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On derived de Rham cohomology

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Title: On Derived de Rham cohomology

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Abstract: The derived de Rham complex has been introduced by Illusie in 1972 as a natural consequence of the definition of the cotangent complex for a scheme morphism. This theory seems to have been forgot until the recents works by Bhatt and Beilinson, who gave several applications, in particular in p-adic Hodge Theory. On the other hand, the derived de Rham cohomology has a crucial role in a conjecture by Flach-Morin about special values of zeta functions for arithmetic schemes. The aim of this thesis is to study and compute the Hodge completed derived de Rham complex in some cases.

Keywords: de Rham cohomology, derived geometry.

Titre: Sur la cohomologie de de Rham derivée

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Résumé: La cohomologie de de Rham dérivée a été introduite par Luc Illusie en 1972, suite à ses travaux sur le complexe cotangent. Cette théorie semble avoir été oubliée jusqu'aux travaux récents de Bhatt et Beilinson, qui ont donné diverses applications, notamment en théorie de Hodge p-adique. D'autre part, la cohomologie de Rham dérivée intervient de manière cruciale dans une conjecture de Flach-Morin sur les valeurs spéciales des fonctions zêta des schémas arithmétiques. Dans cette thèse, on se propose d'étudier et de calculer la cohomologie de de Rham dérivée dans certains cas.

Mots Clées: cohomologie de de Rham, geometrie derivée

Titolo: La coomologia di de Rham derivata

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Abstract: La coomologia di de Rham derivata é stata introdotta da Luc Illusie nel 1972, in seguito ai suoi lavori sul complesso cotangente. Tale teoria sembra essere stata dimenticata fino ai lavori recenti di Bhatt e Beilinson, i quali ne hanno fornito diverse applicazioni, in particolare nella teoria p-adica di Hodge. D'altra parte, la coomologia di de Rham derivata interviene in maniera cruciale in una congettura di Flach-Morin sui valori speciali della funzione zeta di schemi aritmetici. In questa tesi ci proponiamo di studiare e calcolare la coomologia di de Rham derivata in certi casi.

Parole chiave: coomologia di de Rham, geometria derivata

Contents

Introduction

The derived de Rham complex has been introduced by Illusie [26, Ch. VIII] and follows from the notion of the cotangent complex. This theory seems to have been forgotten until the recents works by Beilinson [6] and Bhatt [10],[9]. They gave several applications in particular in p-adic Hodge Theory, but such object turns out to be very versatile. It provides a fruitful generalization of the de Rham theory for singular varieties (see for example [9]) as well as a new construction of Fontaine's period rings ([6]) or numerical invariants for special values of zeta functions of varieties over finite fields ([35],[36]). In particular Bhatt studied the Hodge completion in characteristic zero [9] and the (derived) p-adic completion in characteristic p. The aim of this thesis is to study and compute the Hodge completed derived de Rham complex in positive and mixed characteristic, as well as the derived de Rham complex relative to $\mathrm{Spec}(\mathbb{Z})$, inspired by some results of Morin [36].
At first our work focus on computing the Hodge completed derived de Rham complex of the map $\mathbb{Z} \longrightarrow k$, where k is a perfect ring of characteristic p (i.e. the Frobenius map is an automorphism). It turns out that, being $W = W(k)$ the ring of Witt vectors, there is the following equivalence

Theorem 1. *Let k be a perfect ring of characteristic p and $W(k)$ its ring of Witt vectors. Then there exist a quasi-isomorphism of pro- commutative differential graded algebras*

$$L\widehat{\Omega}^*_{k/\mathbb{Z}} \cong \left(\frac{W\langle x\rangle}{(x)^{[N]}} \xrightarrow{\cdot(x-p)} \frac{W\langle x\rangle}{(x)^{[N]}} \right)_{N\in\mathbb{N}}$$

where the complex on the right has cohomology concentrated in degree zero.

Bhatt computed the (not Hodge completed) derived de Rham complex p-adically completed in the same case (see [10, Corollary 8.6]). Our computations rely on the base change lemma applied to the (crucial) simple case where $k = \mathbb{F}_p$. Similar results may be obtained by means of crystalline theory computations, see [26, Ch.VIII Proposition 2.2.8]. In the present work we give a more direct and elementary proof, which takes into account the multiplicative structure differential graded algebras (say also $E_\infty-$algebras).

In order to apply the previous result to a larger class of objects, we generalize the well known Künneth formula for the Hodge completed derived de Rham complex when it is seen in the category of pro-cdga. In particular we prove the following

Theorem 2. *Given two rings maps $A \longrightarrow B$ and $A \longrightarrow C$ there is a quasi-*

isomorphism of pro-commutative differential graded algebras

$$“\varprojlim_{L\in\mathbb{N}}” \left(\frac{L\Omega^*_{B\otimes_A C/A}}{F^L}\right) \cong “\varprojlim_{N\in\mathbb{N}}” \left(\frac{L\Omega^*_{B/A}}{F^N}\right) \otimes_A “\varprojlim_{M\in\mathbb{N}}” \left(\frac{\Omega^*_{C/A}}{F^M}\right).$$

We then apply such result to schemes over perfect fields $X \longrightarrow k$ in order to exploit our previous computations. Consider cartesian diagrams of the form

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec}(k) & \longrightarrow & \operatorname{Spec}(\mathbb{Z}), \end{array} ,$$

where the morphism of scheme $Y \longrightarrow \operatorname{Spec}(\mathbb{Z})$ is smooth (and then the derived de Rham complex is naturally quasi-isomorphic to the non-derived one, $\Omega^*_{X/Y}$). By means of the Künneth formula, we can then study the complex $L\widehat{\Omega}^*_{X/\mathbb{Z}}$ from those relative to the maps $\mathbb{Z} \longrightarrow k$ and $Y \longrightarrow \operatorname{Spec}(\mathbb{Z})$, for which computations are easier.

These results can be the starting point of further studies. We can for example consider smooth projective varieties over a finite field. Let $X/\mathbb{F}_q$ be such variety, we may investigate the Hodge completed derived de Rham complex of the morphism $X \longrightarrow \operatorname{Spec}(\mathbb{Z})$. It seems to be a close relation between such complex and the de Rham-Witt complex ([27]). We get in particular the following results

Theorem 3. *Let $\mathfrak{X}$ be a smooth scheme over $\operatorname{Spec}(W)$, where $W = W(k)$ for a perfect field k. Consider the following cartesian square*

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{X} \\ \downarrow & \lrcorner & \downarrow \\ \operatorname{Spec}(k) & \longrightarrow & \operatorname{Spec}(W(k)), \end{array} ,$$

where $X := \operatorname{Spec}(k) \times_{\operatorname{Spec}(W(k))} \mathfrak{X}$. Then there is an quasi-isomorphism of pro-complexes

$$L\widehat{\Omega}^*_{X/\mathbb{Z}} \simeq \left(\Omega^*_{\mathfrak{X}/W} \otimes \frac{W\langle x\rangle}{(x)^{[N]}} \xrightarrow{\cdot(x-p)} \Omega^*_{\mathfrak{X}/W} \otimes_W \frac{W\langle x\rangle}{(x)^{[N]}}\right)_N .$$

It would be interesting to replace X with a general separated schemes of finite type over a finite field.

Outline. Chapter 1 is devoted to briefly presents the main tools used along the thesis. Homotopical backgrounds are presented in §1.1 and §1.2. We chose to take on the setting without using model category theory, hoping that the reader who is not into such theory may still be at easy with the exposition. We then briefly recollect some basic results about pro-categories in §1.3. In Chapter 2 we introduce Illusie's cotangent complex (§2.1) and derived de Rham complex (§2.2) and we prove some properties. In §2.3 the original part of the thesis starts. We prove Theorem

1, which will be the cornerstone of the Chapter 4. Chapter 3 is entirely devoted to prove Theorem 2, which is a generalization of a classical cohomology result in the context introduced in §1.3. Finally Chapter 4 aims to apply all the previous results in the context of algebraic varieties over perfect fields and to draw some direction for future investigations.

Chapter 1

Homotopical Algebra Recollection

In order to give a reasonably complete introduction to Illusie's theory of the cotangent complex and the derived de Rham complex ([25],[26]), in the first part of this chapter we give a brief presentation of some simplicial methods, on which the definition of such objects relies. Such methods can be read as formal constructions and accepted without any motivations. For this reason, when presenting some definitions and results, we have chosen to not mention the theory of model category, which underpins all the chapter. As a matter of facts we may say that the strategy leading our exposition aims to extend homological constructions to non-abelian categories. Consider a covariant left exact functor between two abelian categories $F : \mathcal{A} \longrightarrow \mathcal{B}$, supposing $\mathcal{A}$ with enough projectives. Its associated derived functor is defined passing throught the categories of (non negatively graded) chain complexes $\mathrm{Ch}(\mathcal{A}), \mathrm{Ch}(\mathcal{B})$ and the concept of projective resolution. Take for example $\mathcal{A}$ to be the category of R-modules for some commutative unitary ring R. A projective resolution of a R-module M, is a chain complex P^* in $\mathrm{Ch}(\mathcal{A})$, so that each P^n is a projective module and

(1) $H_n(P^*) = 0$ for $n > 0$ and

(2) there is a morphism $P^0 \longrightarrow M$ which induces an isomorphim $H_0(P^*) \cong M$.

Equivalently, if we regard M as a chain complex concentrated in degree zero we can say that there is a morphism of chain complexes $P^* \longrightarrow M$ which induces an isomorphism on homology. Chain complexes of projectives have in particular a "lifting property" which is used to prove the uniqueness of projective resolutions up to chain homotopy. The idea is to extend such machinery to a non-abelian context. A fundamental result we are going to present (Theorem 1.1.31) is the Dold-Kan correspondence which shows that $\mathrm{Ch}(\mathcal{A})$ is equivalent to the category of simplicial objects of $\mathcal{A}$. The idea is then to replace $\mathrm{Ch}(\mathcal{A})$ and all the homological machinery with the category of simplicial objects and the corresponding tools, which can be defined without the condition of abelianity.

The second section is devoted to understand a new structure which arises when applying the Dold-Kan correspondence to the category of simplicial rings or in general simplicial R-algebras, for a unitary commutative ring R. The corresponding complex is endowed with a graded product and the new object is called differential graded algebra. As a particular case of this phenomena, the derived de Rham complex has then a double nature: on one side is a (co)homological object, a complex whose homology frames topological invariants, on the other side is more an algebraic/arithmetic object, a (graded) commutative algebra. In general it is not easy to keep stable this double nature. This is due to the fact that a useful way to consider complexes is up to quasi-isomorphism, that is those morphisms which induce an isomorphism on homology groups. In this way we can consider complexes which share the same homology as equals, in order to chose the representative of some class for our purposes (e.g. projective complexes). This point of view is natural to handle in the category of complexes, since they form a *model category*, but it is not always possible to fit the arithmetic nature in such structure. In the second section we investigate how to deal with this inconveniences, detecting those cases which don't present problems in this sense and providing a general framework where all these nuisances are overcome.

A further structure characterizing the derived de Rham complex is given by the Hodge filtration on it. It induces a projective system of differential graded algebras, which is going to be our main object of investigation. To achieve better flexibility, rather than the category of projective systems, it is convenient to consider the category of pro-dga, where objects are "formal cofiltered limit" of objects of *cdga*. Section 1.3 gives a brief introduction on pro-categories and the translation of some properties of a category $\mathcal{C}$ to the associated pro-category pro$-\mathcal{C}$.

1.1 Simplicial homotopy theory

The origin of simplicial homotopy theory coincides with the beginning of algebraic topology almost a century ago. The thread of ideas started with the work of Poincaré and continued to the middle part of the 20th century in the form of combinatorial topology. The modern period began with the introduction of the notion of simplicial set, by Eilenberg-Zilber in 1950, and evolved into a complete homotopy theory in the work of Kan, beginning in the 1950s, and later Quillen in the 1960s. The theory has always considered simplices with some incidence relations, along with methods for constructing maps and homotopies of maps within these constraints. As such, the methods and ideas are algebraic and combinatorial and, despite the deep connection with the homotopy theory of topological spaces, exist completely outside any topological context. This point of view was effectively introduced by Kan, and later encoded by Quillen in the notion of a closed model category. Simplicial homotopy theory, and more generally the homotopy theories associated to closed model categories, can then be interpreted in a purely algebraic way, which has had substantial applications throughout homological algebra, algebraic geometry, number theory and algebraic K-theory. The point is that homotopy is more than the standard variational principle from topology and analysis: homotopy theories are everywhere, along with functorial methods of relating them.

1.1.1 Definitions and examples

Definition 1.1.1. We denote by Δ the category of non empty finite ordered set with order preserving morphisms, it is usually called the *Simplex category.* A standard way to describe Δ is to consider the objects as $[n] := \{0 < 1 < ... < n\}$, for $n \geq 0$, with non decreasing functions.

Example 1.1.2 (Face and Degeneracy maps)**.** Among the non decreasing functions there are two important families. The *face maps* $\delta_n^i : [n-1] \longrightarrow [n]$, for any $n > 0$ and $i = 0, ..., n$. δ_n^i is the unique injective map whose image misses i, i.e. for $j = 0, ..., n-1$

$$\delta_n^i(j) = \begin{cases} j \text{ if } j < i \\ j+1 \text{ if } j \geq i \end{cases} .$$

Degeneracy maps $\eta_n^i : [n+1] \longrightarrow [n]$, for $n \geq 0$ and $0 \leq i \leq n+1$. The map η_n^i is the unique surjective map with two elements mapped to i, i.e. for $j = 0, ..., n+1$

$$\eta_n^i(j) = \begin{cases} j \text{ if } j \leq i \\ j-1 \text{ if } j > i \end{cases} .$$

When the setting is clear or not really important indices are omitted.

Definition 1.1.3. Given a category $\mathcal{C}$, a *simplicial object* of $\mathcal{C}$ is a functor $X : \Delta^{op} \longrightarrow \mathcal{C}$. A natural transformation of functors of such form is considered as a morphism of simplicial objects. Simplicial objects and their morphisms form a category that in general we denote by juxtaposing an "s" to the relative category, $s\mathcal{C} := Fun(\Delta^{op}, \mathcal{C})$.

Such definition of simplicial object is very neat and easy to manage in the setting of category theory, as it actually denote simplicial objets as a particular class of pre-sheaves. However it will be very useful to handle such tool in a more "concrete" and combinatorial way, within some specific categories. Next lemma will help us in this sense.

Lemma 1.1.4. *Giving a simplicial object $X : \Delta^{op} \longrightarrow \mathcal{C}$ is equivalent to the data of a family of objects $\{X_n\}_{n\geq 0}$ in $\mathcal{C}$ and two families of arrows in $\mathcal{C}$ for each $n \geq 0$*

$$\{\partial_n^i : X_n \longrightarrow X_{n-1}\}_{0\leq i\leq n}, \qquad \{\sigma_n^i : X_n \longrightarrow X_{n+1}\}_{0\leq i\leq n}$$

satisfying the following identities for any $n \geq 0$

$$\begin{aligned} \partial_n^i \partial_{n+1}^j &= \partial_n^{j-1} \partial_{n+1}^i \qquad i < j \\ \sigma_{n+1}^i \sigma_n^j &= \sigma_{n+1}^{j+1} \sigma_n^i \qquad i \leq j \\ \partial_{n+1}^i \sigma_n^j &= \begin{cases} \sigma_{n-1}^{j-1} \partial_n^i & i < j \\ id & i = j \text{ or } i = j+1 \\ \sigma_{n-1}^j \partial_n^{i-1} & i > j+1 \end{cases} \end{aligned} \tag{1.1}$$

Proof. *(Sketch)* The identification is made by the following equalities: $X_n := X([n])$, $\partial_n^i := X(\delta_n^i)$ and $\sigma_n^i := X(\eta_n^i)$. Then the proof relies on the fact that any map $[m] \xrightarrow{f} [n]$ in Δ can be factorized as a composition of an injective and a surjective map $f = \delta \circ \eta$, where δ is the composition of face maps and η is the composition of degeneracy maps. Then it remains to prove that degeneracy maps and face maps satisfies the contravariant version of (1.1) □

Given the above results, we will use both notations for simplicial objects, X or $X_\bullet$, $X([n])$ or X_n and face and degeneracy maps ∂, σ.

Example 1.1.5 (Constant simplex)**.** Given an object A in any category $\mathcal{C}$. There is always a unique constant functor from the category $\underline{\{*\}}$ with one object and only the identity morphism, sending $*$ to A. The constant simplex $A_\bullet$ associated to A is the unique functor $\Delta^{op} \longrightarrow \mathcal{C}$ factoring through the constant functor (sometimes we denote $A_\bullet$ directly as A). More concretely $A_n = A$ for any $n \geq 0$ and all the maps $[m] \longrightarrow [n]$ are sent to the identity morphism.

Example 1.1.6 (Standard q-simplex)**.** For $q \geq 0$ we define the simplicial set $\Delta[q] := \mathrm{Hom}_\Delta(-, [q]) : \Delta^{op} \longrightarrow \underline{Set}$, i.e. the simplicial set defined by $\Delta[q]_n := \mathrm{Hom}_\Delta([n], [q])$, with maps induced by the contravariance of the functor Hom. Such simplicial set is called q-simplex and is a crucial example, with the following universal property. Given any simplicial set X, by the Yoneda lemma

$$\mathrm{Hom}_{s\underline{Set}}(\Delta[q], X) = X_q,$$

i.e. there is a 1-1 correspondence between any element $x \in X_q$ and simplicial morphisms $f : \Delta[q] \longrightarrow X$. In particular f is the unique morphism sending $id_{[q]}$ to x, i.e. any $\alpha \in \Delta[q]_n = \mathrm{Hom}([n], [q])$ is sent to $X(\alpha)(x)$.
Now we compute some basic cases which will be useful in the future.

(1) For any $n \geq 0$, $\Delta[0]_n = \mathrm{Hom}([n], [0]) = \{*\}$, since there is only one trivial map.

(2) For any $n \geq 0$ we can describe $\Delta[1]_n$ as the finite collection of maps $\alpha_n^i : [n] \longrightarrow [1]$, $0 \leq i \leq n+1$, such that $(\alpha_n^i)(j) = 0$ for $j < i$, $(\alpha_n^i)(j) = 1$ otherwise. In particular α_n^{n+1} sends everything to 0 and α_n^0 sends everything to 1. The maps $\partial_n^j = - \circ \delta_n^j$ act as

$$\partial_n^j(\alpha_n^i) = \alpha_n^i \circ \delta_n^j = \begin{cases} \alpha_{n-1}^i \; j > i \\ \alpha_{n-1}^{i-1} \; j \leq i. \end{cases}$$

(3) We can define two maps $e_0, e_1 : \Delta[0] \longrightarrow \Delta[1]$, $(e_0)_n(*) = \alpha_n^{n+1}$, $(e_1)_n(*) = \alpha_n^0$. With respect to the universal property above e_0, e_1 are the morphism corresponding to the elements $(0 \longmapsto 0), (0 \longmapsto 1) \in \Delta[1]_0$.

Example 1.1.7 (Standard boundary q-simplex)**.** Another important example, related to the previous one, is given by the simplicial subset $\Delta^\circ[q] \hookrightarrow \Delta[q]$ (sometimes denoted $\partial\Delta[q]$ and called *boundary of* $\Delta[q]$) defined as the union of the images of the maps

$$\partial^i : \Delta[q-1] \longrightarrow \Delta[q]$$
$$([n] \to [q-1]) \longmapsto ([n] \to [q-1] \xrightarrow{\delta_q^i} [q])$$

It is a simplicial object inheriting the maps from those of $\Delta[q]$ which are defined by pre-composition for any $[m] \to [n]$, so that

$$\Delta^\circ[q]_n \ni \left([n] \to [q-1] \xrightarrow{\delta_q^i} [q]\right) \mapsto \left([m] \to [n] \to [q-1] \xrightarrow{\delta_q^i} [q]\right) \in \Delta^\circ[q]_m.$$

We define lastly $\Delta^\circ[0]$ as the constant simplex defined by the empty set $\emptyset$. It is the initial object of $s\underline{Set}$. The simplicial set $\Delta^\circ[1]$ is made for each $n \geq 0$ of maps $[n] \longrightarrow [1]$ factoring through the maps $\delta_1^0, \delta_1^1 : [0] \longrightarrow [1]$, so that $\Delta^\circ[1]_n$ is always made of two elements.

Example 1.1.8 (Geometric Realization)**.** In order to justify most of the choices of notation I would like to present briefly the relationship between simplicial sets and topological spaces (see for example [18, Ch. I]).
There is a functor from Δ to the category Top of topological spaces. Given $[n]$, we send it to the standard topological n-simplex $\Delta^n := \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} : t_0 + ... + t_n = 1,\ t_i \geq 0\ \forall i\}$. Given a morphism $\phi : [m] \longrightarrow [n]$ of ordered sets, we define $\Delta^m \longrightarrow \Delta^n$ by sending

$$(t_0, ..., t_m) \longmapsto (u_j), \qquad u_j = \sum_{\phi(i)=j} t_i.$$

The empty sum is to be regarded as zero. For instance, the face map δ_n^i will embed Δ^{n-1} as the i^{th} face of Δ^n. Recall that a simplicial set is a pre-sheaf $\Delta^{op} \longrightarrow \underline{\text{Set}}$ and that any pre-sheaf is a colimit of representable functors. Moreover the category Top is cocomplete, so it follows that there is induced a unique colimit-preserving functor

$$|-| : s\,\underline{\text{Set}} \longrightarrow \text{Top}$$

that sends the standard n-simplex $\Delta[n]$ (i.e., the simplicial set corresponding to $[n]$ under the Yoneda embedding) to Δ^n, with the maps $\Delta^n \longrightarrow \Delta^m$ associated to $[n] \longrightarrow [m]$ as before. Such association is then extended by presenting a simplicial set as a colimit of the objects $\Delta[n]$ and taking that colimit in Top. This functor is called *geometric realization.*
This functor has a right adjoint. In fact, this adjoint is none another than the singular simplicial set $\operatorname{Sing} T$ for a topological space T. Given a topological space T consider the set of continuous maps $\operatorname{Sing}_n T = \operatorname{Hom}_{\text{Top}}(\Delta^n, T)$ for every nonnegative integer n. We note that $\operatorname{Sing} T = \{\operatorname{Sing}_n T\}_{n\geq 0}$ has the structure of a simplicial set. For example $\partial^i : \operatorname{Sing}_n T \longrightarrow \operatorname{Sing}_{n-1} T$ carries an n-simplex of T to its i^{th} face. The simplicial set $\operatorname{Sing} T$ is sometimes called the *singular complex* of the topological space T. This object is quite familiar: if we apply the free abelian group functor levelwise to $\operatorname{Sing}(T)$, we form a simplicial abelian group $\mathbb{Z}[\operatorname{Sing}(T)]$. If we take

the alternating sum of the face maps we can extract from this a chain complex $C^*(\mathbb{Z}[Sing(T)])$ (see also Definition 1.1.25), and by definition

$$H_i(C^*(\mathbb{Z}[\mathrm{Sing}(T)])) \cong H_i^{\mathrm{sing}}(T, \mathbb{Z}),$$

i.e. we obtain the singular homology of first-year algebraic topology.

There is an "equivalence" between $s\,\underline{\mathrm{Set}}$ and the category of topological spaces, at the level of homotopy categories. This means we specify a notion of "weak equivalence" on each side (in topological spaces it is the usual notion, where in maps inducing isomorphisms of π_* are weak equivalences), and the "localization" of each side with respect to these are equivalent. We then pull back the notion of weak equivalence along this functor, as well as other classical algebraic topology constructions. This equivalence clarifies in some sense why most of the notation of this section is "copied" from algebraic topology.

Example 1.1.9 (Nerve of a category)**.** The following example is really important for for higher category theory (see for example [17, §2.1]). For every category $\mathcal{C}$ and every integer $n \geq 0$, let C_n denote the set of all composable chains of morphisms

$$C_0 \longrightarrow C_1 \longrightarrow \dots \longrightarrow C_n$$

of length n. The collection of sets $\{C_n\}_{n\geq 0}$ has the structure of a simplicial set, which is called the *nerve* of the category $\mathcal{C}$, and it determines C up to isomorphism. For example, the objects of $\mathcal{C}$ are simply the elements of C_0, and the morphisms in $\mathcal{C}$ are the elements of C_1.

1.1.2 Kan complexes and Homotopy groups

One of the most celebrated invariants in algebraic topology is the fundamental group: given a topological space X with a base point x, the fundamental group $\pi_1(X, x)$ is defined to be the set of paths in X from x to itself, taken modulo homotopy. The language of category theory allows us to package this information together in a very convenient form. In fact, it is possible to develop the theory of algebraic topology in entirely combinatorial terms, using simplicial sets as surrogates for topological spaces. However, not every simplicial set behaves like the singular complex of a space; it is therefore necessary to single out a class of "good" simplicial sets to work with.

Definition 1.1.10. A morphism of simplicial sets $f : X \longrightarrow Y$ has the *right lifting property* (RLP) with respect to the inclusion $\Delta^\circ[n] \hookrightarrow \Delta[n]$ if given a commutative square

$$\begin{array}{ccc} \Delta^\circ[n] & \longrightarrow & X \\ \big\downarrow & & \big\downarrow f \\ \Delta[n] & \longrightarrow & Y, \end{array}$$

then there exist a unique map $\Delta[n] \longrightarrow X$ such that the triangles commute,

$$\begin{array}{ccc}\Delta^\circ[n] & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \Delta[n] & \longrightarrow & Y.\end{array}$$

If f satisfies the RLP with respect to $\Delta^\circ[n] \hookrightarrow \Delta[n]$ for any $n \geq 0$ we call it a *trivial (Kan) fibration* (see [38, §2, Definition 1]). In particular if $X \longrightarrow *$ is a trivial fibration we say that X is a *Kan complex*.

Remark 1.1.11. *Being a Kan complex is equivalent to having the following property: for any $y_0, ..., y_n \in X_{n-1}$ such that $\partial^i_{n-1}(y_j) = \partial^{j-1}_{n-1}(y_i)$ if $i < j$, there exists $y{\in}X_n$ such that $\partial^i_n(y) = y_i$. The condition on $y_0, ..., y_n$ corresponds to the fact that the map*

$$\begin{aligned}\Delta^\circ[n]_{n-1} &\longrightarrow X_{n-1} \\ \delta^i_n &\longmapsto y_i\end{aligned}$$

actually defines a map of simplicial sets $\Delta^\circ[n] \longrightarrow X$. Under this correspondence, the RLP is equivalent to the existence of $y \in X_n$ as there exists a 1-1 correspondence between maps $\Delta[n] \longrightarrow X$ and elements $y \in X_n$ as shown in example 1.1.6.

Proposition 1.1.12. *Any simplicial group is a Kan complex as simplicial set*

Proof. Let G be a simplicial group and consider $x_0, ..., x_n \in G_{n-1}$ such that $\partial^i_{n-1}(x_j) = \partial^{j-1}_{n-1}(x_i)$ for $i < j$. We want to construct by induction an element $g_r \in X_n$ such that $\partial^i_n(g_r) = x_i$ for $i = 0, ..., r$, thus $y = g_n$ will complete the proof. Let $g_{-1} := 1_{G_n}$. Put $u := x_r^{-1}(\partial^r g_{r-1})$, then for $i < r$

$$\begin{aligned}\partial^i(u) &= \partial^i(x_r)^{-1}\partial^i\partial^r(g_{r-1}) \\ &= \partial^i(x_r)^{-1}\partial^{r-1}\partial^i(g_{r-1}) \\ &= \partial^i(x_r)^{-1}\partial^{r-1}(x_i) \\ &= \partial^i(x_r)^{-1}\partial^i(x_r) = 1.\end{aligned}$$

Thus $1 = \sigma^{r-1}\partial^i(u) = \partial^i\sigma^r(u)$. The element $g_r := g_{r-1}\sigma^r(u)^{-1}$ is such that for $i \leq r$

$$\begin{aligned}\partial^i g_r &= \partial^i(g_{r-1}\sigma^r(u)^{-1}) \\ &= \partial^i(g_{r-1})\partial^i(\sigma^r(u)^{-1}) \\ &= \begin{cases}\partial^i(g_{r-1}), & i < r \\ \partial^r(g_{r-1})u^{-1}, & i = r\end{cases} \\ &= \begin{cases}x_i, & i < r \\ (\partial^r g_{r-1})x_r(\partial^r g_{r-1})^{-1}, & i = r\end{cases} \\ &= x_i.\end{aligned}$$

Hence g_r satisfies the inductive hypothesis and we are done. □

The definition of homotopy groups for a Kan complex can be made in several ways. One possibility is to recover it directly from the topology homotopy groups by means of the correspondence of topological spaces and simplicial sets (recall example 1.1.8). Another way starts from the simplicial alter-ego of the topological tools (e.g. maps from $\Delta[0]$ and from $\Delta[1]$ for points and paths, simplicial homotopy etc.) as in [24, Definition 3.4.4] of [18, §I.7]. Moving further away from the topological world is the definition given in [16, §VI.3] and used by Illusie [25, (I.2.1.1)]. Finally there is a totally combinatorial definition, firstly enlightened by Kan [31]. We are going to use the latter, then we provide an equivalence with the second one in Proposition 1.1.21.

Construction 1.1.13. Let X be a Kan complex. Let $* \in X_0$, we write by abuse of notation $* = \sigma_n^0(*) \in X_n$ for any $n \geq 0$. Set

$$Z_n := \{x \in X_n : \partial_n^i(x) = * \text{ for all } i = 0, ..., n\}.$$

Moreover we say that $x, x' \in Z_n$ are *homotopic* if there exists $y \in X_{n+1}$ such that

$$\partial^i(y) = \begin{cases} * \text{ if } i < n \\ x \text{ if } i = n \\ x' \text{ if } i = n+1 \end{cases}.$$

The element y is called *homotopy* and we write $x \sim x'$.

Lemma 1.1.14. *In the setting above, $\sim$ is an equivalence relation.*

Proof. See [44][Lemma 8.3.1] or [31]. □

Definition 1.1.15. The set $\pi_n(X) := Z_n/\sim$ is called *n-th simplicial homotopy group.*

Remark 1.1.16. *As the definition of homotopy group for a general simplicial set relies on the choice of the* base point $* \in X_0$*, for simplicial abelian group such any choice is equivalent up to isomorphism and we canonically consider the unit of the group as base point. More precisely if $x \in X_0$ is the base point, then the right-translation given by yx^{-1} induces an isomorphism of groups.*

1.1.3 Simplicial Homotopies

Definition 1.1.17. Given a simplicial object $X_\bullet$ of a category $\mathcal{C}$ with coproducts, together with a simplicial object $U_\bullet$ of the category of non empty simplicial finite sets. We define the simplicial object $(X \times U)_\bullet$ of $\mathcal{C}$ as

$$(X \times U)_n := \coprod_{u \in U_n} X_n$$

and given $\gamma : [n] \longrightarrow [m]$ we get

$$\begin{aligned}(X \times U)(\gamma) : (X \times U)_m &\longrightarrow (X \times U)_n \\ X_m^{(u)} &\xrightarrow{X(\gamma)} X_n^{(U(\gamma)(u))}\end{aligned}$$

Definition 1.1.18 (Simplicial Homotopy)**.** Assume $\mathcal{C}$ to be a category with finite coproducts, let $f, g : X_\bullet \longrightarrow Y_\bullet$ be morphisms in $s\mathcal{C}$. A *simplicial homotopy* between f and g is a morphism

$$h : X_\bullet \times \Delta[1]_\bullet \longrightarrow Y_\bullet$$

such that $f = h \circ e_0$ and $g = h \circ e_1$, where $e_0, e_1 : X_\bullet \cong X_\bullet \times \Delta[0]_\bullet \longrightarrow X_\bullet \times \Delta[1]_\bullet$ are simplicial maps induced by the ones of example 1.1.6.

We say that f, g are *homotopic* if there exists a simplicial homotopy between them, we write $f \simeq g$.

Remark 1.1.19. *Fix $n \geq 0$, we have that $h_n : (X \times \Delta[1])_n = \coprod_{\alpha_n^i} X_n \longrightarrow Y_n$ corresponds to a family of $n+2$ maps indexed over the $\alpha_n^i \in \Delta[1]_n$ (recall remark 1.1.6)*

$$\{h_n^i : X_n^{(\alpha_n^i)} \longrightarrow Y_n\}_{i=0,\dots,n+1}.$$

The conditions $f = h \circ e_0$ and $g = h \circ e_1$ corresponds to the fact that $f_n = h_n \circ e_0 = h_n^0$ and $g_n = h_n \circ e_1 = h_n^{n+1}$. Moreover, with respect to the previous definition, for $\delta_n^i : [n-1] \longrightarrow [n]$, we have face maps of this form

$$(X \times U)(\delta_n^i) = \partial_{X \times U,n}^i : X_n^{(u)} \xrightarrow{\partial_{X,n}^i} X_{n-1}^{(\partial_{U,n}^i(u))}$$

and, in the case of simplicial homotopy with $U = \Delta[1]$, $\partial_{A \times \Delta[1],n}^j : A_n^{(\alpha_n^i)} \longrightarrow A_{n-1}^{(\alpha_n^i \delta_n^j)}$. If we compute the corresponding map in $\Delta[1]_{n-1}$ we get

$$\alpha_n^i \delta_n^j = \begin{cases} \alpha_{n-1}^i, & i \leq j \\ \alpha_{n-1}^{i-1}, & i > j \end{cases} .$$

This means that at a closer look the rule for maps of simplicial objects $h_{n+1}\partial_n^j = \partial_n^j h_n$ acts on the components as

$$\partial_{n+1}^j h_{n+1}^i = \begin{cases} h_n^i \partial_{n+1}^j, & i \leq j \\ h_n^{i-1} \partial_{n+1}^j, & i > j \end{cases} .$$

As a matter of fact, it can be proved ([44, proof of Theorem 8.3.12]) that a simplicial homotopy map h is equivalent to the data of $m+2$ maps, for any $m \geq 0$,

$$\{h_m^j : \Delta[n]_m \longrightarrow X_m\}_{j=0,\dots,m+1}$$

such that $h_m^0 = f$, $h_m^{m+1} = g$, and

$$\partial_m^i h_m^j = \begin{cases} h_{m-1}^{j-1} \partial_m^i, & i > j \\ h_{m-1}^j \partial_m^i, & i \leq j \end{cases} \qquad \sigma_m^i h_m^j = \begin{cases} h_{m+1}^{j+1} \partial_m^i, & i > j \\ h_{m+1}^j \partial_m^i, & i \leq j \end{cases}$$

Remark 1.1.20. *Simplicial homotopy in general does not define an equivalence relation, since it is not symmetric. However, it can be proved that for simplicial objects of an abelian category this is the case (further is an additive equivalence relation).*

The notation we used in this subsections recalls (and somehow confuses) the one we used in the previous subsection, when dealing with simplicial homotopy groups. Next proposition clarifies our choice.

Proposition 1.1.21. *Let X be a trivial Kan complex in a category with finite coproducts. The simplicial homology group $\pi_n(X)$ corresponds to the set of morphisms $\alpha : \Delta[n] \longrightarrow X$ such that the following diagram commutes*

$$\begin{array}{ccc} \Delta[n] & \xrightarrow{\alpha} & X \\ \uparrow & & \uparrow \\ \Delta^\circ[n] & \longrightarrow & * \end{array}$$

modulo simplicial homotopies (constant on $\Delta^\circ[n]$).

Proof. Recall (remark 1.1.6) that $\mathrm{Hom}(\Delta[n], X) = X_n$, so that for any element $x \in X_n$ we can define a map $\alpha_x : \Delta[n] \longrightarrow X$ and viceversa. In particular

$$\begin{array}{ccc} \Delta[n]_m & \xrightarrow{\alpha_{x,m}} & X_m \\ \downarrow{\scriptstyle \partial^i_m = - \circ \delta^i_m} & & \downarrow{\scriptstyle \partial^i_m} \\ \Delta[n]_{m-1} & \xrightarrow{\alpha_{x,m-1}} & X_{m-1} \end{array}$$

i.e. $\alpha_{x,m}(v \circ \delta^i_m) = \partial^i_m \alpha_{x,m-1}(v)$ for $v : [m] \longrightarrow [n]$. Suppose $\partial^i_n(x) = *$ for $i = 0, ..., n$ and take the previous diagram for the case $m = n$. Then $* = \partial^i_n(x) = \partial^i_n \alpha_{x,n}(id_{[n]}) = \alpha_{x,n-1}(\delta^i_n)$. This means that $\delta^i_n \in \Delta^\circ[n]_{n-1} \subseteq \Delta[n]_{n-1}$ is sent to $*$, so that the composition $\Delta^\circ[n] \hookrightarrow \Delta[n] \xrightarrow{\alpha} X$ factors through $*$.
It remains to show that if $x \sim x'$, then $\alpha_x \simeq_{\Delta^\circ[n]} \alpha_{x'}$ and viceversa.
The first condition is equivalent to the existence of $y \in X_{n+1}$ such that $\partial^i_{n+1} y = *$ $i = 0, ..., n-1$, $\partial^n_{n+1}(y) = x$ and $\partial^{n+1}_{n+1}(y) = x'$. It corresponds to a map $\beta : \Delta[n+1] \longrightarrow X$ such that $\beta(\delta^i_{n+1}) = \partial^i_{n+1}(y)$ for $i = 0, ..., n+1$. On the other hand, a simplicial homotopy between $\alpha_x, \alpha_{x'}$ is a map $h : \Delta[n] \times \Delta[1] \longrightarrow X$, such that $\alpha_x = h \circ e_0$ and $\alpha_{x'} = h \circ e_1$. The condition of being constant of $\Delta^\circ[n]$ corresponds to the fact that the following diagram commutes

$$\begin{array}{ccc} \Delta[n] \times \Delta[1] & \xrightarrow{h} & X \\ {\scriptstyle i \times id}\uparrow & & \uparrow{\scriptstyle \alpha_x|_{\Delta^\circ[n]} = \alpha_{x'}|_{\Delta^\circ[n]}} \\ \Delta^\circ[n] \times \Delta[1] & \xrightarrow{pr_{\Delta^\circ[n]}} & \Delta^\circ[n] \end{array}$$

Or equivalently, since $\alpha_x|_{\Delta^\circ[n]} = \alpha_{x'}|_{\Delta^\circ[n]}$ factors through $*$,

$$\begin{array}{ccc} \Delta[n] \times \Delta[1] & \xrightarrow{h} & X \\ {\scriptstyle i \times id}\uparrow & & \uparrow \\ \Delta^\circ[n] \times \Delta[1] & \longrightarrow & * \end{array}$$

Consider the map $h : \Delta[1] \times \Delta[n] \longrightarrow \Delta[n+1]$ defined by

$$\begin{aligned} \Delta[1]_n \times \Delta[n]_n &\longrightarrow \Delta[n+1]_n \\ (\alpha_n^j, id_{[n]}) &\longmapsto \delta_{n+1}^j = h_n^j(id_{[n]}). \end{aligned}$$

Recalling remark 1.1.19, it turns out that $\beta \circ h$ defines a homotopy map. In fact, the maps $\delta^i \circ \beta : \Delta[n] \longrightarrow X$ are maps of simplicial sets such that $\delta_{n+1}^{n+1}\beta = \alpha_x$ and $\delta_{n+1}^n \beta = \alpha_{x'}$.
Now fix $q \geq 0$, we have

$$\begin{aligned} (\alpha_q^j, f) &\overset{\partial_q^i}{\mapsto} (\alpha_q^j \circ \delta_q^i, f \circ \delta_q^i) \\ &= \begin{cases} (\alpha_{q-1}^j, f \circ \delta_q^i) & i \geq j, \\ (\alpha_{q-1}^{j-1}, f \circ \delta_q^i) & i < j \end{cases} \\ &\overset{h}{\mapsto} \begin{cases} \delta_{n+1}^j \circ f \circ \delta_q^i & i \geq j, \\ \delta_{n+1}^{j-1} \circ f \circ \delta_q^i & i < j \end{cases} \end{aligned}$$

$$\begin{aligned} (\alpha_q^j, f) &\overset{h}{\mapsto} (\delta_{n+1}^j \circ f) \\ &\overset{\partial_q^i}{\mapsto} \delta_{n+1}^j \circ f \circ \delta_q^i \end{aligned}$$

and the same holds for the degeneracy maps. □

Given a map on Kan complexes $f : X \longrightarrow Y$ such that $f(*_X) = *_Y$, there is an induced map $\pi_*(f) : \pi_*(X) \longrightarrow \pi_*(Y)$.

Proposition 1.1.22. *Simplicial homotopic maps $f, g : X \longrightarrow Y$ induces the same map on simplicial homology group $\pi_*(f) = \pi_*(g) : \pi_*(X) \longrightarrow \pi_*(Y)$.*

Proof. The results comes from the new definition of simplicial homotopy groups and the fact that homotopy relation is compatible with function composition in the following sense: if $f_1, g_1 : X \longrightarrow Y$ are homotopic, and $f_2, g_2 : Y \longrightarrow Z$ are homotopic, then their compositions $f_2 \circ f_1, g_2 \circ g_1 : X \longrightarrow Z$ are also homotopic. □

1.1.4 Simplices and Complexes

Last part of this section recollects all the previous results in order to define and structure a bridge between the world of simplices and the world of complexes. Such correspondence is fundamental from several points of view which will be emphasized in the following section.
Recall that given an abelian category $\mathcal{A}$ (e.g. abelian group or R-modules for a commutative ring R), a chain complex in $\mathcal{A}$ is a (possibly finite) sequence of objects and morphisms (called *differentials*) indexed by consecutive integers

$$\ldots \longrightarrow A^n \xrightarrow{d^n} A^{n-1} \longrightarrow \ldots$$

such that the composition of two consecutive maps is the zero map, i.e. $d^n \circ d^{n-1}$ or briefly $d \circ d = 0$.

There are in particular two ways to construct a complex from a simplicial object in an abelian category $\mathcal{A}$.

Definition 1.1.23 (Normalized complex)**.** Let $\mathcal{A}$ be an abelian category, $s\mathcal{A}$ the category of simplicial objects of $\mathcal{A}$ and $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ the category of non-negatively graded cochain complexes of $\mathcal{A}$. We may define the following functor

$$N : s\mathcal{A} \longrightarrow \mathrm{Ch}(\mathcal{A})$$

such that, given a simplicial object $A_\bullet$ in $\mathcal{A}$, the *associated normalized complex* is the complex

$$\dots \xrightarrow{d^{n+2}} NA_{n+1} \xrightarrow{d^{n+1}} NA_n \xrightarrow{d^n} NA_{n-1} \xrightarrow{d^{n-1}} \dots \xrightarrow{d^1} NA_0 \xrightarrow{d^0} 0$$

where $NX_n := \bigcap_{i=0}^{n-1} \ker(\partial_n^i : A_n \longrightarrow A_{n-1})$ and $d^n := (-1)^n \partial_n^n$.

Remark 1.1.24. *The complex is well defined as for $x \in A_n$ such that $\partial_n^i(x) = 0$ for $i = 0, ..., n-1$ we have that $d^n(x) = (-1)^n \partial_n^n \in A_{n-1}$. Suppose $j = 0, ..., n-2$, then by (1.1) we have $\partial_{n-1}^j \partial_n^n(x) = \partial_{n-1}^{n-1} \partial_n^j(x) = 0$, so that $d^n(x) \in \ker \partial_{n-1}^j$ for $j = 0, ..., n-2$, i.e. $d^n(x) \in NA_{n-1}$.*
Furthermore, again by (1.1), $\partial_n^n \partial_{n+1}^{n+1} = \partial_n^n \partial_{n+1}^n$; thus for $x \in NA_{n+1}$ the composition of two consecutive differentials gives $(-1)^n \partial_n^n \circ (-1)^{n+1} \partial_{n+1}^{n+1}(x) = (-1) \partial_n^n \partial_{n+1}^n(x) = 0$ as $x \in \ker \partial_{n+1}^i$ for $i = 0, ..., n$.

Definition 1.1.25 (Unnormalized complex)**.** Let $\mathcal{A}$ be an abelian category, we may define the following functor

$$\int : s\mathcal{A} \longrightarrow \mathrm{Ch}_{\geq 0}(\mathcal{A})$$

such that, given a simplicial object $A_\bullet$ in $\mathcal{A}$, the *associated (unnormalized) complex* is the complex

$$\dots \xrightarrow{d^{n+2}} A_{n+1} \xrightarrow{d^{n+1}} A_n \xrightarrow{d^n} A_{n-1} \xrightarrow{d^{n-1}} \dots \xrightarrow{d^1} A_0 \xrightarrow{d^0} 0$$

where $d^n := \sum_{i=0}^{n} (-1)^i \partial_n^i$. Sometimes is called *Moore complex.*

Remark 1.1.26. *The differential of the associated complex is well defined, in par-*

ticular

$$\begin{aligned}
d^n \circ d^{n+1} &= \left(\sum_{i=0}^{n}(-1)^i\partial_n^i\right) \circ \left(\sum_{j=0}^{n+1}(-1)^j\partial_{n+1}^j\right) \\
&= \left(\sum_{i=0}^{n}(-1)^i\partial_n^i\right) \circ \left(\sum_{j=i+1}^{n+1}(-1)^j\partial_{n+1}^j + \sum_{j=0}^{i}(-1)^j\partial_{n+1}^j\right) \\
&= \sum_{i=0}^{n}\sum_{j=i+1}^{n+1}(-1)^{i+j}\partial_n^i \circ \partial_{n+1}^j + \sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^{i+j}\partial_n^i \circ \partial_{n+1}^j \\
&= \sum_{i=0}^{n}\sum_{j=i+1}^{n+1}(-1)^{i+j}\partial_n^{j-1} \circ \partial_{n+1}^i + \sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^{i+j}\partial_n^i \circ \partial_{n+1}^j \\
&= \sum_{i=0}^{n}\sum_{j=i}^{n}(-1)^{i+j+1}\partial_n^j \circ \partial_{n+1}^i + \sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^{i+j}\partial_n^i \circ \partial_{n+1}^j \\
&= -\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^{i+j}\partial_n^j \circ \partial_{n+1}^i + \sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^{i+j}\partial_n^i \circ \partial_{n+1}^j = 0.
\end{aligned}$$

Remark 1.1.27. *Note that by definition there's a natural inclusion of complexes* $NA^* \hookrightarrow \int A^*$*. In particular it yields a map of functors* $N \longrightarrow \int$*.*

Remark 1.1.28. *Recall Definition 1.1.15 of simplicial homotopy groups and remark 1.1.16 for simplicial abelian groups. Let* A *be a simplicial abelian group and let* $0 \in A_0$ *be the base point; it's easy to see that* $Z_n = \ker(d^n : NA_n \longrightarrow NA_{n-1})$*. On the other hand*

$$\begin{aligned}
\ker(Z_n \longrightarrow \pi_n(A)) &= \{x \in Z_n : x \sim 0\} \\
&= \{x \in Z_n : \exists y \in A_{n+1},\ \partial_{n+1}^{n+1}(y) = x,\ \partial_{n+1}^{n}(y) = 0,\ \partial_{n+1}^{i}(y) = 0\ i < n\} \\
&= \{x \in Z_n : \exists y \in Z_{n+1}\ \partial_{n+1}^{n+1}(y) = x\}.
\end{aligned}$$

Last set corresponds to the image of $d^n : NA_{n+1} \longrightarrow NA_n$*. To sum up, we proved that simplicial homotopy of a simplicial abelian group corresponds to (co)homology of the associated normalized complex*

$$\pi_n(A) = H^n(NA^*).$$

These results can be generalized to the case of any group, by-passing the lost of abelian category structure (see [44] pages 264-265).

Last remark is the meeting point of non-abelian homological algebra, which can be performed in terms of simplicial homology groups (see REF) and homology for general abelian category, without underlying set structure.

Remark 1.1.29. *It is possible to prove that* π *is a (non abelian) homological* δ*-functor, see [44, exercise 8.3.3 and remark 8.3.5].*

Next lemma may look similar to Proposition 1.1.22, but here we are dealing within abelian categories, while previously the setting was more general. On the other hand, if the previous Proposition gives a quasi-isomorphism of Kan complex, the following Lemma gives chain homotopic maps.

Lemma 1.1.30. *Let $\mathcal{A}$ be an abelian category and $f, g : A \longrightarrow B$ two maps in $s\mathcal{A}$. Suppose f, g are simplicially homotopic, then $Nf, Ng : NA \longrightarrow NB$ are chain homotopic maps in* $\mathrm{Ch}(\mathcal{A})$.

Proof. Since being simplicially homotopic is an additive equivalence relation (remark [44, Exercise 8.3.6]), we may replace g by $g - f$ and assume $f = 0$ without loss of generality. Recall remark 1.1.19 about the simplicial homotopy map, then define the map $s_n : A_n \longrightarrow B_{n+1}$ as $s_n := \sum_{i=0}^{n}(-1)^{i+1}h_{n+1}^{i+1}\sigma_n^i$. We want to prove that the $\{s_n\}_{n\geq 0}$ defines a chain homotopy between Ng and the 0-map. Let $x \in NA_n$, we compute for $j = 0, ..., n$

$$\begin{aligned}\partial_{n+1}^j s_n(x) &= \sum_{i=0}^{n}(-1)^{i+1}\partial_{n+1}^j h_{n+1}^{i+1}\sigma_n^i(x)\\ &= \sum_{i<j}(-1)^{i+1}h_n^{i+1}\partial_{n+1}^j\sigma_n^i(x) + \sum_{i\geq j}(-1)^{i+1}h_n^i\partial_{n+1}^j\sigma_n^i(x)\\ &= \sum_{i+1<j}(-1)^i h_n^{i+1}\sigma_{n-1}^i\partial_n^{j-1}(x) + (-1)^j h_n^j(x) + (-1)^{j+1}h_n^j(x) + \sum_{i>j}(-1)^i h_n^i\sigma_{n-1}^i\partial_n^{j-1}(x).\end{aligned}$$

The two single addendums in the last list cancel out each other. Then, as $x \in NA_n$, $\partial_n^j(x) = 0$ for $j = 0, ..., n-1$, the two sums equal 0. Thus $s_n(x) \in \ker\delta_{n+1}^j$ for $j = 0, ..., n$, i.e. $s_n(x) \in NB_{n+1}$. So we have that the induced map

$$s_n : NA_n \longrightarrow NB_{n+1}$$

is well defined for all n. It remains to prove $d^{n+1}s_n - s_{n-1}d^n = (-1)^{n+1}g$, so that $\{(-1)^{n+1}s_n\}_n$ is a chain homotopy between 0-map and g.

$$\begin{aligned}\partial_{n+1}^{n+1}s_n &= \sum_{i=0}^{n}(-1)^{i+1}\partial_{n+1}^{n+1}h_{n+1}^{i+1}\sigma_n^i\\ &= \sum_{i=0}^{n}(-1)^{i+1}h_n^{i+1}\partial_{n+1}^{n+1}\sigma_n^i\\ &= \sum_{i=0}^{n-1}(-1)^{i+1}h_n^{i+1}\sigma_{n-1}^i\partial_n^n + (-1)^{n+1}h_n^{n+1}.\end{aligned}$$

Then

$$\partial_{n+1}^{n+1}s_n - s_{n-1}\partial_n^n = (-1)^{n+1}h_n^{n+1} = (-1)^{n+1}g.$$

So we get

$$(-1)^{n+1}d^{n+1}s_n + (-1)^n s_{n-1}d^n = (-1)^{n+1}(d^{n+1}s_n - s_{n-1}d^n) = (-1)^{2n+2}g = g$$

And we are done. □

All these results somehow let us foreshadow a strong connection between the simplices and the complexes world. The most explanatory result in such sense is the following theorem.

Theorem 1.1.31 (Dold-Kan Correpondence)**.** *Fon any abelian category $\mathcal{A}$, the normalized chain complex functor $N : s\mathcal{A} \longrightarrow \mathrm{Ch}_{\geq 0}(\mathcal{A})$ is an equivalence of categories. Under this correspondence, simplicial homotopy corresponds to (co)homology and simplicially homotopic morphisms correspond to chain homotopic maps.*

Proof. A reference may be [14, Theorem 3.6]. Note that the second part of the statement has been already proved in the previous Lemma. □

Corollary 1.1.32. *The functor N and its quasi-inverse K are exacts, i.e. a sequence $S = (0 \longrightarrow X \longrightarrow X' \longrightarrow X'' \longrightarrow 0)$ in $s\mathcal{A}$ (resp. $\mathrm{Ch}_{\geq 0}(\mathcal{A})$) is exact if and only if NS (resp. KS) is an exact sequence.*

Corollary 1.1.33. *Given an object A in an abelian category $\mathcal{A}$. The inclusion $NA^* \hookrightarrow \int A^*$ induces an isomorphism in cohomology $H^*(NA) = H^*(\int A)$.*

Remark 1.1.34. *Why, since functor N needs not the degeneracy maps, we need them to define a simplicial object? From the proof of the Dold-Kan correspondence turns out that simplicial objects with the same face maps belong to the same homotopy class.*

In (co)homology theory projective resolutions are a fundamental tool for computing (co)homology. They are complexes of projective modules with trivial (co)homology but in degree zero, where they corresponds to the object "resolved". Projectivity is particularly useful, since its lifting property guarantees the unicity of the resolution up to chain homotopy. We want to translate such setting in the simplicial world.

Definition 1.1.35 (Augmentation)**.** Given an object B and a simplicial object $X_\bullet$ in a category $\mathcal{C}$, we define an *augmentation* $\varepsilon : X_\bullet \longrightarrow B$ to be a morphism $X_\bullet \longrightarrow B_\bullet$ in $s\mathcal{C}$, where $B_\bullet$ is the constant simplex.

Lemma 1.1.36. *An augmentation $\epsilon_\bullet : X_\bullet \longrightarrow B$ is equivalent to the data of a morphism $\epsilon_0 : X_0 \longrightarrow B$ satisfying the identity $\epsilon_0 \partial_1^1 = \epsilon_0 \partial_1^0$.*

Proof. Given a map $\epsilon : X_\bullet \longrightarrow B_\bullet$ of simplicial objects, the degree zero component ϵ_0 satisfies this identity by definition. Conversely, given ϵ_0 as in the statement, we may choose an arbitrary morphism $\alpha : [0] \longrightarrow [n]$ and set $\epsilon_n := \epsilon_0 \circ X(\alpha)$. This does not depend on the choice of α, because for a different choice $\beta : [0] \longrightarrow [n]$ we may find a morphism $\gamma : [1] \longrightarrow [n]$ such that both α and β factor through γ, from which the identity $\epsilon_0 \partial_1^1 = \epsilon_0 \partial_1^0$ implies that the resulting maps ϵ_n are the same. The sequence ϵ_n indeed defines an augmentation. □

Definition 1.1.37 (Simplicial Resolution)**.** An augmented object $X_\bullet \longrightarrow B$ is a *simplicial resolution* if $\pi_n(X_\bullet) = 0$ for $n > 0$ and $\pi_0(X_\bullet) = B$.

Remark 1.1.38. *In an abelian category this is equivalent to the assertion that the associated complex NX and $\int X$ are resolution of B.*

Proposition 1.1.39. *If the underlying structure in s$\underline{Set}$ of an augmentation has simplicial homotopic inverse, then it is a resolution.*

Proof. Given a simplicial homotopic inverse as simplicial set, then by Proposition 1.1.22 the simplicial homology groups are isomorphic, which gives the definition of simplicial resolution above. □

1.2 Simplicial Rings, Differential graded Algebras and E_∞-algebras

Consider a (commutative unitary) ring A, some complexes of A-modules have an additional algebraic structure, which yields a richer category.

Definition 1.2.1. A *differential graded algebra* (dga) over some commutative unitary ring A is given by a complex of A-modules C^* and a map of cochain complexes

$$C^* \otimes_A C^* \longrightarrow C^*,$$

unital and associative in the obvious sense. Further a dga is called *(graded) commutative* if it is endowed with a map of cochain complexes

$$\begin{aligned} C^* \otimes_A C^* &\longrightarrow C^* \otimes_A C^* \\ x \otimes y &\longmapsto (-1)^{\deg x \deg y} y \otimes x, \end{aligned}$$

with $\deg x = n$ if and only if $x \in C^n$, such that if we compose it with the previous map it gives a commutative triangle. From now on we consider commutative differential graded algebras, so we may omit to specify it when it is clear by the context. We denote as *cdga* the corresponding category.

We can give a more "concrete" definition of differential graded algebra. As a complex of A-modules endowed with an operation such that, given $x \in C^n$ and $y \in C^m$, the product $x \cdot y \in C^{n+m}$. This corresponds to the above mentioned map of complexes and the relationship with the differentials gives us the Liebnitz rule

$$d(x \cdot y) = dx \cdot y + (-1)^n x \cdot dy.$$

Being unital and associative means that $\oplus_{i \geq 0} C^i$ is a graded A-algebra.

Proposition 1.2.2. *Given a differential graded algebra C^*, its operation induces a structure of graded algebra on cohomology groups $H^*(C^*) := \oplus_i H^i(C^*)$.*

Proof. (Sketch). Being a differential graded algebra implies:

1) If u, v are cocycles, then so is $u \cdot v$.

2) If u, v are cocycles that differ by a coboundary, and w is a cocycle, then uw and vw differ by a coboundary, and similarly for wu and wv.

It follows that multiplication is well-defined on $H^*(C)$ by the formula $[u][v] = [u \cdot v]$: uv is indeed a cocycle by 1), and by 2) changing the representatives for $[u], [v]$ doesn't change the class $[u \cdot v]$. Another proof may pass through the following remark: the chain map $C \otimes C$ induces a graded map $H^*(C \otimes C) \longrightarrow H^*(C)$, and then you have a canonical map $H^*(C) \otimes H^*(C) \longrightarrow H^*(C \otimes C)$: it sends $[u] \otimes [v]$ to $[u \otimes v]$: it can be checked that it is well-defined with a similar argument as above. □

Commutative differential graded algebras are a wonderful arithmetic-geometry object, but difficult to handle in homotopy theory. A first attempt to control its structure is to formalize it.

1.2.1 Tensor products

It will be useful the notion of monoidal structure on a category $\mathcal{C}$. It is a way to formalize the existence of a "tensor product" and a "unit object" satisfying some natural "algebraic operation" axioms.

Definition 1.2.3. A *(symmetric) monoidal category* is a category $\mathcal{C}$ endowed with an object $\mathbf{1}$, a functor $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, and the unital, associative, (commutative) constraints, which are natural equivalences satisfying certain coherence axioms.

Example 1.2.4. The category of sets has a monoidal structure induced by the Cartesian product and a fixed singleton as units.

Example 1.2.5. The ordinary tensor product makes vector spaces, abelian groups, R-modules, or R-algebras into monoidal categories (the unit object would be the initial object of each category).

Example 1.2.6. Given a monoidal category $(\mathcal{C}, - \otimes -, \mathbf{1})$, the associated category of simplicial objects $s\mathcal{C}$ inherits a monoidal structure given by $(X \otimes Y)_n = X_n \otimes Y_n$ and $\mathbf{1}_n = \mathbf{1}$.

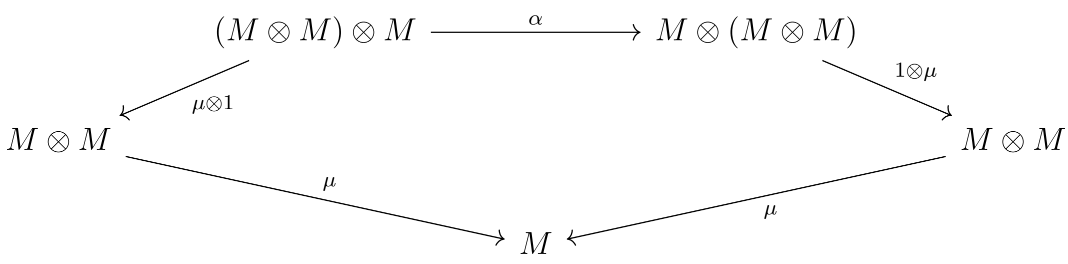
Example 1.2.7. Let R be a ring, the category of chain complexes of R-modules has a monoidal structure given by *Koszul product*, where given X^*, Y^*

$$(X \otimes Y)^n := \oplus_{i+j=n} X^i \otimes_R Y^j$$

and $\mathbf{1}^*$ is the trivial complex with the ring R concentrated in degree zero.

Associated to the notion of monoidal structure is the one of monoid object.

Definition 1.2.8. A *monoid object* (or *monoid*) (M, μ, η) in a monoidal category $(C, \otimes, \mathbf{1})$ is an object M together with a *multiplication* morphism $\mu : M \otimes M \longrightarrow M$ and a *unit* morphism $\eta : \mathbf{1} \longrightarrow M$, such that the following diagrams commutes



$$\begin{array}{ccccc} \mathbf{1} \otimes M & \xrightarrow{\eta \otimes 1} & M \otimes M & \xleftarrow[1 \otimes \eta]{} & M \otimes \mathbf{1} \\ & \searrow & \downarrow{\scriptstyle \mu} & \swarrow & \\ & & M & & \end{array}$$

M is a *symmetric monoid object* if multiplication map is compatible with some symmetry axioms induced by a *twist map*, which is a natural map such that for any couple of objects A, B defines an isomorphism $\tau_{A,B} : A \otimes B \longrightarrow B \otimes A$.

As you see, a monoid object is a generalization of the concept of (algebraic) monoid, i.e. an object where is defined an operation with unit.

Example 1.2.9. A monoid object in the monoidal category of sets is a set with an additional structure which makes it an algebraic monoid in the usual sense. As a matter of facts it is a set M with a special element $e \in M$ determined by a map of sets $\{*\} \longrightarrow M$ and a multiplication map $M \times M \longrightarrow M$ defined as $(x, y) \longmapsto y =: xy$ satisfying the identities $xe = x = xe$ and $(xy)z = x(yz)$ for all $x, y, z \in M$.

Example 1.2.10. A monoid in the category of abelian groups is a ring. For R-modules is an R-algebra.

Example 1.2.11. Given a monoidal category $(\mathcal{C}, - \otimes -, \mathbf{1})$, monoids in the associated category of simplicial objects $s\mathcal{C}$ are the simplicial version of the corresponding monoid.

Example 1.2.12. Monoid object in the category of chain complexes of R-modules are differential graded algebras. Symmetric monoid are commutative differential graded algebras.

Example 1.2.13 (Derived Tensor Product)**.** Let R be a commutative ring. Let $D(R)$ be the category of unbounded chain complex of R-modules (which is a symmetric monoidal category) where the quasi-isomorphisms are formally inverted. The category $D(R)$ is called the *derived category* of R-modules. We can define a tensor product in such category as follows. Two objects A, B of $D(R)$ can be seen as classes of complexes of R-modules up to quasi-isomorphism. Let A', B' be two representatives of such classes that are projective complexes (recall that the category of complexes of R-modules has enough projectives, that is for any complex A we can always find a projective resolution A', which is a projective complex quasi-isomorphic to A). We define the *derived tensor product* $A \otimes_R^L B := A' \otimes_R B'$. It can be proved that the definition does not depend on the choice of the projective representative. With such definition the category $D(R)$ is a symmetric monoidal category (unit object given by the class of the unit object of $C(R)$). The details can be found in Chapter 4 of [24]. From the construction it is clear that $H^i(A \otimes_R^L B) = \mathrm{Tor}_i(A, B)$, in particular $H^0(A \otimes_R^L B) = A \otimes_R B$.

1.2.2 Simplicial Rings to differential graded algebras.

Let $\mathcal{A}$ be an abelian category with a symmetric monoidal structure. We know that both $s\mathcal{A}$ and $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ have a symmetric monoidal structure induced by that of $\mathcal{A}$. Since they are equivalent as categories by the Dold-Kan correspondence, we may be interested in understanding how their monoidal structures and monoids relate under such correspondence.

The unit objects are preserved under the normalization functor and its inverse. However, the two tensor products for chain complexes and simplicial abelian groups are different in an essential way, i.e., the equivalence of categories given by normalization does not take one tensor product to the other. Another way of saying this is that if we use the normalization functor and its inverse to transport the tensor product of simplicial abelian groups to the category of connective chain complexes, we obtain a second monoidal product (sometimes called the shuffle product of complexes) which is non-isomorphic, and significantly bigger than, the tensor product.

All these considerations follow from the Eilenberg-Zilber theorem, which we are going to present briefly after some setting definition.

Definition 1.2.14. A *bisimplicial object* in a category $\mathcal{C}$ is a functor $\Delta^{op}\times\Delta^{op} \longrightarrow \mathcal{C}$, or equivalently a simplicial object in the category of simplicial objects of $\mathcal{C}$, i.e. an object of $ss\mathcal{C} = Fun(\Delta^{op}, Fun(\Delta^{op}, \mathcal{C}))$.

Alternatively a bisimplicial object A is a bigraded sequence of objects $A_{p,q}$ $(p, q \geq 0)$, together with horizontal and vertical face and degeneracy maps ${}^h\partial^i : A_{p,q} \longrightarrow A_{p-1,q}$, ${}^h\sigma^i : A_{p,q} \longrightarrow A_{p+1,q}$, ${}^v\partial^i : A_{p,q} \longrightarrow A_{p,q-1}$, ${}^v\sigma^i : A_{p,q} \longrightarrow A_{p,q+1}$. This maps must satisfy simplicial identities (1.1) horizontally and vertically. Finally every horizontal map must commute with every vertical map.
Given a bisimplicial object A in an abelian category $\mathcal{A}$, we can extend the functor $\int$ and get a first quadrant double complex $\int A = \{A_{p,q}\}_{(p,q)}$ with horizontal maps $d^h = \sum_i (-1)^i\, {}^h\partial^i$ and vertical maps $d^v = (-1)^p \sum_i (-1)^i\, {}^v\partial^i : A_{p,q} \longrightarrow A_{p,q-1}$.
Another interesting functor defined on $ss\mathcal{A}$ is the *diagonal functor* $\mathrm{diag}(-)$, which associates to a bisimplicial object $A : (\Delta\times\Delta)^{op} \longrightarrow \mathcal{A}$ the simplicial object obtained by precomposition with the diagonal functor $\Delta \longrightarrow \Delta\times\Delta$. Hence $\mathrm{diag}(A)_n := A_{n,n}$ and face maps are $\partial^i = {}^h\partial^i\, {}^v\partial^i$ and degeneracy maps $\sigma^i = {}^h\sigma^i\, {}^v\sigma^i$. With these functors we can state the Eilenberg-Zilber theorem.

Theorem 1.2.15 (Eilenberg-Zilber Theorem)**.** *Let A be a bisimplicial object in an abelian category $\mathcal{A}$. Then there is a natural isomorphism*[1]

$$\pi_* \,\mathrm{diag}(A) \cong H^*\, \mathrm{Tot}\Big(\int A\Big).$$

An enlightening application of such theorem is when we consider as bisimplicial object $(A_m \otimes_R B_n)_{(m,n)}$, where $A_\bullet, B_\bullet$ are simplcial R-modules. In this specific

[1] Recall that given a double complex $\{A_{p,q}\}$, the *total complex* is given by $\mathrm{Tot}(A)_n := \oplus_{p+q=n} A_{p,q}$.

case we get a classical result of cohomology theory. Note that the associated total complex $\mathrm{Tot}(\int(A_\bullet \otimes_R B_\bullet))$ is the tensor product of complexes $\int A_\bullet \otimes_R \int B_\bullet$. On the other hand, the diagonal of the bisimplicial object we defined is the tensor product of simplicial R-modules. In this situation the Eilenberg–Zilber theorem gives:

Corollary 1.2.16 (Kunneth Formula)**.**

$$\pi_* \operatorname{diag}(A_\bullet \otimes_R B_\bullet) \cong H^* \mathrm{Tot}(\int A_\bullet \otimes_R \int B_\bullet).$$

From the corollary we see that the Dold-Kan correspondence (remember that the functors N and $\int$ gives quasi-isomorphic complexes) does not take one tensor product to the other in general, but they share the same (co)homology.

Eilenberg-Zilber isomorphism is built on two specific maps: the *shuffle map* and the *Alexander-Whitney map*.

Definition 1.2.17 (Shuffle Map)**.** For simplicial object A, B we define the shuffle map

$$\nabla : \int A \otimes \int B \longrightarrow \int (A \otimes B)$$

as follows; for $a \in A_i$ and $b \in B_j$

$$\nabla(a \otimes b) = \sum_{(\mu,\nu)} \operatorname{sgn}(\mu,\nu)(\sigma_\nu(a) \otimes \sigma_\mu(b)) \in A_{i+j} \otimes B_{i+j}.$$

where the sum runs over all the (i,j)-shuffles[2]: given such a shuffle $(\mu,\nu) = (\mu_1 \ldots \mu_i\ \nu_1 \ldots \nu_j)$, we put $\sigma_\mu = \sigma_{\mu_i - 1}\sigma_{\mu_{i-1}-1} \ldots \sigma_{\mu_1 - 1}$ and $\sigma_\nu = \sigma_{\nu_j - 1}\sigma_{\nu_{j-1}-1} \ldots \sigma_{\nu_1 - 1}$ in order to shift a, b from A_i, B_j to A_{i+j}, B_{i+j} via a path of degeneracy maps σ_n.

Definition 1.2.18 (Alexander-Whitney map)**.** The Alexander-Whitney map

$$AW : \int (A \otimes B) \longrightarrow \int A \otimes \int B$$

goes in the direction opposite to the shuffle map; it is defined for $a \in A_n$ and $b \in B_n$ by

$$AW(a \otimes b) = \bigoplus_{i+j=n} d^i a \otimes d^j b$$

where the *front face* map $d^i : A_{i+j} \longrightarrow A_i$ and the *back face* map $d^j : B_{i+j} \longrightarrow B_j$ are induced by the injective monotone maps $\delta^i : [i] \longrightarrow [i+j]$ and $\delta^j : [j] \longrightarrow [i+j]$ defined by $\delta^i(k) = k$ and $\delta^j(k) = i + k$.

Remark 1.2.19. *Both maps factor over normalized chain complexes. Moreover on the level of normalized complexes, the composite $AW \circ \nabla$ is the identity transformation. The composite of shuffle and Alexander-Whitney maps in the other order are naturally chain homotopic to the identity transformation. In particular,*

[2]Recall that a (i,j)-shuffle consists of a permutation $(\mu,\nu) := (\mu_1 \ldots \mu_i \nu_1 \ldots \nu_j) \in S_{i+j}$, such that $\mu_1 < \ldots < \mu_i$ and $\nu_1 < \ldots < \nu_j$.

the shuffle map the Alexander-Whitney map and their normalized versions are all quasi-isomorphisms of chain complexes.
The shuffle map (normalized and not) respects also the symmetric monoidal structure of $s\mathcal{A}$ and $\mathrm{Ch}_{\geq 0}(\mathcal{A})$. That is not the case for the Alexander-Whitney map.

From the previous setting we can deduce the following result (for a proof see [25, §I.3.1.3])

Proposition 1.2.20. *The functor N and its quasi-inverse K induce the functors $N : sCRing \longrightarrow s(cdga)$ and $K : s(dga) \longrightarrow sRing$.*

More than simplicial rings and differential graded algebras, we are interested in the categories of commutative simplicial rings and commutative differential graded rings. The normalization functor is symmetric monoidal with respect to the shuffle map. Hence it takes commutative simplicial rings to commutative (in the graded sense) differential graded rings. But the Alexander-Whitney map is not symmetric, and so K does not induce a functor backwards. Without a characteristic zero assumption, not every commutative differential graded ring is quasi-isomorphic to the normalization of a commutative simplicial ring: if A is a commutative simplicial ring, then every element x of odd degree in the homology algebra $H^*(NA)$ satisfies $x^2 = 0$; but in a general commutative differential graded algebra we can only expect the relation $2x^2 = 0$. More generally, the homology algebra $H^*(NA)$, for A a commutative simplicial ring, has divided power ([13]) which need not be supported by a general commutative differential graded algebra. Moreover, in general the forgetful functor from differential graded algebras to chain complexes does not create a model structure and there is no homotopically meaningful way to go from differential graded to simplicial algebras in a way that preserves commutativity. In arbitrary characteristic, one should consider the categories of E_∞-algebras instead of the commutative algebras.

We can extend the previous Proposition 1.2.20 to simplicial differential graded algebras

Lemma 1.2.21. *There exists a functor*

$$\int : Fun(\Delta^{op}, cdga) \longrightarrow cdga,$$

where we note as dga the category of commutative differential graded A-algebras.

Proof. Let us consider $(B^*_\bullet, d_\bullet, \wedge_\bullet)$ a simplicial differential graded algebra, which can be seen as an object of $Fun(\Delta^{\mathrm{op}}, dga)$. We associate to $B^*_\bullet$, or better to its associated bicomplex (we are going to use the same notation whether it is clear what we mean by the context), the cochain complex defined as follows

$$\left(\int B^*_\bullet\right)_n := \bigoplus_{p-i=n} B^p_i, \tag{1.2}$$

with the associated differentials $D_n := \bigoplus_{p-i=n} (\partial_i \oplus d_i^p)$. Further we can again follow "Illusie's method" and define the following map of complexes

$$CB^*_\bullet \otimes_A CB^*_\bullet \xrightarrow{\text{shuffle}} C(B^* \otimes B^*)_\bullet \xrightarrow{\wedge_\bullet} CB^*_\bullet \quad \text{(composite: } \cdot \text{)}, \tag{1.3}$$

which corresponds on each degree to the map of A-modules

$$\bigoplus_{p+q-i-j=n} B_i^p \otimes_A B_j^q \longrightarrow \bigoplus_{r-k=n} B_k^r$$

obtained by the direct sums of the following morphisms of A-modules

$$\begin{aligned} \cdot : B_i^p \otimes_A B_j^q &\longrightarrow B_{i+j}^{p+q} \\ \omega \otimes \eta &\longmapsto \omega \cdot \eta = (-1)^{pj} \sum_{(\mu,\nu)} \operatorname{sgn}(\mu,\nu) \sigma_\nu \omega \wedge_{i+j} \sigma_\mu \eta, \end{aligned}$$

where $\wedge_{i+j}$ is the multiplication defined on the complex B_{i+j}^*.

Remark 1.2.22. *I would like to thank T. Szamuely and G. Zábrádi for the fundamental suggestion of changing the leading sign in the definition of the product, for which every computation works. However it is still not clear to me, which is the "naturality" of such choice.*

We want to prove that the following diagram commutes

$$\begin{array}{ccc} \bigoplus_{p-i=n} B_i^p \otimes_A \bigoplus_{q-j=m} B_j^q & \xrightarrow{\quad \cdot \quad} & \bigoplus_{r-k=n+m} B_k^r \\ \Big\downarrow {\scriptstyle D_n \otimes 1 + (-1)^n 1 \otimes D_m} & & \Big\downarrow {\scriptstyle D_{n+m}} \\ \bigoplus_{p-i=n+1} B_i^p \otimes_A \bigoplus_{q-j=m} B_j^q & & \\ \oplus \bigoplus_{p-i=n} B_i^p \otimes_A \bigoplus_{q-j=m+1} B_j^q & \xrightarrow{\quad \cdot \quad} & \bigoplus_{r-k=n+m+1} B_k^r \end{array} .$$

We may prove it by just proving the commutativity of the following square

$$\begin{array}{ccc} B_i^p \otimes_A B_j^q & \xrightarrow{\qquad \cdot \qquad} & B_{i+j}^{p+q} \\ \Big\downarrow {\scriptstyle \partial_i + (-1)^i d_i^p + (-1)^{p-i} \partial_j + (-1)^{p-i-j} d_j^q} & & \Big\downarrow {\scriptstyle \partial_{i+j} + (-1)^{i+j} d_{i+j}^{p+q}} \\ B_{i-1}^p \otimes B_j^q \oplus B_i^{p+1} \otimes B_j^q & & \\ \oplus B_i^p \otimes B_{j-1}^q \oplus B_i^p \otimes B_j^{p+1} & \xrightarrow{\quad \cdot \quad} & B_{i+j-1}^{p+q} \oplus B_{i+j}^{p+q+1} \end{array} ,$$

which corresponds to prove the following equalities, for $\omega \in B_i^p, \eta \in B_j^q$,

$$(-1)^{pj} \sum_{(\mu,\nu)} \operatorname{sgn}(\mu,\nu) \partial_{i+j} (\sigma_\nu \omega \wedge_{i+j} \sigma_\mu \eta) = (-1)^{pj} \sum_{(\mu',\nu')} \operatorname{sgn}(\mu',\nu') \sigma_{\nu'} \partial_i \omega \wedge_{i+j-1} \sigma_{\mu'} \eta + \tag{1.4}$$

$$+ (-1)^{p(j-1)} (-1)^{p-i} \sum_{(\mu'',\nu'')} \operatorname{sgn}(\mu'',\nu'') \sigma_{\nu''} \omega \wedge_{i+j-1} \sigma_{\mu''} \partial_j \eta \tag{1.5}$$

$$(-1)^{pj}(-1)^{i+j}d^{p+q}_{i+j}(\sigma_\nu\omega\wedge_{i+j}\sigma_\mu\eta) = (-1)^{(p+1)j}(-1)^{i}\sigma_\nu d^p_i\omega\wedge_{i+j}\sigma_\mu\eta + (-1)^{pj}(-1)^{p-i-j}\sigma_\nu\omega\wedge_{i+j}\sigma_\mu d^q_j\eta. \tag{1.6}$$

They result to be correct, since the following squares are all commutative

$$\begin{array}{ccccc}
B^p_{i-1}\otimes_A B^q_j \oplus B^p_i\otimes_A B^q_{j-1} & \xrightarrow{\text{shuffle}} & B^p_{i+j-1}\otimes_A B^q_{i+j-1} & \xrightarrow{\wedge_{i+j-1}} & B^{p+q}_{i+j-1} \\
\uparrow {\scriptstyle \partial_i\otimes 1+(-1)^i 1\otimes\partial_j} & & \uparrow {\scriptstyle \partial_{i+j}\otimes\partial_{i+j}} & & \uparrow {\scriptstyle \partial_{i+j}\otimes\partial_{i+j}} \\
B^p_i\otimes_A B^q_j & \xrightarrow{\text{shuffle}} & B^p_{i+j}\otimes_A B^q_{i+j} & \xrightarrow{\wedge_{i+j}} & B^{p+q}_{i+j} \\
\downarrow {\scriptstyle d^p_i\otimes 1+(-1)^p 1\otimes d^q_j} & & \downarrow {\scriptstyle d^p_{i+j}\otimes 1+(-1)^p 1\otimes d^q_{i+j}} & & \downarrow {\scriptstyle d^{p+q}_{i+j}} \\
B^{p+1}_i\otimes_A B^q_j \oplus B^p_i\otimes_A B^{p+1}_j & \xrightarrow{\text{shuffle}} & B^{p+1}_{i+j}\otimes_A B^q_{i+j} \oplus B^p_{i+j}\otimes_A B^{p+1}_{i+j} & \xrightarrow{\wedge_{i+j}} & B^{p+q+1}_{i+j}
\end{array},$$

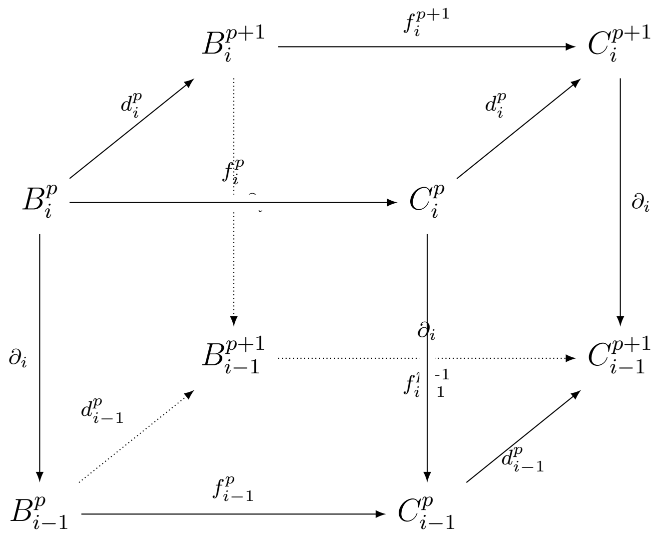
respectively by the fact that we have dga structure induced by the shuffle map, each ∂ is a graded morphism of algebras, shuffle map is a map of complexes (in this case from $B^*_i\otimes_A B^*_j$ to $B^*_{i+j}\otimes_A B^*_{i+j}$), we have a dga structure on B^*_{i+j}.

So we have that $\int B^*_\bullet$ is a commutative dg algebra. We now want to prove that such construction is functorial.

Let us consider a morphism in $Fun(\Delta^{\text{op}}, dga)$

$$f : B^*_\bullet \longrightarrow C^*_\bullet,$$

which gives a collection of commutative cubes of A-modules



,

so that the map induced by the direct sum of f^p_i

$$\int f := \bigoplus_{p-i=n} f^p_i : \bigoplus_{p-i=n} B^p_i \longrightarrow \bigoplus_{p-i=n} C^p_i$$

is a map of cochain complexes, which induces a graded morphism of algebra

$$\bigoplus_{n\geq 0}\bigoplus_{p-i=n} B_i^p \longrightarrow \bigoplus_{n\geq 0}\bigoplus_{p-i=n} C_i^p$$

indeed

$$\begin{aligned}
f_{i+j}^{p+q}(\omega\cdot\eta) &= (-1)^{pj}\sum_{(\mu,\nu)}\operatorname{sgn}(\mu,\nu) f_{i+j}^{p+q}\left(\sigma_\nu\omega\wedge_{i+j}\sigma_\mu\eta\right)\\
&= (-1)^{pj}\sum_{(\mu,\nu)}\operatorname{sgn}(\mu,\nu) f_{i+j}^{p}\left(\sigma_\nu\omega\right)\wedge_{i+j} f_{i+j}^{q}\left(\sigma_\mu\eta\right)\\
&= (-1)^{pj}\sum_{(\mu,\nu)}\operatorname{sgn}(\mu,\nu)\sigma_\nu(f_i^p\omega)\wedge_{i+j}\sigma_\mu(f_j^q\eta)\\
&= f_i^p(\omega)\cdot f_j^q(\eta),
\end{aligned}$$

where firstly we use the fact that f_{i+j}^* is a graded morphism of algebra (with respect to the product $\wedge_{i+j}$), then the fact that f is a functor from Δ^{op}, so it commutes with morphisms of such category. And we are done.

□

1.2.3 Truncations of differential graded algebras

Let C^* be a commutative differential graded algebra (*cdga*). Recall that for any complex C^* we can define the associated canonical truncation complexes

(a) $t_{[n}C^*$ defined as $(0\longrightarrow C^n/d^{n-1}(C^{n-1})\longrightarrow C^{n+1}\longrightarrow ...)$;

(b) $t_{n]}C^*$ defined as $(...\longrightarrow C^{n-1}\longrightarrow \ker d^n\longrightarrow 0)$.

Canonically we have the complex morphisms $C^*\longrightarrow t_{[n}C^*$ and $t_{n]}C^*\longrightarrow C^*$.

Remark 1.2.23. *The "naturality" of such definitions has to be found in the fact that the previous morphisms induces an isomorphism in cohomology for degree greater than n (case (a)) or lower than n (case (b)). In particular, suppose C^* has trivial cohomology group in degree lower (resp. greater) than n, then the morphism $C^*\longrightarrow t_{[n}C^*$ (resp. $t_{n]}C^*\longrightarrow C^*$) is a quasi-isomorphism.*

We wonder if $t_{[n}C^*$ and $t_{n]}C^*$ are still *cdga* and if the canonical maps preserve such structure.

Remark 1.2.24. *Recall that for a graded algebra $\oplus C^n$ the module of elements of degree zero C^0 is actually a ring. In particular $\mathbb{Z}\subseteq C^0$ in a differential graded algebra. The same must happen for the truncation, so that $n\leq 0$ for case (a), viceversa for case (b) $n\geq 0$.*

Remark 1.2.25. *In case (a), consider $a,b\in t_{[n}C^*$, $\deg(a)=n$, so that we may write $a+d(a')$ for some $a'\in t_{[n}C^{n-1}=C^{n-1}$. Then $(a+d(a'))\cdot b=a\cdot b+d(a')\cdot b$ is not well defined as an element of $C^{n+\deg b}$. This force us to consider only the case in which the complex is concentrated only in degree 0.*

Remark 1.2.26. *In case (b), if $n > 0$ it may happen that, given $a, b \in t_{n]}C^*$, $\deg(a) + \deg(b) = n$ and one of them is of degree different from n. In such case $d(a \cdot b) = d(a) \cdot b + (-1)^{\deg(a)} a \cdot d(b) \neq 0$, so that $a \cdot b \notin \ker d^n = t_{n]}C^n$. This force us to consider only the case $n = 0$.*

It turns out that the only cases which preserve the structure of *cdga* are actually the following

$$t_{0]}C^* \longrightarrow C^*, \tag{1.7}$$

$$t_{0]}C^* \longrightarrow t_{[0}t_{0]}C^*. \tag{1.8}$$

Proposition 1.2.27. *Given a commutative differential graded algebra C^*, the complexes $t_{0]}C^*$ and $t_{[0}t_{0]}C^*$ are differential graded algebras.*

Proof. Define on $t_{0]}C^*$ the multiplication as

$$a \cdot b = \begin{cases} ab, & \text{if } \deg(a), \deg(b) \leq 0, \\ 0 & \text{otherwise} \end{cases} ;$$

where ab is given by the multiplication on the *cgda* C^*. Since $\deg(a), \deg(b) \leq 0$ implies $\deg(a) + \deg(b) \leq 0$ and it equals 0 if and only if $\deg(a) = \deg(b) = 0$, the operation is trivially well defined for each degree but the latter case. Let $a, b \in \ker d^0$, then $d^0(ab) = d^0(a)b + ad^0(b) = 0$, so that $a \cdot b \in \ker d^0$. Such operation inherits the properties of associativity and (graded) commutativity from the multiplication defined on C^*. We now want to prove that it satisfy Liebnitz condition. The only non-trivial case are when at least one of the elements is of degree 0. Let us call δ the differential on $t_{0]}C^*$, then

- for $\deg a = 0$ and $\deg b < 0$, $\delta(a \cdot b) = d(ab) = d(a)b + ad(b) = ad(b) = a \cdot \delta(b)$, $\delta(a) \cdot b = d(a)b = 0$, $a \cdot \delta(b) = ad(b)$;
- for $\deg a < 0$ and $\deg b = 0$, $\delta(a \cdot b) = d(ab) = d(a)b + (-1)^{\deg a} ad(b) = d(a)b = \delta(a) \cdot b$, $\delta(a) \cdot b = d(a)b$, $a \cdot \delta(b) = ad(b) = 0$;
- if both has degree 0, everything equals 0.

And in each case Liebnitz condition

$$\delta(a \cdot b) = \delta(a) \cdot b + (-1)^{\deg a} a \cdot \delta(b)$$

is satisfied. In conclusion, $t_{0]}C^*$ is a commutative differential graded algebra.
Now consider the map of complexes (in fact it is a monomorphism)

$$t_{0]}C^* \longrightarrow C^*.$$

We want to show that it is a map of *cdga* actually. In particular we have to show that the following diagram commutes but this is trivial to verify.
On $t_{[0}t_{0]}C^*$ we define the following operation

$$a \star b = \begin{cases} \overline{a \cdot b}, & \text{if } \deg(a) = \deg(b) = 0, \\ 0 & \text{otherwise} \end{cases} ;$$

where $\overline{a \cdot b}$ is the multiplication in $t_{0]}C^*$ modulo $d^{-1}C^{-1}$. To show that the multiplication in well defined we write two elements on $\ker d^0/d^{-1}C^{-1}$ as $a+d(a')$, $b+d(b')$, then $(a+d(a')) \star (b+d(b')) = ab + d(a')b + ad(b') + d(a')d(b')$. In particular

- $d(a')b = d(a'b) + a'd(b) = d(a'b)$ since $b \in \ker d^0$,
- $ad(b') = d(ab') - d(a)b' = d(ab')$ since $a \in \ker d^0$,
- $d(a')d(b') = d(a'd(b'))$.

So that the product is well defined. In this case Liebnitz condition is trivial, moreover associativity and commutativity work trivially.
Consider the map (epimorphism) of complexes

$$t_{0]}C^* \longrightarrow t_{[0}t_{0]}C^*.$$

It induces the diagram for which, supposing $\deg(a) \neq 0$, $\overline{a \cdot b} = 0 = \overline{a} \star \overline{b}$, and, for $\deg(a) = \deg(b) = 0$, $\overline{a \cdot b} = \overline{a} \star \overline{b}$. Hence the diagram commutes, so we have a map of commutative differential algebras. □

Corollary 1.2.28. *Suppose that C^* is a commutative differential graded algebra acyclic but in degree zero. Then the maps (1.7) and (1.8) are quasi-isomorphisms.*

Remark 1.2.29. *In the setting of last Corollary, we see that C^* is quasi-isomorphic to a ring $H^0(C^*)$ seen as a commutative differential graded algebra concentrated in degree zero. This is the key point of most of next chapter computations. The idea is that in this case we can manage a commutative ring as a cdga and viceversa, whether the cohomology is concentrated in degree zero. Actually several problems arise in this case, which get the formalization of this phenomena very complicated. For example the maps (1.7) and (1.8) don't give a quasi-isomorphism $C^* \longrightarrow H^0(C^*)$ as they are oriented in different directions. It is possible to define a category where all quasi-isomorphism of cdga are invertible, but such category will somehow lose the relationship with the corresponding category for complexes (the forgetful functor is no longer well defined). This problems are related to those we pointed out when talking about the Dold-Kan correspondence for the monoidal structure of the category $s\mathcal{A}$ and $\mathrm{Ch}(\mathcal{A})$ and they are solved both in the context of E_∞-algebras. We are going to give just a hint in the next paragraph for completeness, however what is important is that there is a "black box" through which we are allowed to consider rings and differential graded algebras as quite the same thing.*

1.2.4 E_∞-algebras

An E_∞-algebra over a ring R is an analogue of commutative differential graded algebras with strict commutativity of the diagrams replaced by homotopies, themselves subject to higher homotopies, and so on. There are several ways to define E_∞-algebras and all definitions are quite technical and, since for these work we just need to keep this idea in mind, they do not add anything that would help the comprehension of the topic.

We may give some hints about the "non commutative" version of E_∞-algebras: the A_∞-algebras. They may give a further insight of this tools, even if we are not going to use it.

An A_∞-algebra presents an explicit formulation thanks to Steenrod operations. Recall that an associative differential graded algebra over A is a $\mathbb{Z}$-graded R-module A endowed with graded R-linear maps $d = m_1 : A \longrightarrow A$ of degree 1 and $m = m_2 : A \otimes_R A \longrightarrow A$ of degree 0 satisfying the following conditions

- $m_1 m_1 = 0$ (i.e. m_1 is a differential),
- $m_1 m_2 = m_2(m_1 \otimes id_A + id_A \otimes m_1)$ (i.e. m_1 is a derivation with respect to the multiplication m_2),
- $m_1(id_A \otimes m_2 - m_2 \otimes id_A) = 0$ (associativity of m_2).

In the definition of A_∞-algebras, we replace the zero in the associativity of m_2 by the boundary of a homotopy m_3. More precisely, we include a graded map $m_3 : A \otimes A \otimes A \longrightarrow A$ of degree -1 as part of the data and we impose the condition

$$m_1(id_A \otimes m_2 - m_2 \otimes id_A) = m_1 m_3 + m_3(m_1 \otimes id_A \otimes id_A + id_A \otimes m_1 \otimes id_A + id_A \otimes id_A \otimes m_1).$$

The coherence of m_3 is further governed by a map $m_4 : A \otimes A \otimes A \otimes A \longrightarrow A$ of degree -2. It is not difficult to write down all the homotopies: $m_n : A^{\otimes n} \longrightarrow A$ R-linear map of degree $2 - n$ such that

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(id_A^{\otimes r} \otimes m_s \otimes id_A^{\otimes t}) = 0.$$

Now that we have some more insights about E_∞-algebras we can consider the following list of facts (see [40]):

Facts 1.2.30.

1. Any commutative differential graded algebra defines an E_∞-algebra.
2. A morphism of commutative differential graded algebras defines a "morphism" of E_∞-algebras.
3. A quasi-isomorphism of commutative differential graded algebras corresponds to an equivalence of E_∞-algebras. In particular the corresponding "morphism" from one E_∞-algebra to the other has a "morphism" in the opposite direction.

1.3 Pro-categories

We recall some results on pro-categories.

Definition 1.3.1. We say that a category I is *cofiltered* if the following conditions are satisfied:

(1) I is non-empty.

(2) For every pair of objects $i, j \in I$, there exists an object $k \in I$, together with morphisms $k \longrightarrow i$ and $k \longrightarrow j$.

(3) For every pair of morphisms $f, g : i \longrightarrow j$ in I, there exists a morphism $h : k \longrightarrow i$ in I, such that $f \circ h = g \circ h$.

Example 1.3.2 (Directed Sets)**.** A *directed set* is a nonempty set A together with a reflexive and transitive binary relation $\leq$ (that is, a preorder), with the additional property that every pair of elements has an upper bound. In other words, for any a and b in A there exists c in A with $a \leq c$ and $b \leq c$. Therefore a directed set defines a cofiltered category, where objects are elements of A and $a \to b$ if and only if $b \leq a$. We may use both set theoretic and categorical language to discuss cofiltered categories; hence "$a \geq b$" and "$a \to b$" mean the same thing when the indexing category is actually a directed set.

Definition 1.3.3. A functor $J \longrightarrow I$ is *cofinal* if J is a cofiltering full subcategory I and for every i in I, there exists some j in J and an arrow $j \to i$ in I.

Example 1.3.4. Consider the ordered set of natural numbers $\mathbb{N}$ (which is a directed set). A subset $A \subseteq \mathbb{N}$ is cofinal if for every $n \in \mathbb{N}$ there exist $m \in A$ such that $n \leq m$.

Recall that a category is said to be *small* if its objects and its arrows are sets.

Definition 1.3.5. Given a category $\mathcal{C}$, the corresponding *pro-category* (usually noted as $pro(\mathcal{C})$) is the category which has small cofiltered[3] systems $X : I \longrightarrow \mathcal{C}$ as objects (called *pro-objects*). Given two pro-objects X, Y for a category $\mathcal{C}$, we define

$$\operatorname{Hom}(X,Y) = \varprojlim_{j \in J} \varinjlim_{i \in I} \operatorname{Hom}_{\mathcal{C}}(X_i, Y_j).$$

Equivalently X is a pro-system on a small filtered category.

Note that the index categories are not assumed equal. We may use other less compact notations. In particular $X = (X_i)_{i \in I}$ if we want to specify the objects of the projective system and their relationship with the index category. On the other hand $X = \text{"}\varprojlim\text{"}_{i \in I} X_i$ if we want to underline the formal cofiltered limit nature of the pro-object X(see remark 1.3.11). Each element $X(i) = X_i$ for $i \in I$ is called *component element*, while images of the maps $i \longrightarrow j$ in I are called *transition morphisms.*

Remark 1.3.6 (On the morphism of pro-objects)**.** *In order to easily "see" a morphism $f : X \longrightarrow Y$ in $pro(\mathcal{C})$ we first consider Y as a constant projective system, so that we have just to consider $f \in \varinjlim_{i \in I} \operatorname{Hom}_{\mathcal{C}}(X_i, Y)$. Recall that $\operatorname{Hom}_{\mathcal{C}}(-, Y)$ is contravariant, so we have a direct system*

[3] A functor $I \longrightarrow \mathcal{C}$ is a small cofiltered system if the category I, usually called *index category*, is small cofiltered.

$$
\begin{array}{ccc}
i & X_i & \mathrm{Hom}_{\mathcal{C}}(X_i, Y) \\
\downarrow \leq & \uparrow p^i_{i'} & \downarrow - \circ p^i_{i'} \\
i' & X_{i'} & \mathrm{Hom}_{\mathcal{C}}(X_{i'}, Y)
\end{array}
$$

and the direct limit is given by all families $(f_i : X_i \longrightarrow Y)_{i \in I}$ modulo the equivalence relation

$$f_i \sim f_{i'} \iff \exists i'' \geq i, i' \text{ such that } f_i \circ p_i^{i''} = f_{i'} \circ p_{i'}^{i''}.$$

So f is given by one of such equivalence classes.

Now consider $f : X \longrightarrow Y$ in the general case. We have an inverse system

$$
\begin{array}{ccc}
j & Y_j & \varinjlim_I \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j) \\
\downarrow \leq & \uparrow q^j_{j'} & \uparrow q^j_{j'} \circ - \\
j' & Y_{j'} & \varinjlim_I \mathrm{Hom}_{\mathcal{C}}(X_i, Y_{j'})
\end{array}
$$

and the limit on all over $j \in J$ gives us a set whose elements are given by sequences of $f_j : X \longrightarrow Y_j$ such that $f_j = q^i_{j'} \circ f_{j'}$ for $j \leq j'$. In particular we can always define a map of indices $i : J \longrightarrow I$, such that for every $j \in J$, we can find a corresponding representing map for the morphism f_j, that is $f_{i(j)} : X_i \longrightarrow Y_j$ such that $[f_{i(j)}]_\sim = f_j$.

We now presents in such form two fundamental aspects of the morphisms in a category: the identity morphism and the composition.
The identity morphism $1_X : X \longrightarrow X$ is given by the family $\{p_i : X \longrightarrow X_i\}_{i \in I}$, where each class p_i is represented by the identity morphism 1_{X_i}. In particular each p_i corresponds to the canonical projections of the direct limit.
Let us consider two morphisms $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$, where $f = \{f_j\}_{j \in J}$ and $g = \{g_k\}_{k \in K}$. Then $g \circ f$ is given by the family $\{g_{j(k)} \circ f_{j(k)} : X \longrightarrow Z_k\}_{k \in K}$. More precisely $g_{j(k)} \circ f_{j(k)} = [g_{j(k)} \circ f_{i(j(k))}]_\sim$.

Example 1.3.7. For any object $A \in \mathcal{C}$, the constant projective system, indexed by the category $\underline{\{*\}}$ with one object and one map (the identity), defines a *constant pro-object.*

Remark 1.3.8. *The previous example induces a natural inclusion $\iota : \mathcal{C} \hookrightarrow pro(\mathcal{C})$, which makes $\mathcal{C}$ a full subcategory of $pro(\mathcal{C})$. If $\mathcal{C}$ has small filtered projective limits, the functor* lim $: pro(\mathcal{C}) \longrightarrow \mathcal{C}$ *which sends $(X_i)_{i \in I}$ to $\varprojlim_{i \in I} X_i$ is the right adjoint of ι, see [32, Proposition 6.3.1].*

Lemma 1.3.9. *Let $X = (X_j)_{j \in J}$ be a pro-object of a category $\mathcal{C}$ and let $\phi : I \longrightarrow J$ be a cofinal functor. Then the pro-object $X_\phi = (X_{\phi(i)})_{i \in I}$ is isomorphic to X.*

Proof. See [5, Appendix Corollary 2.5] or [43, Proposition 1]. □

Remark 1.3.10. *Last results reflects in the pro-category world the well known fact that, in the same setting, supposing $\mathcal{C}$ provided with cofiltered limits, the canonical map*

$$\varprojlim_{i\in I} X_{\phi(i)} \longrightarrow \varprojlim_{j\in J} X_j$$

is an isomorphism [5, Appendix Proposition 1.8].

It is easy to see that a morphism of inverse systems (a morphism of functor from an index category I to the category $\mathcal{C}$) satisfies the whole previous setting of morphisms of pro-objects, provided they are defined over the same index category I. As a matter of fact, we frequently consider maps between two pro-objects with the same index categories. In this setting, a *level map* $X \longrightarrow Y$ between pro-objects indexed by I is given by maps $X_s \longrightarrow Y_s$ for all s in I. Up to isomorphism, every map is a level map [5, Appendix 3.2].

Remark 1.3.11. *The category $pro(\mathcal{C})$ is the universal category with cofiltered limits receiving the functor $\iota : \mathcal{C} \hookrightarrow pro(\mathcal{C})$: if $\mathcal{D}$ is any other category with cofiltered limits, let $Fun'(pro(\mathcal{C}), \mathcal{D})$ be the collection of functors $pro(\mathcal{C}), \mathcal{D}$ which preserve cofiltered limit, then there is a 1:1 correspondence*

$$Fun'(pro(\mathcal{C}), \mathcal{D}) \xrightarrow{-\circ\iota} Fun(\mathcal{C}, \mathcal{D}).$$

In this sense one may consider $pro(\mathcal{C})$ as the category obtained by freely adding cofiltered limits to $\mathcal{C}$ (see Proposition 1.3.12).

Proposition 1.3.12. *For any category $\mathcal{C}$, the category $pro(\mathcal{C})$ is complete.*

Proof. See [37, §1 pag. 12], or [?]. □

Another, equivalent, definition is to consider $pro(\mathcal{C})$ to be the full subcategory of the opposite category of presheaves[4], i.e. $\mathrm{PSh}(\mathcal{C})^{op}$, determined by those functors which are cofiltered limits of representables (see [37], [22], [32, Definition 6.1.1]). This is reasonable since $\mathrm{PSh}(\mathcal{C})$ is the free completion of $\mathcal{C}$, so $pro(\mathcal{C})$ is the "free completion of $\mathcal{C}$ under cofiltered limits" (see [32]).

Given a category $\mathcal{C}$, the category of ind-objects $Ind(\mathcal{C})$ can be identified with a subcategory $\mathrm{PSh}(\mathcal{C})$ of presheaves over $\mathcal{C}$ preserving small filtered colimits ([32, Theorem 6.1.8]). On the other hand $\mathrm{PSh}(\mathcal{C})$ is the completion of $\mathcal{C}$ under colimits (by the Yoneda embedding and the fact that presheaves of sets are a cocomplete category, see remark 2.2.2 in Daniel Dugger, Sheaves and Homotopy Theory, https://ncatlab.org/nlab/files/cech.pdf for example). This means that for any presheaf F, we have an isomorphism

$$F \cong \varinjlim_{X\in\mathcal{C}/F} yX,$$

where $y : \mathcal{C} \hookrightarrow \mathrm{PSh}(\mathcal{C})$ is the Yoneda embedding, and $\mathcal{C}/F$ denotes the category of couples (X, s), where X is an object of $\mathcal{C}$ and $s : yX \longrightarrow F$ in $\mathrm{PSh}(\mathcal{C})$ (N.B. by

[4]Recall that a presheaf on a category $\mathcal{C}$ is simply the category of functors $\mathcal{C}^{op} \longrightarrow \underline{\mathrm{Set}}$

Yoneda embedding this is equivalent to have an elements $s \in F(X)$).

As a consequence we have that $pro(\mathcal{C})$ can be seen as the opposite category of ind-objects over the opposite category of $\mathcal{C}$, i.e. $pro(\mathcal{C}) = (Ind(\mathcal{C}^{op}))^{op}$ (see [32, Chapter 6]). Last remark may be used to prove the following result (see also [15, §11])

Proposition 1.3.13. *Let $\mathcal{C}$ be a symmetric monoidal category, then we get a symmetric monoidal structure on $pro(\mathcal{C})$, where tensor products are defined as*

$$(X_i)_{i\in I} \otimes (Y_j)_{j\in J} = (X_i \otimes Y_j)_{(i,j)\in I\otimes J}$$

and the unit element is the constant pro-object of the unit element of $\mathcal{C}$, we get a symmetric monoidal structure on $pro(\mathcal{C})$.

Sketch. We recall the characterization of $pro(\mathcal{C})$ as the opposite category of ind-objects over the opposite category of $\mathcal{C}$, i.e. $pro(\mathcal{C}) = (Ind(\mathcal{C}^{op}))^{op}$. The category of ind-objects has a symmetric monoidal structure if taken over a symmetric monoidal category. Given two presheaves F, G, we may define in a unique way a tensor product in $\mathrm{PSh}(\mathcal{C})$, by posing $yX \otimes yY := y(X \otimes Y)$ for two objects X, Y in $\mathcal{C}$. Recall that in this setting colimits and tensor products commutes, so that

$$\begin{aligned} F \otimes G &\cong \varinjlim_{X\in\mathcal{C}/F} yX \otimes \varinjlim_{Y\in\mathcal{C}/G} yY \\ &\cong \varinjlim_{X\in\mathcal{C}/F} \left(yX \otimes \varinjlim_{Y\in\mathcal{C}/G} yY \right) \\ &\cong \varinjlim_{X\in\mathcal{C}/F} \varinjlim_{Y\in\mathcal{C}/G} yX \otimes yY \end{aligned}$$

is uniquely defined. Such definition (together with the unity object yI, with I unity object of $\mathcal{C}$) gives to $Ind(\mathcal{C})$ the structure of symmetric monoidal category.

Passing to $pro(\mathcal{C})$, we have that "$\varprojlim$"$_i X_i \otimes$ "$\varprojlim$"$_j Y_j$ in $pro(\mathcal{C}) = (Ind(\mathcal{C}^{op}))^{op}$ corresponds to "$\varinjlim_i$"$X_i \otimes$ "$\varinjlim_j$"Y_j in $Ind(\mathcal{C}^{op})$. As we saw before "$\varinjlim_i$"$X_i \otimes$ "$\varinjlim_j$"$Y_j \cong$ "$\varinjlim_{i,j}$"$X_i \otimes Y_j$. But the latter corresponds to "$\varprojlim$"$_{i,j} X_i \otimes Y_j$ in $(Ind(\mathcal{C}^{op}))^{op} = pro(\mathcal{C})$. ☐

Chapter 2

Derived de Rham complex

2.1 The Cotangent Complex

The cotangent complex is the result of combined works of several authors, around questions related to deformation theory of rings and schemes. The key problem is the following: let $A \longrightarrow B$ be a map of commutative ring or, in a more geometric form, a map of ring sheaves over a topological space X (or a topos). The aim is to classify the extension of $A-$algebras, i.e. the exact sequences of the form

$$0 \longrightarrow I \longrightarrow B' \longrightarrow B \longrightarrow 0,$$

where $I \subseteq B'$ is an ideal such that $I^2 = 0$. The geometric side of the problem is, given a map of scheme $X \longrightarrow Y$, the classification of $Y-$schemes X' with an immersion of order 1 $i : X \hookrightarrow X'$. On the algebraic side, André and Quillen introduced a homology theory for commutative rings, now called André-Quillen homology ([1], [39]). On the algebraic geometry side, Grothendieck ([19]) and later Illusie ([25],[26]) globalized the definition of André and Quillen and introduced the cotangent complex of a morphism between schemes.

The leading principle is that affine smooth schemes have a very simple deformation theory. The deformation theory of a general scheme should then be understood by performing an approximation by smooth affine schemes. Algebraically, this approximation can be realized by simplicial resolving of commutative algebras by smooth algebras. As we saw in Chapter 1 this is in some sense a multiplicative analogue of resolving a module by projective modules. Fix a field k (we are going actually to work in a more general setting), for a commutative $k-$algebra A, we can choose a smooth algebra A_0 and a surjective morphism $A_0 \to A$, for instance by choosing A_0 to be a polynomial algebra. We can furthermore find another smooth algebra A_1 and two algebra maps $A_1 \rightrightarrows A_0$ in a way that A becomes the coequalizer of the above diagram of commutative $k-$algebras. This process can be continued further and provides a simplicial object $A_\bullet$, made out of smooth and commutative $k-$algebras A_n, together with an augmentation $A_\bullet \longrightarrow A$. This augmentation map is a resolution as we saw previously in Chapter 1. The deformation theory of A is then understood by considering the deformation theory of the simplicial diagram of smooth algebras $A_\bullet$, for which we know that each individual algebra A_n possesses a very simple deformation theory. For this, the key construction is the complex associated with the simplicial modules of Kähler differentials

$$L_A := \int (n \longmapsto \Omega^1_{A_n}).$$

Up to a quasi-isomorphism this complex can be realized as a complex of $A-$modules and is shown to be independent of the choice of the simplicial resolution $A_\bullet$ of A. The object L_A is the cotangent complex of A, and is shown to control the deformation theory of A: there is a bijective correspondence between infinitesimal deformations of A as a commutative algebra and $\mathrm{Ext}^1_A(L_A, A)$. Moreover, the obstruction to extend an infinitesimal deformation of A to an order three deformation (i.e. to pass from a family over $k[x]/x^2$ to a family over $k[x]/x^3$) lies in $\mathrm{Ext}^2(L_A, A)$. The algebraic construction of the cotangent complex has been globalised for general schemes by Grothendieck ([19]) and Illusie ([25]).

The above construction involving simplicial resolutions can be applied to the structure sheaf $\mathcal{O}_X$ of a scheme X. To put things differently: a general scheme is approximated in two steps, first by covering it by affine schemes and then by resolving the commutative algebras corresponding to these affine schemes. The important issue of how these local constructions are glued together is dealt with by the use of standard simplicial resolutions involving infinite dimensional polynomial algebras. For a scheme X (say over the base field k), the result of the standard resolution is a sheaf of simplicial commutative $k-$algebras $\mathcal{A}_\bullet$, together with an augmentation $\mathcal{A}_\bullet \longrightarrow \mathcal{O}_X$ having the property that over any open affine $U = \mathrm{Spec}\, A \subset X$, the corresponding simplicial algebra $\mathcal{A}_\bullet(U)$ is a resolution of A by polynomial $k-$algebras (possibly with an infinite number of generators). Taking the total complex of Kähler differentials yields a complex of $\mathcal{O}_X-$modules L_X, called the cotangent complex of the scheme X. As in the case of commutative algebras, it is shown that L_X controls deformations of the scheme X. For instance, first order deformations of X are in bijective correspondence with $\mathrm{Ext}^1(L_X, \mathcal{O}_X)$, which is a far reaching generalization of the Kodaira-Spencer identification of the first order deformations of a smooth projective complex manifolds with $H^1(X, T_X)$ (see [33]). In a similar fashion the second extension group $\mathrm{Ext}^2(L_X, \mathcal{O}_X)$ receives obstructions to extend first order deformations of X to higher order formal deformations.
In this context we deal with the affine definition.

2.1.1 Definitions

We recall definition and some basic fact about Kähler differentials. Let $A \longrightarrow B$ be a homomorphism of rings. Define the of *relative differentials* (or *Kälher differentials*) $\Omega^1_{B/A}$ as the $B-$module generated by elements of the form db for each $b \in B$, subject to the relations $d(a_1b_1 + a_2b_2) - a_1db_1 - a_2db_2$ and $d(b_1b_2) - b_1db_2 - b_2db_1$ for $a_i \in A$ and $b_i \in B$.

Facts 2.1.1. The module of Kähler differentials satisfies the following basic properties:

(1) (Base change) For an $A-$algebra A' one has $\Omega^1_{B\otimes_A A'/A'} \cong \Omega^1_{B/A} \otimes_A A'$.

(2) (Localization) Given a multiplicative subset S of B, one has $\Omega^1_{BS^{-1}/A} \cong \Omega^1_{B/A} \otimes_B BS^{-1}$.

(3) (First exact sequence) A sequence of ring homomorphisms $A \longrightarrow B \longrightarrow C$ gives rise to an exact sequence of $C-$modules

$$C \otimes_B \Omega^1_{B/A} \longrightarrow \Omega^1_{C/A} \longrightarrow \Omega^1_{C/B} \longrightarrow 0.$$

(4) (Second exact sequence) A surjective morphism $B \longrightarrow C$ of $A-$algebras with kernel I gives rise to an exact sequence of $C-$modules

$$I/I^2 \longrightarrow C \otimes_B \Omega^1_{B/A} \longrightarrow \Omega^1_{C/A} \longrightarrow 0,$$

where the map on the right sends a class x mod I^2 to $1 \otimes dx$. (Note that the $B-$module structure on I/I^2 induces a $C-$module structure).

(5) (Künneth) For to ring maps $A \longrightarrow B$ and $A \longrightarrow C$ there is an isomorphism of $B \otimes_A C-$modules $\Omega^1_{B\otimes_A C} \cong \Omega^1_{B/A} \otimes_A C \oplus B \otimes_A \Omega^1_{C/A}$.

(6) (Inductive limits) Given a direct system of ring maps $\{A_n \longrightarrow B_n\}_{n\in N}$ there is a canonical isomorphism

$$\varinjlim_n \Omega^1_{B_n/A_n} \cong \Omega^1_{\varinjlim_n B_n/\varinjlim_n A_n}.$$

(7) (Functoriality) Suppose that

$$\begin{array}{ccc} B & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \longrightarrow & A'. \end{array}$$

is a commutative diagram of rings. In this case there is a natural map of modules of differentials fitting into the commutative diagram

$$\begin{array}{ccc} \Omega^1_{B/A} & \longrightarrow & \Omega^1_{B'/A'} \\ \downarrow & & \downarrow \\ B & \longrightarrow & B'. \end{array}$$

Equivalently there is a map of $B'-$modules $\Omega^1_{B/A} \otimes_B B' \longrightarrow \Omega^1_{B'/A'}$.

(8) (Extension to scheme) Given a map of schemes $X \longrightarrow Y$, the sheaf of $\mathcal{O}_X-$modules $\Omega^1_{X/Y} = \Omega^1_{\mathcal{O}_X/f^{-1}(\mathcal{O}_Y)}$ is called *sheaf of modules of Kähler differentials.*

For all these facts, and more about $\Omega^1_{B/A}$ a standard reference nowadays is [34, §6.1], but also Illusie provides a large recap in [25, §II.1.1]. Exact sequence (3) above can be extended by 0 on the left under a smoothness assumption on the map $B \longrightarrow C$. However, in general exactness on the left fails. One motivation for introducing the cotangent complex is to remedy this defect.

Construction 2.1.2. We start constructing the so called *standard simplicial resolution* $P_\bullet = P(B)_\bullet$ for an A-algebra B as follow. Let $P_0 := A[B]$ be the free A-algebra whose generators x_b are indexed by the elemets of B. Then we define recursively the free A-algebras $P_{i+1} := A[P_i]$ for $i \geq 0$. Face maps and degeneracy maps of $A-$algebras are defined starting from

$$\begin{array}{ccc} A[B] & \underset{\kappa_B}{\overset{\tau_B}{\rightleftarrows}} & B \\ [b] & \longleftarrow & b \\ \sum a_{b,I}[b_1]^{i_1}...[b_k]^{i_k} & \longmapsto & \sum a_{b,I} b_1^{i_1}...b_k^{i_k} \end{array}.$$

In particular for $0 \leq j \leq i$, maps $\partial_i^j : P_i \longrightarrow P_{i-1}$ are induced by the $A-$algebra homomorphisms κ_{P_j}. In the other direction $\sigma_i^j : P_i \longrightarrow P_{i+1}$ are induced by the maps of sets τ_{P_j}.

Remark 2.1.3. *Recall Definition 1.1.37 of a simplicial resolution and Proposition 1.1.39. A particular way to define a resolution of algebras is to consider their underlying structure of modules. As a matter of fact, since algebras over a ring A are not an abelian category, we may defines being a resolution for an A−algebras as being a resolution as A−module.*

Proposition 2.1.4. *The object $P_\bullet$ in $s\,\underline{\mathrm{Alg}}_A$ is a simplicial resolution for the $A-$algebra B.*

Proof. We need to prove that a) $P_\bullet$ is a simplicial object, b) $P_\bullet \longrightarrow B$ is an augmentation, c) the associated chain complex is acyclic, but in degree zero, where has cohomology isomorphic to B.
First we focus a little on face maps and degeneracy maps. As an example we just describe the first two algebras of the resolution (we use Einsten notation for the sums and we avoid the notation for the product of free elements)

$$
\begin{array}{ccccc}
A[A[A[B]]] & \longrightarrow & A[A[B]] & \longrightarrow & A[B] \\
 & & a_{b,i_1,i_2}[a_{b,i_1}[b]^{i_1}]^{i_2} & \xmapsto{\partial_1^0} & a_{b,i_1,i_2}[a_{b,i_1}b^{i_1}]^{i_2} \\
 & & a_{b,i_1,i_2}[a_{b,i_1}[b]^{i_1}]^{i_2} & \xmapsto{\partial_1^1} & a_{b,i_1,i_2}(a_{b,i_1}[b]^{i_1})^{i_2}
\end{array}
$$

$$
\begin{array}{ccc}
a_{b,i_1,i_2,i_3}[a_{b,i_1,i_2}[a_{b,i_1}[b]^{i_1}]^{i_2}]^{i_3} & \xmapsto{\partial_2^0} & a_{b,i_1,i_2,i_3}[a_{b,i_1,i_2}[a_{b,i_1}b^{i_1}]^{i_2}]^{i_3} \\
a_{b,i_1,i_2,i_3}[a_{b,i_1,i_2}[a_{b,i_1}[b]^{i_1}]^{i_2}]^{i_3} & \xmapsto{\partial_2^1} & a_{b,i_1,i_2,i_3}[a_{b,i_1,i_2}(a_{b,i_1}[b]^{i_1})^{i_2}]^{i_3} \\
a_{b,i_1,i_2,i_3}[a_{b,i_1,i_2}[a_{b,i_1}[b]^{i_1}]^{i_2}]^{i_3} & \xmapsto{\partial_2^2} & a_{b,i_1,i_2,i_3}(a_{b,i_1,i_2}[a_{b,i_1}[b]^{i_1}]^{i_2})^{i_3}
\end{array}
.
$$

In general face maps remove one couple of square brackets (and replace it with round brackets). On the other side, degeneracy maps double one couple of square brackets[1]

$$
\begin{array}{ccccc}
A[A[A[B]]] & \longleftarrow & A[A[B]] & \longleftarrow & A[B] \\
 & & a_{b,i_1}[[b]]^{i_1} & \xleftarrow{\sigma_0^0} & a_{b,i_1}[b]^{i_1} \\
a_{b,i_1,i_2}[a_{b,i_1}[[b]]^{i_1}]^{i_2} & \xleftarrow{\sigma_1^0} & a_{b,i_1,i_2}[a_{b,i_1}[b]^{i_1}]^{i_2} & & \\
a_{b,i_1,i_2}[[a_{b,i_1}[b]^{i_1}]]^{i_2} & \xleftarrow{\sigma_1^1} & a_{b,i_1,i_2}[a_{b,i_1}[b]^{i_1}]^{i_2} & &
\end{array}
.
$$

Now we can prove the points we stated at the beginning of the proof.
a) To prove that $P_\bullet$ is a simplicial object, we just need to prove that face maps and degeneracy maps satisfies the identities of (1.1). This is straightforward when considering the description above for σ_n^i, ∂_m^j.
b) We use Lemma 1.1.36, so that we need to prove just that $\kappa_B \partial_1^1 = \kappa_B \partial_1^0$, but again, it is straightforward from the notation above.
c) Consider the maps (of sets)

$$
\begin{aligned}
f_n : B &\longrightarrow P_n \\
b &\longmapsto [...[b]...]
\end{aligned}
$$

$$
\begin{aligned}
g_n : P_n &\longrightarrow B \\
a_{b,I}[...[b]^{i_1}...]^{i_n} &\longmapsto a_{b,I}(...(b)^{i_1}...)^{i_n}.
\end{aligned}
$$

If we consider $f = (f_n)_{n\geq 0} : B_\bullet \longrightarrow P_\bullet$ and $g = (g_n)_{n\geq 0} : P_\bullet \longrightarrow B_\bullet$, it is easy to see that they are simplicial maps, since $f_n = \tau_{P_{n-1}} \circ ... \circ \tau_{P_0} \circ \tau_B = \tau_{P_{n-1}} \circ f_{n-1}$ and $g_n = \kappa_B \circ \kappa_{P_0} \circ ... \circ \kappa_{P_{n-1}} = g_{n-1} \circ \kappa_{P_{n-1}}$.

$$
\begin{array}{ccc}
A[P_n] & \underset{\kappa_{P_n}}{\overset{\tau_B}{\rightleftarrows}} & P_n \\
{\scriptstyle \tau_{P_n} \circ f_n}\uparrow & & \uparrow{\scriptstyle f_n} \\
B & \xleftarrow[id_B]{} & B
\end{array}
\qquad
\begin{array}{ccc}
A[P_n] & \underset{\kappa_{P_n}}{\overset{\tau_B}{\rightleftarrows}} & P_n \\
{\scriptstyle g_n \circ \kappa_{P_n}}\downarrow & & \downarrow{\scriptstyle g_n} \\
B & \xrightarrow[id_B]{} & B
\end{array}
.
$$

[1]We do not consider maps like $b \mapsto [b]$, since they are just maps of sets.

We want to show that they define a simplicial homotopy, i.e. that $f \circ g$ and $g \circ f$ are homotopic to the simplicial maps $id_{P_\bullet}$ and $id_{B_\bullet}$. First of all $g \circ f = id_{B_\bullet}$ trivially, so that we need to focus only on $f \circ g$. Recall the description of $\Delta[1]$ in Example 1.1.6; we define for each α_n^i a map

$$h_n(\alpha_n^i, -) : P_n \xrightarrow{\kappa_{P_{n-i}} \circ \ldots \circ \kappa_{P_{n-1}}} P_{n-i} \xrightarrow{\tau_{P_{n-1}} \circ \ldots \circ \tau_{P_{n-i}}} P_n.$$

Recollecting all h_n we get a map of simplicial sets $h : \Delta[1] \times P_\bullet \longrightarrow P_\bullet$. If we compose it with $e_0 \times id_{P_\bullet}$, we get, for each $n \geq 0$, $h_n(\alpha_n^{n+1}, id_{P_\bullet}) = f_n \circ g_n$. On the other hand, composing with $e_1 \times id_{P_\bullet}$, we get $h_n(\alpha_n^0, id_{P_\bullet}) = id_{P_n}$. This means that h is a simplicial homotopy between $f \circ g$ and $id_{P_\bullet}$. We proved that $P_\bullet \longrightarrow B$ induces an homotopy equivalence as simplicial sets.

An augmentation of $A-$algebras which induces an homotopy equivalence of simplicial sets is equivalent to the fact that it is a quasi-isomorphism on the associated complexes of $A-$modules (Proposition 1.1.39), i.e. it is a resolution of B over A. □

Remark 2.1.5. *Suppose B a polynomial algebra. Then the map h defined in the previous proof, is actually a simplicial homotopy of simplicial $A-$algebras.*

Remark 2.1.6. *From its definition, it is easy to see that the construction of the standard simplicial resolution is functorial and this is the main motivation for having defined it, since any free resolution could be used (see Theorem 2.1.10). Moreover it commutes with direct limits. See [25, Chapter II §(1.2.1.1) and §(1.2.1.3)].*

We now can give the first fundamental definition.

Definition 2.1.7. Given an A-algebra B and its standard resolution $P_\bullet \longrightarrow B$ defined above. Consider the constant simplicial ring $B_\bullet$ as a simplicial $P_\bullet$-algebra via the augmentation map and the simplicial A-module $\Omega^1_{P_\bullet/A}$. The *cotangent complex* is the complex of B-modules associated to the simplicial object $B_\bullet \otimes_{P_\bullet} \Omega^1_{P_\bullet/A}$.

When it is clear from the context, we abuse the notation and we identify a simplicial object $A_\bullet$ with the associated complex $\int A_\bullet$.

Remark 2.1.8. *Such definition depends functorially on $A \longrightarrow B$ [25, §II.1.2.3.2], thanks to remark 2.1.6, and commutes with inductive limits [25, §II.1.2.3.4].*

First of all we want to allow other ways to compute the cotangent complex. The standard simplicial resolution is useful for proving functoriality, however we can replace the standard simplicial resolution of B with any free simplicial resolution and get the same object up to quasi-isomorphism. Before proving this result we need some technical lemmas.

Lemma 2.1.9. *Given a simplicial ring $A_\bullet$, a morphism of $A_\bullet$-modules $E_\bullet \longrightarrow F_\bullet$ and a termwise flat $A_\bullet$-module $L_\bullet$. Suppose $E_\bullet \longrightarrow F_\bullet$ induces a quasi-isomorphism on the associated complexes, then the induced map $E_\bullet \otimes_{A_\bullet} L_\bullet \longrightarrow F_\bullet \otimes_{A_\bullet} L_\bullet$ also induces a quasi-isomorphism.*

Proof. See [25, Lemma 3.3.2.1]. □

In particular, given a free terms simplicial resolution $Q_\bullet \longrightarrow B_\bullet$, we can apply such lemma to the case $A_\bullet = E_\bullet = Q_\bullet$, $F_\bullet = B_\bullet$ and $L_\bullet = \Omega^1_{Q_\bullet/A}$. The latter is indeed a termwise flat $Q_\bullet$-module and, since $Q_\bullet \longrightarrow B_\bullet$ yields a quasi-isomorphism on the associated complex (by definition of resolution), we have that

$$\Omega^1_{Q_\bullet/A} \cong Q_\bullet \otimes_{Q_\bullet} \Omega^1_{Q_\bullet/A} \overset{\text{q.}}{\cong} B_\bullet \otimes_{Q_\bullet} \Omega^1_{Q_\bullet/A}$$

as complexes (we indicate with $\overset{\text{q.}}{\cong}$ a quasi-isomorphism).

Now we can prove the following.

Theorem 2.1.10. *Let B be an A-algebra and let $Q_\bullet \longrightarrow B$ be a simplicial resolution of B, whose terms are free A-algebras. The B-modules complex associated to $B_\bullet \otimes_{Q_\bullet} \Omega^1_{Q_\bullet/A}$ is quasi-isomorphic to $L_{B/A}$.*

Proof. Consider the simplicial resolution $Q_\bullet \longrightarrow B$ and the bisimplicial A-algebra $P_\bullet(Q_\bullet)$ defined by the standard resolution $P_\bullet(Q_n) \longrightarrow Q_n$ for each level n. Let $\mathrm{Tot}(\Omega^1_{P_\bullet(Q_\bullet)/A})$ the total complex associated to the double complex arising from applying the functor $\Omega^1_{-/A}$ to $P_\bullet(Q_\bullet)$. It can be proved (see Lemma 2.10 of [41]) that[2]

$$\mathrm{Tot}(\Omega^1_{P_\bullet(Q_\bullet)/A}) \overset{\text{q.}}{\cong} \Omega^1_{P_\bullet(B)/A},$$

which is quasi-isomorphic to $B_\bullet \otimes_{P_\bullet(B)} \Omega^1_{P_\bullet(B)/A} = L_{B/A}$, since $P_\bullet(B)$ is a free terms resolution.
On the other hand, since each Q_n is a free algebra, the standard resolution $P_\bullet(Q_n) \longrightarrow (Q_n)_\bullet$ is a homotopy equivalence (see Lemma 2.4 in [41]), so the same holds for $\Omega^1_{P_\bullet(Q_n)/A} \longrightarrow (\Omega^1_{Q_n/A})_\bullet$, i.e. the associated complexes are quasi-isomorphic. In particular $\Omega^1_{P_\bullet(Q_n)/A}$ is an acyclic resolution of $\Omega^1_{Q_n/A}$. Thus we have

$$\mathrm{Tot}(\Omega^1_{P_\bullet(Q_\bullet)/A}) \overset{\text{q.}}{\cong} \Omega^1_{Q_\bullet/A}$$

and the term on the right is quasi-isomorphic to $B_\bullet \otimes_{Q_\bullet} \Omega^1_{Q_\bullet/A}$, since $Q_\bullet$ is a free terms resolution. ☐

Now we presents some results which will be very useful in order to compute some examples. Moreover they "prove" in some sense that the cotangent complex is a genuine generalization of the module of Kähler differentials.

Proposition 2.1.11. *We have a natural isomorphism of B-modules*

$$H_0(L_{B/A}) \cong \Omega^1_{B/A}.$$

Proof. Consider the augmentation $\epsilon_\bullet : P_\bullet \longrightarrow B_\bullet$. It yields a canonical map of complexes $L_B/A \longrightarrow \Omega^1_{B/A}$. Moreover the fact that it induces a quasi-isomorphism of complexes of $A-$modules means in particular that B is the cokernel of the double-map $\partial^0_1, \partial^1_1 : P_1 \longrightarrow P_0$. Since $\Omega^1_{-/-}$ commutes with inductive limits, we have

$$H^0(L_{B/A}) = \text{coker}\left(\Omega^1_{P_1/A} \rightrightarrows \Omega^1_{P_0/A}\right) \cong \Omega^1_{\text{coker}(P_1 \rightrightarrows P_0)/A} = \Omega^1_{B/A}.$$

☐

[2]Recall that the *total complex* of a bicomplex is given by the complex whose terms are computed by the direct sums along the diagonals of the bicomplex.

We saw (Facts 2.1.1(1)) that for any A-algebra A', there exists a canonical isomorphism of $A' \otimes_A B$-modules $\Omega^1_{A'\otimes_A B/A'} \cong \Omega^1_{B/A} \otimes_B (A' \otimes_A B)$. Let $P_\bullet \longrightarrow B$ and $P'_\bullet \longrightarrow A'$ be the standard simplicial resolutions, then the map $B \longrightarrow A' \otimes_A B$ induces a simplicial map $P_\bullet \longrightarrow P'_\bullet \otimes_A P_\bullet$. In this setting we have a natural base change morphism[3]

$$A' \otimes^L_A L_{B/A} \longrightarrow L_{A'\otimes_A B/A'}$$

but a (quasi-)isomorphism holds only for a particular class of A-algebras. Recall that the i-th torsion group $\mathrm{Tor}^A_i(A', B)$ is defined as the i-th cohomological group for the tensor functor $A' \otimes_A -$ (see example 2.1.13).

Definition 2.1.12. Two A-algebras A' and B are Tor-*independent* if $\mathrm{Tor}^A_i(A', B) = 0$ for $i > 0$.

In particular if A' is flat over A, then it is Tor-independent with any A-algebras. Now we can state the following

Lemma 2.1.13 (Base Change). *If A' and B are* Tor-*independent the base change induces a quasi-isomorphism of complexes of $A' \otimes_A B$-modules*

$$A' \otimes^L_A L_{B/A} \xrightarrow{\sim} L_{A'\otimes_A B/A'}.$$

Proof. [25, Proposition II.2.2.1 and §II.2.2.2]. This reduces to the case of polynomial algebras by passage to resolutions. The Tor −independence hypothesis gets used in concluding that if $P_\bullet \longrightarrow B$ is a polynomial $A-$algebra resolution, then $P_\bullet \otimes_A C \longrightarrow B \otimes_A C$ is again a polynomial $A-$algebra resolution. □

With a similar argument we can prove the following.

Lemma 2.1.14 (Künneth Formula). *Let $A \longrightarrow B$, $A \longrightarrow C$ be morphism of rings* Tor *−independent. Then there is a quasi-isomorphism of complexes of $B \otimes_A C-$modules*

$$L_{B\otimes_A C} \xrightarrow{\sim} L_{B/A} \otimes^L_A C \oplus L_{C/A} \otimes^L_A B.$$

Recall that a sequence of rings homomorphisms $A \longrightarrow B \longrightarrow C$ induces an exact sequence (again [34] Proposition 6.1.8)

$$C \otimes_B \Omega^1_{B/A} \longrightarrow \Omega^1_{C/A} \longrightarrow \Omega^1_{C/B} \longrightarrow 0.$$

This results can be interpreted as a "right-exactness" of the functor $\Omega^1_{-/-}$, although we are not in the right category to talk about "exactness", and can be interesting to complete the sequence on the left. Next theorem will provide a sort of generalization of this fact.

[3] Recall example 2.1.13 for the definition of the derived tensor product. By the Dold-Kan correspondence (Theorem 1.1.31) and the characterization of simplicial resolution for algebras and modules (remark 2.1.3) the standard simplicial resolution provides a free complex resolution, in particular projective, for any module.

Theorem 2.1.15. *A sequence of rings maps $A \longrightarrow B \longrightarrow C$ induces an exact triangle in the derived category of complexes of C-modules*

$$C \otimes^L_B L_{B/A} \longrightarrow L_{C/A} \longrightarrow L_{C/B} \longrightarrow C \otimes^L_B L_{B/A}[1].$$

Proof. [25, §II.2.1]. □

This result, together with our information about the 0-th homology group of the cotangent complex, leads to the following

Corollary 2.1.16. *In the above setting, there is a long exact sequence*

$$\ldots \longrightarrow H_1(C\otimes^L_B L_{B/A}) \longrightarrow H_1(L_{C/A}) \longrightarrow H_1(L_{C/B}) \longrightarrow C\otimes_B \Omega^1_{B/A} \longrightarrow \Omega^1_{C/A} \longrightarrow \Omega^1_{C/B} \longrightarrow 0;$$

which supports the previous idea of completing the right-exactness of the differential functor and the its generalization (see also Proposition 2.1.28).

Now recall the definition of étale map of ring, [34, Definition 4.3.17], and the fact that the module of Kähler differentials is trivial for this map [34, Corollary 6.2.3].

Proposition 2.1.17. *(Étale maps) Let $A \longrightarrow B$ be an étale map of rings. Then $L_{B/A}$ is acyclic (i.e. with trivial cohomology).*

Proof. It is a particular case of Proposition III.3.1.1 of [25]. □

The definition of cotangent complex of a ring morphism $A \longrightarrow B$ is easily generalized to a morphism of schemes $f : X \longrightarrow Y$ as a sheaf of $\mathcal{O}_X-$modules $L_{X/Y}$, such that $L_{X/Y}(U) := L_{\mathcal{O}_X(U)/f^{-1}\mathcal{O}_S(U)}$ for any open affine subset $U = \mathrm{Spec}(B) \subset X$. See the chapter on [42] about cotangent complex (as well as obviously [25]) about this and more general settings like ringed topos. All affine results have a global statement as well.

Proposition 2.1.18. *Let $f : X \longrightarrow Y$ a scheme morphism. Homology sheaves of $L_{X/Y}$ are quasi-coherent. Moreover they are coherent if Y is locally noetherian and f locally of finite type.*

Proof. See [25, Corollary II.2.3.7]. □

The first application of cotangent complex, which motivated the work of Grothendieck and Illusie, is related to first-order thickenings of algebras. Given an $A-$algebra B, a *first-order thickening* of B is given by an exact sequence of $A-$algebras

$$0 \longrightarrow I \longrightarrow Y \longrightarrow B \longrightarrow 0,$$

where I is an ideal satisfying $I^2 = 0$. Note that the condition $I^2 = 0$ implies that the natural $Y-$module structure on I induces a $B-$module structure. Two first-order thickenings Y_1, Y_2 of B by the same ideal I are *equivalent* if there is a morphism $Y_1 \longrightarrow Y_2$ inducing the identity map on B and I. A Baer sum construction defines an abelian group structure on equivalence classes, denoted by $\mathrm{Exal}_A(B, I)$. A fundamental theorem in [25] is the following.

Theorem 2.1.19. *Given a ring morphism $A \longrightarrow B$ and a $B-$module M, then there is a functorial isomorphism*

$$\mathrm{Exal}_A(B,M) \cong \mathrm{Ext}^1_B(L_{B/A}, M)$$

.

Proof. See [25, §III.1.2], in particular Theorem 1.2.3. □

2.1.2 Computations

In this section we provide some computational tools followed by some examples of application.

Proposition 2.1.20 (Polynomial algebras)**.** *If B is a free A-algebra, then $L_{B/A}$ is acyclic (i.e. with trivial cohomology) in nonzero degrees.*

Proof. Recall remark 2.1.5, so we have a homotopy equivalence between the constant simplicial algebra $B_\bullet$ and its standard resolution $P_\bullet$. If we apply the functor $\Omega^1_{-/A}$ we get a homotopy equivalence between se simplicial $B-$modules $\Omega^1_{P_\bullet/A} \longrightarrow \Omega^1_{B_\bullet/A}$, whence a quasi-isomorphism on associated chain complexes. But $\Omega^1_{B_\bullet/A}$ is a complex of free modules that is acyclic in nonzero degrees. □

Example 2.1.21. Now via Proposition 2.1.20 we can compute the cotangent complex for a ring of polynomials $A[T_1, ..., T_n]$, which is

$$L_{A[T_1,...,T_n]/A} \overset{\text{q.}}{\cong} \Omega^1_{A[T_1,...,Tn]/A} \cong \bigoplus_{i=1}^{n} A[T_1, ..., T_n]dT_i,$$

considered as a trivial complex concentrated in degree 0.

Example 2.1.22 (Crucial)**.** We want to apply Theorem 2.1.15 to the sequence $\mathbb{Z} \longrightarrow \mathbb{Z}[X_1, ..., X_r] \longrightarrow \mathbb{Z}$ and we get the exact triangle

$$L_{\mathbb{Z}[X_1,...,X_r]/\mathbb{Z}} \otimes^L_{\mathbb{Z}[X_1,...,X_r]} \mathbb{Z} \longrightarrow L_{\mathbb{Z}/\mathbb{Z}} \longrightarrow L_{\mathbb{Z}/\mathbb{Z}[X_1,...,X_r]} \longrightarrow L_{\mathbb{Z}[X_1,...,X_r]/\mathbb{Z}} \otimes^L_{\mathbb{Z}[X_1,...,X_r]} \mathbb{Z}[1]. \tag{2.1}$$

As a particular case of the previous example

$$L_{\mathbb{Z}/\mathbb{Z}} \overset{\text{q.}}{\cong} \Omega^1_{\mathbb{Z}/\mathbb{Z}} \cong 0 \qquad \text{and} \qquad L_{\mathbb{Z}[X_1,...,X_r]/\mathbb{Z}} \overset{\text{q.}}{\cong} \bigoplus \mathbb{Z}[X_1, ..., X_r]dX_i.$$

Thus, considering the long exact sequence associated to (2.1), we get

$$\begin{array}{l} \dots \longrightarrow 0 \longrightarrow H_1(L_{\mathbb{Z}/\mathbb{Z}[X_1,...,X_r]}) \longrightarrow \\ \underbrace{\bigoplus \mathbb{Z}[X_1, ..., X_r]dX_i \otimes_{\mathbb{Z}[X_1,...,X_r]} \mathbb{Z}}_{\cong \mathbb{Z}^r} \longrightarrow 0 \longrightarrow \Omega^1_{\mathbb{Z}/\mathbb{Z}[X_1,...,X_r]} \longrightarrow 0 \end{array}$$

Hence we get that $L_{\mathbb{Z}/\mathbb{Z}[X_1,...,X_r]}$ is acyclic outside degree 1. For the computation of $H_1(L_{\mathbb{Z}/\mathbb{Z}[X_1,...,X_r]})$ we can consider the exact sequence used in Facts 2.1.1(4) for the map $\mathbb{Z}[X_1, ..., X_r] \longrightarrow \mathbb{Z}$,

$$(X_1, ..., X_r)/(X_1, ..., X_r)^2 \xrightarrow{\delta} \mathbb{Z} \otimes_{\mathbb{Z}[X_1,...,X_r]} \Omega^1_{\mathbb{Z}[X_1,...,X_r]/\mathbb{Z}} \longrightarrow \underbrace{\Omega^1_{\mathbb{Z}/\mathbb{Z}}}_{=0} \longrightarrow 0$$

and the map $\delta : X_i \longmapsto 1 \otimes dX_i$ turns out to be injective, hence an isomorphism which, together with the previous long exact sequence, yields an isomorphism $(X_1, ..., X_r)/(X_1, ..., X_r)^2 \cong H_1(L_{\mathbb{Z}/\mathbb{Z}[X_1,...,X_r]})$. To some up, we have that $L_{\mathbb{Z}/\mathbb{Z}[X_1,...,X_r]} \overset{\text{q.}}{\cong} (X_1, ..., X_r)/(X_1, ..., X_r)^2[1]$, where the right hand side of the quasi-isomorphism is a complex concentrated in degree 1.

Proposition 2.1.23. *Given a surjective ring homomorphism $A \longrightarrow B$ with kernel I, generated by a nonzerodivisor $f \in A$, the cotangent complex is quasi-isomorphic to the complex $I/I^2[1]$.*

Proof. Consider the map $\mathbb{Z}[X] \ni X \longmapsto f \in A$, so that A is a $\mathbb{Z}[X]$-module. In particular given a free resolution of $\mathbb{Z}$

$$0 \longrightarrow \mathbb{Z}[X] \xrightarrow{\cdot x} \mathbb{Z}[X] \longrightarrow \mathbb{Z} \longrightarrow 0,$$

tensoring by $\otimes_{\mathbb{Z}[X]} A$ gives

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}[X] \otimes_{\mathbb{Z}[X]} A & \xrightarrow{\cdot x \otimes 1} & \mathbb{Z}[X] \otimes_{\mathbb{Z}[X]} A & \longrightarrow & \mathbb{Z} \otimes_{\mathbb{Z}[X]} A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle x \otimes 1 \mapsto f \cdot 1} & & \downarrow{\scriptstyle x \otimes 1 \mapsto f \cdot 1} & & \downarrow & & \\
 & & A & \longrightarrow & A & \longrightarrow & B & &
\end{array},$$

which is exact since f is regular. Thus $\mathrm{Tor}^i_{\mathbb{Z}[X]}(\mathbb{Z}, A) = 0$ per $i > 0$, i.e. A and $\mathbb{Z}$ are Tor-independent. So we can apply Lemma 2.1.13 and we get

$$L_{\mathbb{Z}/\mathbb{Z}[X]} \otimes^L_{\mathbb{Z}[X]} A \overset{\text{q.}}{\cong} L_{\mathbb{Z} \otimes_{\mathbb{Z}[X]} A/A} \cong L_{B/A}.$$

□

Recall that for a commutative ring A, the sequence of elements $f_1, ..., f_r \in A$ is called *regular* if f_i is a non-zero divisor in $A/(f_1, ..., f_{i-1}, f_{i+1}, ..., f_r)$.

Corollary 2.1.24. *Suppose that, in the setting of the previous proposition, the ideal I is generated by $f_1, ..., f_r \in A$ regular elements. Then $L_{B/A} \overset{\text{q.}}{\cong} I/I^2[1]$.*

Proof. It can be proved that $\mathrm{Tor}_i^{\mathbb{Z}[X_1,...,X_r]}(\mathbb{Z}, A) = 0$ for $i > 0$ (recall that given two ideals $I, J \subseteq A$ $\mathrm{Tor}^A_1(A/I, A/J) = (I \cap J)/IJ$) and then apply Lemma 2.1.13 and we get

$$L_{\mathbb{Z}/\mathbb{Z}[X_1,...,X_r]} \otimes^L_{\mathbb{Z}[X_1,...,X_r]} A \overset{\text{q.}}{\cong} L_{\mathbb{Z} \otimes_{\mathbb{Z}[X_1,...,X_r]} A/A} \cong L_{B/A}.$$

□

Example 2.1.25. Let $L|K$ be a finite extension of fields, then we want to show that the cotangent complex $L_{\mathcal{O}_L/\mathcal{O}_K}$ is acyclic in positive degrees.
We know[4] that $\mathcal{O}_L = \mathcal{O}_K[x]/(f)$ for some polynomial $f \in \mathcal{O}_K[x]$. Let us consider the sequence of maps $\mathcal{O}_K \longrightarrow \mathcal{O}_K[x] \longrightarrow \mathcal{O}_L$, which yields

$$L_{\mathcal{O}_K[x]/\mathcal{O}_K} \otimes^L_{\mathcal{O}_K[x]} \mathcal{O}_L \longrightarrow L_{\mathcal{O}_L/\mathcal{O}_K} \longrightarrow L_{\mathcal{O}_L/\mathcal{O}_K[x]} \longrightarrow .$$

Here we have, by Proposition 2.1.20 $L_{\mathcal{O}_K[x]/\mathcal{O}_K} \overset{\text{q.}}{\cong} \Omega^1_{\mathcal{O}_K[x]/\mathcal{O}_K}$ and, by Proposition 2.1.23, $L_{\mathcal{O}_L/\mathcal{O}_K[x]} \overset{\text{q.}}{\cong} (f)/(f^2)[1]$. Thus $L_{\mathcal{O}_L/\mathcal{O}_K}$ is necessarily acyclic for degrees greater 1 and we only need to study $H_1(L_{\mathcal{O}_L/\mathcal{O}_K})$. The induced long exact sequence presents the following piece

$$0 \longrightarrow H_1(L_{\mathcal{O}_L/\mathcal{O}_K}) \longrightarrow \underbrace{H_1(L_{\mathcal{O}_L/\mathcal{O}_K[x]})}_{=(f)/(f^2)} \overset{\delta}{\longrightarrow} \underbrace{\Omega^1_{\mathcal{O}_K[x]/\mathcal{O}_K} \otimes_{\mathcal{O}_K[x]} \mathcal{O}_L}_{=\mathcal{O}_K[x]dx \otimes_{\mathcal{O}_K[x]} \mathcal{O}_L},$$

where $\delta(\overline{f}) = df \otimes 1$ is injective, thus $0 = \ker \delta = H_1(L_{\mathcal{O}_L/\mathcal{O}_K})$ as it injects into $H_1(L_{\mathcal{O}_L/\mathcal{O}_K[x]})$.

Example 2.1.26. A straightforward consequence of the previous example is related to the cotangent complex of the ring of integers for the algebraic closure of a finite extension $K|F$. First of all recall the fact that colimits commute the cotangent complex, i.e. $\Omega^1_{\varinjlim_i (R_i \longrightarrow S_i)} = \varinjlim_i \Omega^1_{R_i/S_i}$, for a system of ring maps over a directed set $(R_i \longrightarrow S_i)$

$$L_{\varinjlim_i (R_i \longrightarrow S_i)} = \varinjlim_i L_{R_i/S_i}$$

Thus, since $\mathcal{O}\overline{K}$ can be seen as the direct limits of the rings of integers for each finite subextension $K \subset L \subset \overline{K}$, we have the isomorphisms

$$L_{\mathcal{O}\overline{K}/\mathcal{O}K} = \varinjlim L_{\mathcal{O}L/\mathcal{O}K} \overset{\text{q.}}{\cong} \varinjlim \Omega^1_{\mathcal{O}_L/\mathcal{O}_K} \cong \Omega^1_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K}.$$

Example 2.1.27. Another easy exercise is to compute $L_{L/K}$ for a finite separable extension $L|K$ of arbitrary fields. By the primitive element Theorem we have that $L = K(\alpha)$ for some element $\alpha \in L$, in particular if $f \in K[X]$ is the minimal polynomial of α, we have

$$L \cong \frac{K[X]}{f}.$$

Now consider the sequence of ring maps $K \longrightarrow K[X] \longrightarrow L$, with the second arrow the quotient map, which yields

$$L_{K[X]/K} \otimes^L_{K[X]} L \longrightarrow L_{L/K} \longrightarrow L_{L/K[X]} \longrightarrow .$$

As before we have $L_{K[X]/K} \overset{\text{q.}}{\cong} \Omega_{K[X]/K}$ and $L_{L/K[X]} \overset{\text{q.}}{\cong} (f)/(f^2)[1]$. And by the same computation of the previous example we get $L_{L/K}$ is a complex concentrated only in degree 0, i.e. $L_{L/K} \overset{\text{q.}}{\cong} \Omega^1_{L/K}$. Further we know by classical commutative algebra (see [34] Lemma 6.1.13) that, since the extension is separable, $\Omega^1_{L/K} = 0$.

[4] Reference

Recall the definition of smooth morphism [20, 17.3.1].

Proposition 2.1.28 (Smooth morphism). *Given a smooth map of rings $A \longrightarrow B$, then the canonical augmentation map $L_{B/A} \longrightarrow \Omega^1_{B/A}$ is a quasi-isomorphism*

Proof. See [25, Proposition III.3.1.2]. Since $A \longrightarrow B$ is smooth, there is an étale map $B_0 := A[x_1, ..., x_n] \longrightarrow B$. We know that $L_{B_0/A} \cong \Omega^1_{B_0/A}$ (Proposition 2.1.20) and $L_{B/B_0} = 0$ by Proposition 2.1.17. By Theorem 2.1.15, it follows that $L_{B/A} \overset{\text{q.}}{\cong} L_{B_0/A} \otimes_{B_0} B \overset{\text{q.}}{\cong} \Omega^1_{B/A}[0]$. □

Example 2.1.29. We now want to compute the cotangent complex for the morphisms

1. $\mathbb{Z} \longrightarrow \mathbb{F}_p$;
2. $\mathbb{Z}_p \longrightarrow \mathbb{F}_p$;
3. $\mathbb{Z} \longrightarrow \mathbb{F}_q[T]$, $q = p^n$;
4. $\mathbb{Z} \longrightarrow B$, where B is a smooth $\mathbb{F}_p$-algebra.

For the case 1. and 2. we can apply again Proposition 2.1.23, since $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}_p/p\mathbb{Z}_p$ and we get

$$L_{\mathbb{F}_p/\mathbb{Z}} \overset{\text{q.}}{\cong} p\mathbb{Z}/p^2\mathbb{Z}[1] \cong \mathbb{F}_p[1] \qquad \text{and} \qquad L_{\mathbb{F}_p/\mathbb{Z}_p} \overset{\text{q.}}{\cong} p\mathbb{Z}_p/p^2\mathbb{Z}_p[1] \cong \mathbb{F}_p[1].$$

For case 3. we first of all need to study the cotangent complex relative to $\mathbb{Z} \longrightarrow \mathbb{F}_q$. Recalling that the extension $\mathbb{F}_q|\mathbb{F}_p$ is separable, hence $L_{\mathbb{F}_q/\mathbb{F}_p}$ is the trivial complex as in example 2.1.27, we can consider the sequence $\mathbb{Z} \longrightarrow \mathbb{F}_p \longrightarrow \mathbb{F}_q$, which gives us

$$L_{\mathbb{F}_p/\mathbb{Z}} \otimes^L_{\mathbb{F}_p} \mathbb{F}_q \longrightarrow L_{\mathbb{F}_q/\mathbb{Z}} \longrightarrow \underbrace{L_{\mathbb{F}_q/\mathbb{F}_p}}_{\overset{\text{q.}}{\cong} 0} \longrightarrow .$$

So we can deduce that

$$L_{\mathbb{F}_q/\mathbb{Z}} \overset{\text{q.}}{\cong} p\mathbb{Z}/p^2\mathbb{Z} \otimes_{\mathbb{F}_p} \mathbb{F}_q[1].$$

Now we consider the sequence $\mathbb{Z} \longrightarrow \mathbb{F}_p \longrightarrow \mathbb{F}_p[x]$ and the associated triangle

$$L_{\mathbb{F}_p/\mathbb{Z}} \otimes^L_{\mathbb{F}_p} \mathbb{F}_p[x] \longrightarrow L_{\mathbb{F}_p[x]/\mathbb{Z}} \longrightarrow L_{\mathbb{F}_p[x]/\mathbb{F}_p} \longrightarrow .$$

It corresponds to the following diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & L^2_{\mathbb{F}_p[x]/\mathbb{Z}} & \longrightarrow & 0 & \longrightarrow & p/p^2 \otimes_{\mathbb{F}_p} \mathbb{F}_p[x] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
p/p^2 \otimes_{\mathbb{F}_p} \mathbb{F}_p[x] & \longrightarrow & L^1_{\mathbb{F}_p[x]/\mathbb{Z}} & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & L^0_{\mathbb{F}_p[x]/\mathbb{Z}} & \longrightarrow & \Omega^1_{\mathbb{F}_p[x]/\mathbb{F}_p} & \longrightarrow & 0
\end{array}.$$

The bottom line tells us that $L^0_{\mathbb{F}_p[x]\mathbb{Z}} \cong \Omega^1_{\mathbb{F}_p[x]/\mathbb{F}_p}$ and, since $H^0(L_{\mathbb{F}_p[x]\mathbb{Z}}) \cong \Omega^1_{\mathbb{F}_p[x]/\mathbb{Z}} \cong \Omega^1_{\mathbb{F}_p[x]/\mathbb{F}_p}$, the differential map $L^1_{\mathbb{F}_p[x]\mathbb{Z}} \longrightarrow L^0_{\mathbb{F}_p[x]\mathbb{Z}}$ is the zero map. Further $L_{\mathbb{F}_p[x]\mathbb{Z}}$ is acyclic in degree greater than 1 and, since $H^1(L_{\mathbb{F}_p[x]\mathbb{Z}}) \cong p/p^2[x]$, we finally have $L_{\mathbb{F}_p[x]\mathbb{Z}} \cong p/p^2[x]$. Case 4. goes in the same way, since $L_{B/\mathbb{F}_p} \simeq \Omega^1_{B/\mathbb{F}_p}$ as it is a smooth $\mathbb{F}_p-$algebra.

Example 2.1.30. Assume A has characteristic p. Let $A \longrightarrow B$ be a flat map that is *relatively perfect*, i.e. the relative Frobenius $F_{B/A} : B^{(1)} := B \otimes_{A,F_A} A \longrightarrow B$ is an isomorphism. Then $L_{B/A} = 0$.
As a matter of fact, for any $A-$algebra B, the relative Frobenius induces the zero map $L_{F_{B/A}} : L_{B^{(1)}/A} \longrightarrow L_{B/A}$: this is clear when B is a polynomial A-algebra (as $d(x^p) = 0$), and thus follows in general by passage to the canonical resolutions. Now if $A \longrightarrow B$ is relatively perfect, then $L_{F_{B/A}}$ is also an isomorphism by functoriality. Thus, the zero map $L_{B^{(1)}/A} \longrightarrow L_{B/A}$ is an isomorphism, so $L_{B/A} = 0$.

2.2 Derived de Rham complex, definition and properties

Here we introduce the central object of our research. Before we start with main definitions and results, we recall some about the classic (algebraic) de Rham complex. It has been defined by Hartshorne [23] in order to extend the Poincaré Lemma (actually Volterra Lemma, see [28]) to the case of algebraic varieties. Let $A \longrightarrow B$ a rings morphism, we put

$$\Omega^p_{B/A} := \bigwedge_B^p \Omega^1_{B/A}.$$

Further we define the differential map

$$d : \Omega^p_{B/A} \longrightarrow \Omega^{p+1}_{B/A}$$

as the unique map such that

(i) $d \circ d = 0$,

(ii) $d : B \longrightarrow \Omega^1_{B/A}$ is the morphism defined for the Kähler differentials module,

(iii) for $\omega \in \Omega^p_{B/A}, \eta \in \Omega^q_{B/A}$, the following holds $d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^p \omega \wedge d(\eta)$.

By Theorem 16.6.2 in [20] such d exists and it is unique.
Furthermore, since $d \circ d = 0$, it is well defined the following sequence

$$\Omega^{\cdot}_{B/A} = B \longrightarrow \Omega^1_{B/A} \longrightarrow \Omega^2_{B/A} \longrightarrow ... \longrightarrow \Omega^i_{B/A} \longrightarrow \Omega^{i+1}_{B/A} \longrightarrow ...$$

called *de Rham complex*, $\Omega^*_{B/A}$. For a morphism of schemes $X \longrightarrow Y$ we recall that given a sheaf of modules $\mathcal{F}$, we define $\bigwedge^p \mathcal{F}$ as the sheafification of the presheaf

$$U \subseteq X \longmapsto \bigwedge_{\mathcal{O}_X(U)}^p \mathcal{F}(U).$$

Definition 2.2.1. Given an A-algebra B, as before we can consider the associated standard simplicial resolution $P_\bullet \longrightarrow B$, by applying the functors $\Omega^i_{-/A}$ and taking the associated chain complex on the horizontal lines we get the double complex $\Omega^*_{P_\bullet/A}$. The associated total complex (direct sum convention) $L\Omega^*_{B/A} := \mathrm{Tot}(\Omega^*_{P_\bullet/A})$

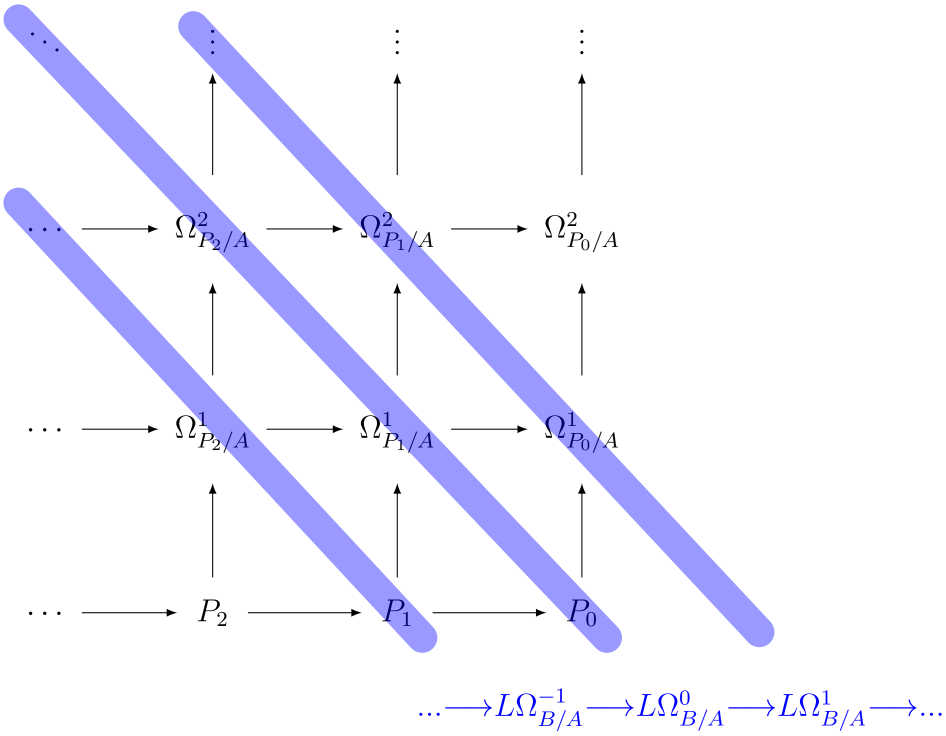


Figure 2.1: The Derived de Rham complex

is the *derived de Rham complex of B* (see Figure 2.2.1).

Remark 2.2.2. *Our construction can be represented as the sequence of functors*

$$P_\bullet \overset{\Omega^*_{-/A}}{\longmapsto} \Omega^*_{P_\bullet/A} \overset{\int}{\longmapsto} \int \Omega^*_{P_\bullet/A}.$$

It is important to not confuse that with the inverse construction "$\Omega^*_{-/A} \circ \int$", which does not make sense: the functor $\Omega^*_{-/A}$ only applies over $A-$algebras (complexes, simplicials o simple), while the functor $\int$ transforms maps of algebras of the simplex $P_\bullet$ into maps of modules (e.g. $(\partial_0 - \partial_1)1_A = \partial_0 1_A - \partial_1 1_A = 0$).

Associated to the derived de Rham complex there's a canonical filtration. Let $N \geq 0$ be an integer. The de Rham complex of any $A-$algebra P carries a natural filtration called *Hodge filtration*

$$F^N\Omega^*_{P/A} := \Omega^{\geq N}_{P/A} = \left[\Omega^N_{P/A} \longrightarrow \Omega^{N+1}_{P/A} \longrightarrow \ldots\right],$$

which induces a filtration on the previous double complex.

Definition 2.2.3. We define on $L\Omega^*_{B/A}$ the *Hodge filtration* $F^iL\Omega^*_{B/A}$ (see Figure 2.2) as the filtration induced by

$$F^i\Omega^*_{P_\bullet/A} = \bigoplus_{q\geq i} \Omega^q_{P_\bullet/A}.$$

Definition 2.2.4. The associated *derived de Rham complex* modulo F^N is defined as

$$\frac{L\Omega^*_{B/A}}{F^N} := \mathrm{Tot}(\Omega^{<N}_{P_\bullet/A}),$$

The *Hodge-completed derived de Rham complex of B over A* is the projective system of complexes of $A-$modules defined by the derived de Rham complexes modulo the Hodge filtration

$$L\widehat{\Omega}^*_{B/A} := (L\Omega^*_{B/A}/F^N)_{N\in\mathbb{N}}.$$

Remark 2.2.5. *In this work we aim to consider the pro-object structure of $L\widehat{\Omega}_{B/A}$. When we want to denote the projective limits of such pro-system, it will be specified.*

Remark 2.2.6. *The Hodge completed derived de Rham complex and the non-completed version $L\Omega^*_{B/A}$ are two very different objects. For example, while the second one is useless in characteristic 0 (see [10] Corollary 2.5), the first one (in the projective limit form) has been proved to provide the right cohomology for (singular) varieties in characteristic 0 (see [9] in general, more precisely Corollary 4.27).*

Usually we use the terminology of "de Rham *algebra*", since $L\Omega^*_{B/A}$ can be equipped with a structure of a commutative differential graded algebra over A, compatible with the Hodge filtration, so that $L\widehat{\Omega}^*_{B/A}$ turns out to be a projective system of differential graded algebras. It is a special case of Lemma 1.2.21. We define the product as

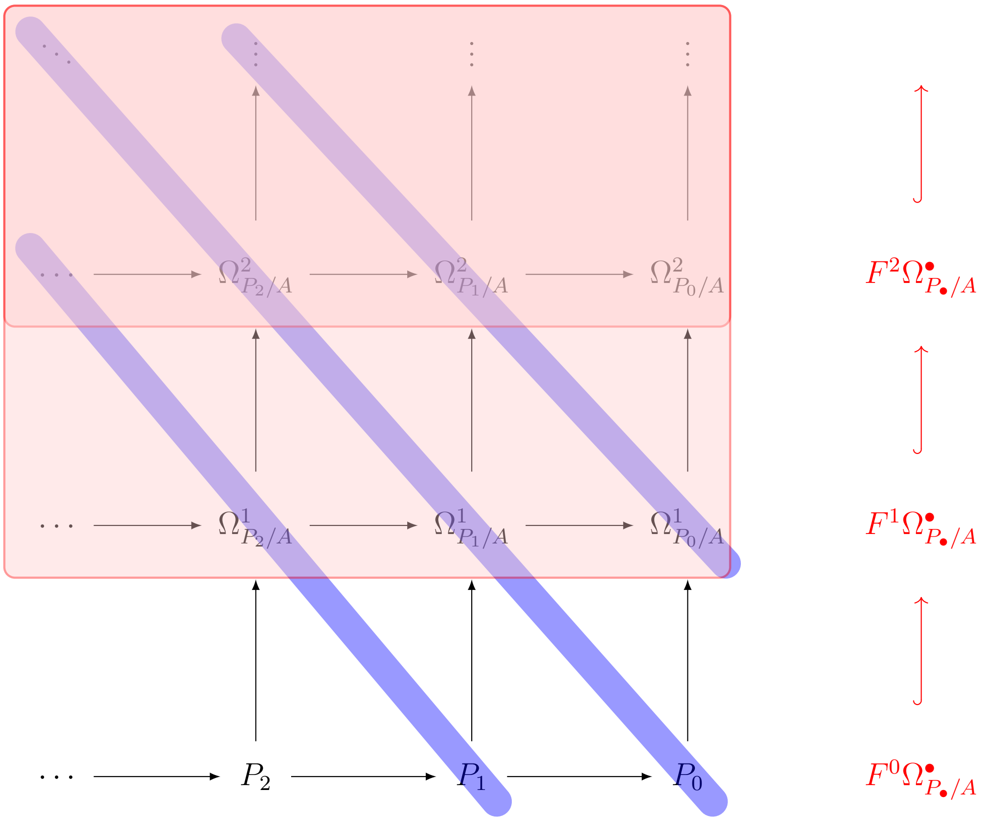
$$\begin{aligned}
&\Omega^h_{P_i/A} \times \Omega^k_{P_j/A} \longrightarrow \Omega^{h+k}_{P_{i+j}/A}\\
&(fdx_H, gdy_K) \longmapsto fdx_H \cdot gdy_K = (-1)^{jp} \sum_{(\mu,\nu)} \mathrm{sgn}(\mu,\nu)(\sigma_\nu f)(\sigma_\mu g)\ d(\sigma_\nu x_H) \wedge d(\sigma_\mu y_K).
\end{aligned}$$

Such map sends $\alpha \otimes \beta \in P_i \otimes P_j$ to

$$\sum_{(\mu,\nu)} \mathrm{sgn}(\mu,\nu)(\sigma_\nu(\alpha) \otimes \sigma_\mu(\beta)) \in C(P_{i+j} \otimes_A P_{i+j})$$

where the sum runs over all the $(i,j)-$shuffles[5]: given such a shuffle $(\mu,\nu) = (\mu_1 \ldots \mu_i\ \nu_1 \ldots \nu_j)$, we put $\sigma_\mu = \sigma_{\mu_i-1}\sigma_{\mu_{i-1}-1}\ldots\sigma_{\mu_1-1}$ and $\sigma_\nu = \sigma_{\nu_j-1}\sigma_{\nu_{j-1}-1}\ldots\sigma_{\nu_1-1}$ in order to shift α, β from P_i, P_j to P_{i+j} via a path of degeneracy maps σ_n. We extend then the product we defined to the whole $(L\Omega^*_{B/A}/F^N, D)$. With such definition the whole machinery gives us a commutative unitary associative product, so that $(L\Omega^*_{B/A}/F^N, D)$ is a differential graded algebra over A. Moreover it is compatible with projection maps of the associated projective system $(L\Omega^*_{B/A}/F^N)_{N\in\mathbb{N}}$, so that also $L\widehat{\Omega}^*_{B/A}$ is a pro-system of $A-$dga.

[5] Recall that a $(i,j)-$shuffle consists of a permutation $(\mu,\nu) := (\mu_1 \ldots \mu_i \nu_1 \ldots \nu_j) \in S_{i+j}$, such that $\mu_1 < \ldots < \mu_i$ and $\nu_1 < \ldots < \nu_j$.



$$F^2 L\Omega_{B/A} : \quad \ldots \longrightarrow \bigoplus_{\substack{p+q=1\\ q\geq 2}} \Omega^q_{P_{-p}/A} \longrightarrow \bigoplus_{\substack{p+q=0\\ q\geq 2}} \Omega^q_{P_{-p}/A} \longrightarrow \bigoplus_{\substack{p+q=-1\\ q\geq 2}} \Omega^q_{P_{-p}/A} \longrightarrow \ldots$$

$$F^1 L\Omega_{B/A} : \quad \ldots \longrightarrow \bigoplus_{\substack{p+q=1\\ q\geq 1}} \Omega^q_{P_{-p}/A} \longrightarrow \bigoplus_{\substack{p+q=0\\ q\geq 1}} \Omega^q_{P_{-p}/A} \longrightarrow \bigoplus_{\substack{p+q=-1\\ q\geq 1}} \Omega^q_{P_{-p}/A} \longrightarrow \ldots$$

$$F^0 L\Omega_{B/A} : \quad \ldots \longrightarrow \bigoplus_{\substack{p+q=1\\ q\geq 0}} \Omega^q_{P_{-p}/A} \longrightarrow \bigoplus_{\substack{p+q=0\\ q\geq 0}} \Omega^q_{P_{-p}/A} \longrightarrow \bigoplus_{\substack{p+q=-1\\ q\geq 0}} \Omega^q_{P_{-p}/A} \longrightarrow \ldots.$$

Figure 2.2: Filtration on Derived de Rham complex

Remark 2.2.7. *As we defined it, $L\widehat{\Omega}^*_{B/A}$ is a (pro-system of) differential graded algebra over A. However it has been recently considered in a broader context as an E_∞-algebra (see [9, Remark 4.2] or [6, 1.1]). This is also due to the fact that if we replace the simplicial standard resolution $P_\bullet$ in the definition of $L\widehat{\Omega}^*_{B/A}$ with any simplicial A-algebra resolution $P'_\bullet \longrightarrow B$ whose terms are free, the output is naturally quasi-isomorphic to the Hodge completed derived de Rham complex (see Theorem 2.2.9). As we saw in Chapter 1 §1.3, any commutative differential graded algebra may be seen as an E_∞-algebra and a morphism of cdga defines a map between the corresponding E_∞-algebras. Further, a quasi-isomorphism of cdga induces an equivalence of E_∞-algebras. Through this pages we try to work as much as possible in the context of classical category theory (as in [41]), considering maps of differential graded algebras; however statements will be presented in the more elegant formalism of E_∞-algebra theory.*

We continue this introduction to the derived de Rham complex and we provide some theoretical results, before computing some examples. This result gives us a useful tool to compute the derived de Rham algebra from the cotangent complex.

Proposition 2.2.8. *There is a quasi-isomorphism of complexes of A-modules*

$$\mathrm{gr}^i_F L\Omega^*_{B/A} \overset{\text{q.}}{\cong} L\wedge^i L_{B/A}[-i],$$

where $L\wedge$ is the derived exterior power (see [41, §A.5 and §A.6])

Proof. Looking at the previous diagrams we can deduce[6] that

$$\mathrm{gr}^i_F L\Omega^*_{B/A} = \frac{F^i L\Omega^*_{B/A}}{F^{i+1} L\Omega^*_{B/A}} \cong \Omega^i_{P_\bullet/A} = \left(\ldots \longrightarrow \Omega^i_{P_j/A} \longrightarrow \ldots \longrightarrow \Omega^i_{P_1/A} \longrightarrow \Omega^i_{P_0/A}\right).$$

Now we know that the resolution $P_\bullet \longrightarrow B$ can be viewed as a map between $P_\bullet$ and the constant complex $B_\bullet$, which in particular is a quasi-isomorphism. Since $\Omega^i_{P_\bullet/A}$ is a free-terms $P_\bullet$-module, we can apply Lemma 2.1.9 and we get that $P_\bullet \overset{\text{q.}}{\cong} B_\bullet$ implies $P_\bullet \otimes_{P_\bullet} \Omega^i_{P_\bullet/A} \overset{\text{q.}}{\cong} B_\bullet \otimes_{P_\bullet} \Omega^i_{P_\bullet/A}$. Now recall the fact that $\Omega^i_{-/-} := \bigwedge^i \Omega^1_{-/-}$ ([23]) and that, by universal property of exterior power,

$$\left(\bigwedge^i \Omega^1_{P_\bullet/A}\right) \otimes_{P_\bullet} B_\bullet \cong \bigwedge^i \left(\Omega^1_{P_\bullet/A} \otimes_{P_\bullet} B_\bullet\right)$$

[6] In particular

$$\frac{F^i L\Omega^*_{B/A}}{F^{i+1} L\Omega^*_{B/A}} = \left(\frac{\bigoplus_{\substack{p+q=j \\ q\geq i}} \Omega^q_{P_{-p}/A}}{\bigoplus_{\substack{p+q=j \\ q\geq i+1}} \Omega^q_{P_{-p}/A}}\right)_j = \left(\bigoplus_{\substack{p+q=j \\ q=i}} \Omega^q_{P_{-p}/A}\right)_j = \left(\Omega^i_{P_{i-j}/A}\right)_j.$$

as $P_\bullet$-modules. Now we simply recollect all these results

$$\begin{aligned}
\mathrm{gr}_F^i L\Omega^*_{B/A} &= \Omega^i_{P_\bullet/A}[-i] \\
&= P_\bullet \otimes_{P_\bullet} \Omega^i_{P_\bullet/A}[-i] \\
&\overset{\text{q.}}{\cong} B_\bullet \otimes_{P_\bullet} \Omega^i_{P_\bullet/A}[-i] \\
&= \left(\bigwedge^i \Omega^1_{P_\bullet/A} \right) \otimes_{P_\bullet} B_\bullet[-i] \\
&\cong \bigwedge^i \left(\Omega^1_{P_\bullet/A} \otimes_{P_\bullet} B_\bullet \right)[-i] \\
&= L \bigwedge^i L_{B/A}[-i].
\end{aligned}$$

□

This shows that the relation between the derived de Rham complex and the cotangent complex is analogous to the relation between the de Rham complex and the sheaf of Kähler differentials. As a matter of fact we can say that the derived de Rham complex is defined as

$$L\Omega^\bullet_X := \mathrm{Tot}\left(\mathcal{O}_X \longrightarrow L_X \longrightarrow \wedge^2 L_X \longrightarrow ... \right),$$

while the non derived case is

$$\Omega^\bullet_X = (\mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow \Omega^2_X \longrightarrow ...),$$

where $\Omega^n_X := \wedge^n \Omega^1_X$. Next result gives the homotopical nature to the derived de Rham complex, allowing it to be defined from any free simplicial resolution. See [41, Theorem 2.25 and Remark 2.26] for an explicit proof.

Theorem 2.2.9. *Let $Q_\bullet \longrightarrow B$ be a simplicial resolution of the $A-$algebra B whose terms are free $A-$algebras. Then we have a quasi-isomorphism of complexes*

$$L\Omega^*_{B/A} \overset{\text{q.}}{\cong} \mathrm{Tot}(\Omega^*_{Q_\bullet/A})$$

compatible with the product structure and the Hodge filtration.

Remark 2.2.10. *As in remark 2.2.2, we should be careful that the construction of the derived de Rham complex is of simplicial nature. So if we have a free complex resolution F^* of B, in order to compute $L\Omega^*_{B/A}$ we have first to compute the associated simplicial object $K_\bullet F^*$ (which is going to be a free simplicial resolution of $B_\bullet$ by Dold-Kan correspondence) and then, thanks to last theorem, take the total complex of $\int \Omega^*_{K_\bullet F^*/A}$.*

2.3 Hodge completed derived de Rham complex for perfect rings

Our main result is the following equivalence of pro-systems of E_∞-$\mathbb{Z}$-algebras

$$L\widehat{\Omega}^*_{k/\mathbb{Z}} \simeq \left(\frac{W\langle x \rangle}{(x)^{[N]}} \xrightarrow{\cdot(x-p)} \frac{W\langle x \rangle}{(x)^{[N]}} \right)_{N \in \mathbb{N}},$$

where k is a perfect ring of characteristic $p > 2$ and $W = W(k)$ is the associated ring of Witt vectors.

Such result implies in particular that the Hodge completed derived de Rham algebra relative to $\mathbb{Z} \longrightarrow k$ "contains" the ring of Witt vectors $W(k)$. Bhatt computed the (not Hodge completed) derived de Rham complex p-adically completed in the same case (see [10] Corollary 8.6). In particular he showed that when k is perfect, the ring $W(k)$ may be obtained as the largest separated torsion-free quotient of the p-adically completed derived de Rham complex (ibidem Remark 8.7).

Outline of the proof. The theorem relies on the base change lemma applied to the (crucial) simple case where $k = \mathbb{F}_p$. Similar results may be obtained by means of crystalline theory computations, see [26, Ch.VIII Proposition 2.2.8]. In the present paper we give a more direct and elementary proof, which takes into account the multiplicative structure differential graded algebras (say also E_∞-algebras). As we said, it is crucial to compute the Hodge-completed derived de Rham complex for the map

$$\mathbb{Z}_p \longrightarrow \mathbb{F}_p, \tag{2.2}$$

which is a particular case of perfect ring and its associated Witt vectors. Once dealt with this step (see Theorem 2.3.13) it is easy to afford the general case by base change (Theorem 2.3.26). So most of the efforts are devoted to the crucial case, which next paragraph is about. The strategy is to consider the map $\mathbb{Z}_p[x] \longrightarrow \mathbb{Z}_p$ and do computations for this morphism. Some of them have already been taken in other works in similar contexts (see [10, Corollary 3.40], [41, proof of Proposition 3.17], [2, Example 6.2]), here we present a really detailed proof. The idea is to exploit the Hodge filtration and the fact that it is easy to compute its graded parts by means of Proposition 2.2.8 and Proposition 2.1.23. We use the graded pieces as bricks to rebuild the quotients $L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N$ (Lemma 2.3.10). This is easy since there is no p-torsion. The case of (2.2) is much challenging because of p-torsion. Finally we can compute the crucial case again by base change (Lemma 2.3.12).

2.3.1 The complex $L\widehat{\Omega}_{\mathbb{F}_p/\mathbb{Z}}$

We start this part giving some basic lemmas.

Lemma 2.3.1. *Let $M \longrightarrow N$ be a morphism of filtered $A-$modules, such that the filtration is decreasing and $F^0M = M$, $F^0N = N$. If the induced map on the graded pieces is an isomorphism, then*

$$\frac{M}{F^nM} \cong \frac{N}{F^nN}, \qquad \textit{for all } n \geq 0.$$

Moreover, if the filtration is finite[7]*, then $M \cong N$.*

[7] Recall that a filtration over an A-module M is *finite* if there exists m, n such that $F^mM = 0$ and $F^nM = M$.

Proof. Using $\mathrm{gr}^0 M \cong \mathrm{gr}^0 N$ as step 0 we want to prove the statement by induction on n. Considering the induced morphisms between the following two short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \dfrac{F^{n-1}M}{F^nM} & \longrightarrow & \dfrac{F^0M}{F^nM} & \longrightarrow & \dfrac{F^0M}{F^{n-1}M} & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & \dfrac{F^{n-1}N}{F^nN} & \longrightarrow & \dfrac{F^0N}{F^nN} & \longrightarrow & \dfrac{F^0N}{F^{n-1}N} & \longrightarrow & 0 \end{array},$$

where the first and third vertical arrows are isomorphism by hypothesis on the graded pieces and by induction hypothesis respectively. Thus by five lemma, the vertical arrow in the middle is an iso as well and we are done. The second statement is straightforward. □

The following Lemma 2.3.2 and Lemma 2.3.4 are similar (but proofs are a little different), they allow us to localize in some case the "ring of coefficients" along an étale map.

Lemma 2.3.2. *Let k be a perfect ring of characteristic p and $W := W(k)$ its Witt vectors ring. Given a ring homomorphism $W \longrightarrow B$, then the canonical map $L\widehat{\Omega}^*_{B/\mathbb{Z}_p} \longrightarrow L\widehat{\Omega}^*_{B/W}$ is a quasi-isomorphism.*

Proof. See [41, Lemma 3.28]. □

Remark 2.3.3. *The result holds also for the non-completed case. Bhatt proved it by means of the conjugate filtration, which yields a spectral sequence convergent to the non completed derived de Rham complex in [10, Proposition 2.3 and Lemma 8.3(5)], see also [2, Remark 4.15].*

Lemma 2.3.4. *Given a sequence of morphisms of rings $\mathbb{Z} \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{F}_p \longrightarrow B$, there exists a quasi-isomorphism between the Hodge completed derived de Rham (differential graded) algebras*

$$L\widehat{\Omega}^*_{B/\mathbb{Z}} \longrightarrow L\widehat{\Omega}^*_{B/\mathbb{Z}_p}.$$

Proof. We claim that there is a quasi-isomorphism

$$L_{B/\mathbb{Z}} \longrightarrow L_{B/\mathbb{Z}_p}. \tag{2.3}$$

For $B = \mathbb{F}_p$, such result can be easily proved by direct computation. From the sequence of morphisms in the statement we get the following diagram of exact triangles

$$
\begin{array}{ccccccc}
L_{\mathbb{F}_p/\mathbb{Z}} \otimes^L_{\mathbb{F}_p} B & \longrightarrow & L_{B/\mathbb{Z}} & \longrightarrow & L_{B/\mathbb{F}_p} & \longrightarrow & L_{\mathbb{F}_p/\mathbb{Z}} \otimes^L_{\mathbb{F}_p} B[1] \\
\downarrow \cong & & \downarrow & & \downarrow = & & \downarrow \cong \\
L_{\mathbb{F}_p/\mathbb{Z}_p} \otimes^L_{\mathbb{F}_p} B & \longrightarrow & L_{B/\mathbb{Z}_p} & \longrightarrow & L_{B/\mathbb{F}_p} & \longrightarrow & L_{\mathbb{F}_p/\mathbb{Z}_p} \otimes^L_{\mathbb{F}_p} B[1]
\end{array}
$$

and we obtain (2.3). The quasi-isomorphism on the level of the cotangent complex is enough to conclude the statement.

□

Computing $L\widehat{\Omega}_{\mathbb{Z}_p/\mathbb{Z}_p[x]}$.

As we explained previously, we consider, as step zero, the quotient map $\mathbb{Z}_p[x] \longrightarrow \mathbb{Z}_p$, which has kernel generated by the element x, which is not a zero-divisor. We are going to prove the following result.

Theorem 2.3.5. *The derived de Rham algebra $L\widehat{\Omega}_{\mathbb{Z}_p/\mathbb{Z}_p[x]}$ has cohomology concentrated in degree zero, in particular*

$$H^0\left(\frac{L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}}{F^N}\right) \cong \frac{\mathbb{Z}_p\langle x\rangle}{(x)^{[N]}},$$

with Hodge filtration on the left corresponding to the filtration induced by the divided powers on the right[8]*.*

Before giving the proof we need some lemmas. As we anticipated, we are interested in the graded parts of the derived de Rham complex $L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}$.

Lemma 2.3.6. *For any $N \geq 0$, graded parts of $L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N$ are quasi-isomorphic to a complex acyclic but in degree zero, in particular there is an isomorphism of $\mathbb{Z}_p$-modules for $n < N$*

$$H^0\,\mathrm{gr}^n_F\left(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N\right) \cong \mathbb{Z}_p.$$

If $n \geq N$ the cohomology group is trivial.

Proof. The map $\mathbb{Z}_p[x] \xrightarrow{x\mapsto 0} \mathbb{Z}_p$ is surjective. Its kernel equals $p\mathbb{Z}$, i.e. it is generated by a regular element. Thus cotangent complex $L_{\mathbb{Z}_p/\mathbb{Z}_p[x]}$ is quasi-isomorphic to the trivial complex $\dfrac{x\mathbb{Z}_p[x]}{x^2\mathbb{Z}_p[x]}[1]$ concentrated in cohomological degree -1 (see for example [41] Proposition 2.16).

[8] Recall that $(x)^{[n]}$ is the ideal of generators $\{x^{[i]},\ i \geq n\}$ (see [7, §I.3.1]).It is easy to see that in general $x^{[n]}\mathbb{Z}_p\langle x\rangle$ differs from $(x)^{[n]}$. In particular $x^{[n+1]} = (1/n)x^{[1]}x^{[n]}$ may not be in $x^{[n]}\mathbb{Z}_p\langle x\rangle$ (for n not invertible), so that the first module does not define a filtration.

Note that $\dfrac{x\mathbb{Z}_p[x]}{x^2\mathbb{Z}_[x]} \cong \mathbb{Z}_p\overline{x}$ is a free $\mathbb{Z}_p$-module of rank 1 whose generator is the class of x.

We get the following quasi-isomorphisms

$$\mathrm{gr}^n\left(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N\right) \overset{\text{q.}}{\cong} \left(L\wedge^n\left(\frac{x\mathbb{Z}_p[x]}{x^2\mathbb{Z}_p[x]}[1]\right)\right)[-n] \quad (2.4)$$

$$\overset{\text{q.}}{\cong} \left(\left(L\Gamma^n\left(\frac{x\mathbb{Z}_p[x]}{x^2\mathbb{Z}_p[x]}\right)\right)[n]\right)[-n] \quad (2.5)$$

$$\overset{\text{q.}}{\cong} \Gamma^n(\mathbb{Z}_p\overline{x})[0] \quad (2.6)$$

for $n < N$ (otherwise it equals 0). Here (2.4) follows from (2.1.1.5) in [26], (2.5) is an application of Quillen shift formula [25, Ch. I Proposition 4.3.2.1.] and for (2.6) we simply note that a free $\mathbb{Z}_p$-module is Γ^n-acyclic. Thus the right hand side of (2.6) is the trivial complex $\mathbb{Z}_p\gamma_n(\overline{x})$ concentrated in degree zero. □

The following lemma comes quite straightforward from the previous one, but it has an important meaning, since it allows us to consider the complex $L\widehat{\Omega}^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}$ as a classic filtered algebra or ring.

Lemma 2.3.7. *The derived de Rham algebra $L\widehat{\Omega}^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}$ has cohomology concentrated in degree zero.*

Proof. Since every graded piece is concentrated in degree zero, the complex $L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N$ is concentrated in degree zero as well. □

Remark 2.3.8. *Recall §1.2.3, the following maps of cdga, which arise from applying the canonical truncations $t_{[0}, t_{0]}$*

$$t_{0]}L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N \longrightarrow L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N$$

$$t_{0]}L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N \longrightarrow t_{[0}t_{0]}\left(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N\right) = H^0(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N)$$

are quasi-isomorphisms of commutative differential graded algebras. As E_∞-algebras, this means that we have an equivalence $L\Omega_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N \longrightarrow H^0(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N)$.

Remark 2.3.9. *The ring $H^0(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N)$ inherits a filtration from the Hodge filtration on the derived de Rham complex (modulo filtration), for $i < N$*

$$Fil^iH^0(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N) := \frac{F^iZ^0(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N)}{F^iB^0(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N)}$$

$$= \frac{\ker D^0 \cap F^i\left(L\Omega^0_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N\right)}{D^{-1}\left(L\Omega^{-1}_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N\right) \cap F^i\left(L\Omega^0_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N\right)}$$

and $Fil^N H^0(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N) = 0$*.*

Consider the spectral sequence associated to the Hodge filtration

$$E_1^{n,m} = H^{n+m}\left(\mathrm{gr}^n\left(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N\right)\right) \Longrightarrow H^{n+m}\left(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N\right).$$

We have $E_1^{n,m} = 0$ *for* $m+n \neq 0$*, so the sequence degenerates and we have*

$$H^0\left(\mathrm{gr}^n_F\left(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N\right)\right) \cong \mathrm{gr}^n_{Fil} H^0\left(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N\right) \tag{2.7}$$

for $n < N$ *and* 0 *otherwise.*

Next two lemmas plays a central role in the proof of Theorem 2.3.5. We prove first that there is a filtered map from $\mathbb{Z}_p\langle x\rangle/(x)^{[N]}$ and $L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N$, then, once we know that such map induces an isomorphism on the graded parts we can recover an isomorphism on the graded rings by means of Lemma 2.3.1.

Lemma 2.3.10. *For any* $N \geq 0$ *there is a filtered ring morphism*

$$\widetilde{\varphi} : \mathbb{Z}_p\langle x\rangle/(x)^{[N]} \longrightarrow H^0(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N),$$

where on the left we consider filtration induced by the Hodge filtration, while on the right is given by the divided power ideals $(x)^{[n]}$ *for* $n \geq 0$*.*

Proof. There exists a morphism of differential graded algebras (the polynomial ring seen as a complex concentrated in degree zero) $\mathbb{Z}_p[x] \longrightarrow L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N$, which induces a graded morphism in cohomology $H^*(\mathbb{Z}_p[x]) \longrightarrow H^*(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N)$ and the corresponding ring morphism in degree zero gives us a ring morphism

$$\varphi : \mathbb{Z}_p[x] \longrightarrow H^0(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N).$$

Thus we can consider $H^0(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N)$ as a $\mathbb{Z}_p[x]$-algebra.

There exists a lifting of φ to

$$\widetilde{\varphi} : \mathbb{Z}_p\langle x\rangle \longrightarrow H^0(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N). \tag{2.8}$$

such that $\widetilde{\varphi}(x^{[n]}) = \varphi(x)^n/n!$ (see within the [41, proof of Proposition 3.17], but it can also be deduced by the previous remark about the shuffle product).

Finally we show that the ring morphism (2.8) induces a filtered ring morphism. Let $\kappa : L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N \longrightarrow \mathbb{Z}_p = gr^0_F(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N)$ be the augmentation map. Then, since $H^0(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N)$ is a $\mathbb{Z}_p[x]$-algebra, there is a commutative diagram of rings morphisms

$$\begin{array}{ccc} H^0(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N) & \xrightarrow{\kappa} & \mathbb{Z}_p \\ & {\scriptstyle\varphi}\nwarrow \quad \nearrow{\scriptstyle\varphi_0} & \\ & \mathbb{Z}_p[x] & \end{array}.$$

In particular $\varphi(x) \in \ker \kappa = Fil^1$. Further

- $\varphi(x)^n \in Fil^n$, for $n \leq N$, since the product on $H^0(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N)$ is graduated;
- $\varphi(x)^n = n!\widetilde{\varphi}(x^{[n]}) \in Fil^n$, by definition;
- take $i \leq n$ to be the minimum integer such that $\widetilde{\varphi}(x^{[n]}) \in Fil^i$;
- suppose $i < n$,
- then the class of $\widetilde{\varphi}(x^{[n]})$ in gr^i_{Fil} is non-zero;
- but $n!\widetilde{\varphi}(x^{[n]}) \in Fil^n$, thus its class in gr^i_{Fil} is zero;

This is impossible, since each graded part is $\mathbb{Z}_p$-free, hence torsion free (Lemma 2.3.6). Then $\widetilde{\varphi}(x^{[n]}) \in Fil^n$. In particular the map $\widetilde{\varphi}$ respects filtrations. □

Again note that a crucial point of the proof is the fact that, since we are working on the map $\mathbb{Z}_p[x] \longrightarrow \mathbb{Z}_p$, we are avoiding p-torsion problems.

Lemma 2.3.11. *The map $\widetilde{\varphi}$ induces an isomorphism on the graded parts.*

Proof. We prove that the class of $\widetilde{\varphi}(x^{[n]})$ in gr^n_{Fil} is a generator for $0 \leq n \leq N$. Suppose this is not the case, i.e. there exists $n \geq 1$ such that the map

$$\mathrm{gr}^n \widetilde{\varphi} : (x)^{[n]}/(x)^{[n+1]} \longrightarrow \mathrm{gr}^n_{Fil}(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N)) \cong \mathbb{Z}_p\gamma_n(\overline{x})$$

sends the generator $x^{[n]}$ within $p\mathbb{Z}_p\gamma_n(\overline{x})$. If we tensorize by $\otimes^L_{\mathbb{Z}_p}\mathbb{F}_p$ we get the $0-$map. But $(\mathrm{gr}^n \widetilde{\varphi}) \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p = \mathrm{gr}^n(\widetilde{\varphi} \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p)$ and it is easy to see that $\widetilde{\varphi} \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p : \mathbb{F}_p\langle x\rangle \longrightarrow L\Omega^*_{\mathbb{F}_p/\mathbb{F}_p[x]}$ is the isomorphism of [10, Lemma 3.29]. So $(\mathrm{gr}^n \widetilde{\varphi}) \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p$ should be an isomorphism as well, which is absurd. □

Proof. (Theorem 2.3.5) . We recollect all previous results. Each $L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N$ has cohomology concentrated in degree zero as their graded parts have (Lemma 2.3.7 and Lemma 2.3.6). This will provide an equivalence $L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N \longrightarrow H^0(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N)$ as $E_\infty-$algebras (remark 2.3.8). Then we compute $H^0(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N)$ as a filtered ring and we get the isomorphism $H^0(L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N) \cong \mathbb{Z}_p\langle x\rangle/(x)^{[N]}$ from its graded parts (Lemma 2.3.1). □

Computing $L\widehat{\Omega}_{\mathbb{F}_p/\mathbb{Z}_p}$.

In the previous section we computed the Hodge completed derived de Rham algebra $L\widehat{\Omega}^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}$. We want to exploit such result in order to compute the same object relative to the morphism $\mathbb{Z}_p \longrightarrow \mathbb{F}_p$. The following lemma will show us the connection between the two cases, it is just a base change result.

Lemma 2.3.12. *Consider the rings maps $\varphi_p : \mathbb{Z}_p[x] \overset{x\mapsto p}{\longrightarrow} \mathbb{Z}_p$ and $\varphi_0 : \mathbb{Z}_p[x] \overset{x\mapsto 0}{\longrightarrow} \mathbb{Z}_p$, which give two different $\mathbb{Z}_p[x]$-algebra structure to $\mathbb{Z}_p$, then*

$$L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N \otimes^L_{\mathbb{Z}_p[x],\varphi_p} \mathbb{Z}_p \simeq L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N$$

Proof. We call the different structures on $\mathbb{Z}_p$ φ_p-structure and φ_0-structure respectively. See that the tensor product over the ring $\mathbb{Z}_p[x]$ of the two algebras is isomorphic to $\mathbb{F}_p$. Furthermore consider the following exact sequence (which is a free $\mathbb{Z}_p[x]$-resolution of $\mathbb{Z}_p$ as φ_0-module)

$$0 \longrightarrow \mathbb{Z}_p[x] \xrightarrow{x} \mathbb{Z}_p[x] \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

and tensorize it with the φ_p-module $\mathbb{Z}_p$. We get the sequence

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \longrightarrow \mathbb{F}_p \longrightarrow 0,$$

which is again an exact sequence. This means that the two algebras are Tor-independent. Therefore, by base change, $L\widehat{\Omega}^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]} \otimes^L_{\mathbb{Z}_p[x]} \mathbb{Z}_p \simeq L\widehat{\Omega}^*_{\mathbb{F}_p/\mathbb{Z}_p}$ as projective systems of differential graded algebras. The same holds for each term of the system: $L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N \otimes^L_{\mathbb{Z}_p[x]} \mathbb{Z}_p \simeq L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N$. □

Finally we are able to compute the Hodge completed derived de Rham complex for the map 2.2.

Theorem 2.3.13. *For any $N \geq 0$ there is a quasi-isomorphism of commutative differential graded algebras*

$$L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N \simeq \frac{\mathbb{Z}_p\langle x\rangle}{(x)^{[N]}} \xrightarrow{\cdot(x-p)} \frac{\mathbb{Z}_p\langle x\rangle}{(x)^{[N]}}, \tag{2.9}$$

where the right hand side is in degree -1 and 0. Moreover $\cdot(x-p)$ is an injective map and $L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N$ has cohomology concentrated in degree zero.

Proof. As we saw before $L\widehat{\Omega}^*_{\mathbb{F}_p/\mathbb{Z}_p} \simeq L\widehat{\Omega}^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]} \otimes^L_{\mathbb{Z}_p[x]} \mathbb{Z}_p$. We replace $\mathbb{Z}_p$ with a $\mathbb{Z}_p[x]$-free resolution in order to compute $\otimes^L_{\mathbb{Z}_p[x]}$ (Recall example 1.2.13 and the fact that a free resolution of an A-module B is a complex of free A-modules P^* acyclic but in degree zero, where its cohomology is isomorphic to B). See that the cokernel of the map $\cdot(x-p)$ is exactly the $(x \mapsto p)$-module $\mathbb{Z}_p$, thus $\mathbb{Z}_p \simeq (\mathbb{Z}_p[x] \xrightarrow{\cdot(x-p)} \mathbb{Z}_p[x])$. Then, by Proposition 2.3.5,

$$\begin{aligned}
L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N &\simeq L\Omega^*_{\mathbb{Z}_p/\mathbb{Z}_p[x]}/F^N \otimes^L_{\mathbb{Z}_p[x]} \mathbb{Z}_p \\
&\simeq \mathbb{Z}_p\langle x\rangle/(x)^{[N]} \otimes_{\mathbb{Z}_p[x]} (\mathbb{Z}_p[x] \xrightarrow{\cdot(x-p)} \mathbb{Z}_p[x]) \\
&\simeq (\mathbb{Z}_p\langle x\rangle/(x)^{[N]} \xrightarrow{\cdot(x-p)} \mathbb{Z}_p\langle x\rangle/(x)^{[N]}).
\end{aligned}$$

We proved the first part of the statement. In order to prove that $H^0(L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N)$ is the cokernel of the map (2.9), we need to prove that the latter is injective.
Let $\omega = \sum_{i=0}^{N-1} a_i x^{[i]}$ be an element of $\mathbb{Z}_p\langle x\rangle/(x)^{[N]}$ (it is easy to see that $1, x, x^{[2]}, ..., x^{[N-1]}$ provide a $\mathbb{Z}_p$-basis). Then

$$(x-p)\cdot\omega = \underbrace{Na_{N-1}x^{[N]}}_{=0} + \sum_{i=1}^{N-1}(ia_{i-1} - pa_i)x^{[i]} - pa_0.$$

Suppose $(x-p)\cdot\omega = 0$, then $pa_0 = 0$ and $ia_{i-1} - pa_i = 0$ for $1 \le i \le N-1$, since they are a $\mathbb{Z}_p$-linear combination equals to zero. The first equality implies $a_0 = 0$, since $\mathbb{Z}_p\langle x\rangle/(x)^{[N]}$ has no $\mathbb{Z}_p$-torsion. By induction on i, we get $0 = pa_i$ for $1 \le i \le N-1$, so that $a_i = 0$. To sum up, for all $i = 0, ..., N-1$ we have $a_i = 0$. Thus (2.9) is injective. □

We are going to extend such result to a wider class of maps, see Section 2.3.2 and Chapter 4. In what follows we describe $H^0(L\widehat{\Omega}_{\mathbb{F}_p/\mathbb{Z}_p})$. The next lemma is due to Bhatt ([10, Corollary 8.6]), who computed the cokernel of the map $\mathbb{Z}_p\langle x\rangle \xrightarrow{\cdot(x-p)} \mathbb{Z}_p\langle x\rangle$ as $\mathbb{Z}_p$-module, which is closed to our case.

Lemma 2.3.14 (Bhatt)**.** *There is a short exact sequence*

$$0 \longrightarrow \mathbb{Z}_p\langle x\rangle \xrightarrow{\cdot(x-p)} \mathbb{Z}_p\langle x\rangle \xrightarrow{f} \mathbb{Z}_p \oplus \bigoplus_{i\ge 1} \mathbb{Z}_p/i\mathbb{Z}_p \longrightarrow 0 \tag{2.10}$$

where the map of $\mathbb{Z}_p$-modules $f : \mathbb{Z}_p\langle x\rangle \longrightarrow \mathbb{Z}_p \oplus \bigoplus_{i\ge1} \mathbb{Z}_p/i\mathbb{Z}_p$ is defined as

$$\sum_{i=0}^{n} a_i (x-p)^{[i]} \longmapsto (a_0; \quad (a_i \mod i)_{i\ge1}).$$

Proof. It is easy to see that the set $\{(x-p)^{[i]}\}_{i\ge0}$ is a basis for the free $\mathbb{Z}_p$-module $\mathbb{Z}_p\langle x\rangle$.

The map $\cdot(x-p)$ is injective. Take any element $\sum_{i=0}^{n} a_i(x-p)^{[i]} \in \mathbb{Z}_p\langle x\rangle$, then

$$(x-p)\sum_{i=0}^{n} a_i(x-p)^{[i]} = \sum_{i=0}^{n} a_i(i+1)(x-p)^{[i+1]}$$

and if it equals zero, then $a_i(i+1) = 0$ for $i = 0, ..., n$, since it is a $\mathbb{Z}_p$-linear combination of elements of the basis. But $\mathbb{Z}_p$ is p-torsion free, so that $a_i = 0$ for $i = 0, ..., n$. Hence the map $\cdot(x-p)$ is injective.

The map f is surjective. Take any $a_0 \in \mathbb{Z}_p$ and any finite sequence $(\overline{a_1}, ..., \overline{a_n}) \in \bigoplus \mathbb{Z}_p/i\mathbb{Z}_p$. Chose any lifting a_i of $\overline{a_i}$ in $\mathbb{Z}_p$. Then $\sum_{i=0}^{n} a_i(x-p)^{[i]}$ is sent to $(a_0; \overline{a_1}, ..., \overline{a_n}, 0, ...)$.

We have the inclusion $\operatorname{Im}\cdot(x-p) \subseteq \ker f$ as

$$\begin{aligned} f\left((x-p)\sum_{i=0}^{n} a_i(x-p)^{[i]}\right) &= f\left(\sum_{i=0}^{n} a_i(i+1)(x-p)^{[i+1]}\right) \\ &= (0; (a_i(i+1) \mod i+1)) = (0; (0)). \end{aligned}$$

On the other hand $\ker f \subseteq \operatorname{Im}\cdot(x-p)$. Suppose $f\left(\sum_{i=0}^{n} a_i(x-p)^{[i]}\right) = (0; (0))$. Then $a_0 = 0$ and $a_i \equiv_i 0$ for $i = 1, ..., n$. Hence there exist $b_1, ..., b_n \in \mathbb{Z}_p$, such that

$$\sum_{i=0}^{n} a_i(x-p)^{[i]} = \sum_{i=1}^{n} ib_i(x-p)^{[i]} = (x-p)\sum_{i=0}^{n-1} b_{i+1}(x-p)^{[i]}.$$

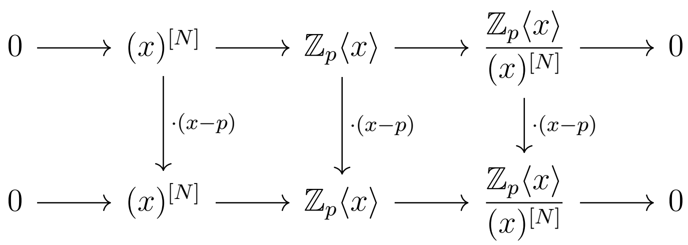
All the things we said prove that $\mathbb{Z}_p \oplus \bigoplus_{i\geq 1} \mathbb{Z}_p/i\mathbb{Z}_p$ is the cokernel of $\mathbb{Z}_p\langle x\rangle \xrightarrow{\cdot(x-p)} \mathbb{Z}_p\langle x\rangle$ via the map f. $\square$

Theorem 2.3.15. *For any $N \geq 0$ the complex $L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N$ cohomologically concentrated in degree zero and $H^0(L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N)$ is isomorphic as $\mathbb{Z}_p$-module to the quotient of the $\mathbb{Z}_p$-module $\mathbb{Z}_p \oplus \bigoplus_{i\geq 1} \mathbb{Z}_p/i\mathbb{Z}_p$ by the sub-module I generated by elements of the form*

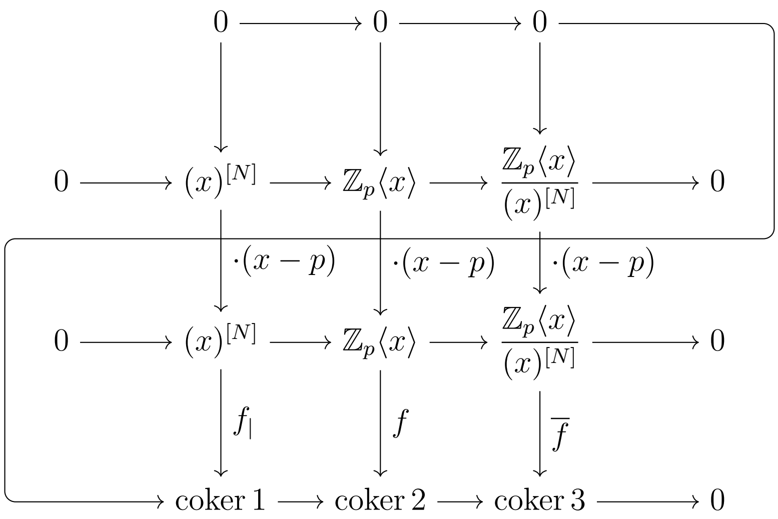
$$f(x^{[n]}) = \left(p^{[n]}; (p^{[n-i]} \mod i)_{i=1}^{n}, 0, ...\right) \tag{2.11}$$

for $n \geq N$.

Proof. By Lemma 2.3.13 we need to compute the cokernel of the map (2.9). Consider the diagram



Recall remark 2.3.1, so that all vertical maps are injective. Apply snake Lemma and we get the following diagram



Recall that by Lemma 2.3.14 we know coker 2 of the middle sequence. The exact sequence of cokernels shows that coker 3 is the quotient coker 2/ coker 1. Therefore it remains to describe the sub-$\mathbb{Z}_p$-module coker 1, but this is generated by the image via f of the ideal $(x)^{[N]}$. The generators of such ideal are of the form $x^{[n]}$ for $n \geq N$. Then $x^{[n]} = \sum_{i=0}^{n} p^{[n-i]}(x-p)^{[i]}$ and $f(x^{[n]})$ is of the form (2.11). $\square$

Remark 2.3.16. *The description of* coker 1 *in the last lemma is complex. The main problem is that it seems impossible to arrange the generators in order to provide an* *equivalence relation on each $\mathbb{Z}_p/i\mathbb{Z}_p$ independently. For example, for any integer $k \geq 1$,*

$$f(x^{[p^k]}) = (p^{[p^k]}; 0, ..., 0, 1 \mod p^k, 0, ...) \quad .$$

Remark 2.3.17. *Let us consider the projection*

$$\pi : \mathbb{Z}_p \oplus \bigoplus_{i\geq 1} \frac{\mathbb{Z}_p}{i\mathbb{Z}_p} \longrightarrow \frac{\mathbb{Z}_p}{(p\mathbb{Z}_p)^{[N]}} \oplus \bigoplus_{i=1}^{N} \frac{\mathbb{Z}_p/i\mathbb{Z}_p}{(p\mathbb{Z}_p)^{[N-i]}}.$$

If we recall the sequence of cokernels in Theorem 2.3.15, we have that $I = \operatorname{coker}\left((x)^{[N]} \xrightarrow{\cdot(x-p)} (x)^{[N]}\right)$ is sent to zero by π. Thus there is a unique factorization of π through $\operatorname{coker} 3 = H^0(L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N)$, in particular we have the following diagram

$$\begin{array}{ccc} I & & \\ \big\downarrow & & \\ \mathbb{Z}_p \oplus \bigoplus_{i\geq 1} \mathbb{Z}_p/i\mathbb{Z}_p & \xrightarrow{\pi} & \frac{\mathbb{Z}_p}{(p\mathbb{Z}_p)^{[N]}} \oplus \bigoplus_{i=1}^{N} \frac{\mathbb{Z}_p/i\mathbb{Z}_p}{(p\mathbb{Z}_p)^{[N-i]}} \\ \big\downarrow & \nearrow_{\overline{\pi}} & \\ H^0(L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N) & & \end{array}$$

Note that the map $\overline{\pi}$ is a map of finite abelian groups.

Computations as pro-objects.

We have already seen that the Hodge completed derived de Rham complex has a pro-complex structure, starting from diagram (2.3.17), we want to define some new pro-objects and to confront them with $L\widehat{\Omega}_{\mathbb{F}_p/\mathbb{Z}_p} = \left(H^0(L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N)\right)_{N\in\mathbb{N}}$.

Lemma 2.3.18. *Let $\mathcal{J}$ be the set of all sub-$\mathbb{Z}_p$-modules $J \subseteq \mathbb{Z}_p \oplus \bigoplus_{i\geq 1} \mathbb{Z}_p/i\mathbb{Z}_p$ such that the quotient $\left(\mathbb{Z}_p \oplus \bigoplus_{i\geq 1} \mathbb{Z}_p/i\mathbb{Z}_p\right)/J$ is a finite abelian group. Consider the associated pre-order category such that $J' \to J$ if and only if $J' \subseteq J$. Then $\mathcal{J}$ is cofiltered.*

Proof. We want to show that given two ideals $J, J' \in \mathcal{J}$, there exists an ideal $K \in \mathcal{J}$ such that $K \to J$ and $K \to J'$, i.e. $K \subseteq J, J'$. We would like to chose $K = J \cap J'$, but we need to prove that it still belongs to $\mathcal{J}$. Suppose it is not, that is $\left(\mathbb{Z}_p \oplus \bigoplus_{i\geq 1} \mathbb{Z}_p/i\mathbb{Z}_p\right)/J \cap J'$ is non finite. This cannot be possible since, it should injects into the finite order module $\left(\left(\mathbb{Z}_p \oplus \bigoplus_{i\geq 1} \mathbb{Z}_p/i\mathbb{Z}_p\right)/J\right) \oplus \left(\left(\mathbb{Z}_p \oplus \bigoplus_{i\geq 1} \mathbb{Z}_p/i\mathbb{Z}_p\right)/J'\right)$. □

Remark 2.3.19. *Note that the cofiltering $\mathcal{J}$ has no initial object, in particular $(0) \notin \mathcal{J}$.*

Let $\widehat{\oplus \mathbb{Z}_p/i} = \left(\mathbb{Z}_p \oplus \bigoplus_{i\geq 1} \mathbb{Z}_p/i\mathbb{Z}_p/J\right)_{J\in\mathcal{J}}$ be the pro-object associated to the cofiltering of the previous lemma. Note that its completion is a profinite group. We want the previous diagram (2.3.17) of projections to induce a triangle of pro-objects.

Proposition 2.3.20. *There exists a commutative diagram of pro-$\mathbb{Z}_p$-modules*

$$\begin{array}{ccc} & \widehat{\oplus \mathbb{Z}_p/i} & \\ \swarrow & & \searrow \\ \left(H^0(L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N)\right)_N & \longrightarrow & \left(\frac{\mathbb{Z}_p}{(p\mathbb{Z}_p)^{[N]}} \oplus \bigoplus_{i=1}^N \frac{\mathbb{Z}_p/i\mathbb{Z}_p}{(p\mathbb{Z}_p)^{[N-i]}}\right)_N \end{array}.$$

Proof. Firstly, for any integer $N \geq 0$, the maps

$$\overline{\pi} : H^0(L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N) \longrightarrow \frac{\mathbb{Z}_p}{(p\mathbb{Z}_p)^{[N]}} \oplus \bigoplus_{i=1}^{N} \frac{\mathbb{Z}_p/i\mathbb{Z}_p}{(p\mathbb{Z}_p)^{[N-i]}} \tag{2.12}$$

are maps of projective systems, hence they induce a map of pro-objects. On the other hand, the other two maps are some finite quotients of $\mathbb{Z}_p \oplus \bigoplus_{i\geq 1} \mathbb{Z}_p/i\mathbb{Z}_p$, thus there are two natural maps

$$\begin{array}{ccc} & \mathbb{Z}_p \oplus \bigoplus_{i\geq 1} \mathbb{Z}_p/i\mathbb{Z}_p & \\ \swarrow\!\!\swarrow & & \searrow\!\!\searrow \\ \left(H^0(L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N)\right)_N & & \left(\frac{\mathbb{Z}_p}{(p\mathbb{Z}_p)^{[N]}} \oplus \bigoplus_{i=1}^N \frac{\mathbb{Z}_p/i\mathbb{Z}_p}{(p\mathbb{Z}_p)^{[N-i]}}\right)_N \end{array}$$

which induce, by the universal property of the pro-completion, two natural maps of pro-finite groups

$$\begin{array}{ccc} & \widehat{\oplus \mathbb{Z}_p/i} & \\ \swarrow & & \searrow \\ \left(H^0(L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N)\right)_N & & \left(\frac{\mathbb{Z}_p}{(p\mathbb{Z}_p)^{[N]}} \oplus \bigoplus_{i=1}^N \frac{\mathbb{Z}_p/i\mathbb{Z}_p}{(p\mathbb{Z}_p)^{[N-i]}}\right)_N \end{array}.$$

Finally, this diagram commutes with the pro-morphism induced by (2.12). □

Computations as finite group.

Lemma 2.3.21. *For any $N \geq 0$, graded parts of $L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N$ are quasi-isomorphic to a complex acyclic but in degree zero, in particular there is an isomorphism of $\mathbb{F}_p$-modules*

$$H^0 \operatorname{gr}^n_F \left(L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N \right) \cong \mathbb{F}_p.$$

if $n < N$, and 0 else.

Proof. The argument is the same of Lemma 2.3.6, since $\mathbb{Z}_p \longrightarrow \mathbb{F}_p$ is a surjective morphism with kernel equal to $p\mathbb{Z}$, i.e. generated by a regular element. The cotangent complex will be $L_{\mathbb{F}_p/\mathbb{Z}} \cong p\mathbb{Z}/p^2\mathbb{Z}[1] \cong \mathbb{F}_p[1]$, i.e. a free $\mathbb{F}_p$-module concentrated in degree 1. Thus we get

$$H^0 \left(\operatorname{gr}^n \left(L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N \right) \right) \cong \mathbb{F}_p \gamma_n(\overline{p}) \tag{2.13}$$

if $n < N$, and 0 else. In particular, since every graded pieces is concentrated in degree zero, the derived de Rham complex is concentrated in degree (as we already know). $\square$

Proposition 2.3.22. *$H^0(L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^N)$ is an abelian group of order p^N for any $N > 0$.*

Proof. We prove this by induction. We have $H^0(L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}/F^1) = H^0(\operatorname{gr}^0 L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}) \cong \mathbb{F}_p$, so we may assume that the statement holds for $N > 1$. Consider the following short exact sequence induced in cohomology by the Hodge filtration

$$0 \longrightarrow H^0 \left(\frac{F^N L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}}{F^{N+1}} \right) \longrightarrow H^0 \left(\frac{L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}}{F^{N+1}} \right) \longrightarrow H^0 \left(\frac{L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}}{F^N} \right) \longrightarrow 0. \tag{2.14}$$

By (2.3.21) the first group has order p. The third group is of order p^N by assumption. Thus the one in the middle must have order p^{N+1}, which proves our claim. $\square$

Computation for $N < p$

We can attribute the computation problems arising in remark 2.3.16 to the fact that the map $v_p(p^N/N!)$ is not monotone. This is due to the fact that, considering $k \in \mathbb{Z}$, the $p-$adic valuation of p^k grows linearly each step by $+1$, while the $p-$adic valuation of $k!$ is constant for k not dividing p and grows by $+h$ when $k = p^h$ (see Figure 2.3). As a matter of fact for $N < p$ the map is still monotone, hence we can make some easy computations.

Remark 2.3.23. *Let us consider a generic element of $\mathbb{Z}_p\langle x\rangle/x^{[N]}$, $\omega = \sum_{i=0}^{N-1} a_i\gamma_i(x)$, with $a_i \in \mathbb{Z}_p$, and we compute the formal division by $x - p$, we get*

- *quotient* $Q = \sum_{i=1}^{N-1} \left(\sum_{j=i}^{N-1} \frac{p^{j-i}(i-1)!}{j!} a_j \right) \gamma_{i-1}(x)$ *and*
- *remainder* $R = \sum_{i=0}^{N-1} \frac{p^i}{i!} a_i$.

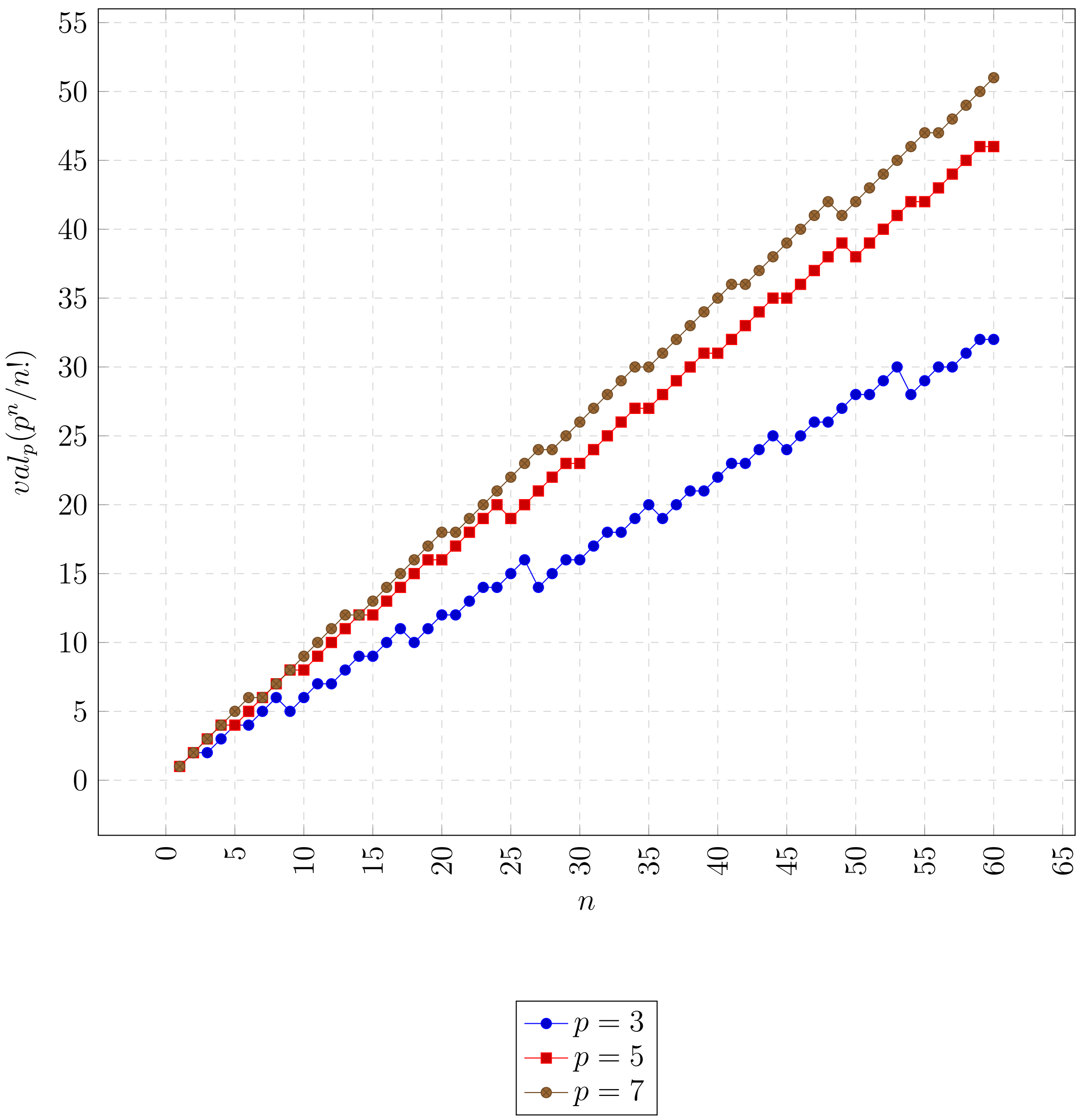


Figure 2.3: Note that the growth is by +1 in general, it is 0 in correspondence of exact multiples of p, -1 for exact multiples of p^2 and so on.

The remainder is well defined as a p-adic integer ($p > 2$), the quotient on the other hand may not be defined for i divided by a power of p, since the internal sum presents the addendum a_i/i.

Proposition 2.3.24. *For $N < p$ there is an isomorphism of rings*

$$H^0\left(\frac{L\Omega^*_{\mathbb{F}_p/\mathbb{Z}_p}}{F^N}\right) \cong \frac{\mathbb{Z}_p}{p^N\mathbb{Z}_p}.$$

Proof. Recall Theorem 2.3.13, so that we just need to compute the cokernel of the map

$$\frac{\mathbb{Z}_p\langle x\rangle}{(x)^{[N]}} \xrightarrow{\cdot(x-p)} \frac{\mathbb{Z}_p\langle x\rangle}{(x)^{[N]}}.$$

We want to prove that the evaluation map

$$\begin{aligned} ev : \frac{\mathbb{Z}_p\langle x\rangle}{(x)^{[N]}} &\longrightarrow \frac{\mathbb{Z}_p}{p^N\mathbb{Z}_p} \\ x &\longmapsto p \\ \gamma_i(x) &\longmapsto p^i/i! \end{aligned}$$

defines the cokernel map we are looking for. The map is clearly surjective, so we need to show that its kernel equals the image of the multiplication by $(x-p)$-map. Clearly any element of the form $(x-p)\omega$ is sent to zero by ev. Note that, given an element $\omega = \sum_{i=0}^{N-1} a_i\gamma_i(x)$ in $\mathbb{Z}_p\langle x\rangle/x^{[N]}$, $ev(\omega) = \sum_{i=0}^{N-1} \frac{p^i}{i!}a_i$, so that any ω in the kernel of ev, must have $\sum_{i=0}^{N-1} \frac{p^i}{i!}a_i = 0$. By the previous remark, we can write it as $\omega = R + (x-p)Q$ ($p^{j-i}(i-1)!/j!$ is a p-adic integer for $1 \le i \le j \le N-1 < p$). and, if $\omega \in \ker ev$, $R = 0$, hence $\omega = (x-p)Q$ is in the image of the multiplication map. $\square$

2.3.2 The case of perfect rings

Given a perfect $\mathbb{F}_p$-algebra k, there is the following diagram of rings

$$\begin{array}{ccc} W & \longrightarrow & k = W \otimes_{\mathbb{Z}_p} \mathbb{F}_p \\ \uparrow & & \uparrow \\ \mathbb{Z}_p & \longrightarrow & \mathbb{F}_p \end{array},$$

where $W = W(k)$. Thus we can apply the base change property (see lemma 2.3.5) once we proved the Tor-independence.

Lemma 2.3.25. *The $\mathbb{Z}_p$-algebras $\mathbb{F}_p$ and W are* Tor*-independent.*

Proof. Consider the following exact sequence, coming from the free resolution of $\mathbb{F}_p$, as $\mathbb{Z}_p$-module,

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{\cdot p} \mathbb{Z}_p \longrightarrow \mathbb{F}_p \longrightarrow 0.$$

If we tensorize by $\otimes_{\mathbb{Z}_p} W$, we get

$$0 \longrightarrow W \xrightarrow{\cdot p} W \longrightarrow k \longrightarrow 0. \qquad (*)$$

It is again an exact sequence, so that $0 = H^i(*) = \mathrm{Tor}^i_W(k, \mathbb{F}_p)$, for any $i > 0$, which proves the Tor-independence. $\square$

Theorem 2.3.26. *Let k a perfect ring and $W = W(k)$ its ring of Witt vectors. For any $N \ge 0$ there is an equivalence*

$$L\Omega^*_{k/\mathbb{Z}}/F^N \simeq L\Omega^*_{k/W}/F^N \simeq \frac{W\langle x\rangle}{(x)^{[N]}} \xrightarrow{\cdot(x-p)} \frac{W\langle x\rangle}{(x)^{[N]}}, \qquad (2.15)$$

where the right hand side is in degree -1 and 0. In particular $L\Omega^*_{k/Z}/F^N$ has cohomology concentrated in degree zero.

Proof. The first equivalence is due to Lemma 2.3.4 and Lemma 2.3.2. The canonical base-change map is induced by the fact that, since k is perfect, $k = \mathbb{F}_p \otimes_{\mathbb{Z}_p} W$, furthermore $\mathrm{Tor}^i_{\mathbb{Z}_p}(\mathbb{F}_p, W) = 0$ for $i > 0$ by Lemma 2.3.25. Hence we can apply the base change lemma and we get the following equivalences

$$\begin{aligned} L\widehat{\Omega}^*_{k/W} &= L\widehat{\Omega}^*_{W\otimes_{\mathbb{Z}_p}\mathbb{F}_p/W\otimes_{\mathbb{Z}_p}\mathbb{Z}_p} \\ &\simeq L\widehat{\Omega}^*_{\mathbb{F}_p/\mathbb{Z}_p} \otimes^L_{\mathbb{Z}_p} W \qquad (2.16) \\ &\simeq \left(\frac{\mathbb{Z}_p\langle x\rangle}{(x)^{[N]}} \xrightarrow{\cdot(x-p)} \frac{\mathbb{Z}_p\langle x\rangle}{(x)^{[N]}} \right) \otimes^L_{\mathbb{Z}_p} W \qquad (2.17) \\ &\simeq \left(\frac{W\langle x\rangle}{(x)^{[N]}} \xrightarrow{\cdot(x-p)} \frac{W\langle x\rangle}{(x)^{[N]}} \right). \end{aligned}$$

See that in (2.16) we applied Theorem 2.3.13. Moreover the derived tensor product in (2.17) is a standard tensor product, since $L\widehat{\Omega}^*_{\mathbb{F}_p/\mathbb{Z}_p}$ is replaced by a complex of $\mathbb{Z}_p$-free modules. □

Chapter 3

Künneth Formula for pro-complexes

Künneth formula is a classical result which relates the (co)homology of two objects to the (co)homology of their product. In particular, in the de Rham context, we have that the cohomology of the product of two smooth varieties is isomorphic to the tensor product of the cohomology of the single ones (see for example [42, Section 0FM9]). In the same spirit there are Künneth formulas holding in the derived case. Bhatt gives a sketch of the proof in [10, Proposition 2.7] for the (non completed) derived de Rham complex as well as for the p-adic completed derived de Rham complex in [10, Proposition 8.3(3)]. Further a Hodge-completed version seems to be generally known to the experts (see Introduction of [9] or [2, Proposition 6.8]), although we haven't found a specific reference for the proof. What is seems to miss is a Künnet formula for the Hodge completed derived de Rham complex seen as pro-complex. In this chapter we give a detailed proof of this result, which could be very useful for many computations.

3.1 Künneth Formula

We want to prove the Künneth formula for the Hodge completed derived de Rham complex seen as pro-complex, which is an improvement of [10, Proposition 2.7]. As we said, we want to prove that the Hodge completed derived de Rham complex of the tensor product of algebras is *isomorphic* to the tensor product of the complexes of the single algebras as pro-objects. Such isomorphism is compatible with the structure of commutative differential graded algebras. As a matter of fact we prove a more specific result: there is an isomorphism of pro-functors

$$\left(\frac{\Omega^*_{-\otimes_A -/A}}{F^L}\right)_{L\in\mathbb{N}} \longrightarrow \left(\frac{\Omega^*_{-/A}}{F^N} \otimes_A \frac{\Omega^*_{-/A}}{F^M}\right)_{(M,N)\in\mathbb{N}^2}.$$

We then apply this result to standard resolutions of algebras, seen as functors $\Delta^{op} \longrightarrow \underline{\mathrm{FAlg}}_A$.

Remark 3.1.1. *It is important to notice the fact that the statement does not hold if we remain in the category of projective systems of complexes, in particular there is no isomorphism of the form*

$$\frac{\Omega^*_{B\otimes_A C/A}}{F^L} \longrightarrow \frac{\Omega^*_{B/A}}{F^N} \otimes_A \frac{\Omega^*_{C/A}}{F^M}.$$

As a matter of fact the isomorphism is realized once you take the completion over the Hodge filtration. In the context of the pro-categories we are then able to compromise between this two facts.

Proposition 3.1.2. *Given two rings morphisms $A \longrightarrow B$, $A \longrightarrow C$, consider the (standard) resolutions $P_\bullet \longrightarrow B$, $Q_\bullet \longrightarrow C$. There is an isomorphism of pro-bisimplicial commutative differential graded A-algebras*

$$\left(\frac{\Omega^*_{P_\bullet \otimes_A Q_\bullet/A}}{F^L}\right)_{L\in\mathbb{N}} \cong \left(\frac{\Omega^*_{P_\bullet/A}}{F^N}\right)_{N\in\mathbb{N}} \otimes_A \left(\frac{\Omega^*_{Q_\bullet/A}}{F^M}\right)_{M\in\mathbb{N}}.$$

We start working in a non-derived context, in the case of polynomials rings. Then we construct two maps of projective objects and we show that they are one inverse of the other (in the pro-category). Finally we apply such results to prove the main one. We choose to display the proof by splitting it in some lemmas.

Lemma 3.1.3 (Polynomials)**.** *Given two free A-algebras B and C, there is a natural isomorphism of commutative differential graded algebras*

$$\Omega^*_{B\otimes_A C/A} \longrightarrow \Omega^*_{B/A} \otimes_A \Omega^*_{C/A}.$$

Proof. We first consider the case where B and C are finite free A-algebras, $B = A[T_1, ..., T_N]$ and $C = A[T_{N+1}, ..., T_{N+M}]$. We now define a morphism of $B \otimes_A C$-modules

$$\bigoplus_{p+q=n} \Omega^p_{B/A} \otimes_A \Omega^q_{C/A} \longrightarrow \Omega^n_{B\otimes_A C/A}$$

as follows

$$\sum_{\substack{I=(i_1,...,i_p)\\ 1\leq i_1<...<i_p\leq N}} b_I\ dT_I \otimes \sum_{\substack{J=(j_1,...,j_q)\\ 1\leq j_1<...<j_q\leq N+M}} c_J\ dT_J \longmapsto \sum_{I,J} (b_I \otimes c_J) dT_J \wedge dT_J$$

where $\sum_{\substack{I=(i_1,...,i_p)\\ 1\leq i_1<...<i_p\leq n}} a_I\ dT_I := \sum_{\substack{I=(i_1,...,i_p)\\ 1\leq i_1<...<i_p\leq n}} a_I\ dT_{i_1} \wedge ... \wedge dT_{i_p}$.

Such map is functorial (it is induced by the universal property of the tensor product) and it is an isomorphism of $B \otimes_A C$-modules (it is clearly surjective and injective, since we are considering polynomial algebras)[1], thus of A-modules. In particular, since it is compatible with the differential, it yields an isomorphism of complex of A-modules (recall that the differential is an A-linear map)

$$\Omega^*_{B\otimes_A C/A} \longrightarrow \Omega^*_{B/A} \otimes_A \Omega^*_{C/A}. \tag{3.1}$$

[1]We can also provide an inverse which works as

$$\Omega^n_{B\otimes_A C/A} \longrightarrow \bigoplus_{p+q=n} \Omega^p_{B/A} \otimes_A \Omega^q_{C/A}$$
$$dT_{i_1} \wedge ... \wedge dT_{i_n} \longmapsto dT_{i_1} \wedge ... \wedge dT_{i_p} \otimes dT_{i_{p+1}} \wedge ... \wedge dT_{i_n},$$

with $1 \leq i_p \leq N$ and $N+1 \leq i_{p+1} \leq N+M$.

Moreover, since we are just playing with wedge products, it is easy to see that such isomorphism is compatible with the differential graded structures on both sides[2]. □

Remark 3.1.4. *The previous statement holds also for arbitrary free A-algebras, by using the fact that differentials commute with direct limits.*

Remark 3.1.5. *The functoriality of (3.1) allows us to view it as an isomorphism of functors from* $\underline{\mathrm{FAlg}}_A \times \underline{\mathrm{FAlg}}_A$ *to the category of complex of A-modules (in particular A-cdga),*

$$\Omega^*_{-\otimes_A -/A} \longrightarrow \Omega^*_{-/A} \otimes_A \Omega^*_{-/A}. \tag{3.2}$$

Lemma 3.1.6 (Construction of pro-objects morphisms)**.** *There are two families of natural maps of commutative differential graded algebras, for any positive integers* M, N

$$\frac{\Omega^*_{B\otimes_A C/A}}{F^{N+M}} \longrightarrow \frac{\Omega^*_{B/A}}{F^N} \otimes_A \frac{\Omega^*_{C/A}}{F^M}$$
$$\frac{\Omega^*_{B/A}}{F^N} \otimes_A \frac{\Omega^*_{C/A}}{F^M} \longrightarrow \frac{\Omega^*_{B\otimes_A C/A}}{F^{\min(N,M)}}$$

which induce two maps of projective systems of cdga.

Proof. Consider the following diagram

$$\begin{array}{ccc} \Omega^*_{B\otimes_A C/A} & \xrightarrow{\sim} & \Omega^*_{B/A} \otimes_A \Omega^*_{C/A} \\ \downarrow & & \downarrow \\ \dfrac{\Omega^*_{B\otimes_A C/A}}{F^{N+M}} & & \dfrac{\Omega^*_{B/A}}{F^N} \otimes_A \dfrac{\Omega^*_{C/A}}{F^M} \end{array},$$

where the horizontal arrow is (3.1) and the vertical ones are the canonical projection on the Hodge-graded pieces, with $N, M \geq 0$. These are morphisms between complexes and, if we look at the n-th level, by (3.1) we have the following correspondence

$$\begin{aligned} \Omega^n_{B\otimes_A C/A} &\longrightarrow \Omega^p_{B/A} \otimes_A \Omega^{n-p}_{C/A} \\ dT_{i_1} \wedge \ldots \wedge dT_{i_n} &\longmapsto dT_{i_1} \wedge \ldots \wedge dT_{i_p} \otimes dT_{i_{p+1}} \wedge \ldots \wedge dT_{i_n}. \end{aligned}$$

Now, if we suppose $n \geq N + M$ (i.e. we consider when the inclusion of the filtered part is 0) we have that either $p \geq N$ or $q \geq M$ (otherwise their sum is $n = p + q < N + M$). Thus by the universal property of the cokernel there exists a canonical arrow (compatible with differential graded algebra structures) such that

[2]Recall that on the tensor product of two cochain complexes the associated product is defined as

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(a')\deg(b)} aa' \otimes bb'$$

$$\begin{array}{ccc} \Omega^*_{B\otimes_A C/A} & \xrightarrow{\sim} & \Omega^*_{B/A}\otimes_A \Omega^*_{C/A} \\ \downarrow & & \downarrow \\ \frac{\Omega^*_{B\otimes_A C/A}}{F^{N+M}} & \longrightarrow & \frac{\Omega^*_{B/A}}{F^N}\otimes_A \frac{\Omega^*_{C/A}}{F^M} \end{array}$$

commutes. Now consider the inverse situation,

$$\begin{array}{ccc} \Omega^*_{B/A}\otimes_A \Omega^*_{C/A} & \xrightarrow{\sim} & \Omega^*_{B\otimes_A C/A} \\ \downarrow & & \downarrow \\ \frac{\Omega^*_{B/A}}{F^N}\otimes_A \frac{\Omega^*_{C/A}}{F^M} & & \frac{\Omega^*_{B\otimes_A C/A}}{F^{\min(N,M)}} \end{array}.$$

Again, if we look at the n-th level, we have the following correspondence

$$\begin{aligned} \Omega^p_{B/A}\otimes_A \Omega^{n-p}_{C/A} &\longrightarrow \Omega^n_{B\otimes_A C/A} \\ dT_{i_1}\wedge ... \wedge dT_{i_p}\otimes dT_{i_{p+1}}\wedge ... \wedge dT_{i_n} &\longmapsto dT_{i_1}\wedge ... \wedge dT_{i_p}\wedge dT_{i_{p+1}}\wedge ... \wedge dT_{i_n}. \end{aligned}$$

Suppose $n \geq N$ or $n \geq M$, then $n < \min(N, M)$. Thus, again, by the universal property of the cokernel there exists a canonical arrow (compatible with differential graded algebra structures) such that

$$\begin{array}{ccc} \Omega^*_{B/A}\otimes_A \Omega^*_{C/A} & \xrightarrow{\sim} & \Omega^*_{B\otimes_A C/A} \\ \downarrow & & \downarrow \\ \frac{\Omega^*_{B/A}}{F^N}\otimes_A \frac{\Omega^*_{C/A}}{F^M} & \longrightarrow & \frac{\Omega^*_{B\otimes_A C/A}}{F^{\min(N,M)}} \end{array}.$$

□

Remark 3.1.7. *In the same spirit of remark 3.1.5 we can see that, since our construction is functorial, the previous commutative diagrams can be translated as morphism of functors. Then, if we consider each side (left and right) as a whole, with N, M running on the naturals, we have two pro-functor and a map of inverse systems between them*

$$\left(\frac{\Omega^*_{-\otimes_A -/A}}{F^{N+M}}\right)_{(M,N)} \longrightarrow \left(\frac{\Omega^*_{-/A}}{F^N}\otimes_A \frac{\Omega^*_{-/A}}{F^M}\right)_{(M,N)}. \tag{3.3}$$

and in the opposite direction

$$\left(\frac{\Omega^*_{-/A}}{F^N}\otimes_A \frac{\Omega^*_{-/A}}{F^M}\right)_{(M,N)} \longrightarrow \left(\frac{\Omega^*_{-\otimes_A -/A}}{F^{\min(N,M)}}\right)_{(M,N)}. \tag{3.4}$$

Lemma 3.1.8 (Isomorphisms of pro-objects.). *The pro-functors*

$$\left(\frac{\Omega^*_{-\otimes_A -/A}}{F^L}\right)_{L\in\mathbb{N}} : \underline{\mathrm{FAlg}}_A \times \underline{\mathrm{FAlg}}_A \longrightarrow pro\text{-}A\text{-}cdga$$

$$\left(\frac{\Omega^*_{-/A}}{F^N} \otimes_A \frac{\Omega^*_{-/A}}{F^M}\right)_{(M,N)\in\mathbb{N}^2} : \underline{\mathrm{FAlg}}_A \times \underline{\mathrm{FAlg}}_A \longrightarrow pro\text{-}A\text{-}cdga$$

are isomorphic.

Proof. We want to prove that pro-natural morphisms in remark (3.1.7) are one the inverse of the other in the category of pro-functors. Let us call (3.3) $\underline{f}$ and (3.4) $\underline{g}$. They are clearly morphisms in $pro(Fun(\underline{\mathrm{FAlg}}_A \times \underline{\mathrm{FAlg}}_A, cdga))$, since they are morphisms of inverse systems. They are represented by the maps

$$\left\{ f_{N,M} : \frac{\Omega^*_{-\otimes_A -/A}}{F^{N+M}} \longrightarrow \frac{\Omega^*_{-/A}}{F^N} \otimes_A \frac{\Omega^*_{-/A}}{F^M} \right\}_{(N,M)\in\mathbb{N}\times\mathbb{N}}$$

and

$$\left\{ g_L : \frac{\Omega^*_{-/A}}{F^L} \otimes_A \frac{\Omega^*_{-/A}}{F^L} \longrightarrow \frac{\Omega^*_{-\otimes_A -/A}}{F^L} \right\}_{L\in\mathbb{N}}$$

and if we compose them, we get

$$\underline{g} \circ \underline{f} = \left\{ \frac{\Omega^*_{-\otimes_A -/A}}{F^{L+L}} \longrightarrow \frac{\Omega^*_{-/A}}{F^L} \otimes_A \frac{\Omega^*_{-/A}}{F^L} \longrightarrow \frac{\Omega^*_{-\otimes_A -/A}}{F^L} \right\}_{L\in\mathbb{N}}$$

and

$$\underline{f} \circ \underline{g} = \left\{ \frac{\Omega^*_{-/A}}{F^{N+M}} \otimes_A \frac{\Omega^*_{-/A}}{F^{N+M}} \longrightarrow \frac{\Omega^*_{-\otimes_A -/A}}{F^{N+M}} \longrightarrow \frac{\Omega^*_{-/A}}{F^N} \otimes_A \frac{\Omega^*_{-/A}}{F^M} \right\}_{(N,M)\in\mathbb{N}\times\mathbb{N}},$$

which are both represented by the canonical projection. This means that the two compositions are both the identity morphisms, which proves our claim. □

Proof Proposition 3.1.2. We compose the (pro-)functors

$$\left(\frac{\Omega^*_{-\otimes_A -/A}}{F^L}\right)_{L\in\mathbb{N}} \quad \text{and} \quad \left(\frac{\Omega^*_{-/A}}{F^N} \otimes_A \frac{\Omega^*_{-/A}}{F^M}\right)_{(M,N)\in\mathbb{N}^2}$$

with the product of two simplicial free A-modules

$$P, Q : \Delta^{\mathrm{op}} \longrightarrow \underline{\mathrm{FAlg}}_A$$

we obtain

$$\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \xrightarrow{(P,Q)} \underline{\mathrm{FAlg}}_A \times \underline{\mathrm{FAlg}}_A \xrightarrow{(3.3)\cong(3.4)} cdga$$

which can be seen in the pro-category $pro(Fun(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, cdga))$. They correspond to two isomorphic bisimplicial pro-complexes of A-modules (or bisimplicial pro$-A-$cdga)

$$\left(\frac{\Omega^*_{P_\bullet \otimes_A Q_\bullet/A}}{F^L}\right)_L \cong \left(\frac{\Omega^*_{P_\bullet/A}}{F^N} \otimes_A \frac{\Omega^*_{Q_\bullet/A}}{F^M}\right)_{N,M}. \tag{3.5}$$

This proves the statement. □

Theorem 3.1.9. *Let $A \longrightarrow B$ and $A \longrightarrow C$ be ring maps. Then we have the Künneth Formula given by the following equivalence of Hodge completed derived de Rham algebras*

$$L\widehat{\Omega}_{B\otimes^L_A C/A} \simeq L\widehat{\Omega}_{B/A} \otimes^L_A L\widehat{\Omega}_{C/A}.$$

Proof. If we suppose P, Q to be the standard resolution of B, C we have that $(P \otimes Q)_\bullet$ is a free resolution[3] of $B \otimes_A C$, thus it can be used to compute the derived de Rham complex. Now we consider the functor (see Lemma 1.2.21)

$$\int : Fun(\Delta^{\mathrm{op}}, cdga) \longrightarrow cdga$$

and, since $Fun(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, cdga) = Fun(\Delta^{\mathrm{op}}, Fun(\Delta^{\mathrm{op}}, cdga))$, we can construct the following functor

$$Fun(\Delta^{\mathrm{op}}, Fun(\Delta^{\mathrm{op}}, cdga)) \xrightarrow{\int \circ -} Fun(\Delta^{\mathrm{op}}, cdga) \xrightarrow{\int} cdga, \tag{3.6}$$

which sends[4]

$$\frac{\Omega^*_{P_\bullet \otimes_A Q_\bullet/A}}{F^L} \longmapsto \bigoplus_{n-i-j=*} \frac{\Omega^n_{P_i \otimes_A Q_j/A}}{F^L} \overset{\text{q.}}{\cong} \frac{L\Omega^*_{B\otimes^L_A C/A}}{F^L}$$

and

$$\begin{aligned}
\frac{\Omega^*_{P_\bullet/A}}{F^N} \otimes_A \frac{\Omega^*_{Q_\bullet/A}}{F^M} \longmapsto \bigoplus_{n-i-j=*} \bigoplus_{p+q=n} \left(\frac{\Omega^p_{P_i/A}}{F^N} \otimes_A \frac{\Omega^q_{Q_j/A}}{F^M} \right) &= \bigoplus_{p+q-i-j=*} \left(\frac{\Omega^p_{P_i/A}}{F^N} \otimes_A \frac{\Omega^q_{Q_j/A}}{F^M} \right) \\
&= \bigoplus_{h+k=*} \bigoplus_{p-i=h} \bigoplus_{q-j=k} \left(\frac{\Omega^p_{P_i/A}}{F^N} \otimes_A \frac{\Omega^q_{Q_j/A}}{F^M} \right) \\
&= \bigoplus_{h+k=*} \left(\bigoplus_{p-i=h} \frac{\Omega^p_{P_i/A}}{F^N} \otimes_A \bigoplus_{q-j=k} \frac{\Omega^q_{Q_j/A}}{F^M} \right) \\
&\overset{\text{q.}}{\cong} \bigoplus_{h+k=*} \frac{L\Omega^h_{B/A}}{F^N} \otimes^L_A \frac{L\Omega^k_{C/A}}{F^M} \\
&= \frac{L\Omega^*_{B/A}}{F^N} \otimes^L_A \frac{L\Omega^*_{C/A}}{F^M}.
\end{aligned}$$

In particular the choice of the order for the "integration" of the simplicial index does not change the outcome, modulo isomorphism of cdga. Taking the pro-object

[3]It may be seen as a consequence of the Eilember-Zilber Theorem 1.2.15 applied to the bisimplicial object $P_\bullet \otimes_A Q_\bullet$. It implies that the map $\int (P \otimes_A Q)_\bullet \longrightarrow \int P_\bullet \otimes \int Q_\bullet$ is a quasi-isomorphism. The latter is a free resolution of $B \otimes_A C$, since it is the total complex associated to a bicomplex whose columns are free resolution of $\int P_\bullet \otimes_A C$ and whose rows are free resolution of $B \otimes_A \int Q_\bullet$.

[4]Note that

$$\bigoplus_{n-i-j=*} \frac{\Omega^n_{P_i \otimes_A Q_j/A}}{F^L} = \bigoplus_{n-j=*} \bigoplus_{m-i=n} \frac{\Omega^n_{P_i \otimes_A Q_j/A}}{F^L} = \bigoplus_{n-j=*} \frac{L\Omega^n_{B \otimes_A Q_j/A}}{F^L} \overset{\text{q.}}{\cong} \frac{L\Omega^*_{B\otimes^L_A C/A}}{F^L}.$$

since $B \otimes_A Q_\bullet$ is a free resolution of $B \otimes_A C$.

associated, we get an equivalence of E_∞- algebra in the derived (∞-)category of pro-A-modules

$$“\varprojlim_{L}” \frac{L\Omega^*_{B\otimes^L_A C/A}}{F^L} \cong “\varprojlim_{N,M}” \left(\frac{L\Omega^*_{B/A}}{F^N} \otimes^L_A \frac{L\Omega^*_{C/A}}{F^M} \right), \tag{3.7}$$

which corresponds to the final statement of the lemma, since tensor product of pro-objects is defined as the pro-object of tensor products. □

Chapter 4

Application to varieties

4.1 Smooth varieties

We saw in Proposition 2.1.28 that the cotangent complex of a smooth morphism $X \longrightarrow S$ is quasi-isomorphic to the Kähler differentials module, seen as a complex concentrated in degree zero. Such result allows us to consider $L_{X/S}$ as a sort of generalization of the sheaf $\Omega^1_{X/S}$ to the case of non necessarily smooth morphisms. Such point of view gets stronger when considering what happens with de derived de Rham complex. First of all an analogous of Proposition 2.1.28 holds in this case.

Proposition 4.1.1. *Given a smooth map of rings $A \longrightarrow B$, then the canonical augmentation map $L\Omega^*_{B/A} \longrightarrow \Omega^*_{B/A}$ is a quasi-isomorphism. The same holds for the Hodge-completed version of the previous map $L\widehat{\Omega}^*_{B/A} \longrightarrow \widehat{\Omega}^*_{B/A}$.*

Proof. See for example [26, Corollary VIII.2.2.8]. For the second part of the statement we can also apply the previous Proposition 2.2.8. As a matter of fact the canonical map $L\Omega^*_{B/A} \longrightarrow \Omega^*_{B/A}$ induces on graded pieces the quasi-isomorphism $\mathrm{gr}^i_F L\Omega^*_{B/A} \overset{\text{q.}}{\cong} L\wedge^i L_{B/A}[-i] \overset{\text{q.}}{\cong} L\wedge^i \omega^1_{B/A}[-i] \overset{\text{q.}}{\cong} \Omega^i_{B/A}$ (recall Proposition 2.1.28). This graded (quasi-)isomorphism induces a quasi-isomorphism on the quotients by the Hodge filtration

$$L\Omega^*_{B/A}/F^N \longrightarrow \Omega^*_{B/A}/F^N,$$

which yields the quasi isomorphism on the completed complexes. □

As a matter of fact $\Omega^1_{X/S}$ and $\Omega^*_{X/S}$ are still computable in a non-smooth case, but the geometrical non-regularity gets "translated" in a non-regularity of the associated algebraic objects (in particular $\Omega^1_{X/S}$ is no longer locally free), which again gets translated in cohomology, losing a topological meaning. The sense of being topologically meaningful is given by the existence of a comparison isomorphism with a cohomology theory, like the Betti cohomology, of exclusively topological nature.

Theorem (Grothendieck [21]). *Given a smooth variety X, there is an isomorphism in cohomology*

$$H^*(X, \Omega^\bullet_X) \cong H^*(X^{\text{an}}, \mathbb{C}) =: H^*_{\text{Betti}}(X).$$

We lose such result when considering non-smooth morphisms (see [3, Example 4.4] for example). There are several ways to recover this comparison isomorphism, one of them is through (Hodge completed) derived de Rham cohomology, thanks to Bhatt [9], who extended a results of Illusie [26, Corollary VIII.2.2.8] for varieties defined over $\mathbb{Q}$.

We want to exploit what has been developed in the previous chapters to investigate what may say (Hodge completed) derived de Rham cohomology relative to, eventually singular, varieties over fields of positive characteristic. In this sense, recently more important and meaningful results have been reached by Antieau, Bhatt, Lurie, Mathew, Morrow, Scholze and others (see for example [2], [11], [12]). In this chapter our general strategy will be using Künneth formula of last chapter in order to split computations relative to a variety in positive characteristic in two parts, one without p-torsion and the other recalling our previous results on the derived de Rham complex for perfect rings (in this sense see also [2, Construction 7.12 and proof of Theorem 7.13]).

4.2 Lifting of varieties over perfect fields

Consider Theorem 1.1.1 and Theorem 1.1.2 of [11], we want to provide a similar result replacing the Hodge completed derived de Rham complex ([26] VIII 2.1.3.3) to the de Rham-Witt complex ([27] and [11]). As far as the proofs of the original theorems is concerned, the authors indicate [27] as reference, Theorem 1.1.1 in Chapter I, Corollary 3.16 and Theorem 1.1.2 within the proof of Theorem 1.4 in Chapter II.

Remark 4.2.1. *Recall Lemma 2.3.2, $L\widehat{\Omega}^*_{B/\mathbb{Z}_p} \simeq L\widehat{\Omega}^*_{B/W}$. In particular take the sequence of schemes $X \longrightarrow \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(\mathbb{Z})$, by Lemma 2.3.2 and Lemma 2.3.4, the natural maps $W \longrightarrow L\widehat{\Omega}^*_{k/\mathbb{Z}} \longrightarrow L\widehat{\Omega}^*_{X/\mathbb{Z}}$ give to the Hodge completed de Rham complex of X a structure of $E_\infty-$algebra over W.*

Remark 4.2.2. *Recall that given a smooth variety X over a ring A, the (Hodge completed) derived de Rham complex $L\widehat{\Omega}^*_{X/A} = \left(L\Omega^*_{X/A}/F^N\right)_N$ is quasi-isomorphic to the standard de Rham (pro) complex $\widehat{\Omega}^*_{X/A} = \left(\Omega^*_{X/A}/F^N\right)_N$ (see 4.1.1). Moreover the Hodge filtration over such complex is finite, that is there exists $N_0 > 0$, such that $F^N\Omega^*_{X/A} = 0$ for $N > N_0$. This means that for $N > N_0$ $\Omega^*_{X/A}/F^N = \Omega^*_{X/A}$. The subset $\{N_0+1, N_0+2, ...\} \subseteq \mathbb{N}$ induces a cofinal functor, so that the pro-complex $\left(\Omega^*_{X/A}/F^N\right)_N$ is isomorphic (as pro-object) to the constant pro-complex $\Omega^*_{X/A}$.*

Here we present the proof for Proposition Proposition 4.2.3. The main point is the fact that, thanks to the pro-version of the Künneth Formula, we can sometimes split the computation of the derived de Rham complex of a variety over a perfect field in two simpler computations: one relative to a smooth variety and the other relative to the base field.

Theorem 4.2.3. *Let $\mathfrak{X}$ be a smooth scheme over* $\mathrm{Spec}(W)$*, where* $W = W(k)$ *for a perfect field* k*. Consider the following cartesian square*

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{X} \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(W(k)), \end{array} ,$$

where $X := \mathrm{Spec}(k) \times_{\mathrm{Spec}(W(k))} \mathfrak{X}$*. Then there is an quasi-isomorphism of pro-complexes*

$$L\widehat{\Omega}^*_{X/\mathbb{Z}} \simeq \left(\Omega^*_{\mathfrak{X}/W} \otimes \frac{W\langle x \rangle}{(x)^{[N]}} \xrightarrow{\cdot(x-p)} \Omega^*_{\mathfrak{X}/W} \otimes_W \frac{W\langle x \rangle}{(x)^{[N]}} \right)_N .$$

Proof. Consider the theorem locally, so that we are dealing with a setting like the following

$$\begin{array}{ccc} \mathrm{Spec}(A_0) & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spf}(W(k)), \end{array} ,$$

where $A_0 = A/I$ and $I = pA$ is the admissible ideal of A. Then we have the following equivalences of projective systems of complexes

$$\begin{aligned} L\widehat{\Omega}^*_{A_0/\mathbb{Z}} &\simeq L\widehat{\Omega}^*_{A_0/\mathbb{Z}_p} \simeq L\widehat{\Omega}^*_{A_0/W} && (4.1)\\ &\simeq L\widehat{\Omega}^*_{A\otimes_W k/W} \\ &\simeq L\widehat{\Omega}^*_{A/W} \otimes^L_W L\widehat{\Omega}^*_{k/W} && (4.2)\\ &\simeq L\widehat{\Omega}^*_{A/W} \otimes^L_W \left(\frac{W\langle x \rangle}{(x)^{[N]}} \xrightarrow{\cdot(x-p)} \frac{W\langle x \rangle}{(x)^{[N]}} \right)_N && (4.3)\\ &\simeq \Omega^*_{A/W} \otimes_W \left(\frac{W\langle x \rangle}{(x)^{[N]}} \xrightarrow{\cdot(x-p)} \frac{W\langle x \rangle}{(x)^{[N]}} \right)_N && (4.4)\\ &\simeq \left(\Omega^*_{A/W} \otimes \frac{W\langle x \rangle}{(x)^{[N]}} \xrightarrow{\cdot(x-p)} \Omega^*_{A/W} \otimes_W \frac{W\langle x \rangle}{(x)^{[N]}} \right)_N && (4.5) \end{aligned}$$

where the first line (4.1) is due to Lemmas (2.3.2) and (2.3.4),(4.2) is Künneth formula, (4.3) is Theorem 2.3.26, (4.4) is due to remark 4.2.2. Once we proved the statement locally, it is easy to get it global (one way is to prove that all the maps that we considered locally before are canonical so that they glue on the intersections). □

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