)$.

Let us now assume that $a = -\frac{3}{8}\sqrt[3]{2}$ (the cases $a = 3\left(\sqrt[3]{2} \pm i\sqrt{3}\sqrt[3]{2}\right)/16$ are analogous). In this case the rational elliptic fibration has fibers of types $I_8 + II + 2I_1$. If the fiber over μ_1 is smooth nothing changes in the Néron–Severi of the K3 surface with respect to the general case. If the fiber over μ_1 is I_1 nothing changes in the Néron–Severi of the K3 surface with respect to the choice of a different value of a, and μ_1 to be a solution of $\Delta = 0$. But if the fiber over μ_1 is of type II, then something changes. In particular we have the following table.

	$a = -\frac{3}{8}\sqrt[3]{2}$			
μ_1	$T_{X_{\mu_1}}$	fibrations		
		$\mathcal{E}_i$	fibers	MW
$-\frac{3}{2}\sqrt[3]{4}$	$\langle 2\rangle \oplus \langle 6\rangle$	$\mathcal{E}_1$	$I_{16}+IV+4I_1$	$\mathbb{Z}$
		$\mathcal{E}_2$	$III^*+I_4^*+I_3+I_2$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_3$	$I_6^*+I_2^*+I_3+I_1$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_4$	$I_{10}^*+I_3+2I_2+I_1$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_5$	$II^*+III^*+I_2+I_3$	$\{1\}$

Remark 6.1. In [5], the authors classify all the elliptic fibrations on the unique K3 surface with transcendental lattice $\langle 2\rangle \oplus \langle 6\rangle$. Here we recover some of the elliptic fibrations on the mentioned K3 surface, in particular the ones numbered as 17, 12, 10, 5 and 2 in the aforementioned article. Their equations are easily obtained here, just specializing the ones of $\mathcal{E}_i$ on X. One observes that in the case of the fibration 17, which corresponds here to $\mathcal{E}_1$, the lattice A_2 in the trivial lattice corresponds not to a fiber of type I_3 (as in general happens) but to a fiber of type IV.

6.2.2. $\boldsymbol{\delta = 0}$. We consider the 1-dimensional family of rational elliptic surfaces, parametrized by a, associated to the pencil of cubics:

$$x^2y + z^2x + xy^2 + az^2y + \mu xyz.$$

The Weierstrass equation of the rational elliptic fibration $\mathcal{E}_R\colon R_a \to \mathbb{P}^1_\mu$ is

$$Y^2 = X^3 + \Big(-\frac{1}{48}\mu^4 + \frac{1}{6}\mu^2 a + \frac{1}{6}\mu^2 - \frac{1}{3}a^2 + \frac{1}{3}a - \frac{1}{3}\Big)X + \frac{1}{864}\left(4a+4-\mu^2\right)\left(16a^2 - 8\mu^2 a - 40a + 16 + \mu^4 - 8\mu^2\right)$$

and the discriminant is

$$\Delta := -\frac{1}{16}a^2\left(4a-4-\mu^2+4\mu\right)\left(4a-4-\mu^2-4\mu\right).$$

To obtain the K3 surface X_{a,μ_1} with the elliptic fibration $\mathcal{E}_X : X_{a,\mu_1} \to \mathbb{P}^1_\tau$ we apply the base change $\mu = \tau^2 + \mu_1$ which is branched at infinity and $\mu = \mu_1$.

On X_{a,μ_1} there are 6 elliptic fibrations. They are:

(1) $\mathcal{E}_1$ induced by the elliptic fibration $\mathcal{E}_R$ on R_a whose discriminant is

$$\Delta_1 := -\frac{1}{16}a^2(4a-4-(\tau^2+\mu_1)^2+4(\tau^2+\mu_1))(4a-4-(\tau^2+\mu_1)^2-4(\tau^2+\mu_1)).$$

The singular fibers are $I_{16}+8I_1$ and $\mathrm{MW} = \mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$;

(2) $\mathcal{E}_2$ induced by the conic bundle $x - my$, whose discriminant is

$$\Delta_2 := m^9(1+m)^2(a+m)^2(4ma+4a+4m^2+4m-m\mu_1^2).$$

The singular fibers are $2III^*+2I_2+2I_1$ and $\mathrm{MW} = \mathbb{Z}/2\mathbb{Z}$.

(3) $\mathcal{E}_3$ induced by the conic bundle $y - mz$, whose discriminant is

$$\Delta_3 := a^2 m^{10}(-m^4 - 2m^3\mu_1 - m^2\mu_1^2 - 2m^2 + 4am^2 - 2m\mu_1 - 1).$$

The singular fibers are $2I_4^* + 4I_1$ and $\mathrm{MW} = \mathbb{Z}/2\mathbb{Z}$;

(4) $\mathcal{E}_4$ induced by the conic bundle $x + y + mz$, whose discriminant is

$$\Delta_4 := -a^2 m^6 (m^4 - 2m^3\mu_1 + m^2\mu_1^2 + 2am^2 + 2m^2 - 2m\mu_1 - 2m\mu_1 a + 1 - 2a + a^2).$$

The singular fibers are $I_8^* + I_0^* + 4I_1$ and $\mathrm{MW} = \mathbb{Z}/2\mathbb{Z}$;

(5) $\mathcal{E}_5$ induced by the conic bundle $xy + y^2 + dxz$, whose discriminant is

$$\Delta_5 := d^{10} \left(4d^2 - 8da\mu_1 - d^2\mu_1^2 + 2a + 2d\mu_1 - 6d^2a^2 + 4a^3d^2 + 18d^3a\mu_1 \right. \\ \left. + 10\, d^2a\mu_1^2 - d^2a^2\mu_1^2 + 2a^2d\mu_1 - 4d^3a\mu_1^3 - 6d^2a - 1 + 18d^3a^2\mu_1 + 27d^4a^2 - a^2\right).$$

The singular fibers are $II^* + I_4^* + 4I_1$ and $\mathrm{MW} = \{1\}$;

(6) $\mathcal{E}_6$ induced by the conic bundle $x^2y + 2xy^2 + y^3 + z^2x + hyz(x + y)$, whose discriminant is

$$\begin{aligned}\Delta_6 := -4096(&4a + a^2h^6 + 2a^3h^4 + a^4h^2 + 8h^5\mu_1 a + 12a^2 - 2\mu_1 h^3 + 12a^3 \\ &+ 4a^4 - 27\mu_1^2a^2 - 2h^6a + h^4\mu_1^2 - 2h^5\mu_1 + h^6 + 2h^4 + 6a^2h^3\mu_1 - 6a^3h\mu_1 \\ &+ 6ah^3\mu_1 - 9a^2h^2 + 42a^2h\mu_1 + 10a^3h^2 + 10h^2a + h^2 + a^2h^4\mu_1^2 - 2a^2h^5\mu_1 \\ &- 2a^3h^3\mu_1 - 6ah^2\mu_1^2 - 6a^2h^2\mu_1^2 + 4ah^3\mu_1^3 - 10ah^4\mu_1^2 - 6ah\mu_1)a^2.\end{aligned}$$

The singular fibers are $I_{12}^* + 6I_1$ and $\mathrm{MW} = \{1\}$. In order to compute the Weierstrass equation as in the algorithm we perform a change of coordinates sending the point $(1 : -1 : 0)$ to $(0 : 1 : 0)$.

We observe that $a \neq 0$, otherwise the pencil of cubics defining the rational elliptic surface does not contain smooth fibers.

If $\mu_1 = 2(\pm 1 \pm \sqrt{a})$, then the branch fibers of the cover $X \to R$ are I_8 and I_1.

If $\mu_1 = 0$ the elliptic fibration $\mathcal{E}_1$ becomes $Y^2 = X^3 + \frac{1}{3}m^3(1+m)(a+m)X$ which is an isotrivial fibration whose general fiber is isometric to the elliptic curve with complex multiplications of order 4. In this case there is an extra automorphism of order 4, but the Néron–Severi does not change. The order 4 automorphism is purely non–symplectic and the presence of such an automorphism reduces the dimension of the family of K3 surface from 2 to 1, see [1]. The specializations of the elliptic fibrations $\mathcal{E}_i$ on X_{a,μ_1} for specific values of a and μ_1 are summarized in the following table.

specialization	$T_{X_{a,\mu_1}}$	$\mathcal{E}_i$	fibers	MW
$\mu_1 = 2(\pm 1 \pm \sqrt{a})$	$\langle 2\rangle^2 \oplus \langle -2\rangle$	$\mathcal{E}_1$	$I_{16} + I_2 + 6I_1$	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_2$	$2III^* + 3I_2$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_3$	$2I_4^* + I_2 + 2I_1$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_4$	$I_8^* + I_0^* + I_2 + 2I_1$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_5$	$II^* + I_4^* + I_2 + 2I_1$	$\{1\}$
		$\mathcal{E}_6$	$I_{12}^* + I_2 + 4I_1$	$\{1\}$
$a = 1$	$U \oplus \langle 4\rangle$	$\mathcal{E}_1$	$I_{16} + 8I_1$	$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_2$	$2III^* + I_4 + 2I_1$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_3$	$2I_4^* + 4I_1$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$
		$\mathcal{E}_4$	$I_8^* + I_1^* + 3I_1$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_5$	$II^* + I_5^* + 3I_1$	$\{1\}$
		$\mathcal{E}_6$	$I_{12}^* + 6I_1$	$\mathbb{Z}$
$a = 1,\ \mu_1 = 0$	$\langle 2\rangle^2$	$\mathcal{E}_1$	$I_{16} + I_4 + 4I_1$	$\mathbb{Z}/4\mathbb{Z}$
		$\mathcal{E}_2$	$2III^* + I_0^*$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_3$	$2I_4^* + 2I_2$	$(\mathbb{Z}/2\mathbb{Z})^2$
		$\mathcal{E}_4$	$I_8^* + I_2^* + 2I_1$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_5$	$II^* + I_6^* + 2I_1$	$\{1\}$
		$\mathcal{E}_6$	$I_{14}^* + 4I_1$	$\mathbb{Z}$
$a = 1,\ \mu_1 = \pm 4$	$\langle 2\rangle \oplus \langle 4\rangle$	$\mathcal{E}_1$	$I_{16} + 3I_2 + 2I_1$	$\mathbb{Z}/4\mathbb{Z}$
		$\mathcal{E}_2$	$2III^* + I_2 + I_4$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_3$	$2I_4^* + I_2 + 2I_1$	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_4$	$I_8^* + I_1^* + I_2 + I_1$	$\mathbb{Z}/2\mathbb{Z}$
		$\mathcal{E}_5$	$II^* + I_5^* + I_2 + I_1$	$\{1\}$
		$\mathcal{E}_6$	$I_{13}^* + I_2 + 3I_1$	$\{1\}$

Remark 6.2. The elliptic fibrations on the K3 surface with transcendental lattice $\langle 2\rangle \oplus \langle 4\rangle$ are classified in [4], where their Weierstrass equations are also given. The fibrations $\mathcal{E}_i$ $i = 1, \dots, 6$ corresponds to the fibrations described lines 18, 12, 14, 22, 25, 23 of the last table in [4] respectively. In our context the equations are just obtained by the application of the algorithm with the assumption $a = 1$, $\mu_1 = \pm 4$.

7. Higher genus curve in the fixed locus of ι

In what follows, we assume that X is a K3 surface and ι is a non-symplectic involution on X whose fixed locus contains a curve C of genus higher than 1 and k rational curves. We assume that ι^* acts as the identity on the Néron–Severi group.

Proposition 7.1. *Let (X, ι) be as in Assumption 2.1 and let $\mathcal{E}\colon X \to \mathbb{P}^1$ be an elliptic fibration on X. Let C be a genus $g > 1$ curve fixed by ι.*

Then the elliptic fibrations on X, necessarily of type 1 with respect to ι, are given in the Appendix.

Moreover each elliptic fibration induces a rational fibration on X/ι (and thus a pencil of rational curves on any birational model of X/ι).

Proof. By Theorem 2.6, each elliptic fibration on X is of type 1, so by Proposition 2.7 one has the list of the reducible fibers which can appear on elliptic fibrations on X. Moreover, by Proposition 2.5, the Mordell–Weil group of any fibration on X is contained in $\mathbb{Z}/2\mathbb{Z}$ and it is trivial if $g > 3$. This allows us to produce the list of elliptic fibrations exactly as in Proposition 2.9. Since the involution ι is the cover involution, it induces a fibration on X/ι whose fibers are the quotient of the fibers of the fibration on X by the involution in fibers obtained from ι. Since ι is the involution $[-1]$ on a fiber, the fibration on X/ι has rational fibers.

All the fibrations listed have Mordell–Weil rank 0, and they exist since they appear in the list in [22]. □

Remark 7.2. The classifications of the elliptic fibrations given in Section 8 can be also obtained by Nishiyama's method, considering the following lattices:

g, k	T	g, k, δ	T
$g = 2,\ k = 1, \dots, 9$	$E_8 \oplus A_1^{9-k}$	$g = 2,\ k = 5,\ \delta = 0$	$D_8 \oplus D_4$
$g = 3,\ k = 1, \dots, 6$	$E_8 \oplus D_4 \oplus A_1^{6-k}$	$g = 3,\ k = 2,\ \delta = 0$	$D_8 \oplus D_4^2$
$g = 4,\ k = 1, \dots, 5$	$E_8 \oplus D_6 \oplus A_1^{5-k}$	$g = 4,\ k = 3,\ \delta = 0$	$E_8 \oplus D_4 \oplus D_4$
$g = 5,\ k = 1, \dots 5$	$E_8 \oplus E_7 \oplus A_1^{5-k}$	$g = 5,\ k = 4,\ \delta = 0$	D_{16}
$g = 6,\ k = 1, \dots 5$	$E_8 \oplus E_8 \oplus A_1^{5-k}$	$g = 7,\ k = 1, 2$	$E_8 \oplus E_8 \oplus D_4 \oplus A_1^{2-k}$
$g = 8,\ k = 1$	$E_8 \oplus E_8 \oplus D_6$	$g = 9,\ k = 1$	$E_8 \oplus E_8 \oplus E_7$
$g = 10,\ k = 1$	$E_8 \oplus E_8 \oplus E_8$		

The K3 surfaces X as in Proposition 7.1 admit two different very natural geometric descriptions: one is as double cover of a minimal model of X/ι (which is rational and in some cases $\mathbb{P}^2$), the other is $\varphi_{|C|}(X) \subset \mathbb{P}^g$, if $g > 2$.

7.1. The K3 surfaces X as double covers of $\mathbb{P}^2$

Let us consider the double cover of $\mathbb{P}^2$ branched on a possibly reducible sextic. The minimal model of this double cover is a K3 surface X, admitting a non-symplectic involution ι (the cover involution) and for sufficiently generic choices of the sextic, ι acts trivially on the Néron–Severi group of X. Hence the Néron–Severi group of X can be deduced by the Nikulin classification of the non-symplectic involutions and depends on the number and the genus of the components of the branch sextic (whose normalization is isomorphic to the fixed locus of ι). On the other hand the choice of the Néron–Severi group of a K3 surface which admits a non-symplectic involution acting trivially on the Néron–Severi group, determines a K3 surface which satisfies the hypothesis of Proposition 7.1, if the fixed curve with highest genus has genus at least 2.

By Proposition 7.1, one obtains that the elliptic fibrations on X are induced by "generalized conic bundles", i.e., by pencils of rational curves passing through a certain numbers of singular points of the branch sextic. This allows to reproduce all the computations done in the case $g = 1$, also in these highest genus cases, if one is able to describe explicitly the branch sextic.

If $g = 6$ and $k = 1$, then the K3 surface X is generic in the unique family of K3 surfaces admitting an involution ι fixing one rational curve and a curve of genus 6. It can be realized as minimal model of the double cover of $\mathbb{P}^2$ branched on a line l and an smooth quintic q.

Similarly, different values of g and k can be obtained modifying the branch curve. So some of the surfaces in Proposition 7.1 can be realized as minimal models of double covers of $\mathbb{P}^2$ branched on a sextic s as in the table:

sextic s	g	k
line l+quintic q, q has $0 \leq \alpha \leq 4$ nodes	$6-\alpha$	1
line l+quintic q, q has $\alpha = 1, 2$ nodes, l through the node of q	$6-\alpha$	2
line l+line m+quartic q, q has $\alpha = 0, 1$ node	$3-\alpha$	2
line l+line m+quartic q, q has $\alpha = 0, 1$ node, $l \cap m \cap q \neq \emptyset$	$3-\alpha$	3
line l+line m+quartic q, q has 1 node, $l \cap m \cap q \neq \emptyset$, m through the node of q	2	4

We observe that if the curve q has a node, then the pencil of lines through this node induces an elliptic fibration on the K3 surface X, double cover of $\mathbb{P}^2$ branched on the sextic s. This provides a geometric description, for example, of the elliptic fibrations obtained listed in the Appendix for $g = 2, \ldots 6$, $k = 1$.

7.2. The model associated to $|C|$, $g = 2, 3$

The curve C fixed by ι is a smooth irreducible curve on X, so its linear system defines a map to a projective space, in particular to $\mathbb{P}^g$. For low values of g, this model is quite clear and sometime also provide an equation for X. Here we collect some results on this.

Proposition 7.3. *Let (X, ι) be as in Assumption 2.1, let $\mathcal{E}\colon X \to \mathbb{P}^1$ be an elliptic fibration on X, and let $C \in \mathrm{Fix}_\iota$ be a curve of genus 2. Then, up to a choice of the coordinates of $\mathbb{P}^2$, $\varphi_{|C|}\colon X \to \mathbb{P}^2$ is a double cover of $\mathbb{P}^2$ branched over the sextic*

$$x_0^4 f_2(x_1 : x_2) + x_0^2 f_4(x_1 : x_2) + f_6(x_1 : x_2),$$

where $f_i \in \mathbb{C}_i[x_1 : x_2]$, and ι is induced on X by $\iota_{|\mathbb{P}^2} : (x_0, x_1, x_2) \to (-x_0, x_1, x_2)$. Denote by α the cover involution. The following hold:

i) *If C is a trisection of $\mathcal{E}$, the fibers of the elliptic fibration $\mathcal{E}$ are mapped to curves of degree 3 in $\mathbb{P}^2$ which pass through $(1 : 0 : 0)$ and intersect the branch sextic in every intersection point with even multiplicity, and α is an involution of the basis of $\mathcal{E}$.*

ii) *If C is a bisection of $\mathcal{E}$, the fibers of the elliptic fibration $\mathcal{E}$ are mapped to lines in $\mathbb{P}^2$ which pass through $(1 : 0 : 0)$. Let σ be the symplectic involution which is the translation by the 2-torsion section of $\mathcal{E}$. Then $\alpha = \iota \circ \sigma$.*

Proof. Since the curve C is a genus 2 curve, the map $\varphi_{|C|}\colon X \to \mathbb{P}^2$ is a double cover. The cover involution α is not ι, since α fixes the branch locus of the cover $X \to \mathbb{P}^2$ and ι fixes the pullback of a generic the hyperplane section, i.e., C. The automorphism ι preserves the map $X \to \mathbb{P}^2$, and fixes the pull-back of a line, so it descends to an automorphism $\iota_{\mathbb{P}^2}$ of $\mathbb{P}^2$ which fixes the branch locus of the cover $X \to \mathbb{P}^2$. Up to a choice of coordinates we can assume that $\iota_{\mathbb{P}^2}$ is $(x_0 : x_1 : x_2) \mapsto (-x_0 : x_1 : x_2)$. This fixes the line $x_0 = 0$ and the point $(1 : 0 : 0)$. The sextics invariant for $\iota_{\mathbb{P}^2}$ are given by

$$ax_0^6 + x_0^4 f_2(x_1 : x_2) + x_0^2 f_4(x_1 : x_2) + f_6(x_1 : x_2).$$

Since X admits an elliptic fibration, ι fixes not only C but at least on rational section, i.e., a rational curve R_1 orthogonal to C. So $\varphi_{|C|}$ contracts at least one rational curve, fixed by ι. The branch sextic of $\varphi_{|C|}\colon X \to \mathbb{P}^2$ has a singularity in the fixed point of $\iota_{\mathbb{P}^2}$. Hence $a = 0$ and the branch sextic is

$$x_0^4 f_2(x_1 : x_2) + x_0^2 f_4(x_1 : x_2) + f_6(x_1 : x_2),$$

the inverse image of $x_0 = 0$ on X is exactly the smooth genus 2 curve C, the curve(s) resolving the singularity of the branch sextic in $(1 : 0 : 0)$ are rational curve(s) possibly fixed by ι.

If C is a trisection, and F denotes the class of the fiber of $\mathcal{E}$, then $C \cdot F = 3$, hence the image of each curve in F has degree 3 in $\mathbb{P}^2$ and the curves in $|\varphi_{|C|}(F)|$ split in the double cover (otherwise $C \cdot F$ should be an even number). Hence α is the involution which switches the two disjoint curves in the inverse image of the curves in $|\varphi_{|C|}(F)|$. So it is an involution which preserves the class of the fiber but switches pairs of fibers of $\mathcal{E}$, and it is an involution of the basis of $\mathcal{E}$. Moreover, $F \cdot R_1 = 1$, so the curves in $|\varphi_{|C|}(F)|$ pass through $\varphi_{|C|}(R_1) = (1 : 0 : 0)$.

If C is a bisection, $C \cdot F = 2$, and there are two possibilities: either $\varphi_{|C|}(F)$ is a curve of degree 2 in $\mathbb{P}^2$ and it splits in the double cover, or $\varphi_{|C|}(F)$ is a curve of degree 1 in $\mathbb{P}^2$ which does not split. If a conic splits in the double cover, its inverse image consists of two copies of a rational curve. This can not be the case, since $|F|$ is a 1-dimensional system of genus 1 curves. So $\varphi_{|C|}(F)$ is a line (which does not split in the double cover). The lines in $|\varphi_{|C|}(F)|$ pass through the node, because $F \cdot R_1 = 1$.

Since C is a bisection of $\mathcal{E}$ passing through some of the 2-torsion points, there exists also a 2-torsion section T in $\mathcal{E}$, which is fixed by ι and contracted by $\varphi_{|C|}$ since $C \cdot T = 0$. In particular the singularity $(1 : 0 : 0)$ in the branch sextic is worst than a simple node. Due to the symmetry $(x_0 : x_1 : x_2) \to (-x_0 : x_1 : x_2)$, the inverse image on X of point $(1 : 0 : 0)$ is not simply an A_2 configuration of rational curves but it is at least an A_3 configuration of rational curves. The two rational curves at the extreme of the trees of rational curves of these A_3 are switched by α and they are sections of the elliptic fibration $\mathcal{E}$. These two curves correspond to the zero section and the 2-torsion section of $\mathcal{E}$, both are fixed by ι. The involutions α and ι preserve each fiber of the fibration, so the same holds for their composition $\alpha \circ \iota = \sigma$, which is then a symplectic involution on $\mathcal{E}$, preserving each fiber and mapping the zero section to the 2-torsion section. This implies that σ is a translation by the 2-torsion section. $\square$

Remark 7.4. By Proposition 7.3, it follows that the K3 surface X as in the hypothesis admits not only a non-symplectic involution ι, but also a symplectic involution σ. These K3 surfaces were studied in [9], where numeric conditions on g and k which imply that K3 surfaces with a non-symplectic involution necessarily admit also a symplectic one are established. The conditions in Proposition 3.1 of [9] imply (with just one exception) that the fixed curves have genus at most 2. Hence, Proposition 7.3 gives a geometric interpretation of the involutions considered in [9].

Let us now briefly discuss the case $g = 3$, naturally associated to a map to $\mathbb{P}^3$. We observe that there is just one case in which this map is $2:1$ onto the image, and not $1:1$. This is exactly the case mentioned in the previous remark, where one has a curve of genus 3 in the fixed locus of an involution ι acting trivially on the Néron–Severi group, but it is still true that every K3 surface with this property also admits a symplectic involution.

Proposition 7.5. *Let (X, ι) be as in Assumption 2.1, let $\mathcal{E}\colon X \to \mathbb{P}^1$ be an elliptic fibration on X, and let $C \in \mathrm{Fix}_\iota$ be a curve of genus 3.*

If C is a trisection, then $\varphi_{|C|}\colon X \to \mathbb{P}^3$ is a generically $1:1$ map to a singular quartic with equation

$$x_0^2 f_2(x_1 : x_2 : x_3) + f_4(x_1 : x_2 : x_4).$$

The eight lines connecting the singular point $(1:0:0:0)$ to the eight points $f_2 \cap f_4 \subset \mathbb{P}^2_{(x_1:x_2:x_3)}$ are contained in $\varphi_{|C|}(X)$.

If C is a bisection, then $\varphi_{|C|}\colon X \to \mathbb{P}^3$ is a generically $2:1$ map to a quadric.

Proof. We assume that X admits an elliptic fibration $\mathcal{E}$ and that C is a trisection of this fibration. Then C in not hyperelliptic and by [20] the map $\varphi_{|C|}\colon X \to \mathbb{P}^3$ exhibits $\varphi_{|C|}(X)$ as quartic hypersurface in $\mathbb{P}^3$. Denote by F and O respectively the fiber and the zero section of $\mathcal{E}$. We have $C \cdot O = 0$, so the curve O is contracted by $\varphi_{|C|}$ and thus the quartic has a node. Moreover, the involution ι descends to an involution of $\mathbb{P}^3$, fixing the node. Up to a choice of coordinates, one can assume that the node is $(1:0:0:0)$ and the involution induced by ι on $\mathbb{P}^3$ is $(x_0 : x_1 : x_2 : x_3) \mapsto (-x_0 : x_1 : x_2 : x_3)$. Since $F \cdot C = 3$, the fibers of $\mathcal{E}$ are mapped to curves of degree 3 in $\mathcal{E}$, passing through the node (since $F \cdot O = 1$). By the equation of the quartic one immediately checks that there are eight lines contained in $\varphi_{|C|}(X)$ passing through O. Let l be one of these lines and L be the class in $\mathrm{NS}(X)$ corresponding to the strict transform of l after blowing up the node. We have $L^2 = -2$, $L \cdot C = 1$, $L \cdot O = 1$. The pencil of hyperplanes through l cuts on $\varphi_{|C|}(X)$ a pencil of genus 1 curves, passing through the node, so it induces on X an elliptic fibration, whose class is $C - L$. Generically this pencil has 7 reducible fibers, corresponding to the hyperplane through l which contains another line of $\varphi_{|C|}(X)$. Indeed generically the reducible fibers of $\mathcal{E}$ are 7 fibers of type I_2 (see Table 4, $k = 1$).

If C is a bisection, it is a hyperelliptic curve and by [20] the map $\varphi_{|C|}\colon X \to \mathbb{P}^3$ is $2:1$ to a quadric. $\square$

8. Appendix

In this section we list the elliptic fibrations on a K3 surface admitting an involution ι which acts trivially on the Néron–Severi group and such that the highest genus g of a fixed curve is greater than 1. The lists are obtained according to the results of Theorem 2.6 and Propositions 2.5 and 2.7 similarly to what is done in the proof of Proposition 2.9. Also the notation is the same as in Proposition 2.9.

- The fibrations on K3 surfaces admitting an involution as ι such that $g = 2$ are given in Table 3;
- the ones such that $g = 3$ in Table 4;
- the ones such that $g = 4$ in Table 5;
- the ones such that $g = 5$ in Table 6;
- the ones such that $g = 6$ in Table 7;
- the ones such that $g = 7$ in Table 8;
- the ones such that $g = 8$ in Table 9, $b = 0$;
- the ones such that $g = 9$ in Table 9, $b = 1$;
- the ones such that $g = 10$ in Table 9, $b = 2$.

TABLE 3. Case $g = 2$.

$g = 2, k = 9, r = 18, a = 0,\ (\delta = 0)$

trivial lattice	$16 = \sum c_i + \text{rank}(MW)$	$9 = k = \sum s_i + \#\text{sections}$	$MW(\mathcal{E})$
$U \oplus E_8 \oplus E_8$	$8 + 8$	$4 + 4 + 1$	$\{1\}$
$U \oplus D_{16}$	16	$7 + 2$	$\mathbb{Z}/2\mathbb{Z}$

$g = 2,\ k = 8, r = 17,\ a = 1,\ (\delta = 1)$

trivial lattice	$15 = \sum c_i + \text{rank}(MW)$	$8 = k = \sum s_i + \#\text{sections}$	$MW(\mathcal{E})$
$U \oplus E_8 \oplus E_7$	$8 + 7$	$4 + 3 + 1$	$\{1\}$
$U \oplus D_{14} \oplus A_1$	$14 + 1$	$6 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$

$g = 2,\ k = 7,\ r = 16,\ a = 2,\ (\delta = 1)$

trivial lattice	$14 = \sum c_i + \text{rank}(MW)$	$7 = k = \sum s_i + \#\text{sections}$	$MW(\mathcal{E})$
$U \oplus E_7 \oplus E_7$	$7 + 7$	$3 + 3 + 1$	$\{1\}$
$U \oplus E_8 \oplus D_6$	$8 + 6$	$4 + 2 + 1$	$\{1\}$
$U \oplus D_{14}$	14	$6 + 1$	$\{1\}$
$U \oplus D_{12} \oplus A_1^2$	$12 + 1 + 1$	$5 + 0 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$

$g = 2,\ k = 6,\ r = 15,\ a = 3,\ (\delta = 1)$

trivial lattice	$13 = \sum c_i + \text{rank}(MW)$	$6 = k = \sum s_i + \#\text{sections}$	$MW(\mathcal{E})$
$U \oplus E_7 \oplus D_6$	$7 + 6$	$3 + 2 + 1$	$\{1\}$
$U \oplus E_8 \oplus D_4 \oplus A_1$	$8 + 4 + 1$	$4 + 1 + 0 + 1$	$\{1\}$
$U \oplus D_{12} \oplus A_1$	$12 + 1$	$5 + 0 + 1$	$\{1\}$
$U \oplus D_{10} \oplus A_1^3$	$10 + 1 + 1 + 1$	$4 + 0 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$

$g = 2,\ k = 5, r = 14,\ a = 4,\ \delta = 0$ or 1

trivial lattice	$12 = \sum c_i$+rank(MW)	$5 = k = \sum s_i$ + #sections	MW($\mathcal{E}$)	δ
$U \oplus D_6 \oplus D_6$	$6 + 6$	$2 + 2 + 1$	$\{1\}$	1
$U \oplus E_7 \oplus D_4 \oplus A_1$	$7 + 4 + 1$	$3 + 1 + 0 + 1$	$\{1\}$	1
$U \oplus D_{10} \oplus A_1^2$	$10 + 1 + 1$	$4 + 0 + 0 + 1$	$\{1\}$	1
$U \oplus E_8 \oplus A_1^4$	$8 + 1 + 1 + 1 + 1$	$4 + 0 + 0 + 0 + 0 + 1$	$\{1\}$	1
$U \oplus D_8 \oplus A_1^4$	$8 + 1 + 1 + 1 + 1$	$3 + 0 + 0 + 0 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$	1
$U \oplus D_8 \oplus D_4$	$8 + 4$	$3 + 1 + 1$	$\{1\}$	0
$U \oplus E_7 \oplus A_1^5$	$7 + 1 + 1 + 1 + 1 + 1$	$3 + 0 + 0 + 0 + 0 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$	0

$g = 2, k = 4, r = 13,\ a = 5\ (\delta = 1)$

trivial lattice	$11 = \sum c_i$+rank(MW)	$4 = k = \sum s_i$ + #sections	MW($\mathcal{E}$)
$U \oplus D_6 \oplus D_4 \oplus A_1$	$6 + 4 + 1$	$2 + 1 + 0 + 1$	$\{1\}$
$U \oplus D_8 \oplus A_1^3$	$8 + 1 + 1 + 1$	$3 + 0 + 0 + 1$	$\{1\}$
$U \oplus E_7 \oplus A_1^4$	$7 + 1 + 1 + 1 + 1$	$3 + 0 + 0 + 0 + 0 + 1$	$\{1\}$
$U \oplus D_6 \oplus A_1^5$	$6 + 1 + 1 + 1 + 1 + 1$	$2 + 0 + 0 + 0 + 0 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$

$g = 2,\ k = 3, r = 12,\ a = 6\ (\delta = 1)$

trivial lattice	$10 = \sum c_i$+rank(MW)	$3 = k = \sum s_i$ + #sections	MW($\mathcal{E}$)
$U \oplus D_4 \oplus D_4 \oplus A_1^2$	$4 + 4 + 1 + 1$	$1 + 1 + 0 + 0 + 1$	$\{1\}$
$U \oplus D_6 \oplus A_1^4$	$6 + 1 + 1 + 1 + 1$	$2 + 0 + 0 + 0 + 0 + 1$	$\{1\}$
$U \oplus D_4 \oplus A_1^6$	$4+1+1+1+1+1+1$	$1+0+0+0+0+0+0+2$	$\mathbb{Z}/2\mathbb{Z}$

$g = 2,\ k = 2, r = 11,\ a = 7\ (\delta = 1)$

trivial lattice	$9 = \sum c_i$ + rank(MW)	$2 = k = \sum s_i$ + #sections	MW($\mathcal{E}$)
$U \oplus D_4 \oplus A_1^5$	$4 + 1 + 1 + 1 + 1 + 1$	$1 + 0 + 0 + 0 + 0 + 0 + 1$	$\{1\}$
$U \oplus A_1^9$	$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$	$0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 2$	$\mathbb{Z}/2\mathbb{Z}$

$g = 2, k = 1, r = 10,\ a = 8\ (\delta = 1)$

trivial lattice	$8 = \sum c_i$ + rank(MW)	$1 = k = \sum s_i$ + #sections	MW($\mathcal{E}$)
$U \oplus A_1^8$	$1+1+1+1+1+1+1+1$	$0+0+0+0+0+0+0+0+1$	$\{1\}$

TABLE 4. Case $g = 3$

$g = 3, k = 6, r = 14, a = 2\ (\delta = 0)$

trivial lattice	$12 = \sum c_i$+rank(MW)	$6 = k = \sum s_i$ + #sections	MW($\mathcal{E}$)
$U \oplus D_{12}$	12	$5 + 1$	$\{1\}$
$U \oplus E_8 \oplus D_4$	$8 + 4$	$4 + 1 + 1$	$\{1\}$

$g = 3, k = 5, r = 13,\ a = 3,\ (\delta = 1)$

trivial lattice	$11 = \sum c_i$+rank(MW)	$5 = k = \sum s_i$ + #sections	MW($\mathcal{E}$)
$U \oplus E_7 \oplus D_4$	$7 + 4$	$3 + 1 + 1$	$\{1\}$
$U \oplus D_{10} \oplus A_1$	$10 + 1$	$4 + 0 + 1$	$\{1\}$
$U \oplus E_8 \oplus A_1^3$	$8 + 1 + 1 + 1$	$4 + 0 + 0 + 0 + 1$	$\{1\}$

$g = 3, k = 4, r = 12,\ a = 4,\ (\delta = 1)$

trivial lattice	$10 = \sum c_i$+rank(MW)	$4 = k = \sum s_i$ + #sections	MW($\mathcal{E}$)
$U \oplus D_6 \oplus D_4$	$6 + 4$	$2 + 1 + 0 + 1$	$\{1\}$
$U \oplus D_8 \oplus A_1^2$	$8 + 1 + 1$	$3 + 0 + 0 + 1$	$\{1\}$
$U \oplus E_7 \oplus A_1^3$	$7 + 1 + 1 + 1$	$3 + 0 + 0 + 0 + 1$	$\{1\}$

$g = 3, k = 3, r = 11,\ a = 5,\ (\delta = 1)$

trivial lattice	$9 = \sum c_i$ + rank(MW)	$3 = k = \sum s_i$ + #sections	MW($\mathcal{E}$)
$U \oplus D_4 \oplus D_4 \oplus A_1$	$4 + 4 + 1$	$1 + 1 + 0 + 1$	$\{1\}$
$U \oplus D_6 \oplus A_1^3$	$6 + 1 + 1 + 1$	$2 + 0 + 0 + 0 + 1$	$\{1\}$

$g = 3, k = 2, r = 10,\ a = 6,\ \delta = 0$ or 1

trivial lattice	$8 = \sum c_i$ + rank(MW)	$2 = k = \sum s_i$ + #sections	MW($\mathcal{E}$)	δ
$U \oplus D_4 \oplus A_1^4$	$4 + 1 + 1 + 1 + 1$	$1 + 0 + 0 + 0 + 1$	$\{1\}$	1
$U \oplus A_1^8$	$1+1+1+1+1+1+1+1$	$0+0+0+0+0+0+0+0+2$	$\mathbb{Z}/2\mathbb{Z}$	0

$g = 3, k = 1, r = 9,\ a = 7,\ (\delta = 1)$

trivial lattice	$8 = \sum c_i$ + rank(MW)	$1 = k = \sum s_i$ + #sections	MW($\mathcal{E}$)
$U \oplus A_1^7$	$1+1+1+1+1+1+1$	$0+0+0+0+0+0+0+1$	$\{1\}$

TABLE 5. Case $g = 4$

$g = 4,\ k = 5, r = 12, a = 2,\ (\delta = 1)$

trivial lattice	$10 = \sum c_i$+rank(MW)	$5 = k = \sum s_i$ + #sections	MW($\mathcal{E}$)
$U \oplus D_{10}$	10	$4 + 1$	$\{1\}$
$U \oplus E_8 \oplus A_1^2$	$8 + 1 + 1$	$4 + 0 + 0 + 1$	$\{1\}$

$g = 4, k = 4, r = 11,\ a = 3,\ (\delta = 1)$

trivial lattice	$9 = \sum c_i$ + rank(MW)	$4 = k = \sum s_i$ + #sections	MW($\mathcal{E}$)
$U \oplus E_7 \oplus A_1^2$	$7 + 1 + 1$	$3 + 0 + 0 + 1$	$\{1\}$
$U \oplus D_8 \oplus A_1$	$8 + 1$	$3 + 0 + 1$	$\{1\}$

$g = 4, k = 3, r = 10,\ a = 4,\ \delta = 0$ or 1

trivial lattice	$8 = \sum c_i$ + rank(MW)	$3 = k = \sum s_i$ + #sections	MW($\mathcal{E}$)	δ
$U \oplus D_6 \oplus A_1^2$	$6 + 1 + 1$	$2 + 0 + 0 + 1$	$\{1\}$	1
$U \oplus D_4 \oplus D_4$	$4 + 4$	$1 + 1 + 0 + 1$	$\{1\}$	0

$g = 4, k = 2, r = 9,\ a = 5,\ (\delta = 1)$

trivial lattice	$7 = \sum c_i + \mathrm{rank(MW)}$	$2 = k = \sum s_i + \#\mathrm{sections}$	$\mathrm{MW}(\mathcal{E})$
$U \oplus D_4 \oplus A_1^3$	$4+1+1+1$	$1+0+0+0+1$	$\{1\}$

$g = 4, k = 1, r = 8,\ a = 6,\ (\delta = 1)$

trivial lattice	$6 = \sum c_i + \mathrm{rank(MW)}$	$1 = k = \sum s_i + \#\mathrm{sections}$	$\mathrm{MW}(\mathcal{E})$
$U \oplus A_1^6$	$1+1+1+1+1+1$	$0+0+0+0+0+0+1$	$\{1\}$

Table 6. Case $g = 5$.

$g = 5, k = 5, r = 11, a = 1,\ (\delta = 1)$

trivial lattice	$9 = \sum c_i + \mathrm{rank(MW)}$	$5 = k = \sum s_i + \#\mathrm{sections}$	$\mathrm{MW}(\mathcal{E})$
$U \oplus E_8 \oplus A_1$	$8+1$	$4+0+1$	$\{1\}$

$g = 5, k = 4, r = 10,\ a = 2,\ \delta = 0$ or 1

trivial lattice	$8 = \sum c_i + \mathrm{rank(MW)}$	$4 = k = \sum s_i + \#\mathrm{sections}$	$\mathrm{MW}(\mathcal{E})$	δ
$U \oplus E_7 \oplus A_1$	$7+1$	$3+0+1$	$\{1\}$	1
$U \oplus D_8$	8	$3+1$	$\{1\}$	0

$g = 5, k = 3, r = 9,\ a = 3,\ (\delta = 1)$

trivial lattice	$7 = \sum c_i + \mathrm{rank(MW)}$	$3 = k = \sum s_i + \#\mathrm{sections}$	$\mathrm{MW}(\mathcal{E})$
$U \oplus D_6 \oplus A_1$	$6+1$	$2+0+1$	$\{1\}$

$g = 5, k = 2, r = 8,\ a = 4,\ (\delta = 1)$

trivial lattice	$6 = \sum c_i + \mathrm{rank(MW)}$	$2 = k = \sum s_i + \#\mathrm{sections}$	$\mathrm{MW}(\mathcal{E})$
$U \oplus D_4 \oplus A_1^2$	$4+1+1$	$1+0+0+1$	$\{1\}$

$g = 5, k = 1, r = 7,\ a = 5$

trivial lattice	$5 = \sum c_i + \mathrm{rank(MW)}$	$1 = k = \sum s_i + \#\mathrm{sections}$	$\mathrm{MW}(\mathcal{E})$
$U \oplus A_1^5$	$1+1+1+1+1$	$0+0+0+0+0+1$	$\{1\}$

Table 7. Case $g = 6$.

$g = 6, k = 5, r = 10, a = 0,\ (\delta = 0)$

trivial lattice	$8 = \sum c_i + \mathrm{rank(MW)}$	$5 = k = \sum s_i + \#\mathrm{sections}$	$\mathrm{MW}(\mathcal{E})$
$U \oplus E_8$	8	$4+1$	$\{1\}$

$g = 6, k = 4, r = 9,\ a = 1,\ (\delta = 1)$

trivial lattice	$7 = \sum c_i + \mathrm{rank(MW)}$	$4 = k = \sum s_i + \#\mathrm{sections}$	$\mathrm{MW}(\mathcal{E})$
$U \oplus E_7$	7	$3+1$	$\{1\}$

$g = 6, k = 3, r = 8,\ a = 2,\ (\delta = 1)$

trivial lattice	$6 = \sum c_i + \mathrm{rank(MW)}$	$3 = k = \sum s_i + \#\mathrm{sections}$	$\mathrm{MW}(\mathcal{E})$
$U \oplus D_6$	6	$2+1$	$\{1\}$

$g = 6, k = 2, r = 7,\ a = 3,\ (\delta = 1)$

trivial lattice	$5 = \sum c_i + \mathrm{rank(MW)}$	$2 = k = \sum s_i + \#\mathrm{sections}$	$\mathrm{MW}(\mathcal{E})$
$U \oplus D_4 \oplus A_1$	$4 + 1$	$1 + 0 + 1$	$\{1\}$

$g = 6, k = 1, r = 6,\ a = 4,\ (\delta = 1)$

trivial lattice	$4 = \sum c_i + \mathrm{rank(MW)}$	$1 = k = \sum s_i + \#\mathrm{sections}$	$\mathrm{MW}(\mathcal{E})$
$U \oplus A_1^4$	$1 + 1 + 1 + 1$	$0 + 0 + 0 + 0 + 1$	$\{1\}$

Table 8. Case $g = 7$.

$g = 7, k = 2, r = 6,\ a = 2,\ (\delta = 1)$

trivial lattice	$4 = \sum c_i + \mathrm{rank(MW)}$	$2 = k = \sum s_i + \#\mathrm{sections}$	$\mathrm{MW}(\mathcal{E})$
$U \oplus D_4$	4	$1 + 1$	$\{1\}$

$g = 7, k = 1, r = 5,\ a = 3,\ (\delta = 1)$

trivial lattice	$4 = \sum c_i + \mathrm{rank(MW)}$	$1 = k = \sum s_i + \#\mathrm{sections}$	$\mathrm{MW}(\mathcal{E})$
$U \oplus A_1^3$	$1 + 1 + 1$	$0 + 0 + 0 + 1$	$\{1\}$

Table 9. Case $g = 8$, 9, 10.

$g = 8 + b,\ k = 1,\ r = 4 - b,\ a = 2 - b$, with $0 \le b \le 2$ ($\delta = 1$ if $b \le 1$, $\delta = 0$ if $b = 2$)

trivial lattice	$2 - b = \sum c_i + \mathrm{rank(MW)}$	$1 = k = \sum s_i + \#\mathrm{sections}$	$\mathrm{MW}(\mathcal{E})$
$U \oplus A_1^{2-b}$	$2 - b$	$0 + 1$	$\{1\}$

Acknowledgments. The authors would like to acknowledge the organization of Women in Numbers Europe I and II, where they met and started their collaboration and the referee for a careful and throughout reading of the paper which resulted in many suggestions that improved it. The authors would like to thank F. Balestrieri, J. Desjardins, C. Maistret, I. Vogt while the second author also thanks J. Top and M. Schuett for many interesting discussions.

References

[1] Artebani, M. and Sarti, A.: Symmetries of order four on K3 surfaces. *J. Math. Soc. Japan* **67** (2015), no. 2, 503–533.

[2] Balestrieri, F., Desjardins, J., Garbagnati, A., Maistret, C., Salgado, C. and Vogt, I.: Elliptic fibrations on covers of the elliptic modular surface of level 5. In *Women in numbers Europe II*, 159–197. Assoc. Women Math. Ser. 11, Springer, Cham, 2018.

[3] Beauville, A.: Les familles stables de courbes elliptiques sur P^1 admettant quatre fibres singulières. *C. R. Acad. Sci. Paris Sér. I Math.* **294** (1982), no. 19, 657–660.

[4] Bertin, M. J. and Lecacheux, O.: Elliptic fibrations on the modular surface associated to $\Gamma_1(8)$. In *Arithmetic and geometry of K3 surfaces and Calabi–Yau threefolds,* 153–199. Fields Inst. Commun. 67, Springer, New York, 2013.

[5] Bertin, M. J., Garbagnati, A., Hortsch, R., Lecacheux, O., Mase, M., Salgado, C. and Whitcher, U.: Classification of elliptic fibrations of a singular K3 surface. In *Women in numbers Europe,* 1–28. Assoc. Women Math. Ser. 2, Springer, Cham, 2015.

[6] Comparin, P. and Garbagnati, A.: Van Geemen–Sarti involutions and elliptic fibrations on K3 surfaces double cover of $\mathbb{P}^2$. *J. Math. Soc. Japan* **66** (2014), no. 2, 479–522.

[7] Fusi, D.: Construction of linear pencils of cubic curves with Mordell–Weil rank six and seven. *Comm. Math. Univ. St. Pauli* **55** (2006), no. 2, 195–205.

[8] Garbagnati, A. and Salgado, C.: Linear systems on rational elliptic surfaces and elliptic fibrations on K3 surfaces. *J. Pure Appl. Algebra* **223** (2019), no. 1, 277–300.

[9] Garbagnati, A. and Sarti, A.: On symplectic and non-symplectic automorphisms on K3 surfaces. *Rev. Mat. Iberoam.* **29** (2013), no. 1, 135–162.

[10] van Geemen, B. and Top, J.: An isogeny of K3 surfaces. *Bull. London Math. Soc.* **38** (2006), no. 2, 209–223.

[11] Kloosterman, R.: Classification of all Jacobian elliptic fibrations on certain K3 surfaces. *J. Math. Soc. Japan* **58** (2006), no. 3, 665–680.

[12] Kumar, A.: Elliptic fibrations on a generic Jacobian Kummer surface. *J. Algebraic Geom.* **23** (2014), no. 44, 599–667.

[13] Kuwata, M. and Shioda, T.: Elliptic parameters and defining equations for elliptic fibrations on a Kummer surface. In*Algebraic geometry in East Asia (Hanoi 2005),* 177–215. Adv. Stud. Pure Math. 50, Math. Soc. Japan, Tokyo, 2008.

[14] Lecacheux, O.: Weierstrass equations for all elliptic fibrations on the modular K3 surface associated to $\Gamma_1(7)$. *Rocky Mountain J. Math.* **45** (2015), no. 5, 1481–1509.

[15] Nikulin, V. V.: Factor groups of groups of the automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections. *Soviet Math. Dokl.* **20** (1979), 1156–1158.

[16] Nishiyama, K.: The Jacobian fibrations on some K3 surfaces and their Mordell–Weil groups. *Japan. J. Math. (N.S.)* **22** (1996), no. 2, 293–347.

[17] Oguiso, K.: On Jacobian fibrations on the Kummer surfaces of the product of nonisogenous elliptic curves. *J. Math. Soc. Japan* **41** (1989), no. 4, 651–680.

[18] Pastro, V.: Construction of linear pencils of cubics with Mordell–Weil rank four. *Comment. Math. Univ. St. Pauli* **61** (2012), no. 1, 29–42.

[19] Persson, U.: Configurations of Kodaira fibers on rational elliptic surfaces. *Math. Z.* **205** (1990), no. 1, 1–47.

[20] Saint Donat, B.: Projective models of K3 surfaces. *Amer. J. Math.* **96** (1974), 602–639.

[21] Salgado, C.: Construction of linear pencils of cubics with Mordell–Weil rank five. *Comment. Math. Univ. St. Pauli* **58** (2010), no. 2, 95–104.

[22] Shimada, I.: On elliptic K3 surfaces. *Michigan Math. J.* **47** (2000), no. 3, 423–446. ArXiv version with the complete table: `math/0505140`, 2006.

[23] Schütt, M. and Shioda, T.: Elliptic surfaces. In *Algebraic geometry in East Asia (Seoul 2008),* 51–160. Adv. Stud. Pure Math. 60, Math. Soc. Japan, Tokyo, 2010.

[24] Top, J. and Yui, N.: Explicit equations of some modular elliptic surfaces. *Rocky Mountain J. Math.* **37** (2007) no. 2, 663–687.

[25] Utsumi, K.: Weierstrass equations for Jacobian fibrations on a certain K3 surface. *Hiroshima Math. J.* **42** (2012), no. 3, 355–383.

[26] Vinberg, É. B.: The two most algebraic K3 surfaces. *Math. Ann.* **265** (1983), no. 1, 1–21.

[27] Zhang, D. Q.: Quotients of K3 surfaces modulo involutions. *Japan J. Math. (N.S.)* **24** (1998), no. 2, 335–366.

Received July 3, 2018; revised January 15, 2019. Published online March 18, 2020.

Alice Garbagnati: Dipartimento di Matematica, Univ. Statale di Milano, Milan, Italy.
E-mail: alice.garbagnati@unimi.it

Cecília Salgado: Instituto de Matemática, Univ. Federal do Rio de Janeiro, Rio de Janeiro, Brazil.
E-mail: salgado@im.ufrj.br

The first author is partially supported by FIRB 2012 “Moduli Spaces and their Applications”. The second author is partially supported by Cnpq- Brasil grant nb: 305743/2014-7, FAPERJ E-26/203.205/2016, CAPES via the support of Proex to the graduate programme in Mathematics of UFRJ and by the Serrapilheira Institute (grant number Serra-1709-17759).