ON GROUPS HAVING A p-CONSTANT CHARACTER

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Dedicated to Carlo Casolo

ABSTRACT. Let G be a finite group, and p a prime number; a character of G is called *p-constant* if it takes a constant value on all the elements of G whose order is divisible by p. This is a generalization of the very important concept of characters of p-defect zero. In this paper, we characterize the finite p-solvable groups having a faithful irreducible character that is p-constant and not of p-defect zero, and we will show that a non-p-solvable group with this property is an almost-simple group.

1. INTRODUCTION

Given a finite group G and a prime number p, an irreducible character of G is said to be *of p-defect zero* if its degree is a multiple of the full p-part of the order of G; as well known, these characters play an important role in both ordinary and local Representation Theory of finite groups, being a key ingredient for several fundamental problems in this research area.

According to Brauer-Nesbitt Theorem (see [5, Theorem 4.6]), irreducible characters of p-defect zero take the value 0 on every *p-singular* element of the group (i.e., on every element whose order is divisible by p), thus, in particular, they are constant on p-singular elements.

Taking this into account, M.A. Pellegrini and A. Zalesski recently introduced and studied in [10] the more general class of *p-constant* characters, defined as the characters taking a constant value on the p-singular elements of the group (the relevant constant value is actually an integer; see [10, Lemma 2.1], or Lemma 3.1). This concept is naturally linked to modular Representation Theory, as a character χ of a finite group G is p-constant if and only if there exist a complex number c and integers a_φ such that

$$\chi - c1_G = \sum_{\varphi \in \mathrm{IBr}_p(G)} a_\varphi \Phi_\varphi,$$

where the Φ_φ are the projective indecomposable characters, and 1_G is the principal character of G. In [10], the authors focus essentially on finite non-abelian simple groups, describing the pairs (G, τ) where G is an alternating group, or a sporadic simple group, or a simple group of Lie type in characteristic p, and τ is an irreducible p-constant character; their analysis shows, among other things, that the constant value associated to an irreducible p-constant character of a non-abelian simple group lies in the set $\{-2, -1, 0, 1\}$, unless the group is of Lie type in characteristic different from p and it has a non-cyclic Sylow p-subgroup (note that, for instance, $\mathrm{PSL}_2(7)$ has an irreducible 3-constant character with constant value 2). In the same spirit, M.A. Pellegrini studies other classes of finite groups such as reflection groups or nilpotent groups (see [9]), and he conjectures that, for every irreducible p-constant character of a finite perfect group, the relevant constant value lies in $\{0, \pm 1, \pm 2\}$ ([9, Conjecture 4.4]).

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We note that the non-abelian simple groups in the GAP library have an abundance of irreducible p-constant characters which are not of p-defect zero, in particular for primes that are larger than 3. Also, a faithful irreducible character can be p-constant and not of p-defect zero for different primes: for instance, any of the irreducible characters of degree 56 of the first Janko group is both 11-constant and 19-constant (note that the same characters are also of 2-defect zero and 7-defect zero). However, as will follow from our main results, this cannot happen if the group is solvable.

The present paper is a contribution to this research field. Our main results can be summarized by saying that *a finite group having a faithful irreducible character which is p-constant and not of p-defect zero is either p-solvable or almost-simple; moreover, in the p-solvable context, a complete characterization is provided.* (In the following statement, as customary, $\mathbf{O}_p(G)$ denotes the largest normal subgroup of G having p-power order.)

Theorem A. *Let p be a prime number and let G be a finite group. Then the following properties are equivalent.*

(a) *$\mathbf{O}_p(G)$ is non-trivial, and G has an irreducible character χ that is faithful and p-constant.*
(b) *G has a unique minimal normal subgroup M, which is a Sylow p-subgroup of G, and a p-complement H of G transitively permutes (acting by conjugation) the non-identity elements of M.*

In this case, $\chi = (1_H)^G - 1_G$ is unique, and the constant value of χ on p-singular elements is -1.

The result above should be compared with Theorem B of [7], where the authors derive similar conclusions for finite groups G having a faithful, irreducible, non-linear character whose values on the p-singular elements of G are all roots of unity, under the assumption that $\mathbf{O}_p(G)$ is non-trivial.

Theorem A can be viewed as a characterization of finite p-solvable groups having a faithful irreducible character that is p-constant *and not of p-defect zero*, because, as explained in Remark 4.2, these conditions are equivalent to those in (a) of the above statement. On the other hand, again in Remark 4.2 we will see that the existence of a faithful irreducible character of p-defect zero doesn't constrain the structure of a p-solvable group significantly.

Finally, we note that the groups as in Theorem A, being 2-transitive permutation groups whose socle is a Sylow p-subgroup, are well understood. In particular, the p-complement H of such a group is either a group of semilinear maps (hence, it is metacyclic), or H belongs to a finite list of exceptions; among the exceptional cases (that are all of even order), we find solvable groups with derived length at most 4, or extensions of $\mathrm{SL}_2(5)$. We refer the reader to Theorem 4.5 for a more detailed statement.

Assume next that, for a given prime p, the finite non-p-solvable group G has a faithful irreducible character that is p-constant and not of p-defect zero. By Theorem A, in this situation we have necessarily $\mathbf{O}_p(G) = 1$ and, as we see with the following result, the investigation in this research area reduces to the class of almost-simple groups.

Theorem B. *Let p be a prime number, and let G be a finite group having an irreducible character that is faithful, p-constant and not of p-defect zero. Assume also that $\mathbf{O}_p(G) = 1$. Then G is an almost-simple group.*

The requirement of faithfulness in the statements of Theorem A and Theorem B is necessary in order to prevent the kernel of the relevant p-constant character from containing all p-singular elements, so avoiding trivial situations. At any rate, since a p-constant character χ of G is also a p-constant character of $G/\ker(\chi)$, Theorem A and Theorem B can be applied to $G/\ker(\chi)$.

Finally, a general observation concerning the degree of irreducible p-constant characters with non-zero constant value.

Theorem C. *Let p be a prime number and let G be a finite group. If $\chi \in \mathrm{Irr}(G)$ is p-constant and not of p-defect zero, then $\chi(1)$ is not divisible by p.*

This of course shows a very different behaviour of these characters with respect to irreducible characters of p-defect zero. As remarked in Section 3, Theorem C holds in fact under the weaker assumption that χ takes a non-zero constant value on the non-trivial p-elements of G.

Throughout the following discussion, every group is assumed to be a finite group.

2. Preliminary results and notation

We start by defining the central concept of this paper, that was already presented in the Introduction.

Definition 2.1. Let G be a group, and let p be a prime number. We say that a character χ of G is *p-constant* if χ takes a constant value, that we will denote by c_χ, on all the elements of G whose order is a multiple of p.

As we have seen, irreducible characters of p-defect zero are particular p-constant characters; nonetheless, they turn out to behave differently from p-constant characters with non-zero constant value, and we will see a first instance of this fact in Lemma 2.2.

Given a group G, let G^0 denote the set of *p-regular* elements (i.e., elements whose order is not divisible by p) of G; in the proof of Lemma 2.2, we will use the following characterization of p-blocks for ordinary characters. Consider the graph whose vertex set is $\mathrm{Irr}(G)$, and where two vertices ϕ, ψ are adjacent if and only if $\sum_{x \in G^0} \phi(x)\overline{\psi(x)} \neq 0$: it can be shown that the irreducible characters ϕ and ψ lie in the same p-block of G if and only if they lie in the same connected component of this graph (see Theorem (3.9) of [6]).

Lemma 2.2. *Let G be a group, and let p be a prime number. If $\chi \in \mathrm{Irr}(G)$ is a p-constant character which is not of p-defect zero, then χ lies in the principal p-block of G.*

Proof. By our assumptions, we have $\chi(x) = c_\chi \neq 0$ for every element x in $G - G^0$; note that $G \neq G^0$, as χ is not of p-defect zero. We will obtain the desired conclusion by showing that χ is adjacent to the principal character 1_G in the graph defined above. In fact, as we can clearly assume $\chi \neq 1_G$, we have

$$0 = [\chi, 1_G] = \frac{1}{|G|} \sum_{x \in G} \chi(x) 1_G(x) = \frac{1}{|G|} \big(|G - G^0| c_\chi + \sum_{x \in G^0} \chi(x) 1_G(x) \big) .$$

Hence $\sum_{x \in G^0} \chi(x) 1_G(x) = -|G - G^0| c_\chi \neq 0$, as claimed. □

Remark 2.3. Let G be a group and p a prime number: we observe that if G has a non-trivial normal p-subgroup M, then G does not have any irreducible character of p-defect zero. In fact, let χ be in $\mathrm{Irr}(G)$ and let θ be an irreducible constituent of χ_M; Clifford Theory yields that $\chi(1)/\theta(1)$ is a divisor of $|G/M|$, and it is therefore easy to see that the p-part of $\chi(1)$ is strictly smaller than that of $|G|$.

Now, assume (as above) that $\mathbf{O}_p(G) > 1$, and that G has an irreducible character χ that is faithful and p-constant. Then Lemma 2.2, and the observation in the paragraph above, yield that χ lies in the principal p-block of G, whence its kernel contains $\mathbf{O}_{p'}(G)$; but, χ being faithful, this means that $\mathbf{O}_{p'}(G) = 1$ (and $\mathbf{O}_p(G) = \mathbf{F}(G)$).

We also remark that if, for a character χ, one considers the weaker condition that χ takes a constant value *on the non-trivial p-elements* rather than on the whole set of p-singular elements, then the conclusions of Lemma 2.2 and of the above paragraph are no longer true in general; we refer the reader to Section 3 for some more comments.

In the following proposition we recall some well-known facts concerning the theory of permutation group (conclusions (a), (b) and (c)), also adding an observation which is relevant in the present context (conclusion (d)). We recall that, given a group G acting on a finite set Ω and setting, for $g \in G$, $\pi(g)$ to be the number of fixed points of g in Ω, the class function $\chi = \pi - 1_G$ is a character of G, called the *deleted permutation character*, and that χ is irreducible if and only if G acts 2-transitively on Ω ([3, Corollary (5.17)]).

Proposition 2.4. *Let H be a group acting faithfully (by automorphisms) on a non-trivial group M, and let $G = MH$ be the corresponding semidirect product. Also, let Ω denote the set $\{Hm \mid m \in M\}$ of the right cosets of H in G. If H acts transitively on the set of non-identity elements of M, then the following conclusions hold.*

(a) *The order of M is a p-power for a suitable prime p, and M is the unique minimal normal subgroup of G.*
(b) *H transitively permutes (acting by right multiplication) the set $\Omega - \{H\}$ of non-trivial cosets.*
(c) *G is a 2-transitive permutation group on Ω (via right multiplication), having M as a regular normal subgroup.*
(d) *If $|H|$ is coprime to p, then the deleted permutation character $\chi \in \mathrm{Irr}(G)$ associated to the action described in* (c) *is p-constant, with constant value $c_\chi = -1$.*

Proof. Since, for every pair of non-trivial elements in M, there exists an automorphism of M (induced by an element of H) mapping one element to the other, we have that the non-trivial elements of M have all the same order, which is then necessarily a prime number p. Also, by the same reason, it is clear that M does not have any H-invariant proper non-trivial subgroup, which makes M a minimal normal subgroup of G. To conclude the proof of (a), observe now that every normal subgroup of G whose order is divisible by p contains M, whereas every normal p'-subgroup of G should lie in $\mathbf{C}_H(M)$, which is trivial by our assumptions.

Part (b) immediately follows from the fact that, as easily checked, the action of H by right multiplication on the set $\Omega - \{H\}$ is equivalent to the action of H on the non-trivial elements of M.

As regards (c), observe that the core of H in G (i.e., $\mathbf{C}_H(M)$) is trivial, thus G is actually a permutation group on the set Ω. As the stabilizer H of a point (the trivial coset) transitively permutes the remaining elements of Ω, the relevant action of G on Ω is 2-transitive. The last claim, concerning the fact that M acts regularly on Ω, can be easily verified.

Finally, if χ is the deleted permutation character associated to the action of (c), then we have $(1_H)^G = 1_G + \chi$. Under our coprimality assumption, H is a p-complement of G, and therefore every p-singular element x of G does not lie in any conjugate of H; now, as $(1_H)^G$ takes the value 0 on every element of this kind, we immediately get $\chi(x) = -1$, which finishes the proof of (d). □

To close this preliminary section, we discuss the structure of the p-solvable 2-transitive groups having a non-trivial normal p-subgroup. In the following, we denote by $\Gamma(p^n)$, where p is a prime and n a positive integer, the group of the semilinear transformations of $\mathrm{GF}(p^n)$ over $\mathrm{GF}(p)$, i.e. the maps of $\mathrm{GF}(p^n)$ onto itself of the form $x \mapsto ax^\sigma$, where $a \in \mathrm{GF}(p^n)$, $a \neq 0$ and $\sigma \in \mathrm{Gal}(\mathrm{GF}(p^n)|\mathrm{GF}(p))$. Moreover, we denote by $\Gamma_0(p^n)$ the cyclic subgroup of $\Gamma(p^n)$ consisting of the maps $x \mapsto ax$, with $a \in \mathrm{GF}(p^n)$, $a \neq 0$.

Theorem 2.5 ([8, Theorem I])**.** *Let G be a 2-transitive permutation group of degree d, let H be a point stabilizer of G and p a prime. If G is p-solvable and $\mathbf{O}_p(G) \neq 1$, then $d = p^n$, for some positive integer n, and we have one of the following.*

(i) *H is (permutation) isomorphic to a subgroup of the semilinear group $\Gamma(p^n)$;*
(ii) *H is solvable and $p^n \in \{3^2, 5^2, 7^2, 11^2, 23^2, 3^4\}$;*
(iii) *H is non-solvable and $p^n \in \{11^2, 19^2, 29^2, 59^2\}$.*

3. On the degree of p-constant characters

Recall that, given a prime number p, an irreducible character of p-defect zero of the group G has (by definition) a degree that is divisible by the full p-part of the order of G. Our aim in this section is to observe that, on the other hand, irreducible p-constant characters with non-zero constant value have a degree that is coprime to p. This is clearly another feature that separate characters of p-defect zero from the other p-constant characters (unless of course the order of G is coprime to p).

Another interesting difference is that, as shown by Theorem 3.2, if an irreducible character of a group G takes the value 0 on the non-trivial p-elements of G, then it takes this value on the whole set of p-singular elements. On the other hand, easy examples show that this fails for a non-zero constant value. However, the results of this section (as well as part of those in the following one) hold under the weaker assumption that the relevant irreducible character takes a constant value on the non-trivial p-elements. Thus we will state them in full generality.

The following result is essentially Lemma 2.1 of [10].

Lemma 3.1. *Let p be a prime number, let G be a group whose order is divisible by p, and let P be a Sylow p-subgroup of G. If $\chi \in \mathrm{Irr}(G)$ takes a constant value c_χ on the elements of $P - \{1\}$, then c_χ is an integer congruent to $\chi(1)$ modulo $|P|$.*

Proof. Let α be a linear character in $\mathrm{Irr}(P) - \{1_P\}$. Then

$$[\chi_P, \alpha] = \frac{1}{|P|}\sum_{x\in P}\chi(x)\overline{\alpha(x)} = \frac{1}{|P|}\Big(\chi(1)\alpha(1) + \sum_{x\in P-\{1\}} c_\chi\,\overline{\alpha(x)}\Big) =$$

$$\frac{1}{|P|}(\chi(1)\alpha(1) - c_\chi\,\alpha(1)) + c_\chi\,[1_P, \alpha] = \frac{1}{|P|}(\chi(1) - c_\chi).$$

Hence $c_\chi = \chi(1) - |P| \cdot [\chi_P, \alpha]$, and the claim follows. □

Next, we recall the aforementioned fundamental result about irreducible characters of p-defect zero.

Theorem 3.2 ([5, Corollary 4.7])**.** *Let p be a prime number, and let χ be an irreducible character of a group G. Then the following conditions are equivalent.*

(a) *χ is of p-defect zero.*
(b) *$\chi(g) = 0$ for every p-singular element $g \in G$.*
(c) *$\chi(g) = 0$ for every non-trivial p-element $g \in G$.*

Actually, condition (c) in the above statement can be replaced by χ taking the value 0 *on the elements of G having order p.* This follows by a theorem due to R. Knörr, that we state in a slightly modified form.

Theorem 3.3. *Let p be a prime number and let G be a group. If $\chi \in \mathrm{Irr}(G)$ is such that $\sum_{o(x)=p}\chi(x)$ is congruent to 0 modulo p, then χ is of p-defect zero.*

Proof. The original hypothesis of this theorem, which can be found as Theorem 4.8 in [5], is that $\sum_{o(x)=p}\chi(x)$ is actually 0. Nevertheless, the proof as given in [5] (due to J. Murray) goes through identically with the weaker assumption considered here. □

We are now in a position to obtain the desired information (i.e., Theorem C) about the degree of a p-constant character that is not of p-defect zero.

Theorem 3.4. *Let p be a prime number and let G be a group. If $\chi \in \mathrm{Irr}(G)$ takes a constant value $c_\chi \neq 0$ on the non-trivial p-elements of G, then $\chi(1)$ is not divisible by p.*

Proof. We can clearly assume that the order of G is divisible by p. If our conclusion is false, then Lemma 3.1 yields that c_χ is congruent to 0 modulo p, and so is clearly $\sum_{o(x)=p}\chi(x)$. Theorem 3.3 implies now that χ is of p-defect zero, contradicting (via Brauer-Nesbitt's Theorem) the fact that c_χ is not 0. $\square$

4. When $\mathbf{O}_p(G)$ is non-trivial

In this section we prove Theorem A, which provides a characterization of groups having a non-trivial normal p-subgroup and a faithful irreducible character that is p-constant. We state it again here for the convenience of the reader.

Theorem 4.1. *Let p be a prime number and let G be a group. Then the following properties are equivalent.*

(a) *$\mathbf{O}_p(G)$ is non-trivial, and G has an irreducible character that is faithful and p-constant.*
(b) *G has a unique minimal normal subgroup M, which is a Sylow p-subgroup of G, and a p-complement H of G transitively permutes (acting by conjugation) the non-identity elements of M.*

In this case, $\chi = (1_H)^G - 1_G$ is unique, and the constant value of χ on p-singular elements is -1.

This result is an immediate consequence of Proposition 2.4, together with Propositions 4.3 and Proposition 4.4 that will be proved after the following remark.

Remark 4.2. Note that if the group G is p-solvable, and it has a faithful irreducible character χ that is p-constant but *not of p-defect zero* (i.e., $c_\chi \neq 0$), then G is as in (a) of the above statement. In fact, χ lies in the principal p-block of G by Lemma 2.2, so its kernel contains $\mathbf{O}_{p'}(G)$, which is then trivial because χ is faithful. As clearly condition (b) implies the p-solvability of G, we conclude that Theorem 4.1 yields a characterization of the p-solvable groups having a faithful irreducible character which is p-constant but not of p-defect zero; see also Theorem 4.5.

On the other hand, the assumption of having a faithful irreducible character of p-defect zero does not seem to constrain the structure of a p-solvable group. To justify the above claim, consider a prime p and any p-solvable group H: for any choice of a prime q not dividing the order of H, it is easily seen that there exists an elementary abelian q-group Q, on which H acts faithfully by automorphism, inducing a regular orbit (i.e., there exists $x \in Q$ such that $\mathbf{C}_H(x) = 1$). Denoting by G the corresponding (p-solvable) semidirect product QH, by coprimality there exists $\lambda \in \mathrm{Irr}(Q)$ such that $I_G(\lambda) = Q$, hence $\chi = \lambda^G$ is an irreducible character of G having p-defect zero, whose kernel lies in Q. So, for instance, the p-length of $G/\ker(\chi)$ can be arbitrarily large.

We move now to the proof of the two propositions that yield Theorem 4.1.

Proposition 4.3. *Let p be a prime number, and let G be a group having a faithful irreducible character χ that takes a constant value on the non-trivial p-elements of G; assume also that $M = \mathbf{O}_p(G)$ is non-trivial. Then M is a Sylow p-subgroup of G. Furthermore, denoting by H a p-complement of G, we have $\mathbf{C}_H(M) = 1$, and H transitively permutes (acting by conjugation) the non-identity elements of M.*

Proof. Let c_χ be the constant value of χ on the non-trivial p-elements of G (recall that, by Theorem 3.2 and Remark 2.3, we have $c_\chi \neq 0$). Observe that $\chi_M - c_\chi 1_M$ is a class function of M which vanishes on all the non-trivial elements of M but not on the identity, as otherwise the kernel of χ_M would contain M, against the fact that χ is faithful; therefore, as well known, $\chi_M - c_\chi 1_M$ is a multiple of the regular character of M by the scalar $b_M = (\chi(1) - c_\chi)/|M|$. Now we get

$$\chi_M = (b_M + c_\chi)1_M + b_M \sum_{\theta \in \mathrm{Irr}(M)-\{1_M\}} \theta.$$

Since, by Clifford Theory, the irreducible constituents of χ_M are pairwise conjugate (and since 1_M does not appear among these constituent), we deduce that $b_M = -c_\chi$, and that all the non-principal irreducible characters of M lie in the same orbit under the action of G.

Next, let P be a Sylow p-subgroup of G. As above, the class function $\chi_P - c_\chi 1_P$ being identically zero on the non-trivial elements of P, we have that $\chi_P - c_\chi 1_P$ is a multiple of the regular character of P by the scalar $b_P = (|M|/|P|)b_M = -(|M|/|P|)c_\chi$, and so we get

$$\chi_P = (b_P + c_\chi)1_P + \Delta,$$

where Δ is a linear combination of (all the) non-principal irreducible characters of P. But again, since $\chi_M = (\chi_P)_M$ does not have the principal character 1_M among its irreducible constituents, the coefficient $b_P + c_\chi = -c_\chi(|M|/|P| - 1)$ is forced to be 0. As a consequence we get $|M|/|P| = 1$, whence $|M| = |P|$ (i.e., M is a Sylow p-subgroup of G).

Finally, taking H to be a p-complement of G, we have $\mathbf{C}_H(M) = 1$ because, by Remark 2.3, $\mathbf{O}_{p'}(G)$ is trivial. Moreover, H transitively permutes the non-principal irreducible characters of M, because so does $G = MH$ and clearly M stabilizes all the elements in $\mathrm{Irr}(M)$; since, by coprimality, the action of H on $\mathrm{Irr}(M) - \{1_M\}$ is equivalent to the conjugation action of H on the set $M - \{1\}$, we conclude that also the latter action is transitive. □

The substantial part of the following proposition is the "uniqueness part"; here the full strength of Definition 2.1 (i.e., the requirement for a p-constant character to be constant on all the p-singular elements of the group) will be crucial.

Proposition 4.4. *Let H be a group acting faithfully and coprimely (by automorphisms) on a non-trivial group M, and let $G = MH$ be the corresponding semidirect product. If H acts transitively on the set of non-identity elements of M, then M is an elementary abelian p-group for a suitable prime p, and G has a unique irreducible character χ which is faithful and p-constant. Moreover, we have $\chi(1) = |M| - 1$ and $c_\chi = -1$.*

Proof. Observe that we are under the hypotheses of Proposition 2.4 and, in view of that, we already know that M is an elementary abelian p-group for a suitable prime p, say of order p^n; also, G is a 2-transitive permutation group on $\Omega = \{Hm \mid m \in M\}$ via right multiplication (recall that the latter action is equivalent to the action of G on M where M acts regularly by right multiplication, and H acts by conjugation), and G does have a faithful p-constant irreducible character, namely, the deleted permutation character. Therefore we will work to establish the uniqueness part of the conclusion.

Since G is p-solvable and $M = \mathbf{O}_p(G) \neq 1$, by Theorem 2.5 we have one of the following three cases:

(a): H is a subgroup of the semilinear group $\Gamma(p^n)$; or
(b): $n = 2$ and $p \in \{3, 5, 7, 11, 19, 23, 29, 59\}$; or
(c): $p^n = 3^4$.

Let us consider a non-identity element x of M and let $A = \mathbf{C}_H(x)$ be its centralizer (i.e. stabilizer, if we use the language of permutation actions) in H. We claim that A is a cyclic group. In fact: in case (a) A intersects trivially the group $\Gamma_0(p^n)$ of multiplication maps, so A is isomorphic to a subgroup of the cyclic group $\Gamma(p^n)/\Gamma_0(p^n)$; in case (b), H can be seen as a subgroup of $\mathrm{GL}_2(p)$ and $A \cap \mathrm{SL}_2(p) = 1$ (as all elements of $\mathrm{SL}_2(p) = 1$ having eigenvalue 1 are p-elements); in case (c), we refer to the structure of H as given in [2], Example XII.7.4: $H = NCA$ where $N = \mathbf{F}(H)$ is a central product of D_8 and Q_8, C is cyclic of order 5 (acting irreducibly on N/N') and A is any of the nontrivial subgroups of a cyclic group B of order 8. (for completeness, we mention that $B \cap N$ is a non-central subgroup of order 2 of N).

Let χ be a faithful character in $\mathrm{Irr}(G)$ (which exists, as G has a unique minimal normal subgroup), and let ϕ be an irreducible constituent of χ_M. Clearly, ϕ is not the principal character 1_M. By Clifford Theory, all the H-conjugates of ϕ are constituents of χ_M and, since H transitively permutes the elements of $\mathrm{Irr}(M) - \{1_M\}$, we see that *every* non-principal irreducible character of M is a constituent of χ_M. Also, by Brauer Permutation Lemma, ϕ can be chosen so that $I_G(\phi) = MA$, where $I_G(\phi)$ is the inertia subgroup of ϕ in G.

The linear character ϕ extends to its inertia subgroup MA because A is cyclic (or also because MA splits over M), and Clifford Correspondence yields that $\chi = \theta^G$ where $\theta \in \mathrm{Irr}(MA)$ is an extension of ϕ; in particular, we get $\chi(1) = |G : MA| = |H : A| = p^n - 1$.

Our aim will be to evaluate χ on the p-singular elements of G, so let $g = g_p g_{p'}$ be such an element, decomposed as a product of its p-part g_p and its p'-part $g_{p'}$ (g_p and $g_{p'}$ are the uniquely defined powers of g such that g_p is a p-element and $g_{p'}$ has order coprime to p; clearly, $g_p \neq 1$ as g is p-singular). Up to conjugation in H, again in view of the transitivity of H on $M - \{1\}$, we can assume that g_p is the element x considered above; but, up to conjugation by a suitable element of M, we can also assume that $y = g_{p'}$ lies in H, hence in $A = \mathbf{C}_H(x)$. In other words, if we want to control the values that χ takes on p-singular elements of G, it will be enough to compute $\chi(xy)$ where y runs in A.

Let $Y = \langle y \rangle$ and $M_Y = \mathbf{C}_M(Y)$. We claim that $\mathbf{N}_H(Y)$ acts transitively on the set $M_Y - \{1\}$. In fact, observe that $x \in M_Y$ and that, for any $x_1 \in M_Y$ there exists an element $h \in H$ such that $x_1 = x^h$. So $Y \leq \mathbf{C}_H(x_1) = (\mathbf{C}_H(x))^h = A^h$ and hence both Y and $Y^{h^{-1}}$ are subgroups (of the same order) of the cyclic group A; as a consequence we get $Y = Y^{h^{-1}}$, thus $h \in \mathbf{N}_H(Y)$.

Let T be a right transversal of A in H; so T is also a transversal of MA in G. Let $T_Y = T \cap \mathbf{N}_H(Y)$. The map $f : T_Y \to M_Y - \{1\}$ such that, for $t \in T_Y$, $f(t) = x^{t^{-1}}$, is a bijection. In fact, f is surjective by the previous paragraph and it is injective as $x^{t^{-1}} = x^{t'^{-1}}$, for $t, t' \in T$, implies $t = t'$ (as $t^{-1}t' \in \mathbf{C}_H(x) = A$).

Let θ° be the class function of G such that $\theta^\circ(g) = \theta(g)$ if $g \in MA$ and $\theta^\circ(g) = 0$ if $g \notin MA$. For $t \in T$, we claim that $txyt^{-1}$ lies in MA if and only if t lies in $\mathbf{N}_H(Y)$, i.e. if and only if $t \in T_Y$. In fact, $txyt^{-1} = (txt^{-1})(tyt^{-1})$ lies in MA if and only if $tyt^{-1} \in A$, which is equivalent to $Y^{t^{-1}} \leq A$; but A is cyclic, so this happens if and only if $Y^{t^{-1}} = Y$, i.e. if and only if $t \in \mathbf{N}_H(Y)$.

Finally, let us consider the linear character $\alpha = \theta_A \in \mathrm{Irr}(A)$. In the cases (a) and (b) above, A intersects trivially a normal subgroup with abelian (cyclic) factor group, so $A \cap H' = 1$ and this implies that $\mathbf{N}_H(Y) = \mathbf{C}_H(Y)$; hence, $tyt^{-1} = y$ for every $t \in T_Y$. For the case (c), one can easily check (using the matrix presentation in [2]) that if $H = NCA$ where A is a subgroup of a cyclic of order 8, then for $Y \leq A$ we have $\mathbf{N}_H(Y) = \mathbf{C}_H(Y)$ if $|Y| \in \{1, 2, 8\}$ and $[\mathbf{N}_H(Y) : \mathbf{C}_H(Y)] = 2$ if $|Y| = 4$. This implies that $\alpha(tyt^{-1}) = \alpha(y)$ if $o(y) \neq 4$ and that $\alpha(tyt^{-1}) \in \{\alpha(y), \overline{\alpha(y)}\}$ if $o(y) = 4$.

So, assuming $o(y) \neq 4$ in case (c), we have

$$\chi(xy) = \sum_{t \in T} \theta^\circ(txyt^{-1}) = \sum_{t \in T_Y} \alpha(tyt^{-1})\theta(txt^{-1}) =$$

$$= \alpha(y) \sum_{t \in T_Y} \phi(txt^{-1}) \ .$$

We work next to show that $\sum_{t \in T_Y} \phi(txt^{-1}) = -1$. We recall that the map $f : T_Y \to M_Y - \{1\}$ such that, for $t \in T_Y$, $f(t) = x^{t^{-1}}$, is a bijection. We also observe that M_Y is not contained in the kernel of the linear character ϕ: in fact, as Y stabilizes ϕ, it follows that $[M, Y] \leq \ker(\phi)$, $M = [M, Y] \times M_Y$ and $\phi \neq 1_M$. Thus, $[\phi_{M_Y}, 1_{M_Y}] = 0$ and so we have

$$\sum_{t\in T_Y}\phi(txt^{-1}) = \sum_{z\in M_Y-\{1\}}\phi(z) = |M_Y|[\phi_{M_Y}, 1_{M_Y}] - \phi(1) = -\phi(1) = -1.$$

Our conclusion so far is that $\chi(xy) = -\alpha(y)$ for every $y \in A$ (in particular, also when $y = 1$), with the only exception of case (c) and $o(y) = 4$.

In view of this fact, if χ is p-constant, then $-\alpha(y) = \chi(xy) = c_\chi$ for all $y \in A$ (so $\alpha(y) = \alpha(1) = 1$) in all cases except in case (c) when $o(y) = 4$; but in this case $y^2 \in \ker(\alpha)$, so $\alpha(y) = \overline{\alpha(y)}$ and with the same argument we still get $\alpha(y) = 1$. Conversely, if $\alpha(y) = 1$ for every $y \in A$, then χ is p-constant.

We are now in a position to finish the proof. Rephrasing the paragraph above, we have that the faithful irreducible character χ of G, whose degree we already know to be $p^n - 1$, is p-constant if and only if it is induced by an extension θ of ϕ to MA such that $\ker(\theta)$ contains A, and in this case the constant value c_χ is -1. Now, there exists a unique extension with this property: it is the canonical extension of ϕ to MA (see Lemma 13.3 of [3]). This concludes the proof. □

The solvable 2-transitive groups were first determined by B. Huppert in [1] (see also [4, Theorem 6.8]). We use this knowledge and Passman's Theorem 2.5 in order to give a detailed description of the p-solvable groups that have a faithful irreducible p-constant character, not of p-defect zero, for a prime number p.

Theorem 4.5. *Let G be a p-solvable group, P a Sylow p-subgroup of G and H a p-complement of G. Assume that G has a faithful irreducible character χ which is p-constant and not of p-defect zero. Then P is the unique minimal normal subgroup of G and, setting $p^n = |P|$, we have the following properties;*

(a): *if H is non-solvable, then $n = 2$, $p \in \{11, 19, 29, 59\}$ and $H = KZ$ is a central product of $K \cong \mathrm{SL}_2(5)$ and Z cyclic of order dividing $p - 1$;*

(b): *either $H \leq \Gamma(p^n)$, so H is metacyclic, or*

- **(b1):** *$n = 2$, $p \in \{5, 7, 11, 23\}$, $\mathbf{F}(H)$ is a central product of a quaternion group of order 8 and a cyclic group of order dividing $p - 1$ and $H/\mathbf{F}(H)$ is isomorphic to either the cyclic group C_3 or the symmetric group* $\mathrm{Sym}(3)$*;*
- **(b2):** *$p^n = 3^4$ and H is an extension of an extraspecial group of order 2^5 by a subgroup, of order multiple of* 5*, of the Frobenius group of order* 20.

Proof. By Proposition 2.4, Remark 4.2 and Proposition 4.3, G is a 2-transitive permutation group, P is its unique minimal normal subgroup and H is a point stabilizer. We can hence apply Theorem 2.5. If H is non-solvable, then $|P| = p^2$, $p \in \{11, 19, 29, 59\}$; so, in particular, we can identify H with a subgroup of $\mathrm{GL}_2(p)$. Recalling the subgroup structure of $\mathrm{SL}_2(p)$ (see for instance Theorem 6.17 in Chapter 3 of [11]), we see that $K = H \cap \mathrm{SL}_2(p)$ is isomorphic to $\mathrm{SL}_2(5)$. One can check that the normalizer in $\mathrm{GL}_2(p)$ of K is the product KZ, with $Z = \mathbf{Z}(\mathrm{GL}_2(p))$ and hence, by Dedekind's Law, we have (a).

If, on the other hand, H is solvable and not a subgroup of the semilinear group $\Gamma(p^n)$, then by Theorem 2.5 either $p^n = 3^4$ or $n = 2$ and $p \in \{3, 5, 7, 11, 23\}$. By Theorem 6.8 of [4], we have that H has the structure described in (b1) and (b2); note that $p \neq 3$, as otherwise H would be isomorphic to either $\mathrm{SL}_2(3)$ or $\mathrm{GL}_2(3)$, against $(p, |H|) = 1$. □

5. When $\mathbf{O}_p(G)$ is trivial: a proof of Theorem B

In this section, we consider a situation that is complementary to the previous one: we assume that a group G *for which $\mathbf{O}_p(G)$ is trivial* has an irreducible character that is faithful, p-constant and not of p-defect zero. As already observed, the faithfulness of the relevant character forces $\mathbf{O}_{p'}(G)$ to be trivial as well, thus our group has in fact a trivial Fitting subgroup.

We will first show in Lemma 5.1 that, under our assumptions, G has a unique minimal normal subgroup. Then in Theorem 5.2, which is Theorem B, this minimal normal subgroup is proved to be simple.

Lemma 5.1. *Let p be a prime number, and let G be a group having an irreducible character that is faithful, p-constant and not of p-defect zero. Assume also that $\mathbf{O}_p(G) = 1$. Then G has a unique minimal normal subgroup.*

Proof. For a proof by contradiction, assume that G has two distinct minimal normal subgroups M_1, M_2; denoting by χ the character as in our hypotheses, set $M = M_1 \times M_2$, and let $\theta_1 \times \theta_2$ be an irreducible constituent of χ_M (where θ_1 and θ_2 are suitable irreducible characters of M_1 and M_2, respectively). If T is a right transversal for the inertia subgroup of $\theta_1 \times \theta_2$ in G, setting $e = [\theta_1 \times \theta_2, \chi_M]$, by Clifford Theory we get

$$c_\chi = \chi(xy) = e \sum_{t \in T} (\theta_1 \times \theta_2)^t(xy) = e \sum_{t \in T} \theta_1^t(x)\theta_2^t(y),$$

whenever x in M_1 or y in M_2 is p-singular. Note that, as $\mathbf{O}_{p'}(G) = 1$, the order of M_1 is certainly divisible by p; therefore we can choose x_0 to be an element of M_1 whose order is divisible by p, and the hypothesis of χ being p-constant yields that the function

$$\sum_{t \in T} \theta_1^t(x_0)\theta_2^t - \left(\sum_{t \in T} \theta_1^t(x_0)\theta_2(1) \right) 1_{M_2}$$

takes the value 0 on every y in M_2.

Observe that the above function is expressed as a linear combination of the irreducible charaters of M_2, in which the principal character 1_{M_2} appears with the coefficient $\sum_{t \in T} \theta_1^t(x_0)\theta_2(1)$ (this follows taking into account that, as χ is faithful, θ_2 and all its G-conjugates are non-principal). But now, the linear independence of the elements in $\mathrm{Irr}(M_2)$ forces $\sum_{t \in T} \theta_1^t(x_0)\theta_2(1) = 0$, i.e., $\chi(x_0) = 0$. Therefore, as χ is p-constant, we get $c_\chi = 0$ against the fact that χ is not a character of p-defect zero. This contradiction completes the proof. □

Theorem 5.2. *Let p be a prime number, and let G be a group having an irreducible character that is faithful, p-constant and not of p-defect zero. Assume also that $\mathbf{O}_p(G) = 1$. Then G is an almost-simple group.*

Proof. By the previous lemma, we know that G has a unique minimal normal subgroup M, and our aim is to show that M is a simple group. Let S be a simple subnormal subgroup of G with $S \leq M$, and let $\{x_1, x_2, \ldots, x_t\}$ be a right transversal of $\mathbf{N}_G(S)$ in G, with $x_1 = 1$. Then $M = S^{x_1} \times S^{x_2} \times \cdots \times S^{x_t}$. We assume, working by contradiction, that $t \geq 2$, i.e. $\mathbf{N}_G(S) \neq G$.

For $x \in G$ we have

$$S^{x_i x^{-1}} = S^{x_{\sigma_x(i)}}$$

for a permutation σ_x, depending on x, of the set $\Omega = \{1, 2, \ldots, t\}$. We observe that this defines a transitive action of G on Ω and that $x \in \mathbf{N}_G(S)$ if and only if $\sigma_x(1) = 1$.

Let $\theta_i \in \mathrm{Irr}(S)$, for $i \in \Omega$, and let $x \in G$. Then it can be checked that

$$(\theta_1^{x_1} \times \cdots \times \theta_t^{x_t})^x = \left(\theta_{\sigma_x(1)}\right)^{x_{\sigma_x(1)}x} \times \cdots \times \left(\theta_{\sigma_x(t)}\right)^{x_{\sigma_x(t)}x}.$$

Now, let $\chi \in \mathrm{Irr}(G)$ be as in our hypotheses, and let $\psi = \theta_1^{x_1} \times \cdots \times \theta_t^{x_t}$ be an irreducible constituent of χ_M, where $\theta_i \in \mathrm{Irr}(S)$. Replacing ψ by a suitable G-conjugate, we may assume that

$\theta_1 \neq 1_S$, since $\psi \neq 1_M$ and the action of G on Ω is transitive. By Clifford's Theorem we have

$$\chi_M = v \sum_{x \in T} \psi^x ,$$

for some $v > 0$ and a subset T of G (this T is a right transversal of the inertia subgroup $I_G(\psi)$ of ψ in G). Let $U = S^{x_2} \times \cdots \times S^{x_t}$. We fix a p-singular element $x_0 \in S$ (recall that $|M|$ is divisible by p because $\mathbf{O}_{p'}(G) = 1$) and we define $\chi_0(u) = \chi(x_0 u)$ for $u \in U$. Then $\chi_0 = k 1_U$, for some constant $k \neq 0$ (by Theorem 3.2). However, for $u \in U$ we have

$$k = \chi_0(u) = \chi(x_0 u) = \sum_{x \in T} v(\theta_{\sigma_x(1)})^{x_{\sigma_x(1)} x}(x_0)[(\theta_{\sigma_x(2)})^{x_{\sigma_x(2)} x} \times \cdots \times (\theta_{\sigma_x(t)})^{x_{\sigma_x(t)} x}](u) .$$

Thus

$$k 1_U = \sum_{x \in T} c_x \varphi_x , \tag{1}$$

where $c_x = v(\theta_{\sigma_x(1)})^{x_{\sigma_x(1)} x}(x_0)$ and

$$\varphi_x = (\theta_{\sigma_x(2)})^{x_{\sigma_x(2)} x} \times \cdots \times (\theta_{\sigma_x(t)})^{x_{\sigma_x(t)} x} \in \mathrm{Irr}(U) \tag{2}$$

for $x \in T$. Then

$$k 1_U = \sum_{x \in T} c_x \varphi_x = \sum_{\substack{x \in T \\ \varphi_x \neq 1_U}} c_x \varphi_x + \sum_{\substack{x \in T \\ \varphi_x = 1_U}} c_x 1_U .$$

Hence

$$\sum_{\substack{x \in T \\ \varphi_x \neq 1_U}} c_x \varphi_x = 0 ,$$

by the linear independence of characters.

Let $x \in T$ such that $\varphi_x = 1_U$ (observe that such an element exists by (1), as $k \neq 0$). Then

$$(\theta_{\sigma_x(i)})^{x_{\sigma_x(i)} x} = 1_{S^{x_i}}$$

for $i \geq 2$. Therefore $\theta_{\sigma_x(i)} = 1_S$ for all $i \geq 2$. So

$$\psi = \theta_1^{x_1} \times 1_{S^{x_2}} \times \cdots \times 1_{S^{x_t}} .$$

Since $\theta_1 \neq 1_S$, we have that $I_G(\psi) \leq \mathbf{N}_G(S)$. Moreover, if $x \in T$ and $\varphi_x = 1_U$, then $\sigma_x(i) \geq 2$ for all $i \geq 2$, so $\sigma_x(1) = 1$ and $x \in \mathbf{N}_G(S)$. Let $T_0 = T \cap \mathbf{N}_G(S)$ and observe that $T_0 \neq T$ because $G \neq \mathbf{N}_G(S)$. Observing that if $x \in T_0$, then $\varphi_x = 1_U$ by (2), we conclude that for $x \in T$ we have $\varphi_x = 1_U$ if and only if $x \in T_0$. Finally, if $x \in T - T_0$, then $\sigma_x(1) > 1$, $\theta_{\sigma_x(1)} = 1_S$ and $c_x = v$. We conclude that

$$0 = \sum_{x \in T - T_0} c_x \varphi_x = v \left(\sum_{x \in T - T_0} \varphi_x \right) .$$

Evaluating in 1, we get a contradiction. □

As the final remark, assume that G is an almost-simple group with socle M and that, for a given prime number p, the character $\chi \in \mathrm{Irr}(G)$ is p-constant and not of p-defect zero. It might be worth observing that the irreducible constituents of the restriction χ_M can be all non-p-constant. This happens for instance considering $G = \mathrm{Sym}(5)$ and its irreducible character of degree 6, which is 5-constant but, when restricted to $\mathrm{Alt}(5)$, has two irreducible constituents that are both non-5-constant.

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