Frontiers of open quantum system dynamics

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Summary. We briefly examine recent developments in the field of open quantum system theory, devoted to the introduction of a satisfactory notion of memory for a quantum dynamics. In particular, we will consider a possible formalization of the notion of non-Markovian dynamics, as well as the construction of quantum evolution equations featuring a memory kernel. Connections will be draw to the corresponding notions in the framework of classical stochastic processes, thus pointing to the key differences between a quantum and classical formalization of the notion of memory effects.

1 Introduction

The theory of open quantum systems denotes the application of quantum mechanics to situations in which the dynamics of the quantum system of interest is influenced by other degrees of freedom, that we are neither interested nor capable to take into account in detail. While in principle any system has to be considered open, since a perfect shielding from the environment is never feasible, in simple situations isolation can be considered as a good approximation. In many realistic settings however, the effect of an external quantum environment on the system dynamics cannot be neglected. This is the case e.g. in many instances of quantum optics, condensed matter physics and quantum chemistry. For the case of an open system dynamics, many interesting physical effects and mathematical structures do appear [1]. From the physical viewpoint, with respect to a closed dynamics we are faced with new phenomena like dissipation and decoherence, which only have partial analog in the classical setting. Such phenomena play a crucial role in many relevant recent fields of research, such as quantum computation and quantum thermodynamics [2, 3, 4].

In this research field many problems are still open, which have important connections to mathematics. In this contribution we will try to highlight, in a concise way, some recent developments in the field of open quantum systems,

connected to some of the presently most active research lines, relevant for the mathematical formulation of the theory. After a brief description of the framework of open quantum systems in Sect. 2, we will address in Sect. 3 the delicate question of the definition of non-Markovian quantum processes. This implies introducing a notion of non-Markovian dynamics, which is quite different form the classical one, though the two can be naturally connected. In Sect. 4 we will point to the derivation of equations of motions allowing to introduce memory effects, focusing in particular on master equations with a memory kernel, for which again a natural connection to a class of non-Markovian classical processes can be considered.

2 Open quantum system dynamics

Let us first introduce some basic elements of open quantum system theory [1], which actually consists in considering the dynamics of a quantum system described on the Hilbert space $\mathcal{H}_S$ without assuming it to be isolated, so that it interacts with an external environment described on a Hilbert space $\mathcal{H}_E$ by menas of unitary operators $U(t)$ acting on $\mathcal{H}_S \otimes \mathcal{H}_E$. The fact that the system is not closed brings with itself two important new aspects. On the one hand, the reduced dynamics of the isolated system only is not described by a Liouville von-Neumann equation, and purity of the state is not preserved during the evolution, On the other hand, even a factorized system-environment state develops correlations, so that the latter play a major role in the time evolution. The tensor product structure of the underlying Hilbert space, on its turn, brings in two important aspects. On the one hand, states can exhibit correlations which are of non-classical nature, such as entanglement. On the other hand, the evolution of the reduced system as a function of time is described by a collection of transformations which have the property of being completely positive, a property strictly connected to the non-commutativity of the space of observables. Assuming that the state at the initial time is factorized

$$\rho_{SE}(0) = \rho_S(0) \otimes \rho_E, \tag{1}$$

we have that the reduced state of the system at a later time is given by

$$\rho_S(t) = \text{Tr}_E\{U(t)\rho_S(0) \otimes \rho_E U(t)^\dagger\}, \tag{2}$$

where Tr_E denotes the partial trace with respect to the environmental degrees of freedom. This state contains all the information relevant for the description of the dynamics of the system observables. In particular this transformation defines a linear map

$$\Phi(t, 0)[\rho_S(0)] \equiv \text{Tr}_E\{U(t)\rho_S(0) \otimes \rho_E U(t)^\dagger\}, \tag{3}$$

which considering an orthogonal resolution for the state of the environment $\rho_E = \sum_\xi \lambda_\xi P_{\varphi_\xi}$, and introducing an orthogonal basis $\{\varphi_\eta\}$ in $\mathcal{H}_E$, admits the following representation

$$\Phi(t,0)[\rho_S(0)] = \sum_{\xi,\eta} \lambda_\xi \langle \varphi_\eta | U(t)\varphi_\xi \rangle \rho_S(0)(\langle \varphi_\eta | U(t)\varphi_\xi \rangle)^\dagger$$

$$= \sum_{\xi,\eta} K_{\xi\eta}(t)\rho_S(0)K_{\xi\eta}^\dagger(t), \qquad (4)$$

where we have introduced so-called Kraus operators $K_{\xi\eta} = \sqrt{\lambda_\xi}\langle \varphi_\eta | U(t)\varphi_\xi \rangle$ acting on $\mathcal{H}_S$. This representation warrants complete positivity of the map. A map Φ defined on the space of trace class operators $\mathcal{T}(\mathcal{H}_S)$ is said to be completely positive if its extension $\Phi \otimes \mathbb{1}_n$ to $\mathcal{T}(\mathcal{H}_S \otimes \mathbb{C}^n)$ defined on operators in tensor product form as

$$\Phi \otimes \mathbb{1}_n[A \otimes B] = \Phi[A] \otimes B$$

is a positive map for any $n \in \mathbb{N}$. Otherwise stated, the trivial extension of a completely positive map acting on some system, to a larger set of degrees of freedom the system is not interacting with, remains positive. It can be shown that any such map admits the representation Eq. (4), and viceversa [5]. For an initial state in factorized form as in Eq. (1) it is thus possible to define a reduced dynamics, described by the time dependent collection of completely positive trace preserving maps $\Phi(t,0)$ given by Eq. (3), as shown in Fig. (2). Two natural questions appear at this stage. On the one hand, now that re-

$$\rho(0) = \rho_S(0) \otimes \rho_E \xrightarrow{\ U(t)\ } \rho(t) = e^{-\frac{i}{\hbar}Ht}(\rho_S(0) \otimes \rho_E)e^{+\frac{i}{\hbar}Ht}$$

$$Tr_E \downarrow \qquad\qquad\qquad\qquad \downarrow Tr_E$$

$$\rho_S(0) \qquad \xrightarrow{\ \Phi(t,0)\ } \qquad \rho_S(t) = \Phi(t,0)[\rho_S(0)]$$

Fig. 1. Commutative diagram showing the existence of a reduced dynamics for an initial system-environment state in factorized from. The reduced state of the system at time t can be equivalently obtained by taking the marginal with respect to the environmental degrees of freedom of the unitarily evolved total state, or by applying the completely positive trace preserving map $\Phi(t,0)$ to the initial state of the system.

versibility of the dynamics warranted by the unitary evolution has got lost, it is interesting to ascertain whether such maps do describe memory effects. On the other hand, one would like to know the possible expression of maps $\Phi(t,0)$ describing a well defined dynamics, as well as the general structure of evolution equations for the statistical operator admitting such collection of maps

as solution. These two aspects have been the object of extensive research, and some recent developments in this respect will be discussed Sects. 3 and 4.

3 Characterization of dynamics with memory

The existence of the reduced dynamics for an open quantum system implies that its time evolution can be described by evolution equations which, on top of a coherent quantum dynamics as can be obtained by a Liouville-von Neumann equation, do exhibit stochasticity. The stochastic contribution to the dynamics is due to the interaction with the unobserved quantum degrees of freedom of the environment. A quite natural question in this setting, in analogy with what happens for a classical stochastic dynamics, is therefore whether the obtained quantum dynamics can exhibit effects which can be reasonably termed memory effects. In the description of classical systems random features in the dynamics are described by the mathematically well established notion of stochastic process. The notion of lack of memory for a stochastic process is enforced by asking a suitable constraint on the conditional probabilities determining the process. Roughly speaking, a process is defined to be Markovian, that is without memory, if the only relevant conditioning of the probability densities for the outcomes of the considered stochastic process is with respect to the last ascertained value of the time dependent random variable considered, and not with respect to values at previous times. In such a way a notion of memory is naturally introduced (see e.g. [6] for a proper formalization of this notion). For a Markov process the notion of lack of memory is therefore naturally linked to the neglecting of knowledge of values taken by the random variable in the past.

In the quantum framework, random variables have to be described by self-adjoint operators acting in the Hilbert space $\mathcal{H}_S$ of the considered system. However, in order to obtain the probability distribution for the values taken by such random variables at a given time one has to perform a measurement. At variance with the classical case, knowledge of the value of the random variable will thus affect the subsequent evolution of the system in a non negligible way, depending on the way the measurement is performed. The external intervention necessary for the measurement thus influences the value of multitime probability densities, which do not admit an obvious definition as in the classical case. The definition of a quantum Markovian process along the lines of the classical viewpoint, pursued in the first systematic studies on the characterisation of open quantum system dynamics [7, 8, 9], thus encounters major difficulties.

More recently, different approaches have been considered, which tackle the issue considering features of the dynamics determined by quantities depending on a single point in time, rather than on multitime probability densities (see [10, 11, 12, 13] for reviews). In such a way one can overcome the difficulties related to the measurement problem and allow for a direct experimental

verification of the property of Markovianity. The connection between these approaches and the notion of classical Markovian process has been discussed in [14]. In this contribution we will concentrate on a approach which connects non-Markovianity to a reversible exchange of information between system and environment, started with the seminal paper [15]. Let us consider a reduced dynamics on $\mathcal{H}_S$ defined by a collection of completely positive trace preserving maps $\Phi(t,0)$ sending the initial system state to the state at time t

$$\Phi \equiv \{\Phi(t,0)\}_{t \in \mathbb{R}_+},$$

and suppose that the experimenter can prepare two distinct initial states of the system, say $\rho_S^1(0)$ and $\rho_S^2(0)$, with the same probability $p = 1/2$. If an observer has to guess which state has actually been prepared by performing a single measurement, as shown in [16] the maximal probability of success, obtained by performing an optimal measurement, is given by the expression

$$P(0) = \frac{1}{2}(1 + D(\rho_S^1(0), \rho_S^2(0))), \tag{5}$$

where

$$D(\rho, \sigma) = \frac{1}{2}\|\rho - \sigma\|_1$$

denotes the trace distance between two statistical operators $\rho, \sigma \in \mathcal{T}(\mathcal{H}_S)$, namely the normalized distance built by means of the trace norm $\|\cdot\|_1$. If the observer tries to distinguish the states at a later time t, after interaction of the system with the environment, the success probability is now given by

$$P(t) = \frac{1}{2}(1 + D(\rho_S^1(t), \rho_S^2(t))), \tag{6}$$

where $\rho_S^{1,2}(t) = \Phi(t,0)[\rho_S^{1,2}(0)]$ and due to a crucial property of the trace distance we have

$$P(t) \leqslant P(0).$$

Indeed the trace distance is a contraction under the action of an arbitrary positive, and therefore in particular completely positive, trace preserving transformation Λ [17]

$$D(\Lambda[\rho], \Lambda[\sigma]) \leqslant D(\rho, \sigma).$$

The effect of the interaction with the environment is thus a reduction of the capability to distinguish quantum states of the system by an observer performing measurements on the system only. For the case in which the dynamics is characterised by a semigroup composition law so that

$$\Phi(t,0) = e^{t\mathcal{L}}, \tag{7}$$

with $\mathcal{L}$ a suitable generator, corresponding to dynamics typically called quantum Markov processes, one further has

$$P(t) \leqslant P(s) \qquad \forall t \geqslant s,$$

so that there is a monotonic decrease in time of the distinguishability between system states. This feature is taken as the defining property of a quantum Markovian dynamics. Accordingly, a quantum dynamics described by a collection of completely positive trace preserving maps $\Phi(t,0)$ is said to be non-Markovian if there are revivals in time in the success probability $P(t)$ or equivalently in the trace distance $D(\rho_S^1(t), \rho_S^2(t))$, which contains the relevant part of the information, so that

$$\dot{D}(\rho_S^1(t), \rho_S^2(t)) \geqslant 0, \tag{8}$$

for at least a point in time and a couple of initial states, where we have denoted by $\dot{D}$ the time derivative of the trace distance between the evolved initial system states.

These revivals do generally depend on the choice of initial states, so that a suitable quantifier of non-Markovianity of the dynamics has been introduced according to the expression

$$\mathcal{N}(\Phi) = \max_{\rho_S^{1,2}(0)} \int_{\dot{D}>0} dt \, \dot{D}(\rho_S^1(t), \rho_S^2(t)). \tag{9}$$

It immediately appears that this definition of non-Markovian quantum dynamics only requires to observe the state of the system at different times and starting from different system initial conditions, rather than on multitime quantities, so that an experimental assessment of non-Markovianity can be obtained by means of a tomographic procedure [18].

To substantiate the interpretation of this notion of non-Markovianity as information back flow from the environment to the system, let us introduce the following quantities [2, 12]

$$\mathcal{I}_{\mathrm{int}}(t) = D(\rho_S^1(t), \rho_S^2(t)) \tag{10}$$

and

$$\mathcal{I}_{\mathrm{ext}}(t) = D(\rho_{SE}^1(t), \rho_{SE}^2(t)) - D(\rho_S^1(t), \rho_S^2(t)), \tag{11}$$

where $\mathcal{I}_{\mathrm{int}}(t)$ is used to quantify the internal information, that is the information accessible by performing measurements on the system only, while $\mathcal{I}_{\mathrm{ext}}(t)$ denotes the external information, which can only be obtained by performing measurements in the Hilbert space of both system and environment $\mathcal{H}_S \otimes \mathcal{H}_E$, minus the internal one. If the overall dynamics is unitary the sum of the two quantities $\mathcal{I}_{\mathrm{tot}}(t) = \mathcal{I}_{\mathrm{int}}(t) + \mathcal{I}_{\mathrm{ext}}(t)$ is conserved, so that in particular

$$\frac{\mathrm{d}}{\mathrm{dt}} D(\rho_S^1(t), \rho_S^2(t)) = \frac{\mathrm{d}}{\mathrm{dt}} \mathcal{I}_{\mathrm{int}}(t)$$
$$= -\frac{\mathrm{d}}{\mathrm{dt}} \mathcal{I}_{\mathrm{ext}}(t).$$

This equality shows that an increase in time of the trace distance corresponds to a decrease in the external information, which being overall conserved can only flow from the environment into the system. To understand in which sense information can be stored outside the system, namely it cannot be retrieved by performing measurements on the system only, it is enlightening to consider the following bound, first introduced in Ref. [19] in connection to detection of initial correlations

$$D(\rho_S^1(t), \rho_S^2(t)) - D(\rho_S^1(s), \rho_S^2(s)) \leqslant D(\rho_{SE}^1(s), \rho_S^1(s) \otimes \rho_E^1(s)) \qquad (12)$$
$$+ D(\rho_{SE}^2(s), \rho_S^2(s) \otimes \rho_E^2(s))$$
$$+ D(\rho_E^1(s), \rho_E^2(s)),$$

where it is assumed that $t \geqslant s$ and, at variance with [19], $\rho_{SE}^{1,2}(0) = \rho_S^{1;2}(0) \otimes \rho_E(0)$, where $\rho_S^{1;2}(0)$ and $\rho_E(0)$ are the marginal states obtained by taking the partial trace with respect to the degrees of freedom of environment and system respectively, so as to ensure the existence of a reduced dynamics. As discussed above the trace distance can be naturally understood as a quantifier of distinguishability among quantum states, so that in particular, if the two statistical operators are a state on a bipartite space and the product of its marginals as in the r.h.s. of Eq. (12), it provides a quantifier of correlations in the overall state. The l.h.s. corresponds to the change over time in trace distance, which can only be positive if at least one of the quantities at the r.h.s. is different from zero, that is after interacting for a time s either system and environment have become correlated or the environmental state has changed in different ways depending on the initial system state.

This definition of non-Markovianity of a quantum dynamics based on the notion of information back flow between system and environment is strictly connected to an alternative notion relying on a mathematical property of the collection of completely positive trace preserving maps describing the reduced dynamics. Indeed such a collection is called P-divisible if the following identity holds [20]

$$\Phi(t, 0) = \Phi(t, s)\Phi(s, 0) \qquad \forall t \geqslant s \geqslant 0, \qquad (13)$$

with $\Phi(t, s)$ positive maps for any $t \geqslant s \geqslant 0$, while it is called CP-divisible if the maps $\Phi(t, s)$ are in particular completely positive for any $t \geqslant s \geqslant 0$, as in the case e.g. of the quantum dynamical semigroup considered in Eq. (7). It immediately appears that both CP-divisible and P-divisible are Markovian according to the trace distance criterion defined above, while a monotonic decrease of the trace distance in general does not warrant neither kind of divisibility. The composition law Eq. (13) tells us that, in order to predict the

time evolution of the system forward in time, we only need to know the state at a given time, thus naturally inducing a formalization of lack of memory, and indeed its violation was proposed as a definition of non-Markovian dynamics in [21].

3.1 Generalized non-Markovianity measure

Recently different refinements of the definition and quantification of non-Markovianity as given by formulae Eq. (8) and Eq. (9) have been considered [22, 23, 24], importantly always supporting the seminal interpretation of non-Markovianity as information back flow from environment to system. An important and natural generalization, first suggested in [25], consists in considering the discrimination problem between quantum states, used to connect trace distance and distinguishability, in the more general setting in which the two known states $\rho_S^1(0)$ and $\rho_S^2(0)$ can be prepared with different weights, say p_1 and p_2. In this case the optimal strategy can be shown to lead to the following success probability

$$P(0) = \frac{1}{2}(1 + \Delta(\rho_S^1(0), \rho_S^2(0); p_1, p_2)), \tag{14}$$

where the expression $\Delta(\rho_S^1(0), \rho_S^2(0); p_1, p_2) = \|p_1\rho_S^1(0) - p_2\rho_S^2(0)\|$ is also known as norm of the Helstrom matrix. Non-Markovianity is then identified with a revival in time of the norm of the Helstrom matrix

$$\dot{\Delta}(\rho_S^1(t), \rho_S^2(t); p_1, p_2) \geqslant 0, \tag{15}$$

for at least a point in time, a couple of initial states and a choice of weights, which provide apriori information on the prepared state. Note that the class of processes which are non-Markovian is thus enlarged, including situations which where previously not encompassed [26, 23]. Accordingly, a generalized measure of non-Markovianity can be considered, given by the expression

$$\mathcal{N}(\Phi) = \max_{p_{1,2}, \rho_S^{1,2}(0)} \int_{\dot{\Delta}>0} dt \, \dot{\Delta}(\rho_S^1(t), \rho_S^2(t); p_1, p_2). \tag{16}$$

An important result of this generalization of the initial definition is the fact that it allows for a clearcut connection with the notion of divisibility considered in Eq. (13), which in its mathematical formulation does not immediately show a link to information flow, since the latter can only be formulated by introducing a quantifier of distinguishability among states. Indeed, thanks to a result by Kossakowski connecting the positivity property of a trace preserving map with its contractivity when acting on an arbitrary hermitian observable [27], monotonicity in time of the behavior of the norm of the Helstrom matrix $\Delta(\rho_S^1(t), \rho_S^2(t); p_1, p_2)$ can be shown to be equivalent to P-divisibility of the collections of time evolution maps in the sense of Eq. (13), provided the

time evolution map is invertible as a linear map on the space of operators. Most importantly, this extension is still compatible with the notion of information back flow as characterizing a non-Markovian dynamics. This fact can be shown considering suitable generalizations of the notion of internal and external information as considered in Eq. (10) and Eq. (11), as well as a generalisation of the bound Eq. (12), thus pointing to the general validity of the starting definition of quantum non-Markovianity [24, 28].

4 Non-Markovian evolution equations

As discussed in Sect. 2, considering an initially factorized state of system and environment is sufficient to warrant the existence of a reduced dynamics, which will depend both on the state of the environment and the unitary interaction, according to expression Eq. (2). However, in the general case the evaluation of the exact dynamics is utterly unfeasible, so that it is of utmost importance to have access to approximate methods. On the one hand, one can consider perturbation expansions; on the other hand, one can look for phenomenological expressions. In both cases one major difficulty is warranting that the obtained time evolutions indeed provide a well-defined dynamics, corresponding to a completely positive trace preserving transformation. In particular, the requirement of complete positivity, which warrants connection to an underlying microscopic dynamics, is difficult to be enforced and is typically lost at intermediate steps in a perturbative approach. A fundamental result has been obtained for the situation in which the time evolution, instead of obeying a group evolution law as in the case of a reversible unitary dynamics, can be described by a semigroup, thus introducing a preferred direction in time. In this case the collection of maps is called quantum dynamical semigroup and is determined by a generator $\mathcal{L}$ according to Eq. (7). A fundamental theorem of open quantum system theory [14, 15] states that this generator has to be in the so-called Gorini-Kossakowksi-Sudarshan-Lindblad form

$$\mathcal{L}[\rho_S] = -\frac{i}{\hbar}[H, \rho_S] + \sum_k \gamma_k \left[L_k \rho_S L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho_S\} \right], \qquad (17)$$

with H a self-adjoint operator corresponding to an effective Hamiltonian, γ_k positive rates and L_k system operators also called Lindblad operators. It provides a generalization of the Liouville von-Neumann equation to include both decoherence and dissipative effects. Solutions of the time evolution equation

$$\frac{\mathrm{d}}{\mathrm{dt}}\rho_S(t) = \mathcal{L}[\rho_S(t)] \qquad (18)$$

together with a suitable initial condition $\rho_S(0)$ do define a collection of completely positive trace preserving maps obeying a semigroup composition law. As discussed in Sect. 3, the obtained dynamics is Markovian and provides

the quantum analog of a classical semigroup evolution. To describe memory effects more general dynamics have to be considered. To this aim one can either consider time dependent generalizations of the generator considered in Eq. (17), or move to evolution equations explicitly featuring a memory kernel. In both cases one has to ensure that the solutions of such equations do provide a collection of time dependent completely positive trace preserving maps, thus describing a well defined dynamics. In the case of so-called time local evolution equations, one has to replace rates and operators appearing in Eq. (17) by time dependent quantities, looking for conditions warranting complete positivity. Considering master equations in integrodifferential form

$$\frac{\mathrm{d}}{\mathrm{dt}}\rho_S(t) = \int_0^t \mathrm{d}\tau \mathcal{K}(t-\tau)[\rho_S(\tau)] \tag{19}$$

the corresponding task is to envisage conditions on the operator kernel $\mathcal{K}(t)$ warranting preservation of positivity and trace of the solutions of the integrodifferential equation. In both cases the most general solution to the problem is not known, even not heuristically, while partial results have been recently obtained [29, 30, 31, 32, 33, 34, 35, 34, 36]. In particular we will consider how to obtain well-defined quantum memory kernels $\mathcal{K}(t)$. While quantum dynamical semigroups can be seen as the quantum counterpart of classical Markov semigroups, the class of considered memory kernels can be taken as the quantum analogue of a class of non-Markovian processes known as semi-Markov process [37].

We consider as starting point an expression for the exact solution of Eq. (17), which can be written as

$$\rho_S(t) = \Phi(t,0)[\rho_S(0)] \tag{20}$$
$$= \mathcal{R}(t)[\rho_S(0)]$$
$$+ \sum_{k=1}^{\infty} \int_0^t \mathrm{dt}_k \dots \int_0^{t_2} \mathrm{dt}_1 \, \mathcal{R}(t-t_k)\mathcal{J} \dots \mathcal{R}(t_2-t_1)\mathcal{J}\mathcal{R}(t_1)[\rho_S(0)],$$

where we have introduced the contraction semigroup

$$\mathcal{R}(t)[\rho] = \mathrm{e}^{-\frac{i}{\hbar}Ht - \frac{1}{2}\sum_k \gamma_k L_k^\dagger L_k t} \rho \, \mathrm{e}^{+\frac{i}{\hbar}Ht - \frac{1}{2}\sum_k \gamma_k L_k^\dagger L_k t}$$

and the completely positive map

$$\mathcal{J}[\rho] = \sum_k \gamma_k L_k \rho L_k^\dagger.$$

As a result the solution is expressed as a sum of contributions characterised by a given number of insertions of the completely positive map $\mathcal{J}$ with an intermediate trace decreasing evolution in between. Complete positivity of the overall evolution is warranted by the fact that we are considering sum and composition of completely positive maps, which form a convex cone. The

solution is expressed in a space of "trajectories" determined by the number of jumps or insertions of the map $\mathcal{J}$ and the points in time at which these jumps happen [38, 39]. Both maps $\mathcal{R}(t)$ and $\mathcal{J}$ are determined by the rates γ_k and the Lindblad operators L_k. To consider a more general situation one can define a collection of linear maps in analogy with Eq. (20), introducing the replacement of jump operator and contraction semigroup by means of an arbitrary completely positive trace preserving transformation $\mathcal{E}$ and a collection of time dependent completely positive trace preserving maps $\mathcal{F}(t)$, according to the scheme

$$\rho_S(t) = g(t)\mathcal{F}(t)[\rho_S(0)] \tag{21}$$
$$+ \sum_{k=1}^{\infty} \int_0^t \mathrm{dt}_k \ldots \int_0^{t_2} \mathrm{dt}_1 \, f(t - t_k)\mathcal{F}(t - t_k)\mathcal{E} \ldots$$
$$\times \ldots f(t_2 - t_1)\mathcal{F}(t_2 - t_1)\mathcal{E}g(t_1)\mathcal{F}(t_1)[\rho_S(0)].$$

In the representation Eq. (21) we have further inserted the functions $f(t)$ and $g(t)$. The function $f(t)$ has to be positive and normalized to one over the interval $[0, \infty)$, so as to be interpreted as a waiting time distribution. Accordingly, the function $g(t)$ is determined by $\dot{g}(t) = -f(t)$, together with $g(0) = 1$, so that it can be interpreted as the associated survival probability. It can be easily seen that these properties are sufficient to identify the linear assignment $\rho_S(0) \to \rho_S(t)$ obtained through Eq. (21) as a collection of completely positive trace preserving maps. These evolutions correspond to a situation in which in between the evolution given by the maps $\mathcal{F}(t)$, the system undergoes a transformation described by the completely positive trace preserving map $\mathcal{J}$. It is however not obvious the existence of a closed evolution equation for the statistical operator of the system $\rho_S(t)$, so as to connect the transformations to a continuous dynamics. To this aim one considers the expression of the Laplace transform of Eq. (21), which thanks to the presence of convolutions takes the simple form

$$\widehat{\rho}_S(u) = (\mathbb{1} - \widehat{f\mathcal{F}}(u)\mathcal{E})^{-1}\widehat{g\mathcal{F}}(u)\rho_S(0),$$

where the hat denotes the Laplace transform, so that by a suitable rearrangement one has

$$u\hat{\rho}_S(u) - \rho_S(0) = \left[\frac{1}{\widehat{g\mathcal{F}}(u)}\widehat{f\mathcal{F}}(u)\mathcal{E} - \left(\frac{1}{\widehat{g\mathcal{F}}(u)} - u \right) \right] \hat{\rho}_S(u), \tag{22}$$

allowing to identify the memory kernel $\mathcal{K}(t)$ in Eq. (19) with the inverse Laplace transform of the operator

$$\widehat{\mathcal{K}}(u) = \frac{1}{\widehat{g\mathcal{F}}(u)}\widehat{f\mathcal{F}}(u)\mathcal{E} - \left(\frac{1}{\widehat{g\mathcal{F}}(u)} - u \right), \tag{23}$$

thus showing in particular that indeed the transformation Eq. (21) describes a closed dynamics. Actually it can be shown that the kernel Eq. (23) despite its complex expression does have a simple and natural interpretation and allows for a connection with a class of non-Markovian processes known as semi-Markov [40, 41]. These generally non-Markovian classical processes describe a dynamics in a discrete state space, in which jumps from site m to site n take place with probability given by the elements of a stochastic matrix π_{nm}, at times distributed according to the waiting time distribution $f_n(t)$. For these processes one can introduce a generalized master equation obeyed by the one-point probability density $P_n(t)$ given by [42, 37]

$$\frac{d}{dt}P_n(t) = \int_0^t d\tau \sum_m [W_{nm}(\tau)P_m(t-\tau) - W_{mn}(\tau)P_n(t-\tau)],$$

whose expression in Laplace transform reads

$$u\hat{P}_n(u) - P_n(0) = \sum_m \left[\pi_{nm}\frac{\hat{f}_m(u)}{\hat{g}_m(u)} - \delta_{nm}\left(\frac{1}{\hat{g}_m(u)} - u\right)\right]\hat{P}_m(u). \quad (24)$$

A natural correspondence can be drawn between Eq. (24) and Eq. (22). The stochastic matrix π_{nm} is replaced by the completely positive trace preserving map $\mathcal{E}$, while the collection of waiting time distributions $f_n(t)$ goes over to $f(t)\mathcal{F}(t)$, product of waiting time distribution and completely positive trace preserving maps. Classical functions are therefore now replaced by operators. The classical dynamics corresponding to jumps between sites with probabilities determined by a given stochastic transition matrix and at times dictated by given waiting time distributions, is replaced by a piecewise quantum dynamics in the space of statistical operators. In this quantum dynamics transformations described by a completely positive trace preserving map $\mathcal{E}$, at times described by a fixed waiting time distribution, are interspersed with a continuous time evolution described by the collection of completely positive trace preserving maps $\mathcal{F}(t)$. It immediately appears that in the correspondence from Eq. (24) to Eq. (22) an important and typically quantum feature appears, namely the relevance of operator ordering. Indeed Eq. (22) can have different quantum counterparts, and another operator ordering leads to an alternative expression for the kernel

$$\widehat{\mathcal{K}}(u) = \mathcal{E}\widehat{f\mathcal{F}}(u)\frac{1}{\widehat{g\mathcal{F}}(u)} - \left(\frac{1}{\widehat{g\mathcal{F}}(u)} - u\right), \quad (25)$$

which substituted in Eq. (19) still leads to a well-defined dynamics. Indeed it turns out that the two combinations describe different microscopic modelling of a quantum piecewise dynamics. The microscopic dynamics formalised by Eq. (23) corresponds to the physics of the micromaser [43, 44, 45], while the kernel Eq. (25) naturally appears in so-called collision models [46, 47].

5 Conclusions and outlook

We have briefly exposed recent work within the framework of open quantum system theory, aiming at the definition and quantification of the so-called non-Markovianity, to be understood as the capability of a quantum dynamics to feature memory effects. In particular, we have pointed to a notion of non-Markovian dynamics connected to an information exchange between the considered system and the surrounding environment, whose generalization can be naturally connected to a notion of divisibility of quantum maps. We have further considered a possible extension of a known class of master equations describing a completely positive trace preserving dynamics to include memory effects by means of the introduction of a memory kernel.

Great efforts are presently being put in the endeavour to understand the relevance of the proposed notions of non-Markovian quantum dynamics for the description of relevant physical systems (see in this respect the recent reviews [10, 11, 12, 13]). A critical and important open issue is, in particular, whether it captures distinctive features of the dynamics, or if a non-Markovian evolution brings with itself advantages in performing relevant tasks, e.g. in quantum information or quantum thermodynamics.

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