Quantum fluctuations and entanglement in the collective atomic recoil laser using a Bose-Einstein condensate

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We present a quantum description of the interaction between a Bose-Einstein condensate and a single-mode quantized radiation field in the presence of a strong far-off-resonant pump laser. In the linear regime, the atomic medium is described approximately by two momentum states coupled to the radiation mode. We calculate the evolution of the operators in the Heisenberg picture and their expectation values, such as average and variance of the occupation numbers, atom-atom and atom-field correlations, and two-mode squeezing parameters. Then, we disentangle the evolution operator and obtain the exact evolution of the state vector in the linear regime. This allows us to demostrate that the system can be atom-atom or atom-field thermally entangled. We define the quasiclassical and the quantum recoil limits, for which explicit expressions of the average population numbers are obtained.

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I. INTRODUCTION

The realization of Bose-Einstein condensates (BECs) of dilute alkali-metal gases [1] has made it possible to study the coherent interaction between light and an ensemble of atoms prepared in a single quantum state. Among the multitude of the experiments studying the behavior of a BEC under the action of external laser beams, only a small number have been devoted to the active role caused by the atoms in the condensate on the radiation [2]. In particular, collective light scattering and matter-wave amplification caused by coherent center-of-mass motion of atoms in a condensate illuminated by a far-off-resonant laser were recently observed and interpreted [3–5] as super-radiant Rayleigh scattering. The basic mechanism of the collective light scattering observed in these experiments is described by the collective atomic recoil laser (CARL), proposed by Bonifacio and co-workers [6–8]. However, the original CARL theory which treats the atomic center-of-mass motion classically fails when the temperature of the atomic sample is below the recoil temperature $T_r = 2\hbar k^2/Mk_B$, where k is the wave vector of the pump-laser field back scattered by the atom, M is the atomic mass, and k_B is the Boltzmann constant.

An extension of the semiclassical CARL Hamiltonian model to include a quantum-mechanical description of the center-of-mass motion of the atoms has been presented by Bonifacio [9] and successively by Moore and Meystre [10], by Berman [11], and by Ling *et al.* [12]. Furthermore, a more detailed description of the exponential instability in a BEC has been given in Ref. [13]. However, the previous analyses in Refs. [10–13] were restricted to a general description of the exponential regime where the collective instability takes place without studying explicitly the quasiclassical and quantum recoil limits of the CARL in a BEC. Instead, this point has been addressed in Ref. [14], where, starting from the same model of Ref. [10], the full nonlinear regime and the quasiclassical and quantum regimes of the CARL were studied. In particular, in Ref. [14], the nonlinear evolution of the system has been studied numerically, solving the Heisenberg equations for the operators treated as classical variables, i.e., neglecting quantum correlations and fluctuations. Analytical results were also presented in Ref. [14] for the dynamical evolution of the field and the atoms in the linear regime and in the "two-level approximation" for the atomic motion in the quantum recoil limit.

The objective of this article is to further investigate analytically some properties of the quantum system BEC radiation in the linear regime, such as quantum fluctuations and entanglement. Although these topics have been already studied in Ref. [13], we explicitly calculate the statistical properties for atoms and photons in the quasiclassical and quantum recoil limits and evaluating explicitly the state of the coupled BEC-light system evolved from vacuum. This allows to study more carefully the entanglement properties of the system. More precisely, it results that, in the limit of undepleted atomic ground state and unsaturated probe field, the quantum CARL Hamiltonian reduces to that for three coupled modes, the first two modes corresponding to atoms having lost or gained a quantum recoil momentum $2\hbar k$ in the two-photon Bragg scattering between the pump and the probe, and the third mode corresponding to the photons of the probe field. Calculating the exact evolution of the state from the vacuum of the three modes, $|0_1,0_2,0_3\rangle$, we demonstrate [see Eq. (47)] that the evolved state is

$$|\Psi\rangle = \frac{1}{\sqrt{1+\langle n_1\rangle}} \sum_{m,n=0}^{\infty} (r_a e^{i\phi_a})^m (r_b e^{i\phi_b})^n \sqrt{\frac{(m+n)!}{m!n!}} \times |m+n,n,m\rangle, \tag{1}$$

where $r_a=[\langle n_3\rangle/(1+\langle n_1\rangle)]^{1/2}$, $r_b=[\langle n_2\rangle/(1+\langle n_1\rangle)]^{1/2}$ and $\langle n_1\rangle=\langle n_2\rangle+\langle n_3\rangle$. This shows that, in general, the system is entangled and the distribution function over the different occupation numbers, n_1, n_2 and n_3, is thermal. In particular, the state reduces to a two-mode squeezed state [15] for $\langle n_3\rangle \ll \langle n_1\rangle \sim \langle n_2\rangle$, showing entanglement between atoms with different momenta, or $\langle n_2\rangle \ll \langle n_1\rangle \sim \langle n_3\rangle$, showing entanglement between atoms and photons. We demonstrate that these two kinds of entanglement may be obtained in the quasiclassical and quantum recoil regimes, respectively. Furthermore, we show that in the exponential regime, the maximum average number of photons scattered by N atoms in the quasiclassical recoil limit is $\rho N/2$ (where ρ is the collective CARL parameter proportional to $N^{1/3}$ [6]), whereas in the quantum recoil limit is N.

The paper is organized as follows. In Sec. II, we give a full quantum description of the interaction between a BEC, considered as an ideal gas, and a single-mode quantized radiation field. Then, we reduce the model to a system of linear equations for three coupled harmonic oscillator operators. In Sec. III, we calculate the approximate solution valid for short times. In Sec. IV, we give the exact solution of linear Heisenberg equations and we deduce from it some general results. In particular, we evaluate the average occupation numbers $\langle n_i\rangle$ $(i=1,2,3)$ and their variances, the correlation functions, and the expectation value for the linearized bunching operator and the two-mode squeezing parameters, proportional to the variance of the difference between two-mode occupation numbers. Finally, we calculate the exact evolution of the state vector writing a Baker-Hausdorff equation for the evolution operator $U(\tau)=\exp(-iH\tau)$. In Sec. V, we obtain explicit expressions of the average occupation numbers of the three modes in the exponential regime and in the quasiclassical and quantum limits. In Sec. VI, we show that the CARL mechanism in a BEC can provide a valuable source for atom-atom or atom-field entanglement. Conclusions are finally summarized in Sec. VII.

II. THE HAMILTONIAN MODEL

Our starting point is the classical model of equations for N two-level atoms exposed to an off-resonant pump laser, whose electric field $\vec{E}_0=\hat{e}\mathcal{E}_0\cos(\vec{k}_2\cdot\vec{x}-\omega_2 t)$ is polarized along $\hat{e}$, propagates along the direction of $\vec{k}_2$, and has a frequency $\omega_2=ck_2$ with a detuning from the atomic resonance, $\Delta_{20}=\omega_2-\omega_0$, much larger than the natural linewidth of the atomic transition, γ. We assume the presence of a scattered field (''probe beam'') with frequency ω_1, wave number $\vec{k}_1$ making an angle ϕ with $\vec{k}_2$, and electric field $\vec{E}=(\hat{e}/2)[\mathcal{E}(t)e^{i(\vec{k}_1\cdot\vec{x}-\omega_1 t)}+\text{c.c.}]$ with the same polarization of the pump field. In the absence of an injected probe field, the emission starts from fluctuations and the propagation direction of the scattered field is determined either by the geometry of the condensate (as in the experiment of Ref. [3], where the condensate has a cigar shape) or by the presence of an optical resonator tuned on a selected longitudinal mode. By adiabatically eliminating the internal atomic degree of freedom, the following semiclassical CARL equations have been derived [6–8]:

$$\frac{d\theta_j}{d\tau}=\overline{p_j}, \tag{2}$$

$$\frac{d\overline{p_j}}{d\tau}=-[\tilde{A}e^{i\theta_j}+\text{c.c.}] \tag{3}$$

$$\frac{d\tilde{A}}{d\tau}=\frac{1}{N}\sum_{j=1}^{N}e^{-i\theta_j}+i\tilde{\Delta}\tilde{A}, \tag{4}$$

where $\tau=\rho\omega_r t$ is the interaction time in units of the collective recoil bandwidth, $\rho\omega_r$, $\omega_r=\hbar q^2/2M$ is the recoil frequency, M is the atomic mass, $q=|\vec{q}|$ and $\vec{q}=\vec{k}_1-\vec{k}_2$ is the difference between the scattered and the incident wave vectors. In Eqs. (2)–(4), $\theta_j=\vec{q}\cdot\vec{x}_j=qz_j$ and $\overline{p_j}=qv_{zj}/\rho\omega_r$ are the dimensionless position and velocity of the jth atom along the axis $\hat{z}$ directed along $\vec{q}$, $\tilde{A}=-i(\epsilon_0/n_s\hbar\omega_2\rho)^{1/2}\mathcal{E}(\tau)e^{i\tilde{\Delta}\tau}$, $\tilde{\Delta}=(\omega_2-\omega_1)/\rho\omega_r$, $\rho=(\Omega_0/2\Delta_{20})^{2/3}(\omega_2\mu^2 n_s/\hbar\epsilon_0\omega_r^2)^{1/3}$ is the collective CARL parameter, $\Omega_0=\mu\mathcal{E}_0/\hbar$ is the Rabi frequency of the pump, $n_s=N/V$ is the average atomic density of the sample (containing N atoms in a volume V), μ is the dipole matrix element, and ϵ_0 is the permittivity of the free space. Where it was possible, we have assumed above $\omega_2\approx\omega_1$. In particular, the recoil momentum can be written as $\hbar q\approx 2\hbar k_2|\sin(\phi/2)|$. Equations (2)–(4) are formally equivalent to those of the free electron laser (FEL) model [16].

In order to quantize both the radiation field and the center-of-mass motion of the atoms, we consider θ_j, $p_j=(\rho/2)\overline{p_j}=Mv_{zj}/\hbar q$ and $a=(N\rho/2)^{1/2}\tilde{A}$ as quantum operators satisfying the canonical commutation relations $[\theta_j,p_{j'}]=i\delta_{jj'}$, and $[a,a^\dagger]=1$. With these definitions, Eqs. (2)–(4) are the Heisenberg equations of motion associated with the Hamiltonian [9]

$$H=\sum_{j=1}^{N}\left[\frac{p_j^2}{\rho}+ig(a^\dagger e^{-i\theta_j}-\text{H.c.})-\frac{\tilde{\Delta}}{N}a^\dagger a\right]=\sum_{j=1}^{N}H_j(\theta_j,p_j,a,a^\dagger), \tag{5}$$

where $g=\sqrt{\rho/2N}$. We note that $[H,Q]=0$, where $Q=a^\dagger a+\Sigma_{j=1}^{N}p_j$ is the total momentum in units of $\hbar q$. The single-particle Hamiltonian H_j in Eq. (5) can be second quantized as [17]

$$\hat{H}=\int_0^{2\pi}d\theta\hat{\Psi}^\dagger(\theta)H\left(\theta,-i\frac{\partial}{\partial\theta},a,a^\dagger\right)\hat{\Psi}(\theta), \tag{6}$$

where the atomic-field operator $\hat{\Psi}(\theta)$ obeys the bosonic equal-time commutation relations $[\hat{\Psi}(\theta),\hat{\Psi}^\dagger(\theta')]=\delta(\theta-\theta')$, $[\hat{\Psi}(\theta),\hat{\Psi}(\theta')]=[\hat{\Psi}^\dagger(\theta),\hat{\Psi}^\dagger(\theta')]=0$ and the normalization condition $\int_0^{2\pi}d\theta\hat{\Psi}(\theta)^\dagger\hat{\Psi}(\theta)=N$. We introduce creation and annihilation operators for the atoms of a definite momentum p, i.e., $\hat{\Psi}(\theta)=\Sigma_m c_m\langle\theta|m\rangle$, where $p|m\rangle=m|m\rangle$ (with $m=-\infty,\ldots,\infty$), $\langle\theta|m\rangle=(2\pi)^{-1/2}\exp(im\theta)$ and c_m are bosonic operators obeying the commutation relations $[c_m,c_{m'}^\dagger]=\delta_{mm'}$ and $[c_m,c_{m'}]=0$. This description of the atomic motion in a BEC assumes that the atoms are delocalized inside the condensate and that, at zero temperature, the momentum uncertainty $\sigma_{p_z}\approx\hbar/\sigma_z$ can be neglected with respect to $\hbar q$. When $\sigma_z\approx L$, where L it the size of the condensate, this approximation is valid for $L \gg \lambda$. The Hamiltonian (6) becomes [10,14]

$$\hat{H} = \sum_{n=-\infty}^{\infty} \left\{ \frac{n^2}{\rho} c_n^\dagger c_n + ig(a^\dagger c_n^\dagger c_{n+1} - \text{H.c.}) \right\} - \tilde{\Delta} a^\dagger a \quad (7)$$

and the Heisenberg equations for c_n and a are

$$\frac{dc_n}{d\tau} = -i[c_n, \hat{H}] = -i \frac{n^2}{\rho} c_n + g(a^\dagger c_{n+1} - a c_{n-1}), \quad (8)$$

$$\frac{da}{d\tau} = -i[a, \hat{H}] = i\tilde{\Delta} a + g \sum_{n=-\infty}^{\infty} c_n^\dagger c_{n+1}. \quad (9)$$

Equations (8) and (9) have been obtained in a similar way by Moore and Meystre in Ref. [10], treating the electromagnetic field classically. They are formally equivalent to the CARL equations (2)–(4), but they describe more conveniently the CARL when the atoms are initially in the ground state with a definite center-of-mass momentum, as for a BEC. The source of the field equation (9) is the bunching operator, defined by $\hat{B} = (1/N)\int_0^{2\pi} d\theta \hat{\Psi}(\theta)^\dagger e^{-i\theta} \hat{\Psi}(\theta) = (1/N)\Sigma_n c_n^\dagger c_{n+1}$. We note that Eqs. (8) and (9) conserve the number of atoms, i.e., $\Sigma_n c_n^\dagger c_n = N$, and the total momentum $Q = a^\dagger a + \Sigma_n n c_n^\dagger c_n$.

Let us now consider the equilibrium state with no probe field and all the atoms in the same initial momentum state n_0, i.e., $c_n \approx \sqrt{N} e^{-in^2\tau/\rho} \delta_{n,n_0}$. This is equivalent to assuming that the temperature of the system is equal to zero and all the atoms are moving with the same momentum $n_0 \hbar \vec{q}$, without spread. The system is unstable for certain values of the detuning $\tilde{\Delta}$. In fact, by linearizing Eqs. (8) and (9) around the equilibrium state, the only equations depending linearly on the radiation field are these for c_{n_0-1} and c_{n_0+1}. Hence, in the linear regime, the only transitions involved are those from the state n_0 towards the levels $n_0 - 1$ and $n_0 + 1$. Introducing the operators

$$a_1 = c_{n_0-1} e^{i(n_0^2 \tau/\rho + \tilde{\Delta}\tau)}, \quad (10)$$

$$a_2 = c_{n_0+1} e^{i(n_0^2 \tau/\rho - \tilde{\Delta}\tau)}, \quad (11)$$

$$a_3 = a e^{-i\tilde{\Delta}\tau}, \quad (12)$$

the atomic-field operator reduces to

$$\hat{\Psi}(\theta) \approx \frac{1}{\sqrt{2\pi}} \{ \sqrt{N} + a_1(\tau) e^{-i(\theta + \tilde{\Delta}\tau)} + a_2(\tau) e^{i(\theta + \tilde{\Delta}\tau)} \} e^{i(n_0\theta - in_0^2\tau/\rho)} \quad (13)$$

and Eqs. (8) and (9) reduce to the linear equations for three coupled harmonic-oscillator operators [18]

$$\frac{da_1^\dagger}{d\tau} = -i\delta_- a_1^\dagger + \sqrt{\rho/2} a_3, \quad (14)$$

$$\frac{da_2}{d\tau} = -i\delta_+ a_2 - \sqrt{\rho/2} a_3, \quad (15)$$

$$\frac{da_3}{d\tau} = \sqrt{\rho/2}(a_1^\dagger + a_2), \quad (16)$$

with the Hamiltonian

$$H = \delta_+ a_2^\dagger a_2 - \delta_- a_1^\dagger a_1 + i\sqrt{\rho/2}[(a_1^\dagger + a_2) a_3^\dagger - (a_1 + a_2^\dagger) a_3], \quad (17)$$

where $\delta_\pm = \delta \pm 1/\rho$ and $\delta = \tilde{\Delta} + 2n_0/\rho = (\omega_2 - \omega_1 + 2n_0\omega_r)/\rho\omega_r$. Hence, the dynamics of the system is that of three parametrically coupled harmonic oscillators a_1, a_2, and a_3 [19], which obey the commutation rules $[a_i, a_j] = 0$ and $[a_i, a_j^\dagger] = \delta_{ij}$ for $i,j = 1,2,3$. Note that the Hamiltonian (17) commutates with the constant of motion

$$C = a_2^\dagger a_2 - a_1^\dagger a_1 + a_3^\dagger a_3. \quad (18)$$

It is worth to introduce also the atomic density operator, $\hat{n}(\theta) \equiv \hat{\Psi}(\theta)^\dagger \hat{\Psi}(\theta)$, which by using Eq. (13), has the following linearized form:

$$\hat{n}(\theta) \approx \frac{N}{\sqrt{2\pi}} (1 + Be^{-i\theta} + B^\dagger e^{i\theta}), \quad (19)$$

where $B = e^{i\delta\tau}(a_1^\dagger + a_2)/\sqrt{N}$ is the linearized bunching operator. Furthermore, it is easy to show that the variance of the atomic density is

$$[\Delta(\hat{n})]^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 = 2\left(\frac{N}{2\pi}\right)^2 (\langle B^\dagger B \rangle - \langle B^\dagger \rangle \langle B \rangle). \quad (20)$$

Then, the expectation values $\langle B \rangle$ and $\langle B^\dagger B \rangle$ of the bunching operator are related to the average and variance of the atomic density, respectively.

III. SPONTANEOUS EMISSION AND SMALL-GAIN REGIME

Before solving exactly the linear equations (14)–(16), we calculate the perturbative solution valid for short times τ. Solving Eqs. (14) and (15) keeping $a_3(\tau) \approx a_3(0)$ constant, we obtain

$$a_1^\dagger(\tau) \approx e^{-i\delta_-\tau} \{ a_1^\dagger(0) + a_3(0)\sqrt{\rho/2}\tau \ \operatorname{sinc}(\delta_-\tau/2) \}, \quad (21)$$

$$a_2(\tau) \approx e^{-i\delta_+\tau} \{ a_2(0) + a_3(0)\sqrt{\rho/2}\tau \ \operatorname{sinc}(\delta_+\tau/2) \}. \quad (22)$$

Where $\operatorname{sinc}(x) = \sin(x)/x$. If the radiation field is initially in a coherent state with amplitude α and the atoms are in the vacuum for the state $n_0 - 1$ and $n_0 + 1$, i.e., if $|\psi(0)\rangle = |0,0,\alpha\rangle$, then

$$\langle n_1 \rangle = \langle a_1^\dagger a_1 \rangle = (1 + |\alpha|^2) S_-, \quad (23)$$

$$\langle n_2 \rangle = \langle a_2^\dagger a_2 \rangle = |\alpha|^2 S_+, \quad (24)$$

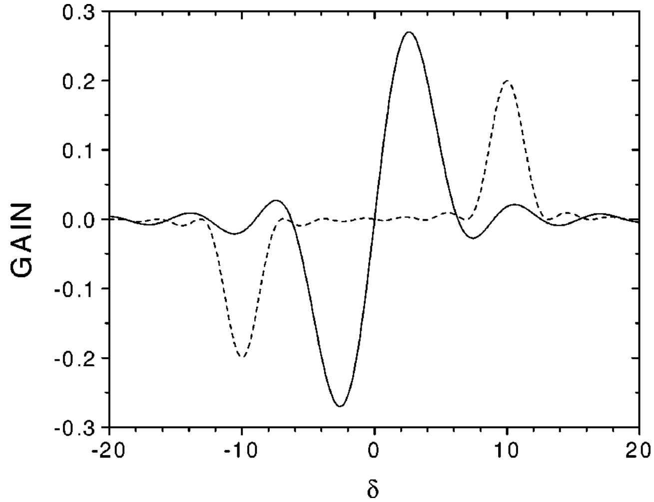

FIG. 1. Small-gain regime: $\mathcal{G}$ vs δ for $1/\rho=0$ and $\tau=1$ (continuous line) and for $1/\rho=10$ and $\tau=2$ (dashed line).

$$\langle n_3\rangle=\langle a_3^\dagger a_3\rangle=S_-+|\alpha|^2[1+\mathcal{G}], \tag{25}$$

where $S_\pm=(\rho\tau^2/2)\mathrm{sinc}^2(\delta_\pm\tau/2)$ is the dimensionless spontaneous emission line shape and $\mathcal{G}=S_--S_+$ is the gain. In Eq. (25) we used the constant of motion (18) to obtain $\langle n_3\rangle=|\alpha|^2+\langle n_1\rangle-\langle n_2\rangle$. The first term S_- in Eq. (25) is the average number of photons emitted spontaneously along the z axis with the usual line shape centered at $\delta=1/\rho$, i.e. at $\omega_1=\omega_2+(2n_0-1)\omega_r$. The second term of Eq. (25) is the stimulated contribution with the familiar antisymmetric dependence on the detuning [8]. In the limit $\rho\gg1$, we obtain

$$\mathcal{G}\approx-(\tau^3/2)\frac{d}{dx}[\mathrm{sinc}^2(x)]_{x=\delta\tau/2}=\frac{4\tau^3}{\delta^3}[1-\cos(\delta\tau)-(\delta\tau/2)\sin(\delta\tau)]. \tag{26}$$

Expression (26) is known in the FEL literature as the "Madey gain" [20] and was originally obtained by Madey (for an electron beam traveling along a wiggler magnetic field) as the difference between the emission and absorption rates, shifted by the recoil frequency. In the limit of small ρ, the solutions (23)–(25) is valid for $\rho\tau^2\ll1$, whereas, in the limit $\rho\gg1$, the Madey gain (26) is valid for $\tau\ll1$. Figure 1 shows $\mathcal{G}$ as a function of δ for $1/\rho=0$ and $\tau=1$ (continuous line) and for $1/\rho=10$ and $\tau=2$ (dashed line). We observe that the Madey gain (26) does not depend explicitly on ρ and has the maximum value $\mathcal{G}_{max}\approx0.27\tau^3$ at $\delta\tau=2.6$. Instead, in the limit $\rho\ll1$, the maximum gain is $\mathcal{G}_{max}=\rho\tau^2/2$, occurring at the center of the spontaneous emission line $\delta=1/\rho$.

IV. SOLUTION OF THE LINEAR QUANTUM REGIME

The exact solution of Eqs. (14)–(16), obtained using the Laplace transform, is the following [9,19]:

$$a_1^\dagger(\tau)=e^{-i\delta\tau}[g_1(\tau)a_1^\dagger(0)+g_2(\tau)a_2(0)+g_3(\tau)a_3(0)], \tag{27}$$

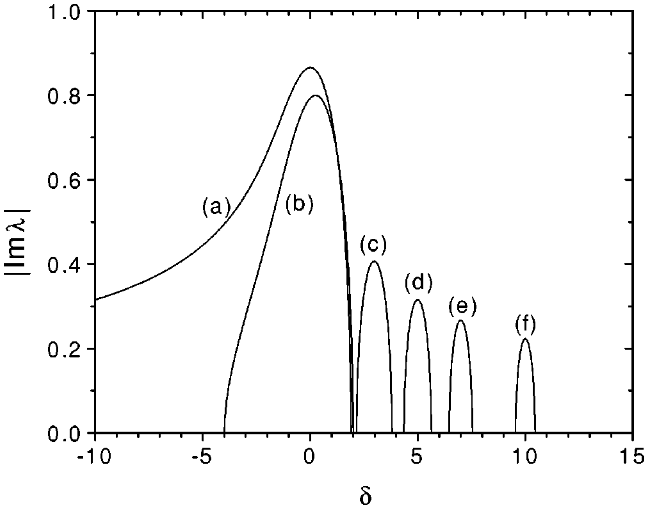

FIG. 2. Imaginary part of the unstable root of the cubic equation (30) vs δ for $1/\rho=0$ (a), 0.5 (b), 3 (c), 5 (d), 7 (e), and 10 (f).

$$a_2(\tau)=e^{-i\delta\tau}[h_1(\tau)a_1^\dagger(0)+h_2(\tau)a_2(0)+h_3(\tau)a_3(0)], \tag{28}$$

$$a_3(\tau)=e^{-i\delta\tau}[f_1(\tau)a_1^\dagger(0)+f_2(\tau)a_2(0)+f_3(\tau)a_3(0)], \tag{29}$$

where the explicit expressions of f_i, g_i and h_i are given in Appendix A, while the initial values $f_i(0)=\delta_{i3}$, $g_i(0)=\delta_{i1}$ and $h_i(0)=\delta_{i2}$ verify the initial conditions for a_i. The functions f_i, g_i and h_i are the sum of three terms proportional to $e^{i\lambda_j\tau}$, where λ_j are the complex roots of the cubic equations:

$$(\lambda-\delta)(\lambda^2-1/\rho^2)+1=0. \tag{30}$$

The characteristic equation (30) has either three real solutions, or one real and a pair of complex-conjugate solutions. In the first case, the system is stable and exhibits only small oscillations around its initial state. In the second case, the system is unstable and grows exponentially, even from noise. In Fig. 2 we plot the imaginary part of λ as a function of δ for different values of ρ. The classical limit is obtained for $\rho\gg1$ [see Fig. 2(a)]. In this case, the system is unstable for $\delta<3/2^{1/3}$ with a maximum instability rate at $\delta=0$ and an unstable root $\lambda_3=(1-i\sqrt{3})/2$. When $\rho<1$ [see Figs. 2(c)–2(f)], the instability rate decreases and the peak of $\mathrm{Im}(\lambda)$ moves at $\delta=1/\rho$. This can be seen explicitly observing that the characteristic equation (30) has two resonances, one at $\delta=-1/\rho$ and the other at $\delta=1/\rho$, corresponding to a mismatch between the probe and the pump field frequencies equal to $\omega_1-\omega_2=(2n_0\mp1)\omega_r$. In the first case, the photon is absorbed from the probe and emitted into the pump, raising the atomic energy from $n_0^2\hbar\omega_r$ to $(n_0+1)^2\hbar\omega_r$. In the second case, the photon is absorbed from the pump and emitted into the probe, decreasing the atom energy to $(n_0-1)^2\hbar\omega_r$. The study of the solutions of Eq. (30) shows that the only unstable process is the second one. When $\rho\gg1$, the gain bandwidth $(\delta\omega_1)_{Gain}=\rho\omega_r\sigma_\delta$ [where σ_δ is defined as the interval of δ, for which $\mathrm{Im}(\lambda)$ is at half of its maximum value] is of the order of $\rho\omega_r \gg \omega_r$ and the shift due to the energy recoil $\hbar\omega_r$ is negligible, also if the system is initially at the zero temperature, i.e., without thermal broadening. Because for $\rho\gg 1$ the gain bandwidth is larger than the separation between the emission and absorption line-shape centers, each atom may absorb the photon either from the pump or from the probe. In this case, the probe field is amplified because, as we will see later, the average number of photons scattered from the pump into the probe is larger than that absorbed from the probe and emitted into the pump.

On the other hand, the quantum recoil effect becomes visible when $\rho<1$. In this case, the unstable root of the characteristic equation (30) is approximately $\lambda\approx 1/\rho+\delta_-/2-(1/2)\sqrt{\delta_-^2-2\rho}$. The imaginary part of λ reaches the maximum value $\sqrt{\rho/2}$ for $\delta_-=0$ (i.e., $\delta=1/\rho$), with a gain bandwidth $(\delta\omega_1)_{Gain}=(2\rho)^{3/2}\omega_r$ less than ω_r. In this case, the absorption line-shape center, $\delta=-1/\rho$, is outside of the gain bandwidth, and the atom can only absorb the photon from the pump and emit it into the probe, whereas the inverse process is not allowed due to the energy conservation. This can be seen explicitly assuming that in the exponential regime $a_1^\dagger$ and a_2 are proportional to $\exp[i(\lambda_3-\delta)\tau]$, where λ_3 is the unstable root of Eq. (30) with negative imaginary part. Then, from Eqs. (14) and (15), it follows that:

$$a_2\sim\left(\frac{1-\rho\lambda_3}{1+\rho\lambda_3}\right)a_1^\dagger. \tag{31}$$

In the case $\rho\gg 1$, $a_2\sim a_1^\dagger$, and the atoms have almost the same probability of transition from the momentum level n_0 to n_0+1 or n_0-1, absorbing or emitting a photon, respectively. On the contrary, in the case $\rho\ll 1$, $\lambda_3\sim 1/\rho$ and Eq. (31) shows that $a_2\ll a_1^\dagger$: the atoms can only emit a photon into the probe due to the energy conservation. This makes the CARL in the quantum recoil limit $\rho\ll 1$ an interesting example of two-level system coupled to a radiation mode, initially inverted and without incoherent spontaneous decay.

In the remaining part of this section, we derive some general analytical results from the solution of the quantum linear model, whereas the quasiclassical ($\rho\gg 1$) and quantum recoil ($\rho<1$) limits will be discussed in the Sec. V.

We assume that the initial state is $|0,0,\alpha\rangle$, i.e., the vacuum state for a_1 and a_2 and a coherent state with amplitude α for a_3. The average occupation numbers $\langle n_i\rangle$ and the number variance $\sigma^2(n_i)=\langle n_i^2\rangle-\langle n_i\rangle^2$ can be calculated from Eqs. (27)–(29) yielding

$$\langle n_i\rangle=\langle n_i\rangle_{\rm sp}+\langle n_i\rangle_{\rm st}, \tag{32}$$

$$\sigma^2(n_i)=\langle n_i\rangle(1+\langle n_i\rangle)-\epsilon_i\langle n_i\rangle_{\rm st}^2 \tag{33}$$

for $i=1,2,3$, where

$$\langle n_1\rangle_{\rm sp}=|g_2|^2+|g_3|^2,\ \ \langle n_1\rangle_{\rm st}=|\alpha g_3|^2, \tag{34}$$

$$\langle n_2\rangle_{\rm sp}=|h_1|^2,\quad \langle n_2\rangle_{\rm st}=|\alpha h_3|^2, \tag{35}$$

$$\langle n_3\rangle_{\rm sp}=|f_1|^2,\quad \langle n_3\rangle_{\rm st}=|\alpha f_3|^2 \tag{36}$$

and $\epsilon_i=1+\delta_{i1}/|\alpha|^2$. The average photon number $\langle n_3\rangle$ and the average number of atoms with momentum $\vec{p}=(n_0\mp 1)\hbar\vec{q}$, i.e., $\langle n_1\rangle$ and $\langle n_2\rangle$, are the sum of two terms representing the spontaneous and the stimulated emission contributions. The first term originates from the fluctuations of the vacuum state and it is the only contribution in the absence of the initial probe field. We observe that the statistics of the spontaneous emission (for $\alpha=0$) is that of a chaotic (i.e., thermal) state, with number variance $\sigma_i^2\approx\langle n_i\rangle_{\rm sp}(1+\langle n_i\rangle_{\rm sp})$. Instead, if the stimulated emission dominates the spontaneous emission ($\langle n_i\rangle_{\rm st}\gg\langle n_i\rangle_{\rm sp}$), then $\sigma_i^2\approx\langle n_i\rangle_{\rm st}[1+2\langle n_i\rangle_{\rm sp}+(1-\epsilon_i)\langle n_i\rangle_{\rm st}]$. We note that we obtain the Poisson statistics for a coherent field, i.e., $\sigma_i^2\approx\langle n_i\rangle_{\rm st}$, only for $\langle n_i\rangle_{\rm sp}\ll 1$.

We also calculate the expectation value $\langle B^\dagger B\rangle$ for the linearized bunching operator B,

$$\langle B^\dagger B\rangle=\frac{1}{N}[|g_1+h_1|^2+|\alpha|^2|g_3+h_3|^2], \tag{37}$$

from which, using Eq. (20), we obtain the atomic density fluctuations [13]

$$\Delta(\hat{n})=\frac{1}{\pi}\sqrt{\frac{N}{2}}|g_1+h_1|. \tag{38}$$

We calculate now the equal-time intensity correlation and cross-correlation functions [13,18] defined, respectively, as

$$g_i^{(2)}=\frac{\langle a_i^\dagger(\tau)a_i^\dagger(\tau)a_i(\tau)a_i(\tau)\rangle}{\langle n_i(\tau)\rangle^2}, \tag{39}$$

$$g_{i,j}^{(2)}=\frac{\langle n_i(\tau)n_j(\tau)\rangle}{\langle n_i(\tau)\rangle\langle n_j(\tau)\rangle}, \tag{40}$$

with $i=1,2,3$ and $i\neq j$. For a classical field, there is an upper limit to the second-order equal-time cross-correlation function given by the Cauchy-Schwartz inequality

$$g_{i,j}^{(2)}(\tau)\leqslant[g_i(\tau)]^{1/2}[g_j(\tau)]^{1/2}.$$

Quantum-mechanical fields, however, can violate this inequality and are instead constrained by

$$g_{i,j}^{(2)}(\tau)\leqslant\left[g_i^{(2)}(\tau)+\frac{1}{\langle n_i\rangle}\right]^{1/2}\left[g_j^{(2)}(\tau)+\frac{1}{\langle n_j\rangle}\right]^{1/2}, \tag{41}$$

which reduces to the classical results in the limit of large occupation numbers. We obtain the following expressions:

$$g_i^{(2)}=2-\epsilon_i\frac{\langle n_i\rangle_{\rm st}^2}{\langle n_i\rangle^2}, \tag{42}$$

$$g_{1,2}^{(2)}=2+\frac{\langle n_2\rangle_{\rm sp}+\langle n_1\rangle_{\rm st}(1-\epsilon_1\langle n_2\rangle_{\rm st})-|\alpha|^2\langle n_3\rangle_{\rm sp}}{\langle n_1\rangle\langle n_2\rangle}, \tag{43}$$

$$g_{1,3}^{(2)}=2+\frac{\langle n_3\rangle_{\rm sp}+\langle n_1\rangle_{\rm st}(1-\epsilon_1\langle n_3\rangle_{\rm st})-|\alpha|^2\langle n_2\rangle_{\rm sp}}{\langle n_1\rangle\langle n_3\rangle}, \tag{44}$$

$$g_{2,3}^{(2)}=2+\frac{\langle n_1\rangle_{\rm st}-\langle n_2\rangle_{\rm st}\langle n_3\rangle_{\rm st}-|\alpha|^2\langle n_1\rangle_{\rm sp}}{\langle n_2\rangle\langle n_3\rangle}. \tag{45}$$

When the system builds up from noise ($\alpha=0$), $g_i^{(2)}=2$, as expected for a thermal or chaotic field. More interesting informations are obtained from the cross-correlation functions. In the spontaneous case $\alpha=0$, $g_{1,2}^{(2)}=g_{1,3}^{(2)}=2+1/\langle n_1\rangle$, and $g_{2,3}^{(2)}=2$. Hence, both $g_{1,2}^{(2)}$ and $g_{1,3}^{(2)}$ violate the Cauchy-Schwartz inequality, while $g_{2,3}^{(2)}$ is consistent with it. Furthermore, because $\langle n_1\rangle=\langle n_2\rangle+\langle n_3\rangle$, $g_{1,2}^{(2)}$ and $g_{1,3}^{(2)}$ are consistent with the quantum inequality (41), and close to their upper value for $\langle n_3\rangle=0$ and $\langle n_2\rangle=0$, respectively [13,18]. Hence, we expect the existence of nonclassical correlations, as for instance, two-mode entanglement between the modes 1 and 2 or between the modes 1 and 3 when the average occupation numbers of the two modes are equal.

In order to study the entanglement of atoms and photons, it can be useful to calculate also the two-mode relative number squeezing parameter [21],

$$\xi_{i,j}=\frac{\sigma^2(n_i-n_j)}{\langle n_i\rangle+\langle n_j\rangle}, \tag{46}$$

although, as pointed correctly in Ref. [21], squeezing in the relative number of particles between states is *not* equivalent to entanglement. In fact, if the two states are independent and coherent, then $\xi_{i,j}=1$, whereas if they are squeezed, $\sigma^2(n_i-n_j)=0$, which implies $\xi_{i,j}=0$ [22]. Hence, it is expected that when $\xi_{i,j}$ decreases, the entanglement between the two states could improve. However, the condition $\sigma^2(n_i-n_j)=0$ is not a sufficient one for a signature for entanglement, showing only that the state is two-mode squeezed or, more generally, nonclassically correlated. In order to demonstrate the existence of the entanglement, it is necessary to show that the state of the system can not be factorized into the product of the single-mode states [22].

Now, we show that the CARL mechanism applied on a BEC can provide a valuable source for entanglement. In fact, because in the spontaneous case $\alpha=0$, $\xi_{1,2}=\langle n_3\rangle(\langle n_3\rangle+1)/[\langle n_1\rangle+\langle n_2\rangle]$, and $\xi_{1,3}=\langle n_2\rangle(\langle n_2\rangle+1)/[\langle n_1\rangle+\langle n_3\rangle]$, maximum entanglement between the modes 1 and 2 or the modes 2 and 3 should occur when $\langle n_3\rangle=0$ or $\langle n_2\rangle=0$, respectively. To demonstrate that this is not only a necessary condition, but also a sufficient one, we have calculated the exact state vector $|\psi(\tau)\rangle=U(\tau)|0,0,0\rangle$ at the time τ when the system evolves from vacuum. At this point, we have disentangled the evolution operator $U(\tau)=\exp(-iH\tau)$, where H is given by Eq. (17), writing it as the product of individual operators. The calculation, reported in detail in Appendix B, yields

$$|\psi(\tau)\rangle=\frac{1}{\sqrt{1+\langle n_1\rangle}}\sum_{n,m=0}^{\infty}\alpha_1^m\alpha_2^n\sqrt{\frac{(m+n)!}{m!n!}}|m+n,n,m\rangle, \tag{47}$$

where

$$\alpha_1=\frac{f_1g_1^*}{1+\langle n_1\rangle}, \tag{48}$$

$$\alpha_2=\frac{h_1g_1^*}{1+\langle n_1\rangle}. \tag{49}$$

We observe that $|\alpha_{1,2}|^2=\langle n_{3,2}\rangle/(1+\langle n_1\rangle)$. Similar results have been recently obtained also in Ref. [23]. Ideally squeezed states between the modes 1 and 2 or the modes 1 and 3 can be obtained when $\langle n_3\rangle=0$ or $\langle n_2\rangle=0$, respectively, giving

$$|\psi_{1,2}\rangle=\frac{1}{\sqrt{1+\langle n_1\rangle}}\sum_{n=0}^{\infty}\alpha_2^n|n,n,0\rangle, \tag{50}$$

$$|\psi_{1,3}\rangle=\frac{1}{\sqrt{1+\langle n_1\rangle}}\sum_{n=0}^{\infty}\alpha_1^n|n,0,n\rangle. \tag{51}$$

The states (50) and (51) are pure bipartite states and are entangled because the two modes are only generated in pairs. It is known that the von Neumann entropy is a good measure of entanglement for bipartite pure states [24]. So, for these states we can give a measure of the degree of entanglement. If we consider the reduced density operators $\rho_i={\rm Tr}_1[\rho_{1i}]$, where $\rho_{1i}=|\psi_{1i}\rangle\langle\psi_{1i}|$ and $i=2,3$, we obtain the thermal state

$$\rho_i=\frac{1}{1+\langle n_i\rangle}\sum_m\left(\frac{\langle n_i\rangle}{1+\langle n_i\rangle}\right)^m|m\rangle\langle m|, \tag{52}$$

for which the entropy $S_i={\rm Tr}[\rho_i\ln\rho_i]$ is maximum, so that the states (50) and (51) are maximally entangled. The presence of the third mode, in general, reduces the entanglement between the other two modes [21]. We observe also that no two-mode entanglement is possible between the states 2 and 3.

V. HIGH-GAIN REGIME

We discuss the above results in the high-gain regime, i.e., for $|{\rm Im}\lambda_3|\tau\gg1$, where λ_3 is the root of Eq. (30) with negative imaginary part. In this limit, the term proportional to $\exp(i\lambda_3\tau)$ will be dominant in the sum of the expressions (A1)–(A6).

A. The quasiclassical recoil limit $\rho\gg1$

For $\rho\gg1$ and $\delta=0$, $\lambda_3\approx(1-i\sqrt{3})/2$ and

$$\langle n_1\rangle\approx\frac{1}{18}[\rho^2/2+\rho+|\alpha|^2(\rho+1)]e^{\sqrt{3}\tau}, \tag{53}$$

$$\langle n_2\rangle\approx\frac{1}{18}[\rho^2/2+|\alpha|^2(\rho-1)]e^{\sqrt{3}\tau}, \tag{54}$$

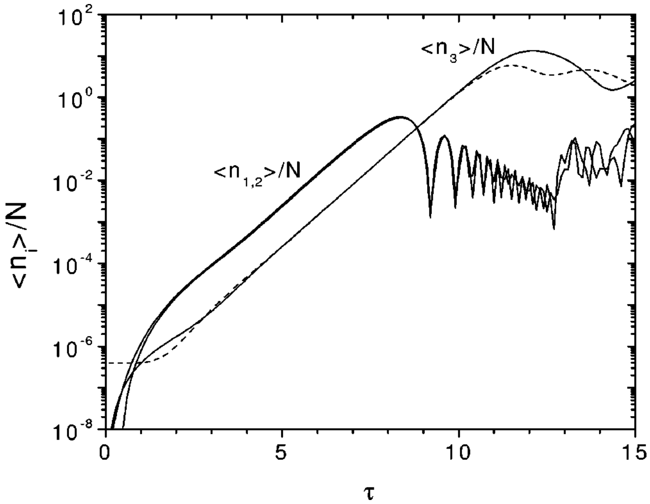

FIG. 3. Nonlinear evolution of the population fractions $\langle n_i\rangle$ for $i=1,2,3$ (continuous lines) and of $(\rho/2)\langle B^+B\rangle$ (dashed line), for $\rho=20$ and $\delta=0$.

$$\langle n_3\rangle \approx \frac{1}{9}(\rho/2+|\alpha|^2)e^{\sqrt{3}\tau}. \tag{55}$$

We observe that $\langle n_1\rangle-\langle n_2\rangle=\langle n_3\rangle$, in agreement with Eq. (18). The stimulated emission dominates the spontaneous emission when $|\alpha|^2\gg\rho/2$. Furthermore, $\langle n_1\rangle\approx\langle n_2\rangle$ and $\langle n_3\rangle\approx(2/\rho)\langle n_1\rangle$, so that the number of emitted photons is much smaller than the occupation number of the motional states. The expectation value of the bunching parameter is

$$\langle B^\dagger B\rangle \approx \frac{1}{9N}[1+(2/\rho)|\alpha|^2]e^{\sqrt{3}\tau}\approx\frac{2}{\rho N}\langle n_3\rangle. \tag{56}$$

Assuming that $\langle B^\dagger B\rangle$ approaches a maximum value of the order of one, then the maximum average number of emitted photons is about $\rho N/2$. Hence, $\rho/2$ can be interpreted as the average number of photons emitted per atom. In order to check the validity of the asymptotic expressions (53)–(56), we have solved numerically the nonlinear Eqs. (8) and (9), treating c_n and a as c numbers. Figure 3 shows the average population fractions $\langle n_i\rangle/N$ with $i=1,2,3$ (continuous lines) and $(\rho/2)\langle B^+B\rangle$ (dashed line), as they result from the simulation with $\rho=20$, $\delta=0$ and a small initial seed simulating a small bunching at $\tau=0$. We observe that Eq. (56) is in a good agreement with the simulation until $\tau\approx12$, up to a maximum value of 0.77 of the expectation value of the bunching operator $\langle B^+B\rangle$. Instead, Eqs. (53) and (54) fit well the numerical result only until $\tau\approx8$, up to the maximum value $\langle n_{1,2}\rangle\approx0.34N$, in agreement with the constraint $\langle n_{1,2}\rangle\leq N/2$ required by the conservation of the atomic number.

When the exponentially growing terms dominate, the relative uncertainty of the occupation numbers reaches a steady-state value given by

$$\frac{\sigma(n_i)}{\langle n_i\rangle}\approx\frac{\sqrt{\rho(\rho+4|\alpha|^2)}}{\rho+2|\alpha|^2} \tag{57}$$

for $i=1,2,3$. We see that for $|\alpha|^2\ll\rho$, $\sigma(n_i)/\langle n_i\rangle\approx1$, as should be for a thermal field when $\langle n_i\rangle\gg1$. Instead, for $|\alpha|^2\gg\rho$, Eq. (57) yields $\sigma(n_i)/\langle n_i\rangle\approx\sqrt{\rho}/|\alpha|\ll1$. Hence, the presence of an initial coherent field decreases the relative uncertainty of the occupation number [9,13]. Finally we note that in the high-gain regime and in the quasiclassical limit $\rho\gg1$, the intensity correlation function for the three modes, Eq. (42), yields

$$g_i^{(2)}=2-\epsilon_i\left(\frac{|\alpha|^2}{\rho/2+|\alpha|^2}\right)^2, \tag{58}$$

which tends to 1 for $|\alpha|^2\gg\rho$ [13].

B. The quantum recoil limit $\rho\leq1$

For $\rho\leq1$, the maximum rate of instability occurs at $\delta=1/\rho$, with $\lambda_3\approx1/\rho-i\sqrt{\rho/2}$ and

$$\langle n_1\rangle\approx\frac{1}{4}[1+(\rho/2)^3+|\alpha|^2]e^{\sqrt{2\rho}\tau}, \tag{59}$$

$$\langle n_2\rangle\approx\frac{1}{4}\left(\frac{\rho}{2}\right)^3(1+|\alpha|^2)e^{\sqrt{2\rho}\tau}, \tag{60}$$

$$\langle n_3\rangle\approx\frac{1}{4}(1+|\alpha|^2)e^{\sqrt{2\rho}\tau}. \tag{61}$$

In this case, the stimulated emission dominates the spontaneous emission when $|\alpha|\gg1$. Furthermore, $\langle n_1\rangle\approx\langle n_3\rangle$ and $\langle n_2\rangle\approx(\rho/2)^3\langle n_1\rangle$, so that the number of atoms $\langle n_2\rangle$ which absorb a photon from the probe gaining a recoil momentum $\hbar q$ is much smaller than the number of atoms $\langle n_1\rangle$ which emit a photon into the probe loosing a recoil momentum. Hence, in the quantum recoil limit, emission dominates absorbtion.

The expectation value of the bunching parameter is

$$\langle B^\dagger B\rangle\approx\frac{1}{4N}[1+|\alpha|^2]e^{\sqrt{2\rho}\tau}=\frac{1}{N}\langle n_3\rangle. \tag{62}$$

When $\langle B^\dagger B\rangle$ reaches a maximum value of the order of one, then the maximum average number of emitted photons is about N, i.e., all the atoms are transferred from the ground motional state n_0 to the side-mode state n_0-1.

In the asymptotic limit $\sqrt{2\rho}\tau\gg1$, the relative uncertainty of the occupation numbers are

$$\frac{\sigma(n_1)}{\langle n_1\rangle}\approx\frac{1}{\sqrt{1+|\alpha|^2}}, \tag{63}$$

$$\frac{\sigma(n_2)}{\langle n_2\rangle}\approx\frac{\sigma(n_3)}{\langle n_3\rangle}\approx\frac{\sqrt{1+2|\alpha|^2}}{1+|\alpha|^2}, \tag{64}$$

whereas

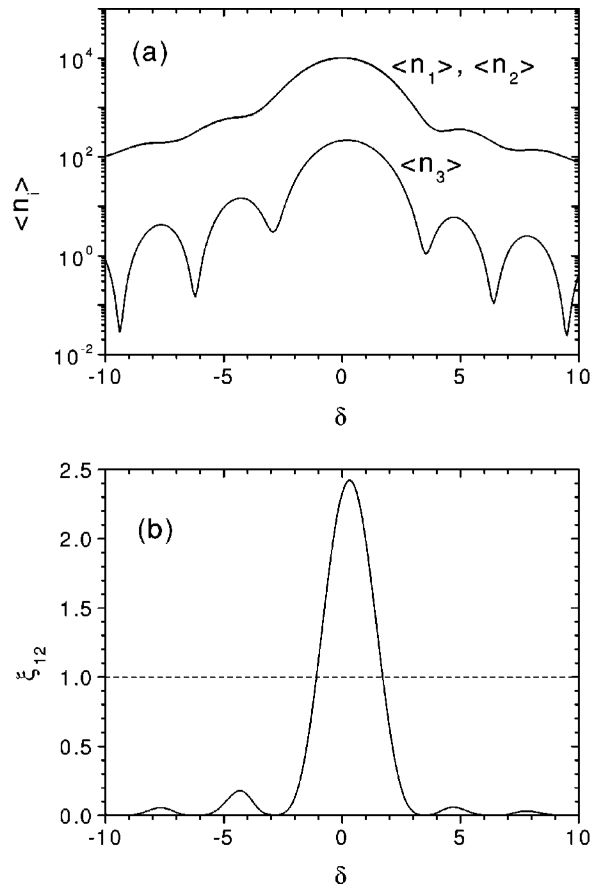

FIG. 4. Atom-atom entanglement: occupation numbers $\langle n_i \rangle$ ($i=1,2,3$) (a) and two-mode squeezing parameter $\xi_{1,2}$ (b) as a function of δ for $\rho=100$, $\tau=2$ and $\alpha=0$.

$$g_i^{(2)}=2-\epsilon_i\left(\frac{|\alpha|^2}{1+|\alpha|^2}\right)^2, \tag{65}$$

tending again to 1 for $|\alpha|^2 \gg 1$ [13].

VI. ATOM-ATOM AND ATOM-PHOTON ENTANGLEMENT

Now, we show the existence of two different regimes of the CARL, in which the initial vacuum state evolves into an entangled state. In particular, atom-atom entanglement can be obtained only in the limit $\rho \gg 1$ and in a detuned, not fully exponential regime. On the contrary, in the limit $\rho < 1$, atom-photon entanglement can be obtained in the exponential regime if the average occupation number $\langle n_2 \rangle$ remains smaller than one. Relative atom-photon number squeezing has been considered also in Refs. [21,25]. In these works it has been noted that, for typical experimental parameters, the squeezing can be strongly limited by collisions and dissipative processes.

As an example of atom-atom entanglement, we show in Fig. 4 the average occupation numbers $\langle n_i \rangle$ ($i=1,2,3$) (a) and the two-mode squeezing parameter $\xi_{1,2}$ (b) as a function of δ for $\rho=100$, $\alpha=0$, and $\tau=2$. Since $\langle n_3 \rangle \ll \langle n_1 \rangle \approx \langle n_2 \rangle$, there exists a region where $\xi_{1,2}$ is less than one, and the state has a form similar to that of Eq. (50). In general, this kind of entanglement is not very efficient because the number of atoms in each quantum state does not grow exponentially and it remains of the order of ρ.

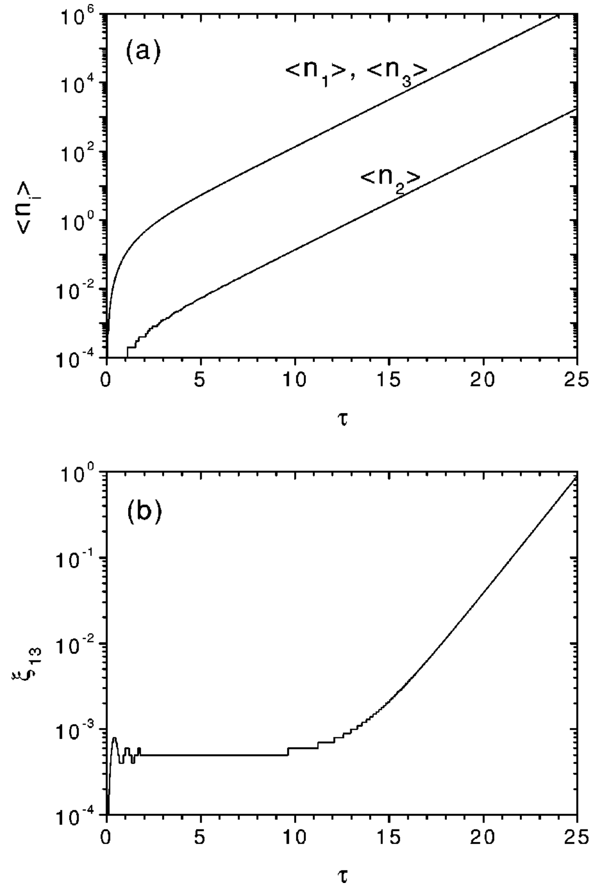

FIG. 5. Atom-photon entanglement: occupation numbers $\langle n_i \rangle$ ($i=1,2,3$) (a), and two-mode squeezing parameter $\xi_{1,3}$ (b), as a function of τ for $1/\rho=\delta=10$ and $\alpha=0$.

Atom-photon entanglement is more easily obtained in the limit $\rho < 1$ because in this case $\langle n_2 \rangle \ll \langle n_1 \rangle \approx \langle n_3 \rangle$ also in the exponential regime. In fact, in this case

$$\xi_{1,3}=\frac{\sigma^2(n_2)}{\langle n_1 \rangle+\langle n_3 \rangle}=\frac{\langle n_2 \rangle(1+\langle n_2 \rangle)}{\langle n_1 \rangle+\langle n_3 \rangle}, \tag{66}$$

where $\langle n_i \rangle$ are given by Eqs. (59)–(61). Since $\langle n_i \rangle$, for $i=1,2,3$, grow exponentially, $\xi_{1,3}$ remains smaller than one only for $\langle n_2 \rangle < 1$. Hence, from Eq. (60), it follows that atom-photon entanglement occurs for

$$\tau < \frac{1}{\sqrt{2\rho}}(5 \ln 2 - 3 \ln \rho). \tag{67}$$

In Fig. 5, we show $\langle n_i \rangle$, ($i=1,2,3$) (a) and the two-mode squeezing parameter $\xi_{1,3}$ (b) between atoms in the momentum state n_0-1 and photons as a function of τ for $1/\rho=\delta=10$ and $\alpha=0$. We observe that the average number of atoms in the momentum state n_0+1, $\langle n_2 \rangle$, is more than three decades smaller than the average number of atoms in the momentum state n_0-1. Hence, the state of the system reduces from the three-mode entangled state (47) to the two-mode entangled state described by Eq. (51). As a confirmation of this, $\xi_{1,3}$ in Fig. 5(b) remains much smaller than one, also in the exponential regime, until $\langle n_2\rangle$ becomes larger than one.

VII. CONCLUSIONS

We have presented a complete quantum description of the collective atomic recoil laser, in which a Bose-Einstein condensate driven by a detuned laser field is coupled to a single-mode quantized radiation field. We have calculated analytically the temporal evolution of the Heisenberg operators and of the state when the system starts from vacuum, and we have investigated the statistical properties of the atom and photon distributions. The calculation of the evolution of the state allows for a detailed description of the entanglement between atoms and photons. In particular, we have shown that the general state may reduce to a thermal atom-atom and atom-photon entangled state. Finally, we have calculated the asymptotic expression of the average numbers of photon and atoms in the exponential regime and in the quasiclassical and quantum limits.

The present theory is valid only in the linear regime, when the atomic ground-state depletion and saturation of the radiation mode are neglected. Furthermore, we have neglected inhomogeneous effects due to the finite spatial extension of the atomic cloud and two-atom collisions. The effect of collisions has been considered recently in Ref. [25], showing that it can seriously limit the coherence of the scattering process and the entanglement. Following the approach of Ref. [25], it will be of considerable interest to extend the analytical description of the linear regime of the quantum CARL to include this and other possible sources of decoherence. Moreover, the CARL Hamiltonian model may be extended to include a dissipative mechanism for the radiation mode, allowing for a quantum description of the statistical properties of the super-radiant regime, recently observed experimentally [3,5].

ACKNOWLEDGMENTS

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APPENDIX A

The expressions of the quantities f_i, g_i and h_i ($i=1,2,3$) which appear in the general solution of the linear problem, Eqs. (27)–(29), are

$$f_1(\tau)=-i\sqrt{\frac{\rho}{2}}\sum_{j=1}^{3}(\lambda_j+1/\rho)\frac{e^{i\lambda_j\tau}}{\Delta_j}=g_3(\tau), \quad \text{(A1)}$$

$$f_2(\tau)=-i\sqrt{\frac{\rho}{2}}\sum_{j=1}^{3}(\lambda_j-1/\rho)\frac{e^{i\lambda_j\tau}}{\Delta_j}=-h_3(\tau), \quad \text{(A2)}$$

$$f_3(\tau)=\sum_{j=1}^{3}(\lambda_j^2-1/\rho^2)\frac{e^{i\lambda_j\tau}}{\Delta_j}; \quad \text{(A3)}$$

$$g_1(\tau)=\sum_{j=1}^{3}[(\lambda_j-\delta)(\lambda_j+1/\rho)-\rho/2]\frac{e^{i\lambda_j\tau}}{\Delta_j}, \quad \text{(A4)}$$

$$g_2(\tau)=-\frac{\rho}{2}\sum_{j=1}^{3}\frac{e^{i\lambda_j\tau}}{\Delta_j}=-h_1(\tau), \quad \text{(A5)}$$

$$h_2(\tau)=\sum_{j=1}^{3}[(\lambda_j-\delta)(\lambda_j-1/\rho)+\rho/2]\frac{e^{i\lambda_j\tau}}{\Delta_j}, \quad \text{(A6)}$$

where $\Delta_j=\lambda_j(3\lambda_j-2\delta)-1/\rho^2$ and λ_1, λ_2 and λ_3 are the roots of the cubic Eq. (30). Since the commutation rules for the operators a_i and $a_i^\dagger$ are preserved at the time τ, they imply the following relations between the functions f_i, g_i, and h_i:

$$1+|f_1|^2=|f_2|^2+|f_3|^2, \quad \text{(A7)}$$

$$|g_1|^2=1+|g_2|^2+|g_3|^2, \quad \text{(A8)}$$

$$1+|h_1|^2=|h_2|^2+|h_3|^2; \quad \text{(A9)}$$

$$g_1f_1^*=g_2f_2^*+g_3f_3^*, \quad \text{(A10)}$$

$$g_1h_1^*=g_2h_2^*+g_3h_3^*, \quad \text{(A11)}$$

$$h_1f_1^*=h_2f_2^*+h_3f_3^*. \quad \text{(A12)}$$

APPENDIX B

We show that the evolution operator $U(\tau)=\exp(-iH\tau)$, where H is given by Eq. (17), can be disentangled into those of individual operators. This allows us to calculate how the state $|\psi(\tau)\rangle$ evolves from the vacuum state $|0,0,0\rangle$.

Introducing the five operators $J_1=a_1a_1^\dagger+a_3^\dagger a_3$, $J_2=a_3^\dagger a_3-a_2^\dagger a_2$, $J=a_2a^\dagger{}_3$, $K=a_1a_3$ and $M=a_1a_2$, the Hamiltonian (17) can be written in the following form:

$$H=C'+\frac{\delta}{3}(J_2-J_1)+\frac{1}{\rho}(J_1+J_2)+i\sqrt{\rho/2}(J-K-J^\dagger+K^\dagger), \quad \text{(B1)}$$

where $C'=(\delta/3)(1+2C)-1/\rho$ and C is the constant of motion (18). The operators J_1, J_2, J, K, and M satisfy the following commutation relations:

$$[J,J^\dagger]=-J_2, \quad \text{(B2)}$$

$$[K,K^\dagger]=J_1, \quad \text{(B3)}$$

$$[M,M^\dagger]=J_1+J_2, \quad \text{(B4)}$$

$$[J_1,J]=J, \quad \text{(B5)}$$

$$[J_2,J]=-2J, \quad \text{(B6)}$$

$$[K,J_1]=2K, \tag{B7}$$

$$[K,J_2]=-K, \tag{B8}$$

$$[M,J_1]=[M,J_2]=M, \tag{B9}$$

$$[K,J]=M, \tag{B10}$$

$$[M,J^\dagger]=K, \tag{B11}$$

$$[M,K^\dagger]=J, \tag{B12}$$

and $[J_1,J_2]=[J^\dagger,K]=[J,M]=[K,M]=0$. Hence, they form a closed algebra. We observe that the operators $J_x=i(M-M^\dagger)$, $J_y=i(K-K^\dagger)$ and $J_z=i(J-J^\dagger)$ are the generators of SU(1,1) Lie algebra, whose statistical properties have been extensively discussed in Refs. [26,27]. Since C' commutes with H, we can write $H=C'+H'$ and neglect the inessential phase factor $e^{-iC'\tau}$ in the evolution operator $U(\tau)$, that can be written in the form of a Baker-Hausdorff equation

$$U(\tau)=e^{-iH'\tau}=e^{\alpha_1K^\dagger}e^{\alpha_2M^\dagger}e^{\alpha_3J^\dagger}e^{\alpha_4J_1}e^{\alpha_5J_2}e^{\alpha_6J}e^{\alpha_7K}e^{\alpha_8M}, \tag{B13}$$

where α_i $(i=1,\ldots 8)$ are complex functions depending on τ. Applying $U(\tau)$ to the vacuum state, we obtain

$$|\psi\rangle=U(\tau)|0,0,0\rangle=e^{\alpha_4}e^{\alpha_1K^\dagger}e^{\alpha_2M^\dagger}|0,0,0\rangle$$
$$=e^{\alpha_4}\sum_{m,n=0}^{\infty}\alpha_1^m\alpha_2^n\sqrt{\frac{(m+n)!}{m!n!}}|m+n,n,m\rangle. \tag{B14}$$

The constant α_4 can be obtained from the normalization condition

$$\langle\psi|\psi\rangle=1=e^{2\,\mathrm{Re}(\alpha_4)}\sum_{m,n=0}^{\infty}\frac{(m+n)!}{m!n!}|\alpha_1|^{2m}|\alpha_2|^{2n}$$
$$=\frac{e^{2\mathrm{Re}(\alpha_4)}}{1-|\alpha_1|^2-|\alpha_2|^2}, \tag{B15}$$

where we used the formula

$$\sum_{n=0}^{\infty}\Gamma(a+n)\frac{z^n}{n!}=\Gamma(a)(1-z)^{-a} \tag{B16}$$

and $\Gamma(a)$ is the Gamma function. Hence, neglecting the imaginary part of α_4,

$$e^{\alpha_4}=\sqrt{1-|\alpha_1|^2-|\alpha_2|^2}. \tag{B17}$$

The calculation of the average occupation numbers for the modes 2 and 3, using the state (B14), yield

$$\langle n_2\rangle=\frac{|\alpha_2|^2}{1-|\alpha_1|^2-|\alpha_2|^2}, \tag{B18}$$

$$\langle n_3\rangle=\frac{|\alpha_1|^2}{1-|\alpha_1|^2-|\alpha_2|^2}, \tag{B19}$$

which, once inverted and because $\langle n_1\rangle=\langle n_2\rangle+\langle n_3\rangle$, gives

$$e^{\alpha_4}=\frac{1}{\sqrt{1+\langle n_1\rangle}}. \tag{B20}$$

The two functions α_1 and α_2 can be obtained by calculating the expectation values of a_1a_3 and $a_2^\dagger a_3$ and using the Heisenberg picture of the operators, Eqs. (27)–(29),

$$\langle a_1a_3\rangle=f_2g_2^*+f_3g_3^*=f_1g_1^*, \tag{B21}$$

$$\langle a_2^\dagger a_3\rangle=f_1h_1^*, \tag{B22}$$

where we used Eq. (A10). Conversely, evaluating these expectation values using the state (B14), we obtain

$$\langle a_1a_3\rangle=e^{2\alpha_4}\alpha_1, \tag{B23}$$

$$\langle a_2^\dagger a_3\rangle=e^{2\alpha_4}\alpha_1\alpha_2^*. \tag{B24}$$

Finally, combining the two results, we obtain after some algebra:

$$\alpha_1=\frac{f_1g_1^*}{1+\langle n_1\rangle}, \tag{B25}$$

$$\alpha_2=\frac{h_1g_1^*}{1+\langle n_1\rangle}. \tag{B26}$$

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