SIGN-CHANGING TOWER OF BUBBLES FOR THE BREZIS-NIRENBERG PROBLEM

ALESSANDRO IACOPETTI AND GIUSI VAIRA

ABSTRACT. In this paper, we prove that the Brezis-Nirenberg problem

$$-\Delta u = |u|^{p-1}u + \epsilon u \qquad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is a symmetric bounded smooth domain in $\mathbb{R}^N$, $N \geq 7$ and $p = \frac{N+2}{N-2}$, has a solution with the shape of a tower of two bubbles with alternate signs, centered at the center of symmetry of the domain, for all $\epsilon > 0$ sufficiently small.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

In this paper we are interested in the construction of solutions to the following problem

$$\begin{cases} -\Delta u = |u|^{p-1}u + \epsilon u & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ with $N \geq 7$, ϵ is supposed to be small and positive while $p+1 = \frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding of $H^1_0(\Omega)$ into $L^{p+1}(\Omega)$.

The pioneering paper on equation (1.1) was written by Brezis and Nirenberg [9] in 1983 where the authors showed that for $N \geq 4$ and $\epsilon \in (0, \lambda_1)$, the problem (1.1) has at least one positive solution where λ_1 denotes the first eigenvalue of $-\Delta$ on Ω.
In the case $N = 3$, a similar result was proved in [9] but only for $\epsilon \in (\lambda^*, \lambda_1)$ with $\lambda^* = \lambda^*(\Omega) > 0$. Moreover by using a version of the Pohozaev Identity the authors showed that $\lambda^*(\Omega) = \frac{1}{4}\lambda_1$ if Ω is a ball and that no positive solutions exist for $\epsilon \in (0, \frac{1}{4}\lambda_1)$.
Note that, by using again Pohozaev Identity, it is easy to check that problem (1.1) has no non-trivial solutions when $\epsilon \leq 0$ and Ω is star-shaped.
Since then, there has been a considerable number of papers on problem (1.1).
We briefly recall some of the main ones.
Han, in [22], proved that the solution found by Brezis and Nirenberg blows-up at a critical point of the Robin's function as ϵ goes to zero. Conversely, Rey in [30] and in [31] proved that any C^1- stable critical point of the Robin's function generates a family of positive solutions which blows-up at this point as ϵ goes to zero.
After the work of Brezis and Nirenberg, Capozzi, Fortunato and Palmieri [12] showed that for $N = 4$, $\epsilon > 0$ and $\epsilon \notin \sigma(-\Delta)$ (the spectrum of $-\Delta$) problem (1.1) has a nontrivial solution. The same holds if $N \geq 5$ for all $\epsilon > 0$ (see also [21]).

The first multiplicity result was obtained by Cerami, Fortunato and Struwe in [14], in which they proved that the number of nontrivial solutions of (1.1), for $N \geq 3$, is bounded below by the number of eigenvalues of $(-\Delta, \Omega)$ belonging to $(\epsilon, \epsilon + S|\Omega|^{-2/N})$, where S is the best constant

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for the Sobolev embedding $D^{1,2}(\mathbb{R}^N)$ into $L^{p+1}(\mathbb{R}^N)$ and $|\Omega|$ is the Lebesgue measure of Ω. Moreover, if $N \geq 4$, then for any $\epsilon > 0$ and for a suitable class of symmetric domain Ω, problem (1.1) has infinitely many solutions of arbitrarily large energy (see Fortunato and Jannelli [20]). If $N \geq 7$ and Ω is a ball, then for each $\epsilon > 0$, problem (1.1) has infinitely many sign-changing radial solutions (see Solimini [33]).
In the papers [20, 33], the radial symmetry of the domain plays an essential role, therefore their methods do not work for general domains.
Concerning sign-changing solutions, Cerami, Solimini and Struwe showed in [15] that if $N \geq 6$ and $\epsilon \in (0, \lambda_1)$, problem (1.1) has a pair of least energy sign-changing solution. In the same paper the authors studied the multiplicity of nodal solutions proving the existence of infinitely many radial solutions when Ω is a ball centered at the origin.
On the other side, for $3 \leq N \leq 6$ and when Ω is a ball, it can be proved that there is a $\lambda^* > 0$ such that (1.1) has no sign-changing radial solutions for $\epsilon \in (0, \lambda^*)$ (see Atkinson, Brezis and Peletier [2]).
Moreover, Devillanova and Solimini in [18] showed that, if $N \geq 7$ and Ω is an open regular subset of $\mathbb{R}^N$, problem (1.1) has infinitely many solutions for each $\epsilon > 0$.
For low dimensions, namely $N = 4, 5, 6$ and in an open regular subset of $\mathbb{R}^N$, in [19], Devillanova and Solimini proved the existence of at least $N + 1$ pairs of solutions provided ϵ is small enough. In [16], Clapp and Weth extended this last result to all $\epsilon > 0$.
Neither in [18, 19] nor in [16] there is information on the kind of sign-changing solutions obtained. Recently, in [32], Schechter and Wenming Zou showed that in any bounded and smooth domain, for $N \geq 7$ and for each fixed $\epsilon > 0$, problem (1.1) has infinitely many sign changing solutions.

Concerning the profile of sign-changing solutions some results have been obtained in [5], [6] for low energy solutions, namely solutions u_ϵ such that $\int_\Omega |\nabla u_\epsilon|^2 \, dx \to 2S^{\frac{N}{2}}$, as $\epsilon \to 0$, S being the Sobolev constant for the embedding of $H^1_0(\Omega)$ into $L^{p+1}(\Omega)$. More precisely in [5] it is proved that for $N = 3$ these solutions concentrate and blow-up in two different points of Ω, as $\epsilon \to 0$, and have the asymptotic profile of two separate bubbles. A similar result is proved in [6] for $N \geq 4$ but assuming that the blow-up rate of the positive and negative part of u_ϵ is the same. Existence of nodal solutions with two nodal regions concentrating in two different points of the domain Ω as $\epsilon \to 0$ has been obtained in [13], [24] and [4]. So none of these solutions look like tower of bubbles, i.e. superposition of two bubbles with opposite sign concentrating at the same point, as $\epsilon \to 0$. Such a type of solutions is shown to exist for other semilinear problems like the almost critical Lane-Emden problem (see [7], [28], [27]) but not, to our knowledge, for the Brezis-Nirenberg problem with the exception of the case of the ball. If Ω is a ball, and $N \geq 7$, in a recent paper [23] the asymptotic behaviour as $\epsilon \to 0$ of the least energy nodal radial solution v_ϵ is analysed and among other things, it is shown that the positive and negative part of v_ϵ concentrate at the origin. Moreover they have the asymptotic profile of a positive and negative solution of the critical problem in $\mathbb{R}^N$ and the concentration speeds are different.
Hence [23] provides the first example of bubble of towers for the Brezis-Nirenberg problem.
Then the natural question is whether these kind of solutions exist in bounded domains other than the ball.
In the present paper we answer positively this question constructing a sign-changing solution of (1.1) in any bounded domain symmetric with respect to N orthogonal axis.

We next state our result.

Theorem 1.1. *Let $N \geq 7$ and let Ω be a smooth bounded domain in $\mathbb{R}^N$ such that Ω is symmetric with respect to $x_1, \ldots, x_N$ and $0 \in \Omega$. There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ there exist positive numbers $d_{j\epsilon}$, $j = 1, 2$ and a solution u_ϵ of problem (1.1) of the form*

$$u_\epsilon(x) = \alpha_N \left[\left(\frac{d_{1\epsilon} \epsilon^{\frac{1}{N-4}}}{d_{1\epsilon}^2 \epsilon^{\frac{2}{N-4}} + |x|^2} \right)^{\frac{N-2}{2}} - \left(\frac{d_{2\epsilon} \epsilon^{\frac{3N-10}{(N-4)(N-6)}}}{d_{2\epsilon}^2 \epsilon^{2\frac{3N-10}{(N-4)(N-6)}} + |x|^2} \right)^{\frac{N-2}{2}} \right] + \Phi_\epsilon, \tag{1.2}$$

where $\alpha_N := [N(N-2)]^{\frac{N-2}{4}}$, $d_{j\epsilon} \to \bar{d}_j > 0$, as $\epsilon \to 0$, $\Phi_\epsilon \to 0$ in $H^1(\Omega)$, as $\epsilon \to 0$. Moreover u_ϵ is even with respect to the variables $x_1, \ldots, x_N$.

We remark that the assumption $N \geq 7$ in our proof is crucial. We believe that it is possible to extend our result to a general domain Ω with some suitable modifications.

In the case the remainder term converges to zero also in $L^\infty_{loc}(\Omega)$, then, the asymptotic expansion and some energy estimates derived in the course of the proof allow to draw interesting consequences concerning the number and shape of the nodal domains of the solution u_ϵ.

Theorem 1.2. *Let $N \geq 7$ and assume that the remainder term Φ_ϵ, appearing in Theorem 1.1, is such that $\Phi_\epsilon \to 0$ uniformly in compact subsets of Ω. Then, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, the solution u_ϵ constructed in Theorem 1.1 has precisely two nodal domains Ω^1_ϵ, Ω^2_ϵ such that Ω^1_ϵ contains the sphere $\mathcal{S}^1_\epsilon := \left\{ x \in \mathbb{R}^N \ : \ |x| = \epsilon^{\frac{1}{N-4}} \right\}$, Ω^2_ϵ contains the sphere $\mathcal{S}^2_\epsilon := \left\{ x \in \mathbb{R}^N \quad |x| = \epsilon^{\frac{3N-10}{(N-4)(N-6)}} \right\}$ and $u_\epsilon > 0$ on Ω^1_ϵ and $u_\epsilon < 0$ on Ω^2_ϵ.*
Consequently, $0 \in \Omega^2_\epsilon$ and Ω^1_ϵ is the only nodal domain of u_ϵ which touches $\partial\Omega$.

Remark 1.3. *Under the assumptions of Theorem 1.2 it follows that the sign-changing tower of bubble u_ϵ constructed in Theorem 1.1 has two nodal domains and its nodal set does not touch $\partial\Omega$. By this we mean that, denoting by*

$$Z_\epsilon := \{x \in \Omega \ : \ u_\epsilon(x) = 0\}$$

the nodal set of u_ϵ then $\overline{Z}_\epsilon \cap \partial\Omega = \emptyset$.

The proof of Theorem 1.1 is based on the Lyapunov-Schmidt reduction.
To describe the procedure and explain the difficulties which arise when looking for bubble towers of the Brezis-Nirenberg problem, we introduce the functions

$$\mathcal{U}_\delta(x) = \alpha_N \frac{\delta^{\frac{N-2}{2}}}{(\delta^2 + |x|^2)^{\frac{N-2}{2}}}, \qquad \delta > 0 \tag{1.3}$$

with $\alpha_N := [N(N-2)]^{\frac{N-2}{4}}$. Is is well known (see [3], [11], [34]) that (1.3) are the only radial solutions of the equation

$$-\Delta u = u^p \qquad \text{in } \mathbb{R}^N. \tag{1.4}$$

We define φ_δ to be the unique solution to the problem

$$\begin{cases} \Delta \varphi_\delta = 0 & \text{in } \Omega \\ \varphi_\delta = \mathcal{U}_\delta & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

and let

$$\mathcal{P}\mathcal{U}_\delta := \mathcal{U}_\delta - \varphi_\delta \tag{1.6}$$

be the projection of $\mathcal{U}_\delta$ onto $H^1_0(\Omega)$, i.e.

$$\begin{cases} -\Delta \mathcal{P}\mathcal{U}_\delta = \mathcal{U}_\delta^p & \text{in } \Omega \\ \mathcal{P}\mathcal{U}_\delta = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Finally, let $G(x,y)$ be the Green's function associated to $-\Delta$ with Dirichlet boundary conditions and $H(x,y)$ be its regular part, namely

$$H(x,y)=\frac{1}{|x-y|^{N-2}}-\frac{1}{\gamma_N}G(x,y),\qquad \forall\, x,y\in\Omega,\qquad \text{with } \gamma_N=\frac{1}{N(N-2)\omega_N},$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$.
The function $\tau(x):=H(x,x)$, $x\in\Omega$ is called *Robin's function.*

It is well-known that the following expansions holds (see [30])

$$\varphi_\delta(x)=\alpha_N\delta^{\frac{N-2}{2}}H(0,x)+O(\delta^{\frac{N+2}{2}})\qquad \text{as } \delta\to 0. \tag{1.8}$$

Moreover, from elliptic estimates it follows that

$$0<\varphi_\delta(x)<c\delta^{\frac{N-2}{2}},\qquad \text{in } \Omega, \tag{1.9}$$

$$|\varphi_\delta|_{q,\Omega}\le C\delta^{\frac{N-2}{2}},\qquad q\in\left(\frac{p+1}{2},p+1\right] \tag{1.10}$$

and

$$|\nabla\varphi_\delta|_{2,\Omega}\le C_1\delta^{\frac{N-2}{2}} \tag{1.11}$$

see for instance [30], [35] and references therein.

We look for an approximate solution to problem (1.1) which is a superposition of two standard bubbles with two different scaling parameters, namely we take $\delta_1>\delta_2$ and we look for a solution to (1.1) of the form

$$u_\epsilon(x)=\mathcal{P}\mathcal{U}_{\delta_1}-\mathcal{P}\mathcal{U}_{\delta_2}+\Phi_\epsilon(x) \tag{1.12}$$

where the remainder term Φ_ϵ is a small function which is even with respect to the variables $x_1,\dots,x_N$.

The Lyapunov-Schmidt reduction allows us to reduce the problem of finding blowing-up solutions to (1.1) to the problem of finding critical points of a functional (the reduced energy) which depends only on the concentration parameters.
As announced before in our case some difficulties arise which need some modification of the standard procedure to be overcome.
Indeed, first we remark that the solutions of problem (1.1) are the critical points of the functional $J_\epsilon: H^1_0(\Omega)\to\mathbb{R}$ defined as

$$J_\epsilon(u)=\frac12\int_\Omega|\nabla u|^2\,dx-\frac{1}{p+1}\int_\Omega|u|^{p+1}\,dx-\frac{\epsilon}{2}\int_\Omega u^2\,dx,\qquad u\in H^1_0(\Omega). \tag{1.13}$$

If we apply directly the reduction method looking for a solution of the form (1.12) we get that the remainder term is such that

$$\|\Phi_\epsilon\|=O\left(\epsilon^{\frac{N-2}{N-4}+\sigma}\right)\qquad \sigma>0$$

where $\|\cdot\|$ denotes the $H^1_0(\Omega)$-norm, and that the reduced energy

$$\textit{Reduced Energy}\sim J_\epsilon(\mathcal{P}\mathcal{U}_{\delta_1}-\mathcal{P}\mathcal{U}_{\delta_2})=C+C_1\tau(0)\delta_1^{N-2}-C_2\epsilon\delta_1^2+C_3\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}+H.O.T.$$

where C, C_i are some known positive constants.
Since δ_1, δ_2 are proper power of ϵ of the form $\delta_j = \epsilon^{\gamma_j} d_j$, $d_j > 0$, after some easy computations, in order to find a critical point of the reduced energy, we get that

$$\textit{Reduced Energy} \sim C + \epsilon^{\frac{N-2}{N-4}} \left[C_1 \tau(0) d_1^{N-2} - C_2 d_1^2 + C_3 \left(\frac{d_2}{d_1} \right)^{\frac{N-2}{2}} \right] + o(\epsilon^{\frac{N-2}{N-4}})$$

with

$$\gamma_1 := \frac{1}{N-4}; \qquad \gamma_2 := \frac{3}{N-4}.$$

However the function

$$\Psi(d_1, d_2) = C_1 \tau(0) d_1^{N-2} - C_2 d_1^2 + C_3 \left(\frac{d_2}{d_1} \right)^{\frac{N-2}{2}}$$

has a critical point in d_1 but not in d_2 and hence in this way we cannot find a solution of our problem.

Hence we use a new idea. We split the remainder term Φ_ϵ in two parts:

$$\Phi_\epsilon(x) = \phi_{1,\epsilon}(x) + \phi_{2,\epsilon}(x)$$

such that

$$\|\phi_{2,\epsilon}\| = o(\|\phi_{1,\epsilon}\|), \quad \text{as } \epsilon \to 0.$$

Usually, the remainder term Φ_ϵ, solution of the auxiliary equation, is found with a fixed point argument. Here we have to use the Contraction Mapping Theorem twice, since we split the auxiliary equation in a system of two equations. The first one depends only on ϕ_1 while the second one depends on both ϕ_1, ϕ_2. So we solve the first equation in ϕ_1 and then the second one finding ϕ_2. Then we obtain the remainder term Φ_ϵ which consists of two terms of different orders. Then we study the finite-dimensional problem, namely the reduced energy that consists of two functions of different orders. The lower term depends only on d_1 while the term of higher order depends on d_1, d_2. At the end we look for a critical point of this new type of reduced energy. We believe that our strategy can be used also in other contexts.

The outline of the paper is the following: in Section 2 we explain the setting of the problem. In Section 3 we look for the remainder term Φ_ϵ in a suitable space. In Section 4 we study the reduced energy and finally Theorem 1.1 and Theorem 1.2 are proved in Section 5.

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2. Setting of the problem

In what follows we let

$$(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad \|u\| := \left(\int_\Omega |\nabla u|^2 \, dx \right)^{\frac{1}{2}}$$

as the inner product in $H_0^1(\Omega)$ and its corresponding norm while we denote by $(\cdot, \cdot)_{H^1(\mathbb{R}^N)}$ and by $\| \cdot \|_{H^1(\mathbb{R}^N)}$ the scalar product and the standard norm in $H^1(\mathbb{R}^N)$. Moreover we denote by

$$|u|_r := \left(\int_\Omega |u|^r \, dx \right)^{\frac{1}{r}}$$

the $L^r(\Omega)$-standard norm for any $r \in [1, +\infty)$. When $A \neq \Omega$ is any Lebesgue measurable subset of $\mathbb{R}^N$, or, when $A = \Omega$ and we need to specify the domain of integration, we will use the alternative notations $\|u\|_A$, $|u|_{r,A}$.

From now on we assume that Ω is a bounded open set with smooth boundary of $\mathbb{R}^N$, symmetric with respect to $x_1, \dots, x_N$ and which contains the origin. Moreover we assume that $N \geq 7$.

We define then

$$H_{sim} := \left\{ u \in H_0^1(\Omega) \ : \ u \text{ is symmetric with respect to each variable } x_k, \ k = 1, \dots, N \right\},$$

and for $q \in [1, +\infty)$

$$L^q_{sim} := \{ u \in L^q(\Omega) \ : \ u \text{ is symmetric with respect to each variable } x_k, \ k = 1, \dots, N \}.$$

Let $i^* : L_{sim}^{\frac{2N}{N+2}} \to H_{sim}$ be the adjoint operator of the embedding $i : H_{sim}(\Omega) \to L_{sim}^{\frac{2N}{N-2}}$, namely if $v \in L_{sim}^{\frac{2N}{N+2}}$ then $u = i^*(v)$ in H_{sim} is the unique solution of the equation

$$-\Delta u = v \qquad \text{in } \Omega \qquad u = 0 \qquad \text{on } \partial\Omega.$$

By the continuity of i it follows that

$$\|i^*(v)\| \leq C|v|_{\frac{2N}{N+2}} \qquad \forall v \in L_{sim}^{\frac{2N}{N+2}} \tag{2.1}$$

for some positive constant C which depends only on N.
Hence we can rewrite problem (1.1) in the following way

$$\begin{cases} u = i^* \left[f(u) + \epsilon u \right] \\ u \in H_{sim} \end{cases} \tag{2.2}$$

where $f(s) = |s|^{p-1}s$, $p = \frac{N+2}{N-2}$.

We next describe the shape of the solution we are looking for.

Let $\delta_j = \delta_j(\epsilon)$, for $j = 1, 2$ be positive parameters defined as proper powers of ϵ, multiplied by a suitable positive constant to be determined later, namely

$$\delta_j = \epsilon^{\alpha_j} d_j \qquad \text{with } \ d_j > 0 \tag{2.3}$$

and $\alpha_1 := \frac{1}{N-4}$; $\alpha_2 := \frac{3N-10}{(N-4)(N-6)}$.
Fixed a small $\eta > 0$ we impose that the parameters d_j will satisfy

$$\eta < d_j < \frac{1}{\eta} \qquad \text{for } \ j = 1, 2. \tag{2.4}$$

Hence, it is immediate to see that

$$\frac{\delta_2}{\delta_1} = \epsilon^{\frac{2(N-2)}{(N-4)(N-6)}} \frac{d_2}{d_1} \to 0 \qquad \text{as } \epsilon \to 0.$$

We construct solutions to problem (1.1), as predicted by Theorem 1.1, which are superpositions of copies of the standard bubble defined in (1.3) with alternating signs, properly modified (namely we consider the projection of the original bubble into $H_0^1(\Omega)$), centered at the origin which is the center of symmetry of Ω with parameters of concentrations δ_j. Such an object has the shape of a tower of two bubbles.

Hence the solution to problem (1.1) will be of the form

$$u_\epsilon(x) = V_\epsilon(x) + \Phi_\epsilon(x) \tag{2.5}$$

where

$$V_\epsilon(x) := \mathcal{P}\mathcal{U}_{\delta_1}(x) - \mathcal{P}\mathcal{U}_{\delta_2}(x). \tag{2.6}$$

The term Φ_ϵ has to be thought as a remainder term of lower order, which has to be described accurately.
Let Z_j the following functions

$$Z_j(x) := \partial_{\delta_j}\mathcal{U}_{\delta_j}(x) = \alpha_N \frac{N-2}{2}\delta_j^{\frac{N-4}{2}} \frac{|x|^2 - \delta_j^2}{\left(\delta_j^2 + |x|^2\right)^{\frac{N}{2}}}, \quad j = 1, 2. \tag{2.7}$$

We remark that the functions Z_j solve the problem (see [8])

$$-\Delta z = p|\mathcal{U}_\delta|^{p-1}z, \qquad \text{in } \mathbb{R}^N. \tag{2.8}$$

Let $\mathcal{P}Z_j$ the projection of Z_j onto $H_0^1(\Omega)$. Elliptic estimates give

$$\mathcal{P}Z_j(x) = Z_j(x) - \alpha_N \frac{N-2}{2}\delta_j^{\frac{N-4}{2}} H(0,x) + O(\delta_j^{\frac{N}{2}}), \quad j = 1, 2, \tag{2.9}$$

uniformly in Ω.

Let us consider

$$\mathcal{K}_1 := \text{span}\left\{\mathcal{P}Z_1\right\} \subset H_{sim}; \qquad \mathcal{K} := \text{span}\left\{\mathcal{P}Z_j \ : \ j = 1, 2\right\} \subset H_{sim}$$

and

$$\mathcal{K}_1^\perp := \left\{\phi \in H_{sim} \ : \ \langle \phi, \mathcal{P}Z_1\rangle = 0\right\}; \qquad \mathcal{K}^\perp := \left\{\phi \in H_{sim} \ : \ \langle \phi, \mathcal{P}Z_j\rangle = 0, \ j = 1, 2\right\}.$$

Let $\Pi_1 : H_{sim} \to \mathcal{K}_1$, $\Pi : H_{sim} \to \mathcal{K}$ and $\Pi_1^\perp : H_{sim} \to \mathcal{K}_1^\perp$, $\Pi^\perp : H_{sim} \to \mathcal{K}^\perp$ be the projections onto $\mathcal{K}_1$, $\mathcal{K}$ and $\mathcal{K}_1^\perp$, $\mathcal{K}^\perp$, respectively.
In order to solve problem (1.1) we will solve the couple of equations

$$\Pi^\perp\left\{V_\epsilon + \Phi_\epsilon - i^*\left[f(V_\epsilon + \Phi_\epsilon) + \epsilon(V_\epsilon + \Phi_\epsilon)\right]\right\} = 0 \tag{2.10}$$

$$\Pi\left\{V_\epsilon + \Phi_\epsilon - i^*\left[f(V_\epsilon + \Phi_\epsilon) + \epsilon(V_\epsilon + \Phi_\epsilon)\right]\right\} = 0. \tag{2.11}$$

For any (d_1, d_2) satisfying condition (2.4), we solve first the equation (2.10) in $\Phi_\epsilon \in \mathcal{K}^\perp$ which is the lower order term in the description of the ansatz.

We start with solving the auxiliary equation (2.10). As anticipated in the introduction, we split the remainder term as

$$\Phi_\epsilon = \phi_{1,\epsilon} + \phi_{2,\epsilon}$$

with

$$\|\phi_{2,\epsilon}\| = o(\|\phi_{1,\epsilon}\|), \quad \text{as } \epsilon \to 0.$$

In order to find $\phi_{1,\epsilon}$ and $\phi_{2,\epsilon}$ we solve the following system of equations

$$\begin{cases} \mathcal{R}_1 + \mathcal{L}_1(\phi_1) + \mathcal{N}_1(\phi_1) = 0, \\ \\ \mathcal{R}_2 + \mathcal{L}_2(\phi_2) + \mathcal{N}_2(\phi_1, \phi_2) = 0, \end{cases} \tag{2.12}$$

where

$$\mathcal{R}_1 := \Pi_1^\perp\left\{\mathcal{P}\mathcal{U}_{\delta_1} - i^*\left[f(\mathcal{P}\mathcal{U}_{\delta_1}) + \epsilon\mathcal{P}\mathcal{U}_{\delta_1}\right]\right\}, \tag{2.13}$$

$$\mathcal{R}_2 := \Pi^\perp\left\{-\mathcal{P}\mathcal{U}_{\delta_2} - i^*\left[f(V_\epsilon) - f(\mathcal{P}\mathcal{U}_{\delta_1}) - \epsilon\mathcal{P}\mathcal{U}_{\delta_2}\right]\right\}, \tag{2.14}$$

$$\mathcal{L}_1(\phi_1) := \Pi_1^{\perp}\left\{\phi_1 - i^*\left[f'(\mathcal{P}\mathcal{U}_1)\phi_1 + \epsilon\phi_1\right]\right\}, \tag{2.15}$$

$$\mathcal{L}_2(\phi_2) := \Pi^{\perp}\left\{\phi_2 - i^*\left[f'(V_\epsilon)\phi_2 + \epsilon\phi_2\right]\right\}, \tag{2.16}$$

$$\mathcal{N}_1(\phi_1) := \Pi_1^{\perp}\{-i^*[f(\mathcal{P}\mathcal{U}_{\delta_1} + \phi_1) - f(\mathcal{P}\mathcal{U}_{\delta_1}) - f'(\mathcal{P}\mathcal{U}_{\delta_1})\phi_1]\}, \tag{2.17}$$

and

$$\mathcal{N}_2(\phi_1, \phi_2) := \Pi^{\perp}\{-i^*[f(V_\epsilon + \phi_1 + \phi_2) - f(V_\epsilon) - f'(V_\epsilon)\phi_2 - f(\mathcal{P}\mathcal{U}_{\delta_1} + \phi_1) + f(\mathcal{P}\mathcal{U}_{\delta_1})]\}. \tag{2.18}$$

We remark that it is not restrictive to consider $\mathcal{R}_1, \mathcal{L}(\phi_1), \mathcal{N}_1(\phi_1) \in \mathcal{K}_1^{\perp}$ since only δ_1 appears and it is clear that a solution of (2.12) gives a solution of (2.10).

Therefore we solve the first equation in (2.12) finding a solution $\bar{\phi}_1 = \bar{\phi}_1(\epsilon, d_1)$ and after that we solve the second equation in (2.12) (with $\phi_1 = \bar{\phi}_1$) finding also $\bar{\phi}_2 = \bar{\phi}_2(\epsilon, d_1, d_2)$.

Finally let us recall some useful inequality that we will use in the sequel. Since these are known results, we omit the proof.

Lemma 2.1. *Let α be a positive real number. If $\alpha \le 1$ there holds*

$$(x+y)^\alpha \le x^\alpha + y^\alpha,$$

for all $x, y > 0$. If $\alpha \ge 1$ we have

$$(x+y)^\alpha \le 2^{\alpha-1}(x^\alpha + y^\alpha),$$

for all $x, y > 0$.

Lemma 2.2. *Let q be a positive real number. There exists a positive constant c, depending only on q, such that for any $a, b \in \mathbb{R}$*

$$||a+b|^q - |a|^q| \le \begin{cases} c(q)\min\{|b|^q, |a|^{q-1}|b|\} & \text{if } 0 < q < 1, \\ c(q)(|a|^{q-1}|b| + |b|^q) & \text{if } q \ge 1. \end{cases} \tag{2.19}$$

Moreover if $q > 2$ then

$$\left||a+b|^q - |a|^q - q|a|^{q-2}ab\right| \le C\left(|a|^{q-2}|b|^2 + |b|^q\right). \tag{2.20}$$

Lemma 2.3. *Let $N \ge 7$. There exists a positive constant c, depending only on p, such that for any $a, b \in \mathbb{R}$*

$$|f(a+b) - f(a) - f'(a)b| \le c|b|^p. \tag{2.21}$$

Lemma 2.4. *There exists a positive constant c, depending only on p, such that for any $a, b \in \mathbb{R}$*

$$|f(a-b) - f(a) + f(b)| \le c(p)(|a|^{p-1}|b| + |b|^p), \tag{2.22}$$

or

$$|f(a-b) - f(a) + f(b)| \le c(p)(|b|^{p-1}|a| + |a|^p). \tag{2.23}$$

Lemma 2.5. *Let $N \ge 7$. There exists a positive constant c depending only on p such that for any $a, b_1, b_2 \in \mathbb{R}$ we get*

$$|f(a+b_1) - f(a+b_2) - f'(a)(b_1 - b_2)| \le C\left(|b_1|^{p-1} + |b_2|^{p-1}\right)|b_1 - b_2|. \tag{2.24}$$

3. The auxiliary equation: solution of the system (2.12)

We first define

$$\theta_1 := \frac{N-2}{N-4}; \qquad \theta_2 := \frac{(N-2)^2}{(N-4)(N-6)}. \tag{3.1}$$

We observe that θ_2 is well defined since $N \geq 7$. We also remark that having defined δ_j as in (2.3), $j = 1, 2$, the functions $\mathcal{U}_{\delta_j}$ depend on the parameters d_j, $j = 1, 2$.

In this section we solve system (2.12). More precisely, the aim is to prove the following result.

Proposition 3.1. *Let $N \geq 7$. For any $\eta > 0$, there exist $\epsilon_0 > 0$ and $c > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, for all $(d_1, d_2) \in \mathbb{R}^2_+$ satisfying* (2.4)*, there exists a unique $\bar{\phi}_1 = \bar{\phi}_1(\epsilon, d_1) \in \mathcal{K}_1^\perp$ solution of the first equation of* (2.12) *such that*

$$\|\bar{\phi}_1\| \leq c\epsilon^{\frac{\theta_1}{2}+\sigma}$$

and there exists a unique solution $\bar{\phi}_2 = \bar{\phi}_2(\epsilon, d_1, d_2) \in \mathcal{K}^\perp$ of the second equation of (2.12) *(with $\phi_1 = \bar{\phi}_1$) such that*

$$\|\bar{\phi}_2\| \leq c\, \epsilon^{\frac{\theta_2}{2}+\sigma},$$

for some positive real number σ whose choice depends only on N. Furthermore, $\bar{\phi}_1$ does not depend on d_2 and it is continuously differentiable with respect to d_1, $\bar{\phi}_2$ is continuously differentiable with respect to (d_1, d_2).

In order to prove Proposition 3.1 let us first consider the linear operator

$$\mathcal{L}_1 : \mathcal{K}_1^\perp \to \mathcal{K}_1^\perp$$

defined as in (2.15).
The next result provides an a-priori estimate for solutions $\phi \in \mathcal{K}_1^\perp$ of $\mathcal{L}_1(\phi) = h$, for some right-hand side h with bounded $\|\cdot\|-$ norm.

Lemma 3.2. *Let $N \geq 7$. For any $\eta > 0$, there exists $\epsilon_0 > 0$ and $c > 0$ such that for all $d_1 \in \mathbb{R}_+$ satisfying* (2.4) *for $j = 1$, for all $\phi \in \mathcal{K}_1^\perp$ and for all $\epsilon \in (0, \epsilon_0)$ it holds*

$$\|\mathcal{L}_1(\phi)\| \geq c\|\phi\|.$$

Proof. For the proof it suffices to repeat with small changes the proof of Lemma 3.1 of [27]. □

Next result states the invertibility of the operator $\mathcal{L}_1$ and provides a uniform estimate on the inverse of the operator $\mathcal{L}_1$.

Proposition 3.3. *Let $N \geq 7$. For any $\eta > 0$, there exists $\epsilon_0 > 0$ and $c > 0$ such that the linear operator $\mathcal{L}_1$ is invertible and $\|\mathcal{L}_1^{-1}\| \leq c$ for all $\epsilon \in (0, \epsilon_0)$, for all $d_1 \in \mathbb{R}_+$ satisfying* (2.4) *for $j = 1$.*

Proof. For the proof it suffices to repeat with small changes the proof of Proposition 3.2 of [27]. □

For the linear operator $\mathcal{L}_2$ we state analogous results.

Lemma 3.4. *Let $N \geq 7$. For any $\eta > 0$, there exists $\epsilon_0 > 0$ and $c > 0$ such that for all $(d_1, d_2) \in \mathbb{R}^2_+$ satisfying* (2.4)*, for all $\phi \in \mathcal{K}^\perp$ and for all $\epsilon \in (0, \epsilon_0)$ it holds*

$$\|\mathcal{L}_2(\phi)\| \geq c\|\phi\|.$$

Proof. For the proof see Lemma 3.1 of [27]. □

Proposition 3.5. *Let $N \geq 7$. For any $\eta > 0$, there exists $\epsilon_0 > 0$ and $c > 0$ such that the linear operator $\mathcal{L}_2$ is invertible and $\|\mathcal{L}_2^{-1}\| \leq c$ for all $\epsilon \in (0, \epsilon_0)$, for all $(d_1, d_2) \in \mathbb{R}^2_+$ satisfying* (2.4).

Proof. For the proof see Proposition 3.2 of [27]. □

The strategy is to solve the first equation of (2.12) by a fixed point argument, finding a unique $\bar{\phi}_1$ and then, substituting $\bar{\phi}_1$ in the second equation of (2.12), we obtain an equation depending only on the variable ϕ_2. Hence, using again a fixed point argument, we solve the second equation of (2.12) uniquely.

3.1. **The solution of the first equation of** (2.12)**.** The aim is to prove the following proposition.

Proposition 3.6. *Let $N \geq 7$. For any $\eta > 0$, there exists $\epsilon_0 > 0$ and $c > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, for all $d_1 \in \mathbb{R}_+$ satisfying condition* (2.4) *for $j = 1$, there exists a unique solution $\bar{\phi}_1 = \bar{\phi}_1(\epsilon, d_1)$, $\bar{\phi}_1 \in \mathcal{K}_1^\perp$ of the first equation in* (2.12) *which is continuously differentiable with respect to d_1 and such that*

$$\|\bar{\phi}_1\| \leq c\epsilon^{\frac{\theta_1}{2}+\sigma}, \tag{3.2}$$

where θ_1 is defined in (3.1) *and σ is some positive real number whose choice depends only on N.*

In order to prove Proposition 3.6 we have to estimate the error term $\mathcal{R}_1$ defined in (2.13). It holds the following result.

Proposition 3.7. *Let $N \geq 7$. For any $\eta > 0$, there exists $\epsilon_0 > 0$ and $c > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, for all $d_1 \in \mathbb{R}_+$ satisfying condition* (2.4) *for $j = 1$, we have*

$$\|\mathcal{R}_1\| \leq c\ \epsilon^{\frac{\theta_1}{2}+\sigma},$$

for some positive real number σ whose choice depends only on N.

Proof. By continuity of $\Pi_1^\perp$, by using (2.1) and since $\mathcal{P}\mathcal{U}_{\delta_1}$ weakly solves $-\Delta\mathcal{P}\mathcal{U}_{\delta_1} = \mathcal{U}_{\delta_1}^p$ in Ω, it follows that

$$\begin{aligned} \|\mathcal{R}_1\| &= \|\Pi_1^\perp \left\{\mathcal{P}\mathcal{U}_{\delta_1} - i^*\left[f(\mathcal{P}\mathcal{U}_{\delta_1}) + \epsilon\mathcal{P}\mathcal{U}_{\delta_1}\right]\right\}\| \leq C_1\|\mathcal{P}\mathcal{U}_{\delta_1} - i^*\left[f(\mathcal{P}\mathcal{U}_{\delta_1}) + \epsilon\mathcal{P}\mathcal{U}_{\delta_1}\right]\| \\ &\leq C_2\,|f(\mathcal{U}_{\delta_1}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) - \epsilon\mathcal{P}\mathcal{U}_{\delta_1}|_{\frac{2N}{N+2}} \leq C\underbrace{|f(\mathcal{U}_{\delta_1}) - f(\mathcal{P}\mathcal{U}_{\delta_1})|_{\frac{2N}{N+2}}}_{(I)} + \underbrace{\epsilon\,|\mathcal{P}\mathcal{U}_{\delta_1}|_{\frac{2N}{N+2}}}_{(II)}. \end{aligned}$$

Let us fix $\eta > 0$. We estimate the terms $(I), (II)$.

Claim 1:

$$(I) = O(\epsilon^{\frac{N+2}{2(N-4)}}). \tag{3.3}$$

By using (1.9), (1.10) and by elementary inequalities we get

$$\begin{aligned} \int_\Omega |(\mathcal{P}\mathcal{U}_{\delta_1})^p - \mathcal{U}_{\delta_1}^p|^{\frac{2N}{N+2}}\,dx &\leq c_1\int_\Omega |\mathcal{U}_{\delta_1}^{p-1}\varphi_{\delta_1}|^{\frac{2N}{N+2}}\,dx + c_2\int_\Omega |\varphi_{\delta_1}|^{p+1}\,dx \\ &\leq c_3\delta_1^{\frac{N-2}{2}\frac{2N}{N+2}}\int_\Omega \left(\frac{\delta_1^2}{(\delta_1^2+|x|^2)^2}\right)^{\frac{2N}{N+2}}\,dx + c_2|\varphi_{\delta_1}|_{p+1,\Omega}^{p+1} \\ &= c_3\delta_1^{\frac{N(N-2)}{N+2}}\int_\Omega \left(\frac{\delta_1^2}{(\delta_1^2+|x|^2)^2}\right)^{\frac{2N}{N+2}}\,dx + c_4\delta_1^N. \end{aligned}$$

Now for $N \geq 7$ we have

$$\int_\Omega \left(\frac{\delta_1^2}{(\delta_1^2+|x|^2)^2}\right)^{\frac{2N}{N+2}}\,dx = O\left(\delta_1^{\frac{4N}{N+2}}\right).$$

Indeed:

$$\int_\Omega \left(\frac{\delta_1^2}{(\delta_1^2+|x|^2)^2}\right)^{\frac{2N}{N+2}} dx = \delta_1^{\frac{4N}{N+2}} \int_\Omega \frac{1}{(\delta_1^2+|x|^2)^{\frac{4N}{N+2}}} dx \le \delta_1^{\frac{4N}{N+2}} \int_\Omega \frac{1}{|x|^{\frac{8N}{N+2}}} dx,$$

and the last integral is finite since $N > 6$, which implies $\frac{8N}{N+2} < N$. Finally, since $\frac{4N^2(N-2)}{(N+2)^2} > N$, for any $N > 4$, we deduce that

$$\int_\Omega |(\mathcal{P}\mathcal{U}_{\delta_1})^p - \mathcal{U}_{\delta_1}^p|^{\frac{2N}{N+2}} dx = O\left(\delta_1^N\right),$$

and hence

$$|(\mathcal{P}\mathcal{U}_{\delta_1})^p - \mathcal{U}_{\delta_1}^p|_{\frac{2N}{N+2}} = O\left(\delta_1^{\frac{N+2}{2}}\right). \tag{3.4}$$

Since $\delta_1 = d_1\epsilon^{\frac{1}{N-4}}$ and d_1 satisfies (2.4), we get that $|(\mathcal{P}\mathcal{U}_{\delta_1})^p - \mathcal{U}_{\delta_1}^p|_{\frac{2N}{N+2}} = O\left(\epsilon^{\frac{N+2}{2(N-4)}}\right)$ and Claim 1 is proved.

Claim 2:

$$(II) = O(\epsilon^{\frac{N-2}{N-4}}). \tag{3.5}$$

$$\begin{aligned}\int_\Omega \mathcal{P}\mathcal{U}_{\delta_1}^{\frac{2N}{N+2}} dx &\le \int_\Omega \mathcal{U}_{\delta_1}^{\frac{2N}{N+2}} dx = \alpha_N^{\frac{2N}{N+2}} \int_\Omega \frac{\delta_1^{-\frac{N(N-2)}{N+2}}}{(1+|\frac{x}{\delta_1}|^2)^{\frac{N(N-2)}{N+2}}} dx \\ &= \alpha_N^{\frac{2N}{N+2}} \delta_1^{\frac{4N}{N+2}} \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N(N-2)}{N+2}}} dy + o(\delta_1^{\frac{4N}{N+2}}).\end{aligned} \tag{3.6}$$

Thus, since $\delta_1 = d_1\epsilon^{\frac{1}{N-4}}$ and d_1 satisfies (2.4), we get that

$$\int_\Omega \mathcal{P}\mathcal{U}_{\delta_1}^{\frac{2N}{N+2}} dx = O\left(\epsilon^{\frac{4N}{(N+2)(N-4)}}\right),$$

and hence

$$\epsilon\left(\int_\Omega P\mathcal{U}_{\delta_1}^{\frac{2N}{N+2}} dx\right)^{\frac{N+2}{2N}} = \epsilon O\left(\epsilon^{\frac{2}{N-4}}\right) = O\left(\epsilon^{\frac{N-2}{N-4}}\right).$$

The proof of Claim 2 is complete.

Hence, by (3.3) and (3.5), we deduce that there exist a constant $c = c(\eta) > 0$ and $\epsilon_0 = \epsilon_0(\eta) > 0$ sufficiently small such that, for all $\epsilon \in (0, \epsilon_0)$ and $d_1 \in \mathbb{R}_+$ satisfying (2.4) (with $j = 1$)

$$\|\mathcal{R}_1\| \le c\ (\epsilon^{\frac{N+2}{2(N-4)}} + \epsilon^{\frac{N-2}{N-4}}) \le c\epsilon^{\frac{\theta_1}{2}+\sigma},$$

with σ such that $0 < \sigma < \frac{2}{N-4}$. □

We are ready to prove Proposition 3.6.

Proof of Proposition 3.6. Let us fix $\eta > 0$ and define $\mathcal{T}_1 : \mathcal{K}_1^\perp \to \mathcal{K}_1^\perp$ as

$$\mathcal{T}_1(\phi_1) := -\mathcal{L}_1^{-1}[\mathcal{N}_1(\phi_1) + \mathcal{R}_1].$$

Clearly solving the first equation of (2.12) is equivalent to solving the fixed point equation $\mathcal{T}_1(\phi_1) = \phi_1$.

Let us define the ball

$$B_{1,\epsilon} := \{\phi_1 \in \mathcal{K}_1^\perp;\ \|\phi_1\| \le r\ \epsilon^{\frac{\theta_1}{2}+\sigma}\} \subset \mathcal{K}_1^\perp$$

with $r>0$ sufficiently large and $\sigma>0$.
We want to prove that, for ϵ small, $\mathcal{T}_1$ is a contraction in the proper ball $B_{1,\epsilon}$, namely we want to prove that, for ϵ sufficiently small

(1) $\mathcal{T}_1(B_{1,\epsilon})\subset B_{1,\epsilon}$;
(2) $\|\mathcal{T}_1\|<1$.

By Lemma 3.2 we get:

$$\|\mathcal{T}_1(\phi_1)\|\le c(\|\mathcal{N}_1(\phi_1)\|+\|\mathcal{R}_1\|) \tag{3.7}$$

and

$$\|\mathcal{T}_1(\phi_1)-\mathcal{T}_1(\psi_1)\|\le c(\|\mathcal{N}_1(\phi_1)-\mathcal{N}_1(\psi_1)\|), \tag{3.8}$$

for all $\phi_1,\psi_1\in\mathcal{K}_1^\perp$. Thanks to (2.1) and the definition of $\mathcal{N}_1$ we deduce that

$$\|\mathcal{N}_1(\phi_1)\|\le c|f(\mathcal{P}\mathcal{U}_{\delta_1}+\phi_1)-f(\mathcal{P}\mathcal{U}_{\delta_1})-f'(\mathcal{P}\mathcal{U}_{\delta_1})\phi_1|_{\frac{2N}{N+2}}, \tag{3.9}$$

and

$$\|\mathcal{N}_1(\phi_1)-\mathcal{N}_1(\psi_1)\|\le c|f(\mathcal{P}\mathcal{U}_{\delta_1}+\phi_1)-f(\mathcal{P}\mathcal{U}_{\delta_1}+\psi_1)-f'(\mathcal{P}\mathcal{U}_{\delta_1})(\phi_1-\psi_1)|_{\frac{2N}{N+2}}. \tag{3.10}$$

Now we estimate the right-hand term in (3.7). Thanks to Lemma 2.3 we have the following inequality:

$$|f(\mathcal{P}\mathcal{U}_{\delta_1}+\phi_1)-f(\mathcal{P}\mathcal{U}_{\delta_1})-f'(\mathcal{P}\mathcal{U}_{\delta_1})\phi_1|\le c|\phi_1|^p. \tag{3.11}$$

Since $p\frac{2N}{N+2}=\frac{2N}{N-2}$ and $|\phi_1^p|_{\frac{2N}{N+2}}=|\phi_1|^p_{\frac{2N}{N-2}}$, from (3.11) and the Sobolev inequality we deduce the following:

$$|f(\mathcal{P}\mathcal{U}_{\delta_1}+\phi_1)-f(\mathcal{P}\mathcal{U}_{\delta_1})-f'(\mathcal{P}\mathcal{U}_{\delta_1})\phi_1|_{\frac{2N}{N+2}}\le c_1|\phi_1|^p_{\frac{2N}{N-2}}\le c_2\|\phi_1\|^p. \tag{3.12}$$

Thanks to (3.7), Proposition 3.7, (3.9), (3.12) and since $p>1$, then, there exist $c=c(\eta)>0$ and $\epsilon_0=\epsilon_0(\eta)>0$ such that

$$\|\phi_1\|\le c\epsilon^{\frac{\theta_1}{2}+\sigma}\Rightarrow\|\mathcal{T}_1(\phi_1)\|\le c\epsilon^{\frac{\theta_1}{2}+\sigma},$$

for all $\epsilon\in(0,\epsilon_0)$, for all $d_1\in\mathbb{R}_+$ satisfying (2.4) (with $j=1$), for some positive real number σ, whose choice depends only on N. In other words $\mathcal{T}_1$ maps the ball $B_{1,\epsilon}$ into itself and (1) is proved.

We want to show that $\mathcal{T}_1$ is a contraction. By using Lemma 2.5 we get that for any $\phi_1,\psi_1\in B_{1,\epsilon}$

$$|f(\mathcal{P}\mathcal{U}_{\delta_1}+\phi_1)-f(\mathcal{P}\mathcal{U}_{\delta_1}+\psi_1)-f'(\mathcal{P}\mathcal{U}_{\delta_1})(\phi_1-\psi_1)|\le C\left(|\phi_1|^{p-1}+|\psi_1|^{p-1}\right)|\phi_1-\psi_1|.$$

By direct computation $(p-1)\frac{2N}{N+2}=\frac{8N}{(N-2)(N+2)}$, so, since $|\phi_1|^{(p-1)\frac{2N}{N+2}},|\psi_1|^{(p-1)\frac{2N}{N+2}}\in L^{\frac{N+2}{4}}$, $|\phi_1-\psi_1|^{\frac{2N}{N+2}}\in L^p$ and $1=\frac{4}{N+2}+\frac{N-2}{N+2}$ by Hölder inequality we get that

$$\begin{aligned}\left|\left(|\phi_1|^{p-1}+|\psi_1|^{p-1}\right)(\phi_1-\psi_1)\right|_{\frac{2N}{N+2}}&\le\left[\left(|\phi_1|^{\frac{4}{N-2}}_{\frac{2N}{N-2}}+|\psi_1|^{\frac{4}{N-2}}_{\frac{2N}{N-2}}\right)^{\frac{2N}{N+2}}\left(|\phi_1-\psi_1|^{\frac{2N}{N-2}}_{\frac{2N}{N-2}}\right)^{\frac{N-2}{N+2}}\right]^{\frac{N+2}{2N}}\\&=\left(|\phi_1|^{\frac{4}{N-2}}_{\frac{2N}{N-2}}+|\psi_1|^{\frac{4}{N-2}}_{\frac{2N}{N-2}}\right)|\phi_1-\psi_1|_{\frac{2N}{N-2}}.\end{aligned} \tag{3.13}$$

Hence by (3.8), (3.10), (3.13) and Sobolev inequality we get that there exists $L\in(0,1)$ such that

$$\|\phi_1\|\le c\epsilon^{\frac{\theta_1}{2}+\sigma},\|\psi_1\|\le c\epsilon^{\frac{\theta_1}{2}+\sigma}\Rightarrow\|\mathcal{T}_1(\phi_1)-\mathcal{T}_1(\psi_1)\|\le L\|\phi_1-\psi_1\|.$$

Hence by the Contraction Mapping Theorem we can uniquely solve $\mathcal{T}_1(\phi_1)=\phi_1$ in $B_{1,\epsilon}$. We denote by $\bar\phi_1\in B_{1,\epsilon}$ this solution. A standard argument shows that $d_1\to\bar\phi_1(d_1)$ is a C^1-map (see also [27]). The proof is then concluded. □

3.2. **The proof of Proposition 3.1.** Before proving Proposition 3.1 we need some preliminary results, in particular we need to improve the estimate on the solution $\bar{\phi}_1$ of the first equation of (2.12) found in Proposition 3.6.

The first preliminary result is an estimate on the error term $\mathcal{R}_2$ defined in (2.14).

Proposition 3.8. *For any $\eta > 0$, there exists $\epsilon_0 > 0$ and $c > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, for all $(d_1, d_2) \in \mathbb{R}^2_+$ satisfying (2.4), we have*

$$\|\mathcal{R}_2\| \le c\, \epsilon^{\frac{\theta_2}{2}+\sigma},$$

for some positive real number σ, whose choice depends only on N.

Proof. By continuity of $\Pi^\perp$ and by using (2.1) we deduce that

$$\begin{aligned} \|\mathcal{R}_2\| &\le c|f(\mathcal{U}_{\delta_2}) + f(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) - \epsilon \mathcal{P}\mathcal{U}_{\delta_2}|_{\frac{2N}{N+2}} \\ &\le \underbrace{c|f(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) + f(\mathcal{P}\mathcal{U}_{\delta_2})|_{\frac{2N}{N+2}}}_{(I)} + \underbrace{c|f(\mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{U}_{\delta_2})|_{\frac{2N}{N+2}}}_{(II)} \\ &\quad + \underbrace{c\epsilon|\mathcal{P}\mathcal{U}_{\delta_2}|_{\frac{2N}{N+2}}}_{(III)}. \end{aligned} \tag{3.14}$$

Let us fix $\eta > 0$. We begin estimating (I). Let $\rho > 0$ so that $B(0,\rho) \subset \Omega$. We decompose the domain Ω as $\Omega = A_0 \sqcup A_1 \sqcup A_2$, where $A_0 := \Omega \setminus B(0,\rho)$, $A_1 := B(0,\rho) \setminus B(0, \sqrt{\delta_1\delta_2})$ and $A_2 := B(0, \sqrt{\delta_1\delta_2})$. We evaluate (I) in every set of this decomposition.

Thanks to Lemma 2.4 there exists a positive constant c (depending only on p) such that

$$|f(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) + f(\mathcal{P}\mathcal{U}_{\delta_2})| \le c(\mathcal{P}\mathcal{U}_{\delta_1}^{p-1}\mathcal{P}\mathcal{U}_{\delta_2} + \mathcal{P}\mathcal{U}_{\delta_2}^{p}). \tag{3.15}$$

Integrating on A_0 and using the usual elementary inequalities (see Lemma 2.1) we get that

$$\begin{aligned} &\int_{A_0} |f(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) + f(\mathcal{P}\mathcal{U}_{\delta_2})|^{\frac{2N}{N+2}}\, dx \\ &\le C_1 \int_{A_0} (\mathcal{P}\mathcal{U}_{\delta_1}^{(p-1)(\frac{2N}{N+2})}\mathcal{P}\mathcal{U}_{\delta_2}^{\frac{2N}{N+2}} + \mathcal{P}\mathcal{U}_{\delta_2}^{p+1})\, dx \\ &\le C_2 \int_{A_0} \frac{\delta_1^{\frac{4N}{N+2}}}{(\delta_1^2 + |x|^2)^{\frac{4N}{N+2}}} \frac{\delta_2^{\frac{N(N-2)}{N+2}}}{(\delta_2^2+|x|^2)^{\frac{N(N-2)}{N+2}}}\, dx + C_3 \int_{A_0} \frac{\delta_2^N}{(\delta_2^2+|x|^2)^N}\, dx \\ &\le C_4 \frac{\delta_1^{\frac{4N}{N+2}}}{\rho^{\frac{8N}{N+2}}} \frac{\delta_2^{\frac{N(N-2)}{N+2}}}{\rho^{2\frac{N(N-2)}{N+2}}} + C_5 \frac{\delta_2^N}{\rho^{2N}} \end{aligned} \tag{3.16}$$

and hence we deduce that (recall the choice of δ_1, δ_2 (see (2.3)))

$$|f(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) + f(\mathcal{P}\mathcal{U}_{\delta_2})|_{\frac{2N}{N+2}, A_0} \le c\epsilon^{\frac{3N^2-12N-4}{2(N-4)(N-6)}} \le c\epsilon^{\frac{\theta_2}{2}+\sigma} \tag{3.17}$$

where c depends on η (and also on Ω, ρ, N), σ is some positive real number (to be precise we can choose $0 < \sigma \le \frac{N^2-4N-4}{(N-4)(N-6)}$).

We evaluate now (I) in A_1. By (3.15) and the usual elementary inequalities we deduce the following:

$$\int_{A_1} |f(P\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) + f(\mathcal{P}\mathcal{U}_{\delta_2})|^{\frac{2N}{N+2}}\, dx \le c \int_{A_1} (\mathcal{P}\mathcal{U}_{\delta_1}^{(p-1)(\frac{2N}{N+2})}\mathcal{P}\mathcal{U}_{\delta_2}^{\frac{2N}{N+2}} + \mathcal{P}\mathcal{U}_{\delta_2}^{p+1})\, dx. \tag{3.18}$$

Let us estimate every term:

$$\begin{aligned}
\int_{A_1} \mathcal{P}\mathcal{U}_{\delta_1}^{(p-1)(\frac{2N}{N+2})}\mathcal{P}\mathcal{U}_{\delta_2}^{\frac{2N}{N+2}}\,dx &\le \int_{A_1} \mathcal{U}_{\delta_1}^{(p-1)(\frac{2N}{N+2})}\mathcal{U}_{\delta_2}^{\frac{2N}{N+2}}\,dx = \alpha_N^{p+1}\int_{A_1} \frac{\delta_1^{\frac{4N}{N+2}}}{(\delta_1^2+|x|^2)^{\frac{4N}{N+2}}}\frac{\delta_2^{\frac{N(N-2)}{N+2}}}{(\delta_2^2+|x|^2)^{\frac{N(N-2)}{N+2}}}\,dx \\
&= c_1\int_{\sqrt{\delta_1\delta_2}}^{\rho} \frac{\delta_1^{\frac{4N}{N+2}}}{(\delta_1^2+r^2)^{\frac{4N}{N+2}}}\frac{\delta_2^{\frac{N(N-2)}{N+2}}}{(\delta_2^2+r^2)^{\frac{N(N-2)}{N+2}}} r^{N-1}dr = c_1\int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}} \frac{\delta_1^{\frac{4N}{N+2}}}{(\delta_1^2+\delta_2^2 s^2)^{\frac{4N}{N+2}}}\frac{\delta_2^{-\frac{N(N-2)}{N+2}}}{(1+s^2)^{\frac{N(N-2)}{N+2}}}\delta_2^N s^{N-1}ds \\
&= c_1\int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}} \frac{\delta_1^{-\frac{4N}{N+2}}}{\left[1+\left(\frac{\delta_2}{\delta_1}\right)^2 s^2\right]^{\frac{4N}{N+2}}}\frac{\delta_2^{\frac{4N}{N+2}}}{(1+s^2)^{\frac{N(N-2)}{N+2}}} s^{N-1}ds \le c_1\left(\frac{\delta_2}{\delta_1}\right)^{\frac{4N}{N+2}}\int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}} \frac{1}{(1+s^2)^{\frac{N(N-2)}{N+2}}} s^{N-1}ds \qquad (3.19)\\
&\le c_1\left(\frac{\delta_2}{\delta_1}\right)^{\frac{4N}{N+2}}\int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}} \frac{1}{s^{\frac{N^2-5N+2}{N+2}}}ds = c_2\left(\frac{\delta_2}{\delta_1}\right)^{\frac{4N}{N+2}}\left[\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N^2-6N}{2(N+2)}} - \left(\frac{\delta_2}{\rho}\right)^{\frac{N^2-6N}{(N+2)}}\right] \\
&\le c_3\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}.
\end{aligned}$$

Moreover

$$\int_{A_1} \mathcal{P}\mathcal{U}_{\delta_2}^{p+1}\,dx \le \int_{A_1} \mathcal{U}_{\delta_2}^{p+1}\,dx \le C_1\int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}} \frac{r^{N-1}}{(1+r^2)^N}\,dr \le C_2\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}. \qquad (3.20)$$

Thanks to the choice of δ_1, δ_2 we have

$$\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}} = O(\epsilon^{\frac{N(N-2)}{(N-4)(N-6)}}). \qquad (3.21)$$

Hence, from (3.18), (3.19), (3.20) and (3.21) we deduce that

$$|f(\mathcal{P}\mathcal{U}_{\delta_1}-\mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) + f(\mathcal{P}\mathcal{U}_{\delta_2})|_{\frac{2N}{N+2},A_1} \le c\epsilon^{\frac{(N-2)(N+2)}{2(N-4)(N-6)}} \le c\epsilon^{\frac{\theta_2}{2}+\sigma}, \qquad (3.22)$$

where c depends on η, σ is some positive real number (to be precise we can choose $0 < \sigma \le \frac{2(N-2)}{(N-4)(N-6)}$).

Now we evaluate (I) in A_2. To do this we apply (2.23) of Lemma 2.4, so there exists a constant $c > 0$ such that

$$|f(\mathcal{P}\mathcal{U}_{\delta_1}-\mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) + f(\mathcal{P}\mathcal{U}_{\delta_2})| \le c(\mathcal{P}\mathcal{U}_{\delta_2}^{p-1}\mathcal{P}\mathcal{U}_{\delta_1} + \mathcal{P}\mathcal{U}_{\delta_1}^{p}). \qquad (3.23)$$

Thanks to (3.23) and the usual elementary inequalities we deduce the following:

$$\int_{A_2} |f(\mathcal{P}\mathcal{U}_{\delta_1}-\mathcal{P}\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_1}) + f(\mathcal{P}\mathcal{U}_{\delta_2})|^{\frac{2N}{N+2}}\,dx \le c\int_{A_2} (\mathcal{P}\mathcal{U}_{\delta_2}^{(p-1)(\frac{2N}{N+2})}\mathcal{P}\mathcal{U}_{\delta_1}^{\frac{2N}{N+2}} + \mathcal{P}\mathcal{U}_{\delta_1}^{p+1})\,dx. \qquad (3.24)$$

We estimate the first term

$$
\begin{aligned}
\int_{A_2} \mathcal{PU}_{\delta_2}^{(p-1)(\frac{2N}{N+2})}\mathcal{PU}_{\delta_1}^{\frac{2N}{N+2}}dx \le \int_{A_2} \mathcal{U}_{\delta_2}^{(p-1)(\frac{2N}{N+2})}\mathcal{U}_{\delta_1}^{\frac{2N}{N+2}}dx = \alpha_N^{p+1}\int_{A_2} \frac{\delta_2^{\frac{4N}{N+2}}}{(\delta_2^2+|x|^2)^{\frac{4N}{N+2}}}\frac{\delta_1^{\frac{N(N-2)}{N+2}}}{(\delta_1^2+|x|^2)^{\frac{N(N-2)}{N+2}}}\,dx \\
= \ c_1\int_0^{\sqrt{\frac{\delta_2}{\delta_1}}} \frac{\delta_2^{\frac{4N}{N+2}}}{(\delta_2^2+\delta_1^2s^2)^{\frac{4N}{N+2}}}\frac{\delta_1^{-\frac{N(N-2)}{N+2}}}{(1+s^2)^{\frac{N(N-2)}{N+2}}}\delta_1^N s^{N-1}ds = c_1\int_0^{\sqrt{\frac{\delta_2}{\delta_1}}} \frac{\delta_2^{\frac{4N}{N+2}}}{\left[\left(\frac{\delta_2}{\delta_1}\right)^2+s^2\right]^{\frac{4N}{N+2}}}\frac{\delta_1^{-\frac{4N}{N+2}}}{(1+s^2)^{\frac{N(N-2)}{N+2}}}s^{N-1}ds \\
\le \ c_1\left(\frac{\delta_2}{\delta_1}\right)^{\frac{4N}{N+2}}\int_0^{\sqrt{\frac{\delta_2}{\delta_1}}}\frac{1}{s^{\frac{8N}{N+2}}(1+s^2)^{\frac{N(N-2)}{N+2}}}s^{N-1}ds \le c_1\left(\frac{\delta_2}{\delta_1}\right)^{\frac{4N}{N+2}}\int_0^{\sqrt{\frac{\delta_2}{\delta_1}}}\frac{s^{\frac{N^2-7N-2}{N+2}}}{(1+s^2)^{\frac{N(N-2)}{N+2}}}ds \\
\le \ c_1\left(\frac{\delta_2}{\delta_1}\right)^{\frac{4N}{N+2}}\int_0^{\sqrt{\frac{\delta_2}{\delta_1}}} s^{\frac{N^2-7N-2}{N+2}}ds = c_2\left(\frac{\delta_2}{\delta_1}\right)^{\frac{4N}{N+2}}\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N^2-6N}{2(N+2)}} \\
= \ c_2\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}.
\end{aligned} \tag{3.25}
$$

By making similar computations as before we get that

$$\int_{A_2}\mathcal{PU}_{\delta_1}^{p+1}\,dx \le c_3\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}. \tag{3.26}$$

So from (3.24) and (3.25) we deduce that

$$|f(\mathcal{PU}_{\delta_1}-\mathcal{PU}_{\delta_2}) - f(\mathcal{PU}_{\delta_1}) + f(\mathcal{PU}_{\delta_2})|_{\frac{2N}{N+2},A_2} \le c\epsilon^{\frac{(N+2)(N-2)}{2(N-4)(N-6)}} \le c\epsilon^{\frac{\theta_2}{2}+\sigma}, \tag{3.27}$$

where c depends on η, σ is some positive real number (to be precise we can choose $0<\sigma\le \frac{2(N-2)}{(N-4)(N-6)}$). Hence from (3.17), (3.22) and (3.27) we deduce that

$$(I)\le c\epsilon^{\frac{\theta_2}{2}+\sigma}, \tag{3.28}$$

for some positive constant c, for some positive real number σ depending only on N.
Now by making similar computations as for (I) of Proposition 3.7 (see (3.4)) we get that

$$(II) = O\left(\delta_2^{\frac{N+2}{2}}\right),$$

and hence we deduce that

$$(II)\le c\epsilon^{\frac{(3N-10)(N+2)}{2(N-4)(N-6)}} \le c\epsilon^{\frac{\theta_2}{2}+\sigma},$$

where c, $0<\sigma\le\frac{N^2-12}{(N-4)(N-6)}$.

It remains to estimate (III).

From (3.6), exchanging δ_1 with δ_2 we get:

$$\int_\Omega \mathcal{PU}_{\delta_2}^{\frac{2N}{N+2}}\ dx \ \le\ \alpha_N^{\frac{2N}{N+2}}\delta_2^{\frac{4N}{N+2}}\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{\frac{N(N-2)}{N+2}}}\ dy.$$

Hence we deduce that $(III)\le c\,\epsilon\delta_2^2$, and thanks to the choice δ_2, by an elementary computation, we get that:

$$(III)\le c\ \epsilon^{\frac{(N-2)^2}{(N-4)(N-6)}} \le c\epsilon^{\frac{\theta_2}{2}+\sigma},$$

where c, $0 < \sigma \le \frac{(N-2)^2}{2(N-4)(N-6)}$. Finally, putting together all these estimates we deduce that there exist a positive constant $c = c(\eta) > 0$ and $\epsilon_0 = \epsilon_0(\eta) > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, for all $(d_1, d_2) \in \mathbb{R}^2_+$ satisfying (2.4)

$$\|\mathcal{R}_2\| \le c\epsilon^{\frac{\theta_2}{2}+\sigma},$$

for some positive real number σ (whose choice depends only on N). The proof is complete. □

Now we prove a technical result on the behavior of the L^∞-norm of $\bar{\phi}_1$, which will be useful in the sequel.

Lemma 3.9. *Let η be a small positive real number and let $\bar{\phi}_1 \in \mathcal{K}_1^\perp$ be the solution of the first equation in* (2.12)*, found in Proposition 3.6. Then, as $\epsilon \to 0^+$, we have*

$$|\bar{\phi}_1|_\infty = o(\epsilon^{-\frac{N-2}{2(N-4)}}),$$

uniformly with respect to d_1 satisfying (2.4) *for $j = 1$.*

Proof. Let us fix a small $\eta > 0$ and remember that $\delta_1 = \epsilon^{\frac{1}{N-4}} d_1$ (see (2.3)), with d_1 satisfying (2.4) for $j = 1$. We observe that by definition, since $\bar{\phi}_1 \in \mathcal{K}_1^\perp$ solves the first equation of (2.12), then, for all $\epsilon > 0$ sufficiently small, there exists a constant c_ϵ (which depends also on d_1) such that $\bar{\phi}_1$ weakly solves

$$-\Delta\bar{\phi}_1 = \epsilon\bar{\phi}_1 + \epsilon\mathcal{P}\mathcal{U}_{\delta_1} + f(\mathcal{P}\mathcal{U}_{\delta_1} + \bar{\phi}_1) - f(\mathcal{U}_{\delta_1}) - c_\epsilon\Delta\mathcal{P}Z_1. \tag{3.29}$$

Testing (3.29) with $\mathcal{P}Z_1$, taking into account that $\bar{\phi}_1 \in \mathcal{K}_1^\perp$ and the definition of $\mathcal{P}Z_1$, we have that

$$\begin{aligned} c_\epsilon \int_\Omega pU_{\delta_1}^{p-1}\mathcal{P}Z_1 Z_1\,dx \;&=\; -\epsilon\int_\Omega \bar{\phi}_1\mathcal{P}Z_1\,dx - \epsilon\int_\Omega \mathcal{P}\mathcal{U}_{\delta_1}\mathcal{P}Z_1\,dx - \int_\Omega \left[f(\mathcal{P}\mathcal{U}_{\delta_1}) - f(\mathcal{U}_{\delta_1})\right]\mathcal{P}Z_1\,dx \\ &\quad - \int_\Omega \left[f(\mathcal{P}\mathcal{U}_{\delta_1} + \bar{\phi}_1) - f(\mathcal{P}\mathcal{U}_{\delta_1})\right]\mathcal{P}Z_1\,dx. \end{aligned} \tag{3.30}$$

By definition, if we set $\psi := Z_1 - PZ_1$, then ψ is an harmonic function and $\psi = Z_1$ on $\partial\Omega$, therefore, by elementary elliptic estimates, for all sufficiently small $\epsilon > 0$, for any $d_1 \in]\eta, \frac{1}{\eta}[$ we have that $|\psi|_{\infty,\Omega} \le C\delta_1^{\frac{N-4}{2}}$, for some positive constant $C = C(N, \Omega)$ depending only on N and Ω, and hence

$$\int_\Omega pU_{\delta_1}^{p-1}\mathcal{P}Z_1 Z_1\,dx = \int_\Omega pU_{\delta_1}^{p-1} Z_1^2\,dx - \int_\Omega pU_{\delta_1}^{p-1}\psi Z_1\,dx.$$

Now

$$\begin{aligned} \int_\Omega pU_{\delta_1}^{p-1} Z_1^2\,dx \;&=\; c_N\delta_1^{N-4}\delta_1^2 \int_\Omega \frac{1}{(\delta_1^2 + |x|^2)^2}\frac{(|x|^2 - \delta_1^2)^2}{(\delta_1^2 + |x|^2)^N}\,dx \\ &=\; c_N\delta_1^{-2}\int_{\mathbb{R}^N} \frac{(|y|^2 - 1)^2}{(1 + |y|^2)^{N+2}}\,dy + O(\delta_1^{N-2}) \\ &=\; A_N\delta_1^{-2} + o(1), \quad \text{as } \epsilon \to 0. \end{aligned}$$

By using the property $|\psi|_{\infty,\Omega} \le C\delta_1^{\frac{N-4}{2}}$, by the same computations, we see that

$$\int_\Omega pU_{\delta_1}^{p-1}\psi Z_1\,dx = O(\delta_1^{N-4}), \quad \text{as } \epsilon \to 0.$$

Therefore, we get that

$$\int_\Omega pU_{\delta_1}^{p-1}\mathcal{P}Z_1 Z_1\,dx = A_N\delta_1^{-2} + o(1), \quad \text{as } \epsilon \to 0. \tag{3.31}$$

Moreover, reasoning as before, we have

$$\begin{aligned}\int_\Omega Z_1^2\,dx &= c_N\delta_1^{N-4}\int_\Omega \frac{(|x|^2-\delta_1^2)^2}{(\delta_1^2+|x|^2)^N}\,dx\\ &= c_N\int_{\mathbb{R}^N}\frac{(|y|^2-1)^2}{(1+|y|^2)^N}\,dy+O(\delta_1^{N-2})\\ &= B_N+o(1),\quad \text{as } \epsilon\to 0,\end{aligned}$$

and, by an analogous computation

$$|Z_1|_{\frac{2N}{N-2}}\le c_N\left[\int_\Omega \delta_1^{\frac{N(N-4)}{N-2}}\frac{||x|^2-\delta_1^2|^{\frac{2N}{N-2}}}{(\delta_1^2+|x|^2)^{\frac{N^2}{N-2}}}\,dx\right]^{\frac{N-2}{2N}}\le C_N\delta_1^{-1},$$

and hence, since $\mathcal{P}Z_1=Z_1-\psi$, by elementary estimates, we get that for all sufficiently small $\epsilon>0$

$$|\mathcal{P}Z_1|_2^2\le 2B_N,\quad |\mathcal{P}Z_1|_{\frac{2N}{N-2}}\le 2C_N\delta_1^{-1}. \tag{3.32}$$

Thanks to (3.32), applying Hölder inequality, Poincarè inequality, taking into account of (3.4), the asymptotic expansion of $|\mathcal{P}\mathcal{U}_{\delta_1}|_2$ (see Lemma 4.6 and its proof), the choice of δ_1 (see (2.3)) and since $\bar\phi_1\in B_{1,\epsilon}$, we have the following inequalities

$$\epsilon\int_\Omega|\bar\phi_1||\mathcal{P}Z_1|\,dx\le\epsilon|\bar\phi_1|_2|\mathcal{P}Z_1|_2\le c_1\epsilon\|\bar\phi_1\||\mathcal{P}Z_1|_2\le c_2\epsilon^{\frac{\theta_1}{2}+1+\sigma}$$

$$\epsilon\int_\Omega\mathcal{P}\mathcal{U}_{\delta_1}|\mathcal{P}Z_1|\,dx\le\epsilon|\mathcal{P}\mathcal{U}_{\delta_1}|_2|\mathcal{P}Z_1|_2\le c\epsilon\delta_1,$$

$$\int_\Omega|f(\mathcal{P}\mathcal{U}_{\delta_1})-f(\mathcal{U}_{\delta_1})||\mathcal{P}Z_1|\,dx\le|f(\mathcal{P}\mathcal{U}_{\delta_1})-f(\mathcal{U}_{\delta_1})|_{\frac{2N}{N+2}}|\mathcal{P}Z_1|_{\frac{2N}{N-2}}\le c\delta_1^{\frac{N+2}{2}}\delta_1^{-1}=c\delta_1^{\frac{N}{2}}.$$

Moreover, taking into account of Lemma 2.3 and Sobolev inequality, we get that

$$\begin{aligned}&\int_\Omega|f(\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1)-f(\mathcal{P}\mathcal{U}_{\delta_1})||\mathcal{P}Z_1|\,dx\\ \le\ & |f(\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1)-f(\mathcal{P}\mathcal{U}_{\delta_1})|_{\frac{2N}{N+2}}|\mathcal{P}Z_1|_{\frac{2N}{N-2}}\\ \le\ & |f(\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1)-f(\mathcal{P}\mathcal{U}_{\delta_1})-f'(\mathcal{P}\mathcal{U}_{\delta_1})\bar\phi_1|_{\frac{2N}{N+2}}|\mathcal{P}Z_1|_{\frac{2N}{N-2}}+|f'(\mathcal{P}\mathcal{U}_{\delta_1})\bar\phi_1|_{\frac{2N}{N+2}}|\mathcal{P}Z_1|_{\frac{2N}{N-2}}\\ \le\ & c\big||\bar\phi_1|^p\big|_{\frac{2N}{N+2}}|\mathcal{P}Z_1|_{\frac{2N}{N-2}}+|f'(\mathcal{P}\mathcal{U}_{\delta_1})\bar\phi_1|_{\frac{2N}{N+2}}|\mathcal{P}Z_1|_{\frac{2N}{N-2}}\\ \le\ & c_1\left(|\bar\phi_1|^{\frac{N+2}{N-2}}_{\frac{2N}{N-2}}|\mathcal{P}Z_1|_{\frac{2N}{N-2}}+|\mathcal{P}\mathcal{U}_{\delta_1}|^{\frac{4}{N-2}}_{\frac{2N}{N-2}}|\bar\phi_1|_{\frac{2N}{N-2}}|\mathcal{P}Z_1|_{\frac{2N}{N-2}}\right)\\ \le\ & c_2\left(\|\bar\phi_1\|^{\frac{N+2}{N-2}}|\mathcal{P}Z_1|_{\frac{2N}{N-2}}+|\mathcal{P}\mathcal{U}_{\delta_1}|^{\frac{4}{N-2}}_{\frac{2N}{N-2}}\|\bar\phi_1\||\mathcal{P}Z_1|_{\frac{2N}{N-2}}\right)\\ \le\ & c_3\epsilon^{\frac{\theta_1}{2}+\sigma}\delta_1^{-1}=c_4\epsilon^{\frac{N-2}{2(N-4)}+\sigma-\frac{1}{N-4}}\le c_4\epsilon^{\frac12}.\end{aligned}$$

Thus, from (3.30), (3.31) and the previous estimates, we get that for all sufficiently small $\epsilon>0$

$$\begin{aligned}|c_\epsilon|\ \le\ & \frac{1}{A\delta_1^{-2}+o(1)}\left[\left|\epsilon\int_\Omega\bar\phi_1\mathcal{P}Z_1\,dx\right|+\left|\epsilon\int_\Omega\mathcal{P}\mathcal{U}_{\delta_1}\mathcal{P}Z_1\,dx\right|\right.\\ &\left.+\left|\int_\Omega\left[f(\mathcal{P}\mathcal{U}_{\delta_1})-f(\mathcal{U}_{\delta_1})\right]\mathcal{P}Z_1\,dx\right|+\left|\int_\Omega\left[f(\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1)-f(\mathcal{P}\mathcal{U}_{\delta_1})\right]\,dx\right|\right]\\ &\le c\epsilon^{\frac{2}{N-4}+\frac12},\end{aligned} \tag{3.33}$$

uniformly with respect to d_1 satisfying $\eta<d_1<\frac{1}{\eta}$.

We observe that $\bar{\phi}_1$ is a classical solution of (3.29). This comes from the fact that $\bar{\phi}_1 \in H^1_0(\Omega)$ weakly solves (3.29), taking into account the smoothness of $\mathcal{P}\mathcal{U}_{\delta_1}$, $\mathcal{U}_{\delta_1}$, $\mathcal{P}Z_1$, from standard elliptic regularity theory and the application of a well-known lemma by Brezis and Kato.

We consider the quantity $\sup_{d_1 \in]\eta, \frac{1}{\eta}[} \left(\frac{|\bar{\phi}_1|_\infty}{|\mathcal{U}_{\delta_1}|_\infty} \right)$, which is defined for all $\epsilon \in (0, \epsilon_0)$, where $\epsilon_0 > 0$ is given by Proposition 3.6. We want to prove that

$$\lim_{\epsilon \to 0^+} \sup_{d_1 \in]\eta, \frac{1}{\eta}[} \left(\frac{|\bar{\phi}_1|_\infty}{|\mathcal{U}_{\delta_1}|_\infty} \right) = 0. \tag{3.34}$$

It is clear that (3.34) implies the thesis. In fact, we recall that, thanks to the definition (1.3) and the choice of δ_1 (see (2.3)), for any $d_1 \in]\eta, \frac{1}{\eta}[$ we have

$$\alpha_N \eta^{\frac{N-2}{2}} \epsilon^{-\frac{N-2}{2(N-4)}} < |\mathcal{U}_{\delta_1}|_\infty < \alpha_N \eta^{-\frac{N-2}{2}} \epsilon^{-\frac{N-2}{2(N-4)}}.$$

Hence, by this estimate and (3.34), we get that

$$0 \leq \sup_{d_1 \in]\eta, \frac{1}{\eta}[} \frac{|\bar{\phi}_1|_\infty}{\epsilon^{-\frac{N-2}{2(N-4)}}} = \sup_{d_1 \in]\eta, \frac{1}{\eta}[} \left(\frac{|\bar{\phi}_1|_\infty}{|\mathcal{U}_{\delta_1}|_\infty} \cdot \frac{|\mathcal{U}_{\delta_1}|_\infty}{\epsilon^{-\frac{N-2}{2(N-4)}}} \right) \leq \sup_{d_1 \in]\eta, \frac{1}{\eta}[} \left(\frac{|\bar{\phi}_1|_\infty}{|\mathcal{U}_{\delta_1}|_\infty} \right) \alpha_N \eta^{-\frac{N-2}{2}} \to 0,$$

as $\epsilon \to 0^+$, and we are done.

In order to prove (3.34) we argue by contradiction. Assume that (3.34) is false. Then, there exists a positive number $\tau \in \mathbb{R}^+$, a sequence $(\epsilon_k)_k \subset \mathbb{R}^+$, $\epsilon_k \to 0$ as $k \to +\infty$, such that

$$\sup_{d_1 \in]\eta, \frac{1}{\eta}[} \left(\frac{|\bar{\phi}_{1,k}|_\infty}{|\mathcal{U}_{\delta_{1,k}}|_\infty} \right) > \tau, \tag{3.35}$$

for any $k \in \mathbb{N}$, where, $\bar{\phi}_{1,k} := \bar{\phi}_1(\epsilon_k, d_1) \in B_{1,\epsilon_k}$ and $\delta_{1,k} := \epsilon_k^{\frac{1}{N-4}} d_1$. We observe that (3.35) contemplates the possibility that $\sup_{d_1 \in]\eta, \frac{1}{\eta}[} \left(\frac{|\bar{\phi}_{1,k}|_\infty}{|\mathcal{U}_{\delta_{1,k}}|_\infty} \right) = +\infty$. From (3.35), for any $k \in \mathbb{N}$, thanks to the definition of sup, we get that there exists $d_{1,k} \in]\eta, \frac{1}{\eta}[$ such that

$$\left(\frac{|\bar{\phi}_{1,k}|_\infty}{|\mathcal{U}_{\delta_{1,k}}|_\infty} \right) (d_{1,k}) > \frac{\tau}{2}.$$

Hence, if we consider the sequence $\left(\frac{|\bar{\phi}_{1,k}|_\infty}{|\mathcal{U}_{\delta_{1,k}}|_\infty} (d_{1,k}) \right)_k$, then, up to a subsequence, as $k \to +\infty$, there are only two possibilities:

- **(a):** $\frac{|\bar{\phi}_{1,k}|_\infty}{|\mathcal{U}_{\delta_{1,k}}|_\infty} (d_{1,k}) \to +\infty$;
- **(b):** $\frac{|\bar{\phi}_{1,k}|_\infty}{|\mathcal{U}_{\delta_{1,k}}|_\infty} (d_{1,k}) \to l$, for some $l \geq \frac{\tau}{2} > 0$.

We will show that (a) and (b) cannot happen.

Assume (a). We point out that, since $\eta > 0$ is fixed, then, $d_{1,k} \in]\eta, \frac{1}{\eta}[$ for all k, in particular this sequence stays definitely away from 0 and from $+\infty$. Hence, in order to simplify the notation of this proof, we omit the dependence from $d_{1,k}$ in $\bar{\phi}_{1,k}(d_{1,k})$ and in $\delta_{1,k}(d_{1,k}) = \epsilon_k^{\frac{1}{N-4}} d_{1,k}$ and thus we simply write $\bar{\phi}_{1,k}$, $\delta_{1,k}$. In particular, we observe that, for any fixed k, $\bar{\phi}_{1,k}$ is a function depending only on the space variable $x \in \Omega$.

Then, for any $k \in \mathbb{N}$, let $a_k \in \Omega$ such that $|\bar{\phi}_{1,k}(a_k)| = |\bar{\phi}_{1,k}|_\infty$ and set $M_k := |\bar{\phi}_{1,k}|_\infty$. Thanks to the assumption (a), since $|\mathcal{U}_{\delta_{1,k}}|_\infty = \alpha_N \delta_{1,k}^{-\frac{N-2}{2}} = \alpha_N \epsilon_k^{-\frac{N-2}{2(N-4)}} d_{1,k}^{-\frac{N-2}{2}}$, we get that $M_k \to +\infty$, as $k \to +\infty$. We consider the rescaled function

$$\widetilde{\phi}_{1,k}(y) := \frac{1}{M_k}\bar{\phi}_{1,k}\left(a_k + \frac{y}{M_k^\beta}\right), \qquad \beta = \frac{2}{N-2}$$

defined for $y \in \widetilde{\Omega}_k := M_k^{\frac{2}{N-2}}(\Omega - a_k)$. Moreover let us set

$$\widetilde{\mathcal{P}\mathcal{U}}_{1,k}(y) := \frac{1}{M_k}\mathcal{P}\mathcal{U}_{\delta_{1,k}}\left(a_k + \frac{y}{M_k^\beta}\right); \quad \widetilde{\mathcal{U}}_{1,k}(y) := \frac{1}{M_k}\mathcal{U}_{\delta_{1,k}}\left(a_k + \frac{y}{M_k^\beta}\right);$$

$$\widehat{\mathcal{P}Z}_{1,k}(y) := \frac{1}{M_k^{2\beta+1}}\mathcal{P}Z_{1,k}\left(a_k + \frac{y}{M_k^\beta}\right).$$

Since we are assuming (a) it is clear that $|\widetilde{\mathcal{P}\mathcal{U}}_{1,k}|_{\infty,\widetilde{\Omega}_k}, |\widetilde{\mathcal{U}}_{1,k}|_{\infty,\widetilde{\Omega}_k} \to 0$, as $k \to +\infty$. Moreover, thanks to the definition of Z_1, and since $\mathcal{P}Z_1 = Z_1 - \psi$, with $|\psi|_{\infty,\Omega} \leq C\delta_1^{\frac{N-4}{2}}$, we have that $|PZ_{1,k}|_\infty \simeq |Z_{1,k}|_\infty \simeq \delta_{1,k}^{-\frac{N}{2}}$, and hence, thanks to (a), we have $\frac{1}{M_k^{2\beta+1}} = o(\delta_{1,k}^{\frac{N+2}{2}})$, which implies that $|\widehat{\mathcal{P}Z}_{1,k}|_{\infty,\widetilde{\Omega}_k} \to 0$, as $k \to +\infty$. In particular, thanks to (3.33), the same conclusion holds for $c_{\epsilon_k}(d_{1,k})\widehat{\mathcal{P}Z}_{1,k}$. Taking into account that $2\beta + 1 = p$, by elementary computations, we see that $\widetilde{\phi}_{1,k}$ solves

$$\begin{cases} -\Delta\widetilde{\phi}_{1,k} = \frac{\epsilon_k}{M_k^{2\beta}}\widetilde{\phi}_{1,k} + \epsilon_k\frac{\widetilde{\mathcal{P}\mathcal{U}_{\delta_{1,k}}}}{M_k^{2\beta}} + f(\widetilde{\mathcal{P}\mathcal{U}}_{\delta_{1,k}} + \widetilde{\phi}_{1,k}) - f(\widetilde{\mathcal{U}}_{\delta_{1,k}}) + c_{\epsilon_k}(d_{1,k})\widehat{\mathcal{P}Z}_{1,k} & \text{in } \widetilde{\Omega}_k, \\ \widetilde{\phi}_{1,k} = 0 & \text{on } \partial\widetilde{\Omega}_k. \end{cases} \tag{3.36}$$

Let us denote by Π the limit domain of $\widetilde{\Omega}_k$. Since $M_k \to +\infty$, as $k \to +\infty$, we have that Π is the whole $\mathbb{R}^N$ or an half-space. Moreover, since the family $(\widetilde{\phi}_{1,k})_k$ is uniformly bounded and solves (3.36), then, by the same proof of Lemma 2.2 of [6], we get that $0 \in \Pi$ (in particular $0 \notin \partial\Pi$), and, by standard elliptic theory, it follows that, up to a subsequence, as $k \to +\infty$, we have that $\widetilde{\phi}_{1,k}$ converges in $C^2_{loc}(\Pi)$ to a function w which satisfies

$$-\Delta w = f(w) \text{ in } \Pi, \quad w(0) = 1 \text{ (or } w(0) = -1), \quad |w| \leq 1 \text{ in } \Pi, \quad w = 0 \text{ on } \partial\Pi. \tag{3.37}$$

We observe that, thanks to the definition of the chosen rescaling, by elementary computations (see Lemma 2 of [23]), it holds $\|\widetilde{\phi}_{1,k}\|^2_{\widetilde{\Omega}_\epsilon} = \|\bar{\phi}_{1,k}\|^2_\Omega$. Now, since $\|\bar{\phi}_{1,k}\| \leq c\epsilon_k^{\frac{\theta_1}{2}+\sigma}$, where c depends only on η and σ is some positive number (see Proposition 3.6), we have $\|\widetilde{\phi}_{1,k}\|^2_{\widetilde{\Omega}_k} = \|\bar{\phi}_{1,k}\|^2_\Omega \to 0$, as $k \to +\infty$. Hence, since $\widetilde{\phi}_{1,k} \to w$ in $C^2_{loc}(\Pi)$, by Fatou's lemma, it follows that

$$\|w\|^2_\Pi \leq \liminf_{k\to+\infty} \|\widetilde{\phi}_{1,k}\|^2_{\widetilde{\Omega}_k} = 0. \tag{3.38}$$

Therefore, since $\|w\|^2_\Pi = 0$ and w is smooth, it follows that w is constant, and from $w(0) = 1$ (or $w(0) = -1$) we get that $w \equiv 1$ (or $w \equiv -1$) in Π. But, since w is constant and solves $-\Delta w = f(w)$ in Π, then necessarily $f(w) \equiv 0$ in Π, and hence w must be the null function, but this contradicts $w \equiv 1$ (or $w \equiv -1$).

Alternatively, if Π is an half-space, by using the boundary condition $w = 0$ on $\partial\Pi$, we contradicts $w \equiv 1$ (or $w \equiv -1$). Hence, the only possibility is $\Pi = \mathbb{R}^N$. In this case, since w solves (3.37) and $\|w\|^2_\Pi \leq 2S^{N/2}$, it is well known that w cannot be sign-changing and hence, assuming without loss of generality that $w(0) = 1$, w must be a positive function of the form $\mathcal{U}_{\delta_N}$ (see

(1.3)), for some δ_N such that $\mathcal{U}_{\delta_N}(0) = 1$, and this contradicts $w \equiv 1$.
Hence (a) cannot happen.

Assume (b). Using the same convention on the notation as in previous case, we deduce that there exist two positive uniform constants c_1, c_2 such that

$$c_1\delta_{1,k}^{-\frac{N-2}{2}} \leq |\bar{\phi}_{1,k}|_\infty \leq c_2\delta_{1,k}^{-\frac{N-2}{2}}, \tag{3.39}$$

for all sufficiently large k. In particular, it still holds that $M_k \to +\infty$, as $k \to +\infty$. We consider the same rescaled functions $\widetilde{\phi}_{1,k}$ as in (a) and, as before, we denote by Π the limit domain of $\widetilde{\Omega}_k$. Now, up to a subsequence, since $\widetilde{\mathcal{PU}}_{1,k}$ and $\widetilde{\mathcal{U}}_{1,k}$ are uniformly bounded we see that they converge in $C^2_{loc}(\Pi)$ to a bounded function which we denote, respectively, by $\overline{\mathcal{PU}}$ and $\overline{\mathcal{U}}$ (one of them or both could be eventually the null function). In fact $\widetilde{\mathcal{U}}_{1,k}$ is uniformly bounded and solves $-\Delta\widetilde{\mathcal{U}}_{1,k} = \widetilde{\mathcal{U}}^p_{1,k}$ on $\widetilde{\Omega}_k$, and so by standard elliptic theory we get that $\widetilde{\mathcal{U}}_{1,k}$ converges in $C^2_{loc}(\Pi)$ to some non-negative bounded function $\overline{\mathcal{U}}$ which solves $-\Delta\overline{\mathcal{U}} = \overline{\mathcal{U}}^p$ in Π. Now, taking into account that $\widetilde{\mathcal{U}}_{1,k} \to \overline{\mathcal{U}}$ in $C^2_{loc}(\Pi)$, the same argument applies to $\widetilde{\mathcal{PU}_{1,k}}$, which solves

$$\begin{cases} -\Delta\widetilde{\mathcal{PU}}_{1,k} = \widetilde{\mathcal{U}}^p_{1,k} & \text{in } \widetilde{\Omega}_k, \\ \widetilde{\mathcal{PU}}_{1,k} = 0 & \text{on } \partial\widetilde{\Omega}_k, \end{cases}$$

and hence $\widetilde{\mathcal{PU}}_{1,k}$ converges in $C^2_{loc}(\Pi)$ to some non-negative bounded function $\overline{\mathcal{PU}}$ satisfying $-\Delta\overline{\mathcal{PU}} = \overline{\mathcal{U}}^p$ in Π, $\overline{\mathcal{PU}} = 0$ on $\partial\Pi$.

We point out that as in (a), but using (3.39), we still have $c_{\epsilon_k}(d_{1,k})|\widehat{\mathcal{P}Z}_{1,k}|_{\infty,\widetilde{\Omega}_k} \to 0$, as $k \to +\infty$. Moreover, by the proof of Lemma 2.2 of [6], it also holds that $0 \in \Pi$.

Hence, by standard elliptic theory, we have that $\widetilde{\phi}_{1,k}$ converges in $C^2_{loc}(\Pi)$ to a function w which solves

$$\begin{cases} -\Delta w = f(\overline{\mathcal{PU}} + w) - f(\overline{\mathcal{U}}) & \text{in } \Pi, \\ \\ w = 0 & \text{on } \partial\Pi, \\ \\ w(0) = 1 \text{ (or } w(0) = -1). \end{cases} \tag{3.40}$$

As in (3.38) we have $\|w\|^2_\Pi = 0$ and hence, since w is smooth, the only possibility is $w \equiv 1$ (or $w \equiv -1$) because of the condition $w(0) = 1$ (or $w(0) = -1$). Moreover, thanks to the definition of the chosen rescaling, it also holds $|\widetilde{\phi}_{1,k}|_{\frac{2N}{N-2},\widetilde{\Omega}_k} = |\bar{\phi}_{1,k}|_{\frac{2N}{N-2},\Omega}$ (for the proof see Lemma 2 of [23]). Therefore, since $|\bar{\phi}_{1,k}|_{\frac{2N}{N-2},\Omega} \to 0$ (because $\|\bar{\phi}_{1,k}\| \leq c\epsilon_k^{\frac{\theta_1}{2}+\sigma}$, where $c > 0$ depends only on η) and $\widetilde{\phi}_{1,k} \to w$ in $C^2_{loc}(\Pi)$, as $k \to +\infty$, then, by Fatou's Lemma, it follows that $|w|_{\frac{2N}{N-2},\Pi} = 0$, and thus it cannot happen that $w \equiv 1$ (or $w \equiv -1$).
Hence (a) and (b) cannot happen, and the proof is then concluded. □

We are now in position to prove Proposition 3.1.

Proof of Proposition 3.1. Let us fix $\eta > 0$ and let $\bar{\phi}_1 \in \mathcal{K}_1^\perp \cap B_{1,\epsilon}$ be the unique solution of the first equation of (2.12) found in Proposition 3.6. We define the operator $\mathcal{T}_2 : \mathcal{K}^\perp \to \mathcal{K}^\perp$ as

$$\mathcal{T}_2(\phi_2) := -\mathcal{L}_2^{-1}[\mathcal{N}_2(\bar{\phi}_1, \phi_2) + \mathcal{R}_2].$$

In order to find a solution of the second equation of (2.12) we solve the fixed point problem $\mathcal{T}_2(\phi_2) = \phi_2$. Let us define the proper ball

$$B_{2,\epsilon} := \{\phi_2 \in \mathcal{K}^{\perp};\ \|\phi_2\| \le r\ \epsilon^{\frac{\theta_2}{2}+\sigma}\}$$

for $r > 0$ sufficiently large and $\sigma > 0$ to be chosen later.
From Lemma 3.4, there exists $\epsilon_0 = \epsilon_0(\eta) > 0$ and $c = c(\eta) > 0$ such that:

$$\|\mathcal{T}_2(\phi_2)\| \le c(\|\mathcal{N}_2(\bar\phi_1, \phi_2)\| + \|\mathcal{R}_2\|), \tag{3.41}$$

and

$$\|\mathcal{T}_2(\phi_2) - \mathcal{T}_2(\psi_2)\| \le c(\|\mathcal{N}_2(\bar\phi_1, \phi_2) - \mathcal{N}_2(\bar\phi_1, \psi_2)\|), \tag{3.42}$$

for all $\phi_2, \psi_2 \in \mathcal{K}^{\perp}$, for all $(d_1, d_2) \in \mathbb{R}^2_+$ satisfying (2.4) and for all $\epsilon \in (0, \epsilon_0)$.

We begin with estimating the right hand side of (3.41).
Thanks to Proposition 3.8 we have that

$$\|\mathcal{R}_2\| \le c\epsilon^{\frac{\theta_2}{2}+\sigma},$$

for all $\epsilon \in (0, \epsilon_0)$, for all $(d_1, d_2) \in \mathbb{R}^2_+$ satisfying (2.4). Thus it remains only to estimate $\|\mathcal{N}_2(\bar\phi_1, \phi_2)\|$. Thanks to (2.1) and the definition of $\mathcal{N}_2$ we deduce:

$$\|\mathcal{N}_2(\bar\phi_1, \phi_2)\| \le c|f(V_\epsilon + \bar\phi_1 + \phi_2) - f(V_\epsilon) - f'(V_\epsilon)\phi_2 - f(\mathcal{P}\mathcal{U}_{\delta_1} + \bar\phi_1) + f(\mathcal{P}\mathcal{U}_{\delta_1})|_{\frac{2N}{N+2}}. \tag{3.43}$$

We estimate the right-hand side of (3.43):

$$\begin{aligned} & |f(V_\epsilon + \bar\phi_1 + \phi_2) - f(V_\epsilon) - f'(V_\epsilon)\phi_2 - f(\mathcal{P}\mathcal{U}_{\delta_1} + \bar\phi_1) + f(\mathcal{P}\mathcal{U}_{\delta_1})|_{\frac{2N}{N+2}} \\ \le\ & |f(V_\epsilon + \bar\phi_1 + \phi_2) - f(V_\epsilon + \bar\phi_1) - f'(V_\epsilon + \bar\phi_1)\phi_2|_{\frac{2N}{N+2}} + |(f'(V_\epsilon + \bar\phi_1) - f'(V_\epsilon))\phi_2|_{\frac{2N}{N+2}} \\ & + |f(V_\epsilon + \bar\phi_1) - f(V_\epsilon) - f(\mathcal{P}\mathcal{U}_{\delta_1} + \bar\phi_1) + f(\mathcal{P}\mathcal{U}_{\delta_1})|_{\frac{2N}{N+2}} \end{aligned}$$

In order to estimate the last three terms, by Lemma 2.2 and Lemma 2.3 we deduce that:

$$|f(V_\epsilon + \bar\phi_1 + \phi_2) - f(V_\epsilon + \bar\phi_1) - f'(V_\epsilon + \bar\phi_1)\phi_2| \le c|\phi_2|^p \tag{3.44}$$

and

$$|(f'(V_\epsilon + \bar\phi_1) - f'(V_\epsilon))\phi_2| \le c|\bar\phi_1|^{p-1}|\phi_2|. \tag{3.45}$$

Since $\frac{2N}{N+2} \cdot p = p + 1$ we get that

$$\int_\Omega |f(V_\epsilon + \bar\phi_1 + \phi_2) - f(V_\epsilon + \bar\phi_1) - f'(V_\epsilon + \bar\phi_1)\phi_2|^{\frac{2N}{N+2}}\, dx \quad \le \quad c\int_\Omega |\phi_2|^{p+1}\, dx,$$

and applying Sobolev inequality we deduce that

$$|f(V_\epsilon + \bar\phi_1 + \phi_2) - f(V_\epsilon + \bar\phi_1) - f'(V_\epsilon + \bar\phi_1)\phi_2|_{\frac{2N}{N+2}} \le c\|\phi_2\|^p. \tag{3.46}$$

By (3.45) we get that

$$\int_\Omega |(f'(V_\epsilon + \bar\phi_1) - f'(V_\epsilon))\phi_2|^{\frac{2N}{N+2}}\, dx \quad \le \quad c\int_\Omega |\bar\phi_1|^{(p-1)\frac{2N}{N+2}} |\phi_2|^{\frac{2N}{N+2}}\, dx.$$

We observe that $\phi_1^{(p-1)\frac{2N}{N+2}} \in L^{\frac{N+2}{4}}$, $\phi_2^{\frac{2N}{N+2}} \in L^p$ and p, $\frac{N+2}{4}$ are conjugate exponents in Hölder inequality. Moreover $(p-1)\frac{2N}{N+2}\frac{N+2}{4} = p + 1$ so

$$|(f'(V_\epsilon + \bar\phi_1) - f'(V_\epsilon))\phi_2|^{\frac{2N}{N+2}}_{\frac{2N}{N+2}} \le c|\bar\phi_1|_{p+1}^{\frac{8N}{(N+2)(N-2)}} |\phi_2|_{p+1}^{\frac{2N}{N+2}},$$

and hence by Sobolev inequality we deduce that

$$|(f'(V_\epsilon + \bar\phi_1) - f'(V_\epsilon))\phi_2|_{\frac{2N}{N+2}} \le c\|\bar\phi_1\|^{\frac{4}{N-2}}\|\phi_2\|. \tag{3.47}$$

It remains to estimate the last term. As in the proof of Proposition 3.8 we make the decomposition of the domain Ω as $\Omega = A_0 \sqcup A_1 \sqcup A_2$. Hence we get that:

$$|f(V_\epsilon + \bar\phi_1) - f(V_\epsilon) - f(\mathcal{PU}_{\delta_1} + \bar\phi_1) + f(\mathcal{PU}_{\delta_1})|_{\frac{2N}{N+2},A_0} \le |f(V_\epsilon + \bar\phi_1) - f(\mathcal{PU}_{\delta_1} + \bar\phi_1)|_{\frac{2N}{N+2},A_0} + |f(V_\epsilon) - f(\mathcal{PU}_{\delta_1})|_{\frac{2N}{N+2},A_0}$$

Then, by using the definition of δ_1, δ_2, the usual elementary inequalities, the computations made in (3.16) and Sobolev inequality, we get that

$$\begin{aligned}
|f(V_\epsilon + \bar\phi_1) - f(\mathcal{PU}_{\delta_1} + \bar\phi_1)|_{\frac{2N}{N+2},A_0} &\le c_1 \left(|\mathcal{PU}_{\delta_2}|^p_{p+1,A_0} + |\mathcal{PU}^{p-1}_{\delta_1}\mathcal{PU}_{\delta_2}|_{\frac{2N}{N+2},A_0} + \Big| |\bar\phi_1|^{p-1}\mathcal{PU}_{\delta_2}\Big|_{\frac{2N}{N+2},A_0} \right) \\
&\le c_2 \left(\delta_2^{\frac{N+2}{2}} + \delta_1^2 \delta_2^{\frac{N-2}{2}} + \|\bar\phi_1\|^{p-1}\delta_2^{\frac{N-2}{2}} \right) \\
&\le c_3 \epsilon^{\frac{\theta_2}{2}+\sigma},
\end{aligned}$$

for some $\sigma > 0$.

Moreover, as in the previous estimate, we get that

$$\begin{aligned}
|f(V_\epsilon) - f(\mathcal{PU}_{\delta_1})|_{\frac{2N}{N+2},A_0} &\le c_1 \left(|\mathcal{PU}^{p-1}_{\delta_1}\mathcal{PU}_{\delta_2}|_{\frac{2N}{N+2},A_0} + |\mathcal{PU}_{\delta_2}|^p_{\frac{2N}{N+2},A_0} \right) \\
&\le c_2 \epsilon^{\frac{\theta_2}{2}+\sigma}.
\end{aligned}$$

In A_1 we argue as in the previous case. The various terms now can be estimated as done in (3.19) and (3.20) and hence the same conclusion holds.

For A_2, by using the usual elementary inequalities, Lemma 3.9 and remembering the choice of δ_1, δ_2, we have:

$$\begin{aligned}
&|f(V_\epsilon + \bar\phi_1) - f(V_\epsilon) - f(\mathcal{PU}_{\delta_1} + \bar\phi_1) + f(\mathcal{PU}_{\delta_1})|_{\frac{2N}{N+2},A_2} \\
\le\ & |f(V_\epsilon + \bar\phi_1) - f(V_\epsilon) - f'(V_\epsilon)\bar\phi_1|_{\frac{2N}{N+2},A_2} + |f(\mathcal{PU}_{\delta_1} + \bar\phi_1) - f(\mathcal{PU}_{\delta_1}) - f'(\mathcal{PU}_{\delta_1})\bar\phi_1|_{\frac{2N}{N+2},A_2} \\
& + |[f'(V_\epsilon) - f'(\mathcal{PU}_{\delta_1})]\bar\phi_1|_{\frac{2N}{N+2},A_2} \\
\le\ & c\Big||\bar\phi_1|^p\Big|_{\frac{2N}{N+2},A_2} + c|\mathcal{PU}^{p-1}_{\delta_2}\bar\phi_1|_{\frac{2N}{N+2},A_2} \\
\le\ & c|\bar\phi_1|^p_\infty \left(\int_{A_2} 1\, dx \right)^{\frac{N+2}{2N}} + c|\bar\phi_1|_\infty \left(\int_{A_2} \mathcal{U}_{\delta_2}^{\frac{8N}{N^2-4}}\, dx \right)^{\frac{N+2}{2N}} \\
\le\ & c_1 \delta_1^{-\frac{N-2}{2}p} \left(\int_0^{\sqrt{\delta_1\delta_2}} r^{N-1}\, dr \right)^{\frac{N+2}{2N}} + c_2|\bar\phi_1|_\infty \left(\int_{A_2} \frac{\delta_2^{\frac{4N}{N+2}}}{(\delta_2^2 + |x|^2)^{\frac{4N}{N+2}}}\, dx \right)^{\frac{N+2}{2N}} \\
\le\ & c_3 \delta_1^{-\frac{N+2}{2}} (\delta_1\delta_2)^{\frac{N+2}{4}} + c_2|\bar\phi_1|_\infty \delta_2^2 \left(\int_{A_2} \frac{1}{|x|^{\frac{8N}{N+2}}}\, dx \right)^{\frac{N+2}{2N}} \\
\le\ & c_3 \left(\frac{\delta_2}{\delta_1} \right)^{\frac{N+2}{4}} + c_4 \delta_1^{-\frac{N-2}{2}} \delta_2^2 \left(\int_0^{\sqrt{\delta_1\delta_2}} r^{\frac{N^2-7N-2}{N+2}}\, dr \right)^{\frac{N+2}{2N}} \\
\le\ & c_3 \left(\frac{\delta_2}{\delta_1} \right)^{\frac{N+2}{4}} + c_5 \delta_1^{-\frac{N-2}{2}} \delta_2^2 (\delta_1\delta_2)^{\frac{N-6}{4}} = c_6 \left(\frac{\delta_2}{\delta_1} \right)^{\frac{N+2}{4}} \le c_7 \epsilon^{\frac{\theta_2}{2}+\sigma}.
\end{aligned}$$

Hence, from these estimates, we have

$$|f(V_\epsilon+\bar\phi_1)-f(V_\epsilon)-f(\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1)+f(\mathcal{P}\mathcal{U}_{\delta_1})|_{\frac{2N}{N+2}}\le c\epsilon^{\frac{\theta_2}{2}+\sigma}. \tag{3.48}$$

Since $\phi_2\in B_{2,\epsilon}$ and thanks to (3.43), (3.46), (3.47) and (3.48) we get that

$$\|\mathcal{T}_2(\phi_2)\|\le c\epsilon^{\frac{\theta_2}{2}+\sigma},\qquad \sigma>0$$

and hence $\mathcal{T}_2$ maps $B_{2,\epsilon}$ into itself .

It remains to prove that $\mathcal{T}_2:B_{2,\epsilon}\to B_{2,\epsilon}$ is a contraction. Thanks to (3.42) it suffices to estimate $\|\mathcal{N}_2(\bar\phi_1,\phi_2)-\mathcal{N}_2(\bar\phi_1,\psi_2)\|$ for any $\psi_2,\phi_2\in B_{2,\epsilon}$. To this end, thanks to (2.1), the definition of $\mathcal{N}_2$ and reasoning as in the proof of Proposition 3.6 we have:

$$\|\mathcal{N}_2(\bar\phi_1,\phi_2)-\mathcal{N}_2(\bar\phi_1,\psi_2)\|\le\epsilon^\alpha\|\phi_2-\psi_2\|,$$

for some $\alpha>0$.

At the end we get that there exists $L\in(0,1)$ such that

$$\|\mathcal{T}_2(\phi_2)-\mathcal{T}_2(\psi_2)\|\le L\|\phi_2-\psi_2\|.$$

Finally, taking into account that $d_1\to\bar\phi_1(d_1)$ is a C^1-map, a standard argument shows that also $(d_1,d_2)\to\bar\phi_2(d_1,d_2)$ is a C^1-map. The proof is complete. □

4. The reduced functional

We are left now to solve (2.11). Let $(\bar\phi_1,\bar\phi_2)\in\mathcal{K}_1^\perp\times\mathcal{K}^\perp$ be the solution found in Proposition 3.1. Hence $V_\epsilon+\bar\phi_1+\bar\phi_2$ is a solution of our original problem (1.1) if we can find $\bar d_\epsilon=(\bar d_{1\epsilon},\bar d_{2\epsilon})$ which satisfies condition (2.4) and solves equation (2.11).

To this end we consider the reduced functional $\tilde J_\epsilon:\mathbb{R}^2_+\to\mathbb{R}$ defined by:

$$\tilde J_\epsilon(d_1,d_2):=J_\epsilon(V_\epsilon+\bar\phi_1+\bar\phi_2),$$

where J_ϵ is the functional defined in (1.13).

Our main goal is to show first that solving equation (2.11) is equivalent to finding critical points $(\bar d_{1,\epsilon},\bar d_{2,\epsilon})$ of the reduced functional $\tilde J_\epsilon(d_1,d_2)$ and then that the reduced functional has a critical point. These facts are stated in the following proposition:

Proposition 4.1. *The following facts hold:*

(i): *If $(\bar d_{1,\epsilon},\bar d_{2,\epsilon})$ is a critical point of $\tilde J_\epsilon$, then the function $V_\epsilon+\bar\phi_1+\bar\phi_2$ is a solution of (1.1).*

(ii): *For any $\eta>0$, there exists $\epsilon_0>0$ such that for all $\epsilon\in(0,\epsilon_0)$ it holds:*

$$\tilde J_\epsilon(d_1,d_2)=\frac{2}{N}S^{N/2}+\epsilon^{\theta_1}\left[a_1\tau(0)d_1^{N-2}-a_2d_1^2\right]+O(\epsilon^{\theta_1+\sigma}), \tag{4.1}$$

with

$$O(\epsilon^{\theta_1+\sigma})=\epsilon^{\theta_1+\sigma}g(d_1)+\epsilon^{\theta_2}\left[a_3\tau(0)\left(\frac{d_2}{d_1}\right)^{\frac{N-2}{2}}-a_2d_2^2\right]+o\left(\epsilon^{\theta_2}\right), \tag{4.2}$$

for some function g depending only on d_1 (and uniformly bounded with respect to ϵ), where θ_1,θ_2 are defined in (3.1), σ is some positive real number (depending only on N), τ is the Robin's function of the domain Ω at the origin and

$$a_1:=\frac12\alpha_N^{p+1}\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}dy;\quad a_2:=\frac12\alpha_N^2\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{N-2}}dy;$$

$$a_3:=\alpha_N^{p+1}\int_{\mathbb{R}^N}\frac{1}{|y|^{N-2}(1+|y|^2)^{\frac{N+2}{2}}}dy.$$

The expansions (4.1), (4.2) *are C^0-uniform with respect to (d_1, d_2) satisfying condition* (2.4).

Remark 4.2. *We point out that the term g appearing in* (4.2) *does not depend on d_2 and this will be used in the sequel, in particular in* (5.5).

The aim of this section is to prove Proposition 4.1. First we prove two lemmas about the C^0-expansion of the reduced functional $\tilde{J}_\epsilon(d_1, d_2) := J_\epsilon(V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2)$, where $\bar{\phi}_1 \in \mathcal{K}_1^\perp \cap B_{1,\epsilon}$ and $\bar{\phi}_2 \in \mathcal{K}^\perp \cap B_{2,\epsilon}$ are the functions given by Proposition 3.1.

Lemma 4.3. *For any $\eta > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ it holds:*

$$J_\epsilon(V_\epsilon + \bar{\phi}_1) = J_\epsilon(V_\epsilon) + O(\epsilon^{\theta_1+\sigma}),$$

with

$$O(\epsilon^{\theta_1+\sigma}) = \epsilon^{\theta_1+\sigma} g_1(d_1) + O\left(\epsilon^{\theta_2+\sigma}\right), \tag{4.3}$$

for some function g_1 depending only on d_1 (and uniformly bounded with respect to ϵ), where θ_1, θ_2 are defined in (3.1), *σ is some positive real number (depending only on N). These expansion are C^0-uniform with respect to (d_1, d_2) satisfying condition* (2.4).

Proof. Let us fix $\eta > 0$. By direct computation we immediately see that

$$\begin{aligned} J_\epsilon(V_\epsilon + \bar{\phi}_1) - J_\epsilon(V_\epsilon) &= \tfrac{1}{2}\int_\Omega |\nabla\bar{\phi}_1|^2\,dx + \int_\Omega \nabla V_\epsilon \cdot \nabla\bar{\phi}_1\,dx - \tfrac{\epsilon}{2}\int_\Omega |\bar{\phi}_1|^2\,dx - \epsilon\int_\Omega V_\epsilon\bar{\phi}_1\,dx \\ &\quad - \tfrac{1}{p+1}\int_\Omega (|V_\epsilon + \bar{\phi}_1|^{p+1} - |V_\epsilon|^{p+1})\,dx. \end{aligned} \tag{4.4}$$

By definition we have

$$\int_\Omega \nabla V_\epsilon \cdot \nabla\bar{\phi}_1\,dx = \int_\Omega \nabla(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2})\cdot\nabla\bar{\phi}_1\,dx = \int_\Omega (\mathcal{U}_{\delta_1}^p - \mathcal{U}_{\delta_2}^p)\bar{\phi}_1\,dx = \int_\Omega [f(\mathcal{U}_{\delta_1}) - f(\mathcal{U}_{\delta_2})]\bar{\phi}_1\,dx,$$

moreover, since $F(s) = \frac{1}{p+1}|s|^{p+1}$ is a primitive of f, we can write (4.4) as

$$\begin{aligned} J_\epsilon(V_\epsilon + \bar{\phi}_1) - J_\epsilon(V_\epsilon) &= \tfrac{1}{2}\|\bar{\phi}_1\|^2 - \tfrac{\epsilon}{2}|\bar{\phi}_1|_2^2 - \epsilon\int_\Omega V_\epsilon\bar{\phi}_1\,dx + \int_\Omega [f(\mathcal{U}_{\delta_1}) - f(\mathcal{U}_{\delta_2})]\bar{\phi}_1\,dx \\ &\quad - \int_\Omega [F(V_\epsilon + \bar{\phi}_1) - F(V_\epsilon)]\,dx \\ &= \tfrac{1}{2}\|\bar{\phi}_1\|^2 - \tfrac{\epsilon}{2}|\bar{\phi}_1|_2^2 - \epsilon\int_\Omega V_\epsilon\bar{\phi}_1\,dx + \int_\Omega [f(\mathcal{U}_{\delta_1}) - f(\mathcal{U}_{\delta_2}) - f(V_\epsilon)]\bar{\phi}_1\,dx \\ &\quad - \int_\Omega [F(V_\epsilon + \bar{\phi}_1) - F(V_\epsilon) - f(V_\epsilon)\bar{\phi}_1]\,dx \\ &\quad A + B + C + D + E. \end{aligned} \tag{4.5}$$

A,B: Thanks to Proposition 3.1, for all sufficiently small ϵ, we have $\|\bar{\phi}_1\| \le c\epsilon^{\frac{\theta_1}{2}+\sigma}$, for some $c > 0$ and for some $\sigma > 0$ depending only on N. Hence we deduce that $A = O(\epsilon^{\theta_1+2\sigma})$, $B = O(\epsilon^{\theta_1+2\sigma+1})$. We point out that, since only $\bar{\phi}_1$ is involved in A and B, these terms depend only on d_1.

C: By definition we have

$$\epsilon\int_\Omega V_\epsilon\bar{\phi}_1\,dx = \epsilon\int_\Omega \mathcal{P}\mathcal{U}_{\delta_1}\bar{\phi}_1\,dx - \epsilon\int_\Omega \mathcal{P}\mathcal{U}_{\delta_2}\bar{\phi}_1\,dx = I_1 + I_2.$$

We observe that in the estimate I_1 only δ_1 and $\bar{\phi}_1$ are involved. Hence I_1 depends only on d_1. Thanks to Hölder inequality, we have the following:

$$|I_1| \le \epsilon|\mathcal{U}_{\delta_1}|_{\frac{2N}{N+2}}|\bar{\phi}_1|_{\frac{2N}{N-2}}$$

Since $N\geq 7$ we have $|\mathcal{U}_{\delta_i}|_{\frac{2N}{N+2}}=O(\delta_i^2)$, for $i=1,2$, so from our choice of δ_i (see (2.3)) and since $\|\bar\phi_1\|\leq c\epsilon^{\frac{\theta_1}{2}+\sigma}$ we deduce that

$$|I_1|\leq c\epsilon(\epsilon^{\frac{2}{N-4}}\epsilon^{\frac{N-2}{2(N-4)}+\sigma})\leq c\epsilon^{\theta_1+\sigma}, \tag{4.6}$$

for all sufficiently small ϵ. For I_2, with similar computations, we get that

$$|I_2|\leq \epsilon|\mathcal{U}_{\delta_2}|_{\frac{2N}{N+2}}|\bar\phi_1|_{\frac{2N}{N-2}}\leq c\epsilon^{1+\frac{2(3N-10)}{(N-4)(N-6)}}\epsilon^{\frac{N-2}{2(N-4)}+\sigma}.$$

Since $N\geq 7$ it is elementary to see that $1+\frac{2(3N-10)}{(N-4)(N-6)}+\frac{N-2}{2(N-4)}>\theta_2$. From this we deduce that

$$|I_2|\leq c\epsilon^{\theta_2+\sigma},$$

for all sufficiently small ϵ.

D: we have

$$\begin{aligned}\int_\Omega[f(\mathcal{U}_{\delta_1})-f(\mathcal{U}_{\delta_2})-f(V_\epsilon)]\bar\phi_1\,dx&=\underbrace{\int_\Omega[f(\mathcal{PU}_{\delta_1})-f(\mathcal{PU}_{\delta_2})-f(V_\epsilon)]\bar\phi_1\,dx}_{I_1}+\\&+\underbrace{\int_\Omega[f(\mathcal{U}_{\delta_1})-f(\mathcal{PU}_{\delta_1})]\bar\phi_1\,dx}_{I_2}+\underbrace{\int_\Omega[f(\mathcal{PU}_{\delta_2})-f(\mathcal{U}_{\delta_2})]\bar\phi_1\,dx}_{I_3}\end{aligned} \tag{4.7}$$

We evaluate separately the three terms.
We divide Ω into the three regions A_0, A_1, A_2 (see the proof of Proposition 3.8 for their definition). Then

$$\begin{aligned}\int_\Omega[f(\mathcal{PU}_{\delta_1})-f(\mathcal{PU}_{\delta_2})-f(V_\epsilon)]\bar\phi_1\,dx&=\underbrace{\sum_{j=0}^{1}\int_{A_j}[f(\mathcal{PU}_{\delta_1})-f(V_\epsilon)]\bar\phi_1\,dx}_{I_1'}\\&-\underbrace{\sum_{j=0}^{1}\int_{A_j}f(\mathcal{PU}_{\delta_2})\bar\phi_1,dx}_{I_1''}+\underbrace{\int_{A_2}[f(\mathcal{PU}_{\delta_1})-f(\mathcal{PU}_{\delta_2})-f(V_\epsilon)]\bar\phi_1\,dx}_{I_1'''}\end{aligned}$$

Now, writing $f(\mathcal{PU}_{\delta_1})-f(V_\epsilon)=f(\mathcal{PU}_{\delta_1})-f(V_\epsilon)+f'(\mathcal{PU}_{\delta_1})\mathcal{PU}_{\delta_2}-f'(\mathcal{PU}_{\delta_1})\mathcal{PU}_{\delta_2}$, applying the usual elementary inequalities, Hölder inequality and taking into account the computations made in (3.16) , (3.19), (3.20), we get that

$$\begin{aligned}|I_1'|&\leq c|\mathcal{PU}_{\delta_2}^p|_{\frac{2N}{N+2},A_0}|\bar\phi_1|_{\frac{2N}{N-2},A_0}+c|\mathcal{PU}_{\delta_2}^p|_{\frac{2N}{N+2},A_1}|\bar\phi_1|_{\frac{2N}{N-2},A_1}\\&\quad+c|\mathcal{PU}_{\delta_1}|^{p-1}_{\frac{2N}{N-2},A_0}|\mathcal{PU}_{\delta_2}|_{\frac{2N}{N+2},A_0}|\bar\phi_1|_{\frac{2N}{N-2},A_0}+c|\mathcal{PU}_{\delta_1}^{p-1}\mathcal{PU}_{\delta_2}|_{\frac{2N}{N+2},A_1}|\bar\phi_1|_{\frac{2N}{N-2},A_1}\\&\leq c_1\left(\delta_2^{\frac{N+2}{2}}\epsilon^{\frac{\theta_1}{2}+\sigma}+\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{4}}\epsilon^{\frac{\theta_1}{2}+\sigma}+\delta_1^2\delta_2^{\frac{N-2}{2}}\epsilon^{\frac{\theta_1}{2}+\sigma}+\left(\frac{\delta_2}{\delta_1}\right)^2\left(\int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}}r^{\frac{-N^2+5N-2}{N+2}}\,dr\right)^{\frac{N+2}{2N}}\epsilon^{\frac{\theta_1}{2}+\sigma}\right)\\&\leq c_2\left(\delta_2^{\frac{N+2}{2}}\epsilon^{\frac{\theta_1}{2}+\sigma}+\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{4}}\epsilon^{\frac{\theta_1}{2}+\sigma}+\delta_1^2\delta_2^{\frac{N-2}{2}}\epsilon^{\frac{\theta_1}{2}+\sigma}+\left(\frac{\delta_2}{\delta_1}\right)^2\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-6}{4}}\epsilon^{\frac{\theta_1}{2}+\sigma}\right)\\&\leq c_3\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{4}}\epsilon^{\frac{\theta_1}{2}+\sigma}\leq c_4\epsilon^{\theta_2+\sigma}.\end{aligned}$$

As before we have

$$|I_1''| \le \sum_{j=0}^{1} |f(\mathcal{P}\mathcal{U}_{\delta_2})|_{\frac{2N}{N+2},A_j} |\bar\phi_1|_{\frac{2N}{N-2},A_j} \le c\epsilon^{\theta_2+\sigma}.$$

Now, by Hölder inequality and reasoning as in (3.24), (3.25), (3.26), we get that

$$\begin{aligned}
|I_1'''| &\le |f(\mathcal{P}\mathcal{U}_{\delta_1}) - f(\mathcal{P}\mathcal{U}_{\delta_2}) - f(V_\epsilon)|_{\frac{2N}{N+2},A_2} |\bar\phi_1|_{\frac{2N}{N-2},A_2} \\
&\le c_1 \left(|\mathcal{P}\mathcal{U}_{\delta_1}^p|_{\frac{2N}{N+2},A_2} + |\mathcal{P}\mathcal{U}_{\delta_2}^{p-1}\mathcal{P}\mathcal{U}_{\delta_1}|_{\frac{2N}{N+2},A_2} \right) |\bar\phi_1|_{\frac{2N}{N-2},A_2} \\
&\le c_2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{4}} \epsilon^{\frac{\theta_1}{2}+\sigma} \le c_3 \epsilon^{\theta_2+\sigma}.
\end{aligned}$$

At the end we conclude that

$$|I_1| \le c\epsilon^{\theta_2+\sigma}.$$

For the remaining two terms of (4.7), reasoning as in the proof of Proposition 3.7, we get that

$$|f(\mathcal{P}\mathcal{U}_{\delta_i}) - f(\mathcal{U}_{\delta_i})|_{\frac{2N}{N+2}} \le c\delta_i^{\frac{N+2}{2}}.$$

Hence

$$|I_2| \le |f(\mathcal{U}_{\delta_1}) - f(\mathcal{P}\mathcal{U}_{\delta_1})|_{\frac{2N}{N+2}} |\bar\phi_1|_{\frac{2N}{N-2}} \le c\epsilon^{\frac{N+2}{2(N-4)}} \epsilon^{\frac{\theta_1}{2}+\sigma} \le c\epsilon^{\theta_1+\sigma},$$

for all sufficiently small ϵ. We remark that I_2 depends only on d_1 and hence it is sufficient that it is of order $\theta_1+\sigma$.
At the end

$$|I_3| \le |f(\mathcal{U}_{\delta_2}) - f(\mathcal{P}\mathcal{U}_{\delta_2})|_{\frac{2N}{N+2}} |\bar\phi_1|_{\frac{2N}{N-2}} \le c\epsilon^{\theta_2+\sigma},$$

for all sufficiently small ϵ.

E: We decompose Ω in the three regions A_j, $j=0,1,2$ used before.
For $j=0,1$ we have

$$\begin{aligned}
&\int_{A_j} \left[|V_\epsilon+\bar\phi_1|^{p+1} - |V_\epsilon|^{p+1} - (p+1)|V_\epsilon|^{p-1}V_\epsilon\bar\phi_1\right] dx \\
&\quad = \underbrace{\int_{A_j} \left[|\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2} + \bar\phi_1|^{p+1} - |\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1|^{p+1} + (p+1)|\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1|^{p-1}(\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1)\mathcal{P}\mathcal{U}_{\delta_2}\right] dx}_{I_1} \\
&\quad \underbrace{- \int_{A_j} \left[|\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}|^{p+1} - \mathcal{P}\mathcal{U}_{\delta_1}^{p+1} + (p+1)\mathcal{P}\mathcal{U}_{\delta_1}^p\mathcal{P}\mathcal{U}_{\delta_2}\right] dx}_{I_2} \\
&\quad \underbrace{+ \int_{A_j} \left[|\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1|^{p+1} - \mathcal{P}\mathcal{U}_{\delta_1}^{p+1} - (p+1)\mathcal{P}\mathcal{U}_{\delta_1}^p\bar\phi_1\right] dx}_{I_3} \\
&\quad -(p+1)\underbrace{\int_{A_j} \left[|\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}|^{p-1}(\mathcal{P}\mathcal{U}_{\delta_1} - \mathcal{P}\mathcal{U}_{\delta_2}) - \mathcal{P}\mathcal{U}_{\delta_1}^p\right]\bar\phi_1\, dx}_{I_4} \\
&\quad -(p+1)\underbrace{\int_{A_j} \left[|\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1|^{p-1}(\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1) - \mathcal{P}\mathcal{U}_{\delta_1}^p\right]\mathcal{P}\mathcal{U}_{\delta_2}\, dx}_{I_5}.
\end{aligned} \tag{4.8}$$

In order to estimate I_1, I_2, I_4 and I_5, applying the usual elementary inequalities, we see that

$$|I_1| \le c\left(\int_{A_j} \mathcal{P}\mathcal{U}_{\delta_2}^{p+1}\,dx + \int_{A_j} \mathcal{P}\mathcal{U}_{\delta_1}^{p-1}\mathcal{P}\mathcal{U}_{\delta_2}^{2}\,dx + \int_{A_j} |\bar\phi_1|^{p-1}\mathcal{P}\mathcal{U}_{\delta_2}^{2}\,dx\right)$$

$$|I_2| \le c\left(\int_{A_j} \mathcal{P}\mathcal{U}_{\delta_2}^{p+1}\,dx + \int_{A_j} \mathcal{P}\mathcal{U}_{\delta_1}^{p-1}\mathcal{P}\mathcal{U}_{\delta_2}^{2}\,dx\right)$$

$$|I_4| \le c\left(\int_{A_j} \mathcal{P}\mathcal{U}_{\delta_2}^{p}|\bar\phi_1|\,dx + \int_{A_j} \mathcal{P}\mathcal{U}_{\delta_1}^{p-1}\mathcal{P}\mathcal{U}_{\delta_2}|\bar\phi_1|\,dx\right)$$

$$|I_5| \le c\left(\int_{A_j} |\bar\phi_1|^{p}\mathcal{P}\mathcal{U}_{\delta_2}\,dx + \int_{A_j} \mathcal{P}\mathcal{U}_{\delta_1}^{p-1}\mathcal{P}\mathcal{U}_{\delta_2}|\bar\phi_1|\,dx\right).$$

Now, as seen in the proof of (3.16) and thanks to (3.20), we have

$$\int_{A_j} \mathcal{P}\mathcal{U}_{\delta_2}^{p+1}\,dx \le c\begin{cases} \delta_2^{N} & \text{if } j=0\\ \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}} & \text{if } j=1,\end{cases}$$

$$\int_{A_0} \mathcal{P}\mathcal{U}_{\delta_2}^{2}\mathcal{P}\mathcal{U}_{\delta_1}^{p-1}\,dx \le c|\mathcal{P}\mathcal{U}_{\delta_1}|_{p+1,A_0}^{p-1}|\mathcal{P}\mathcal{U}_{\delta_2}|_{p+1,A_0}^{2} \le c\delta_1^2\delta_2^{N-2}.$$

Moreover, by analogous computations, we get that

$$\int_{A_1} \mathcal{P}\mathcal{U}_{\delta_2}^{2}\mathcal{P}\mathcal{U}_{\delta_1}^{p-1}\,dx \le c_1\left(\frac{\delta_2}{\delta_1}\right)^2\int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}} \frac{1}{r^{N-3}}\,dr \le c_2\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}},$$

$$\int_{A_j} \mathcal{P}\mathcal{U}_{\delta_2}^{2}|\bar\phi_1|^{p-1}\,dx \le c|\mathcal{P}\mathcal{U}_{\delta_2}|_{p+1,A_j}^{2}\|\bar\phi_1\|^{p-1} \le c\begin{cases} \delta_2^{N-2}\epsilon^{(p-1)(\frac{\theta_1}{2}+\sigma)} & \text{if } j=0\\ \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\epsilon^{(p-1)(\frac{\theta_1}{2}+\sigma)} & \text{if } j=1,\end{cases}$$

$$\int_{A_j} \mathcal{P}\mathcal{U}_{\delta_2}^{p}|\bar\phi_1|\,dx \le c|\mathcal{P}\mathcal{U}_{\delta_2}|_{p+1,A_j}^{p}\|\bar\phi_1\| \le c\begin{cases} \delta_2^{\frac{N+2}{2}}\epsilon^{\frac{\theta_1}{2}+\sigma} & \text{if } j=0\\ \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{4}}\epsilon^{\frac{\theta_1}{2}+\sigma} & \text{if } j=1,\end{cases}$$

$$\int_{A_0} \mathcal{P}\mathcal{U}_{\delta_2}\mathcal{P}\mathcal{U}_{\delta_1}^{p-1}|\bar\phi_1|\,dx \le c|\mathcal{P}\mathcal{U}_{\delta_2}|_{p+1,A_0}|\mathcal{P}\mathcal{U}_{\delta_1}|_{p+1,A_0}^{p-1}\|\bar\phi_1\| \le c\delta_1^2\delta_2^{\frac{N-2}{2}}\epsilon^{\frac{\theta_1}{2}+\sigma},$$

and, thanks to (3.19), we have

$$\begin{aligned}\int_{A_1} \mathcal{P}\mathcal{U}_{\delta_2}\mathcal{P}\mathcal{U}_{\delta_1}^{p-1}|\bar\phi_1|\,dx &\le |\mathcal{P}\mathcal{U}_{\delta_2}\mathcal{P}\mathcal{U}_{\delta_1}^{p-1}|_{\frac{2N}{N+2},A_1}|\bar\phi_1|_{\frac{2N}{N-2},A_1}\\ &\le c_1\left[\int_{A_1}\left(\frac{\delta_2^{\frac{N-2}{2}}\delta_1^2}{(\delta_2^2+|x|^2)^{\frac{N-2}{2}}(\delta_1^2+|x|^2)^2}\right)^{\frac{2N}{N+2}}dx\right]^{\frac{N+2}{2N}}\epsilon^{\frac{\theta_1}{2}+\sigma}\\ &\le c_2\left(\frac{\delta_2}{\delta_1}\right)^2\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-6}{4}}\epsilon^{\frac{\theta_1}{2}+\sigma} = c_2\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{4}}\epsilon^{\frac{\theta_1}{2}+\sigma}\end{aligned}$$

At the end

$$\int_{A_0} \mathcal{P}\mathcal{U}_{\delta_2}|\bar\phi_1|^{p}\,dx \le c_1|\mathcal{P}\mathcal{U}_{\delta_2}|_{p+1,A_0}\|\bar\phi_1\|^{p} \le c_2\delta_2^{\frac{N-2}{2}}\epsilon^{p(\frac{\theta_1}{2}+\sigma)}$$

and, by using Lemma 3.9, we get that

$$\begin{aligned}
\int_{A_1} \mathcal{P}\mathcal{U}_{\delta_2}|\bar\phi_1|^p\,dx &\le c_1|\bar\phi_1|_\infty^{p-1}\left[\int_{A_1}\mathcal{P}\mathcal{U}_{\delta_2}^{\frac{2N}{N+2}}\,dx\right]^{\frac{N+2}{2N}}|\bar\phi_1|_{p+1,A_1}\\
&\le c_2\epsilon^{-\frac{2}{N-4}}\delta_2^2\left[\int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}} r^{\frac{-N^2+5N-2}{N+2}}\,dr\right]^{\frac{N+2}{2N}}\epsilon^{\frac{\theta_1}{2}+\sigma}\\
&= c_3\epsilon^{-\frac{2}{N-4}}\delta_2^2\left[\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-6}{4}}-\delta_2^{\frac{N-6}{2}}\right]\epsilon^{\frac{\theta_1}{2}+\sigma}\\
&\le c_4\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{4}}\epsilon^{\frac{\theta_1}{2}+\sigma}.
\end{aligned}$$

In order to estimate I_3 we observe that

$$\left|\int_{A_0}\left[|\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1|^{p+1}-\mathcal{P}\mathcal{U}_{\delta_1}^{p+1}-(p+1)\mathcal{P}\mathcal{U}_{\delta_1}^p\bar\phi_1\right]dx\right|\le c_1\left(\|\bar\phi_1\|^2|\mathcal{P}\mathcal{U}_{\delta_1}|_{p+1}^{p-1}+\|\bar\phi_1\|^{p+1}\right)\le c_2\epsilon^{\theta_1+\sigma},\tag{4.9}$$

which is sufficient since this term does not depend on d_2.
Moreover

$$\begin{aligned}
&\int_{A_1}\left[|\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1|^{p+1}-\mathcal{P}\mathcal{U}_{\delta_1}^{p+1}-(p+1)\mathcal{P}\mathcal{U}_{\delta_1}^p\bar\phi_1\right]dx=\int_{B(0,\rho)}\left[|\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1|^{p+1}-\mathcal{P}\mathcal{U}_{\delta_1}^{p+1}-(p+1)\mathcal{P}\mathcal{U}_{\delta_1}^p\bar\phi_1\right]dx\\
&-\int_{A_2}\left[|\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1|^{p+1}-\mathcal{P}\mathcal{U}_{\delta_1}^{p+1}-(p+1)\mathcal{P}\mathcal{U}_{\delta_1}^p\bar\phi_1\right]dx.
\end{aligned}$$

We observe that the first integral in the right-hand side of the previous equation depends only on d_1. Hence, as in (4.9), we have

$$\left|\int_{B(0,\rho)}\left[|\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1|^{p+1}-\mathcal{P}\mathcal{U}_{\delta_1}^{p+1}-(p+1)\mathcal{P}\mathcal{U}_{\delta_1}^p\bar\phi_1\right]dx\right|\le c\epsilon^{\theta_1+\sigma}.$$

Furthermore, by using Lemma 3.9, we get that

$$\begin{aligned}
&\left|\int_{A_2}\left[|\mathcal{P}\mathcal{U}_{\delta_1}+\bar\phi_1|^{p+1}-\mathcal{P}\mathcal{U}_{\delta_1}^{p+1}-(p+1)\mathcal{P}\mathcal{U}_{\delta_1}^p\bar\phi_1\right]dx\right|\le c_1\left(|\mathcal{P}\mathcal{U}_{\delta_1}|_{p+1,A_2}^{p-1}|\bar\phi_1|_{p+1,A_2}^2+|\bar\phi_1|_{p+1,A_2}^{p+1}\right)\\
&\le c_2\left(\left[\int_{B(0,\sqrt{\frac{\delta_2}{\delta_1}})}\frac{1}{(1+|y|^2)^N}\,dy\right]^{\frac{2}{N}}|\bar\phi_1|_\infty^2\left[\int_{A_2}1\,dx\right]^{\frac{2}{p+1}}+|\bar\phi_1|_\infty^{p+1}\int_{A_2}1\,dx\right)\\
&\le c_3\left(\left[\int_0^{\sqrt{\frac{\delta_2}{\delta_1}}}r^{N-1}\,dr\right]^{\frac{2}{N}}\epsilon^{-\frac{N-2}{N-4}}\left[\int_0^{\sqrt{\delta_1\delta_2}}r^{N-1}\,dr\right]^{\frac{2}{p+1}}+\epsilon^{-\frac{N}{N-4}}\int_0^{\sqrt{\delta_1\delta_2}}r^{N-1}\,dr\right)\\
&\le c_4\left(\frac{\delta_2}{\delta_1}\epsilon^{-\frac{N-2}{N-4}}(\delta_1\delta_2)^{\frac{N-2}{2}}+\epsilon^{-\frac{N}{N-4}}(\delta_1\delta_2)^{\frac{N}{2}}\right)\\
&\le c_5\epsilon^{\theta_2+\sigma}.
\end{aligned}\tag{4.10}$$

Now, it remains only to estimate the left-hand side of (4.8) for $j = 2$. Hence, thanks to the usual elementary inequalities, we get that

$$\left| \int_{A_2} \left[|V_\epsilon + \bar{\phi}_1|^{p+1} - |V_\epsilon|^{p+1} - (p+1)|V_\epsilon|^{p-1} V_\epsilon \bar{\phi}_1 \right] dx \right|$$
$$\leq c \left(\int_{A_2} |V_\epsilon|^{p-1} \bar{\phi}_1^2 \, dx + \int_{A_2} |\bar{\phi}_1|^{p+1} \, dx \right)$$
$$\leq c \left(\int_{A_2} \mathcal{P}\mathcal{U}_{\delta_1}^{p-1} \bar{\phi}_1^2 \, dx + \int_{A_2} \mathcal{P}\mathcal{U}_{\delta_2}^{p-1} \bar{\phi}_1^2 \, dx + \int_{A_2} |\bar{\phi}_1|^{p+1} \, dx \right)$$

For the first and third integrals in the last right-hand side we can reason as in (4.10). For the second integral, using Lemma 3.9, we have

$$\begin{aligned} \int_{A_2} \mathcal{P}\mathcal{U}_{\delta_2}^{p-1} \bar{\phi}_1^2 \, dx &\leq c_1 |\bar{\phi}_1|_\infty^2 \int_{A_2} \frac{\delta_2^2}{(\delta_2^2 + |x|^2)^2} \, dx \\ &\leq c_2 \delta_1^{-(N-2)} \delta_2^2 \int_{A_2} \frac{1}{|x|^4} \, dx \\ &\leq c_3 \delta_1^{-N+2} \delta_2^2 \int_0^{\sqrt{\delta_1 \delta_2}} r^{N-5} \, dr \\ &\leq c_4 \left(\frac{\delta_2}{\delta_1} \right)^{\frac{N}{2}} \end{aligned}$$

Finally, summing up all the estimates, we conclude that $|\mathbf{E}| = \epsilon^{\theta_1 + \sigma} g(d_1) + O(\epsilon^{\theta_2 + \sigma})$.

□

Lemma 4.4. *For any $\eta > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ it holds:*

$$J_\epsilon(V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2) = J_\epsilon(V_\epsilon + \bar{\phi}_1) + O(\epsilon^{\theta_2 + \sigma}),$$

C^0-uniformly with respect to (d_1, d_2) satisfying condition (2.4)*, for some positive real number σ depending only on N.*

Proof. As we have seen in the proof of Lemma 4.3, by direct computation we get that

$$\begin{aligned} J_\epsilon(V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2) - J_\epsilon(V_\epsilon + \bar{\phi}_1) &= \tfrac{1}{2} \textstyle\int_\Omega |\nabla \bar{\phi}_2|^2 \, dx + \int_\Omega \nabla(V_\epsilon + \bar{\phi}_1) \cdot \nabla \bar{\phi}_2 \, dx \\ &\quad - \tfrac{\epsilon}{2} \textstyle\int_\Omega |\bar{\phi}_2|^2 \, dx - \epsilon \int_\Omega (V_\epsilon + \bar{\phi}_1) \bar{\phi}_2 \, dx - \frac{1}{p+1} \int_\Omega (|V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2|^{p+1} - |V_\epsilon + \bar{\phi}_1|^{p+1}) \, dx \\ &= -\tfrac{1}{2} \|\bar{\phi}_2\|^2 + \tfrac{\epsilon}{2} |\bar{\phi}_2|_2^2 + \textstyle\int_\Omega \nabla(V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2) \cdot \nabla \bar{\phi}_2 \, dx \\ &\quad - \epsilon \textstyle\int_\Omega (V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2) \bar{\phi}_2 \, dx - \int_\Omega f(V_\epsilon + \bar{\phi}_1) \bar{\phi}_2 \, dx \\ &\quad - \textstyle\int_\Omega [F(V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2) - F(V_\epsilon + \phi_1) - f(V_\epsilon + \bar{\phi}_1) \bar{\phi}_2] \, dx \end{aligned} \tag{4.11}$$

Since $\bar{\phi}_1 + \bar{\phi}_2$ is a solution of (2.10) we have

$$\Pi^\perp \{ V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2 - i^* [\epsilon(V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2) + f(V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2)] \} = 0,$$

hence, for some $\psi \in \mathcal{K}$, we get that $V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2$ weakly solves

$$-\Delta(V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2) + \Delta \bar{\psi} - [\epsilon(V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2) + f(V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2)] = 0. \tag{4.12}$$

Choosing $\bar\phi_2$ as test function, since $\bar\phi_2 \in \mathcal{K}^\perp$, $\psi \in \mathcal{K}$ we deduce that

$$\int_\Omega \nabla(V_\epsilon + \bar\phi_1 + \bar\phi_2)\cdot\nabla\bar\phi_2\,dx - \epsilon\int_\Omega (V_\epsilon + \bar\phi_1 + \bar\phi_2)\bar\phi_2\,dx = \int_\Omega f(V_\epsilon + \bar\phi_1 + \bar\phi_2)\bar\phi_2\,dx \tag{4.13}$$

Thanks to (4.13) we rewrite (4.11) as

$$\begin{aligned} J_\epsilon(V_\epsilon + \bar\phi_1 + \bar\phi_2) - J_\epsilon(V_\epsilon + \bar\phi_1) &= -\frac{1}{2}\|\bar\phi_2\|^2 + \frac{\epsilon}{2}|\bar\phi_2|_2^2 + \int_\Omega [f(V_\epsilon + \bar\phi_1 + \bar\phi_2) - f(V_\epsilon + \bar\phi_1)]\bar\phi_2\,dx \\ &\quad - \int_\Omega [F(V_\epsilon + \bar\phi_1 + \bar\phi_2) - F(V_\epsilon + \phi_1) - f(V_\epsilon + \bar\phi_1)\bar\phi_2]\,dx \\ &= A + B + C + D. \end{aligned} \tag{4.14}$$

A, B: Thanks to Proposition 3.1, for all sufficiently small ϵ, we have $\|\bar\phi_2\| \le c\epsilon^{\frac{\theta_2}{2}+\sigma}$, for some $c > 0$ and for some $\sigma > 0$ depending only on N. Hence we deduce that $A = O(\epsilon^{\theta_2+2\sigma})$, $B = O(\epsilon^{\theta_2+2\sigma+1})$.

C: By Lemma 2.2 we get

$$\begin{aligned} \left|\int_\Omega [f(V_\epsilon + \bar\phi_1 + \bar\phi_2) - f(V_\epsilon + \bar\phi_1)]\bar\phi_2\,dx\right| &\le \int_\Omega |\bar\phi_2|^{p+1}\,dx + \int_\Omega |V_\epsilon + \bar\phi_1|^{p-1}\bar\phi_2^2\,dx \\ &\le c\|\bar\phi_2\|^{p+1} + c|V_\epsilon|_{p+1}^{p-1}|\bar\phi_2|_{p+1}^2 + c|\bar\phi_1|_{p+1}^{p-1}|\bar\phi_2|_{p+1}^2 \\ &\le c\epsilon^{\theta_2+\sigma} \end{aligned}$$

for all sufficiently small ϵ.

D: Applying Lemma 2.2 and Hölder inequality we get that

$$\left|\int_\Omega [F(V_\epsilon + \bar\phi_1 + \bar\phi_2) - F(V_\epsilon + \bar\phi_1) - f(V_\epsilon + \bar\phi_1)\bar\phi_2]\,dx\right| \le c|V_\epsilon|_{p+1}^{p-1}|\bar\phi_2|_{p+1}^2 + c|\bar\phi_1|_{p+1}^{p-1}|\bar\phi_2|_{p+1}^2 + c|\bar\phi_2|_{p+1}^{p+1}.$$

Since all the terms from A to D are high order terms with respect to ϵ^{θ_2} the proof is complete. □

In order to prove Proposition 4.1 some further preliminary lemmas are needed.

Lemma 4.5. *Let δ_j as in (2.3) for $j = 1, 2$ and $N \ge 7$. For any $\eta > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, it holds*

$$\frac{1}{2}\int_\Omega |\nabla\mathcal{P}\mathcal{U}_{\delta_j}|^2\,dx - \frac{1}{p+1}\int_\Omega \mathcal{P}\mathcal{U}_{\delta_j}^{p+1}\,dx = \frac{1}{N}S^{N/2} + a_1\tau(0)\delta_j^{N-2} + O(\delta_j^{N-1}),$$

C^0-uniformly with respect to (d_1, d_2) satisfying condition (2.4), where $a_1 := \frac{1}{2}\alpha_N^{p+1}\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy$ and $\tau(0)$ is the Robin's function of the domain Ω at the origin.

Proof. By using (1.6), (1.7) and (1.8) we have that

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla\mathcal{P}\mathcal{U}_{\delta_j}|^2\,dx - \frac{1}{p+1}\int_\Omega \mathcal{P}\mathcal{U}_{\delta_j}^{p+1}\,dx = \frac{1}{2}\int_\Omega \mathcal{U}_{\delta_j}^p\mathcal{P}\mathcal{U}_{\delta_j}\,dx - \frac{1}{p+1}\int_\Omega \mathcal{P}\mathcal{U}_{\delta_j}^{p+1}\,dx \\ &= \frac{1}{2}\int_\Omega \mathcal{U}_{\delta_j}^p(\mathcal{U}_{\delta_j} - \varphi_{\delta_j})\,dx - \frac{1}{p+1}\int_\Omega (\mathcal{U}_{\delta_j} - \varphi_{\delta_j})^{p+1}\,dx \\ &= \frac{1}{2}\int_\Omega \mathcal{U}_{\delta_j}^{p+1}\,dx - \frac{1}{2}\int_\Omega \mathcal{U}_{\delta_j}^p\varphi_{\delta_j}\,dx - \frac{1}{p+1}\int_\Omega \mathcal{U}_{\delta_j}^{p+1}\,dx + \int_\Omega \mathcal{U}_{\delta_j}^p\varphi_{\delta_j}\,dx + O\left(\int_\Omega \mathcal{U}_{\delta_j}^{p-1}\varphi_{\delta_j}^2\,dx\right) \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega \mathcal{U}_{\delta_j}^{p+1}\,dx + \frac{1}{2}\int_\Omega \mathcal{U}_{\delta_j}^p\varphi_{\delta_j}\,dx + O\left(\int_\Omega \mathcal{U}_{\delta_j}^{p-1}\varphi_{\delta_j}^2\,dx\right) \end{aligned}$$

Now it is easy to see that

$$\int_\Omega \mathcal{U}_{\delta_j}^{p+1}\ dx = \int_{\mathbb{R}^N} \frac{\alpha_N^{p+1}}{(1+|y|^2)^N}\ dy + O(\delta_j^N), \tag{4.15}$$

while

$$\begin{aligned}\int_\Omega \mathcal{U}_{\delta_j}^p \varphi_{\delta_j}\ dx &= \int_\Omega \mathcal{U}_{\delta_j}^p \left(\alpha_N \delta_j^{\frac{N-2}{2}} H(0,x) + O(\delta_j^{\frac{N+2}{2}})\right)\ dx \\ &= \alpha_N \delta_j^{\frac{N-2}{2}} \int_\Omega \mathcal{U}_{\delta_j}^p H(0,x)\ dx + O\left(\delta_j^{\frac{N+2}{2}} \int_\Omega \mathcal{U}_{\delta_j}^p\ dx\right) \\ &= \alpha_N^{p+1}\tau(0)\delta_j^{N-2} \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\ dy + O(\delta_j^{N-1}). \end{aligned} \tag{4.16}$$

Moreover

$$O\left(\int_\Omega \mathcal{U}_{\delta_j}^{p-1}\varphi_{\delta_j}^2\ dx\right) = O(\delta_j^{N-1}). \tag{4.17}$$

Indeed, we get

$$\begin{aligned}\int_\Omega \mathcal{U}_{\delta_j}^{p-1}\varphi_{\delta_j}^2\, dx &= \int_{B_{\sqrt{\delta_j}}(0)} \mathcal{U}_{\delta_j}^{p-1}\varphi_{\delta_j}^2\, dx + \int_{\Omega\setminus B_{\sqrt{\delta_j}}(0)} \mathcal{U}_{\delta_j}^{p-1}\varphi_{\delta_j}^2\, dx \\ &\le c_1\delta_j^{N-2}\int_{B_{\sqrt{\delta_j}}(0)} \mathcal{U}_{\delta_j}^{p-1}\, dx + |\varphi_{\delta_j}|_{p+1}^2 \left(\int_{\Omega\setminus B_{\sqrt{\delta_j}}(0)} \mathcal{U}_{\delta_j}^{p+1}\, dx\right)^{\frac{p-1}{p+1}} \\ &\le c_2\delta_j^{2N-4}\int_0^{\frac{1}{\sqrt{\delta_j}}} \frac{r^{N-1}}{(1+r^2)^2}\, dr + c_3\delta_j^{N-2}\left(\int_{\frac{1}{\sqrt{\delta_j}}}^{+\infty} \frac{r^{N-1}}{(1+r^2)^N}\ dr\right)^{\frac{p-1}{p+1}} \\ &\le c_4\delta_j^{\frac{3N-4}{2}} + c_5\delta_j^{N-1} \le c_6\delta_j^{N-1}.\end{aligned}$$

Hence, from (4.15), (4.16), (4.17) we get the thesis. □

Lemma 4.6. *Let δ_j as in (2.3) for $j=1,2$ and $N\ge 7$. For any $\eta>0$ there exists $\epsilon_0>0$ such that for any $\epsilon\in(0,\epsilon_0)$, it holds*

$$\frac{\epsilon}{2}\int_\Omega \mathcal{P}\mathcal{U}_{\delta_j}^2\, dx = a_2\epsilon\delta_j^2 + O(\epsilon\delta_j^{\frac{N}{2}}),$$

C^0-uniformly with respect to (d_1,d_2) satisfying condition (2.4), where $a_2 := \frac{1}{2}\alpha_N^2\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{N-2}}\, dy$.

Proof. From (1.6) we get that

$$\frac{\epsilon}{2}\int_\Omega (\mathcal{P}\mathcal{U}_{\delta_j})^2\ dx = \frac{\epsilon}{2}\int_\Omega (\mathcal{U}_{\delta_j}-\varphi_{\delta_j})^2\ dx = \frac{\epsilon}{2}\int_\Omega \mathcal{U}_{\delta_j}^2\ dx - \epsilon\int_\Omega \mathcal{U}_{\delta_j}\varphi_{\delta_j}\ dx + \frac{\epsilon}{2}\int_\Omega \varphi_{\delta_j}^2\ dx. \tag{4.18}$$

The principal term is the first one, in fact we have:

$$\begin{aligned}\frac{\epsilon}{2}\int_\Omega \mathcal{U}_{\delta_j}^2\,dx &= \frac{\epsilon}{2}\,\alpha_N^2\int_\Omega \frac{\delta_j^{N-2}}{(\delta_1^2+|x|^2)^{N-2}}\,dx = \frac{\epsilon}{2}\,\alpha_N^2\int_\Omega \frac{\delta_j^{-(N-2)}}{(1+|x/\delta_1|^2)^{N-2}}\,dx\\ &= \frac{\epsilon}{2}\,\alpha_N^2\int_{\Omega/\delta_j}\frac{\delta_j^{-(N-2)}}{(1+|y|^2)^{N-2}}\delta_j^N\,dy = \frac{\epsilon}{2}\,\alpha_N^2\delta_j^2\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{N-2}}\,dy+\\ &\quad +O\left(\epsilon\delta_j^2\int_{1/\delta_j}^{+\infty}\frac{r^{N-1}}{(1+r^2)^{N-2}}\,dr\right)\\ &= \frac{\epsilon}{2}\,\alpha_N^2\delta_j^2\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{N-2}}\,dy+O\left(\epsilon\delta_j^{N-2}\right).\end{aligned} \tag{4.19}$$

For the remaining terms, by using also (1.10), we deduce that

$$\epsilon\int_\Omega \mathcal{U}_{\delta_j}\varphi_{\delta_j}\,dx \le \epsilon|\mathcal{U}_{\delta_j}|_2|\varphi_{\delta_j}|_2 \le c\epsilon\delta_j\delta_j^{\frac{N-2}{2}} \le c\epsilon\delta_j^{\frac{N}{2}}. \tag{4.20}$$

Moreover by using again (1.10)

$$\frac{\epsilon}{2}\int_\Omega \varphi_{\delta_j}^2\,dx = \frac{\epsilon}{2}|\varphi_{\delta_j}|_2^2 \le C\epsilon\delta_j^{N-2}$$

and the lemma is proved. □

Lemma 4.7. *Let δ_j as in (2.3) for $j=1,2$ and $N\ge 7$. For any $\eta>0$ there exists $\epsilon_0>0$ such that for any $\epsilon\in(0,\epsilon_0)$ it holds*

$$\epsilon\int_\Omega \mathcal{P}\mathcal{U}_{\delta_1}\mathcal{P}\mathcal{U}_{\delta_2}\,dx = O\left(\epsilon\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\delta_1^2\right),$$

C^0-uniformly with respect to (d_1,d_2) satisfying condition (2.4).

Proof. From (1.6) we get that

$$\begin{aligned}\epsilon\int_\Omega \mathcal{P}\mathcal{U}_{\delta_1}\,\mathcal{P}\mathcal{U}_{\delta_2}\,dx &= \epsilon\int_\Omega(\mathcal{U}_{\delta_1}-\varphi_{\delta_1})(\mathcal{U}_{\delta_2}-\varphi_{\delta_2})\,dx\\ &= \epsilon\int_\Omega \mathcal{U}_{\delta_1}\mathcal{U}_{\delta_2}\,dx-\epsilon\int_\Omega \mathcal{U}_{\delta_1}\varphi_{\delta_2}\,dx-\epsilon\int_\Omega \mathcal{U}_{\delta_2}\varphi_{\delta_1}\,dx+\epsilon\int_\Omega \varphi_{\delta_1}\varphi_{\delta_2}\,dx.\end{aligned} \tag{4.21}$$

We analyze every term.

$$
\begin{aligned}
\epsilon \int_\Omega \mathcal{U}_{\delta_1}\mathcal{U}_{\delta_2}\, dx &= \epsilon\, \alpha_N^2 \int_\Omega \frac{\delta_1^{-\frac{N-2}{2}}}{(1+|x/\delta_1|^2)^{\frac{N-2}{2}}} \frac{\delta_2^{\frac{N-2}{2}}}{(\delta_2^2+|x|^2)^{\frac{N-2}{2}}}\, dx \\
&= \epsilon\, \alpha_N^2 \int_{\Omega/\delta_1} \frac{\delta_1^{\frac{N+2}{2}}}{(1+|y|^2)^{\frac{N-2}{2}}} \frac{\delta_2^{\frac{N-2}{2}}}{(\delta_2^2+\delta_1^2|y|^2)^{\frac{N-2}{2}}}\, dy \\
&= \epsilon\, \alpha_N^2 \int_{\Omega/\delta_1} \frac{\delta_1^{-\frac{N-6}{2}}}{(1+|y|^2)^{\frac{N-2}{2}}} \frac{\delta_2^{\frac{N-2}{2}}}{\left(\left(\frac{\delta_2}{\delta_1}\right)^2+|y|^2\right)^{\frac{N-2}{2}}}\, dy \\
&\le \epsilon\, \alpha_N^2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2 \int_{\Omega/\delta_1} \frac{1}{(1+|y|^2)^{\frac{N-2}{2}}|y|^{N-2}}\, dy \\
&= \epsilon\, \alpha_N^2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2 \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N-2}{2}}|y|^{N-2}}\, dy \\
&+ O\left(\epsilon \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2 \int_{1/\delta_1}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{\frac{N-2}{2}} r^{N-2}}\, dr\right) \\
&= \epsilon\, \alpha_N^2 \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2 \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N-2}{2}}|y|^{N-2}}\, dy + O\left(\epsilon \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^{N-2}\right).
\end{aligned} \tag{4.22}
$$

Hence $\epsilon \int_\Omega \mathcal{U}_{\delta_1}\mathcal{U}_{\delta_2}\, dx = O\left(\epsilon \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2\right)$. Thanks to (4.20) we deduce that $\epsilon \int_\Omega \mathcal{U}_{\delta_1}\varphi_{\delta_2}\, dx = O\left(\epsilon\, \delta_1^{\frac{N-2}{2}} \delta_2^{\frac{N-2}{2}}\right)$, $\epsilon \int_\Omega \mathcal{U}_{\delta_2}\varphi_{\delta_1}\, dx = O\left(\epsilon\, \delta_1^{\frac{N-2}{2}} \delta_2^{\frac{N-2}{2}}\right)$. Moreover it is clear that $\epsilon \int_\Omega \varphi_{\delta_1}\varphi_{\delta_2}\, dx = O\left(\epsilon\, \delta_1^{\frac{N-2}{2}} \delta_2^{\frac{N-2}{2}}\right)$. Since these last three terms are high order terms compared to $\epsilon \left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}} \delta_1^2$, we deduce the thesis, and the proof is complete. □

We are ready to prove Proposition 4.1.

Proof of Proposition 4.1. (i): One can reason as Part 1 of Proposition 2.2 of [27].

(ii): Let us fix $\eta > 0$. From Lemma 4.3 and Lemma 4.4, for all sufficiently small ϵ, we get that

$$J_\epsilon(V_\epsilon + \bar\phi_1 + \bar\phi_2) = J_\epsilon(V_\epsilon) + \epsilon^{\theta_1+\sigma} g(d_1) + O(\epsilon^{\theta_2+\sigma}),$$

for some $\sigma > 0$. We evaluate $J_\epsilon(V_\epsilon) = J_\epsilon(\mathcal{PU}_{\delta_1} - \mathcal{PU}_{\delta_2})$.

$$
\begin{aligned}
J_\epsilon(\mathcal{PU}_{\delta_1} - \mathcal{PU}_{\delta_2}) &= \frac{1}{2}\int_\Omega |\nabla(\mathcal{PU}_{\delta_1} - \mathcal{PU}_{\delta_2})|^2\, dx - \frac{1}{p+1}\int_\Omega |\mathcal{PU}_{\delta_1} - \mathcal{PU}_{\delta_2}|^{p+1}\, dx \\
&- \frac{\epsilon}{2}\int_\Omega (\mathcal{PU}_{\delta_1} - \mathcal{PU}_{\delta_2})^2 dx \\
&= \frac{1}{2}\int_\Omega |\nabla \mathcal{PU}_{\delta_1}|^2\, dx + \frac{1}{2}\int_\Omega |\nabla \mathcal{PU}_{\delta_2}|^2\, dx - \int_\Omega \nabla \mathcal{PU}_{\delta_1}\cdot \nabla \mathcal{PU}_{\delta_2}\, dx \\
&- \frac{1}{p+1}\int_\Omega |\mathcal{PU}_{\delta_1} - \mathcal{PU}_{\delta_2}|^{p+1}\, dx - \frac{\epsilon}{2}\int_\Omega (\mathcal{PU}_{\delta_1})^2\, dx - \frac{\epsilon}{2}\int_\Omega (\mathcal{PU}_{\delta_2})^2\, dx
\end{aligned}
$$

$$
\begin{aligned}
&+\quad \epsilon\int_\Omega \mathcal{P}\mathcal{U}_{\delta_1}\ \mathcal{P}\mathcal{U}_{\delta_2}\ dx\\
&=\quad \underbrace{\sum_{j=1}^{2}\left(\frac{1}{2}\int_\Omega |\nabla \mathcal{P}\mathcal{U}_{\delta_j}|^2\,dx-\frac{1}{p+1}\int_\Omega \mathcal{P}\mathcal{U}_{\delta_j}^{p+1}\,dx\right)}_{(I)}-\underbrace{\sum_{j=1}^{2}\frac{\epsilon}{2}\int_\Omega \mathcal{P}\mathcal{U}_{\delta_j}^2\,dx}_{(II)}\\
&\quad+\underbrace{\epsilon\int_\Omega \mathcal{P}\mathcal{U}_{\delta_1}\mathcal{P}\mathcal{U}_{\delta_2}\,dx}_{(III)}-\underbrace{\int_\Omega \nabla\mathcal{P}\mathcal{U}_{\delta_1}\nabla\mathcal{P}\mathcal{U}_{\delta_2}\,dx}_{(IV)}\\
&\quad\underbrace{-\frac{1}{p+1}\int_\Omega\left[|\mathcal{P}\mathcal{U}_{\delta_1}-\mathcal{P}\mathcal{U}_{\delta_2}|^{p+1}-\mathcal{P}\mathcal{U}_{\delta_1}^{p+1}-\mathcal{P}\mathcal{U}_{\delta_2}^{p+1}\right]dx}_{(IV)}.
\end{aligned}
$$

By Lemma 4.5, Lemma 4.6 and Lemma 4.7 we get

$$
\begin{aligned}
(I) &= \frac{2}{N}S^{N/2}+a_1\tau(0)\delta_1^{N-2}+a_1\tau(0)\delta_2^{N-2}+O(\delta_1^{N-1})+O(\delta_2^{N-1}),\\
(II) &= a_2\epsilon\delta_1^2+a_2\epsilon\delta_2^2+O(\epsilon\delta_1^{\frac{N}{2}})+O(\epsilon\delta_2^{\frac{N}{2}}),\\
(III) &= O\left(\epsilon\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\delta_1^2\right).
\end{aligned}
$$

Now since $-\Delta\mathcal{P}\mathcal{U}_{\delta_2}=\mathcal{U}_{\delta_2}^p$ then $\int_\Omega \nabla\mathcal{P}\mathcal{U}_{\delta_1}\nabla\mathcal{P}\mathcal{U}_{\delta_2}\,dx=\int_\Omega \mathcal{U}_{\delta_2}^p\mathcal{P}\mathcal{U}_{\delta_1}\,dx$ and hence

$$
\begin{aligned}
(IV)\quad &=\quad \underbrace{-\frac{1}{p+1}\int_\Omega\left[|\mathcal{P}\mathcal{U}_{\delta_1}-\mathcal{P}\mathcal{U}_{\delta_2}|^{p+1}-\mathcal{P}\mathcal{U}_{\delta_1}^{p+1}-\mathcal{P}\mathcal{U}_{\delta_2}^{p+1}+(p+1)\mathcal{P}\mathcal{U}_{\delta_2}^p\mathcal{P}\mathcal{U}_{\delta_1}\right]dx}_{I_1}\\
&\quad+\underbrace{\int_\Omega\left[\mathcal{P}\mathcal{U}_{\delta_2}^p-\mathcal{U}_{\delta_2}^p\right]\mathcal{P}\mathcal{U}_{\delta_1}\,dx}_{I_2}.
\end{aligned}
$$

By (1.6) and Lemma 2.2 we deduce that

$$
|I_2|\le C\int_\Omega \mathcal{U}_{\delta_2}^{p-1}\varphi_{\delta_2}\mathcal{P}\mathcal{U}_{\delta_1}\,dx+C\int_\Omega \varphi_{\delta_2}^p\mathcal{P}\mathcal{U}_{\delta_1}\,dx.
$$

Now let $\rho>0$ such that $B(0,\rho)\subset\Omega$.

$$
\begin{aligned}
\int_\Omega \varphi_{\delta_2}^p\mathcal{P}\mathcal{U}_{\delta_1}\,dx &\le \int_\Omega \varphi_{\delta_2}^p\mathcal{U}_{\delta_1}\,dx=\int_{\Omega\setminus B(0,\rho)}\varphi_{\delta_2}^p\mathcal{U}_{\delta_1}\,dx+\int_{B(0,\rho)}\varphi_{\delta_2}^p\mathcal{U}_{\delta_1}\,dx\\
&\le |\varphi_{\delta_2}|_{p+1}^p\left(\int_{\Omega\setminus B(0,\rho)}\mathcal{U}_{\delta_1}^{p+1}\,dx\right)^{\frac{1}{p+1}}+C\delta_2^{\frac{N+2}{2}}\int_{B(0,\rho)}\frac{1}{\left(1+\left|\frac{x}{\delta_1}\right|^2\right)^{\frac{N-2}{2}}}\,dx\\
&\le C_1\delta_2^{\frac{N+2}{2}}\delta_1^{\frac{N-2}{2}}+C_2\delta_2^{\frac{N+2}{2}}\delta_1^N\int_0^{\frac{\rho}{\delta_1}}\frac{r^{N-1}}{(1+r^2)^{\frac{N-2}{2}}}\,dr\\
&\le C_3\left[\delta_2^{\frac{N+2}{2}}\delta_1^{\frac{N-2}{2}}+\delta_2^{\frac{N+2}{2}}\delta_1^{N-2}\right]\le C_3\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N+2}{2}}\delta_1^{\frac{N}{2}}.
\end{aligned}
$$

Moreover, since $\int_\Omega \mathcal{U}_{\delta_j}^p\,dx = O(\delta_j^{\frac{N-2}{2}})$, we get

$$
\begin{aligned}
\int_\Omega \mathcal{U}_{\delta_2}^{p-1}\varphi_{\delta_2}\mathcal{P}\mathcal{U}_{\delta_1}\,dx &\le \|\varphi_{\delta_2}\|_\infty \int_\Omega \mathcal{U}_{\delta_2}^{p-1}\mathcal{U}_{\delta_1}\,dx \le C\delta_2^{\frac{N-2}{2}}\int_\Omega \mathcal{U}_{\delta_2}^{p-1}\mathcal{U}_{\delta_1}\,dx\\
&\le C\delta_2^{\frac{N-2}{2}}\left(\int_\Omega \mathcal{U}_{\delta_2}^{p}\,dx\right)^{\frac{p-1}{p}}\left(\int_\Omega \mathcal{U}_{\delta_1}^{p}\,dx\right)^{\frac{1}{p}}\\
&\le C_1\delta_2^{\frac{N^2+4N-12}{2(N+2)}}\delta_1^{\frac{(N-2)^2}{2(N+2)}} = C_1\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\left(\frac{\delta_2}{\delta_1}\right)^{\frac{2(N-2)}{N+2}}\delta_1^{N-2}\\
&\le C_1\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}\delta_1^{N-2}.
\end{aligned}
$$

Now let $\rho > 0$ and we decompose the domain Ω as $\Omega = A_0 \cup A_1 \cup A_2$ where $A_0 = \Omega \setminus B(0,\rho)$, $A_1 = B(0,\rho)\setminus B(0,\sqrt{\delta_1\delta_2})$, $A_2 = B(0,\sqrt{\delta_1\delta_2})$. Then we define

$$
L_j := -\frac{1}{p+1}\int_{A_j}\left[|\mathcal{P}\mathcal{U}_{\delta_1}-\mathcal{P}\mathcal{U}_{\delta_2}|^{p+1} - \mathcal{P}\mathcal{U}_{\delta_1}^{p+1} - \mathcal{P}\mathcal{U}_{\delta_2}^{p+1} + (p+1)\mathcal{P}\mathcal{U}_{\delta_2}^{p}\mathcal{P}\mathcal{U}_{\delta_1}\right]dx
$$

for $j = 0, 1, 2$.
Now, by using Lemma 2.2 and Hölder inequality, we see that

$$
\begin{aligned}
|L_0| &\le \frac{1}{p+1}\left[\int_{A_0}\left(|\mathcal{P}\mathcal{U}_{\delta_1}-\mathcal{P}\mathcal{U}_{\delta_2}|^{p+1} - \mathcal{P}\mathcal{U}_{\delta_1}^{p+1}\right)dx + \int_{A_0}\mathcal{P}\mathcal{U}_{\delta_2}^{p+1}\,dx + \int_{A_0}\mathcal{P}\mathcal{U}_{\delta_2}^{p}\mathcal{P}\mathcal{U}_{\delta_1}\,dx\right]\\
&\le C\left(\int_{A_0}\mathcal{P}\mathcal{U}_{\delta_1}^{p}\mathcal{P}\mathcal{U}_{\delta_2}\,dx + \int_{A_0}\mathcal{P}\mathcal{U}_{\delta_2}^{p+1}\,dx + \int_{A_0}\mathcal{P}\mathcal{U}_{\delta_2}^{p}\mathcal{P}\mathcal{U}_{\delta_1}\,dx\right)\\
&\le C\left(\int_{A_0}\mathcal{U}_{\delta_1}^{p}\mathcal{U}_{\delta_2}\,dx + \int_{A_0}\mathcal{U}_{\delta_2}^{p+1}\,dx + \int_{A_0}\mathcal{U}_{\delta_2}^{p}\mathcal{U}_{\delta_1}\,dx\right)\\
&\le C\left(\int_{A_0}\mathcal{U}_{\delta_1}^{p+1}\,dx\right)^{\frac{p}{p+1}}\left(\int_{A_0}\mathcal{U}_{\delta_2}^{p+1}\,dx\right)^{\frac{1}{p+1}} + C_1\int_{\frac{\rho}{\delta_2}}^{+\infty}\frac{r^{N-1}}{(1+r^2)^N}\,dr\\
&\quad +C\left(\int_{A_0}\mathcal{U}_{\delta_2}^{p+1}\,dx\right)^{\frac{p}{p+1}}\left(\int_{A_0}\mathcal{U}_{\delta_1}^{p+1}\,dx\right)^{\frac{1}{p+1}}\\
&\le C_2\left(\int_{\frac{\rho}{\delta_1}}^{+\infty}\frac{r^{N-1}}{(1+r^2)^N}\,dr\right)^{\frac{p}{p+1}}\left(\int_{\frac{\rho}{\delta_2}}^{+\infty}\frac{r^{N-1}}{(1+r^2)^N}\,dr\right)^{\frac{1}{p+1}} + C_3\delta_2^N\\
&\quad +C_2\left(\int_{\frac{\rho}{\delta_2}}^{+\infty}\frac{r^{N-1}}{(1+r^2)^N}\,dr\right)^{\frac{p}{p+1}}\left(\int_{\frac{\rho}{\delta_1}}^{+\infty}\frac{r^{N-1}}{(1+r^2)^N}\,dr\right)^{\frac{1}{p+1}}\\
&\le C_4\left(\delta_1^{\frac{N+2}{2}}\delta_2^{\frac{N-2}{2}} + \delta_2^N + \delta_2^{\frac{N+2}{2}}\delta_1^{\frac{N-2}{2}}\right) \le C_5\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\delta_1^N.
\end{aligned}
$$

Now

$$
\begin{aligned}
L_1 &= -\frac{1}{p+1}\int_{A_1}\left[|\mathcal{P}\mathcal{U}_{\delta_1}-\mathcal{P}\mathcal{U}_{\delta_2}|^{p+1} - \mathcal{P}\mathcal{U}_{\delta_1}^{p+1} + (p+1)\mathcal{P}\mathcal{U}_{\delta_1}^{p}\mathcal{P}\mathcal{U}_{\delta_2}\right]dx\\
&\quad +\int_{A_1}\mathcal{P}\mathcal{U}_{\delta_1}^{p}\mathcal{P}\mathcal{U}_{\delta_2}\,dx - \int_{A_1}\mathcal{P}\mathcal{U}_{\delta_2}^{p}\mathcal{P}\mathcal{U}_{\delta_1}\,dx - \frac{1}{p+1}\int_{A_1}\mathcal{P}\mathcal{U}_{\delta_2}^{p+1}\,dx.
\end{aligned}
$$

Applying Lemma 2.2 we get

$$\left|\int_{A_1}\left[|\mathcal{P}\mathcal{U}_{\delta_1}-\mathcal{P}\mathcal{U}_{\delta_2}|^{p+1}-\mathcal{P}\mathcal{U}_{\delta_1}^{p+1}+(p+1)\mathcal{P}\mathcal{U}_{\delta_1}^{p}\mathcal{P}\mathcal{U}_{\delta_2}\right]dx\right|\le C\left(\int_{A_1}\mathcal{P}\mathcal{U}_{\delta_1}^{p-1}\mathcal{P}\mathcal{U}_{\delta_2}^{2}\,dx+\int_{A_1}\mathcal{P}\mathcal{U}_{\delta_2}^{p+1}\,dx\right)$$
$$\le C_1\left(\left(\frac{\delta_2}{\delta_1}\right)^2\int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}}\frac{1}{r^{N-3}}\,dr+\int_{\sqrt{\frac{\delta_2}{\delta_1}}}^{\frac{\rho}{\delta_2}}\frac{r^{N-1}}{(1+r^2)^N}\,dr\right)\le C_2\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}.$$

Thanks to (1.6) and Lemma 2.2 we have

$$\begin{aligned}\int_{A_1}\mathcal{P}\mathcal{U}_{\delta_1}^{p}\mathcal{P}\mathcal{U}_{\delta_2}\,dx &= \int_{A_1}\mathcal{U}_{\delta_1}^{p}\mathcal{P}\mathcal{U}_{\delta_2}\,dx+O\left(\int_{A_1}\mathcal{U}_{\delta_1}^{p-1}\varphi_{\delta_1}\mathcal{P}\mathcal{U}_{\delta_2}\,dx\right)+O\left(\int_{A_1}\varphi_{\delta_1}^{p}\mathcal{P}\mathcal{U}_{\delta_2}\,dx\right)\\ &= \int_{A_1}\mathcal{U}_{\delta_1}^{p}\mathcal{U}_{\delta_2}\,dx+O\left(\int_{\Omega}\mathcal{U}_{\delta_1}^{p}\varphi_{\delta_2}\,dx\right)+O\left(\int_{\Omega}\mathcal{U}_{\delta_1}^{p-1}\varphi_{\delta_1}\mathcal{P}\mathcal{U}_{\delta_2}\,dx\right)+\\ &\quad +O\left(\int_{\Omega}\varphi_{\delta_1}^{p}\mathcal{P}\mathcal{U}_{\delta_2}\,dx\right)\end{aligned}$$

By definition we have:

$$\begin{aligned}\int_{A_1}\mathcal{U}_{\delta_1}^{p}\mathcal{U}_{\delta_2}\,dx &= \alpha_N^{p+1}\int_{A_1}\frac{\delta_1^{-\frac{N+2}{2}}}{\left(1+\left|\frac{x}{\delta_1}\right|^2\right)^{\frac{N+2}{2}}}\frac{\delta_2^{-\frac{N-2}{2}}}{\left(1+\left|\frac{x}{\delta_2}\right|^2\right)^{\frac{N-2}{2}}}\,dx\\ &= \alpha_N^{p+1}\delta_1^{-\frac{N+2}{2}+N}\delta_2^{-\frac{N-2}{2}}\left(\frac{\delta_2}{\delta_1}\right)^{N-2}\int_{\sqrt{\frac{\delta_2}{\delta_1}}\le|x|\le\frac{\rho}{\delta_1}}\frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\frac{1}{|y|^{N-2}}\,dy+o\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\right)\\ &= a_3\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}+o\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\right)\end{aligned}$$

Moreover by using (1.9) we get

$$\int_{\Omega}\mathcal{U}_{\delta_1}^{p}\varphi_{\delta_2}\,dx\le C\delta_2^{\frac{N-2}{2}}\int_{\Omega}\mathcal{U}_{\delta_1}^{p}\,dx\le C_1\delta_1^{\frac{N-2}{2}}\delta_2^{\frac{N-2}{2}}$$

and by using again (1.9) we have

$$\begin{aligned}\int_{\Omega}\mathcal{U}_{\delta_1}^{p-1}\varphi_{\delta_1}\mathcal{P}\mathcal{U}_{\delta_2}\,dx &\le \int_{\Omega}\mathcal{U}_{\delta_1}^{p-1}\varphi_{\delta_1}\mathcal{U}_{\delta_2}\,dx\\ &\le C\delta_1^{\frac{N-2}{2}}\left(\int_{\Omega}\mathcal{U}_{\delta_1}^{p+1}\,dx\right)^{\frac{p-1}{p+1}}\left(\int_{\Omega}\mathcal{U}_{\delta_2}^{\frac{p+1}{2}}\,dx\right)^{\frac{2}{p+1}}\le C_1\delta_1^{\frac{N-2}{2}}\delta_2^{\frac{N-2}{2}}.\end{aligned}$$

Finally

$$\begin{aligned}\int_{\Omega}\varphi_{\delta_1}^{p}\mathcal{P}\mathcal{U}_{\delta_2}\,dx &\le \int_{\Omega}\varphi_{\delta_1}^{p}\mathcal{U}_{\delta_2}\,dx=\int_{B(0,\rho)}\varphi_{\delta_1}^{p}\mathcal{U}_{\delta_2}\,dx+\int_{\Omega\setminus B(0,\rho)}\varphi_{\delta_1}^{p}\mathcal{U}_{\delta_2}\,dx\\ &\le C\delta_1^{\frac{N+2}{2}}\delta_2^{-\frac{N-2}{2}+N}\int_0^{\frac{\rho}{\delta_2}}\frac{r^{N-1}}{(1+r^2)^{\frac{N-2}{2}}}\,dr+|\varphi_{\delta_1}|_{p+1}^{p}\left(\int_{\Omega\setminus B(0,\rho)}\mathcal{U}_{\delta_2}^{p+1}\,dx\right)^{\frac{1}{p+1}}\\ &\le C\delta_1^{\frac{N+2}{2}}\delta_2^{\frac{N+2}{2}}+C_1\delta_1^{\frac{N+2}{2}}\left(\int_{\frac{\rho}{\delta_2}}^{+\infty}\frac{r^{N-1}}{(1+r^2)^N}\,dr\right)^{\frac{N-2}{2N}}\\ &\le C_2\delta_1^{\frac{N+2}{2}}\delta_2^{\frac{N-2}{2}}.\end{aligned}$$

At the end

$$\int_{A_1} \mathcal{PU}_{\delta_2}^p \mathcal{PU}_{\delta_1}\, dx \le \alpha_N^{p+1}\delta_2^{-\frac{N+2}{2}+N}\delta_1^{-\frac{N-2}{2}}\int_{\sqrt{\frac{\delta_1}{\delta_2}}\le|y|\le\frac{\rho}{\delta_2}} \frac{1}{\left(1+\left|\frac{\delta_2}{\delta_1}y\right|^2\right)^{\frac{N-2}{2}}}\frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy$$
$$\le C_1\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\int_{\sqrt{\frac{\delta_1}{\delta_2}}}^{\frac{\rho}{\delta_2}}\frac{r^{N-1}}{(1+r^2)^{\frac{N+2}{2}}}\,dr \le C_2\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}.$$

Finally, thanks to Lemma 2.2 we get that

$$|L_2| \le \frac{1}{p+1}\left\{\left|\int_{A_2}\left[|\mathcal{PU}_{\delta_1}-\mathcal{PU}_{\delta_2}|^{p+1}-\mathcal{PU}_{\delta_2}^{p+1}+(p+1)\mathcal{PU}_{\delta_2}^p\mathcal{PU}_{\delta_1}\right]dx\right|+\int_{A_2}\mathcal{PU}_{\delta_1}^{p+1}\,dx\right\}$$
$$\le C\left(\int_{A_2}\mathcal{PU}_{\delta_2}^{p-1}\mathcal{PU}_{\delta_1}^2\,dx+\int_{A_2}\mathcal{PU}_{\delta_1}^{p+1}\,dx\right)\le C\left(\int_{A_2}\mathcal{U}_{\delta_2}^{p-1}\mathcal{U}_{\delta_1}^2\,dx+\int_{A_2}\mathcal{U}_{\delta_1}^{p+1}\,dx\right)$$
$$\le C_1\left(\left(\frac{\delta_2}{\delta_1}\right)^2\int_0^{\sqrt{\frac{\delta_2}{\delta_1}}}\frac{r^{N-5}}{(1+r^2)^{N-2}}\,dr+\int_0^{\sqrt{\frac{\delta_2}{\delta_1}}}\frac{r^{N-1}}{(1+r^2)^N}\,dr\right)$$
$$\le C_2\left(\left(\frac{\delta_2}{\delta_1}\right)^2\int_0^{\sqrt{\frac{\delta_2}{\delta_1}}}r^{N-5}\,dr+\int_0^{\sqrt{\frac{\delta_2}{\delta_1}}}r^{N-1}\,dr\right)\le C_2\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}.$$

From Lemma 4.5 to Lemma 4.7 summing up all the terms we get that

$$\begin{aligned} J_\epsilon(\mathcal{PU}_{\delta_1}-\mathcal{PU}_{\delta_2}) &= \frac{2}{N}S^{N/2}+a_1\tau(0)\delta_1^{N-2}+a_1\tau(0)\delta_2^{N-2}+O(\delta_1^{N-1})+O(\delta_2^{N-1})\\ &+ O\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\delta_1^2\right)+O\left(\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N}{2}}\right)+a_3\left(\frac{\delta_2}{\delta_1}\right)^{\frac{N-2}{2}}\\ &- a_2\epsilon\delta_1^2+O\left(\epsilon\delta_1^{N-2}\right)-a_2\epsilon\delta_2^2+O\left(\epsilon\delta_2^{N-2}\right),\end{aligned}\tag{4.23}$$

where $a_1=\frac{1}{2}\alpha_N^{p+1}\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy$, $a_2=\frac{1}{2}\alpha_N^2\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{N-2}}\,dy$, $a_3=\alpha_N^{p+1}\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\,dy$.
Recalling the choice of δ_j, $j=1,2$ we get

$$\begin{aligned} J_\epsilon(\mathcal{PU}_{\delta_1}-\mathcal{PU}_{\delta_2}) &= \frac{2}{N}S^{N/2}+a_1\tau(0)d_1^{N-2}\epsilon^{\frac{N-2}{N-4}}+a_1\tau(0)d_2^{N-2}\epsilon^{\frac{(3N-10)(N-2)}{(N-4)(N-6)}}\\ &+ O\left(\epsilon^{\frac{N-1}{N-4}}\right)+O\left(\epsilon^{\frac{(3N-10)(N-1)}{(N-4)(N-6)}}\right)+O\left(\epsilon^{\frac{N+2}{N-6}}\right)+O\left(\epsilon^{\frac{(N-2)N}{(N-4)(N-6)}}\right)\\ &+ a_3\left(\frac{d_2}{d_1}\right)^{\frac{N-2}{2}}\epsilon^{\frac{(N-2)^2}{(N-4)(N-6)}}-a_2d_1^2\epsilon^{\frac{N-2}{N-4}}+O\left(\epsilon^{\frac{2(N-3)}{N-4}}\right)-a_2d_2^2\epsilon^{\frac{(N-2)^2}{(N-4)(N-6)}}\\ &+ O\left(\epsilon^{\frac{2(2N^2-13N+22)}{(N-4)(N-6))}}\right)\\ &= \frac{2}{N}S^{N/2}+[a_1\tau(0)d_1^{N-2}-a_2d_1^2]\epsilon^{\frac{N-2}{N-4}}+O\left(\epsilon^{\frac{N-1}{N-4}}\right)\\ &+ \left[a_3\left(\frac{d_2}{d_1}\right)^{\frac{N-2}{2}}-a_2d_2^2\right]\epsilon^{\frac{(N-2)^2}{(N-4)(N-6)}}+O\left(\epsilon^{\frac{N+2}{N-6}}\right).\end{aligned}\tag{4.24}$$

We point out that the term $O(\epsilon^{\frac{N-1}{N-4}})$ depends only on d_1.

□

5. PROOF OF THEOREMS 1.1 AND 1.2

Proof of Theorem 1.1. Let us set $G_1(d_1) := a_1\tau(0)d_1^{N-2} - a_2d_1^2$, where a_1, a_2 are the positive constants appearing in Proposition 4.1 and $\tau(0)$ is the Robin's function of the domain Ω at the origin, so by definition it follows that $\tau(0)$ is positive. It's elementary to see that the function $G_1 : \mathbb{R}^+ \to \mathbb{R}$ has a strictly local minimum point at $\bar{d}_1 = \left(\frac{2a_2}{(N-2)a_1\tau(0)}\right)^{\frac{1}{N-4}}$.

Since $\bar{d}_1$ is a strictly local minimum for G_1, then, for any sufficiently small $\gamma > 0$ there exists an open interval I_{1,σ_1} such that $\overline{I}_{1,\sigma_1} \subset \mathbb{R}^+$, I_{1,σ_1} has diameter σ_1, $\bar{d}_1 \in I_{1,\sigma_1}$ and for all $d_1 \in \partial I_{1,\sigma_1}$

$$G_1(d_1) \geq G_1(\bar{d}_1) + \gamma. \tag{5.1}$$

Clearly as $\gamma \to 0$ we can choose σ_1 so that $\sigma_1 \to 0$.

We set $G_2(d_1, d_2) := a_3\tau(0)\left(\frac{d_2}{d_1}\right)^{\frac{N-2}{2}} - a_2d_2^2$, $G_2 : \mathbb{R}^2_+ \to \mathbb{R}$, where $a_3 > 0$ is the same constant appearing in Proposition 4.1. If we fix $d_1 = \bar{d}_1$ then $\hat{G}_2(d_2) := G(\bar{d}_1, d_2)$ has a strictly local minimum point at $\bar{d}_2 := \left(\frac{2a_2\bar{d}_1^{\frac{N-2}{2}}}{a_3\tau(0)\frac{N-2}{2}}\right)^{\frac{2}{N-6}}$. As in the previous case there exists an open interval I_{2,σ_2} such that $\overline{I}_{2,\sigma_2} \subset \mathbb{R}^+$, I_{2,σ_2} has diameter σ_2, $\bar{d}_2 \in I_{1,\sigma_1}$ and for all $d_2 \in \partial I_{2,\sigma_2}$

$$\hat{G}_2(d_2) \geq \hat{G}_2(\bar{d}_2) + \gamma. \tag{5.2}$$

As $\gamma \to 0$ we can choose σ_2 so that $\sigma_2 \to 0$.

Let us set $K := \overline{I_{1,\sigma_1} \times I_{2,\sigma_2}}$ and let $\eta > 0$ be small enough so that $K \subset]\eta, \frac{1}{\eta}[\times]\eta, \frac{1}{\eta}[$. Thanks to Proposition 3.1, for all sufficiently small ϵ, $\tilde{J}_\epsilon : \mathbb{R}^2_+ \to \mathbb{R}$ is defined and it is of class C^1. By Weierstrass theorem we know there exists a global minimum point for $\tilde{J}_\epsilon$ in K. Let $(d_{1,\epsilon}, d_{2,\epsilon})$ be that point, we want to show that there exists ϵ_1 such that, for all $\epsilon < \epsilon_1$, $(d_{1,\epsilon}, d_{2,\epsilon})$ lies in the interior of K.

Assume by contradiction there exists a sequence $\epsilon_n \to 0$ such that for all $n \in \mathbb{N}$

$$(d_{1,\epsilon_n}, d_{2,\epsilon_n}) \in \partial K.$$

There are only two possibilities:

- **(a):** $d_{1,\epsilon_n} \in \partial I_{1,\sigma_1}$, $d_{2,\epsilon_n} \in \overline{I}_{2,\sigma_2}$,
- **(b):** $d_{1,\epsilon_n} \in \overline{I}_{1,\sigma_1}$, $d_{2,\epsilon_n} \in \partial I_{2,\sigma_2}$.

Thanks to (ii) of Proposition 4.1 we have the uniform expansion

$$\tilde{J}_\epsilon(d_1, d_2) - \tilde{J}_\epsilon(\bar{d}_1, d_2) = \epsilon^{\theta_1}\left[G_1(d_1) - G_1(\bar{d}_1)\right] + o\left(\epsilon^{\theta_1}\right). \tag{5.3}$$

for all $\epsilon < \epsilon_0$, $(d_1, d_2) \in K$. We point out that we have incorporated the other high order terms in $o\left(\epsilon^{\theta_1}\right)$. Thanks to (5.1) and (5.3), for all sufficiently small ϵ we have

$$\tilde{J}_\epsilon(d_1, d_2) - \tilde{J}_\epsilon(\bar{d}_1, d_2) > 0, \tag{5.4}$$

for all $d_1 \in \partial I_{1,\sigma_1}$, for all $d_2 \in \overline{I}_{2,\sigma_2}$. So for n sufficiently large if (a) holds, since by definition $\tilde{J}_{\epsilon_n}(d_{1,\epsilon_n}, d_{2,\epsilon_n}) = \min_K \tilde{J}_{\epsilon_n}$, then

$$\tilde{J}_{\epsilon_n}(d_{1,\epsilon_n}, d_{2,\epsilon_n}) \leq \tilde{J}_{\epsilon_n}(\bar{d}_1, d_{2,\epsilon_n}),$$

which contradicts (5.4). Assume (b). Thanks to (ii) of Proposition 4.1 (see also Remark 4.2) we have the uniform expansion

$$\tilde{J}_\epsilon(d_1, d_2) - \tilde{J}_\epsilon(d_1, \bar{d}_2) = \epsilon^{\theta_2}\left[G_2(d_1, d_2) - G_2(d_1, \bar{d}_2)\right] + o\left(\epsilon^{\theta_2}\right), \tag{5.5}$$

for all $\epsilon \in (0, \epsilon_0)$, for all $(d_1, d_2) \in K$.

For n sufficiently large so that $\epsilon_n < \epsilon_0$ we have

$$\begin{aligned}\tilde{J}_{\epsilon_n}(d_{1,\epsilon_n}, d_{2,\epsilon_n}) - \tilde{J}_{\epsilon_n}(d_{1,\epsilon_n}, \bar{d}_2) &= \epsilon^{\theta_2}\left[G_2(d_{1,\epsilon_n}, d_{2,\epsilon_n}) - G_2(d_{1,\epsilon_n}, \bar{d}_2)\right] + o\left(\epsilon^{\theta_2}\right) \\ &= \epsilon^{\theta_2}\left[G_2(d_{1,\epsilon_n}, d_{2,\epsilon_n}) - G_2(\bar{d}_1, d_{2,\epsilon_n}) + G_2(\bar{d}_1, d_{2,\epsilon_n}) - G_2(\bar{d}_1, \bar{d}_2)\right. \\ &\qquad \left. G_2(\bar{d}_1, \bar{d}_2) - G_2(d_{1,\epsilon_n}, \bar{d}_2)\right] + o\left(\epsilon^{\theta_2}\right) \\ &= \epsilon^{\theta_2}\left[a_3\tau(0) d_{2,\epsilon_n}^{\frac{N-2}{2}}\left(\frac{1}{d_{1,\epsilon_n}^{\frac{N-2}{2}}} - \frac{1}{\bar{d}_1^{\frac{N-2}{2}}}\right) + G_2(\bar{d}_1, d_{2,\epsilon_n}) - G_2(\bar{d}_1, \bar{d}_2)\right. \\ &\qquad \left. + a_3\tau(0)\bar{d}_2^{\frac{N-2}{2}}\left(\frac{1}{\bar{d}_1^{\frac{N-2}{2}}} - \frac{1}{d_{1,\epsilon_n}^{\frac{N-2}{2}}}\right)\right] + o\left(\epsilon_n^{\theta_2}\right)\end{aligned} \tag{5.6}$$

We observe now that, up to a subsequence, $d_{1,\epsilon_n} \to \bar{d}_1$ as $n \to +\infty$. This is a consequence of the uniform expansion given by (ii) of Proposition 4.1, in fact

$$\tilde{J}_{\epsilon_n}(d_{1,\epsilon_n}, d_{2,\epsilon_n}) - \tilde{J}_{\epsilon_n}(\bar{d}_1, \bar{d}_2) = \epsilon_n^{\theta_1}\left[G_1(d_{1,\epsilon_n}) - G_1(\bar{d}_1)\right] + o\left(\epsilon_n^{\theta_1}\right). \tag{5.7}$$

Since $(d_{1,\epsilon_n}, d_{2,\epsilon_n})$ is the minimum point we have $\tilde{J}_\epsilon(d_{1,\epsilon_n}, d_{2,\epsilon_n}) - \tilde{J}_\epsilon(\bar{d}_1, \bar{d}_2) \leq 0$, hence, dividing (5.7) by $\epsilon_n^{\theta_1}$, for all sufficiently large n we get that $G_1(d_{1,\epsilon_n}) - G_1(\bar{d}_1) \leq -\frac{o(\epsilon_n^{\theta_1})}{\epsilon_n^{\theta_1}}$. On the other side, since $\bar{d}_1$ is the minimum of G_1, we get that $G_1(d_{1,\epsilon_n}) - G_1(\bar{d}_1) \geq 0$. So we have proved that

$$0 \leq G_1(d_{1,\epsilon_n}) - G_1(\bar{d}_1) \leq -\frac{o\left(\epsilon_n^{\theta_1}\right)}{\epsilon_n^{\theta_1}},$$

and passing to the limit we deduce that $\lim_{n\to+\infty} G_1(d_{1,\epsilon_n}) = G_1(\bar{d}_1)$. Hence, up to a subsequence, since $\bar{d}_1$ is a strict local minimum, the only possibility is $d_{1,\epsilon_n} \to \bar{d}_1$.

Since we are assuming (b), from (5.2) we get that

$$G_2(\bar{d}_1, d_{2,\epsilon_n}) - G_2(\bar{d}_1, \bar{d}_2) \geq \gamma.$$

From this last inequality, (5.6) and since $(d_{2,\epsilon_n})_n$ is bounded, then, choosing $\bar{n}$ sufficiently large so that $a_3\tau(0) d_{2,\epsilon_n}^{\frac{N-2}{2}}\left|\frac{1}{\bar{d}_1^{\frac{N-2}{2}}} - \frac{1}{d_{1,\epsilon_n}^{\frac{N-2}{2}}}\right|$ and $a_3\tau(0)\bar{d}_2^{\frac{N-2}{2}}\left|\frac{1}{\bar{d}_1^{\frac{N-2}{2}}} - \frac{1}{d_{1,\epsilon_n}^{\frac{N-2}{2}}}\right|$ are small enough, we deduce that

$$\tilde{J}_{\epsilon_n}(d_{1,\epsilon_n}, d_{2,\epsilon_n}) - \tilde{J}_{\epsilon_n}(d_{1,\epsilon_n}, \bar{d}_2) > 0,$$

for all $n > \bar{n}$. Since $(d_{1,\epsilon_n}, d_{2,\epsilon_n})$ is the minimum point it also holds

$$\tilde{J}_{\epsilon_n}(d_{1,\epsilon_n}, d_{2,\epsilon_n}) - \tilde{J}_{\epsilon_n}(d_{1,\epsilon_n}, \bar{d}_2) \leq 0,$$

and we get a contradiction.

To complete the proof we point out that, as observed before, up to a subsequence $d_{1,\epsilon} \to \bar{d}_1$ as $\epsilon \to 0$. With a similar argument we prove that $d_{2,\epsilon} \to \bar{d}_2$. In fact, from the same argument of

(5.6), since $d_{1,\epsilon} \to \bar{d}_1$ and $(d_{2,\epsilon})_\epsilon$ is bounded, we have

$$\begin{aligned} 0 \geq \frac{\tilde{J}_\epsilon(d_{1,\epsilon}, d_{2,\epsilon}) - \tilde{J}_\epsilon(d_{1,\epsilon}, \bar{d}_2)}{\epsilon^{\theta_2}} &= G_2(d_{1,\epsilon}, d_{2,\epsilon}) - G_2(d_{1,\epsilon}, \bar{d}_2) + \frac{o\left(\epsilon^{\theta_2}\right)}{\epsilon^{\theta_2}} \\ &= a_3\tau(0) d_{2,\epsilon}^{\frac{N-2}{2}} \left(\frac{1}{d_{1,\epsilon}^{\frac{N-2}{2}}} - \frac{1}{\bar{d}_1^{\frac{N-2}{2}}} \right) + G_2(\bar{d}_1, d_{2,\epsilon}) - G_2(\bar{d}_1, \bar{d}_2) \\ &\quad + a_3\tau(0)\bar{d}_2^{\frac{N-2}{2}} \left(\frac{1}{\bar{d}_1^{\frac{N-2}{2}}} - \frac{1}{d_{1,\epsilon}^{\frac{N-2}{2}}} \right) + \frac{o\left(\epsilon^{\theta_2}\right)}{\epsilon^{\theta_2}} \\ &= o(1) + G_2(\bar{d}_1, d_{2,\epsilon}) - G_2(\bar{d}_1, \bar{d}_2). \end{aligned} \tag{5.8}$$

Since $\bar{d}_2$ is a local maximum point for $d_2 \to \hat{G}_2(d_2)$ we have $G_2(\bar{d}_1, d_{2,\epsilon}) - G_2(\bar{d}_1, \bar{d}_2) \geq 0$ and so from (5.8) we get that

$$0 \leq G_2(\bar{d}_1, d_{2,\epsilon}) - G_2(\bar{d}_1, \bar{d}_2) \leq -o(1).$$

Passing to the limit as $\epsilon \to 0$ we deduce that $\hat{G}_2(d_{2,\epsilon}) \to \hat{G}_2(\bar{d}_2)$. Hence, up to a subsequence, since $\bar{d}_2$ is a strict local minimum, the only possibility is $d_{2,\epsilon} \to \bar{d}_2$.
Hence by (i) of Proposition 4.1 we have that $V_\epsilon + \bar{\phi}_1 + \bar{\phi}_2$ is a solution of (1.1). Moreover, taking into account of (1.6), (1.10) and (1.11), we get that the solution obtained is of the form (1.2) and the proof is complete. □

We are ready also to prove Theorem 1.2. We reason as in [28].

Proof of Theorem 1.2. Let u_ϵ be a solution of (1.1) as in Theorem 1.1 and assume that $\Phi_\epsilon \to 0$ uniformly in compact subsets of Ω. We set

$$\begin{aligned} \tilde{u}_\epsilon(x) &:= \left(\frac{d_{1\epsilon}\epsilon^{\frac{1}{N-4}}}{d_{1\epsilon}^2 \epsilon^{\frac{2}{N-4}} + |x|^2} \right)^{\frac{N-2}{2}} - \left(\frac{d_{1\epsilon}\epsilon^{\frac{3N-10}{(N-4)(N-6)}}}{d_{1\epsilon}^2 \epsilon^{2\frac{3N-10}{(N-4)(N-6)}} + |x|^2} \right)^{\frac{N-2}{2}} \\ &= \left(\frac{1}{d_{1\epsilon}\epsilon^{\frac{1}{N-4}} + d_{1\epsilon}^{-1}\epsilon^{-\frac{1}{N-4}}|x|^2} \right)^{\frac{N-2}{2}} - \left(\frac{1}{d_{2\epsilon}\epsilon^{\frac{3N-10}{(N-4)(N-6)}} + d_{2\epsilon}^{-1}\epsilon^{-\frac{3N-10}{(N-4)(N-6)}}|x|^2} \right)^{\frac{N-2}{2}} \end{aligned}$$

Then, by Theorem 1.1 and by using the assumption on the remainder term Φ_ϵ we get

$$u_\epsilon(x) = \alpha_N \tilde{u}_\epsilon(x)(1 + o(1)), \qquad x \in \Omega, \tag{5.9}$$

where $o(1) \to 0$ uniformly on compact subsets of Ω.
We consider the spheres

$$\mathcal{S}_\epsilon^1 := \{x \in \mathbb{R}^N \ : \ |x| = \epsilon^{\frac{1}{N-4}}\}$$

and

$$\mathcal{S}_\epsilon^2 := \{x \in \mathbb{R}^N \ : \ |x| = \epsilon^{\frac{3N-10}{(N-4)(N-6)}}\}.$$

We may fix a compact subset $K \subset \Omega$ such that $\mathcal{S}_\epsilon^j \subset K$, $j = 1, 2$ and $\epsilon > 0$ sufficiently small.
For $x \in \mathcal{S}_\epsilon^1$ we get

$$\begin{aligned}\tilde{u}_\epsilon(x) &= \left(\frac{1}{d_{1\epsilon}\epsilon^{\frac{1}{N-4}} + d_{1\epsilon}^{-1}\epsilon^{\frac{1}{N-4}}}\right)^{\frac{N-2}{2}} - \left(\frac{1}{d_{2\epsilon}\epsilon^{\frac{3N-10}{(N-4)(N-6)}} + d_{2\epsilon}^{-1}\epsilon^{-\frac{N+2}{(N-4)(N-6)}}}\right)^{\frac{N-2}{2}} \\ &= \epsilon^{-\frac{N-2}{2(N-4)}}\left[\left(\frac{1}{d_{1\epsilon} + d_{1\epsilon}^{-1}}\right)^{\frac{N-2}{2}} - \left(\frac{1}{d_{2\epsilon}\epsilon^{\frac{2(N-2)}{(N-4)(N-6)}} + d_{2\epsilon}^{-1}\epsilon^{-\frac{8}{(N-4)(N-6)}}}\right)^{\frac{N-2}{2}}\right] \\ &= \epsilon^{-\frac{N-2}{2(N-4)}}\left[\left(\frac{1}{d_{1\epsilon} + d_{1\epsilon}^{-1}}\right)^{\frac{N-2}{2}} + o(1)\right]\end{aligned}$$

as $\epsilon \to 0$. Hence $\tilde{u}_\epsilon > 0$ on $\mathcal{S}_\epsilon^1$ for ϵ small.
Analogously if $x \in \mathcal{S}_\epsilon^2$ then

$$\tilde{u}_\epsilon(x) = -\epsilon^{-\frac{(3N-10)(N-2)}{2(N-4)(N-6)}}\left[\left(\frac{1}{d_{2\epsilon} + d_{2\epsilon}^{-1}}\right)^{\frac{N-2}{2}} + o(1)\right]$$

as $\epsilon \to 0$ and hence $\tilde{u}_\epsilon < 0$ on $\mathcal{S}_\epsilon^2$ for ϵ small.
Since (5.9) holds, this implies that $u_\epsilon > 0$ on $\mathcal{S}_\epsilon^1$ and $u_\epsilon < 0$ on $\mathcal{S}_\epsilon^2$ for ϵ small.
Then u_ϵ has at least two nodal domains Ω_1, Ω_2 such that Ω_j contains the sphere $\mathcal{S}_\epsilon^j$, $j = 1, 2$.
Next we show that u_ϵ has not more than two nodal domains for ϵ small.
We remark that by (ii) of Proposition 4.1 and by Lemmas 4.3, 4.4 it follows that

$$J_\epsilon(u_\epsilon) \to \frac{2}{N}S^{\frac{N}{2}}, \qquad \text{as } \epsilon \to 0 \tag{5.10}$$

where J_ϵ is defined in (1.13) and S is the best Sobolev constant for the embedding of $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$, namely

$$S := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{\left(\int_\Omega |u|^{p+1}\,dx\right)^{\frac{2}{p+1}}}.$$

We set $c_\epsilon := \inf_{\mathcal{N}_\epsilon} J_\epsilon$, where $\mathcal{N}_\epsilon$ is the Nehari manifold, which is defined by

$$\mathcal{N}_\epsilon := \left\{u \in H_0^1(\Omega) \;:\; \int_\Omega |\nabla u|^2\,dx = \int_\Omega |u|^{p+1}\,dx + \epsilon\int_\Omega u^2\,dx\right\}.$$

It is easy to see that $c_\epsilon \to c_0 = \frac{1}{N}S^{\frac{N}{2}}$ as $\epsilon \to 0$ and therefore, by (5.10), we get that

$$J_\epsilon(u_\epsilon) < 3c_\epsilon \tag{5.11}$$

for ϵ small enough.
We now suppose by contradiction that u_ϵ has at least 3 pairwise different nodal domains $\Omega_1, \Omega_2, \Omega_3$.
Let χ_i be the characteristic function corresponding to the sets Ω_i.
Then $u_\epsilon\chi_i \in H_0^1(\Omega)$ (see [25]). Moreover

$$\begin{aligned}\int_\Omega |\nabla(u_\epsilon\chi_i)|^2\,dx &= \int_\Omega \nabla u_\epsilon \nabla(u_\epsilon\chi_i) = -\int_\Omega \Delta u_\epsilon(u_\epsilon\chi_i)\,dx \\ &= \int_\Omega |u_\epsilon|^p(u_\epsilon\chi_i)\,dx + \epsilon\int_\Omega u_\epsilon \cdot u_\epsilon\chi_i\,dx \\ &= \int_\Omega |u_\epsilon\chi_i|^{p+1}\,dx + \epsilon\int_\Omega (u_\epsilon\chi_i)^2\,dx\end{aligned}$$

so that $u_\epsilon \chi_i \in \mathcal{N}_\epsilon$. Since also $u_\epsilon \in \mathcal{N}_\epsilon$ we obtain

$$\begin{aligned} J_\epsilon(u_\epsilon) &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega |u_\epsilon|^{p+1}\,dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p+1}\right) \sum_{i=1}^{3} \int_\Omega |u_\epsilon \chi_i|^{p+1}\,dx \\ &= \sum_{i=1}^{3} J_\epsilon(\chi_i u_\epsilon) \geq 3c_\epsilon \end{aligned}$$

contrary to (5.11). The contradiction shows that u_ϵ has at most two nodal domains for ϵ small. This completes the proof. □

References

[1] A. Ambrosetti, A. Malchiodi, perturbation Methods and Semilinear Elliptic problems on $\mathbb{R}^N$, Progress in Mathematics Vol. 240, Birkhäuser Verlag (2006).

[2] F. V. Atkinson, H. Brezis, L. A. Peletier, *Nodal solutions of elliptic equations with critical Sobolev exponents*, J. Diff. Eqs **85** (1990), 151–170.

[3] T. Aubin, *Problémes isoperimetriques et spaces de Sobolev*, J. Diff. Geom. **11** (1976), 573–598.

[4] T. Bartsch, A. M. Micheletti, A. Pistoia,*On the existence and the profile of nodal solutions of elliptic equations involving critical growth*, Calc. Var. PDEs **26** (2006), no. 3, 265-282.

[5] M. Ben Ayed, K. El Mehdi, F. Pacella,*Blow-up and nonexistence of sign-changing solutions to the Brezis-Nirenberg problem in dimension three*, Ann. Inst. H. Poincaré, A. Non Linéaire **23** (2006), no. 4, 567–589.

[6] M. Ben Ayed, K. El Mehdi, F. Pacella,*Blow-up and symmetry of the sign-changing solutions to some critical elliptic equations*, J. Diff. Eqs **230** (2006), 771–795.

[7] M. Ben Ayed, K. El Mehdi, F. Pacella,*Classification of the low energy sign-changing solutions of an almost critical problem*, J. Funct. Analysis **250** (2007), no. 2, 347–373.

[8] E. Bianchi, H. Egnell, *A note on the Sobolev inequality*, J. Funct. Anal. **100** (1991), 18–24.

[9] H. Brezis, L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), no. 4, 437-477.

[10] K. J. Brown, N. Stavrakakis, *Global bifurcation results for a semilinear elliptic equation on all of* $\mathbb{R}^N$, Duke Math. J. **85** (1996) 77–94.

[11] L. Caffarelli, B. Gidas, J. Spruck, *Asymptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989), 271–297.

[12] A. Capozzi, D. Fortunato, G. Palmieri, *An existence result for nonlinear elliptic problems involving critical Sobolev exponent,* Ann. Inst. H. Poincaré **2** (1985), no. 6, 463–470.

[13] A. Castro, M. Clapp, *The effect of the domain topology on the number of minimal nodal solutions of an elliptic equation at critical growth in a symmetric domain*, Nonlinearity **16** (2003), no. 2, 579-590.

[14] G. Cerami, D. Fortunato, M. Struwe, *Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents*, Ann. Inst. H. Poincaré - Analyse, Nonlin. **1** (1984), 341–350.

[15] G. Cerami, S. Solimini, M. Struwe, *Some existence results for superlinear elliptic boundary value problems involving critical exponents*, J. Funct. Anal. **69** (1986), 289–306.

[16] M. Clapp, T. Weth, *Multiple solutions for the Brezis-Nirenberg problem*, Adv. Diff. Eqs **10** (2005), 463–480.

[17] M. del Pino, J. Dolbeault, M. Musso, *The Brezis-Nirenberg problem near criticality in dimension 3*, J. Math. Pures Appl. **9** 83 (2004), no. 12, 1405-1456.

[18] G. Devillanova, S. Solimini, *Concentrations estimates and multiple solutions to elliptic problems at critical growth*, Adv. Diff. Eqs. **7** (2002), 1257–1280.

[19] G. Devillanova, S. Solimini, *A multiplicity result for elliptic equations at critical growth in low dimension*, Commun. Contemp. Math. **5** (2003), no. 2, 171-177.

[20] D. Fortunato, E. Jannelli, *Infinitely many solutions for omer nonlinear elliptic problems in symmetrical domains*, Proc. Roy. Soc. Edinburgh, **105A** (1987), 205–213.

[21] H. Ch. Grunau, *Polyharmonische Dirichlet probleme: Positivität, kritische exponenten und kritische dimensionen,* Habilitationsschrift, Universität Bayreuth, 1996.

[22] Z. C. Han, *Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent*, Ann. Inst. H. Poincar Anal. Non Linaire **8** (1991), no. 2, 159-174

[23] A. Iacopetti, *Asymptotic analysis for radial sign-changing solutions of the Brezis-Nirenberg problem,* Annali di Matematica Pura ed Applicata, 10.1007/s10231-014-0438-y.
[24] Micheletti, A. M.; Pistoia, A., *On the effect of the domain geometry on the existence of sign changing solutions to elliptic problems with critical and supercritical growth*, Nonlinearity **17** (2004), no. 3, 851-866.
[25] E. Müller-Pfeiffer, *On the number of nodal domains for elliptic differential operators*, J. London Math. Soc. **31** no. 2 (1985), 91–100.
[26] Musso, M.; Pistoia, A., *Multispike solutions for a nonlinear elliptic problem involving the critical Sobolev exponent*, Indiana Univ. Math. J. **51** (2002), no. 3, 541-579.
[27] M. Musso, A. Pistoia, *Tower of bubbles for almost critical problems in general domains*, J. Math. Pures Appl. **93** (2010) 1–40.
[28] A. Pistoia, T. Weth, *Sign-changing bubble tower solutions in a slightly subcritical semilinear Dirichlet problem* Ann. Inst. H. Poincaré Anal. Non Linéaire **24** no. 2 (2007), 325–340.
[29] S. I. Pohozaev, *On the eigenfunctions of the equation* $\Delta u + f(u) = 0$, Soviet Math. Dokl. **6** (1965), 1408–1411.
[30] O. Rey, *The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent* J. Funct. Anal. **89** (1990), no. 1, 1-52.
[31] O. Rey, *Proof of two conjectures of H. Brzis and L. A. Peletier*, Manuscripta Math. **65** (1989), no. 1, 19-37.
[32] M. Schechter, Wenming Zou, *On the Brezis Nirenberg problem*, Arch. Rat. Mech. Anal., **197** (2010), 337–356.
[33] S. Solimini, *Morse index estimates in minimax Theorems*, Manuscr. Math. **63** (1989), 421–453.
[34] G. Talenti, *Best constants in Sobolev inequality*, Ann. Math. Pura Appl. **110** (1976), 353–372.
[35] G. Vaira, A new kind of blowing-up solutions for the Brezis-Nirenberg problem, to appear on Calc. Var. PDE

(Alessandro Iacopetti) Dipartimento di Matematica e Fisica, Universitá degli Studi di Roma Tre, L.go S. Leonardo Murialdo 1, 00146 Roma, Italy
E-mail address: iacopetti@mat.uniroma3.it

(Giusi Vaira) Dipartimento di Matematica "G. Castelnuovo", Università di Roma "La Sapienza", Piazzale A. Moro 1, 00161 Roma, Italy
E-mail address: vaira@mat.uniroma1.it