MATRICES DROPPING RANK IN CODIMENSION ONE AND CRITICAL LOCI IN COMPUTER VISION

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ABSTRACT. Critical loci for projective reconstruction from three views in four dimensional projective space are defined by an ideal generated by maximal minors of suitable 4×3 matrices, N, of linear forms. Such loci are classified in this paper, in the case in which N drops rank in codimension one, giving rise to reducible varieties. This leads to a complete classification of matrices of size $(n+1) \times n$ for $n \leq 3$, which drop rank in codimension one. Instability of reconstruction near non-linear components of critical loci is explored experimentally.

1. INTRODUCTION

Linear projections from $\mathbb{P}^k$ to $\mathbb{P}^2$, and even from $\mathbb{P}^k$ to $\mathbb{P}^m, m \geq 3$, are of interest to the computer vision community as simple models of the process of taking pictures of particular three-dimensional scenes, see for example [27], [15], [18], [13], [14], [25], [26]. In this framework, the authors have investigated in several works, [8], [4], [2], [3], [1], [5], [6], the algebro-geometric properties of varieties that arise as *critical loci* for projective reconstruction from multiple views, i.e. from multiple projections $\mathbb{P}^k \dashrightarrow \mathbb{P}^2$ (see Sections 4 and 5 for an introduction to the reconstruction problem and the notion of critical locus). In particular, [5] presents a comprehensive treatment of the families of varieties involved, if the number of views is the minimum necessary for reconstruction. In [5] it is shown that, under suitable genericity assumptions, critical loci are either hypersurfaces if the ambient space is odd dimensional, or special determinantal varieties of codimension two if the ambient space is even dimensional. The latter codimension-two varieties are studied in great detail in [6], in the case of 3 projections (views) from $\mathbb{P}^4$ as ambient space, where they are shown to fill the irreducible component of the Hilbert scheme of $\mathbb{P}^4$ whose general element is a classical Bordiga surface.

When genericity assumptions are dropped, one is led to consider a number of degenerate configurations of centers of projection and corresponding degenerate critical loci. In the classical case of projections from $\mathbb{P}^3$ to $\mathbb{P}^2$, Hartley and Kahl described degenerate cases in [19]. This work revisits the case of 3 projections from $\mathbb{P}^4$ to $\mathbb{P}^2$, while dropping the genericity assumptions, in order to conduct a detailed analysis of possible degenerate cases of critical loci, in a proper algebro-geometric setting.

In [6] it is shown that the minimal generators of the ideal of the critical locus for 3 projections from $\mathbb{P}^4$ to $\mathbb{P}^2$ are cubic polynomials, arising as maximal minors of a suitable 4×3 matrix N of linear forms. Genericity assumptions are reflected in the fact that such minors do not share any common factors. Given the focus of this work, one is naturally led to consider $(n+1) \times n$ matrices of linear forms, whose minors have common factors. Matrices of type $(n+1) \times n$ that drop rank in codimension 2 have been intensively studied from a geometrical point of view in the framework of liaison theory (e.g. see Gaeta's Theorem), while, in the framework of commutative algebra, many researchers have contributed to generalize Hilbert–Burch Theorem by deeply studying homological properties of rings and modules.

On the contrary, matrices of type $(n + 1) \times n$ of linear forms that drop rank in codimension 1 do not seem to have been studied with any systematic approach. In order to address the main goal of this work, a classification of canonical forms of such matrices, over any field, for $n \leq 3$ is conducted. The authors believe that this contribution may have relevance in other contexts and be of more general interest. Theorem 2.1 and Theorem 2.2 contain such classification, respectively for minors with a linear or quadratic common factor. Degeneration loci of such matrices are then investigated in Section 3, in the case of interest for this work, when the ambient space is four-dimensional. Leveraging this analysis, a complete geometrical classification of degenerate critical loci for three projections from $\mathbb{P}^4$ to $\mathbb{P}^2$ is presented in Theorem 5.1. For degenerate critical loci in situations in which the reconstruction of the trifocal tensor still makes sense, see Section 6, and the tensor can be uniquely reconstructed, we conduct instability experiments, following [4].

The paper is organized as follows: Section 2 contains the classification of 4×3 matrices whose maximal minors share a common factor, see Theorem 2.1 and Theorem 2.2. In Section 3 the degeneracy loci of such matrices are studied. Section 4 introduces multiview geometry and basic facts on the reconstruction problem in computer vision. Section 5 presents the notion of critical locus for three projections from four-dimensional projective space, and, relating its study to matrices investigated in Section 2, presents a classification of such loci, see Theorem 5.1. In Section 6 we address the actual possibility of reconstructing the trifocal tensor in each situation appearing in Theorem 5.1. Finally, in Section 7, we conduct experiments to investigate the instability of the reconstruction algorithm in a neighborhood of each non linear component of the critical loci obtained in Theorem 5.1.

2. Classification of 4×3 matrices that drop rank in codimension 1

In this section we compute the canonical forms of $(n + 1) \times n$ matrices, with $n \leq 3$, of linear forms whose maximal minors have a non trivial common factor, up to elementary operations on rows and columns. As the maximal minors have degree at most three, the degree of the common factor is either 1 or 2.

As standard notations, R is the polynomial ring in $x_0, \dots, x_r$ over any field K with $char(K) \neq 2$. Let A and B be matrices with entries in R. A^T denotes the transpose of A. $A_{\widehat{i_1 \dots i_r}}$ denotes the matrix obtained from A by deleting the $i_1, \dots, i_r$ rows. Assuming that A and B have compatible sizes, AB denotes the product of the two matrices, while $(A|B)$ denotes matrix concatenation. Let $f_1, \dots f_r$ be polynomials in R; then $\langle f_1, \dots, f_r \rangle$ denotes the ideal generated by the f_is.

In the rest of this work we will often deal with matrices whose entries are linear forms. Unless otherwise explicitly stated, we will always assume that linear forms appearing in columns of matrices involved in our arguments, are as linearly independent as possible, given explicit assumptions made in each instance.

2.1. **Matrices with maximal minors having a common factor of degree** 1. In this subsection, we classify the 4×3 matrices whose maximal minors have a greatest common divisor of degree 1, see Theorem 2.1. To achieve our goal, we analyze first matrices of type 2×1 and 3×2, see paragraph below and Proposition 2.1. Along the way, we prove a general technical result, see Lemma 2.1.

Any 2×1 matrix of linear forms whose maximal minors have a greatest common divisor of degree 1 has entries that differ by a multiplicative constant. Hence it can be reduced, via elementary row operations, to the form

$$\begin{pmatrix} 0 \\ \ell \end{pmatrix}.$$

Lemma 2.1. *Let M be a $(n + 2) \times n$ matrix of linear forms whose $(n + 1) \times n$ submatrices do not drop rank in codimension 1. Let P be a $1 \times (n + 2)$ matrix of forms of degree n such that $PM = 0$.*

Then, the forms in P are linearly dependent, and a free resolution of P is

$$0 \to \begin{matrix} R(-n) \\ \oplus \\ R^n(-n-1) \end{matrix} \xrightarrow{\widetilde{M}} R^{n+2}(-n) \xrightarrow{P} R \to 0,$$

where $\widetilde{M}$ is the concatenation of M and a suitable column of constants.

Proof. Let $D_{ij} = (-1)^{i+j} \det(M_{\widehat{ij}})$ if $i < j$, $D_{ij} = -D_{ji}$ if $i > j$ and $D_{ii} = 0$. Let D be the order $n+2$ square matrix whose elements are D_{ij}. By construction, D is skew–symmetric and homogeneous of degree n. Moreover, $DM = 0$. To see this, note that each row of D consists of maximal minors of $M_{\widehat{i}}$, with the appropriate sign. As such a matrix does not drop rank in codimension 1, we can apply Hilbert–Burch Theorem to get the claim. D drops rank in codimension 3 and so

$$0 \to R^n(-n-2) \xrightarrow{M} R^{n+2}(-n-1) \xrightarrow{D} R^{n+2}(-1) \xrightarrow{M^T} R^n \to 0$$

is a minimal free resolution of M^T.

Let P be a $1 \times (n+2)$ matrix of forms of degree n such that $PM = 0$. Then, there exists a $1 \times (n+2)$ constant matrix X such that $P = XD$. As D is skew–symmetric, $XDX^T = -XDX^T$ and so $PX^T = 0$, that is to say, the elements of P are linearly dependent. Furthermore, $PM = 0$, and so, letting $\widetilde{M} = (X^T|M)$, it is $P\widetilde{M} = 0$. Moreover, it is easy to check that the maximal minors of $\widetilde{M}$ are equal to the elements of P, up to sign. Hence, Hilbert–Burch Theorem gives the free resolution of P, as in the statement. $\qquad\square$

The following additional lemma deals with 2×2 matrices of linear forms, whose determinant is the product of linear forms.

Lemma 2.2. *Let*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

be a square matrix of linear forms such that $\det(A) = uv$ for suitabe linear forms u, v. Then, up to elementary operations on the columns of A, the elements of a column are linearly dependent.

Proof. We can assume that a_{11}, a_{21} are linearly independent. The assumption $\det(A) = uv$ can be rewritten as $-a_{11}a_{22} + a_{21}a_{12} + uv = 0$. First assume that a_{11}, a_{21}, u are linearly independent. Then

$$\begin{pmatrix} a_{22} \\ a_{12} \\ v \end{pmatrix} = \begin{pmatrix} 0 & u & a_{21} \\ -u & 0 & a_{11} \\ a_{21} & a_{11} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

for suitable constants x_1, x_2, x_3, because, in this case, the first syzygy module of $-a_{11}, a_{21}, u$ is generated by the Koszul relations. Hence, $a_{22} = x_2 u + x_3 a_{21}$ and $a_{12} = -x_1 u + x_3 a_{11}$, that is to say,

$$A = \begin{pmatrix} a_{11} & -x_1 u \\ a_{21} & x_2 u \end{pmatrix}$$

after performing the elementary operation $C_2 - x_3 C_1$ on the second column of A.

Now assume instead that $u = \alpha a_{11} + \beta a_{21}$, for suitable constants α, β. The assumption $\det(A) = uv$ can be rewritten as $(-a_{22} + \alpha v)a_{11} + (a_{12} + \beta v)a_{21} = 0$. Hence, there exists a constant γ such that $a_{22} = \alpha v + \gamma a_{21}$, $a_{12} = -\beta v + \gamma a_{11}$ because the syzygies of a_{11}, a_{21} are again generated exclusively by the Koszul relations. Hence, after performing the elementary operation $C_2 - \gamma C_1$ on the second column of A, the elements (linear forms) of the second column of A are linearly dependent, as in the previous case. $\qquad\square$

Proposition 2.1. *Let N be a 3×2 matrix of linear forms, whose maximal minors have a greatest common divisor of degree 1. Then, up to elementary operations on rows and columns, it is:*

$$(1) \qquad N = \begin{pmatrix} 0 & n_{12} \\ 0 & n_{22} \\ n_{31} & n_{32} \end{pmatrix} \qquad or \qquad N = \begin{pmatrix} 0 & n_{12} \\ n_{12} & 0 \\ n_{31} & n_{32} \end{pmatrix}$$

.

Proof. As the maximal minors of N have a greatest common divisor of degree 1, there exists a 1×3 matrix P of linear forms such that $PN = 0$. Of course, both columns of N are syzygies of P. Let C be the first column of N.

First assume that the linear forms in C are linearly independent. Then Lemma 2.1 applied to C implies that there exists X such that $P = XD$, and the syzygy matrix of P is $(X^T|C)$. The second column of N is then $\alpha X^T + \beta C$ where α is now a linear form and β is a constant. Therefore, up to elementary operations on rows and columns of the matrix, N is as in (1), case on the left.

One can now assume that the linear forms in the first column C span a subspace of dimension 2. Then one can reduce N to the form

$$N = \begin{pmatrix} 0 & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{pmatrix},$$

with n_{21} and n_{31} linearly independent, and $n_{12} \neq 0$, to avoid trivial cases.

Note that we can also assume that the linear forms in the second column of N span a space of dimension 2, otherwise N would again be as in (1), case on the left. Because, moreover, $n_{12} \neq 0$, we have that n_{22}, n_{32} are linearly dependent modulo n_{12}, and so we can further reduce N by assuming that $n_{22} = 0$. The maximal minors of N are $n_{21}n_{32}, n_{12}n_{31}, -n_{21}n_{12}$. Then, n_{12} divides $n_{21}n_{32}$. We know that n_{12} and n_{32} are linearly independent, and so, n_{12} divides n_{21}. Hence N is as in (1), case on the right.   □

Theorem 2.1. *Let N be a 4×3 matrix of linear forms whose maximal minors have a greatest common divisor of degree 1. Then, up to elementary operations on its rows and columns, N has one of the following forms, where α and β are suitable constants, $(\alpha, \beta) \neq (0,0)$:*

$$(2) \qquad N_A = \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{21} & n_{22} & 0 \\ n_{31} & n_{32} & 0 \\ n_{41} & n_{42} & n_{43} \end{pmatrix}, \qquad N_B = \begin{pmatrix} 0 & 0 & n_{13} \\ 0 & n_{13} & 0 \\ n_{31} & n_{32} & n_{33} \\ n_{41} & n_{42} & n_{43} \end{pmatrix},$$

$$(3) \qquad N_C = \begin{pmatrix} 0 & 0 & \alpha n_{31} + \beta n_{22} \\ 0 & n_{22} & \alpha n_{41} \\ n_{31} & 0 & \beta n_{42} \\ n_{41} & n_{42} & 0 \end{pmatrix}, \qquad N_D = \begin{pmatrix} 0 & 0 & n_{13} \\ 0 & n_{31} & n_{23} \\ n_{31} & 0 & n_{33} \\ n_{41} & n_{42} & n_{43} \end{pmatrix}.$$

Proof. Let N be a 4×3 matrix of linear forms, whose maximal minors have a greatest common divisor of degree 1. Then, up to the common linear factor, the maximal minors of N are 4 quadratic forms. Let P be the row vector whose entries are these quadratic forms, in the right order. Then $PN = 0$, which shows that N is a submodule of the syzygies of P.

Let us first assume that there exist two columns of N, say the first two C_1 and C_2, with the property that every 3×2 submatrix of $N(1,2) = (C_1|C_2)$ has maximal minors with trivial greatest common divisor. Then, as in the proof of Lemma 2.1, it is $P = XD$ for a suitable X of type 1×4, and the syzygy matrix of P is equal to $(X^T|C_1|C_2)$. Hence, the third column of N is $C_3 = \delta X^T + \beta_1 C_1 + \beta_2 C_2$ where δ is a suitable linear form and β_1, β_2 are constants. Thus, up to

elementary operations on rows and columns, it is $N = N_A$ in (2). Notice that if the linear forms of any column of N span a 1 dimensional subspace, then the maximal minors of N have a common factor of degree 1 and N can be reduced as above. Therefore from now on we will assume that the linear spaces spanned by the linear forms of each column have dimension at least two.

From the above discussion, we can now assume that every pair of columns of N has a 3×2 submatrix whose maximal minors have a greatest common divisor of degree 1. Proposition 2.1, up to elementary operations on rows and columns, gives the following possibilities for N :

$$N_1 = \begin{pmatrix} 0 & 0 & n_{13} \\ 0 & 0 & n_{23} \\ n_{31} & n_{32} & n_{33} \\ n_{41} & n_{42} & n_{43} \end{pmatrix}, N_2 = \begin{pmatrix} 0 & 0 & n_{13} \\ 0 & n_{13} & 0 \\ n_{31} & n_{32} & n_{33} \\ n_{41} & n_{42} & n_{43} \end{pmatrix}, N_3 = \begin{pmatrix} 0 & 0 & n_{13} \\ 0 & n_{22} & n_{23} \\ n_{31} & n_{13} & 0 \\ n_{41} & n_{42} & n_{43} \end{pmatrix},$$

$$N_4 = \begin{pmatrix} 0 & n_{12} & n_{13} \\ 0 & n_{22} & n_{23} \\ n_{31} & n_{32} & 0 \\ n_{41} & 0 & n_{32} \end{pmatrix}, N_5 = \begin{pmatrix} 0 & n_{12} & n_{31} \\ n_{12} & 0 & n_{32} \\ n_{31} & n_{32} & 0 \\ n_{41} & n_{42} & n_{43} \end{pmatrix}, N_6 = \begin{pmatrix} 0 & 0 & n_{13} \\ 0 & n_{22} & n_{23} \\ n_{31} & 0 & n_{33} \\ n_{41} & n_{42} & n_{43} \end{pmatrix}.$$

Let us consider N_1. If n_{13} and n_{23} are linearly dependent, then N_1, up to elementary operations, has a row of zeros. Therefore we can assume that n_{13} and n_{23} are linearly independent, and so $n_{31}n_{42} - n_{32}n_{41}$ is reducible. Then, Lemma 2.2, up to elementary operations, implies that the forms of the first or of the second column of N_1 span a space of dimension 1, and hence N_1 is a specialization of N_A.

It is immediate to see that $N_2 = N_B$, and that its maximal minors have n_{13} as a common factor.

Now we consider N_3. Its maximal minors have n_{13} as common factor if n_{13} divides either n_{31} or $n_{22}n_{43} - n_{23}n_{42}$. In the first case N_3 can be reduced to N_B with $n_{32} = 0$. In the second case, by Lemma 2.2, N_3 can be reduced to the form

$$\begin{pmatrix} 0 & 0 & n_{13} \\ 0 & n_{22} & n_{23} \\ n_{31} & n_{13} & 0 \\ n_{41} & 0 & 0 \end{pmatrix}$$

that is a specialization of N_D.

The maximal minors of N_4, up to a sign, are

$$n_{32}\left(n_{41}n_{23} + n_{22}n_{31}\right), n_{32}\left(n_{41}n_{13} + n_{12}n_{31}\right), n_{41}\left(n_{12}n_{23} - n_{13}n_{22}\right), n_{31}\left(n_{12}n_{23} - n_{13}n_{22}\right).$$

As n_{31} and n_{41} are linearly independent, then either n_{32} divides $n_{12}n_{23} - n_{13}n_{22}$, or n_{31} divides both n_{13} and n_{23}, or n_{41} divides both n_{12} and n_{22}, or, finally, $n_{41}n_{23} + n_{22}n_{31}, n_{41}n_{13} + n_{12}n_{32}$, and $n_{12}n_{23} - n_{13}n_{22}$ have a common linear factor. In the first case, a column of the submatrix formed by the first two rows and columns of N_4 contains forms that are multiple of n_{32}, and so a column of N_4 spans a linear space of dimension 1. Hence, we get a special case of N_A. In the second case it is easy to check that we get a specialization of N_D. The third case is identical to the previous one, swapping the role of n_{31} and n_{41}. In the last case, the maximal minors of the matrix

$$\begin{pmatrix} n_{12} & n_{13} \\ n_{22} & n_{23} \\ -n_{41} & n_{31} \end{pmatrix}$$

have a common linear form. By Proposition 2.1, we get a specialization of either N_A or N_B.

The maximal minors of N_5, up to the sign, are

$$2n_{12}n_{31}n_{32}, n_{12}\left(n_{32}n_{41} + n_{31}n_{42} - n_{12}n_{43}\right), n_{31}\left(-n_{32}n_{41} + n_{31}n_{42} - n_{12}n_{43}\right),$$

and

$$n_{32}\left(-n_{32}n_{41} + n_{31}n_{42} + n_{12}n_{43}\right).$$

Assume first that n_{12} is the common factor among them. Then, n_{12} divides $-n_{32}n_{41} + n_{31}n_{42}$, that forces N_5 to be a specialization of N_A. The cases when n_{31} or n_{32} is the common factor are dealt with similarly.

The last remaining case, N_6, requires a slightly more elaborate analysis. The maximal minors of N_6, up to sign, are

$$(4) \qquad n_{13}n_{31}n_{42}, n_{13}n_{22}n_{41}, n_{13}n_{22}n_{31}, n_{23}n_{31}n_{42} + n_{33}n_{22}n_{41} - n_{43}n_{22}n_{31}.$$

We remark that n_{31}, n_{41} and n_{22}, n_{42} are pairwise linearly independent. The maximal minors of N_6 have a common linear form if one of the following condition holds:

(1) n_{31} divides both n_{13} and n_{33};
(2) n_{31} divides n_{22};
(3) n_{42} divides both n_{13} and $n_{33}n_{41} - n_{31}n_{43}$;
(4) n_{13} divides $n_{23}n_{31}n_{42} + n_{33}n_{22}n_{41} - n_{43}n_{22}n_{31}$.

The cases above follow from the analysis of the maximal minor $n_{13}n_{31}n_{42}$. An analogous list could be obtained starting from the second or third maximal minor in (4).

In case (1), N_6 is a specialization of N_B. In cases (2) and (3), N is either N_D or one of its specializations. The rest of this proof is devoted to case (4). First notice that we can assume that the submatrix obtained from the last 3 rows and first two columns of N_6 satisfies the hypothesis of the Hilbert–Burch Theorem. Otherwise we would fall back on case (2). Case (4) can be rewritten as

$$(5) \qquad n_{23}(n_{31}n_{42}) + n_{33}(n_{22}n_{41}) - n_{43}(n_{22}n_{31}) + n_{13}q = 0$$

for a suitable quadratic form q. Let $I = \langle n_{31}n_{42}, n_{22}n_{41}, n_{22}n_{31} \rangle$, and $J = \langle n_{13} \rangle$. Then $I + J$ is generated by all the previous generators, and (5) gives that the transpose of $(n_{23}, n_{33}, -n_{43}, q)$ is a syzygy of the generators of $I + J$.

Case (4.1): n_{13} is regular for I.

Then the syzygy matrix of $I + J$ can be computed by mapping cone procedure, and it is

$$M_1 = \begin{pmatrix} -n_{13} & 0 & 0 & 0 & n_{22} \\ 0 & -n_{13} & 0 & n_{31} & 0 \\ 0 & 0 & -n_{13} & -n_{41} & -n_{42} \\ n_{31}n_{42} & n_{22}n_{41} & n_{22}n_{31} & 0 & 0 \end{pmatrix}.$$

Hence, there exist constants $x_1, \ldots, x_5$ such that

$$\begin{pmatrix} n_{23} \\ n_{33} \\ -n_{43} \\ q \end{pmatrix} = M_1 \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix}$$

that is to say $n_{23} = -x_1n_{13} + x_5n_{22}, n_{33} = -x_2n_{13} + x_4n_{31}, n_{43} = x_3n_{13} + x_4n_{41} + x_5n_{42}$. By substituting in N_6 and performing suitable elementary operations, we get a matrix that is a specialization of N_A.

We remark that $I = \langle n_{31}, n_{41} \rangle \cap \langle n_{22}, n_{42} \rangle \cap \langle n_{22}, n_{31} \rangle$. So, if n_{13} is not regular for I, then it has to belong to one of the three ideals whose intersection is I.

Case (4.2): $n_{13} = \alpha n_{31} + \beta n_{41}$ with $(\alpha, \beta) \neq (0, 0)$.

The syzygy matrix of $I + J$, in this case, is

$$M_2 = \begin{pmatrix} 0 & 0 & n_{22} & -n_{13} \\ -\beta & n_{31} & 0 & 0 \\ -\alpha & -n_{41} & -n_{42} & 0 \\ n_{22} & 0 & 0 & n_{31}n_{42} \end{pmatrix}.$$

Hence, there exists a linear form v and constants x_2, x_3, x_4 such that

$$\begin{pmatrix} n_{23} \\ n_{33} \\ -n_{43} \\ q \end{pmatrix} = M_2 \begin{pmatrix} v \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

and so $n_{23} = x_3 n_{22} - x_4 n_{13}, n_{33} = -\beta v + x_2 n_{31}, n_{43} = \alpha v + x_2 n_{41} + x_3 n_{42}$. By substituting in N_6 and performing suitable elementary operations, we get a matrix that is a specialization of N_D.

Case (4.3): $n_{13} = \alpha n_{22} + \beta n_{42}$ with $(\alpha, \beta) \neq (0,0)$.

This case is completely analogous to (4.2) and leads to a specialization of N_D.

Case (4.4): $n_{13} = \alpha n_{31} + \beta n_{22}$ with $(\alpha, \beta) \neq (0,0)$.

The syzygy matrix of $I + J$, in this case, is

$$M_3 = \begin{pmatrix} 0 & n_{22} & n_{13} & 0 & 0 & \alpha n_{41} \\ n_{31} & 0 & 0 & n_{13} & 0 & \beta n_{42} \\ -n_{41} & -n_{42} & 0 & 0 & n_{13} & 0 \\ 0 & 0 & -n_{31}n_{42} & -n_{22}n_{41} & -n_{22}n_{31} & -n_{41}n_{42} \end{pmatrix}.$$

There exist constants $x_1, \ldots, x_6$ such that, as before, $n_{23} = x_2 n_{22} + x_3 n_{13} + \alpha x_6 n_{41}, n_{33} = x_1 n_{31} + x_4 n_{13} + \beta x_6 n_{42}, n_{43} = x_1 n_{41} + x_2 n_{42} - x_5 n_{13}$. By substituting in N_6 and performing suitable elementary operations, we get the matrix N_C. $\qquad \square$

Remark 2.1. *Each of the matrices N_A, N_B, N_C, N_D is not a specialization of any of the others. In fact, in general cases, the columns of each matrix span a space of maximal dimension, and such dimensions are uniquely associated to each matrix.*

Remark 2.2. *Note that each general matrix N_A, N_B, N_C, N_D can be specialized to a specializations of the remaining three. This statement can be interpreted from a different point of view. The set of 4×3 matrices of linear forms over a given polynomial ring is an affine space, or a projective space, if we exclude the null matrix, and so it is irreducible. The general element of such a space corresponds to a matrix whose maximal minors have a trivial greatest common divisor. The complement of the open set of such matrices is a union of algebraic sets. The statement above shows that the union of the loci whose general elements are $N_A, \ldots, N_D$, respectively, is connected.*

In the next section, where we study the geometry of the degeneration loci of the above matrices, more evidence for the fact that we have four irreducible algebraic sets, one for each matrix $N_A, \ldots, N_D$ will emerge.

2.2. Matrices with maximal minors with a common factor of degree 2.

In this subsection, we classify the 4×3 matrices whose maximal minors have a greatest common divisor of degree 2.

Let N be a 4×3 matrix whose maximal minors have a greatest common divisor q of degree 2. Let $D_i = q l_i$ for suitable linear forms l_i, $i = 1, \ldots, 4$, be the minor, with its proper sign, obtained by removing the i-th row from N. As

$$q(l_1, l_2, l_3, l_4)N = 0,$$

the matrix N is a submodule of the syzygy module of (l_1, l_2, l_3, l_4). The linear forms l_i can span a subspace of dimension $4, 3,$ or 2. In the first case, they are linearly independent, in the second, we assume $l_1 + z_2 l_2 + z_3 l_3 + z_4 l_4 = 0$ for suitable scalars z_j, while in the last case we assume $l_1 + z_{31} l_3 + z_{41} l_4 = l_2 + z_{32} l_3 + z_{42} l_4 = 0$ for suitable scalars z_{ij}. Then the syzygy modules, respectively, are generated by the columns of the following matrices:

$$(6) \qquad S_1 = \begin{pmatrix} 0 & -l_3 & l_2 & 0 & 0 & l_4 \\ l_3 & 0 & -l_1 & 0 & l_4 & 0 \\ -l_2 & l_1 & 0 & l_4 & 0 & 0 \\ 0 & 0 & 0 & -l_3 & -l_2 & -l_1 \end{pmatrix}, S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ z_2 & 0 & -l_4 & l_3 \\ z_3 & l_4 & 0 & -l_2 \\ z_4 & -l_3 & l_2 & 0 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ z_{31} & z_{32} & -l_4 \\ z_{41} & z_{42} & l_3 \end{pmatrix}.$$

Moreover, in the first case there exists a maximal rank 6×3 matrix X_1, with scalar entries, such that $N = S_1 X_1$; in the second case there exists a maximal rank 4×3 matrix X_2, with linear forms on the first row and scalars elsewhere, such that $N = S_2 X_2$; in the last case there exists a generically maximal rank 3×3 matrix X_3, with linear forms on the first two rows and scalars on the last row, such that $N = S_3 X_3$.

The discussion above can now be summarized in the following result:

Theorem 2.2. *Let N be a 4×3 matrix of linear forms that drops rank on a degree 2 hypersurface Q, and let L be the linear subspace where N drops rank, residual of Q. Then, $N = S_i X_i$ where $i = 5 - codim(L)$, and S_i, X_i are as described above.*

3. GEOMETRY OF THE DEGENERATION LOCI

We now turn our attention to degeneration loci of matrices, i.e. we study the loci where matrices classified in Section 2 drop rank, under some mild generality assumptions and in the dimensional context of interest for our computer vision goals. In this section, we assume $R = \mathbb{C}[x_0, \ldots, x_4]$ so that degeneration loci are algebraic schemes in $\mathbb{P}^4 = \mathrm{Proj}(R)$.

We begin our analysis with matrices N_A, N_B, N_C, N_D, as classified in Theorem 2.1.

Proposition 3.1. *The degeneration locus of a general N_A is the union of a hyperplane H_A and a minimal surface S_A of degree 3 in $\mathbb{P}^4$.*

Proof. The defining ideal of the degeneracy locus of such a matrix is

$$I = (n_{43}(n_{11}n_{22} - n_{21}n_{12}), n_{43}(n_{31}n_{12} - n_{11}n_{32}), n_{43}(n_{21}n_{32} - n_{31}n_{22})).$$

Of course, the vanishing locus of n_{43} is a hyperplane in $\mathbb{P}^4$. Furthermore, $I : \langle n_{43} \rangle$ is the ideal generated by the 2×2 minors of

$$\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \\ n_{31} & n_{32} \end{pmatrix}.$$

It is known that such minors define a minimal surface of degree 3 in $\mathbb{P}^4$, that is to say, a rational normal scroll in the general case, as well as a cone over a rational normal curve contained in a suitable hyperplane, or a degeneration of one of the above surfaces. □

Proposition 3.2. *The degeneration locus of a general N_B in $\mathbb{P}^4$ is a hyperplane H_B, and the union of a dimension 2 linear space L_B and a twisted cubic curve $C_B \subseteq H_B$. Moreover, $H_B \cap L_B$ is a line r_B that meets C_B in two points.*

Proof. In this case, n_{31} and n_{41} are linearly independent, and n_{13} is the common factor to all the maximal minors of N_B. Let H_B be the hyperplane defined by $n_{13} = 0$. If I is the ideal generated by the maximal minors of N_B and $J = I : \langle n_{13} \rangle$, we have

$$J = \langle n_{13}n_{31}, n_{13}n_{41}, q_3 = n_{31}n_{42} - n_{41}n_{32}, q_2 = n_{31}n_{43} - n_{41}n_{33} \rangle.$$

The degree 2 forms q_3 and q_2 are two of the three maximal minors of the 2×3 matrix consisting of the last two rows of N_B. For completeness, let $q_1 = n_{32}n_{43} - n_{42}n_{33}$ be the last maximal minor of such a matrix. One can then easily obtain the minimal free resolution of J :

$$0 \to R(-5) \xrightarrow{M_2} \begin{matrix} R^3(-3) \\ \oplus \\ R(-4) \end{matrix} \xrightarrow{M_1} R^4(-2) \to J \to 0,$$

where

$$M_1 = \begin{pmatrix} n_{41} & n_{42} & n_{43} & 0 \\ -n_{31} & -n_{32} & -n_{33} & 0 \\ 0 & -n_{13} & 0 & -q_2 \\ 0 & 0 & -n_{13} & q_3 \end{pmatrix}, \quad M_2 = \begin{pmatrix} q_1 \\ -q_2 \\ q_3 \\ n_{13} \end{pmatrix}.$$

From the resolution, the Hilbert polynomial of the scheme defined by J is $p(t) = \frac{1}{2}t^2 + \frac{9}{2}t$ and so the top dimensional part of such a scheme is a linear space L_B of dimension 2. As $J \subseteq \langle n_{31}, n_{41} \rangle$, then the defining ideal of L_B is $\langle n_{31}, n_{41} \rangle$. The residual scheme is defined by the ideal $J : \langle n_{31}, n_{41} \rangle$. Let $K = \langle n_{13}, q_1, q_2, q_3 \rangle$, then we claim $J : \langle n_{31}, n_{41} \rangle = K$.

To show $K \subseteq J : \langle n_{31}, n_{41} \rangle$ one needs only to verify that $n_{31}q_1$ and $n_{41}q_1$ are in J. This follows from $n_{i1}q_1 - n_{i2}q_2 + n_{i3}q_3 = 0$ for $i = 3, 4$.

To show $K \supseteq J : \langle n_{31}, n_{41} \rangle$, let $f \in J : \langle n_{31}, n_{41} \rangle$, and show $f \in K$. From the assumption, both $fn_{31} \in J$ and $fn_{41} \in J$, thus $n_{31}f = \alpha_1 n_{13}n_{31} + \alpha_2 n_{13}n_{41} + \alpha_3 q_3 + \alpha_4 q_2$ and $n_{41}f = \beta_1 n_{13}n_{31} + \beta_2 n_{13}n_{41} + \beta_3 q_3 + \beta_4 q_2$. Then

$$(n_{41}\alpha_1 - n_{31}\beta_1)n_{13}n_{31} + (n_{41}\alpha_2 - n_{31}\beta_2)n_{13}n_{41} + (n_{41}\alpha_3 - n_{31}\beta_3)q_3 + (n_{41}\alpha_4 - n_{31}\beta_4)q_2 = 0.$$

Then, there exists a matrix C of type 4×1 such that

$$\begin{pmatrix} n_{41}\alpha_1 - n_{31}\beta_1 \\ n_{41}\alpha_2 - n_{31}\beta_2 \\ n_{41}\alpha_3 - n_{31}\beta_3 \\ n_{41}\alpha_4 - n_{31}\beta_4 \end{pmatrix} = M_1 C = M_1 \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}.$$

Standard computations give $f = n_{13}(a + n_{31}d_1 + n_{41}d_2) - bq_1 + d_4q_2 + d_3q_3$, where

$$C = \begin{pmatrix} a \\ 0 \\ 0 \\ b \end{pmatrix} + cM_2, \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ n_{33}b \\ -n_{32}b \end{pmatrix} + n_{31} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix},$$

and

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ n_{43}b \\ -n_{42}b \end{pmatrix} + n_{41} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}.$$

Hence $f \in K$, and thus $K = J : \langle n_{31}, n_{41} \rangle$, as claimed. K is the defining ideal of a twisted cubic curve C_B contained in the hyperplane H_B because q_1, q_2, q_3 are the 2×2 minors of a 2×3 general matrix.

Lastly, $r_B = H_B \cap L_B$ is defined by n_{13}, n_{31}, n_{41}, hence r_B is a line if the linear forms are linearly independent. Moreover, $r_B \cap C_B$ is defined by $n_{13}, n_{31}, n_{41}, q_1$ and so it has degree 2. Hence, r_B is a secant line to C_B. $\square$

Proposition 3.3. *The degeneration locus of a general N_C in $\mathbb{P}^4$ is the union of a hyperplane H_C and two dimension 2 linear spaces L_{C1}, L_{C2} that meet at a point $p \in H_C$.*

Proof. The maximal minors of N_C have the linear factor $\alpha n_{31} + \beta n_{22}$ in common, which defines the hyperplane H_C.

The residual locus is defined by the ideal

$$I = \langle n_{22}n_{31}, n_{22}n_{41}, n_{31}n_{42}, n_{41}n_{42} \rangle.$$

In the general case, we have $I = \langle n_{22}, n_{42} \rangle \cap \langle n_{31}, n_{41} \rangle$. Thus the degeneracy locus is equal to $H_C \cup L_{C1} \cup L_{C2}$, where L_{C1} and L_{C2} are 2-dimensional linear spaces defined by $\langle n_{31}, n_{41} \rangle$ and $\langle n_{22}, n_{42} \rangle$, respectively. Moreover, $L_{C1} \cap L_{C2}$ is defined by $\langle n_{22}, n_{31}, n_{41}, n_{42} \rangle$ and so it is a point in H_C. $\square$

Proposition 3.4. *The degeneration locus of a general N_D in $\mathbb{P}^4$ is a hyperplane H_D, and the union of a quadric surface Q_D and a line r_D in H_D. Moreover, $Q_D \cap r_D$ is a point.*

Proof. The degeneracy locus of N_D is defined by the following ideal:

$$I = \langle n_{13}n_{31}^2, n_{13}n_{31}n_{41}, n_{13}n_{31}n_{42}, n_{31}(n_{31}n_{43} - n_{42}n_{23} - n_{41}n_{33}) \rangle.$$

It is evident that $I \subseteq \langle n_{31} \rangle$, and so the top dimensional part of the degeneracy locus is the hyperplane H_D defined by n_{31}. The residual is defined by the ideal

$$J = I : \langle n_{31} \rangle = \langle n_{13}n_{31}, n_{13}n_{41}, n_{13}n_{42}, n_{31}n_{43} - n_{42}n_{23} - n_{41}n_{33} \rangle =$$
$$= \langle n_{13}, n_{31}n_{43} - n_{42}n_{23} - n_{41}n_{33} \rangle \cap \langle n_{31}, n_{41}, n_{42} \rangle.$$

Let Q_D be the quadric surface defined by $\langle n_{13}, n_{31}n_{43} - n_{42}n_{23} - n_{41}n_{33} \rangle$, and let r_D be the line defined by $\langle n_{31}, n_{41}, n_{42} \rangle$, which is contained in H_D. Then the degeneracy locus of N_D is as in the statement.

The intersection $r_D \cap Q_D$ is defined by the ideal $\langle n_{13}, n_{31}, n_{41}, n_{42} \rangle$ which gives a point, as claimed. $\qquad\square$

Now, we consider the matrices in Theorem 2.2.

Proposition 3.5. *Let N be a general 4×3 matrix of linear forms that drops rank on a degree 2 hypersurface, as in Theorem 2.2.*

(1) *If $N = S_1 X_1$, then its degeneration locus in $\mathbb{P}^4$ is the union of a cone Q over a smooth quadric surface in $\mathbb{P}^3$, and its vertex.*

(2) *If $N = S_2 X_2$, then its degeneration locus in $\mathbb{P}^4$ is a smooth quadric hypersurface Q and a line r in Q.*

(3) *If $N = S_3 X_3$, then its degeneration locus in $\mathbb{P}^4$ is the union of a smooth quadric hypersurface Q and a 2–dimensional linear space.*

Proof. Let us consider first the case in which $N = S_1 X_1$. The maximal minors of N, with appropriate sign, are given by $q\ell_1, \ldots, q\ell_4$, where $\ell_1, \ldots, \ell_4$ are linearly independent linear forms and q is a quadratic form. Let (i, j, k) denote the determinant of the submatrix of X_1 consisting of the i–th, j–th and k–th row. Consider the symmetric matrix:

$$D = \frac{1}{2}\begin{pmatrix} -2(2,3,6) & -(2,3,5)+(1,3,6) & -(2,3,4)-(1,2,3) & (3,4,6)+(2,5,6) \\ -(2,3,5)+(1,3,6) & 2(1,3,5) & (1,3,4)-(1,2,5) & (3,4,5)-(1,5,6) \\ -(2,3,4)-(1,2,3) & (1,3,4)-(1,2,5) & -2(1,2,4) & -(2,4,5)-(1,4,6) \\ (3,4,6)+(2,5,6) & (3,4,5)-(1,5,6) & -(2,4,5)-(1,4,6) & 2(4,5,6) \end{pmatrix}.$$

Standard direct computations show that

$$q = \begin{pmatrix} \ell_1 & \cdots & \ell_4 \end{pmatrix} D \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_4 \end{pmatrix}.$$

Notice that q defines a cone over a quadric of $\mathbb{P}^3$ because its equation depends only on 4 linearly independent linear forms. The vertex is defined by $\ell_1 = \cdots = \ell_4 = 0$. The smoothness of the quadric can be obtained by computing the determinant of D with the help of any software for symbolic computations.

In the second case, the maximal minors of N, with appropriate sign, are still given by $q\ell_1, \ldots, q\ell_4$ but now $\ell_1 = -z_2\ell_2 - z_3\ell_3 - z_4\ell_4$. Moreover one has:

$$q = \ell_2 (1, 3, 4) - \ell_3 (1, 2, 4) + \ell_4 (1, 2, 3).$$

One can verify on a random numerical example that the general q defines a smooth quadratic hypersurface. Apart from the vanishing locus of q, the degeneracy locus of N is given by the three

linearly independent linear forms (ℓ_2, ℓ_3, ℓ_4) defining a line r. From the previous description of q, we get that $r \subseteq Q$.

In the remaining case, the maximal minors with sign of N are once again given by $q\ell_1, \dots, q\ell_4$, but now $\ell_1 = -z_{31}\ell_3 - z_{41}\ell_4$ and $\ell_2 = -z_{32}\ell_3 - z_{42}\ell_4$. Then, the degeneracy locus is defined by

$$\langle q \rangle \cap \langle \ell_3, \ell_4 \rangle.$$

The quadratic form q is equal to $\det(X_3)$ and so it defines a smooth quadratic hypersurface in $\mathbb{P}^4$. The second ideal defines a linear space of dimension 2 because the two linear forms are linearly independent. $\qquad\square$

4. Multiview Geometry for projections from $\mathbb{P}^4$ to $\mathbb{P}^2$

A classical problem in Computer Vision is *reconstruction*: given multiple images of an unknown scene, taken from unknown cameras, try to reconstruct the positions of the cameras and of the scene points. It is not difficult to see that sufficiently many images, and sufficiently many sets of corresponding points in the given images, chosen as images of the same set of test points in space, should allow for a successful projective reconstruction. As mentioned in the introduction, similar problems have been investigated in a more general framework of projections from $\mathbb{P}^k$ to $\mathbb{P}^m$, and we are focusing on the case of three projections from $\mathbb{P}^4$ to $\mathbb{P}^2$.

As above, $\mathbb{P}^k$ denotes the $k-$dimensional complex projective space. Once a projective frame is chosen for $\mathbb{P}^k$, coordinate vectors $\mathbf{X}$ of points in $\mathbb{P}^k$ are written as columns, thus $\mathbf{X}^T = (X_1, X_2, ..., X_{k+1})$. A *camera* P is a linear projection from $\mathbb{P}^4$ onto $\mathbb{P}^2$, from a line C_P, called *center* of projection. The target space $\mathbb{P}^2$ is called a *view*. A *scene* is a set of points $\mathbf{X}_i \in \mathbb{P}^4$.

The camera P is identified with a 3×5 matrix of maximal rank, defined up to a multiplicative constant. Hence C_P comes out to be the right annihilator of P.

If $\mathbf{X}$ is a point in $\mathbb{P}^4$, we denote its image in the projection equivalently as $P(\mathbf{X})$ or $P \cdot \mathbf{X}$.

Given different projections with skew centers, $P_i : \mathbb{P}^4 \setminus C_{P_i} \to \mathbb{P}^2$, $i = 1 \dots m$, the images $E_{r,s} = P_r(C_s)$ $(r, s = 1 \dots m, r \neq s)$ are lines of the view spaces, usually called *epipoles*.

Proper linear subspaces (points or lines), $L_i, i = 1 \dots m$, of different views are said to be *corresponding* if there exists at least one point $\mathbf{X} \in \mathbb{P}^4$ such that $P_i(\mathbf{X}) \in L_i$ for all $i = 1 \dots m$.

Hartley and Schaffalitzky, [15], constructed a set of multiview tensors, called *Grassmann tensors*, encoding the relations between sets of corresponding subspaces in the general settings of multiple projections from $\mathbb{P}^k$ to $\mathbb{P}^{h_i}$. Such tensors, which generalize the notion of *fundamental matrix*, are the key ingredient of the process of reconstruction. Indeed, once such a tensor is obtained, one can proceed to reconstruct cameras and scene points. For our purposes, here we recall the definition of such tensors in the case of three projections from $\mathbb{P}^4$ to $\mathbb{P}^2$.

Consider three projections $P_j : \mathbb{P}^4 \setminus C_{P_j} \to \mathbb{P}^2$, $j = 1, 2, 3$, with centers $C_{P_1}, C_{P_2}, C_{P_3}$ in general position. A *profile* is a partition $(\alpha_1, \alpha_2, \alpha_3)$ of $k + 1 = 5$, i.e. $1 \leq \alpha_j \leq 2$ for all j, and $\sum \alpha_j = 5$. The only possible profiles are: $(\alpha_1, \alpha_2, \alpha_3) = (2, 2, 1), (2, 1, 2)$ or $(1, 2, 2)$.

Let $\{L_1, L_2, L_3\}$ be three general linear subspaces of $\mathbb{P}^2$ of codimension $\alpha_1, \alpha_2, \alpha_3$, respectively. Let S_j be the maximal rank $3 \times (3 - \alpha_j)$-matrix whose columns are a basis for $L_j, j = 1, 2, 3$. By definition, if all the L_js are corresponding subspaces there exists a point $\mathbf{X} \in \mathbb{P}^k$ such that $P_j(\mathbf{X}) \in L_j$ for $j = 1, 2, 3$. In other words there exist three vectors $\mathbf{v_j} \in \mathbb{C}^{3-\alpha_j}$ $j = 1, 2, 3$ such that:

$$(7) \qquad \begin{pmatrix} S_1 & 0 & 0 & P_1 \\ 0 & S_2 & 0 & P_2 \\ 0 & 0 & S_3 & P_3 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{v_1} \\ \mathbf{v_2} \\ \mathbf{v_3} \\ \mathbf{X} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The existence of a non–trivial solution $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{X}\}$ of the system (7) implies that the 9×9 coefficient matrix has determinant zero. This determinant can be thought of as a tri-linear form (tensor) in the Plücker coordinates of the spaces L_j. This tensor is the *Grassmann tensor*.

We explicitly construct such a tensor for the profile $(2, 2, 1)$, others being similar. In the case of the chosen profile, L_1, L_2 are points and L_3 is a line. We denote by $\mathbf{x} = (x_1, x_2, x_3)^T, \mathbf{y} = (y_1, y_2, y_3)^T$ the homogeneous coordinates of L_1 and L_2, respectively, and by $\mathbf{z} = (z_1, z_2, z_3)^T$ and $\mathbf{w} = (w_1, w_2, w_3)^T$ the homogeneous coordinates of two points of L_3. Then the matrix of the coefficients of the linear system (7) becomes:

$$T^{P_1,P_2,P_3}_{L_1,L_2,L_3} = \begin{pmatrix} \mathbf{x} & 0 & 0 & 0 & P_1 \\ 0 & \mathbf{y} & 0 & 0 & P_2 \\ 0 & 0 & \mathbf{z} & \mathbf{w} & P_3 \end{pmatrix}$$

If L_1, L_2 and L_3 are corresponding spaces then the linear system

$$(8) \qquad T^{P_1,P_2,P_3}_{L_1,L_2,L_3} \begin{pmatrix} \lambda \\ \mu \\ \alpha \\ \beta \\ \mathbf{X} \end{pmatrix} = \mathbf{0}$$

has a non trivial solution, and so $\det(T^{P_1,P_2,P_3}_{L_1,L_2,L_3}) = 0$.

The converse is true for general L_1, L_2 and L_3 since we are looking for a non trivial solution of (8) in which $\mathbf{X}$ is a point of $\mathbb{P}^4$ and hence $\mathbf{X} \neq \mathbf{0}$, and $\mathbf{X} \notin C_1 \cup C_2 \cup C_3$.

In particular this happens if $L_1 \notin E_{1,2} \cup E_{1,3}$ and $L_2 \notin E_{2,1} \cup E_{2,3}$. Under this hypothesis, if $\det(T^{P_1,P_2,P_3}_{L_1,L_2,L_3}) = 0$, then L_1, L_2 and L_3 are corresponding spaces as the linear system (8) has a non trivial solution with $\mathbf{X}$ as required. Indeed, if $\mathbf{X} = \mathbf{0}$ were part of the solution, either L_1 or L_2 are not proper points, or L_3 is not a line. Moreover $\mathbf{X} \notin C_1 \cup C_2 \cup C_3$, from our initial hypothesis on the epipoles.

In conclusion, for the chosen profile $(2, 2, 1)$, one sees that $\det(T^{P_1,P_2,P_3}_{L_1,L_2,L_3}) = 0$ is indeed the tri–linear constraint between the coordinates $\mathbf{x}$ and $\mathbf{y}$ of points in the first and second view and the Plücker (i.e. dual) coordinates of lines $< \mathbf{z}, \mathbf{w} >$ in the third view, encoding the fact that L_1, L_2, L_3 are corresponding spaces.

More explicitly, denoting by p_1, p_2, p_3 the dual Plücker coordinates of L_3, with an iterated application of the generalized Laplace expansion, one gets:

$$(9) \qquad det(T^{P_1,P_2,P_3}_{L_1,L_2,L_3}) = \sum_{i,j,k} T_{i,j,k} x_i y_j p_k$$

where $i, j, k = 1, 2, 3$ and where the entries $T_{i,j,k}$ of the tensor are given by:

$$(10) \qquad T_{i,j,k} = (-1)^{(i+j+k+1)} \det \begin{bmatrix} P_{1_{\hat{i}}} \\ P_{2_{\hat{j}}} \\ P_3(4-k) \end{bmatrix}$$

where, as above, $P_{1_{\hat{i}}}$ and $P_{2_{\hat{j}}}$ are obtained by P_1 and P_2 respectively deleting rows i and j, and $P_3(4-k)$ denotes the row $4-k$ of the matrix P_3.

5. CRITICAL LOCI: GENERAL SET UP

As discussed in the previous section, sufficiently many views and sufficiently many sets of corresponding points in the given views, should allow for a successful projective reconstruction. This is generally true, but even in the classical set up of two projections from $\mathbb{P}^3$ to $\mathbb{P}^2$ one can have non projectively equivalent pairs of sets of scene points and of cameras that produce the same images in the view planes, from a projective point of view, thus preventing reconstruction. Such configurations and the loci they describe are referred to as *critical.* In [5] critical loci for projective

reconstruction of camera centers and scene points from multiple views for projections from $\mathbb{P}^k$ to $\mathbb{P}^2$ have been introduced and studied. Here we shortly recall the basic definitions in the case of interest, i.e. three views from $\mathbb{P}^4$ to $\mathbb{P}^2$.

A set of points $\{\mathbf{X}_j\}$, $j = 1, \ldots, N$, $N \geq 7$, in $\mathbb{P}^4$ is said to be a *critical configuration for projective reconstruction from* 3-*views* if there exist a non-projectively equivalent set of N points $\{\mathbf{Y}_j\} \subset \mathbb{P}^4$ and two collections of 3×5 full-rank projection matrices P_i and Q_i, $i = 1, 2, 3$, such that, for all i and j, $P_i \cdot \mathbf{X}_j = Q_i \cdot \mathbf{Y}_j$, up to homography in the image planes. Critical configurations arising from a given pair of triples of projections P_i and Q_i define a scheme called *critical locus*.

As shown in [6], the generators of the ideal of the critical locus $\mathcal{X}$ can be obtained directly making use of the Grassmmann tensor built with the Q_is, $det(T_{L_1,L_2,L_3}^{Q_1,Q_2,Q_3})$, introduced in section 4. The idea is that if $\mathbf{X}$ is in the critical locus, the points $P_1(\mathbf{X})$, $P_2(\mathbf{X})$ and $P_3(\mathbf{X})$ are corresponding not only for the P_is but also for the Q_is. This implies that $\mathbf{X}$ is in the critical locus $\mathcal{X}$ if and only if all the maximal minors of the following matrix vanish:

$$(11) \qquad M = \begin{pmatrix} P_1(\mathbf{X}) & 0 & 0 & Q_1 \\ 0 & P_2(\mathbf{X}) & 0 & Q_2 \\ 0 & 0 & P_3(\mathbf{X}) & Q_3 \end{pmatrix}.$$

5.1. **The general case of critical loci.** As we have seen in the previous section, the critical locus $\mathcal{X}$ turns out to be a determinantal variety, associated to the matrix M as above. In [6] the authors study $\mathcal{X}$ when the matrix M satisfies two key hypothesis:

   i) M does not drop rank in codimension 1;
   ii) the last five columns of M are linearly independent.

Hypothesis ii) has a computer vision interpetation. If these columns were linearly dependent, the centers of projection of the matrices Q_i would intersect in at least one point, in which case the entire ambient space would be critical. Hypothesis i) has important algebro-geometric consequences. Indeed it implies that Hilbert–Burch Theorem can be applied to the ideal of $\mathcal{X}$, enabling the computation of its generators and their first syzygy module, giving a scheme–theoretical description of the critical locus itself. It turns out that the ideal is minimally generated by 4 degree-3 forms that are the maximal minors of a 4×3 matrix $N_{\mathcal{X}}$ with linear entries. Moreover, the entries of each column of $N_{\mathcal{X}}$ define four hyperplanes meeting in a line. These three lines are indeed the centers of projection of the P_i. Hence it follows that the critical locus is a determinantal variety of codimension 2 and degree 6 in $\mathbb{P}^4$, and so it belongs to the irreducible component of the Hilbert scheme containing the Bordiga surfaces. We recall that a Bordiga surface S is the blow–up of $\mathbb{P}^2$ at ten general points, embedded in $\mathbb{P}^4$ via the complete linear system of the quartics through the 10 points.

5.2. **The degenerate cases of critical loci.** In this section we study the same problem analyzed in [6] and recalled above, assuming now that the maximal minors of M have a non trivial common factor, and hence the hypothesis of the Hilbert–Burch Theorem are not satisfied. Obviously we still assume hypothesis ii) as above, i.e. that rank $\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = 5$.

Following [6], we now construct a 4×3 matrix $N_{\mathcal{X}}$, with linear entries, whose maximal minors define the same ideal as those of the matrix M.

Up to elementary row operations, M can be written as the following block matrix:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is of type 4×3 and D is of type 5×5, invertible. A new series of elementary operations on rows and columns can then turn M into the block matrix:

$$\begin{pmatrix} N_{\mathcal{X}} & 0 \\ 0 & I_5 \end{pmatrix}$$

where I_5 is the 5×5 identity matrix, 0 are null matrices of suitable type, and

(12) $$N_{\mathcal{X}} = A - BD^{-1}C.$$

The matrix $N_{\mathcal{X}}$ is a 4×3 matrix of linear forms, the minors of which have a non trivial common factor and hence it is one of the matrices classified in Section 2 whose degeneracy loci are studied in Section 3.

Remark 5.1. *Notice that,in view of (11) and (12), the i-th column of $N_{\mathcal{X}}$ contains linear forms that vanish on the center of projection C_{P_i}. Therefore each column of N can contain at most three linearly independent linear forms.*

Conversely, following [6], for a given a matrix $N_{\mathcal{X}}$ as above, it is possible to recover projection matrices $P_1, P_2, P_3, Q_1, Q_2, Q_3$ whose critical locus is given by the minors of $N_{\mathcal{X}}$.

The above remark gives a necessary condition for one of the matrices classified in section 2 to appear in the setting of multiview geometry. This condition is now checked for all cases appearing in Section 2, and the critical locus is described, following Section 3 for admissible cases.

Theorem 5.1. *With notations as above, let P_1, P_2, P_3, and Q_1, Q_2, Q_3 be two sets of projections from $\mathbb{P}^4$ to $\mathbb{P}^2$ whose critical locus is defined by a matrix $N_{\mathcal{X}}$ whose maximal minors have a non trivial common factor. Then the critical locus is as in one of the following cases:*

   i) *the union of a hyperplane H_A and a minimal surface S_A of degree 3 in $\mathbb{P}^4$. The three centers of projections C_{P_i} are contained in S_A.*
   ii) *a hyperplane H_B, and the union of a 2-space L_B and a twisted cubic curve $C_B \subseteq H_B$. Moreover, $H_B \cap L_B$ is a line r_B that meets C_B in two points. The three centers of projections C_{P_i} are contained in H_B.*
   iii) *a hyperplane H_D, and the union of a quadric surface Q_D and a line r_D in H_D. Moreover, $Q_D \cap r_D$ is a point and the three centers of projection C_{P_i} are contained in a 2-space.*
   iv) *the union of a cone Q over a smooth quadric surface in $\mathbb{P}^3$, and its vertex. The three centers of projection C_{P_i} are contained in Q.*
   v) *a smooth quadric hypersurface Q and a line $r \subset Q$. The three centers of projections C_{P_i} are contained in Q.*
   vi) *the union of a smooth quadric hypersurface Q and a 2-space Π. The three centers of projections C_{P_i} are contained in Π.*

Proof. Our assumptions imply that, up to elementary row and column operations, $N_{\mathcal{X}}$ is one of the 4×3 matrices classified in Section 2. Considering matrices as in Theorem 2.1, one sees that:

   A) $N_{\mathcal{X}}$ can be of the form N_A. Indeed, specialization of the first two columns, and suitable elementary operations on the third column, bring N_A to a matrix whose three columns define lines. From Proposition 3.1, we obtain case $i)$ in the statement.
   B) $N_{\mathcal{X}}$ can be of the form N_B. Indeed, as above, suitable elementary operations on the columns, bring N_B to a matrix whose three columns define lines, all contained in the hyperplane $n_{13} = 0$. From Proposition 3.2 we obtain case ii) in the statement.
   C) $N_{\mathcal{X}}$ can not be of the form N_C. From our general assumptions, the third column defines a line. Swapping the first and third column in N_C, we can see that in this case, in (12), it is

(13) $$A = \begin{pmatrix} \alpha n_{31} + \beta n_{22} & 0 & 0 \\ \alpha n_{41} & 0 & 0 \\ \beta n_{42} & 0 & 0 \\ 0 & n_{42} & 0 \end{pmatrix}$$

and the first column of C is the zero vector. Moreover, all linear forms on the remaining columns of C are linearly independent. As the first row of $N_{\mathcal{X}}$ contains two zeros, it follows that the first row of BD^{-1} must be the zero vector, which implies that the first row of B is the zero vector, which is impossible as projection matrices Q_i are of maximal rank.

D) $N_{\mathcal{X}}$ can be of the form N_D. Adding to the first and second column of N_D suitable multiples of the third one, with non-zero coefficients a and b respectively, one can bring N_D to a matrix whose three columns define lines, the first two of which are contained in the 2-space $n_{13} = n_{31} + bn_{23} + an_{33} = 0$, the first and the third of which are contained in the 2-space $n_{13} = n_{23} = 0$, and where the second and third of which are contained in the 2-space $n_{13} = n_{33} = 0$. From Proposition (3.4) we obtain case iii).

Considering matrices as in Theorem 2.2, one sees that:

1) $N_{\mathcal{X}}$ can be of the form $S_1 X_1$. In this case, the j-th column of N is $(N)_j = S_1 \begin{pmatrix} x_{1j} \\ \vdots \\ x_{6j} \end{pmatrix} = Y_j L$

where $Y_j = \begin{pmatrix} 0 & x_{3j} & -x_{2j} & x_{6j} \\ -x_{3j} & 0 & x_{1j} & x_{5j} \\ x_{2j} & -x_{1j} & 0 & x_{4j} \\ -x_{6j} & -x_{5j} & -x_{4j} & 0 \end{pmatrix}$ and $L = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{pmatrix}$. As Y_j is skewsymmetric, its

rank is even and Remark 5.1 implies that $\mathrm{rk}\,(Y_j) = 2$, hence $x_{3j}x_{4j} + x_{2j}x_{5j} + x_{1j}x_{6j} = 0$. Recalling (11) and (12), this case must come from a matrix M whose rows were permuted. Without loss of generality we can assume that $N = A - BD^{-1}C$, where

(14)
$$A = \begin{bmatrix} P_{11}X & 0 & 0 \\ P_{12}X & 0 & 0 \\ 0 & P_{21}X & 0 \\ 0 & 0 & P_{31}X \end{bmatrix} \qquad C = \begin{bmatrix} P_{13}X & 0 & 0 \\ 0 & P_{22}X & 0 \\ 0 & P_{23}X & 0 \\ 0 & 0 & P_{32}X \\ 0 & 0 & P_{33}X \end{bmatrix}.$$

Let $T = (t_{ij}) = BD^{-1}$. Then the first column of N is

$$Y_1 L = (P_{11}X - t_{11}P_{13}X, P_{12}X - t_{21}P_{13}X, -t_{31}P_{13}X, -t_{41}P_{13}X)^T$$

where $P_{11}X, P_{12}X$, and $P_{13}X$ are linearly independent. Remark 5.1 implies $t_{31} = t_{41} = 0$ and therefore $x_{11} = x_{21} = x_{41} = x_{51} = x_{61} = 0$, and $x_{31} \neq 0$. Thus $P_{11}X = t_{11}P_{13}X + x_{31}l_2$, $P_{12}X = t_{21}P_{13}X - x_{31}l_1$. Notice that the first projection matrix P_1 is therefore completely determined once the linear form $P_{13}X$ and t_{11}, t_{21} are chosen.

Proceeding in a similar fashion for the second and third column of N, with somewhat laborious but standard calculations, one sees that P_2 and P_3 are also completely determined once their third rows are known, i.e. once linear forms $P_{23}X$ and $P_{33}X$ are chosen together with suitable parameters related to t_{ij}.

From $T = BD^{-1}$ one can then retrieve the remaining matrices Q_i. For the convenience of the reader, a numerical example of this case is found in section 7.2. From Proposition (3.5) we obtain case iv).

2) $N_{\mathcal{X}}$ can be of the form $S_2 X_2$. For example, assuming all z_is vanish, one can choose three generic linear forms as first row for X_2 and the 3×3 identity matrix to complete X_2. From Proposition 3.5 we obtain case v) in the statement.

3) $N_{\mathcal{X}}$ can be of the form $S_3 X_3$. Recalling (6), we can assume $z_i = 0$ per $i = 1, \ldots 4$. Remark 5.1 implies that one of the entries of the last row of X_3 must vanish and that the linear forms of the first and second column of X_3 must be linearly dependent. It follows then that the entries of the first two columns of $S_3 X_3$ span lines while those of the last column span a plane. M can then be reconstructed as usual. Similarly to case 1) above, notice that the

third projection matrix is not uniquely determined, as one linear form can be freely chosen. From Proposition (3.5) we obtain case vi).

$\square$

6. Reconstruction of the trifocal tensor

This section is dedicated to the investigation of the actual possibility of reconstructing the trifocal tensor in the situations listed in Theorem 5.1 and, when reconstruction is possible, of instability phenomena. Once the tensor is obtained, then one can further reconstruct cameras and sets of scene points, if needed, by intersecting projecting rays of corresponding spaces. As previously noted, the trifocal tensor encodes triplets of corresponding spaces in the views. Let $L_1 = (x_1, x_2, x_3)$ and $L_2 = (y_1, y_2, y_3)$, be two points in the first two views, respectively, and let $L_3 = < (z_1, z_2, z_3), (w_1, w_2, w_3) >$ be a line in the third view, spanned by two given points. Then (9) identifies triplets L_1, L_2, L_3 of corresponding spaces. Viceversa, given a large enough number of triplets of corresponding spaces L_1, L_2, L_3, repeated use of (9) gives rise to a linear system that, when its matrix M_T has maximal rank rk $(M_T) = 26$, determines, up to a multiplicative constant, the 27 entries of the trifocal tensor T. When cameras are in general configurations, rk (M_T) is indeed maximal. This is reflected in [28] where, roughly speaking in our context, the authors study the map associating to a triplet of projection matrices the variety parameterizing the triplets of corresponding points in the views. The main result of this work is that the general fiber of the map above consists of a single orbit of $PGL(5, \mathbb{C})$, in other words the projection matrices, and thus the tensor, are uniquely determined up to projective equivalence in $\mathbb{P}^4$.

Although projection matrices in the cases of interest in this section are not general, one can verify that, in cases (i, ii, iv, v) the rank of the corresponding matrix M_T is still maximal, regardless of the chosen profile. On the contrary, as we will see below, rk (M_T) in cases iii and vi is not maximal.

Remark 6.1. *Assume now that at least two of the centers of projections, say C_{P_1} and C_{P_2}, are contained in a 2-plane Π, and hence they intersect. In this case the trifocal tensor of profile $(2, 2, 1)$ is no longer uniquely determined by constraints generated by triplets of corresponding spaces. It is interesting to see both from a geometric and an algebraic point of view how infinitely many tensors vanish on the same triplets of corresponding spaces.*

From the geometric point of view, one can argue as follows. Assume that the two lines, center of projections, C_{P_1} and C_{P_2} intersect at a point $\mathbf{V}$. Let $\mathbf{X}$ be a scene point and let us consider two corresponding points $L_1 = P_1(\mathbf{X})$ and $L_2 = P_2(\mathbf{X})$ in the first and second view, respectively. Let L_3 be any line in the third view and consider the two planes, $\pi_1 = < L_1, C_{P_1} >$ and $\pi_2 = < L_2, C_{P_2} >$ which are the projecting rays for L_1 and L_2. Then π_1 and π_2 meet along the line $\lambda = < \mathbf{X}, \mathbf{V} >$, which in turn intersects the $3-$ dimensional space $< L_3, C_{P_3} >$ at a point $\mathbf{Y}$, such that $L_1 = P_1(\mathbf{Y}), L_2 = P_2(\mathbf{Y})$ and $P_3(\mathbf{Y}) \in L_3$.

This shows that L_1, L_2 and L_3 are corresponding spaces for any choice of L_3.

From the algebraic point of view, it is easy to see that the trifocal tensor is not well defined. Indeed, one can assume that the matrices $P_i, i = 1, 2$, have the same first two rows. From the expression (10) of the entries of the trifocal tensor, one sees that $T_{i,j,k}$ vanishes unless $i = 1$ and $j = 2$ for any k or $i = 2$ and $j = 1$ for any k. In other words the trilinear relation in this case is

$$(15) \qquad (T_{1,2,1}x_1y_2 + T_{2,1,1}x_2y_1)p_1 + (T_{1,2,2}x_1y_2 + T_{2,1,2}x_2y_1)p_2 + (T_{1,2,3}x_1y_2 + T_{2,1,3}x_2y_1)p_3,$$

where (x_1, x_2, x_3) and (y_1, y_2, y_3) are chosen coordinates in the first two views and (p_1, p_2, p_3) are Plücker coordinates of lines in the third view. Moreover $T_{1,2,k} = -T_{2,1,k}$, for any k. Now, if $(x_1, x_2, x_3) = P_1(\mathbf{X})$ and $(y_1, y_2, y_3) = P_2(\mathbf{X})$, then $x_1 = y_1$ and $x_2 = y_2$, so that expression (15) vanishes for any choice of p_1, p_2, p_3. Moreover any trilinear relation of the form

$$(ax_1y_2 - ax_2y_1)p_1 + (bx_1y_2 - bx_2y_1)p_2 + (bx_1y_2 - bx_2y_1)p_3$$

vanishes as well. This means that the admissible trifocal tensors for this profile depend on three homogeneous parameters, which shows that in these cases $\mathrm{rk}\,(M_T) = 24$. *On the contrary, assuming that the third center is not contained in* Π, *a direct computation shows that, for the remaining profiles,* $\mathrm{rk}\,(M_T) = 26$.

6.1. **Reconstruction of the trifocal tensor - cases** iii, vi**.** In these cases all three centers of projection C_{P_i} are contained in the same 2-space Π. The above remark shows that there is no profile for which the tensor can be uniquely reconstructed.

6.2. **Reconstruction of the trifocal tensor - cases** i, ii, iv **and** v**.** In cases i, ii, iv and v we consider camera centers for the P_is which are in special position in $\mathbb{P}^4$, but do not intersect one another. Although the critical locus, as seen above, turns out to be reducible and of higher codimension than usual, the reconstruction algorithm for the tensor is unaffected by the degenerate camera configuration. As discussed above, the key element needed to reconstruct the tensor is the existence of an actual constraint as a result of imposing to a triplet of spaces in the three views to be corresponding. The fact that the three center of projections are now lines in special position does not affect the generation of the constraints. Indeed, given any to general points L_1, L_2 in the first and second view,respectively, their projecting rays R_1, R_2 are 2-spaces in $\mathbb{P}^4$ that meet at one point $\mathbf{X} \in \mathbb{P}^4$. The projecting ray R_3 of a line in the third view is a 3-space that, in general, will not contain $\mathbf{X}$. Hence, the condition $\mathbf{X} \in R_3$, needed to impose to L_1, L_2, L_3 to be corresponding, is an actual constraint. This is confirmed by a few numerical tests in which $\mathrm{rk}\,(M_T) = 26$.

7. INSTABILITY RESULTS NEAR CRITICAL LOCI

In this section we intend to study the instability of the reconstruction algorithm in the vicinity of the critical locus. Notice that in cases i and ii the reducible critical locus has a hyperplane as a component. When corresponding points chosen in the views are images of points which all lie in a hyperplane, whether or not this hyperplane is a component of the critical locus, $\mathrm{rk}\,(M_T)$ cannot be maximal. If it were, one could reconstruct uniquely the tensor and hence the set of projection matrices, which is clearly impossible as points lying in a hyperplane can be mapped to the same points by infinitely many different projection matrices. Therefore, we will not concern ourselves with instability phenomena starting from points lying on linear components of the critical loci. We will then only consider the cubic rational scroll S_A in case i, and the hyperquadric in cases iv and v.

As it is natural in applications, we reset the framework in an affine context, assuming that the world scene observed lies entirely within the affine chart given by $x_5 \neq 0$ in $\mathbb{P}^4$. The experimental process to investigate instability relies on algorithms developed in [3], and it is described below.

1 *Generation of Critical Configurations*
Two sets of projection matrices $\{P_i\}$ and $\{Q_i\}$, $i = 0, 1, 2$ of the appropriate type, are obtained from N_X as described in [6]. From projection matrices equations of the components of the critical locus and sets of points $\{\mathbf{X}_i\}$ lying on the non-linear components are obtained with the help of Maple, using elementary geometric arguments described in the next two subsections.

2 *Perturbation of critical configurations*
Points $\{\mathbf{X}_i\}$ are then perturbed with a 4-dimensional noise, normally distributed, with zero mean, and with assigned standard deviation σ, obtaining a new configuration $\{\mathbf{X_i}^{pert}\}$, which is close to being critical. Configuration $\{\mathbf{X_i}^{pert}\}$ is then projected via P_1, P_2, P_3. The resulting images $\mathbf{x_{ij}} = P_j \mathbf{X_i}^{pert}$ are again perturbed with normally distributed 2-dimensional noise with zero mean and standard deviation 0.01 to obtain $\{\mathbf{x_{ij}}^{pert}\}$.

3 *Reconstruction*
The trifocal tensor corresponding to the true reconstruction, T_P, is computed from the P_is using a reconstruction algorithm described in detail in [3], section 4.1, and implemented

in Matlab. An estimated trifocal tensor T is computed from $\{\mathbf{x_{ij}}^{pert}\}$, using an algorithm described in [3], section 4.2 and implemented in Matlab as well.

4 *Estimating instability*

As trifocal tensors are defined up to multiplication by a non-zero constant, T_P, and T are normalized (using Frobenius norm) and the space of trifocal tensors is then identified with a subset of the quotient of the unit sphere in $\mathbb{R}^{27}$, $S^{26}/\simeq$, where $\simeq$ denotes antipodal identification. It is simple to account for antipodal identification when computing distances: for any pair of tensors A and B, with unit Frobenius norm, we set $d(A,B) = min(||A - B||, ||A + B||)$. Using this notion of distance, we estimate whether T is close to T_P, or not, where "close" means within a hypersphere of suitable radius δ. In order to choose a suitable δ we start from a set of random points in $\mathbb{P}^4$, project them with the P_i, perturb images using the same noise as in 2 above, reconstruct the tensor T_{test}, compute $d(T_P, Ttest)$ (as above), repeat this procedure 1000 times, find the mean m of all distances, and select δ as a suitable multiple of m.

The above procedure is then repeated 10 times for every fixed value of σ.

7.1. Instability results in case i.
In this case, suitable projection matrices are chosen as follows:

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, P_3 = \begin{bmatrix} \frac{10158}{25} & 729 & 4050 & \frac{31152}{5} & -3645 \\ 608 & -\frac{13692}{25} & 1900 & 836 & \frac{13692}{5} \\ 288 & 162 & \frac{258}{25} & 396 & -810 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, Q_2 = \begin{bmatrix} 0 & 0 & \frac{55}{51} & -\frac{75}{34} & -\frac{625}{51} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, Q_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A set of 100 points are generated on the cubic scroll S, each of which is obtained as the third intersection of S with a 2-plane spanned by two points chosen on the centers of projection of the P_i and a third point chosen randomly in $\mathbb{P}^4$. The result of the experimental process described above is presented in Figure 7.1, where the frequency with which the reconstructed solution is close or far from the true solution T_P, against the values of σ utilized, is plotted. In this case $m = 0.014$, and $\delta = 0.03$.

The set of parameters utilized in this case is as follows:

- $X_i \in S$, $\delta = .03$, $\sigma \in (10^{-4}, 1)$, every 10^{-2}.

The experiment shows that reconstruction near the cubic scroll is quite unstable. As the standard deviation σ of the perturbation in $\mathbb{P}^4$ approaches 1 the stability of the reconstruction increases as expected, but still not fully stable when $\sigma = 1$.

7.2. Instability results in case iv.
In this case, let us choose projection matrices as follows:

$$P_1 = \begin{bmatrix} 0 & 1 & -6 & -2 & 4 \\ -1 & 0 & -15 & -5 & 10 \\ 0 & 0 & -3 & -1 & 2 \end{bmatrix}, P_2 = \begin{bmatrix} -35 & -42 & -47 & 13 & -32 \\ -8 & -12 & -11 & 4 & -8 \\ -2 & -3 & -2 & 1 & -2 \end{bmatrix}, P_3 = \begin{bmatrix} -2 & 3 & -12 & 6 & 3 \\ 4 & 6 & 0 & -8 & 6 \\ 2 & 3 & 0 & -2 & 3 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} 2 & 1 & -4 & 1 & -2 \\ 5 & -2 & 8 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, Q_2 = \begin{bmatrix} 0 & 5 & -4 & 3 & -6 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, Q_3 = \begin{bmatrix} 0 & 1 & -4 & -2 & 5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$
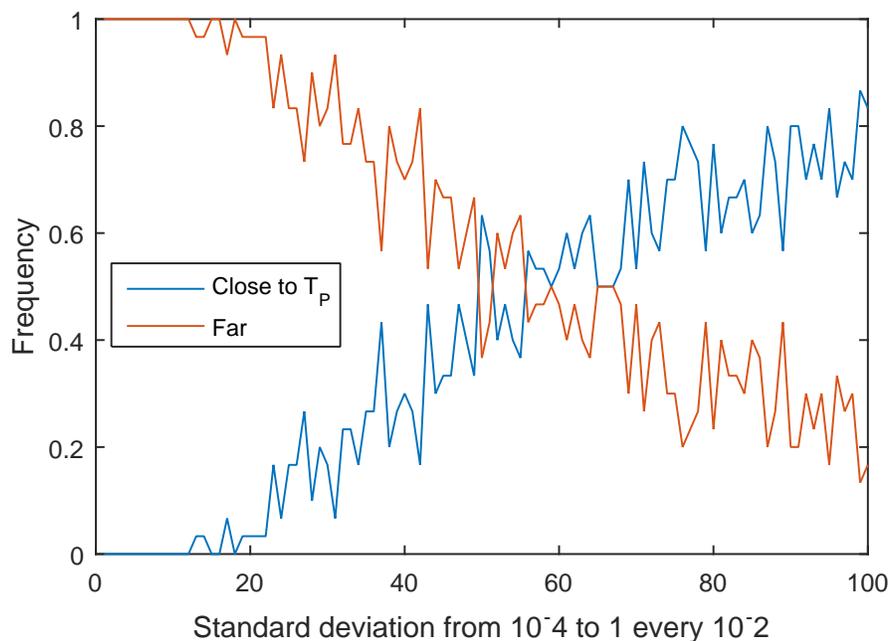
FIGURE 1. Instability of reconstruction of a trifocal tensor as described in 7.1 near S

Recalling (12), (11), and (14), one can check that this case is of type iv with

$$
S_1 = \begin{bmatrix} 0 & -x_3 & x_2 & 0 & 0 & x_4 \\ x_3 & 0 & -x_1 & 0 & x_4 & 0 \\ -x_2 & x_1 & 0 & x_4 & 0 & 0 \\ 0 & 0 & 0 & -x_3 & -x_2 & -x_1 \end{bmatrix}, X_1 = \begin{bmatrix} 0 & 6 & 0 \\ 0 & -3 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 12 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}.
$$

The equation of the critical quadric cone Q is $x_1^2 - 2x_1x_2 + 3x_3x_1 + x_1x_4 - 6x_3x_2$. A set of 100 points are generated on the quadric cone, leveraging the fact that the equation of the quadric can be easily parametrized. The result of the experimental process described above is presented in Figure 7.2, where the frequency with which the reconstructed solution is close or far from the true solution T_P, against the values of σ utilized, is plotted. In this case $m = 0.0012$, and $\delta = 0.015$.

7.3. Instability results in case v.
In this case, suitable projection matrices are chosen as follows:

$$
P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}, P_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}
$$

$$
Q_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}, Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, Q_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

Various experiments on random choices give the results in the Figure 7.3 or similar to it.

In all cases, the experiments were performed as follows. A set of 99 points are generated on the quadric hypersurface Q, each of which is obtained as the second intersection of Q with a line through
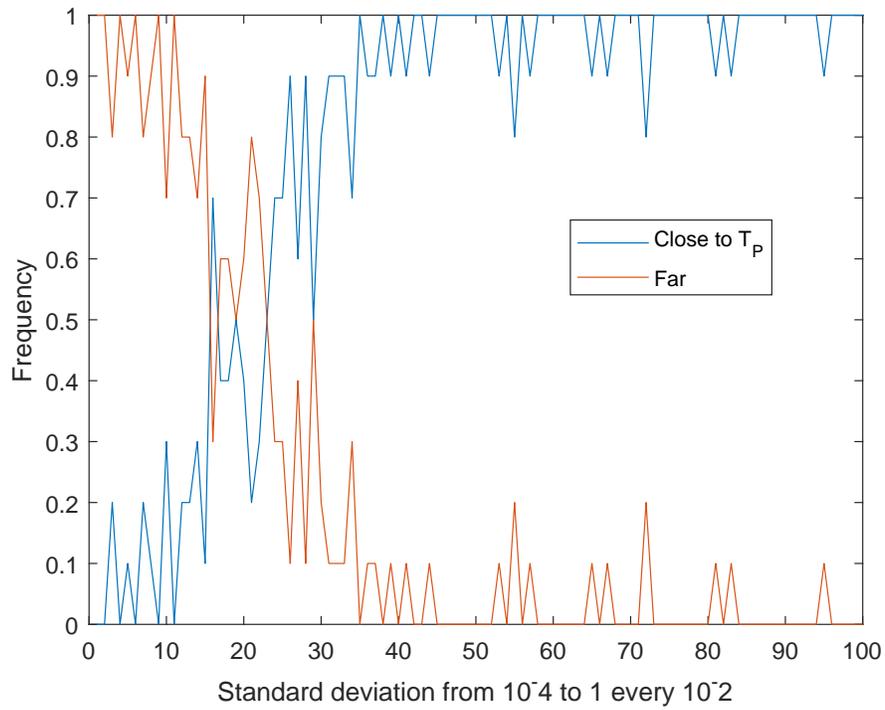
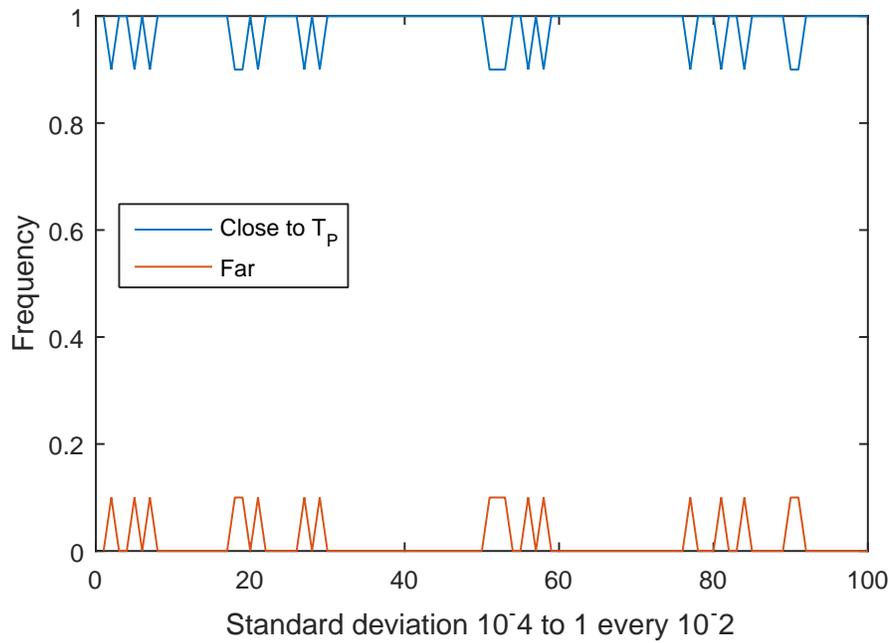FIGURE 2. Instability of reconstruction of a trifocal tensor as described in 7.2 near Q.

FIGURE 3. Instability of reconstruction of a trifocal tensor as described in 7.3 near Q.

a point chosen on one of the centers of projection of the P_i and a second point chosen randomly in $\mathbb{P}^4$. The result of the experimental process described above is presented below in Figure 2, where the frequency with which the reconstructed solution is close or far from the true solution T_P, against the values of σ utilized, is plotted. In this case $m = 0.015$, and $\delta = 0.03$.

The set of parameters utilized in this case is as follows:

- $X_i \in Q$, $\delta = .03$, $\sigma \in (10^{-4}, 1)$, every 10^{-2}.

The experiment shows that reconstruction near the quadric is surprisingly very stable.

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