On the continuation of degenerate periodic orbits via normal form: lower dimensional resonant tori

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Abstract

We consider the classical problem of the continuation of periodic orbits surviving to the breaking of invariant lower dimensional resonant tori in nearly integrable Hamiltonian systems. In particular we extend our previous results (presented in CNSNS, 61:198-224, 2018) for full dimensional resonant tori to lower dimensional ones. We develop a constructive normal form scheme that allows to identify and approximate the periodic orbits which continue to exist after the breaking of the resonant torus. A specific feature of our algorithm consists in the possibility of dealing with degenerate periodic orbits. Besides, under suitable hypothesis on the spectrum of the approximate periodic orbit, we obtain information on the linear stability of the periodic orbits feasible of continuation. A pedagogical example involving few degrees of freedom, but connected to the classical topic of discrete solitons in dNLS-lattices, is also provided.

Keywords: Hamiltonian normal forms, lower dimensional resonant tori, degenerate periodic orbits, linear stability, perturbation theory

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1. Introduction

Consider a canonical system of differential equations with Hamiltonian

$$H(I,\varphi,\xi,\eta,\varepsilon) = H_0(I,\xi,\eta) + \varepsilon H_1(I,\varphi,\xi,\eta;\varepsilon) \ , \quad H_0 = h_0(I) + g_0(\xi,\eta) \ , \tag{1}$$

in $n = n_1 + n_2$ degree of freedom where $(I,\varphi) \in \mathcal{U}(I^*) \times \mathbb{T}^{n_1}$ are angle-action variables defined in a neighbourhood $\mathcal{U}(I^*) \subset \mathbb{R}^{n_1}$ of the action I^*, $(\xi,\eta) \in \mathcal{V}(0)$ are Cartesian variables defined in a neighbourhood $\mathcal{V}(0) \subset \mathbb{C}^{2n_2}$ of the origin and ε is a small parameter. The Hamiltonian (1) is assumed to be analytic in all variables and in the small parameter ε. The unperturbed Hamiltonian H_0 is assumed to be the sum of a generic Hamiltonian $h_0(I)$ that can be expanded in power series of $J = I - I^*$ as

$$h_0(J;I^*) = \langle \hat\omega(I^*), J\rangle + \frac{1}{2}\langle D_I^2 h_0(I^*)J, J\rangle + h.o.t. \ , \qquad \text{with} \quad \hat\omega(I^*) = \frac{\partial h_0}{\partial I}\bigg|_{I=I^*} \ , \tag{2}$$

and an Hamiltonian $g_0(\xi,\eta)$ that has an elliptic equilibrium at the origin, i.e.,

$$g_0(\xi,\eta) = \sum_{j=1}^{n_2} \mathbf{i}\Omega_j \xi_j \eta_j + h.o.t. \ . \tag{3}$$

In this paper we investigate the problem of the continuation of periodic orbits which survive to the breaking of a completely resonant n_1-dimensional torus I^* of (1). A typical example is provided by physical models described by a Hamiltonian (1) made by identical and weakly coupled nonlinear oscillators (see for example the editorial review [30] on Hamiltonian Lattices), with n_1 ones that have been excited and oscillate periodically with the same frequencies $\hat\omega(I^*)$ and n_2 ones are at rest.

The existence of sub-tori surviving to n_1-dimensional partially resonant tori has been widely treated in the literature; see, e.g., [13, 41, 4], where some nondegeneracy assumptions on one or both of the Hessian

$D^2_I h_0(I^*)$ (Kolmogorov nondegeneracy) and of the critical points of the time-averaged[1] perturbation $\langle H_1 \rangle_T$ (Poincaré nondegeneracy) are assumed. Other more recent works investigate the problem when degeneracy of $D^2_I h_0(I^*)$ occurs, like in [42, 25, 14, 15] or in the recent works [43, 44], where degeneracy is due to different time scales in the integrable Hamiltonian (like in problems of Celestial Mechanics).

Differently from the previous literature, our attention is on the problem of continuation of periodic orbits when Poincaré nondegeneracy does not hold; this typically happens when critical points of $\langle H_1 \rangle_T$ are not isolated, being part of a d-parameter family. Our perspective is then to look for a normal form construction which allows us to inspect the Poincaré degeneracy, in the case of completely resonant lower dimensional tori: such a perturbation approach is able to identify unperturbed periodic orbits which are candidates for continuation at $\varepsilon \neq 0$, as well as to show the structure of the linear dynamics around those orbits. In this sense, the results here included represent the natural generalization of [34], where the same problem was faced limiting to resonant tori of maximal dimension, thus extending the original ideas of Poincaré, see [37, 38]. An informal statement that sums up our results is the following

> Consider the Hamiltonian (1) with H_0 as specified in (2) and (3). Take an unperturbed resonant torus carrying periodic orbits with frequency ω, where ω is such that $\hat{\omega}(I^*) = \omega k$ with $k \in \mathbb{Z}^{n_1}$. Assume that the frequency ω and the transverse frequency vector $\Omega = \{\Omega_j\}$ are strongly nonresonant and satisfy the first and second Melnikov condition. Assume also that $h_0(I)$ is nondegenerate. Then, there exists $\varepsilon^* > 0$ such that for $|\varepsilon| < \varepsilon^*$ the following statements hold true:
>
> - the Hamiltonian can be put in normal form up to a finite arbitrary order r by means of an analytic canonical transformation. *Typically* the normal form allows to identify isolated approximate periodic orbits that can survive the breaking of the unperturbed torus;
> - under suitable assumptions on the spectrum of the approximate monodromy matrix, given by the truncated normal form, the approximate periodic orbit can be continued for $\varepsilon \neq 0$;
> - under stricter conditions on the spectrum, the linear stability of the true periodic orbit can be inferred from the approximate one.

In order to illustrate our original approach, we propose a pedagogical example with few degrees of freedom, which is inspired to the problem of the existence of discrete solitons in discrete NLS models. The example is described in the following subsection, and is analyzed in detail in Section 5; the corresponding calculations have been developed with the help of Mathematica Software. Applications of the normal form to proper dNLS models, with a sufficiently large number of sites and a suitable variety of degenerate spatially localized configurations (like vortexes, multi-peaked solutions or different resonances), would require a systematic investigation with longer algebraic manipulations; this will be the object of a distinct and subsequent publication.

1.1. The seagull *example*

Consider a system of 5 coupled anharmonic oscillators with Hamiltonian

$$H = H_0 + \varepsilon H_1 = \sum_{j=-2}^{2} \left(\frac{x_j^2 + y_j^2}{2} + \gamma \left(\frac{x_j^2 + y_j^2}{2} \right)^2 \right) + \varepsilon \sum_{j=-2}^{1} (x_{j+1} x_j + y_{j+1} y_j) , \quad (x, y) \in \mathbb{R}^5 \times \mathbb{R}^5 , \quad (4)$$

with $\gamma \neq 0$ a parameter tuning the nonlinearity; considering the figure below, one could consider in an equivalent way a chain of 7 masses, i.e., $(x, y) \in \mathbb{R}^7 \times \mathbb{R}^7$ and with fixed boundary conditions $x_{-3} = y_{-3} =$

[1] Notice that H_1 is turned into a function of time once evaluated on the periodic flow given by the unperturbed resonant torus I^*; hence it can be time-averaged over the period T of the unperturbed flow.

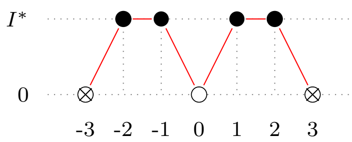
$x_3 = y_3 = 0$. We introduce action-angle variables $(x_j, y_j) = (\sqrt{2I_j}\cos\varphi_j, -\sqrt{2I_j}\sin\varphi_j)$, for the set of indices $j \in \mathcal{I} = \{-2, -1, 1, 2\}$, and the complex canonical coordinates

$$x_0 = \frac{1}{\sqrt{2}}(\xi_0 + \mathbf{i}\eta_0), \quad y_0 = \frac{\mathbf{i}}{\sqrt{2}}(\xi_0 - \mathbf{i}\eta_0) ,$$

for the remaining central one (x_0, y_0), so that the Hamiltonian reads as (1) with

$$\begin{aligned} h_0(I) &= \sum_{j\in\mathcal{I}} \left(I_j + \gamma I_j^2\right) , \\ g_0(\xi, \eta) &= \mathbf{i}\xi_0\eta_0 - \gamma\xi_0^2\eta_0^2 , \\ H_1 &= 2\sqrt{I_{-1}I_{-2}}\cos(\varphi_{-1} - \varphi_{-2}) + 2\sqrt{I_2I_1}\cos(\varphi_2 - \varphi_1) + \\ &\quad + (\xi_0 + \mathbf{i}\eta_0)\Big(\sqrt{I_{-1}}\cos(\varphi_{-1}) + \sqrt{I_1}\cos(\varphi_1)\Big) - \mathbf{i}(\xi_0 - \mathbf{i}\eta_0)\Big(\sqrt{I_{-1}}\sin(\varphi_{-1}) + \sqrt{I_1}\sin(\varphi_1)\Big) . \end{aligned}$$

Consider now the 4-dimensional unperturbed resonant torus $I = I^*$ with $I_j^* = I_l^*$, for $j, l \in \mathcal{I}$, and $\xi_0 = \eta_0 = 0$. The configuration is represented in the following picture, which explains the name seagull



where the central oscillator is free to move, while the first and last one are kept at rest due to the Dirichlet boundary conditions. This case provides a typical and easy mechanism for Poincaré degeneracy, due to the absence of the 1:1 resonance among the nonlinear oscillators I_{-1} and I_1 in the perturbation εH_1 (see also [35]). Indeed these two oscillators interact at order $\mathcal{O}(\varepsilon)$ only with the central one (ξ_0, η_0), which is at rest in the unperturbed dynamics; as a consequence $\langle H_1\rangle_T$ is independent of the phase difference $\varphi_1 - \varphi_{-1}$, and its critical points are not isolated.

In order to reveal a finer structure of the dynamics around the unperturbed low-dimensional torus, we expand H in power series of $J = I - I^*$ and introduce the resonant angles $\hat{q} = (q_1, q)$ and their conjugate actions $\hat{p} = (p_1, p)$ as

$$\begin{cases} q_1 = \varphi_{-2} \\ q_2 = \varphi_{-1} - \varphi_{-2} \\ q_3 = \varphi_1 - \varphi_{-1} \\ q_4 = \varphi_2 - \varphi_1 \end{cases} \qquad \begin{cases} p_1 = J_{-2} + J_{-1} + J_1 + J_2 \\ p_2 = J_{-1} + J_1 + J_2 \\ p_3 = J_1 + J_2 \\ p_4 = J_2 \end{cases} . \tag{5}$$

Besides using the change of coordinates above, we decide to split the Hamiltonian in the form

$$\begin{aligned} H^{(0)} &= \omega p_1 + \mathbf{i}\xi_0\eta_0 + f_4^{(0,0)} \\ &\quad + f_0^{(0,1)} + f_1^{(0,1)} + f_2^{(0,1)} + f_3^{(0,1)} + f_4^{(0,1)} + \sum_{\ell>4} f_\ell^{(0,1)} + \mathcal{O}(\varepsilon^2) , \end{aligned}$$

where $\omega = 1 + 2\gamma I^*$ is the frequency of any periodic orbit on the unperturbed torus $p = 0$ and $f_\ell^{(r,s)}$ is a polynomial of degree l in $\hat{p}$ and degree m in (ξ_0, η_0) with $\ell = 2l + m$ and with coefficients depending on the angles $\hat{q}$. The index $r \geq 0$ identifies the order of normalization ($r = 0$ being the original Hamiltonian), while s keeps track of the order in the small parameter ε. The explicit form of $f_0^{(0,1)}$ is

$$f_0^{(0,1)} = 2I^*\varepsilon\left(\cos(q_2) + \cos(q_4)\right) .$$

The splitting of the Hamiltonian in such a form may seem quite obscure now. However, considering the equation of motion restricted to the lower dimensional torus $p = 0$, $\xi_0 = \eta_0 = 0$ a moment's thought suggests how to put in evidence the relevant terms of the perturbation.

In order to continue the periodic orbit surviving the breaking of the unperturbed lower dimensional torus, the standard approach consists in averaging the leading term of the perturbation, namely $f_0^{(0,1)}$, with respect to the fast angle q_1 and to look for critical points of the averaged function on the torus $\mathbb{T}^3$. In this specific example however no averaging is required as $f_0^{(0,1)}$ does not depend on q_1, due to the rotational symmetry typical of dNLS models. Still, solutions of $\nabla_q f_0^{(0,1)} = 0$ are not isolated and appear as 1-parameter families parameterized by q_3, hence Poincaré degeneracy occurs. Let us remark again that here the degeneracy is due to the lack of the harmonic $(\varphi_{-1} - \varphi_1)$ in the perturbation at order ε, that entails the independence of $f_0^{(0,1)}$ by q_3.

Our aim is to show that only solutions (q_2, q_3, q_4) with $q_j \in \{0, \pi\}$ (the so-called *in* or *out-of* phase solutions) can be continued for $\varepsilon \neq 0$. To this end we implement a normal form construction that is reminiscent of the Kolmogorov algorithm (see also [40, 11]). Indeed, we perform a sequence of canonical transformations in order to remove the terms $f_1^{(0,1)}$ and $f_3^{(0,1)}$ and to average the terms $f_0^{(0,1)}$, $f_2^{(0,1)}$ and $f_4^{(0,1)}$ over the fast angle q_1. In addition, we perform a translation of the actions $\hat{p}$ so as to keep fixed the linear frequency ω.

This procedure brings the Hamiltonian in normal form at order ε. Iterating twice the procedure we get the Hamiltonian in normal form at order ε^2 that reads

$$\begin{aligned}
H^{(2)} = &\omega p_1 + \mathbf{i}\xi_0\eta_0 + f_4^{(2,0)} \\
&+ f_0^{(2,1)} + f_2^{(2,1)} + f_4^{(2,1)} + \sum_{\ell>4} f_\ell^{(2,1)} \\
&+ f_0^{(2,2)} + f_2^{(2,2)} + f_4^{(2,2)} + \sum_{\ell>4} f_\ell^{(2,2)} + \mathcal{O}(\varepsilon^3) \ .
\end{aligned}$$

Considering the normal form truncated at order two, i.e., neglecting terms of order $\mathcal{O}(\varepsilon^3)$, the leading terms of the perturbation[2] are

$$f_0^{(2,1)} + f_0^{(2,2)} = 2I^*\varepsilon\left(\cos(q_2) + \cos(q_4)\right) + \frac{\varepsilon^2}{\gamma}\big(\cos(q_3) - \cos(q_2)\cos(q_2^*) - \cos(q_4)\cos(q_4^*)\big) \ ,$$

and looking for the critical point one gets

$$2I^*\sin(q_j) - \frac{\varepsilon}{\gamma}\sin(q_j)\cos(q_j^*) = 0 \ , \quad \text{for } j = 2, 4 \ , \quad \text{and} \quad \varepsilon\frac{\sin(q_3)}{\gamma} = 0 \ . \tag{6}$$

The normal form at order two, as we see from the previous equations, introduces the dependence on q_3 that was missing at order one. With standard arguments of bifurcation theory (the same already used for example in [36, 35]) it is possible to prove that for ε small enough all families break down and only solutions with $q_j \in \{0, \pi\}$ survive; continuation then follows by means of Newton-Kantorovich fixed point method, since suitable spectral conditions are verified. We refer the reader to Section 5 for details and for interesting results about the role of γ in the linear stability analysis of the solutions: indeed we will show that neither changing the sign in the nonlinear parameter γ (from focusing to defocusing) nor in the coupling parameter ε (from attractive to repulsive) affects the nature of the degenerate eigenspace related to q_3, while nondegenerate directions (as already known from the literature, see [31, 24]) switch from saddle to center depending on the sign of the product $\gamma\varepsilon$.

Let us remark that the previous example might be explored with a different approach, which exploits the dNLS structure of (4), namely its second conserved quantity $\sum_{j=-2}^{2}(x_j^2 + y_j^2)$ and the discrete soliton

[2] Let us stress that the parameters q_2^* and q_4^* allow to select the approximate periodic orbit, see equation (19). These parameters are introduced by the translation of the actions $\hat{p}$ outlined above, see Proposition 2.1 and (35) for more details.

ansatz, which separate time and (discrete) space variables (see [17, 16, 31, 20, 24, 32, 5]). The same problem is investigated also in discrete Klein-Gordon models, but with different perturbation techniques (Lyapunov-Schmidt decomposition as in [33, 35] and Hamiltonian averaging as in [2, 1, 24, 22, 23]). However, up to our knowledge, the existing results are valid for specific configurations (e.g., restricting to consecutive oscillators) and degenerate solutions can be hardly explored (see [6]). Moreover, available methods for nondegenerate solutions (as in [19, 1, 24]) can be recovered by a single step of our normal form approach.

The formal statements of the three main results, i.e., the normal form construction, the continuation theorem and the linear stability theorem, are detailed in Section 2 as Proposition 2.1, Theorem 2.1 and Theorem 2.2, respectively. Sections 3 and 4 provide the description of the normal form construction together with some analytical estimates. Section 5 treats extensively the example (4). A concluding Appendix collects some technical results.

2. Main results

In this section we introduce the analytic setting and we precisely state the main results of the paper.

2.1. Analytic setting

Consider the distinguished classes of functions $\widehat{\mathcal{P}}_{l,m}$, with integers l and m, which can be written as a Taylor-Fourier expansion

$$g(I,\varphi,\xi,\eta) = \sum_{\substack{i\in\mathbb{N}^{n_1}\\ |i|=l}} \sum_{\substack{(m_1,m_2)\in\mathbb{N}^{2n_2}\\ |m_1|+|m_2|=m}} \sum_{k\in\mathbb{Z}^{n_1}} g_{i,m_1,m_2,k}\, I^i \exp(\mathbf{i}\langle k,\,\varphi\rangle)\xi^{m_1}\eta^{m_2} \ , \tag{7}$$

with coefficients $g_{i,m_1,m_2,k} \in \mathbb{C}$. We say that $g \in \mathcal{P}_\ell$ in case

$$g \in \bigcup_{\substack{l\geq 0, m\geq 0\\ 2l+m=\ell}} \widehat{\mathcal{P}}_{l,m} \ .$$

We also set $\mathcal{P}_{-4} = \mathcal{P}_{-3} = \mathcal{P}_{-2} = \mathcal{P}_{-1} = \{0\}$; moreover, we introduce the following notation for those terms which are independent of both actions I and Cartesian variables (ξ,η)

$$f_{l,0} \in \widehat{\mathcal{P}}_{l,0} \ , \qquad\qquad f_{0,m} \in \widehat{\mathcal{P}}_{0,m} \ . \tag{8}$$

Consider the Hamiltonian (1) and select a specific completely resonant elliptic lower dimensional torus for the unperturbed Hamiltonian, i.e., set $\xi = \eta = 0$ and $I = I^*$ such that

$$\hat{\omega}(I^*) = \left.\frac{\partial h_0}{\partial I}\right|_{I=I^*} = \omega k \ , \quad \text{with } \omega\in\mathbb{R} \ , \ k\in\mathbb{Z}^{n_1} \ . \tag{9}$$

Expanding the Hamiltonian in Taylor series of the translated actions $J = I - I^*$ and the Cartesian coordinates (ξ,η), and in Fourier series of the angles φ one has

$$H^{(0)} = \langle\hat{\omega}, J\rangle + \sum_{j=1}^{n_2} \mathbf{i}\Omega_j\xi_j\eta_j + \sum_{\ell>2} f_\ell^{(0,0)}(J,\xi,\eta) + \sum_{s>0}\sum_{\ell\geq 0} f_\ell^{(0,s)}(J,\varphi,\xi,\eta) \ , \tag{10}$$

where $f_\ell^{(0,s)} \in \mathcal{P}_\ell$ and is of order $\mathcal{O}(\varepsilon^s)$. The superscript 0 indicates that the Hamiltonian is the starting one and in the following will be used to keep track of the normalization order.

2.2. The normal form

We define the (n_1-1)-dimensional resonant module associated to the resonant frequency $\hat{\omega}(I^*)$ in (9) as

$$\mathcal{M}_\omega = \left\{h \in \mathbb{Z}^{n_1} : \langle\hat{\omega}(I^*), h\rangle = 0\right\} .$$

In a neighbourhood of the resonant torus, it is useful to introduce the resonant variables $(\hat{p}, \hat{q})$ in place of (J, φ). Without affecting the generality of the result, we will assume $k_1 = 1$ (see (9)); this choice simplifies the interpretation of the new variables. Given $k \in \mathbb{Z}^{n_1}$ defined by (9), the canonical change of coordinates is built with an unimodular matrix which defines the *slow* angles $\hat{q}_j = k_j\varphi_1 - \varphi_j$, for $j = 2, \ldots, n_1$, as the phase differences with respect to the *fast* angle $\hat{q}_1$ of the periodic orbit; the momenta are defined so as to complement the canonical change of coordinates, in particular $\hat{p}_1 = \langle k, J\rangle$.

In order to distinguish the dependence on fast and slow variables in the normal form construction, we introduce the notations $\hat{p} = (p_1, p)$, $\hat{q} = (q_1, q)$ with $p_1 = \hat{p}_1$, $p = (\hat{p}_2, \ldots, \hat{p}_n)$ and correspondingly for q_1 and q. The Hamiltonian (10) then reads

$$H^{(0)} = \omega p_1 + \sum_{j=1}^{n_2} \mathbf{i}\Omega_j\xi_j\eta_j + \sum_{\ell>2} f_\ell^{(0,0)}(\hat{p}, \xi, \eta) + \sum_{s>0}\sum_{\ell\geq 0} f_\ell^{(0,s)}(\hat{p}, \hat{q}, \xi, \eta) \ , \tag{11}$$

where $f_\ell^{(0,s)} \in \mathcal{P}_\ell$ and it is a function of order $\mathcal{O}(\varepsilon^s)$. Indeed, the linear change of coordinates keeps unchanged the classes of functions $\mathcal{P}_\ell$.

We introduce the usual complex domains $\mathcal{D}_{\rho,\sigma,R} = \mathcal{G}_\rho \times \mathbb{T}^{n_1}_\sigma \times \mathcal{B}_R$, namely

$$\begin{aligned} \mathcal{G}_\rho &= \left\{\hat{p} \in \mathbb{C}^{n_1} : \max_{1\leq j\leq n_1} |\hat{p}_j| < \rho\right\} \ , \\ \mathbb{T}^{n_1}_\sigma &= \left\{\hat{q} \in \mathbb{C}^{n_1} : \operatorname{Re}\hat{q}_j \in \mathbb{T}, \ \max_{1\leq j\leq n_1} |\operatorname{Im}\hat{q}_j| < \sigma\right\} \ , \\ \mathcal{B}_R &= \left\{(\xi, \eta) \in \mathbb{C}^{2n_2} : \max_{1\leq j\leq n_2} (|\xi_j| + |\eta_j|) < R\right\} \ . \end{aligned}$$

Given a generic analytic function $g : \mathcal{D}_{\rho,\sigma,R} \to \mathbb{C}$, we define the weighted Fourier norm

$$\|g\|_{\rho,\sigma,R} = \sum_{i\in\mathbb{N}^{n_1}} \ \sum_{(m_1,m_2)\in\mathbb{N}^{2n_2}} \ \sum_{k\in\mathbb{Z}^{n_1}} |g_{i,m_1,m_2,k}|\rho^{|i|}R^{|m_1|+|m_2|}e^{|k|\sigma} \ ;$$

hereafter, we are going to use the shorthand notation $\|\cdot\|_\alpha$ for $\|\cdot\|_{\alpha(\rho,\sigma)}$.

We now state our main result on the normal form construction; the proof is deferred to Section 4.

Proposition 2.1. *Consider the Hamiltonian $H^{(0)}$, expanded as in* (11)*, and being analytic in the domain $\mathcal{D}_{\rho,\sigma,R}$. Assume that*

H1) *there exists a positive constant m such that for every $v \in \mathbb{R}^{n_1}$ one has*

$$m\sum_{i=1}^{n_1}|v_i| \leq \sum_{i=1}^{n_1}\Big|\sum_{j=1}^{n_1} C_{0,ij}v_j\Big| \ , \quad \textit{where} \quad C_{0,ij} = \frac{\partial^2 f_4^{(0,0)}}{\partial\hat{p}_i\partial\hat{p}_j}\bigg|_{\hat{p}=0} \ ; \tag{12}$$

H2) *the terms appearing in the expansion of the Hamiltonian satisfy*

$$\|f_\ell^{(0,s)}\|_1 \leq \frac{E}{2^\ell}\varepsilon^s \ , \qquad \textit{with} \quad E > 0 \ , \ l, s \geq 0 \ . \tag{13}$$

H3) *the frequencies ω defined in* (9) *and Ω_l introduced in* (3) *satisfy the first and second nonresonance Melnikov conditions*

$$k_1\omega \pm \Omega_j \neq 0, \qquad k_1 \in \mathbb{Z} \ , \tag{14}$$

$$k_1\omega \pm \Omega_l \pm \Omega_k \neq 0, \qquad k_1 \in \mathbb{Z}\setminus\{0\} \ . \tag{15}$$

Then, for every integer $r \geq 1$ there exists $\varepsilon_r^* > 0$ such that for $|\varepsilon| < \varepsilon_r^*$ there exists an analytic canonical transformation $\Phi^{(r)}$ satisfying

$$\mathcal{D}_{\frac{1}{4}(\rho,\sigma,R)} \subset \Phi^{(r)}\Big(\mathcal{D}_{\frac{1}{2}(\rho,\sigma,R)}\Big) \subset \mathcal{D}_{\frac{3}{4}(\rho,\sigma,R)} \tag{16}$$

such that the Hamiltonian $H^{(r)} = H^{(0)} \circ \Phi^{(r)}$ has the following expansion in normal form up to order r

$$\begin{aligned} H^{(r)}(\hat{q},\hat{p},\xi,\eta;q^*) =& \omega p_1 + \sum_{j=1}^{n_2} \mathbf{i}\Omega_j \xi_j \eta_j + \sum_{\ell>2} f_\ell^{(r,0)}(\hat{p},\xi,\eta) \\ &+ \sum_{s=1}^{r} \Big(f_0^{(r,s)}(q;q^*) + f_2^{(r,s)}(q,\hat{p},\xi,\eta;q^*) + f_3^{(r,s)}(\hat{q},\xi,\eta;q^*) + f_4^{(r,s)}(\hat{q},\hat{p},\xi,\eta;q^*)\Big) \\ &+ \sum_{s>r}\sum_{\ell=0}^{4} f_l^{(r,s)}(\hat{q},\hat{p},\xi,\eta;q^*) + \sum_{s>0}\sum_{\ell>4} f_l^{(r,s)}(\hat{q},\hat{p},\xi,\eta;q^*) \ , \end{aligned} \tag{17}$$

where $q^* \in \mathbb{T}^{n_1-1}$ is a fixed but arbitrary vector of parameters. The Hamiltonian (17) *is said to be in normal form up to order r since for $s \leq r$ it satisfies*

1. *$f_0^{(r,s)}(q;q^*)$ do not depend on the fast angle q_1;*
2. *$f_1^{(r,s)}(\hat{q},\xi,\eta;q^*)$ do not appear;*
3. *$f_2^{(r,s)}(q,\hat{p},\xi,\eta;q^*)$ do not depend on q_1 and, evaluated at $\xi = \eta = 0$ and $q = q^*$, are equal to zero;*
4. *$f_3^{(r,s)}(\hat{q},\hat{p},\xi,\eta;q^*)$ do not depend on the actions $\hat{p}$;*
5. *$f_4^{(r,s)}(\hat{q},\hat{p},\xi,\eta;q^*)$, evaluated at $\xi = \eta = 0$, do not depend on the fast angle q_1.*

Some comments are in order. Assumption **H1** is needed in order to keep the frequency fixed on the torus, as in the classical Kolmogorov construction. It implies the invertibility of the Hessian C_0, which is equivalent to the invertibility of $D_J^2 h_0(I^*)$ in the original coordinates: this is sometimes known as *twist* condition, or *Kolmogorov nondegeneracy* condition, and encodes the fact that the resonant torus is locally isolated in the space of actions. Assumption **H2** is a typical requirement on the decay of the homogeneous terms of the Taylor expansion in ε. The last ones, **H3**, are the so-called first and second Melnikov conditions, and ensure absence of resonances between the periodic motion and the transverse linear oscillators. Actually, the first Melnikov condition (14) is enough to get existence of the continuation of the periodic orbit, while the second one (15) is needed to exhibit the linear stability of the orbit (see [10]).

The proof of Proposition 2.1 is based on standard arguments in Lie series theory. The key estimates that allow to complete the proof are reported in Lemma 4.4 in Section 4. We do not report here all the (tedious) details since similar results have been already published in, e.g., [7, 9, 10, 11, 12, 39, 34].

2.3. Approximation and continuation of periodic orbits

The Hamiltonian (17), being in normal form up to order r, allows to find approximate periodic orbits. Precisely, consider the normal form approximation $Z^{(r)}$, i.e., $H^{(r)}$ neglecting the terms of order $\mathcal{O}(\varepsilon^{r+1})$,

$$\begin{aligned} Z^{(r)}(\hat{q},\hat{p},\xi,\eta;q^*) =& \omega p_1 + \sum_{j=1}^{n_2} \mathbf{i}\Omega_j \xi_j \eta_j + \sum_{\ell>2} f_\ell^{(r,0)}(\hat{p},\xi,\eta) \\ &+ \sum_{s=1}^{r} \Big(f_0^{(r,s)}(q;q^*) + f_2^{(r,s)}(q,\hat{p},\xi,\eta;q^*) + f_3^{(r,s)}(\hat{q},\xi,\eta;q^*) + f_4^{(r,s)}(\hat{q},\hat{p},\xi,\eta;q^*)\Big) \\ &+ \sum_{s=0}^{r}\sum_{\ell>4} f_l^{(r,s)}(\hat{q},\hat{p},\xi,\eta;q^*) \ , \end{aligned} \tag{18}$$

and take as initial datum $x^* = (q = q^*, \hat{p} = 0, \xi = 0, \eta = 0)$. It is straightforward to see that the canonical equations read

$$\dot{q}_1 = \omega \ , \qquad \dot{q} = 0 \ , \qquad \dot{p}_1 = 0 \ , \qquad \dot{p} = -\sum_{s=1}^{r} \nabla_q f_0^{(r,s)}\Big|_{q=q^*} \ , \qquad \dot{\xi} = 0 \ , \qquad \dot{\eta} = 0 \ .$$

Hence, if $q = q^*$ is chosen as a solution of

$$\sum_{s=1}^{r} \nabla_q f_0^{(r,s)}\big|_{q=q^*} = 0 \ , \tag{19}$$

then $(q_1(0), x^*)$ is the initial datum (modulo the initial phase $q_1(0)$) of a periodic orbit with frequency ω for the truncated normal form, $x = x^*$ being a relative equilibrium[3] for $Z^{(r)}$.

We now introduce the smooth map $\Upsilon(x) : \mathcal{U}(x^*) \subset \mathbb{R}^{2n-1} \to \mathcal{V}(x^*) \subset \mathbb{R}^{2n-1}$ as

$$\Upsilon(x(0); \varepsilon, q_1(0)) = \begin{pmatrix} q_1(T) - q_1(0) - \omega T \\ q(T) - q(0) \\ p(T) - p(0) \\ \xi(T) - \xi(0) \\ \eta(T) - \eta(0) \end{pmatrix} \ , \tag{20}$$

parameterized by the initial phase $q_1(0)$ and ε, with T the period of the periodic orbit. Then, the periodicity condition is equivalent to $\Upsilon(x(0); \varepsilon, q_1(0)) = 0$; notice that in (20) we have neglected the equation for p_1, due to the conservation of the energy, which provides a dependence relation among the $2n$ equations (see for example [28]).

The periodic orbit x^* of the truncated normal form $Z^{(r)}$ turns out to be an approximate periodic orbit of the full Hamiltonian system; indeed it will be shown (see Lemma 4.5 in Section 4) that

$$\|\Upsilon(x^*; \varepsilon, q_1(0))\| \leq c_1 \varepsilon^{r+1} \ ,$$

for some positive constant $c_1(r)$ (growing with r) and ε small enough. A true periodic orbit, close to the approximate one, is then identified by an initial datum $x^*_{\text{p.o.}} \in \mathcal{U}(x^*)$ such that

$$\Upsilon(x^*_{\text{p.o.}}; \varepsilon, q_1(0)) = 0 \ .$$

In order to prove the existence of a true periodic orbit $x^*_{p.o.}$ close to x^* we apply the Newton-Kantorovich method (see [21, 34]).

Proposition 2.2 (Newton-Kantorovich)**.** *Consider a smooth map $\Upsilon \in \mathcal{C}^1(\mathcal{U}(x^*) \times \mathcal{U}(0), V)$. Assume that there exist three positive constants c_1, c_2 and c_3 dependent, for ε small enough, on $\mathcal{U}(x^*) \subset V$ only, and two parameters $0 \leq 2\alpha < \beta$ such that*

$$\|\Upsilon(x^*, \varepsilon)\| \leq c_1 |\varepsilon|^{\beta} \ , \tag{21}$$

$$\left\|[\Upsilon'(x^*, \varepsilon)]^{-1}\right\|_{op} \leq c_2 |\varepsilon|^{-\alpha} \ , \tag{22}$$

$$\|\Upsilon'(z, \varepsilon) - \Upsilon'(x^*, \varepsilon)\|_{op} \leq c_3 \|z - x^*\| \ , \tag{23}$$

where $\|\cdot\|_{op}$ denotes the operator norm. Then there exist positive c_0 and ε^* such that, for $|\varepsilon| < \varepsilon^*$, there exists a unique $x^*_{p.o.}(\varepsilon) \in \mathcal{U}(x^*)$ which fulfills

$$\Upsilon(x^*_{p.o.}, \varepsilon) = 0 \ , \qquad \|x^*_{p.o.} - x^*\| \leq c_0 |\varepsilon|^{\beta - \alpha} \ .$$

Furthermore, Newton's algorithm converges to $x^*_{p.o.}$.

[3] Let us stress that, for $r > 1$, $q^*(\varepsilon)$ actually depends on ε and it is analytic. Indeed, $(\omega t + q_1(0), x^*(\varepsilon))$ is a periodic solution of an analytic Hamiltonian whose flow is analytic in ε.

We stress that the main assumption concerns the invertibility of differential of Υ,

$$M(\varepsilon) = \Upsilon'(x^*;\varepsilon, q_1(0)) \ , \tag{24}$$

being essentially a condition on the minimum eigenvalue, that is vanishing with ε. Indeed, it is extremely difficult to directly verify (22) on an actual application starting from the definition (24).

A way out is given by the so-called variational equations, around a given orbit $\phi^t(x_0)$, where $x_0 = \phi^0(x_0)$ is the initial datum. It turns out that $M(\varepsilon)$ can be derived by the monodromy matrix $\Phi(T; H^{(r)}, x^*)$, which is the evolution at time T of the fundamental matrix of the linear vector field $DX_{H^{(r)}}(\phi^t(x^*))$

$$\frac{d}{dt}\Phi(t; H^{(r)}, x^*) = DX_{H^{(r)}}(\phi^t(x^*))\Phi(t; H^{(r)}, x^*) \ ,$$

where we have denoted by X_H the Hamiltonian vector field given by H. Actually, one can take advantage of the normal normal form construction in order to approximate $M(\varepsilon)$; indeed, by considering the truncated normal form $Z^{(r)}$ in (18), it turns out that the linearization $DX_{Z^{(r)}}(x^*)$ around the relative equilibrium x^* is a constant matrix and, furthermore, it is block diagonal. This is a consequence of properties *4* and *5* in the normal form construction, which allows to split the dependence on the "internal" variables $(q, \hat{p})$ and (ξ, η) in the quadratic part of $Z^{(r)}$. In order to better develop this point, and show how to exploit $DX_{Z^{(r)}}(x^*)$ to verify (22), we need to introduce a convenient notation. Let M be a $2n$-dimensional square matrix. We denote by $M_\flat$ the reduced matrix: the $2(n-1)$ dimensional square matrix obtained from M by removing the first column (related to the fast angle q_1) and the (n_1+1)-th row (related to the momentum p_1).

We can now state the following

Lemma 2.1. *The differential $M(\varepsilon)$ defined in (24) is the reduction of $\Phi(T; H^{(r)}, x^*) - \mathrm{Id}$, namely*

$$M(\varepsilon) = \Big(\Phi(T; H^{(r)}, x^*) - \mathrm{Id}\Big)_\flat \ .$$

Moreover, $M(\varepsilon)$ has the following decomposition

$$M(\varepsilon) = N(\varepsilon) + \mathcal{O}(\varepsilon^{r+1}) \ , \qquad \textit{with} \quad N(\varepsilon) = \begin{pmatrix} N_{11}(\varepsilon) & O \\ O & N_{22}(\varepsilon) \end{pmatrix} \ , \tag{25}$$

where the leading term reads

$$N(\varepsilon) = \Big(\Phi(T; Z^{(r)}, x^*) - \mathrm{Id}\Big)_\flat \ , \qquad \textit{with} \quad \Phi(T; Z^{(r)}, x^*) = \exp\left(DX_{Z^{(r)}}(x^*)T\right) \ .$$

Proof. It is well known that the fundamental matrix $\Phi(T; H^{(r)}, x^*)$ equals the differential of the Hamiltonian flow with respect to the generic initial datum (close to $\phi^t(x^*)$). Since Υ ignores the evolution of p_1 and does not depend on the fast angle q_1, we obtain $M(\varepsilon) = \big(\Phi(T; H^{(r)}, x^*) - \mathrm{Id}\big)_\flat$. Thus, the structure of $\Phi(T; H^{(r)}, x^*) - \mathrm{Id}$ can be investigated exploiting the linearization around the relative equilibrium x^* of the truncated normal form $Z^{(r)}$. Indeed, the matrix $\Phi(T; H^{(r)}, x^*)$ can be approximated by $\Phi(T; Z^{(r)}, x^*)$, the last having a block diagonal structure and being represented by the exponential of the time-independent matrix $DX_{Z^{(r)}}(x^*)$. □

In view of Lemma 2.1, we can focus on the leading term $N(\varepsilon)$, which is expected to be explicitly calculated via the normal form algorithm. Moreover, the nonresonance condition (14) ensures that the matrix $N_{22}(0)$ is diagonal with eigenvalues of order $\mathcal{O}(1)$. Thus, by continuity of the spectrum with respect to ε, the same order of magnitude holds also for the eigenvalues of $N_{22}(\varepsilon)$. Accordingly, only the eigenvalues in $\Sigma(N_{11}(\varepsilon))$ vanish with $\varepsilon \to 0$, and the continuation result can be formulated by assuming suitable conditions on $N_{11}(\varepsilon)$.

Theorem 2.1. *Consider the map Υ defined in (20) in a neighbourhood of the lower dimensional torus $\hat{p} = 0$, $\xi = \eta = 0$ and let $x^*(\varepsilon) = (q^*(\varepsilon), 0, 0, 0)$, with $q^*(\varepsilon)$ satisfying (19). Assume that*

$$\|\Upsilon(x^*(\varepsilon); \varepsilon, q_1(0))\| \leq c_1 \varepsilon^{r+1} \ , \tag{26}$$

where c_1 is a positive constant depending on $\mathcal{U}$ and r. Assume that $N_{11}(\varepsilon)$ in (25) *is invertible and there exists $\alpha > 0$ with $2\alpha < r+1$ such that*

$$|\lambda| \gtrsim |\varepsilon|^\alpha \ , \qquad \textit{for all} \quad \lambda \in \Sigma(N_{11}(\varepsilon)) \ . \tag{27}$$

Then, there exist $c_0 > 0$ and $\varepsilon^* > 0$ such that for any $0 \le |\varepsilon| < \varepsilon^*$ there exists a unique $x^*_{\rm p.o.}(\varepsilon) = (q^*_{\rm p.o.}(\varepsilon), \hat{p}_{\rm p.o.}(\varepsilon), \xi_{\rm p.o.}(\varepsilon), \eta_{\rm p.o.}(\varepsilon)) \in \mathcal{U}$ which solves

$$\Upsilon(x^*_{\rm p.o.}; \varepsilon, q_1(0)) = 0 \ , \qquad \textit{with} \quad \left\| x^*_{\rm p.o.} - x^* \right\| \le c_0 \varepsilon^{r+1-\alpha} \ . \tag{28}$$

Proof. The proof consists in the application of the Newton-Kantorovich method, as stated in the Proposition 2.2. Indeed, condition (21) holds true with $\beta = r$, because of Lemma 4.5. Moreover, condition (23) is also satisfied, in view of the analyticity of the Hamiltonian and its vector field. The third and last hypothesis (22) is about the invertibility of the Jacobian matrix $M(\varepsilon)$ and on the smallness of its eigenvalues. Due to Lemma 2.1, invertibility of $N(\varepsilon)$ requires invertibility of the two blocks $N_{11}(\varepsilon)$ and $N_{22}(\varepsilon)$; but $N_{22}(\varepsilon)$ is invertible because of the first Melnikov condition, hence assuming $N_{11}(\varepsilon)$ invertible ensures invertibility of $N(\varepsilon)$. Then, thanks to $|\lambda| \gtrsim |\varepsilon|^\alpha$ (with $\alpha < r$, which is guaranteed by the hypothesis $2\alpha < r+1$) and by exploiting Proposition A.1 in the Appendix, invertibility is preserved under perturbations of order $\mathcal{O}(\varepsilon^{r+1})$ and $|\nu| \gtrsim |\varepsilon|^\alpha$ is ensured for any $\nu \in \Sigma(M(\varepsilon))$, thus proving the validity of (22). □

Remark 2.1. *Let us note that in the nondegenerate case ($r = 1$) it can be shown (e.g. via a $\sqrt{\varepsilon}$-scaling of the momenta $\hat{p}$, as in [41]) that the eigenvalues are of order $\mathcal{O}(\sqrt{\varepsilon})$ and the existence of periodic orbits follows by direct application of the implicit function theorem, taking $\alpha = \frac{1}{2}$.*

2.4. Approximate and effective linear stability

We come now to the investigation of the linear stability of the approximate periodic orbit x^* via the normal form construction. In plain words, the normal form procedure has removed the time dependence in the $\Phi(t; H^{(r)}, x^*)$ up to order $\mathcal{O}(\varepsilon^r)$, thus reducing the problem of the approximate stability to the computation of the eigenvalues of a constant matrix, in place of the Floquet exponents. To clarify this point, we introduce the variables $(\hat{Q}, \hat{P})$, representing the small displacements around the relative equilibrium

$$Q_1 = q_1 - \omega t - q_1(0) \ , \qquad Q = q - q^* \ , \qquad P_1 = p_1 \ , \qquad P = p \ .$$

Replacing the original variables $(\hat{q}, \hat{p})$ with the new ones $(\hat{Q}, \hat{P})$ in (18), i.e., the truncated Hamiltonian in normal form at order r, and keeping only the quadratic terms, one immediately obtains the Hamiltonian vector field linearized around x^*. Let us stress that, by normal form construction, the quadratic Hamiltonian, and hence the linearized equations, is independent of Q_1.

To represent the linearized Hamiltonian vector field, it is convenient to introduce the matrices

$$\begin{aligned}
B(\varepsilon) &= D^2_q Z^{(r)}(\omega t + q_1(0), x^*) \ , \qquad & G(\varepsilon) &= D^2_\xi Z^{(r)}(\omega t + q_1(0), x^*) \ , \\
D(\varepsilon) &= D^2_{q\hat{p}} Z^{(r)}(\omega t + q_1(0), x^*) \ , \qquad & F(\varepsilon) &= D^2_\eta Z^{(r)}(\omega t + q_1(0), x^*) \ , \\
C(\varepsilon) &= D^2_{\hat{p}} Z^{(r)}(\omega t + q_1(0), x^*) \ , \qquad & E(\varepsilon) &= D^2_{\xi\eta} Z^{(r)}(\omega t + q_1(0), x^*) \ .
\end{aligned}$$

We notice that the above matrices admit the asymptotic (analytic) expansions in ε

$$\begin{aligned}
B(\varepsilon) &= \qquad \varepsilon B_1 + \mathcal{O}(\varepsilon^2) \ , \qquad & G(\varepsilon) &= \qquad \varepsilon G_1 + \mathcal{O}(\varepsilon^2) \ , \\
D(\varepsilon) &= \qquad \varepsilon D_1 + \mathcal{O}(\varepsilon^2) \ , \qquad & F(\varepsilon) &= \qquad \varepsilon F_1 + \mathcal{O}(\varepsilon^2) \ , \\
C(\varepsilon) &= C_0 + \varepsilon C_1 + \mathcal{O}(\varepsilon^2) \ , \qquad & E(\varepsilon) &= E_0 + \varepsilon C_1 + \mathcal{O}(\varepsilon^2) \ , \quad \text{with} \quad E_0 = \text{diag}(\mathbf{i}\Omega_j) \ .
\end{aligned}$$

In order to formally include the Q_1 dependence in the quadratic Hamiltonian, we extend the matrices $B(\varepsilon)$ and $D(\varepsilon)$ (the last one being $(n_1 - 1) \times n_1$ rectangular) to $n_1 \times n_1$ square matrices. Precisely, we denote by $B(\varepsilon)_\#$ the square-matrix obtained by adding a zero row and column, respectively at the top and left of $B(\varepsilon)$.

Similarly, we denote by $D(\varepsilon)_\#$ the square-matrix obtained adding a zero row at top of $D(\varepsilon)$. In this way, the quadratic Hamiltonian that gives the linear approximation of the dynamics close to the approximate periodic orbit reads

$$Z_2^{(r)} = \frac{1}{2}\langle B(\varepsilon)_\#\hat{Q},\hat{Q}\rangle + \langle D(\varepsilon)_\#\hat{Q},\hat{P}\rangle + \frac{1}{2}\langle C(\varepsilon)\hat{P},\hat{P}\rangle + \frac{1}{2}\langle G(\varepsilon)\xi,\eta\rangle + \langle E(\varepsilon)\xi,\eta\rangle + \frac{1}{2}\langle F(\varepsilon)\eta,\eta\rangle \ , \tag{29}$$

and the canonical linear vector field can be represented by a block diagonal matrix $L(\varepsilon)$ as

$$L(\varepsilon) = \begin{pmatrix} L_{11}(\varepsilon) & 0 \\ 0 & L_{22}(\varepsilon)\end{pmatrix} \ , \tag{30}$$

with

$$L_{11}(\varepsilon) = \begin{pmatrix} D(\varepsilon)_\#^\top & C(\varepsilon) \\ -B(\varepsilon)_\# & -D(\varepsilon)_\#\end{pmatrix} \quad \text{and} \quad L_{22}(\varepsilon) = \begin{pmatrix} E(\varepsilon)^\top & F(\varepsilon) \\ -G(\varepsilon) & -E(\varepsilon)\end{pmatrix} \ .$$

As $L(\varepsilon)$ is independent of time, the stability of the approximate periodic orbit reduces to the study of the spectrum of $L(\varepsilon)$, which splits into two distinct components. The first one is $\Sigma(L_{11}(\varepsilon))$, made of $2n_1$ eigenvalues which vanish as $\varepsilon \to 0$, in view of the $n_1 - 1$ resonances on the unperturbed invariant torus $I = I^*$. The second one is $\Sigma(L_{22}(\varepsilon))$. From now on, we assume that $L_{22}(\varepsilon)$ is positive (negative) definite, so that $\Sigma(L_{22}(\varepsilon))$ is generically made of n_2 pairs of conjugate imaginary eigenvalues $\mathbf{i}\Omega_j(\varepsilon)$, at least for ε small enough. The linear stability in the directions transverse to the lower dimensional torus are guaranteed by the assumptions on $L_{22}(\varepsilon)$ to be positive definite, i.e., $\Omega_j(0) > 0$ for $j = 1,\ldots,n_2$. Indeed, the origin is a nondegenerate elliptic equilibrium for the unperturbed Hamiltonian g_0, therefore it persists for ε small enough. As a consequence, the approximate linear stability of the periodic orbit depends only on the *vanishing part* of the spectrum $\Sigma(L_{11}(\varepsilon))$. On the contrary, if some harmonic oscillators $\mathbf{i}\Omega_j\xi_j\eta_j$ are free to rotate with equal frequencies, but in opposite directions, then the transverse instability of the approximate periodic orbit can be produced by collisions of eigenvalues having different Krein signature.

We now investigate to what extent the stability of the *true* periodic orbit can be inferred by the stability of the approximate one, under suitable assumptions on $\Sigma(L_{11}(\varepsilon))$. In fact, the distance between the monodromy matrix $\Phi(T; H^{(r)}, x^*_{\rm p.o.})$ and the matrix $\exp(L(\varepsilon)T)$ is not only due to the normal form remainder, that is of order $\mathcal{O}(\varepsilon^{r+1})$, but also to the approximation of the periodic orbit, which is actually dominant, being of order $\mathcal{O}(\varepsilon^{r+1-\alpha})$, with $2\alpha < r+1$. This is part of the statement claimed in the next Theorem 2.2; this statement also claims that the spectrum $\Sigma(\Phi(T; H^{(r)}, x^*_{\rm p.o.}))$ of the monodromy matrix splits into two different components, which are deformations of the approximate ones.

Theorem 2.2. *Consider the monodromy matrix* $\Phi(T; H^{(r)}, x^*_{\rm p.o.})$ *and its approximation given by* $\exp(L(\varepsilon)T)$, *with* $L_{22}(\varepsilon)$ *positive definite. Then for* $|\varepsilon|$ *small enough the following holds true:*

1. *there exists a positive constant* c_A *such that one has*

$$\Phi(T; H^{(r)}, x^*_{\rm p.o.}) = \exp(L(\varepsilon)T) + A \ , \qquad \textit{with} \quad \|A\|_{op} \le c_A|\varepsilon|^{r+1-\alpha} \ , \tag{31}$$

   where α *is the same as in Theorem 2.1;*

2. $\Sigma(\Phi(T; H^{(r)}, x^*_{\rm p.o.})) = \Sigma_{11} \cup \Sigma_{22}$, *where* Σ_{11} *is close to* $\Sigma(\exp(L_{11}(\varepsilon)T))$ *and includes at least two elements equal to* 1, *while* Σ_{22} *is close to* $\Sigma(\exp(L_{22}(\varepsilon)T))$ *and all its elements lie on the unit circle.*

Proof. First observe that, in view of the continuity and the separation of the two spectra $\Sigma(L_{11}(\varepsilon))$ and $\Sigma(L_{22}(\varepsilon))$, the spectrum of the monodromy matrix can be split into two different parts. Moreover, in view of the Melnikov nonresonance conditions and the fact that transverse linear oscillators have frequencies with the same sign, Krein signature (see for example [45, 26, 1, 27, 29]) ensures that Σ_{22}, which is a deformation of $\Sigma(\exp(L_{22}(\varepsilon)T))$, lies on the unit circle.

In order to obtain the estimate of the error in (31), we exploit the fact that the monodromy matrix is the differential of the flow with respect to the initial datum. Considering the matrix $\Phi(T; Z^{(r)}, x^*_{\rm p.o.})$, we take into account two different sources of approximation: the one of the Hamiltonian $H^{(r)}$ with its normal form $Z^{(r)}$ and the one due to the approximation of the initial datum of the periodic orbit. Hence, the error term consists of the normal form remainder $\mathcal{O}(\varepsilon^{r+1})$ and of the error of the periodic orbit, which is of order $\mathcal{O}(\varepsilon^{r+1-\alpha})$ (with $2\alpha < r+1$), as it follows from Theorem 2.1. The latter is the dominant one and this allows to conclude the proof.

□

We remark that in view of Lemma 2.1, in Theorem 2.1 we can rewrite $M(\varepsilon)$ as

$$M(\varepsilon) = \left(\exp\left(L(\varepsilon)T\right) - \mathrm{Id}\right)_\flat + \mathcal{O}(\varepsilon^{r+1}) \ . \tag{32}$$

As a consequence, the matrix $N_{11}(\varepsilon)$ in (27) reads $N_{11}(\varepsilon) = (\exp(L_{11}(\varepsilon)T) - \mathrm{Id})_\flat$.

We are now ready to state the result on the localization of the eigenvalues for Σ_{11} (which are all close to 1) by exploiting the spectrum of $L_{11}(\varepsilon)$ in the generic case of distinct eigenvalues. The result is the following (see also Lemma 2 in [1])

Theorem 2.3. *Assume that $L_{11}(\varepsilon)$ has $2n_1-2$ distinct non-zero eigenvalues and let $\tilde{c} > 0$ and $\beta < r+1-\alpha$, with $2\alpha < r+1$ as in Theorem 2.1, be such that*

$$|\lambda_j - \lambda_k| > \tilde{c}\varepsilon^\beta \ , \qquad \textit{for all} \quad \lambda_j, \lambda_k \in \Sigma(L_{11}(\varepsilon)) \setminus \{0\} \ . \tag{33}$$

Then there exists $\varepsilon^* > 0$ such that if $|\varepsilon| < \varepsilon^*$ and $\mu = e^{\lambda T} \in \Sigma(\exp(L_{11}(\varepsilon)T))$, there exists one eigenvalue $\nu \in \Sigma_{11}$ inside the complex disk $D_\varepsilon(\lambda) = \left\{z \in \mathbb{C} \, : \, |z-\lambda| < c\varepsilon^{r-\alpha+1}\right\}$, with $c > 0$ a suitable constant independent of μ.

Proof. The proof follows from Proposition A.2 in the Appendix, by exploiting (31) and the fact that the difference between the Floquet multipliers close to 1, $e^{\lambda_j T} - e^{\lambda_k T}$, is, at leading order, the same as the exponents $\lambda_j - \lambda_k$. □

Corollary 2.1. *Under the assumptions of Theorem 2.3 the periodic orbit $x^*_{\mathrm{p.o.}}$ is linearly stable if (and only if) the same holds for the approximate periodic orbit x^*. In the unstable case, the number of hyperbolic directions of the periodic orbit $x^*_{\mathrm{p.o.}}$ is the same as for x^*.*

3. Normal form algorithm

In this section, by using the formalism of Lie series (see [3, 7]), we detail the generic step of the normal form algorithm that takes the Hamiltonian at order r and brings it into normal form up to order $r+1$. We stress here that the normal form algorithm is a completely constructive procedure that can be effectively implemented by means of computer algebra, see, e.g., [8].

The relevant algebraic property of the $\mathcal{P}_\ell$ classes of functions is stated by the following

Lemma 3.1. *Let $f \in \mathcal{P}_{s_1}$ and $g \in \mathcal{P}_{s_2}$, then $\{f, g\} \in \mathcal{P}_{s_1+s_2-2}$.*

The proof is left to the reader, being a trivial consequence of the definition of the Poisson bracket.

3.1. Generic r-th normalization step

We summarize the five stages of a generic r-th normalizing step. The starting Hamiltonian has the form

$$
\begin{aligned}
H^{(r-1)} = \omega p_1 + \sum_{j=1}^{n_2} \mathbf{i}\Omega_j \xi_j \eta_j \\
&+ \sum_{s<r} f_0^{(r-1,s)} + \sum_{s<r} f_2^{(r-1,s)} + \sum_{s<r} f_3^{(r-1,s)} + \sum_{s<r} f_4^{(r-1,s)} \\
&+ f_0^{(r-1,r)} + f_1^{(r-1,r)} + f_2^{(r-1,r)} + f_3^{(r-1,r)} + f_4^{(r-1,r)} \\
&+ \sum_{s>r} f_0^{(r-1,s)} + \sum_{s>r} f_1^{(r-1,s)} + \sum_{s>r} f_2^{(r-1,s)} + \sum_{s>r} f_3^{(r-1,s)} + \sum_{s>r} f_4^{(r-1,s)} \\
&+ \sum_{s\geq 0} \sum_{\ell>2} f_\ell^{(r-1,s)} \ .
\end{aligned}
\tag{34}
$$

where $f_0^{(r-1,s)}$, $f_2^{(r-1,s)}$, $f_3^{(r-1,s)}$ and $f_4^{(r-1,s)}$, for $1 \leq s < r$, are in normal form.

3.1.1. First stage of the r-th normalization step

We average the term $f_0^{(r-1,r)}$ with respect to the fast angle q_1, determining the generating function

$$
\chi_0^{(r)}(\hat{q}) = X_0^{(r)}(\hat{q}) + \langle \zeta^{(r)}, \hat{q} \rangle \qquad \text{with} \qquad \zeta^{(r)} \in \mathbb{R}^{n_1} \ ,
$$

belonging to $\mathcal{P}_0$ and of order $\mathcal{O}(\varepsilon^r)$, by solving the homological equations

$$
\begin{aligned}
&L_{X_0^{(r)}} \omega p_1 + f_0^{(r-1,r)} = \langle f_0^{(r-1,r)} \rangle_{q_1} \ , \\
&L_{\langle \zeta^{(r)}, \hat{q} \rangle} f_4^{(0,0)} \Big|_{\xi=\eta=0} + \Big\langle f_2^{(r-1,r)} \Big|_{\substack{\xi=\eta=0 \\ q=q^*}} \Big\rangle_{q_1} = 0 \ .
\end{aligned}
$$

By considering the Taylor-Fourier expansion

$$
f_0^{(r-1,r)}(\hat{q}) = \sum_k c_{0,0,0,k}^{(r-1,r)} \exp(\mathbf{i}\langle k, \hat{q} \rangle) \ ,
$$

we obtain

$$
X_0^{(r)}(\hat{q}) = \sum_{k_1 \neq 0} \frac{c_{0,0,0,k}^{(r-1,r)}}{\mathbf{i} k_1 \omega} \exp(\mathbf{i}\langle k, \hat{q} \rangle) \ .
$$

The vector $\zeta^{(r)}$ is determined by solving the linear system

$$
\sum_j C_{0,ij} \zeta_j^{(r)} = \frac{\partial}{\partial \hat{p}_i} \Big\langle f_2^{(r-1,r)} \Big|_{\substack{\xi=\eta=0 \\ q=q^*}} \Big\rangle_{q_1} \ . \tag{35}
$$

The transformed Hamiltonian is computed as

$$
\begin{aligned}
H^{(\mathrm{I};r-1)} &= \exp\Big(L_{\chi_0^{(r)}}\Big) H^{(r-1)} = \\
&= \omega p_1 + \sum_{j=1}^{n_2} \mathbf{i}\Omega_j \xi_j \eta_j \\
&+ \sum_{s<r} f_0^{(\mathrm{I};r-1,s)} + \sum_{s<r} f_2^{(\mathrm{I};r-1,s)} + \sum_{s<r} f_3^{(\mathrm{I};r-1,s)} + \sum_{s<r} f_4^{(\mathrm{I};r-1,s)} \\
&+ f_0^{(\mathrm{I};r-1,r)} + f_1^{(\mathrm{I};r-1,r)} + f_2^{(\mathrm{I};r-1,r)} + f_3^{(\mathrm{I};r-1,r)} + f_4^{(\mathrm{I};r-1,r)} \\
&+ \sum_{s>r} f_0^{(\mathrm{I};r-1,s)} + \sum_{s>r} f_1^{(\mathrm{I};r-1,s)} + \sum_{s>r} f_2^{(\mathrm{I};r-1,s)} + \sum_{s>r} f_3^{(\mathrm{I};r-1,s)} + \sum_{s>r} f_4^{(\mathrm{I};r-1,s)} \\
&+ \sum_{s\geq 0} \sum_{\ell>2} f_\ell^{(\mathrm{I};r-1,s)} \ .
\end{aligned}
\tag{36}
$$

The functions $f_{\ell}^{(\mathrm{I};r-1,s)}$ are recursively defined as

$$
\begin{aligned}
f_0^{(\mathrm{I};r-1,r)} &= \langle f_0^{(r-1,r)} \rangle_{q_1} \ , \\
f_{\ell}^{(\mathrm{I};r-1,s)} &= \sum_{j=0}^{\lfloor s/r \rfloor} \frac{1}{j!} L_{\chi_0^{(r)}}^{j} f_{\ell+2j}^{(r-1,s-jr)} \ , \qquad \text{for } \ell = 0,\ s \neq r \ , \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{or } \ell \neq 0\ s \geq 0 \ \ ,
\end{aligned}
\tag{37}
$$

with $f_{\ell}^{(\mathrm{I};r-1,s)} \in \mathcal{P}_{\ell}$.

3.1.2. Second stage of the r-th normalization step

We now remove the term $f_1^{(\mathrm{I};r-1,r)}$ by means of the generating function $\chi_1^{(r)}$, belonging to $\mathcal{P}_1$ and of order $\mathcal{O}(\varepsilon^r)$, by solving the homological equation

$$
L_{\chi_1^{(r)}} \Big(\omega p_1 + \sum_{j=1}^{n_2} \mathbf{i} \Omega_j \xi_j \eta_j \Big) + f_1^{(\mathrm{I};r-1,r)} = 0 \ . \tag{38}
$$

Considering again the Taylor-Fourier expansion

$$
f_1^{(\mathrm{I};r-1,r)}(\hat{q}, \xi, \eta) = \sum_{\substack{|m_1|+|m_2|=1 \\ k}} c_{0,m_1,m_2,k}^{(\mathrm{I};r-1,r)} \exp(\mathbf{i}\langle k,\, \hat{q}\rangle) \xi^{m_1} \eta^{m_2} \ ,
$$

we get

$$
\chi_1^{(r)}(\hat{q}, \xi, \eta) = \sum_{\substack{|m_1|+|m_2|=1 \\ k}} \frac{c_{0,m_1,m_2,k}^{(\mathrm{I};r-1,r)} \exp(\mathbf{i}\langle k,\, \hat{q}\rangle)\, \xi^{m_1} \eta^{m_2}}{\mathbf{i}\big[k_1 \omega + \langle m_1 - m_2,\, \Omega \rangle\big]} \ .
$$

with $\Omega \in \mathbb{R}^{n_2}$.

The transformed Hamiltonian is calculated as

$$
\begin{aligned}
H^{(\mathrm{II};r-1)} &= \exp\Big(L_{\chi_1^{(r)}}\Big) H^{(\mathrm{I};r-1)} = \\
&= \omega p_1 + \sum_{j=1}^{n_2} \mathbf{i} \Omega_j \xi_j \eta_j \\
&\quad + \sum_{s<r} f_0^{(\mathrm{II};r-1,s)} + \sum_{s<r} f_2^{(\mathrm{II};r-1,s)} + \sum_{s<r} f_3^{(\mathrm{II};r-1,s)} + \sum_{s<r} f_4^{(\mathrm{II};r-1,s)} \\
&\quad + f_0^{(\mathrm{II};r-1,r)} + f_2^{(\mathrm{II};r-1,r)} + f_3^{(\mathrm{II};r-1,r)} + f_4^{(\mathrm{II};r-1,r)} \\
&\quad + \sum_{s>r} f_0^{(\mathrm{II};r-1,s)} + \sum_{s>r} f_1^{(\mathrm{II};r-1,s)} + \sum_{s>r} f_2^{(\mathrm{II};r-1,s)} + \sum_{s>r} f_3^{(\mathrm{II};r-1,s)} + \sum_{s>r} f_4^{(\mathrm{II};r-1,s)} \\
&\quad + \sum_{s\geq 0} \sum_{\ell>2} f_{\ell}^{(\mathrm{II};r-1,s)} \ ,
\end{aligned}
\tag{39}
$$

with

$$
\begin{aligned}
f_1^{(\mathrm{II};r-1,r)} &= 0 \ , \\
f_0^{(\mathrm{II};r-1,2r)} &= f_0^{(\mathrm{I};r-1,2r)} + L_{\chi_1^{(r)}} f_1^{(\mathrm{I};r-1,r)} + \frac{1}{2} L_{\chi_1^{(r)}} \Big(L_{\chi_1^{(r)}} f_2^{(\mathrm{I};r-1,0)} \Big) = \\
&= f_0^{(\mathrm{I};r-1,2r)} + \frac{1}{2} L_{\chi_1^{(r)}} f_1^{(\mathrm{I};r-1,r)} \ , \\
f_{\ell}^{(\mathrm{II};r-1,s)} &= \sum_{j=0}^{\lfloor s/r \rfloor} \frac{1}{j!} L_{\chi_1^{(r)}}^{j} f_{\ell+j}^{(\mathrm{I};r-1,s-jr)}, \qquad \text{for } \ell = 0,\ s \neq 2r \ , \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{or } \ell = 1\ s \neq r \ \ , \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{or } \ell \geq 2\ s \geq 0 \ \ ,
\end{aligned}
\tag{40}
$$

where we have exploited (38).

3.1.3. Third stage of the r-th normalization step

We now average the term $f_2^{(\mathrm{II};r-1,r)}$ with respect to the fast angle q_1, determining the generating function $\chi_2^{(r)}$, belonging to $\mathcal{P}_2$ and of order $\mathcal{O}(\varepsilon^r)$, by solving the homological equation

$$L_{\chi_2^{(r)}}\Big(\omega p_1 + \sum_{j=1}^{n_2} \mathbf{i}\Omega_j \xi_j \eta_j\Big) + f_2^{(\mathrm{II};r-1,r)} = \langle f_2^{(\mathrm{II};r-1,r)} \rangle_{q_1} \ . \tag{41}$$

Therefore, considering the Taylor-Fourier expansion

$$f_2^{(\mathrm{II};r-1,r)}(\hat{p},\hat{q},\xi,\eta) = \sum_{\substack{|l|=1\\ k}} c_{l,0,0,k}^{(\mathrm{II};r-1,r)} \hat{p}^l \exp(\mathbf{i}\langle k,\,\hat{q}\rangle) + \sum_{\substack{|m_1|+|m_2|=2\\ k}} c_{0,m_1,m_2,k}^{(\mathrm{II};r-1,r)} \exp(\mathbf{i}\langle k,\,\hat{q}\rangle)\xi^{m_1}\eta^{m_2} \ ,$$

we obtain

$$\chi_2^{(r)}(\hat{p},\hat{q},\xi,\eta) = \sum_{\substack{|l|=1\\ k_1\neq 0}} \frac{c_{l,0,0,k}^{(\mathrm{II};r-1,r)} \hat{p}^l \exp(\mathbf{i}\langle k,\,\hat{q}\rangle)}{\mathbf{i}k_1\omega} + \sum_{\substack{|m_1|+|m_2|=2\\ k_1\neq 0}} \frac{c_{0,m_1,m_2,k}^{(\mathrm{II};r-1,r)} \exp(\mathbf{i}\langle k,\,\hat{q}\rangle)\,\xi^{m_1}\eta^{m_2}}{\mathbf{i}\big[k_1\omega + \langle m_1 - m_2,\,\Omega\rangle\big]} \ .$$

The transformed Hamiltonian is computed as

$$H^{(\mathrm{III};r-1)} = \exp\Big(L_{\chi_2^{(r)}}\Big)\, H^{(\mathrm{II};r-1)}$$

and is in the form (39), replacing the upper index II by III, with $f_\ell^{(\mathrm{III};r-1,s)} \in \mathcal{P}_\ell$ given by

$$\begin{aligned}
f_2^{(\mathrm{III};r-1,r)} &= \langle f_2^{(\mathrm{II};r-1,r)} \rangle_{q_1} \ ,\\
f_2^{(\mathrm{III};r-1,ri)} &= \frac{1}{(i-1)!} L_{\chi_2^{(r)}}^{i-1} \Big(f_2^{(\mathrm{II};r-1,r)} + \frac{1}{i} L_{\chi_2^{(r)}} f_2^{(\mathrm{II};r-1,0)} \Big)\\
&\quad + \sum_{j=0}^{i-2} \frac{1}{j!} L_{\chi_2^{(r)}}^{j} f_2^{(\mathrm{II};r-1,ri-rj)}\\
&= \frac{1}{(i-1)!} L_{\chi_2^{(r)}}^{i-1} \Big(\frac{1}{i} \langle f_2^{(\mathrm{II};r-1,r)} \rangle_{q_1} + \frac{i-1}{i} f_2^{(\mathrm{II};r-1,r)} \Big)\\
&\quad + \sum_{j=0}^{i-2} \frac{1}{j!} L_{\chi_2^{(r)}}^{j} f_2^{(\mathrm{II};r-1,ri-rj)} \ ,\\
f_\ell^{(\mathrm{III};r-1,s)} &= \sum_{j=0}^{\lfloor s/r \rfloor} \frac{1}{j!} L_{\chi_2^{(r)}}^{j} f_\ell^{(\mathrm{II};r-1,s-jr)} \ , \qquad \text{for } \ell = 2,\ s \neq ri \ ,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{or } \ell \neq 2,\ s \geq 0 \ ,
\end{aligned} \tag{42}$$

where we have used the homological equation (41).

3.1.4. Fourth stage of the r-th normalization step

We now remove the term $f_{1,1}^{(\mathrm{III};r-1,r)} \in \mathcal{P}_3$, which depends both on the actions and on the transverse variables ξ, η. We determine the generating function $\chi_3^{(r)}$, belonging to $\mathcal{P}_3$ and of order $\mathcal{O}(\varepsilon^r)$, by solving the homological equation

$$L_{\chi_3^{(r)}}\Big(\omega p_1 + \sum_{j=1}^{n_2} \mathbf{i}\Omega_j \xi_j \eta_j\Big) + f_{1,1}^{(\mathrm{III};r-1,r)} = 0 \ . \tag{43}$$

Hence, considering the Taylor-Fourier expansion

$$f_{1,1}^{(\mathrm{III};r-1,r)}(\hat{q},\hat{p},\xi,\eta) = \sum_{\substack{|l|=1\\ |m_1|+|m_2|=1\\ k}} c_{l,m_1,m_2,k}^{(\mathrm{III};r-1,r)} \exp(\mathbf{i}\langle k,\,\hat{q}\rangle)\hat{p}^l \xi^{m_1}\eta^{m_2} \ ,$$

we get

$$\chi_3^{(r)}(\hat{q},\hat{p},\xi,\eta) = \sum_{\substack{|l|=1\\ |m_1|+|m_2|=1\\ k}} \frac{c_{l,m_1,m_2,k}^{(\mathrm{III};r-1,r)}\ \exp(\mathbf{i}\langle k,\,\hat{q}\rangle)\,\hat{p}^l\xi^{m_1}\eta^{m_2}}{\mathbf{i}\big[k_1\omega + \langle m_1-m_2,\,\Omega\rangle\big]}\ .$$

with $\Omega\in\mathbb{R}^{n_2}$.

The transformed Hamiltonian is computed as

$$H^{(\mathrm{IV};r-1)} = \exp\left(L_{\chi_3^{(r)}}\right) H^{(\mathrm{III};r-1)}$$

and is given in the form (39), replacing the upper index II by IV, with

$$\begin{aligned}
f_3^{(\mathrm{IV};r-1,r)} &= f_3^{(\mathrm{III};r-1,r)}\Big|_{\hat{p}=0}\ ,\\
f_4^{(\mathrm{IV};r-1,2r)} &= f_4^{(\mathrm{III};r-1,2r)} + L_{\chi_3^{(r)}} f_3^{(\mathrm{III};r-1,r)} + \frac{1}{2}L^2_{\chi_3^{(r)}} f_2^{(\mathrm{III};r-1,0)} =\\
&= f_4^{(\mathrm{III};r-1,2r)} + \frac{1}{2}L_{\chi_3^{(r)}} f_3^{(\mathrm{III};r-1,r)} + \frac{1}{2}L_{\chi_3^{(r)}} f_3^{(\mathrm{III};r-1,2)}\Big|_{\hat{p}=0}\ ,\\
f_\ell^{(\mathrm{IV};r-1,s)} &= \sum_{j=0}^{\lfloor s/r\rfloor} \frac{1}{j!} L^j_{\chi_3^{(r)}} f_{\ell-j}^{(\mathrm{III};r-1,s-jr)}, \qquad \text{for } \ell=3,\ s\neq r\ ,\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{or } \ell=4,\ s\neq 2r\ ,\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{or } \ell\neq 3,4,\ s\geq 0\ .
\end{aligned} \tag{44}$$

where we have exploited (43).

3.1.5. Fifth stage of the r-th normalization step

We average the term $f_4^{(\mathrm{IV};r-1,r)}\Big|_{\xi=\eta=0}$ with respect to the fast angle q_1. We determine the generating function $\chi_4^{(r)}$, belonging to $\mathcal{P}_4$ and of order $\mathcal{O}(\varepsilon^r)$, by solving the homological equation

$$L_{\chi_4^{(r)}}\omega p_1 + f_4^{(\mathrm{IV};r-1,r)}\Big|_{\xi=\eta=0} = \langle f_4^{(\mathrm{IV};r-1,r)}\Big|_{\xi=\eta=0}\rangle_{q_1}\ . \tag{45}$$

By considering the Taylor-Fourier expansion

$$f_4^{(\mathrm{IV};r-1,r)}(\hat{p},\hat{q}) = \sum_{\substack{|l|=2\\ k}} c_{l,0,0,k}^{(\mathrm{IV};r-1,r)}\hat{p}^l\exp(\mathbf{i}\langle k,\,\hat{q}\rangle)\ ,$$

we obtain

$$\chi_4^{(r)}(\hat{p},\hat{q}) = \sum_{\substack{|l|=2\\ k_1\neq 0}} \frac{c_{l,0,0,k}^{(\mathrm{IV};r-1,r)}\hat{p}^l\exp(\mathbf{i}\langle k,\,\hat{q}\rangle)}{\mathbf{i}k_1\omega}\ .$$

The transformed Hamiltonian is calculated as

$$H^{(r)} = \exp\left(L_{\chi_4^{(r)}}\right) H^{(\mathrm{IV};r-1)}$$

and is given in the form (34), replacing the upper index $r-1$ by r, with

$$\begin{aligned}
f_4^{(r,r)} &= \langle f_4^{(\mathrm{IV};r-1,r)}\Big|_{\xi=\eta=0}\rangle_{q_1} + \left(f_4^{(\mathrm{IV};r-1,r)} - f_4^{(\mathrm{IV};r-1,r)}\Big|_{\xi=\eta=0}\right)\ ,\\
f_\ell^{(r,s)} &= \sum_{j=0}^{\lfloor s/r\rfloor} \frac{1}{j!} L^j_{\chi_4^{(r)}} f_{\ell-2j}^{(\mathrm{IV};r-1,s-jr)}\ . \qquad \text{for } \ell=4,\ s\neq r\ ,\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{or } \ell\neq 4,\ s\geq 0\ .
\end{aligned} \tag{46}$$

Before closing this section we think it is worth to stress the connection between Theorem 2.1 and the normal form structure. In order to simply state a theorem about the continuation of the periodic orbits, three stages of the normalization step would be enough, the last one consisting in the average of the term $f_{1,0}^{(\mathrm{II};r-1,r)}(\hat{p},\hat{q})$ only. With this minimal normal form construction, the abstract result would require assumptions on the eigenvalues of the whole matrix $N(\varepsilon)$, not only on the block $N_{11}(\varepsilon)$; indeed, three stages would not be enough to split the spectrum of the matrix $N(\varepsilon)$ in the spectrum of two diagonal blocks. As a consequence, with the purpose of getting a more accessible criterion for applications, it is necessary to perform at least a fourth stage in the normalization step, which allows to remove the term $f_{1,1}^{(\mathrm{III};r-1,r)}$, thus achieving the desired structure with null block on the anti-diagonal. Let us stress that the fourth stage does not need a second Melnikov condition. However, with four steps only, the matrix of the linearized system is not independent of time, due to lack of averaging of the terms $f_{0,2}^{(\mathrm{II};r-1,r)}(\xi,\eta)$ and $f_{2,0}^{(\mathrm{II};r-1,r)}(\hat{p},\hat{q})$, namely part of the third stage and the fifth stage. Therefore, we cannot easily deduce the structure of the matrix $M(\varepsilon)$, simply considering the linearization of the relative equilibrium for $Z^{(r)}$. Nevertheless, this allows to simplify the statement, giving a criterion on the eigenvalues of the block $N_{11}(\varepsilon)$ which can represent an easier condition to be checked in applications.

4. Analytic estimates

Our aim now is to turn the formal algorithm into a recursive scheme of estimates. Prior to describing the main results, we must anticipate some useful technical tools.

4.1. Estimates for Poisson brackets and Lie series

We report here some basic Cauchy's estimates which will be needed to bound the transformed Hamiltonian. Since similar estimates have already been presented, see, e.g. [34], we decide to not dwell on it and only include the statement of the Lemmas.

Lemma 4.1. *Let $d\in\mathbb{R}$ such that $0<d<1$ and $g\in\mathcal{P}_\ell$ be an analytic function with bounded norm $\|g\|_1$. Then one has*

$$\left\|\frac{\partial g}{\partial \hat{p}_j}\right\|_{1-d}\le\frac{\|g\|_1}{d\rho}\ ,\qquad \left\|\frac{\partial g}{\partial \hat{q}_j}\right\|_{1-d}\le\frac{\|g\|_1}{ed\sigma}\ ,\qquad \left\|\frac{\partial g}{\partial \xi_j}\right\|_{1-d}\le\frac{\|g\|_1}{dR}\ ,\qquad \left\|\frac{\partial g}{\partial \eta_j}\right\|_{1-d}\le\frac{\|g\|_1}{dR}\ ,$$

Lemma 4.2. *Let $d\in\mathbb{R}$ such that $0<d<1$ and $j\ge1$. Then one has*

$$\left\|L^j_{\chi_0^{(r)}}f\right\|_{1-d-d'}\le\frac{j!}{e}\left(\frac{e\|X_0^{(r)}\|_{1-d'}}{d^2\rho\sigma}+\frac{e|\zeta^{(r)}|}{d\rho}\right)^j\|f\|_{1-d'}\ ,\tag{47}$$

$$\left\|L^j_{\chi_1^{(r)}}f\right\|_{1-d-d'}\le\frac{j!}{e^2}\left(\frac{\|\chi_1^{(r)}\|_{1-d'}}{d^2}\left(\frac{e}{\rho\sigma}+\frac{e^2}{R^2}\right)\right)^j\|f\|_{1-d'}\ ,\tag{48}$$

$$\left\|L^j_{\chi_2^{(r)}}f\right\|_{1-d-d'}\le\frac{j!}{e^2}\left(\frac{\|\chi_2^{(r)}\|_{1-d'}}{d^2}\left(\frac{2e}{\rho\sigma}+\frac{e^2}{R^2}\right)\right)^j\|f\|_{1-d'}\ ,\tag{49}$$

$$\left\|L^j_{\chi_3^{(r)}}f\right\|_{1-d-d'}\le\frac{j!}{e^2}\left(\frac{\|\chi_3^{(r)}\|_{1-d'}}{d^2}\left(\frac{2e}{\rho\sigma}+\frac{e^2}{R^2}\right)\right)^j\|f\|_{1-d'}\ ,\tag{50}$$

$$\left\|L^j_{\chi_4^{(r)}}f\right\|_{1-d-d'}\le\frac{j!}{e^2}\left(\frac{2e\|\chi_4^{(r)}\|_{1-d'}}{d^2\rho\sigma}\right)^j\|f\|_{1-d'}\ ,\tag{51}$$

4.2. Recursive scheme of estimates

Having fixed $d \in \mathbb{R}$, $0 < d \leq 1/4$, we consider a sequence $\delta_{r\geq 1}$ of positive real numbers satisfying

$$\delta_{r+1} \leq \delta_r \ , \quad \sum_{r\geq 1} \delta_r \leq \frac{d}{5} \ , \tag{52}$$

and a further sequence $d_{r\geq 0}$ defined as

$$d_0 = 0 \ , \quad d_r = d_{r-1} + 5\delta_r \ . \tag{53}$$

This sequence allows to control the restrictions of the domain due to the Cauchy's estimate.

The factors entered by the estimate of the norm of the Poisson brackets are bounded by

$$\Xi_r = \max\left(\frac{eE}{\alpha\delta_r^2\rho\sigma} + \frac{eE}{4m\delta_r\rho^2}, 2 + \frac{eE}{\alpha\delta_r^2\rho\sigma}, \frac{E}{\alpha\delta_r^2}\left(\frac{2e}{\rho\sigma} + \frac{e^2}{R^2}\right)\right) \ , \tag{54}$$

with

$$\alpha = \min_{k_1,j,l,k} \left(|\omega|, |k_1\omega \pm \Omega_j| \, , |k_1\omega \pm \Omega_l \pm \Omega_k|\right) \ ,$$

that is strictly greater than zero in view of the the Melnikov conditions.

The number of terms in (37), (40), (42), (44) and (46) is controlled by the five sequences

$$\begin{aligned}
\nu_{0,s} &= 1 && \text{for } s \geq 0\,, \\
\nu_{r,s}^{(\mathrm{I})} &= \sum_{j=0}^{\lfloor s/r \rfloor} \nu_{r-1,r}^{j} \nu_{r-1,s-jr} && \text{for } r \geq 1\,,\ s \geq 0\,, \\
\nu_{r,s}^{(\mathrm{II})} &= \sum_{j=0}^{\lfloor s/r \rfloor} (\nu_{r,r}^{(\mathrm{I})})^j \nu_{r,s-jr}^{(\mathrm{I})} && \text{for } r \geq 1\,,\ s \geq 0\,, \\
\nu_{r,s}^{(\mathrm{III})} &= \sum_{j=0}^{\lfloor s/r \rfloor} (2\nu_{r,r}^{(\mathrm{II})})^j \nu_{r,s-jr}^{(\mathrm{II})} && \text{for } r \geq 1\,,\ s \geq 0\,. \\
\nu_{r,s}^{(\mathrm{IV})} &= \sum_{j=0}^{\lfloor s/r \rfloor} (\nu_{r,r}^{(\mathrm{III})})^j \nu_{r,s-jr}^{(\mathrm{III})} && \text{for } r \geq 1\,,\ s \geq 0\,. \\
\nu_{r,s} &= \sum_{j=0}^{\lfloor s/r \rfloor} (\nu_{r,r}^{(\mathrm{IV})})^j \nu_{r,s-jr}^{(\mathrm{IV})} && \text{for } r \geq 1\,,\ s \geq 0\,.
\end{aligned} \tag{55}$$

Again, since similar estimates has been already presented, see, e.g. [34], we only include the statement of the following Lemma.

Lemma 4.3. *The sequence of positive integers* $\{\nu_{r,s}\}_{r\geq 0\,,\,s\geq 0}$ *defined in* (55) *is bounded by*

$$\nu_{r,s} \leq \nu_{s,s} \leq \frac{2^{14s}}{2^8} \ .$$

The following Lemma collects all the key estimates concerning the generating functions and the transformed Hamiltonians. A detailed proof of the Lemma will take several pages of straightforward (and tedious) calculation. Since the key aspects have already been presented in [34] we omit the proof and leave the adaptation to the willing reader.

Lemma 4.4. *Consider a Hamiltonian $H^{(r-1)}$ expanded as in* (11)*. Let $\chi_0^{(r)}$, $\chi_1^{(r)}$, $\chi_2^{(r)}$, $\chi_3^{(r)}$ and $\chi_4^{(r)}$ be the generating functions used to put the Hamiltonian in normal form at order r, then one has*

$$\begin{aligned}
\|X_0^{(r)}\|_{1-d_{r-1}} &\leq \frac{1}{\alpha}\nu_{r-1,r}\Xi_r^{5r-5}E\varepsilon^r \ , \\
|\zeta^{(r)}| &\leq \frac{1}{4m\rho}\nu_{r-1,r}\Xi_r^{5r-3}E\varepsilon^r \ , \\
\|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r} &\leq \frac{1}{\alpha}\nu_{r,r}^{(\mathrm{I})}\Xi_r^{5r-4}\frac{E}{2}\varepsilon^r \ , \\
\|\chi_2^{(r)}\|_{1-d_{r-1}-2\delta_r} &\leq \frac{1}{\alpha}2\nu_{r,r}^{(\mathrm{II})}\Xi_r^{5r-3}\frac{E}{2^2}\varepsilon^r \ , \\
\|\chi_3^{(r)}\|_{1-d_{r-1}-3\delta_r} &\leq \frac{1}{\alpha}\nu_{r,r}^{(\mathrm{III})}\Xi_r^{5r-2}\frac{E}{2^3}\varepsilon^r \ , \\
\|\chi_4^{(r)}\|_{1-d_{r-1}-4\delta_r} &\leq \frac{1}{\alpha}\nu_{r,r}^{(\mathrm{IV})}\Xi_r^{5r-1}\frac{E}{2^4}\varepsilon^r \ .
\end{aligned}$$

The terms appearing in the expansion of $H^{(r)}$, i.e. in (17)*, are bounded as*

$$\|f_\ell^{(r,s)}\|_{1-d_r} \leq \nu_{r,s}\Xi_r^{5s}\frac{E}{2^\ell}\varepsilon^s \ . \tag{56}$$

Let us stress that the proof of the Lemma actually requires stricter estimates in (56) both for the lower order terms, as is evident from the bounds on the generating functions, and for the intermediate stages of the r-th normalization step. The interested reader can refer to [34], *mutatis mutandis*.

4.3. Estimates for the approximate periodic orbit

Lemma 4.5. *Let $x^* = (q^*, 0, 0)$ be a relative equilibrium for the truncated normal form $Z^{(r)}$ defined in* (18)*, then x^* is an approximate periodic orbit for the Hamiltonian $H^{(r)}$ of order $\mathcal{O}(\varepsilon^r)$. Precisely, there exist $\varepsilon^*(r)$ and $c_1(r)$ such that for $\varepsilon < \varepsilon^*$*

$$\|\Upsilon(x^*; \varepsilon, q_1(0))\| \leq c_1\varepsilon^{r+1} \ . \tag{57}$$

Proof. Consider the remainder $H^{(r)} - Z^{(r)}$, namely $\sum_{s>r}\sum_{\ell\geq 0} f_\ell^{(r,s)}$. Then, for ε small enough, i.e., for $\varepsilon < 1/(2^{14}\Xi_r^5)$, one has

$$\sum_{s>r}\sum_{\ell\geq 0}\|f_\ell^{(r,s)}\| \leq \sum_{s>r}\sum_{\ell\geq 0} 2^{14s}\Xi_r^{5s}\frac{E}{2^\ell}\varepsilon^s \leq 2E\frac{\left(2^{14}\Xi_r^5\varepsilon\right)^{r+1}}{1-2^{14}\Xi_r^5\varepsilon} \ ,$$

where we used the estimates in Lemma 4.4. Applying the Cauchy's estimate for the symplectic gradient and integrating over the period T, we can deduce that there exist a domain $\mathcal{U}$ and a constant $c_1(r)$, dependent on the domain, such that

$$\|\Upsilon(x^*; \varepsilon, q_1(0))\| \leq c_1(r)\varepsilon^{r+1} \ ,$$

i.e., for $\varepsilon^*(r) = 1/(2^{14}\Xi_r^5)$, one can take $c_1(r) = 4E\left(2^{14}\Xi_r^5\right)^{r+1}$. □

5. The *seagull* example

In this Section we study in detail the pedagogical example (4) already presented in the Introduction, focusing on the continuation of degenerate periodic orbits via the normal form construction described in the paper. Although the present example does not represent a dNLS lattice in the proper sense due to the limited number of sites, it is nevertheless suitable to see the benefits of our normal form construction in a direct and easy way, thanks to the simplicity of the change of coordinates between Cartesian and action-angles. Furthermore, it sheds some light onto the role of the nonlinearity in the linear stability of multi-peaked discrete solitons in dNLS lattices. Indeed, at variance with models considered in literature, we notice that a change in the sign of the nonlinear parameter γ does not influence the nature of the degenerate eigenspaces. On the contrary, considering consecutive excited sites as in [31], a change in the sign of γ (at fixed linear interaction ε) produces an exchange of stable and unstable degenerate directions around the periodic solutions; actually the switch from saddle to center depends on the sign of the product $\gamma\varepsilon$.

Let us remark that, in order to apply Theorem 2.1, we have to control the smallest eigenvalue of the matrix $M(\varepsilon)$, see (27). This is a delicate point, particularly in actual applications. Indeed, to numerically verify this assumption one has to investigate the spectrum of the matrix $(\exp(L_{11}(\varepsilon)T) - \mathrm{Id})_\flat$, by interpolating the decay of the smallest eigenvalue with respect to ε.

However, in some specific cases like the example here considered, one can further decompose the quadratic Hamiltonian $Z_2^{(r)}$ in (29) in order to decouple the fast variables (Q_1, P_1) from the slow variables (Q, P) with a linear canonical change of coordinates (see [41]).

Precisely, we decompose the matrix C so as to put in evidence the first row and column vectors, namely

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where C_{11} is the first element, $C_{12} = C_{21}^\top$ is the $(n_1 - 1)$-dimensional row vector and C_{22} is the $(n_1 - 1)$-dimensional square matrix.

Assume now that C_{22} is invertible, then we can introduce the canonical change of coordinates

$$u = Q \ , \quad u_1 = Q_1 - C_{12}C_{22}^{-1}Q \ , \quad P = v - v_1 C_{22}^{-1} C_{21} \ , \quad P_1 = v_1 \ . \tag{58}$$

The transformed quadratic Hamiltonian $Z_2^{(r)}$ in (29) now reads

$$Z_2^{(r)} = \frac{1}{2}c_{11}v_1^2 + \frac{1}{2}[\langle Bu, u\rangle + \langle C_{22}v, v\rangle] + \langle Du, v\rangle + \frac{1}{2}\langle G\xi, \xi\rangle + \langle E\xi, \eta\rangle + \frac{1}{2}\langle F\eta, \eta\rangle \ , \tag{59}$$

where $c_{11} = C_{11} - C_{12}C_{22}^{-1}C_{21}$ and the term $\langle Du, v\rangle$ contains mixed terms in action-angles variables. The main advantage is that, if $D = 0$, then the fast dynamics and the slow one turn out to be decoupled, hence it suffices to investigate the eigenvalues of the matrix

$$\begin{pmatrix} 0 & C_{22} \\ -B & 0 \end{pmatrix} \tag{60}$$

which represents the linear vector fields of the new slow variables (Q, P). Hence (27) can be easily checked, possibly without the needs of numerical interpolation.

Consider the Hamiltonian system (4), that we report here for convenience

$$H = H_0 + \varepsilon H_1 = \sum_{j=-3}^{3} \left(\frac{x_j^2 + y_j^2}{2} + \gamma \left(\frac{x_j^2 + y_j^2}{2} \right)^2 \right) + \varepsilon \sum_{j=-3}^{2} (x_{j+1}x_j + y_{j+1}y_j) \ ,$$

with fixed boundary conditions $x_{-3} = y_{-3} = x_3 = y_3 = 0$.

Following the procedure reported in the Introduction, we introduce action-angle variables $(x_j, y_j) = (\sqrt{2I_j}\cos\varphi_j, -\sqrt{2I_j}\sin\varphi_j)$, for the set of indices $j \in \mathcal{I} = \{-2, -1, 1, 2\}$, and complex canonical coordinates for the remaining central one (x_0, y_0). Then we focus on the 4-dimensional resonant torus $I = I^*$ with $I_j^* = I_l^*$, for $j, l \in \mathcal{I}$, and $\xi_0 = \eta_0 = 0$. Finally, introducing the canonical change of coordinates (5) we get

$$\begin{aligned} H^{(0)} =& \omega p_1 + \mathbf{i}\xi_0\eta_0 + f_4^{(0,0)} \\ &+ f_0^{(0,1)} + f_1^{(0,1)} + f_2^{(0,1)} + f_3^{(0,1)} + f_4^{(0,1)} + \sum_{\ell>4} f_\ell^{(0,1)} + \mathcal{O}(\varepsilon^2) \ , \end{aligned}$$

with $\omega = 1 + 2\gamma I^*$ and

$$
\begin{aligned}
f_4^{(0,0)}(\hat{p}, \xi_0, \eta_0) &= \gamma\left((p_1-p_2)^2 + (p_2-p_3)^2 + (p_3-p_4)^2 + p_4^2\right) - \gamma\xi_0^2\eta_0^2 \ , \\
f_0^{(0,1)}(q_2, q_4) &= \varepsilon\left(2I^*\cos(q_2) + 2I^*\cos(q_4)\right) \ , \\
f_1^{(0,1)}(\hat{q}, \xi_0, \eta_0) &= \varepsilon\bigg((\xi_0 + \mathbf{i}\eta_0)\sqrt{I^*}\cos(q_1+q_2) + (\xi_0 + \mathbf{i}\eta_0)\sqrt{I^*}\cos(q_1+q_2+q_3) + \\
&\qquad - \mathbf{i}(\xi_0 - \mathbf{i}\eta_0)\sqrt{I^*}\sin(q_1+q_2) - \mathbf{i}(\xi_0 - \mathbf{i}\eta_0)\sqrt{I^*}\sin(q_1+q_2+q_3)\bigg) \ , \\
f_2^{(0,1)}(\hat{p}, q) &= \varepsilon\left((p_1-p_3)\cos(q_2) + p_3\cos(q_4)\right) \ , \\
f_3^{(0,1)}(p, \hat{q}, \xi_0, \eta_0) &= \varepsilon\bigg(\frac{(p_2-p_3)(\xi_0+\mathbf{i}\eta_0)\cos(q_1+q_2)}{2\sqrt{I^*}} - \frac{\mathbf{i}(p_2-p_3)(\xi_0-\mathbf{i}\eta_0)\sin(q_1+q_2)}{2\sqrt{I^*}} + \\
&\qquad + \frac{(p_3-p_4)(\xi_0+\mathbf{i}\eta_0)\cos(q_1+q_2+q_3)}{2\sqrt{I^*}} - \frac{\mathbf{i}(p_3-p_4)(\xi_0-\mathbf{i}\eta_0)\sin(q_1+q_2+q_3)}{2\sqrt{I^*}}\bigg) \ , \\
f_4^{(0,1)}(p, q) &= \varepsilon\bigg(-\frac{\left((p_1-p_2)^2 + (p_2-p_3)^2\right)\cos(q_2)}{4I^*} + \frac{(p_1-p_2)(p_2-p_3)\cos(q_2)}{2I^*} + \\
&\qquad - \frac{\left((p_3-p_4)^2 + p_4^2\right)\cos(q_4)}{4I^*} + \frac{p_4(p_3-p_4)\cos(q_4)}{2I^*}\bigg) \ .
\end{aligned}
$$

The transformed Hamiltonian, $H^{(0)}$, is now in a suitable form for applying our normal form procedure. As already said in the Introduction, in order to remove the degeneracy, we need to compute the normal form up to order two. Hence, in the following, we detail all the needed normal form steps.

Since $f_0^{(0,1)}$ does not depend on q_1, the first stage of the first normalization step only consists in the translation of the actions $\hat{p}$ which allows to keep the frequency fixed. Moreover, as $f_2^{(0,1)}$ does not depend on the fast angle q_1, the equation for $\zeta^{(1)}$ reads

$$
\sum_j C_{0,i,j}\zeta_j^{(1)} = \frac{\partial}{\partial \hat{p}_i} f_2^{(0,1)}\Big|_{q=q^*} \ ,
$$

with

$$
C_0 = \gamma\begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 4 \end{pmatrix}
$$

and the solution is

$$
\zeta^{(1)} = \left(\frac{\varepsilon}{\gamma}\left(\cos(q_2^*) + \cos(q_4^*)\right) \ , \ \frac{\varepsilon}{\gamma}\left(\frac{\cos(q_2^*)}{2} + \cos(q_4^*)\right) \ , \ \frac{\varepsilon}{\gamma}\cos(q_4^*) \ , \ \frac{\varepsilon}{2\gamma}\cos(q_4^*)\right) \ . \tag{61}
$$

The generating function $\chi_1^{(1)}$ that kills the term $f_1^{(\mathrm{I};0,1)} = f_1^{(0,1)}$ reads

$$
\chi_1^{(1)} = \varepsilon\left(\mathbf{i}\frac{\sqrt{I^*}\left(e^{-\mathbf{i}(q_1+q_2)} + e^{-\mathbf{i}(q_1+q_2+q_3)}\right)\xi_0}{\omega - 1} + \frac{\sqrt{I^*}\left(e^{\mathbf{i}(q_1+q_2)} + e^{\mathbf{i}(q_1+q_2+q_3)}\right)\eta_0}{\omega - 1}\right) \ .
$$

Since $f_2^{(\mathrm{II};0,1)} = f_2^{(0,1)} + L_{\langle\zeta^{(1)},\hat{q}\rangle} f_4^{(0,0)}$ is again independent of q_1, no further average is required in the third stage of the normalization procedure.

Next, we have to remove the cubic terms which depend both on the actions and on the transverse variables from $f_3^{(\mathrm{III},0,1)} = f_3^{(0,1)} + L_{\chi_1^{(1)}} f_4^{(0,0)}$. This is done via the generating function

$$\begin{aligned}\chi_3^{(1)} =& -\varepsilon e^{-\mathbf{i}(q_1+q_2+q_3)} \frac{\left(-1+e^{\mathbf{i}q_3}\right) p_3 \left(e^{\mathbf{i}(2q_1+2q_2+q_3)}\eta_0 - \mathbf{i}\xi_0\right) + e^{\mathbf{i}q_3} p_2 \left(e^{2\mathbf{i}(q_1+q_2)}\eta_0 + \mathbf{i}\xi_0\right)}{2\sqrt{I^*}\,(\omega-1)} \\ &+ \varepsilon e^{-\mathbf{i}(q_1+q_2+q_3)} \frac{p_4 \left(e^{2\mathbf{i}(q_1+q_2+q_3)}\eta_0 + \mathbf{i}\xi_0\right)}{2\sqrt{I^*}\,(\omega-1)} \ .\end{aligned}$$

Finally, the term $f_4^{(\mathrm{IV},0,1)} = f_4^{(0,1)}$ turns out to be independent of q_1, thus the first normalization step is concluded.

As already noticed, the solution of $\nabla_q f_0^{(1,1)} = 0$, that determines the q^*, and so the approximate periodic orbits, is given by the four one-parameter families $Q_1 = (0,\vartheta,0)$, $Q_2 = (0,\vartheta,\pi)$, $Q_3 = (\pi,\vartheta,0)$, $Q_4 = (\pi,\vartheta,\pi)$, with $\vartheta \in S^1$. Thus, a further normalization step is needed in order to investigate the continuation of these families of periodic orbits. The transformed Hamiltonian at order two reads

$$\begin{aligned}H^{(2)} =& \omega p_1 + \mathbf{i}\xi_0\eta_0 + f_4^{(2,0)} \\ &+ f_0^{(2,1)} + f_2^{(2,1)} + f_4^{(2,1)} + \sum_{\ell>4} f_\ell^{(2,1)} \\ &+ f_0^{(2,2)} + f_2^{(2,2)} + f_4^{(2,2)} + \sum_{\ell>4} f_\ell^{(2,2)} + \mathcal{O}(\varepsilon^3) \ .\end{aligned}$$

The approximate periodic orbits are the solutions q^* of

$$\nabla_q f_0^{(2,1)}(q) + \nabla_q f_0^{(2,2)}(q) = 0 \ , \tag{62}$$

with, neglecting the constant terms,

$$f_0^{(2,2)}(q) = L_{\langle \zeta^{(1)},\hat{q}\rangle} f_2^{(0,1)} + \frac{1}{2} L_{\chi_1^{(1)}} f_1^{(0,1)} = \frac{\varepsilon^2}{\gamma}[\cos(q_3) - \cos(q_2)\cos(q_2^*) - \cos(q_4)\cos(q_4^*)] \ .$$

The system (62) can be written as

$$F(q,\varepsilon) = F_0(q) + \varepsilon F_1(q) = 0 \tag{63}$$

where $F : \mathbb{T}^3 \times \mathcal{U}(0) \to \mathbb{R}^3$ and $F_0(Q_j(\vartheta)) = 0$. We now introduce the matrices $\tilde{B}_{0,j}(\vartheta) = \frac{\partial F_0(Q_j(\vartheta))}{\partial q}$ and remark that the tangent direction to the four families $\partial_\vartheta Q_j = (0,1,0)$ is the Kernel direction of $\tilde{B}_{0,j}(\vartheta)$, for $j = 1,\ldots,4$. By computing

$$\langle F_1(Q_j(\vartheta,0)), \partial_\theta Q_j\rangle = -\frac{\sin(\vartheta)}{\gamma} \qquad j = 1,\ldots,4 \ ,$$

it is possible to deduce from standard bifurcation arguments (see [36, 34]) that, apart from the *in* and *out-of-phase* configurations $(0,0,0)$, $(0,\pi,0)$, $(0,0,\pi)$, $(0,\pi,\pi)$, $(\pi,0,0)$, $(\pi,\pi,0)$, $(\pi,0,\pi)$ and (π,π,π), the four families break down. In order to ensure the continuation of these configurations, we have to verify the condition (27), with (26) taking the form

$$\|\Upsilon(x^*;\varepsilon,q_1(0))\| \le c_1\varepsilon^3 \ .$$

In order to verify (27) we have two options: (i) examine the spectrum of $(\exp(L_{11}(\varepsilon)T) - \mathrm{Id})_\flat$ and numerically interpolate the smallest eigenvalue, getting $|\lambda| \gtrsim \varepsilon$ for each of the eight configuration; (ii) since $q_j = q^* = \{0,\pi\}$, the mixed terms in action-angle variables are missing and we can compute $e^{T\sigma_l} - 1$, with σ_l eigenvalues of the matrix

$$\begin{pmatrix} 0 & C_{22} \\ -B & 0 \end{pmatrix} \ ,$$

with

$$B = \begin{pmatrix} -2\varepsilon I^* \cos(q_2) + \frac{\varepsilon^2}{\gamma} \cos^2(q_2) & 0 & 0 \\ 0 & -\frac{\varepsilon^2}{\gamma} \cos(q_3) & 0 \\ 0 & 0 & -2\varepsilon I^* \cos(q_4) + \frac{\varepsilon^2}{\gamma} \cos^2(q_4) \end{pmatrix} . \tag{64}$$

Hence, C_{22} being definite and of order $\mathcal{O}(1)$ in the limit of small ε, we obtain that condition (27) is plainly verified with $\alpha = 1$ and $r = 2$. Applying the Theorem 2.1 we can infer the existence of a unique $x^*_{\text{p.o.}}(\varepsilon) = (q^*_{\text{p.o.}}(\varepsilon), \hat{p}_{\text{p.o.}}(\varepsilon), \xi_{\text{p.o.}}(\varepsilon), \eta_{\text{p.o.}}(\varepsilon))$, with $q^*_{\text{p.o.}}(\varepsilon) = q^* = \{0, \pi\}$ such that $\left\|x^*_{\text{p.o.}} - x^*\right\| \le c_0\varepsilon^2$, for each candidate for the continuation.

Coming to the linear stability, first we exploit the structure of the quadratic Hamiltonian (59), with $D \equiv 0$, in order to get the approximate linear stability of the continued periodic orbits. It turns out that the stable and unstable directions correspond to the positive or negative eigenvalues of $\gamma C_{22} B$, where the prefactor γ accounts for the positive or negative signature of C_{22} (which is the same of C). Hence, the stability depends on the signature of γB, given by the elements on its diagonal

$$\gamma B = \begin{pmatrix} -2\varepsilon\gamma I^* \cos(q_2) + \varepsilon^2 \cos^2(q_2) & 0 & 0 \\ 0 & -\varepsilon^2 \cos(q_3) & 0 \\ 0 & 0 & -2\varepsilon\gamma I^* \cos(q_4) + \varepsilon^2 \cos^2(q_4) \end{pmatrix} . \tag{65}$$

The degenerate direction depends only on ε^2, thus it is always a saddle at $q_3 = 0$ and always a center at $q_3 = \pi$. Instead, the nondegenerate directions depend on the sign of the product $\gamma\varepsilon$, which converts hyperbolic subspace into center subspace at fixed $q_{2,4} \in \{0, \pi\}$. In particular, by studying the spectrum in the in/out-of-phase configurations and for attractive interactions ($\varepsilon > 0$), we find that the only stable approximate periodic orbit corresponds to the configuration (π, π, π) when $\gamma > 0$ and to $(0, \pi, 0)$ when $\gamma < 0$. In order to derive the effective linear stability, we have to verify (33). Symbolic calculations implemented in Mathematica give

$$\begin{aligned}
\lambda_{1,2} &= \pm\mathbf{i}\left(2\sqrt{2}\sqrt{I^*|\gamma|}\sqrt{\varepsilon} + \frac{\sqrt{2}\varepsilon^{3/2}}{\sqrt{I^*|\gamma|}} + h.o.t.\right) \\
\lambda_{3,4} &= \pm\mathbf{i}\left(2\sqrt{2}\sqrt{I^*|\gamma|}\sqrt{\varepsilon} + \frac{5\varepsilon^{3/2}}{2\sqrt{2}\sqrt{I^*|\gamma|}} + h.o.t.\right) \\
\lambda_{5,6} &= \pm\mathbf{i}\left(\sqrt{2}\varepsilon - \frac{\varepsilon^2}{4\sqrt{2}I^*|\gamma|} + h.o.t.\right) ,
\end{aligned}$$

thus condition (33) holds true with $r + 1 - \alpha = 2$ and $\beta = \frac{3}{2}$.

A. Spectrum deformation under matrix perturbations

In this section we collect some useful results concerning the deformation of the spectrum of a matrix under small perturbations. The results are based on resolvent formalism, we refer to the classical book of Kato [18] for a detailed treatment of the subject and to [26, 1] for the study of the linear stability of breathers and multibreathers.

We first set some notations. Given a matrix M defined on a vector space X, we denote by $\Sigma(M)$ its spectrum and by $\rho(M) = \max_{\lambda \in \Sigma(M)} |\lambda_j|$ its spectral radius. For any $z \notin \Sigma(M)$ it is well defined $R(z) = (M - z)^{-1}$, which is the *resolvent* of M. The inverse of the spectral radius of $R(z)$ measures the distance between $z \in \mathbb{C}$ and the spectrum of M

$$\text{dist}\,(z, \Sigma(M)) = \frac{1}{\rho(R(z))} . \tag{A.1}$$

Let us recall that $\rho(M)$ and the operatorial norm $\|M\|_{\mathsf{op}} = \sup_{x\neq 0} \frac{\|Mx\|}{\|x\|}$ satisfy

$$\rho(M) \leq \|M\|_{\mathsf{op}} \ . \tag{A.2}$$

Moreover, a converse inequality is given by the following

Lemma A.1. *Let $M : X \to X$, then there exists $c_{op} > 1$, depending on the dimension of X only, such that*

$$\|M\|_{op} \leq c_{op} \max\{\rho(M), 1\} \ . \tag{A.3}$$

If M is diagonalizable, or if $\rho(M) \geq 1$, then the above simplifies to

$$\|M\|_{op} \leq c_{op}\rho(M) \ . \tag{A.4}$$

A.0.1. On the minimum eigenvalue

Let us now consider a given matrix N and its perturbation $M(\mu) = N + \mu P$ depending analytically on a small parameter μ. The resolvent $R(z, \mu)$ of $M(\mu)$ is still well defined and holomorphic in the two variables, provided $z \notin \Sigma(M(\mu))$; moreover, it is possible to relate the perturbed and unperturbed resolvents via the following series expansion

$$R(z, \mu) = R_0(z)[\mathrm{Id} + A(z, \mu)R_0(z)]^{-1} \ , \quad R_0(z) = R(z, 0) \ , \tag{A.5}$$

where $R_0(z)$ is the resolvent of the leading term $N = M(0)$, while $A(z, \mu) = R(z, \mu) - R_0(z)$ represents the *deformation* due to the small perturbation. In what follows we collect some useful results relating the *unperturbed spectrum* $\Sigma(N)$ to the *perturbed spectrum* $\Sigma(M(\mu))$.

We collect the results relating the minimum eigenvalue of N and M in the following

Proposition A.1. *Let us consider a matrix $M(\varepsilon) = N(\varepsilon) + \mu(\varepsilon)P(\varepsilon)$, depending on the small parameter $\varepsilon \in \mathcal{U}(0)$. Let us assume that for any $\varepsilon \in \mathcal{U}$ it holds true:*

(1) $N(\varepsilon)$ is invertible and there exist $c_1 > 0$ and $\alpha > 0$ independent of ε such that

$$\left\|N^{-1}\right\|_{op} \leq c_1|\varepsilon|^{-\alpha} \ ;$$

(2) $P(\varepsilon) = P(0) + \mathcal{O}(\varepsilon)$ and there exist $c_2 > 0$ and $\beta > \alpha$ such that

$$|\mu(\varepsilon)| \leq c_2|\varepsilon|^{\beta} \ .$$

Then for ε small enough $M(\varepsilon)$ is invertible and there exists $c_3 > 0$ independent of ε such that

$$|\nu| \geq c_3|\varepsilon|^{\alpha} \ , \qquad \textit{for all} \quad \nu \in \Sigma(M). \tag{A.6}$$

Moreover, the same result holds true if we replace the first assumption with

$$\min_{\lambda\in\Sigma(N(\varepsilon))} \{|\lambda|\} \geq c_1|\varepsilon|^{\alpha} \ . \tag{A.7}$$

Proof. Due to the invertibility of N, we can rewrite M as $M = N(\mathrm{Id} + \mu T)$, with $T = N^{-1}P$. We now show that $(\mathrm{Id} + \mu T)^{-1}$ is well defined, thus the inverse of M is given by $M^{-1} = (\mathrm{Id} + \mu T)^{-1} N^{-1}$. The operator $(\mathrm{Id} + \mu T)$ being a small perturbation of the identity Id, its inverse exists provided $|\mu| \|T\|_{\mathsf{op}} < 1$, hence hypothesis *(1)* implies

$$|\mu| \|T\|_{\mathsf{op}} \leq |\mu| \left\|N^{-1}\right\|_{\mathsf{op}} \|P\|_{\mathsf{op}} \leq c|\varepsilon|^{\beta-\alpha} < 1 \ , \tag{A.8}$$

where $c \approx c_2 c_1 \|P(0)\|_{\mathsf{op}}$ for sufficiently small ε and in such a regime is independent of ε; as a consequence, since $\beta - \alpha > 0$, also $M(\varepsilon)$ is invertible for ε small enough. From the estimate

$$\left\|(\mathrm{Id} + \mu T)^{-1}\right\|_{\mathsf{op}} \leq \sum_{n\geq 0} |\mu|^n \|T\|_{\mathsf{op}}^n \leq \frac{1}{1 - |\mu| \|T\|_{\mathsf{op}}} ,$$

and recalling (A.2) we get

$$\rho(M^{-1}) \leq \left\|M^{-1}\right\|_{\mathsf{op}} \leq \frac{\left\|N^{-1}\right\|_{\mathsf{op}}}{1 - |\mu| \|T\|_{\mathsf{op}}} ,$$

where the spectral radius $\rho(M^{-1}) = \frac{1}{\min\{|\nu_k|\}}$, with $\nu_k \in \Sigma(M)$. Let ν_1 be the minimum, then

$$|\nu_1| \geq \frac{1 - |\mu| \|T\|_{\mathsf{op}}}{\|N^{-1}\|_{\mathsf{op}}} .$$

For ε sufficiently small, condition (A.8) holds and we have

$$|\mu| \|T\|_{\mathsf{op}} \leq \frac{1}{2} , \qquad \text{that implies} \quad |\nu_1| \geq c_3 |\varepsilon|^{\alpha} , \quad \text{with} \quad c_3 = \frac{1}{2c_1} .$$

It is clear that the main point in the proof is the bound of $\left\|N^{-1}\right\|_{\mathsf{op}}$. In fact, the estimate (A.8) can be obtained also replacing the assumption on $\|N\|_{\mathsf{op}}$ with (A.7). Indeed by means of (A.3) one can obtain the following estimate

$$\left\|N^{-1}\right\|_{\mathsf{op}} \leq c_{\mathsf{op}} \max\left\{1, \rho(N^{-1})\right\} = c_{\mathsf{op}} \max\left\{1, \frac{1}{\min_{\lambda\in\Sigma(N(\varepsilon))}\{|\lambda|\}}\right\} \leq c_{\mathsf{op}} c_1^{-1} |\varepsilon|^{-\alpha} ,$$

since $c_1^{-1}|\varepsilon|^{-\alpha} \gg 1$, which allows to conclude

$$|\nu_1| \geq c_3 |\varepsilon|^{\alpha} , \qquad \text{with} \quad c_3 = \frac{1}{2} c_{\mathsf{op}}^{-1} c_1 .$$

□

A.0.2. Deformation of eigenvalues.

We aim at localizing the eigenvalues of $M(\varepsilon) = N(\varepsilon) + \mu(\varepsilon)P(\varepsilon)$, when the spectrum of its leading part $N(\varepsilon)$ is known, provided ε is taken small enough in a small neighbourhood of the origin $\mathcal{U}(0)$. We start with a preliminary Lemma

Lemma A.2. *Let $z \notin \Sigma(N)$ be a complex number satisfying*

$$\mathrm{dist}(z, \Sigma(N)) \geq 4\mu c_{op} \|P\|_{op} . \tag{A.9}$$

Then the following inequality holds true

$$\frac{1}{c_{op}} \mathrm{dist}(z, \Sigma(N)) - \mu \|P\|_{op} \leq \mathrm{dist}(z, \Sigma(M)) \leq c_{op} \mathrm{dist}(z, \Sigma(N)) + \mu c_{op} \|P\|_{op} . \tag{A.10}$$

Proof. To shorten the proof, let us introduce the notations

$$\delta_N(z) = \mathrm{dist}(z, \Sigma(N)) , \qquad \delta_M(z) = \mathrm{dist}(z, \Sigma(M)) .$$

By setting $R_0(z)$ the resolvent of N, from (A.1) we have

$$\rho(R_0(z)) = \frac{1}{\delta_N} .$$

In the rest of the proof we aim at deriving bounds for δ_M by exploiting the perturbation of R_0 given by μP. Thus, let us set R the resolvent of M and recall that

$$\Sigma(R(z)) = \left\{ \frac{1}{\lambda_j - z} \right\}_{\lambda_j \in \Sigma(M)} ;$$

from (A.2) and (A.4) we have

$$\frac{1}{\delta_N} = \rho(R_0(z)) \leq \|R_0(z)\|_{\mathsf{op}} \leq c_{\mathsf{op}} \rho(R_0(z)) = \frac{c_{\mathsf{op}}}{\delta_N} .$$

From the second Neumann series (A.5) we can write

$$R(z) = R_0(z)[\mathrm{Id} + \mu P R_0]^{-1} , \tag{A.11}$$

where, due to (A.9), the product $\mu P R_0$ represents a perturbation of the identity

$$\|\mu P R_0\|_{\mathsf{op}} \leq \mu \|P\|_{\mathsf{op}} \|R_0\|_{\mathsf{op}} \leq \frac{c_{\mathsf{op}} \mu \|P\|_{\mathsf{op}}}{\delta_N} \leq \frac{1}{4} , \tag{A.12}$$

so that $[\mathrm{Id} + \mu P R_0]^{-1}$ is well defined in terms of power series of $\mu P R_0$. As a consequence, from the estimate of $\left\| \sum_{k \geq 0} (-1)^k \mu^k (P R_0)^k \right\|_{\mathsf{op}}$, we get

$$\delta_M(z) = \frac{1}{\rho(R(z))} \geq \frac{1}{\|R\|_{\mathsf{op}}} \geq \frac{1}{\|R_0\|_{\mathsf{op}}} - \mu \|P\|_{\mathsf{op}} \geq \frac{\delta_N(z)}{c_{\mathsf{op}}} - \mu \|P\|_{\mathsf{op}} ,$$

which gives the lower bound in (A.10). Let us consider again (A.5) but reversing the roles of R and R_0

$$R_0(z) = R(z)[\mathrm{Id} - \mu P R]^{-1} ; \tag{A.13}$$

from (A.11) and (A.12) we have

$$\|R\|_{\mathsf{op}} \leq \frac{4}{3} \|R_0\|_{\mathsf{op}} \qquad \text{that implies} \qquad \|\mu P R\|_{\mathsf{op}} \leq \frac{1}{3} ,$$

which provides the upper bound in (A.10)

$$\delta_N(z) = \frac{1}{\rho(R_0(z))} \geq \frac{1}{\|R_0\|_{\mathsf{op}}} \geq \frac{3/2}{\|R\|_{\mathsf{op}}} - \mu \|P\|_{\mathsf{op}} \geq \frac{3}{2c_{\mathsf{op}}} \delta_M(z) - \mu \|P\|_{\mathsf{op}} .$$

□

The localization result is collected in following

Proposition A.2. *Let $M(\varepsilon) \in \mathsf{Mat}(n)$ be decomposed into $M(\varepsilon) = N(\varepsilon) + \mu(\varepsilon) P(\varepsilon)$. Assume that:*

(1) $P(\varepsilon) = P(0) + \mathcal{O}(\varepsilon)$ with $\|P(0)\|_{op} \leq c_P$ and there exists $\beta_1 > 0$ such that

$$|\mu(\varepsilon)| \leq |\varepsilon|^{\beta_1} ;$$

(2) there exists a $c_N > 0$ such that for any couple of distinct eigenvalues $\lambda_i \neq \lambda_j \in \Sigma(N)$

$$|\lambda_i - \lambda_j| \geq c_N \varepsilon^{\beta_2} , \qquad \text{with} \quad \beta_2 < \beta_1 .$$

Then there exists $\varepsilon^* > 0$ (depending on $\Sigma(N)$) such that, given $|\varepsilon| < \varepsilon^*$, for any $\lambda \in \Sigma(N)$ there exist one eigenvalue $\nu \in \Sigma(M)$ inside the disk $D_\varepsilon(\lambda) = \left\{ z \in \mathbb{C} \ : \ |z - \lambda| < c_M |\varepsilon|^{\beta_1} \right\}$, with $c_M > 0$ a suitable constant independent of λ.

Proof. Take an arbitrary eigenvalue $\lambda \in \Sigma(N)$ and consider a complex number $z \in \mathbb{C}$ at distance $\tilde{\delta} = c|\varepsilon|^{\beta_1}$, with c independent of ε to be determined along the proof. In view of *(2)* and for ε small enough, one has that $c_N|\varepsilon|^{\beta_2} \gg c_P|\varepsilon|^{\beta_1}$; hence by defining $\delta_N(z) = \mathrm{dist}(z, \Sigma(N))$ it turns out

$$\delta_N(z) = \tilde{\delta} = c|\varepsilon|^{\beta_1} \ .$$

We want to use upper bound of (A.10) to control $\delta_M(z) = \mathrm{dist}\,(z, \Sigma(M))$; to fulfil the requirements of Lemma A.2 we take $c \geq 4c_{\mathsf{op}}c_P$ and we exploit *(1)*, so that

$$\delta_M(z) \leq c_{\mathsf{op}}(c + c_P)|\varepsilon|^{\beta_1} \ . \tag{A.14}$$

This ensures the existence of an eigenvalue $\nu \in \Sigma(M)$, which depends on the choice of z, whose distance from the initially chosen z is of order $\mathcal{O}(\varepsilon^{\beta_1})$

$$\exists \nu \in \Sigma(M) \ : \ |\nu - z| \leq c_{\mathsf{op}}(c + c_P)|\varepsilon|^{\beta_1} \ ,$$

which provides the final estimate

$$|\nu - \lambda| \leq |\nu - z| + |z - \lambda| \leq c_M|\varepsilon|^{\beta_1} \ , \qquad \text{with} \quad c_M = c + c_{\mathsf{op}}(c + c_P) \ .$$

□

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