ON COMPACT AFFINE QUATERNIONIC CURVES AND SURFACES

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Abstract. This paper is devoted to the study of affine quaternionic manifolds and to a possible classification of all compact affine quaternionic curves and surfaces. It is established that on an affine quaternionic manifold there is one and only one affine quaternionic structure. A direct result, based on the celebrated Kodaira Theorem that classifies all compact complex manifolds in complex dimension 2, states that the only compact affine quaternionic curves are the quaternionic tori. As for compact affine quaternionic surfaces, the study of their fundamental groups, together with the inspection of all nilpotent hypercomplex simply connected 8-dimensional Lie Groups, identifies a path towards their classification.

1. Introduction

The definition of slice regularity for functions of one and several quaternionic variables (see, e.g., [11, 12]) has led to a renewed interest for a direct approach to the study of quaternionic manifolds. Quaternionic manifolds, as spaces locally modelled on $\mathbb{H}^n$ in a slice regular sense, are presented in [9] with the name of *quaternionic regular manifolds*, and the closely related class of quaternionic toric manifolds is studied in [10]. In this setting, the class of *affine quaternionic manifolds* - containing those manifolds that admit a quaternionic affine structure - reveals to be of natural interest, both because of the well established interest for affine complex manifolds, and for the reason that most of the natural quaternionic manifolds already studied are indeed affine quaternionic manifolds.

The main purpose of this paper is to find a path towards a classification of all compact affine quaternionic curves and surfaces.

The well celebrated Kodaira Theorem exhibits a list of all compact complex manifolds in complex dimension 2. Among these manifolds, Vitter [20], Matsushima [16] and Inoue, Kobayashi, Ochiai [13] identify all those admitting a complex affine structure. Since affine quaternionic curves are affine

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complex surfaces, we can then prove that the only 1 dimensional compact affine quaternionic manifolds are the quaternionic tori studied in [4].

In quaternionic dimension 2, the lack of a classification of affine compact complex manifolds of dimension 4 advises us to change point of view in order to classify the compact affine quaternionic surfaces. We adopt in fact the approach used in [7], based on the study of the fundamental groups of compact affine complex surfaces, and prove the following result

Theorem 1.1. *If the subgroup $\Gamma \subseteq Aff(2,\mathbb{H})$ acts freely and properly discontinuously on $\mathbb{H}^2$, and $\mathbb{H}^2/\Gamma$ is compact, then Γ contains a unipotent normal subgroup Γ_0 of finite index such that Γ/Γ_0 is isomorphic to a finite subgroup of $\mathbb{S}^3$.*

which, thanks to a Theorem due to Malcev, [15], has the following consequence

Corollary 1.2. *Let the subgroup $\Gamma \subseteq Aff(2,\mathbb{H})$ act freely and properly discontinuously on $\mathbb{H}^2$, and assume that $\mathbb{H}^2/\Gamma$ is compact. Let $\Gamma_0 \subseteq \Gamma$ be a unipotent normal subgroup of finite index such that Γ/Γ_0 is isomorphic to a finite subgroup of $\mathbb{S}^3$ (see Theorem 1.1). Then Γ_0 is a discrete subgroup of a nilpotent hypercomplex simply connected 8-dimensional Lie Group N such that N/Γ_0 is compact.*

This corollary - together with the classification of all nilpotent hypercomplex simply connected 8-dimensional Lie Groups given by Dotti and Fino, [7] - indicates a path towards the classification of compact affine quaternionic surfaces that we will follow in a forthcoming paper.

2. AFFINE QUATERNIONIC MANIFOLDS

In this setting the Dieudonné determinant $\det_{\mathbb{H}}$ plays a similar role as the usual one. To each quaternionic matrix we can associate a complex matrix via the algebra homomorphism

$$\psi : \mathcal{M}(n,\mathbb{H}) \to \mathcal{M}(2n,\mathbb{C})$$

defined by

$$\psi(A+Bj) = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix},$$

and it turns out that $(\det_{\mathbb{H}}(M))^2 = \det(\psi(M))$, where the right hand term is the usual determinant, see [1]. Hence, the group of quaternionic $n \times n$ invertible matrices $GL(n,\mathbb{H})$ can be introduced in the usual fashion via the Dieudonné determinant.

For $Q = {}^t(q_1, q_2, \ldots, q_n) \in \mathbb{H}^n$, we can define the group of all quaternionic affine transformations

$$Aff(n,\mathbb{H}) = \{Q \mapsto AQ + B : A \in GL(n,\mathbb{H}), B = {}^t(b_1, b_2, \ldots, b_n) \in \mathbb{H}^n\}$$

which is included in the class of (right) slice regular functions, [12]. In complete analogy with what Kobayashi does in the complex case, we give the following:

Definition 2.1. A differentiable manifold M of $4n$ real dimensions has a *quaternionic affine structure* if it admits a differentiable atlas whose transition functions are restrictions of quaternionic affine functions of $\mathit{Aff}(n,\mathbb{H})$.

In particular, differentiable manifolds endowed with a quaternionic affine structure are quaternionic regular [9], [10]. This fact can be used to construct a large class of quaternionic regular manifolds; indeed, for any subgroup $\Gamma \subset \mathit{Aff}(n,\mathbb{H})$ which acts freely and properly discontinuously on $\mathbb{H}^n$, the quotient space

$$M = \mathbb{H}^n/\Gamma$$

admits an atlas whose transition functions are slice regular belonging to $\Gamma \subset \mathit{Aff}(n,\mathbb{H})$, and hence has a quaternionic affine structure. It is worthwhile noticing that the quaternionic manifolds studied by Sommese, [19], all admit a quaternionic affine structure, and hence they are quaternionic regular. Many significant examples can be found in his paper.
The relation between affine complex structures and flat connections in the complex setting has been deeply investigated during the past years. In particular in the complex setting a theorem of Matsushima [16] states that there is a one-to-one correspondence between affine structures and affine holomorphic connections which are torsion-free and flat. Vitter [20], Inoue, Kobayashi e Ochiai [13] gave a classification of all manifolds admitting such connections in complex dimension 1 and 2.
A similar correspondence holds also in the quaternionic setting. But the quaternionic structures are much more rigid. A manifold is said to admit a $GL(n,\mathbb{H})$-structure if it can be endowed with two anticommunting almost complex structures, see [18, page 48]. On such manifolds, also called *almost quaternionic* by Sommese in [19], it is possible to define a connection, the Obata connection, which is torsion free, and it is the only one with this property. Moreover the Obata connection turns out to be flat if and only if the $GL(n,\mathbb{H})$-structure is integrable (if and only if M is quaternionic in the sense Sommese).

For an almost quaternionic manifold, having an integrable $GL(n,\mathbb{H})$-structure is equivalent to the nullity of three tensors which, in the quaternionic setting, play the role of the Nijenhuis tensor, [17]. Summarizing

Proposition 2.2. *A manifold M has an integrable $GL(n,\mathbb{H})$-structure if and only if it is affine quaternionic and equivalently if and only if one can define on it a connection that is torsion free and flat, the Obata connection.*

Moreover

Remark 2.3. An affine quaternionic manifold is *hypercomplex*, since the integrability of the $GL(n,\mathbb{H})$-structure implies that it can be endowed with two anti-commuting complex structures.

Thanks to the one-to-one correspondence between affine structures and flat torsion free holomorphic connections in the complex setting, and the uniqueness of the Obata connection on an affine quaternionic manifold, we obtain that

Corollary 2.4. *On an affine quaternionic manifold there is one and only one affine quaternionic structure.*

In the complex setting the situation is quite different. Indeed a fixed affine compact complex manifold may have a number of distinct affine structures which all induce the given complex structure, that is, it may have affine structures which are complex analytically but not affinely equivalent. As an example of this phenomenon, consider a complex one dimensional torus $T = \frac{\mathbb{C}}{\Lambda}$. The usual affine coordinate on T is the coordinate z of the universal cover of T, defined locally on T. But there are other distinct affine structures on T, in fact, there is an affine structure on T whose coordinate is

$$\frac{1}{a}(e^{az} - 1) = \sum_{k}^{\infty} \frac{a^{k-1} z^k}{k!}$$

where z is the usual affine coordinate mentioned above.

3. Towards a classification of affine quaternionic manifolds in low dimension

In order to classify all the affine quaternionic manifolds in low dimensions, one can try to argue as in the complex case. In the complex setting, however, the Kodaira Theorem gives a list in complex dimension 2 of compact complex manifolds on which one can look at the affine structures or, equivalently, at the affine holomorphic flat connections. In the quaternionic setting, also in low dimensions (in quaternionic dimension 2 for example) a similar list does not exist. For the quaternionic dimension 1, one can argue as follows: if M is affine quaternionic of dimension 1 it is also complex affine, so one can go through Vitter's classification, [20], of affine compact complex manifolds of complex dimension 2, and determine those admitting a quaternionic affine structure. Going through this list it is not difficult to see that there are no examples, a part from the quaternionic tori that have already been found in [4]. These exhaust the affine quaternionic manifolds in quaternionic dimension 1.

In quaternionic dimension 2, some examples of affine quaternionic manifolds are given by slice affine quaternionic Hopf surfaces, [2], and by some tori that one can construct by adapting the strategy in [20] to complex dimension 4.

Since we cannot refer to a classification of affine compact complex manifolds of dimension 4, in order to classify completely the affine quaternionic manifolds in dimension 2, we adopt the approach used in [8], based on the study of the fundamental groups of affine compact complex surfaces.

Our aim is to find necessary conditions on a discrete subgroup Γ of the group of quaternionic affine transformations $\mathit{Aff}(2,\mathbb{H})$ so that its action on $\mathbb{H}^2$ is free and properly discontinuous. In what follows we identify $\mathit{Aff}(2,\mathbb{H})$ with the group of invertible 3×3 matrices with quaternionic entries of the form

$$\begin{pmatrix} a & b & r \\ c & d & s \\ 0 & 0 & 1 \end{pmatrix}$$

In this section we study properties of a subgroup Γ of the group $\mathit{Aff}(2,\mathbb{H})$, acting *freely* on $\mathbb{H}^2$. The action of $A \in \Gamma$ on $\mathbb{H}^2$ on the left maps (x, y) in (x', y') where

$$\begin{cases} x' = ax + by + r \\ y' = cx + dy + s \end{cases}$$

With the usual notation, we denote with $h(A)$ the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $GL(2,\mathbb{H})$, called the *holonomy part* of A.

We first recall a few definitions and well known facts about eigenvalues and eigenvectors in the quaternionic setting. We refer to [6],[14] for an exhaustive treatment of quaternionic linear algebra. In the study of spectral theory in the quaternionic setting one has to define what is an eigenvalue for a matrix A, indeed, once chosen the left action of the matrix, one can state the "right eigenvalue problem" and the "left eigenvalue problem" according to the position of the eigenvalue. We will focus on the right eigenvalue problem.

Definition 3.1. Let A be a $n \times n$-quaternionic matrix. Then $\lambda \in \mathbb{H}$ is a *right eigenvalue* for A if and only if there exists a nonzero $v \in \mathbb{H}^n$ such that

$$Av = v\lambda.$$

In this case v is called an *eigenvector* of A.

Remark 3.2. If $\lambda \in \mathbb{H}$ is an eigenvalue of a quaternionic matrix A, then all the elements in the 2-sphere $S_\lambda = \{u^{-1}\lambda u : 0 \neq u \in \mathbb{H}\}$ of all conjugates of λ turn out to be eigenvalues of A: if $Av = v\lambda$, then $A(vu) = (vu)u^{-1}\lambda u$ for any invertible $u \in \mathbb{H}$.

Remark 3.3. If v is an eigenvector of a quaternionic matrix A, with eigenvalue λ, then $v\mu$ with $\mu \in \mathbb{H}, \mu \neq 0$, is an eigenvector with respect to the eigenvalue $\mu^{-1}\lambda\mu$ in the sphere S_λ.

Proposition 3.4. *Let $M \in \mathcal{M}(n,\mathbb{H})$ be a quaternionic matrix. Then $\lambda \in \mathbb{H}$ is a right eigenvalue of M if and only if there exists a complex $\tilde\lambda \in S_\lambda$ such that*

$$\det_{\mathbb{H}}(M - \tilde\lambda I_n) = 0.$$

Proof. The quaternion λ is a right eigenvalue of M if and only if every element in S_λ is. Let $\tilde{\lambda} \in S_\lambda$ be a complex eigenvalue of M; then $\tilde{\lambda}$ is an eigenvalue of $\psi(M)$, and hence

$$0 = \det(\psi(M) - \tilde{\lambda} I_{2n}) = \det(\psi(M - \tilde{\lambda} I_n)) = (\det_{\mathbb{H}}(M - \tilde{\lambda} I_n))^2.$$

□

We point out that right eigenvalues are shared by similar matrices: if $Av = v\lambda$, then $M^{-1}AM(M^{-1}v) = (M^{-1}v)\lambda$ for any invertible quaternionic matrix M. The same is not true when considering left eigenvalues. In addition, a quaternionic matrix is diagonalisable if and only if its complex representation is diagonalisable.

Lemma 3.5. *Let the subgroup $\Gamma \subseteq Aff(2, \mathbb{H})$ act freely on $\mathbb{H}^2$. Then each element of $h(\Gamma)$ has 1 as an eigenvalue.*

Proof. Let $A \in \Gamma$. The point $(x, y) \in \mathbb{H}^2$ is fixed by

$$A = \begin{pmatrix} a & b & r \\ c & d & s \\ 0 & 0 & 1 \end{pmatrix}$$

if and only if

$$\begin{cases} (a-1)x + by = -r \\ cx + (d-1)y = -s \end{cases}$$

If 1 is not an eigenvalue of $h(A)$, then $(A - I) \in GL(2, \mathbb{H})$ and hence the linear system has a solution; thus the action is not free. □

Let us now define two groups of quaternionic matrices,

$$G_1 = \left\{ \begin{pmatrix} a & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} : a, b, r, s, \in \mathbb{H}, a \neq 0 \right\}$$

and

$$G_2 = \left\{ \begin{pmatrix} 1 & b & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix} : b, r, s, d \in \mathbb{H}, d \neq 0 \right\},$$

which play a key role in the study of subgroups of $Aff(2, \mathbb{H})$ acting freely on $\mathbb{H}^2$.

Proposition 3.6. *Let the subgroup $\Gamma \subseteq Aff(2, \mathbb{H})$ act freely on $\mathbb{H}^2$. Then Γ is conjugate in $Aff(2, \mathbb{H})$ to a subgroup of G_1 or G_2.*

Proof. Suppose first that Γ contains an element A such that $h(A)$ has an eigenvalue $\lambda \neq 1$. Then, we can diagonalise $h(A)$ via a matrix P in $\mathrm{GL}(2, \mathbb{H})$. Suppose

$$B \in \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \Gamma \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$

Write $h(B)=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, then $Ph(A)P^{-1}h(B)=\begin{pmatrix} \lambda & 0\\ 0 & 1\end{pmatrix}h(B)=\begin{pmatrix} \lambda a & \lambda b\\ c & d\end{pmatrix}$.
By the previous lemma, both $h(B)$ and $Ph(A)P^{-1}h(B)$ have 1 as an eigenvalue, so there exist (x,y) and $(z,w)\in\mathbb{H}^2$ such that

$$\begin{cases} ax+by=x\\ cx+dy=y\end{cases} \quad\text{and}\quad \begin{cases} \lambda az+\lambda bw=z\\ cz+dw=w\end{cases}.$$

Suppose first that $y\neq 0$ and $w\neq 0$. Hence, up to a rescaling of the eigenvector (note that, in general, thanks to Remark 3.3, the corresponding eigenvalue changes, remaining in the same sphere; in the present case the real eigenvalue does not change), we can suppose that $y=w$. In this case, subtracting the second equations of the systems, we get $c(x-z)=0$ which implies either $c=0$ or $x=z$.

- If $c=0$ then $d=1$ (since $y\neq 0$);
- If $x=z$ we get $\lambda=1$ (a contradiction) or $x=z=0$. If $x=z=0$ then $b=0$ (and hence again $d=1$ since $y\neq 0$).

Suppose now that $x\neq 0$ and $z\neq 0$; then again we can assume that $x=z$, and with straightforward computations we get $d(y-w)=y-w$; thus $d=1$ or $y=w$.

- If $d=1$ then $c=0$ (since $x\neq 0$);
- if $y=w\neq 0$ we get $\lambda=1$ (a contradiction) or $y=w=0$. If $y=w=0$ then $c=0$ and $a=1$.

If now $y=0$ (and necessarily $x\neq 0$) and $z=0$ (and necessarily $w\neq 0$), we get $c=0$ and $a=1$. If instead $x=0$ (and necessarily $y\neq 0$) and $w=0$ (and necessarily $z\neq 0$, we get $b=0$ and $d=1$.
So the possibilities for $h(B)$ are: if $b=0$,

$$\begin{pmatrix} a & 0\\ c & 1\end{pmatrix};$$

or, if $c=0$

$$\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix};$$

Note that we cannot have both kinds of $h(B)$ occurring, for if both $\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix} a' & 0\\ c' & 1\end{pmatrix}$ where in $Ph(\Gamma)P^{-1}$ with $b\neq 0$ and $c'\neq 0$ also their product $\begin{pmatrix} aa'+bc' & b\\ c' & 1\end{pmatrix}$ would belong to it, but it is easy to prove that this matrix does not have 1 as eigenvalue. Hence we have that

$$\begin{pmatrix} P & 0\\ 0 & 1\end{pmatrix}\Gamma\begin{pmatrix} P & 0\\ 0 & 1\end{pmatrix}^{-1}$$

is contained in G_1 or in the group of all quaternionic matrices of the form $\begin{pmatrix} a & 0 & r \\ c & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$; the latter is conjugate to G_2 via an element of type $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and we are done.

Now suppose every element of $h(\Gamma)$ has both eigenvalues 1. If $h(\Gamma)$ is the identity, we are done, otherwise some conjugate of Γ contains an element of the form $L = \begin{pmatrix} 1 & 1 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}$. Let $C = \begin{pmatrix} a & b & r \\ c & d & s \\ 0 & 0 & 1 \end{pmatrix}$ be an arbitrary element of this conjugate of Γ. Then $h(C) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $h(LC) = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$ have 1 as eigenvalue with molteplicity 2. A direct computation implies again that $c = 0$ and $a = d = 1$. □

In the complex case, G_1 and G_2 turn out to be solvable; we point out that this is not the case in the quaternionic setting due to the non commutativity of $\mathbb{H}$.

Lemma 3.7. *If the subrgroup $\Gamma \subseteq Aff(2, \mathbb{H})$ acts freely on $\mathbb{H}^2$ and $h(\Gamma)$ is abelian then Γ is conjugate in $Aff(2, \mathbb{H})$ to a subgroup of the group of all matrices of the form* $\begin{pmatrix} 1 & 0 & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix}$ *with $d \neq 0$ or to a subgroup of the group of all matrices of the form* $\begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$

Proof. The proof given for the complex case in [Fillmore, Lemma 2.4] can be easily adapted to matrices with quaternionic entries. □

Lemma 3.8. *If* $\begin{pmatrix} a & b & r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *has no fixed points in $\mathbb{H}^2$ then $a = 1$ and $b = 0$.*

Proof. If $b \neq 0$ then $(0, -b^{-1}r, 1)$ is a fixed point. Now, suppose $b = 0$; if $a \neq 1$ then $(-(a-1)^{-1}r, y, 1)$ is a fixed point. Hence the assertion follows. □

Lemma 3.9. *If the subgroup $\Gamma \subseteq G_1$ acts freely on $\mathbb{H}^2$ then $h(\Gamma)$ is abelian and there exists a complex plane containing all a such that*

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in h(\Gamma).$$

Proof. Let

$$A = \begin{pmatrix} a & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} a' & b' & r' \\ 0 & 1 & s' \\ 0 & 0 & 1 \end{pmatrix}$$

be elements of Γ. By direct computation, it is easy to verify that their inverse elements are the quaternionic matrices

$$A^{-1} = \begin{pmatrix} a^{-1} & -a^{-1}b & -a^{-1}r + a^{-1}bs \\ 0 & 1 & -s \\ 0 & 0 & 1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} a'^{-1} & -a'^{-1}b' & -a'^{-1}r' + a'^{-1}b's' \\ 0 & 1 & -s' \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, for some $c \in \mathbb{H}$,

$$ABA^{-1}B^{-1} = \begin{pmatrix} aa'a^{-1}a'^{-1} & aa'(-a^{-1}a'^{-1}b' - a^{-1}b) + ab' + b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From the fact that $ABA^{-1}B^{-1}$ acts without fixed points, applying Lemma 3.8, we get that

$$(ABA^{-1}B^{-1})_{12} = 0 \quad \text{and} \quad aa'a^{-1}a'^{-1} = 1$$

which immediately imply that a and a' belong to the same complex plane, and that $h(\Gamma)$ is abelian. $\square$

Now, combining Lemmas 3.7 and 3.9, we get

Corollary 3.10. *If the subgroup $\Gamma \subseteq Aff(2, \mathbb{H})$ acts freely on $\mathbb{H}^2$, then Γ is conjugate in $Aff(2, \mathbb{H})$) to a subgroup of G_2.*

Lemma 3.11. *If the subgroup $\Gamma \subseteq Aff(2, \mathbb{H})$ is abelian and acts freely on $\mathbb{H}^2$ then it is conjugate in $Aff(2, \mathbb{H})$ to a subgroup of the group of all matrices of the form $\begin{pmatrix} 1 & 0 & r \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $d \neq 0$ or to a subgroup of the group of all matrices of the form $\begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$.*

Proof. Since $h(\Gamma)$ is abelian we can use Lemma 3.7 and conjugate Γ into the group of all $\begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$ and we are done, or into the group of all $\begin{pmatrix} 1 & 0 & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix}$.

In the latter case: if all entries $d = 1$ we are done. Otherwise suppose $d \neq 1$ for some element A in Γ. After a conjugation with

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & (1-d)^{-1}s \\ 0 & 0 & 1 \end{pmatrix},$$

the matrix A is taken to $\begin{pmatrix} 1 & 0 & r \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and all the other elements to $A' = \begin{pmatrix} 1 & 0 & r' \\ 0 & d' & s' \\ 0 & 0 & 1 \end{pmatrix}$. Now, since Γ is abelian, we have that $CAC^{-1}A' = A'CAC^{-1}$ which implies that $d'd = dd'$ (so d and d' belong to the same complex plane), and $ds' = s'$. Thus if there exists $s' \neq 0$ we get $d = 1$, a contradiction. □

In what follows in addiction to the hypothesis that the action of Γ on $\mathbb{H}^2$ is *free*, we will assume that Γ acts *properly discontinuously* and that $\mathbb{H}^2/\Gamma$ is compact. This has important consequences that we collect here.
We recall the First Theorem of Bieberbach, see, e.g., [3, Theorem 1].

Theorem 3.12. *Let G be a subgroup of $Aff(n, \mathbb{C})$, acting freely and properly discontinuously on $\mathbb{C}^n$ and such that $\mathbb{C}^n/G$ is compact. Then the subgroup $\widetilde{G} \subseteq G$ of pure translations is generated by n linearly independent translations and $G/\widetilde{G} \simeq h(G)$ is a finite group.*

As a direct consequence we get

Lemma 3.13. *If the subgroup Γ acts freely, properly discontinuously on $\mathbb{H}^2$ and $\mathbb{H}^2/\Gamma$ is compact, then the set of translational parts (r, s) of elements of the form $\begin{pmatrix} a & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$ of Γ contains a basis for $\mathbb{H}^2$ as a real vector space.*

Proof. The proof easily follows taking into account that $Aff(2, \mathbb{H})$ can be identified as a subgroup of $Aff(4, \mathbb{C})$. □

We can now solve completely the abelian case: indeed, applying the previous Lemma, we understand that the first possibility of Lemma 3.11 does not occur; thus we have

Corollary 3.14. *If the subgroup $\Gamma \subseteq Aff(2, \mathbb{H})$ is abelian and acts freely and properly discontinuously on $\mathbb{H}^2$, and $\mathbb{H}^2/\Gamma$ is compact, then it is conjugate in $Aff(2, \mathbb{H})$ to a subgroup of the group of all matrices of the form $\begin{pmatrix} 1 & b & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$.*

Lemma 3.15. *If the subgroup $\Gamma \subseteq Aff(2,\mathbb{H})$ acts properly discontinuously on $\mathbb{H}^2$, and contains elements*

$$A = \begin{pmatrix} 1 & b & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \mathrm{B} = \begin{pmatrix} 1 & f & u \\ 0 & h & v \\ 0 & 0 & 1 \end{pmatrix}$$

such that $AB \neq BA$, then d is a root of unity.

The proof is similar to the one in the complex case, but the computations are much more complicated, due to the non commutativity of quaternions.

Proof. By direct computation we obtain that for any $n \in \mathbb{N}$

$$A^n = \begin{pmatrix} 1 & b(d-1)^{-1}(d^n-1) & b(d-1)^{-2}(d^n-1)s + nr - bn(d-1)^{-1}s \\ 0 & d^n & (d-1)^{-1}(d^n-1)s \\ 0 & 0 & 1 \end{pmatrix}$$

Hence

$$C_n = A^{-n}BA^nB^{-1} = \begin{pmatrix} 1 & f_n & u_n \\ 0 & h_n & v_n \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$h_n = d^{-n}hd^nh^{-1}$$
$$f_n = f(d^n-1)h^{-1} + b(d-1)^{-1}(d^n-1)(1-d^{-n}hd^n)h^{-1}$$
$$\begin{aligned} u_n = {} & fh^{-1}v - fd^nh^{-1}v - b(d-1)^{-1}(d^n-1)h^{-1}v \\ & + b(d-1)^{-1}(d^n-1)d^{-n}hd^nh^{-1}v + f(d-1)^{-1}(d^n-1)s \\ & - b(d-1)^{-1}(d^n-1)d^{-n}h(d-1)^{-1}(d^n-1)s \\ & - b(d-1)^{-1}(d^n-1)d^{-n}v + b(d-1)^{-2}(d^n-1)^2d^{-n}s \end{aligned}$$
$$v_n = -d^{-n}hd^nh^{-1}v + d^{-n}h(d-1)^{-1}(d^n-1)s + d^{-n}v - d^{-n}(d-1)^{-1}(d^n-1)s$$

Suppose that d is not a root of unity. Let us show that, in this case, $C_n \neq C_m$ for $n \neq m$.

Assume first that $hd \neq dh$. Then $C_n = C_m$ if and only of $h_n = h_m$, that is if and only if $n = m$.

If instead h and d commute, then the entries of C_n become

$$h_n = 1$$
$$f_n = [fh^{-1} + b(d-1)^{-1}(h^{-1}-1)](d^n-1)$$
$$\begin{aligned} u_n = {} & -f(d^n-1)[h^{-1}v - (d-1)^{-1}s] - b(d-1)^{-1}(d^n-1)(h^{-1}v - v) \\ & + b(d-1)^{-2}(2-d^{-n}-d^n)(s-hs) - b(d-1)^{-1}(1-d^{-n})v \end{aligned}$$
$$v_n = (1-d^{-n})[-v + (d-1)^{-1}(hs-s)]$$

Suppose that $C_n = C_m$ for $n \neq m$. Then $f_n = f_m$ and $v_n = v_m$ that is

$$\begin{cases} fh^{-1} + b(d-1)^{-1}(h^{-1}-1) = 0 \\ -v + (d-1)^{-1}(hs-s) = 0 \end{cases}$$

which is equivalent to

$$\begin{cases} f(d-1)+b(1-h)=0 \\ -(d-1)v+(1-h)s=0 \end{cases} \tag{3.1}$$

The first equation in system (3.1) is equivalent to the fact that the holonomies $h(A)$ and $h(B)$ commute. Recalling that $d \neq 1$, we have that $h(A)$ can be diagonalised, thus also $h(B)$ can be diagonalised via the same matrix (direct computation: indeed suppose that $h(B)$ cannot be diagonalised, since they commute they can be simultaneously triangulated, and they still commute, thus $f = fd$ which implies $f = 0$ since $d \neq 1$). Hence, up to conjugation,

$$A = \begin{pmatrix} 1 & 0 & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & u \\ 0 & h & v \\ 0 & 0 & 1 \end{pmatrix}$$

and if the second equation in System (3.1) is satisfied, A and B commute, in contradiction with our hypothesis. Therefore the matrices C_n are all distinct.

Let us show that, in both cases, if d is not a root of unity, the action of Γ is not properly discontinuous.

Suppose first that $|d| = 1$. Consider the sequence of points $(x_n, y_n) \in \mathbb{H}^2$ obtained applying the matrices C_n to the point $(0, 1)$,

$$\begin{aligned} x_n &= f_n + u_n \\ y_n &= d_n + v_n \end{aligned}$$

Up to subsequences, d^n goes to 1 as n goes to infinity, thus h_n goes to 1 and f_n, u_n and v_n go to 0. Hence the orbit of $(0, 1)$ has $(0, 1)$ as an accumulation point.

Suppose now that $|d| > 1$, and consider the orbit of $(0, v - h(d-1)^{-1}s)$. In this case

$$\begin{aligned} x_n &= f_n(v - h(d-1)^{-1}s) + u_n \\ y_n &= d_n(v - h(d-1)^{-1}s) + v_n \end{aligned}$$

Up to subsequences, h_n tends to $a \in \mathbb{H}$, $|a| = 1$ as n tends to infinity (if d and h commute $h_n = 1$ for any n). Hence, with long but straightforward computations, we get

$$\begin{aligned} x_n &= b(d-1)^{-1}[(d-1)^{-1}(d^{-n}-1)s + (1-d^{-n})h(d-1)^{-1}s - (1-d^{-n})v] \\ y_n &= -d^{-n}hd^n(d-1)^{-1}s + d^{-n}h(d-1)^{-1}(d^n-1)s + d^{-n}v - (d-1)^{-1}(1-d^{-n})s \end{aligned}$$

Taking into account the fact that $d^{-n}hd^n$ is bounded, since in modulus equals $|h|$, we obtain that (x_n, y_n) has an accumulation point.

The case where $|d| < 1$ can be treated analogously, considering the orbit through the same point via the matrices $\widetilde{C_n} = A^n B A^{-n} B^{-1}$.

□

In order to state and prove the next result, we define and list all (up to conjugation) finite subgroups of unitary quaternions; to do this we use the notations and approach of the book [5] by Conway and Smith. Let $I, J \in \mathbb{H}$ be purely imaginary unit quaternions, with $I \perp J$, and let $\{1, I, J, IJ = K\}$ be a basis for $\mathbb{H}$ having the usual multiplication rules. For

$$\sigma = \frac{\sqrt{5}-1}{2}, \quad \tau = \frac{\sqrt{5}+1}{2}$$

we consider the unitary quaternions

$$I_{\mathbb{I}} = \frac{I + \sigma J + \tau K}{2}; \qquad I_{\mathbb{O}} = \frac{J+K}{\sqrt{2}}; \quad \omega = \frac{-1+I+J+K}{2};$$
$$I_{\mathbb{T}} = I; \qquad e_n = e^{\frac{\pi I}{n}}.$$

and define the finite subgroups of the sphere $\mathbb{S}^3$ generated as follows:

$$2\mathbb{I} = \langle I_{\mathbb{I}}, \omega\rangle, \quad 2\mathbb{O} = \langle I_{\mathbb{O}}, \omega\rangle, \quad 2\mathbb{T} = \langle I_{\mathbb{T}}, \omega\rangle,$$
$$2D_{2n} = \langle e_n, J\rangle, \quad 2C_n = \langle e_n\rangle, \quad 1C_n = \langle e_{\frac{n}{2}}\rangle \ (n \text{ odd}).$$

The following result holds (see, e.g., [5, Theorem 12, page 33]).

Theorem 3.16. *Every finite subgroup of the sphere $\mathbb{S}^3$ of unitary quaternions is conjugated to a subgroup of the following list:*

$$2\mathbb{I}, \quad 2\mathbb{O}, \quad 2\mathbb{T}, \quad 2D_{2n}, \quad 2C_n, \quad 1C_n(n \text{ odd}).$$

Recall that a group G is said to be *unipotent* if all of its elements are unipotent, i.e., for all $g \in G$, there exists $n \in \mathbb{N}$ such that $(g-1)^n = 0$.
We are now ready to state and prove

Theorem 3.17. *If the subgroup $\Gamma \subseteq Aff(2, \mathbb{H})$ acts freely and properly discontinuously on $\mathbb{H}^2$, and $\mathbb{H}^2/\Gamma$ is compact, then Γ contains a unipotent normal subgroup Γ_0 of finite index such that Γ/Γ_0 is isomorphic to a finite subgroup of $\mathbb{S}^3$ (see Theorem 3.16).*

Proof. We can assume for the previous results that, up to conjugation, Γ is contained in G_2. Suppose first that Γ contains a central element A of the form $\begin{pmatrix} 1 & b & r \\ 0 & d & s \\ 0 & 0 & 1 \end{pmatrix}$ with $d \neq 1$. Conjugate Γ by

$$M = \begin{pmatrix} 1 & b(d-1)^{-1} & 0 \\ 0 & 1 & (1-d)^{-1}s \\ 0 & 0 & 1 \end{pmatrix} \in G_2$$

then $M^{-1}AM = \begin{pmatrix} 1 & 0 & r' \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Since this element is central in $M^{-1}\Gamma M$, all its elements of are of the form $\begin{pmatrix} 1 & 0 & u \\ 0 & h & 0 \\ 0 & 0 & 1 \end{pmatrix}$. And this is in contradiction with

Lemma 3.13 since the translational parts of $\widetilde{\Gamma}$ form a basis for $\mathbb{H}^2$. Thus, for any central element in Γ, $d = 1$. Since Γ is the fundamental group of the compact manifold $\mathbb{H}^2/\Gamma$, it is finitely generated. Let

$$A_i = \begin{pmatrix} 1 & b_i & r_i \\ 0 & d_i & s_i \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{for } 1 \leq i \leq k,$$

be the set of generators of Γ. If A_i is central, $d_i = 1$. If A_i is not central by Lemma 3.15 we have that d_i is a root of unity. Consider then the homomorphism $\varphi : \Gamma \to \mathbb{S}^3$ defined as

$$\varphi(A_i) = \varphi \begin{pmatrix} 1 & b_i & r_i \\ 0 & d_i & s_i \\ 0 & 0 & 1 \end{pmatrix} = d_i.$$

Let Γ_0 denote the kernel of φ. Then Γ_0 is normal and unipotent, and Γ/Γ_0 is isomorphic to a subgroup of $\mathbb{S}^3$. Now recall that if $\widetilde{\Gamma}$ denotes the subgroup of pure translation in Γ, by Theorem 3.12, $\Gamma/\widetilde{\Gamma}$ is finite. Taking into account that $\widetilde{\Gamma} \subseteq \Gamma_0$, we conclude that $\Gamma/\Gamma_0 \subseteq \Gamma/\widetilde{\Gamma}$ is isomorphic to a finite subgroup of $\mathbb{S}^3$. □

Let us give an explicit example of affine quaternionic manifold of quaternionic dimension 2.

Example 3.18. Consider the following transformations in $\mathit{Aff}(2, \mathbb{H})$:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & I \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & 0 & J \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 & \frac{J}{2} \\ 0 & -1 & I \\ 0 & 0 & 1 \end{pmatrix},$$

where $I, J, K, L \in \mathbb{S}$ are imaginary units. If $\Gamma_0 = \langle A, B, C, D \rangle$ and $\Gamma = \langle A, B, C, D, S \rangle$, then $\mathbb{H}^2/\Gamma_0$ and $\mathbb{H}^2/\Gamma$ are affine quaternionic manifolds. In particular, they are the quaternionic analogs of examples (f) and (b) in Vitter's paper.

Now, starting from $\Gamma_0 \subseteq \Gamma \subseteq \mathit{Aff}(2, \mathbb{H})$ (where Γ_0 is finitely generated, and without torsion elements), thanks to a theorem of Malcev [15], Γ_0 may be considered as a discrete subgroup of a unique connected, simply connected nilpotent Lie group N such that N/Γ_0 is compact. Thus Γ_0 acts freely and properly discontinuously on N and its orbit space is compact. Now the fact that Γ/Γ_0 is finite and $\mathbb{H}^2/\Gamma_0$ is compact forces N to have real dimension 8. Moreover, the fact that $\Gamma_0 \subseteq \mathit{Aff}(2, \mathbb{H})$, endows N/Γ_0 with the structure of a affine quaternionic manifold. Therefore N/Γ_0 is hypercomplex, and the same holds for N (see Remark 2.3). Summarizing, N is a nilpotent hypercomplex simply connected Lie Group. Dotti and Fino, in [7], give a classification of all possibilities for N.

Corollary 3.19. *Let the subgroup* $\Gamma \subseteq Aff(2,\mathbb{H})$ *act freely and properly discontinuously on* $\mathbb{H}^2$*, and assume that* $\mathbb{H}^2/\Gamma$ *is compact. Let* $\Gamma_0 \subseteq \Gamma$ *be a unipotent normal subgroup of finite index such that* Γ/Γ_0 *is isomorphic to a finite subgroup of* $\mathbb{S}^3$ *(see Theorem 3.17). Then* Γ_0 *is a discrete subgroup of a nilpotent hypercomplex simply connected* 8*-dimensional Lie Group* N *such that* N/Γ_0 *is compact.*

At this point, the next step would be to find explicitly, and possibly list, all subgroups $\Gamma \subseteq Aff(2,\mathbb{H})$ acting freely and properly discontinuously on $\mathbb{H}^2$, and such that $\mathbb{H}^2/\Gamma$ is compact. In order to do this, one could exploit the classification of Dotti and Fino, [7]: first identify all discrete subgroups Γ_0 of a nilpotent hypercomplex simply connected 8-dimensional Lie Group N such that N/Γ_0 is compact, and then restrict to those such that Γ/Γ_0 is one of the finite subgroups of $\mathbb{S}^3$ listed above.

We will address this topic in a forthcoming paper.

References

[1] Aslaksen, H., *Quaternionic determinants.* Math. Intelligencer **18** (1996), no. 3, 57-65.
[2] Angella D., Bisi C., *Slice-quaternionic Hopf surfaces*, J. Geom. Anal. ISSN:1050-6926 (In Press)
[3] Auslander, L., *Four dimensional compact locally hermitian manifolds*, Trans. Amer. Math. Soc. **84** (1957), 379-391.
[4] Bisi, C.; Gentili, G., *On quaternionic tori and their moduli space*, J. Noncommut. Geom. **12** (2018), no. 2, 473-510.
[5] Conway, J. H., Smith, D. A., On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry. A.K.Peters, Natick, Massachusetts, 2003.
[6] De Leo, S., ; Scolarici, G., Solombrino, L. *Quaternionic eigenvalue problem*, J. Math. Phys. **43** (2002), no. 11, 5815-5829.
[7] Dotti, I. G.; Fino, A., *Hypercomplex eight-dimensional nilpotent Lie groups*, J. Pure Appl. Algebra **184** (2003), no. 1, 41-57.
[8] Fillmore, J.P., Scheuneman, J., *Fundamental groups of compact complete locally affine complex surfaces*, Pacific J. Math. **44** (1973), 487-496.
[9] Gentili, G., Gori, A., Sarfatti, G., *A direct approach to quaternionic manifolds*, Math. Nachr. **290** (2017), no. 2-3, 321-331.
[10] Gentili, G., Gori, A., Sarfatti, G., *Quaternionic toric manifolds*, J. Sympl. Geom., (In Press)
[11] Gentili, G., Stoppato, C., Struppa, D.C., Regular Functions of a Quaternionic Variable, Springer Monographs in Mathematics, Springer, Berlin-Heidelberg, 2013.
[12] Ghiloni R., Perotti A., *Slice regular functions of several Clifford variables* in 9th International Conference on Mathematical Problems in Engineering, Aerospace and Sciences : ICNPAA 2012 : Vienna, Austria.
[13] Inoue, M., Kobayashi, S., Ochiai, T., *Holomorphic affine connections on compact complex surfaces*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 247-264.
[14] Rodman, L., Topics in quaternion linear algebra. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2014. xii+363 pp.
[15] Malcev, A. I., *On a class of homogeneous spaces*, Amer. Math. Soc. Translation, (1951). no. 39, 33 pp.

[16] Matsushima, Y., *Affine structures on complex manifolds* Osaka J. Math. **5** (1968) 215-222.
[17] Obata, M., *Affine connections on manifolds with almost complex, quaternion or Hermitian structure* Jap. J. Math. **26** (1956) 43-77.
[18] Salamon, S. M., *Differential geometry of quaternionic manifolds*, Ann. Sci. École Norm. Sup. (4) **9** (1986), no. 1, 31-55.
[19] Sommese, A., *Quaternionic manifolds*, Math. Ann. **212** (1974/75), 191-214.
[20] Vitter, A., *Affine structures on compact complex manifolds*, Invent. Math. **17** (1972), 231-244.

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